Additional Notes and Solution Manual For: Multigrid Methods by William L. Briggs

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1 Addendums/Clarifications/Derivations

1.1 Assembly of the linear system in two dimensions (Page 3)

Our second order finite difference equation for the two dimensional Poisson's equation is given by

$$\frac{-v_{i-1,j} + 2v_{i,j} - v_{i+1,j}}{h_x^2} + \frac{-v_{i,j-1} + 2v_{i,j} - v_{i,j+1}}{h_y^2} = f_{i,j} \,,$$

with boundary conditions given by $v_{i,j} = 0$ if $i = 0$ or $i = M$ or $j = 0$ or $j = N$. Here we will assume that $v_{i,j}$ is represented in a standard "matrix" notation, with i the row index and j the column index. Focusing our attention on operations that first operate on a fixed *row* or i index we we have have the following rearraingment of the above, where we first consider the index i with no increment or decrement, then the index i decremented by one, and finally the index i incremented by one, giving

$$-\frac{v_{i,j-1}}{h_y^2} + \left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right) v_{i,j} - \frac{v_{i,j+1}}{h_y^2} - \frac{v_{i-1,j}}{h_x^2} - \frac{v_{i+1,j}}{h_x^2} = f_{i,j} \,.$$

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Now focusing on the ith row we can write the difference equation for this row *only* as follows

$$
\begin{pmatrix}
2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 & \cdots & 0 \\
\frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & \cdots & 0 \\
0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-1}{h_y^2}
\end{pmatrix}
\begin{pmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \vdots \\ v_{i,N-1} \end{pmatrix}
$$

$$
+\begin{pmatrix}
-\frac{1}{h_x^2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{h_x^2} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h_x^2} & 0 & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
0 & 0 & 0 & 0 & -\frac{1}{h_x^2}
\end{pmatrix}
\begin{pmatrix} v_{i-1,1} \\ v_{i-1,2} \\ v_{i-1,3} \\ \vdots \\ v_{i-1,N-1} \end{pmatrix}
$$

$$
+\begin{pmatrix}
-\frac{1}{h_x^2} & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{h_x^2} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{h_x^2} & 0 & 0 \\
\vdots & \vdots & & \ddots & \vdots \\
0 & 0 & 0 & 0 & -\frac{1}{h_x^2}
\end{pmatrix}
\begin{pmatrix} v_{i+1,1} \\ v_{i+1,2} \\ v_{i+1,3} \\ \vdots \\ v_{i+1,N-1} \end{pmatrix}
$$

$$
=\begin{pmatrix} f_{i,1} \\ f_{i,2} \\ f_{i,3} \\ \vdots \\ f_{i,N-1} \end{pmatrix} \quad \text{for} \quad i = 1, 2, \ldots, M-1\,.
$$

Now the first matrix above is a tridiagonal matrix of size $N-1$ by $N-1$ with entries of $\frac{1}{h_x^2}$ on sub and superdiagonal and elements

$$
2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right),
$$

on the diagonal. Now defining the vector's $\mathbf{v_i}$ to be all of the unknowns in the i row, i.e. $v_{i,1}, v_{i,2}, \ldots v_{i,N-1}$, (similarly for $\mathbf{f_i}$), we can write the above equations in a block matrix form as

$$
\begin{pmatrix}
A & \tilde{I} & & & \\
\tilde{I} & A & \tilde{I} & & \\
 & & \ddots & & \\
 & & & A & \\
 & & & \tilde{I} & A
\end{pmatrix}
\begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \vdots \\ \mathbf{v_{M-1}} \end{pmatrix}
=
\begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \\ \vdots \\ \mathbf{f_{M-1}} \end{pmatrix}.
$$

which provides the global matrix structure for *all* of the unknowns. In the above we have defined the matrix $\tilde{I}$ as

$$\tilde{I} = -\frac{1}{h_x^2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the matrix A as

$$A = \begin{pmatrix} 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 & \cdots & 0 \\ \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & \cdots & 0 \\ 0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} \end{pmatrix}.$$

1.2 The Weighted Jacobi Method (Page 10)

Using an intermediate state v_j^* given by the direct Jacobi method i.e.

$$v_j^* = \frac{1}{2}(v_{j-1}^{(0)} + v_{j+1}^{(0)} + h^2 f_j)\,,$$

we computing the true update $v_j^{(1)}$ as a weighted update of the original unknown $v_j^{(0)}$ and the intermediate state v_j^* we have

$$v_j^{(1)} = (1-\omega)v_j^{(0)} + \omega v_j^* = (1-\omega)v_j^{(0)} + \frac{\omega}{2}\left(v_{j-1}^{(0)} + v_{j+1}^{(0)} + h^2 f_j\right)$$

When written in matrix form (by remembering the definition of D, L, and U for the test problem considered here) we have

$$v^{(1)} = \left((1-\omega)I + \omega D^{-1}(L+U)\right) v^{(0)} + \omega D^{-1} f\,,$$

which is the matrix update formula for the weighted Jacobian iteration scheme.

1.3 The Weighted Jacobi Method Applied to A (Page 19)

From the manipulations in the book, the iteration matrix for the weighted Jacobi method is given by

$$P_\omega = (1-\omega)I + \omega D^{-1}(L+U)$$

Problem Solutions

Problem 1.1 (a problem with periodic boundary conditions)

Our differential equation for this problem is given by

$$-u''(x) + \sigma u(x) = f(x) \,,$$

with boundary conditions $u'(0) = u'(1) = 0$. A second order, finite difference spatial discretization of this equation in the interior of our domain gives

$$-\frac{v_{i-1} - 2v_i + v_{i+1}}{h_x^2} + \sigma v_i = f_i \quad \text{for } i = 1, 2, \ldots, N-1 \,.$$

Grouping terms this simplifies to

$$-\frac{1}{h_x^2} v_{i-1} + (\frac{2}{h_x^2} + \sigma) v_i - \frac{1}{h_x^2} v_{i+1} = f_i \quad \text{for} \quad i = 1, 2, \ldots, N-1 \,. \tag{1}$$

Using a second order centered approximation to the derivative at $x = x_0$ gives

$$u_x(x_0 = 0) \approx \frac{u(x_1) - u(x_{-1})}{2h_x} = 0$$

which simplifies to give our first discrete boundary condition $v_1 = v_{-1}$. For the node x_N a second order centered difference approximation of the derivative at $x = 1$ will give

$$u_x(x_N = 1) \approx \frac{u(x_{N+1}) - u(x_{N-1})}{2h_x} = 0 \,,$$

which simplifies to give our second discrete boundary condition $v_{N+1} = v_{N-1}$. We note that this problem (as specified) provides no equation for the unknowns v_0 and v_N. If we assume that Equation 1 also holds at $i = 0$ and $i = N$, we can evaluate it at $i = 0$ to obtain

$$-\frac{1}{h_x^2} v_{-1} + \left(\frac{2}{h_x^2} + \sigma \right) v_0 - \frac{1}{h_x^2} v_1 = f_0 \,,$$

with our periodicity condition $v_{-1} = v_1$ the above becomes

$$\left(\frac{2}{h_x^2} + \sigma \right) v_0 - \frac{2}{h_x^2} v_1 = f_0 \,.$$

Evaluating Equation 1 at $i = N$ we have

$$-\frac{1}{h_x^2} v_{N-1} + \left(\frac{2}{h_x^2} + \sigma \right) v_N - \frac{1}{h_x^2} v_{N+1} = f_N$$

which using $v_{N+1} = v_{N-1}$ becomes

$$-\frac{2}{h_x^2} v_{N-1} + \left(\frac{2}{h_x^2} + \sigma \right) v_N = f_N \,.$$

Thus the system of equations to solve for the unknowns $v_0, v_1, \ldots, v_{N-1}, v_N$ is given by

$$\begin{aligned} \left(\frac{2}{h_x^2} + \sigma\right) v_0 - \frac{2}{h_x^2} v_1 &= f_0 \\ -\frac{1}{h_x^2} v_{i-1} + (\frac{2}{h_x^2} + \sigma) v_i - \frac{1}{h_x^2} v_{i+1} &= f_i \quad i = 1, 2, \ldots, N-2, N-1 \\ -\frac{2}{h_x^2} v_{N-1} + \left(\frac{2}{h_x^2} + \sigma\right) v_N &= f_N \end{aligned}$$

This system has $N+1$ unknowns and additional information would have to be provided to uniquely specify a solution.

Problem 1.2 (two dimensional diffusion with an advection term)

The equation we desire to discretize is given by

$$-\epsilon(u_{xx} + u_{yy}) + a u_x = f(x, y) .$$

Note that the terms $u_{xx} + u_{yy}$ have been descretized earlier, see subsection 1.1. Using a second order centered difference for each derivative (including the advection term) we obtain

$$-\epsilon \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2} \right) + a \left(\frac{u_{i+1,j} - u_{i-1,j}}{2h_x} \right) = f(x_i, y_j) ,$$

for indexes i and j that satisfy

$$1 \le i \le M-1 \quad \text{and} \quad 1 \le j \le N-1 ,$$

and boundary conditions $v_{i,j} = 0$ if $i = 0$, $i = M$, $j = 0$, or $j = N$. Grouping the unknowns together the above simplifies to

$$\left(\frac{-\epsilon}{h_x^2} - \frac{a}{2h_x}\right) v_{i-1,j} + \left(\frac{2\epsilon}{h_x^2} + \frac{2\epsilon}{h_y^2}\right) v_{i,j} + \left(-\frac{\epsilon}{h_x^2} + \frac{a}{2h_x}\right) v_{i+1,j} - \frac{\epsilon}{h_y^2} v_{i,j-1} - \frac{\epsilon}{h_y^2} v_{i,j+1} = f(x_i, y_j) .$$

As in subsection 1.1, we write the above difference equation focusing on the ith row *only*. This gives

$$\begin{pmatrix} 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-\epsilon}{h_y^2} & 0 & \cdots & 0 \\ \frac{-\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-\epsilon}{h_y^2} & \cdots & 0 \\ 0 & \frac{-\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-\epsilon}{h_y^2} & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \frac{-\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & \frac{-\epsilon}{h_y^2} \end{pmatrix} \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \vdots \\ v_{i,N-1} \end{pmatrix}$$

$$+ \begin{pmatrix} \left(-\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right) & 0 & 0 & 0 & 0 \\ 0 & \left(-\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right) & 0 & 0 & 0 \\ & 0 & \left(-\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right) & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \left(-\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right) \end{pmatrix} \begin{pmatrix} v_{i-1,1} \\ v_{i-1,2} \\ v_{i-1,3} \\ \vdots \\ v_{i-1,N-1} \end{pmatrix}$$

$$+ \begin{pmatrix} \left(-\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right) & 0 & 0 & 0 & 0 \\ 0 & \left(-\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right) & 0 & 0 & 0 \\ & 0 & \left(-\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right) & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \left(-\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right) \end{pmatrix} \begin{pmatrix} v_{i+1,1} \\ v_{i+1,2} \\ v_{i+1,3} \\ \vdots \\ v_{i+1,N-1} \end{pmatrix}$$

$$= \begin{pmatrix} f_{i,1} \\ f_{i,2} \\ f_{i,3} \\ \vdots \\ f_{i,N-1} \end{pmatrix} \quad \text{for} \quad i = 1, 2, \ldots, M-1\,.$$

Now the first matrix above is the *same* tridiagonal matrix of size $N-1$ by $N-1$ as we found earlier (modulo the factor of ϵ). As before we can defining the vector's $\mathbf{v_i}$ to be all of the unknowns in the i row, i.e. $v_{i,1}, v_{i,2}, \ldots v_{i,N-1}$, (similarly for $\mathbf{f_i}$), with this definition we can write the above equations in a block matrix form as

$$\begin{pmatrix} A & \tilde{I}_u & & & \\ \tilde{I}_l & A & \tilde{I} & & \\ & & \ddots & & \\ & & & A & \tilde{I}_u \\ & & & \tilde{I}_l & A \end{pmatrix} \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \vdots \\ \mathbf{v_{M-1}} \end{pmatrix} = \begin{pmatrix} \mathbf{f_1} \\ \mathbf{f_2} \\ \mathbf{f_3} \\ \vdots \\ \mathbf{f_{M-1}} \end{pmatrix}.$$

which provides the global matrix structure for *all* of the unknowns. This is a block tridiagonal matrix with block dimension $(M-1)\times(M-1)$ and blocks of size $(N-1)\times(N-1)$ giving a total matrix size of $(M-1)(N-1)\times(M-1)(N-1)$. In the above we have defined the matrices $\tilde{I}_l$, and $\tilde{I}_u$ as

$$\tilde{I}_l = -\left(\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right) I_{N-1,N-1} \quad \text{and} \quad \tilde{I}_u = -\left(\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right) I_{N-1,N-1}\,,$$

here $I_{N-1,N-1}$ is the $N-1 \times N-1$ identity matrix, and the matrix A as

$$A = \epsilon \begin{pmatrix} 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 & \cdots & 0 \\ \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & \cdots & 0 \\ 0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \frac{-1}{h_y^2} & 2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) & \frac{-1}{h_y^2} \end{pmatrix} .$$

Note the ϵ in this matrix defintion. This result could also have been obtained simply by noting that the derivative term when appoximated by second order differences as

$$au_x \approx a\frac{u_{i+1,j} - u_{i-1,j}}{2h_x}$$

would add a term proprtionate to

$$\frac{a}{2h_x} ,$$

to each of the diagonal matrices $\tilde{I}$'s in the block tridiagonal representation, as was verified above.

Problem 2.1 (the derivation of the residual equation)

For a linear system $Au = f$ with exact solution u and an approximate solution v the algebraic error is defined as $e = u - v$, so our exact solution u in terms of the error and the approximate solution is

$$u = e + v .$$

Putting this into $Au = f$ we have that $A(e + v) = f$, or $Ae = f - Av$. Using the definition of the residual of $r = f - Av$ we have that

$$Ae = r ,$$

or the residual equation.

Problem 2.2 (the weighted Jacobi method in terms of the residual)

The weighted Jacobi method is given by

$$v^{(1)} = \left((1-\omega)I + \omega D^{-1}(L+U)\right) v^{(0)} + \omega D^{-1} f .$$

This can be simplified by grouping all terms with an ω acting on $v^{(0)}$ as follows

$$v^{(1)} = v^{(0)} - \omega \left(I - D^{-1}(L+U)\right) v^{(0)} + \omega D^{-1} f .$$

From this expression, we will write the matrix operator $I - D^{-1}(L+U)$ (found above) in terms of the residual and in doing so derive the requested expression. Towards this direction we recall the definition of the residual for an approximate solution v given by $r = f - Av$. Solving for Av we have $Av = f - r$. Splitting the coefficient matrix A into its standard diagonal, lower and upper triangular parts as $A = D - L - U$, this equation becomes, $(D - L - U)v = f - r$. Multiplying this by D^{-1} on both sides we have

$$(I - D^{-1}(L+U))v = D^{-1}(f - r)\,.$$

From which we can simplify the definition of the weighted Jacobi update expressed in terms of the operator $I - D^{-1}(L+U)$ giving

$$v^{(1)} = v^{(0)} - \omega D^{-1}(f - r^{(0)}) + \omega D^{-1} f = v^{(0)} + \omega D^{-1} r^{(0)}\,.$$

Which was the expression we desired to obtain.

Problem 2.4-2.5 (eigenvalues and vectors for the model problem)

In general, for banded matrices, where the values on each band are constant, *explicit* formulas for the eigenvalues and eigenvectors can be obtained from the theory of finite differences. We will demonstrate this theory for the -1,2,-1 tridiagonal model problem matrix considered here. Here we will let the unknown vector be denoted by w. In addition, because we will use the symbol i for the imaginary unit ($\sqrt{-1}$), rather than the usual "i" subscript convention we will let our independent variable (ranging over components of the vector) be denoted t. Thus notationally $w_i \equiv w(t)$. Converting our eigenvector equation $Aw = \lambda w$ into a *system* of equations we have that $w(t)$, must satisfy

$$-w(t-1) + 2w(t) - w(t+1) = \lambda w(t) \quad \text{for} \quad t = 1, 2, \ldots, N-1\,,$$

with boundary conditions on $w(t)$ taken such that $w(0) = 0$ and $w(N) = 0$. Then the above equation can be written as

$$w(t-1) - (2-\lambda)w(t) + w(i+1) = 0\,.$$

Substituting $w(t) = m^t$ into the above we get

$$m^2 - (2-\lambda)m + 1 = 0\,.$$

Solving this quadratic equation for m gives

$$m = \frac{(2-\lambda) \pm \sqrt{(2-\lambda)^2 - 4}}{2}$$

From this expression if $|2-\lambda| \geq 2$ the expression under the square root is positive and the two roots are both *real*. With two real roots, the only solution that satisfies the boundary conditions is the trivial one ($w(t) = 0$). If $|2-\lambda| < 2$ then m is a complex number and the boundary conditions can be satisfied non-trivially. To further express this, define θ such that

$$2 - \lambda = 2\cos(\theta)$$

then the expression for m (in terms of θ) becomes

$$m = \frac{2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1}$$

or

$$m = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}$$

from the theory of finite differences the solution $w(t)$ is a linear combination of the two fundamental solutions or

$$w(t) = Ae^{i\theta t} + Be^{-i\theta t}. \tag{2}$$

Imposing the two homogeneous boundary condition we have the following system that must be solved for A and B

$$\begin{aligned} A + B &= 0 \\ Ae^{i\theta N} + Be^{-i\theta N} &= 0 \end{aligned}$$

Putting the first equation into the second gives

$$B(e^{i\theta N} - e^{-i\theta N}) = 0$$

Since B cannot be zero (else the eigenfunction $w(t)$ is identically zero) we must have θ satisfy

$$\sin(\theta N) = 0$$

Thus $\theta N = \pi n$ or

$$\theta = \frac{\pi n}{N} \qquad \text{for} \qquad n = 1, 2, \ldots, N-1$$

Tracing θ back to the definition of λ we have that

$$\lambda = 2 - 2\cos(\theta) = 2 - 2\cos(\frac{\pi n}{N})$$

Using the trigonometric identity

$$1 - \cos(\psi) = 2\sin(\frac{\psi}{2})^2$$

we get

$$\lambda_n = 4\sin(\frac{\pi n}{2N})^2 \qquad \text{for} \quad n = 1, 2, 3, \ldots, N-1$$

For the eigenvalues of the $-1, 2, -1$ discrete one dimensional discrete Laplacian. We note that there are $N-1$ eigenvalues. To evaluate the eigenvectors we go back to Eq. 2 using our new definition of θ. We get that

$$\begin{aligned} w(t) &\propto e^{i\theta t} - e^{-i\theta t} \\ &\propto \sin(\theta t) \\ &\propto \sin(\frac{\pi n}{N} t) \qquad \text{for} \quad n = 1, 2, 3, \ldots, N-1 \end{aligned}$$

Here the range of t is formally given by $t = 1, 2, \ldots, N-1$, but if $w(t)$ is evaluated at the points $t = 0$ and $t = N$, the correct boundary conditions are obtained and thus the domain of the functions $w(t)$ can be extended to include these points.

We can explicity evaluate some of the λ's for $N = 64$ for example we find that

$$\begin{aligned}
\lambda_1 &= 4\sin(\frac{\pi}{2(64)})^2 \approx 0 \\
\lambda_2 &= 9.6\,10^{-3} \\
\lambda_{N-2} &\approx 4.0 \\
\lambda_{N-1} &\approx 4.0
\end{aligned}$$

In the book, the functional form for the eigenvectors was supplied (i.e. $w(t)$ was given) and another approach to evaluating the eigenvalues is to simply substitute the given $w(t)$ into the discrete Laplacian and solve for the resulting λ using trigonametric relationships to simply the combination of trigonametric functions that results.

Problem 2.6 (the eigensystem for the weighted Jacobi method)