A Solution Manual and Notes for: Applied Optimal Estimation by Arthur Gelb.

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Introduction

Here you'll find various notes and derivations of the technical material I made as I worked through this book. There is also quite a complete set of solutions to the various end of chapter problems. I did much of this in hopes of improving my understanding of Kalman filtering and thought it might be of interest to others. I have tried hard to eliminate any mistakes but it is certain that some exit. I would appreciate constructive feedback (sent to the email below) on any errors that are found in these notes. I will try to fix any corrections that I receive. In addition, there were several problems that I was not able to solve or that I am not fully confident in my solutions for. If anyone has any suggestions at solution methods or alternative ways to solve given problems please contact me. Finally, some of the derivations found here can be quite long (since I really desire to fully document exactly how to do each derivation) many of these can be skipped if they are not of interest.

I hope you enjoy this book as much as I have and that these notes might help the further development of your skills in Kalman filtering.

As a final comment, I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

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Chapter 1: Introduction

Notes On The Text

optimal estimation with two measurements of a constant value

We desire our estimate $\hat{x}$ of x to be a linear combination of the two measurements z_i for $i = 1, 2$. Thus we take $\hat{x} = k_1 z_1 + k_2 z_2$, and define $\tilde{x}$ to be our estimate error given by $\tilde{x} = \hat{x} - x$. To make our estimate $\hat{x}$ unbiased requires we set $E[\tilde{x}] = 0$ or

$$\begin{aligned} E[\tilde{x}] &= E[k_1(x + v_1) + k_2(x + v_2) - x] = 0 \\ &= E[(k_1 + k_2)x + k_1 v_1 + k_2 v_2 - x] \\ &= E[(k_1 + k_2 - 1)x + k_1 v_1 + k_2 v_2] \\ &= (k_1 + k_2)x - x = (k_1 + k_2 - 1)x = 0 \,, \end{aligned}$$

thus this requirement becomes $k_2 = 1 - k_1$ which is the same as the books Equation 1.0-4. Next lets pick k_1 and k_2 (subject to the above constraint such that) the error as small as possible. When we take $k_2 = 1 - k_1$ we find that $\hat{x}$ is given by

$$\hat{x} = k_1 z_1 + (1 - k_1) z_2 \,,$$

so $\tilde{x}$ is given by

$$\begin{aligned} \tilde{x} &= \hat{x} - x = k_1 z_1 + (1 - k_1) z_2 - x \\ &= k_1(x + v_1) + (1 - k_1)(x + v_2) - x \\ &= k_1 v_1 + (1 - k_1) v_2 \,. \end{aligned} \tag{1}$$

Next we compute the expected error or $E[\tilde{x}^2]$ and find

$$\begin{aligned} E[\tilde{x}^2] &= E[k_1^2 v_1^2 + 2k_1(1 - k_1) v_1 v_2 + (1 - k_1)^2 v_2^2] \\ &= k_1^2 \sigma_1^2 + 2k_1(1 - k_1) E[v_1 v_2] + (1 - k_1)^2 \sigma_2^2 \\ &= k_1^2 \sigma_1^2 + (1 - k_1)^2 \sigma_2^2 \,, \end{aligned}$$

since $E[v_1 v_2] = 0$ as v_1 and v_2 are assumed to be uncorrelated. This is the books equation 1.0-5. We desire to minimize this expression with respect to the variable k_1. Taking its derivative with respect to k_1, setting the result equal to zero, and solving for k_1 gives

$$2k_1 \sigma_1^2 + 2(1 - k_1)(-1)\sigma_2^2 = 0 \Rightarrow k_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \,.$$

Putting this value in our expression for $E[\tilde{x}^2]$ to see what our minimum error is given by we find

$$\begin{aligned} E[\tilde{x}^2] &= \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right)^2 \sigma_1^2 + \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right)^2 \sigma_2^2 \\ &= \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} \left(\sigma_2^2 + \sigma_1^2 \right) = \frac{\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)} \\ &= \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^{-1} \,, \end{aligned}$$

which is the books equation 1.06. Then our optimal estimate $\hat{x}$ take the following form

$$\hat{x} = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right) z_1 + \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \right) z_2 \, .$$

Some special cases of the above that validate its usefulness are when each measurement contributes the same uncertainty then $\sigma_1 = \sigma_2$ and we see that $\hat{x} = \frac{1}{2} z_1 + \frac{1}{2} z_2$, or the average of the two measurements. As another special case if one measurement is *exact* i.e. $\sigma_1 = 0$, then we have $\hat{x} = z_1$ (in the same way if $\sigma_2 = 0$, then $\hat{x} = z_2$).

Problem Solutions

Problem 1-1 (correlated measurements)

For this problem we are now going to assume that $E[v_1 v_2] = \rho \sigma_1 \sigma_2$ i.e. that the noise v_1 and v_2 are correlated. Recall from above that the condition $E[\tilde{x}] = 0$ requires that our estimate $\hat{x} = k_1 z_1 + k_2 z_2$ requires $k_2 = 1 - k_1$. Next we compute the expected error or $E[\tilde{x}^2]$ and in this case using Equation 1 for $\tilde{x}$ we find

$$\begin{aligned} E[\tilde{x}^2] &= E[k_1^2 v_1^2 + 2k_1(1-k_1)v_1 v_2 + (1-k_1)^2 v_2^2] \\ &= k_1^2 \sigma_1^2 + 2k_1(1-k_1)E[v_1 v_2] + (1-k_1)^2 \sigma_2^2 \\ &= k_1^2 \sigma_1^2 + 2k_1(1-k_1)\rho\sigma_1\sigma_2 + (1-k_1)^2 \sigma_2^2 \, . \end{aligned} \tag{2}$$

To find a minimum variance estimator we will take the derivative of $E[\tilde{x}^2]$ with respect to k_1, set the result equal to zero, and then solve for k_1. We have

$$\frac{dE[\tilde{x}^2]}{dk_1} = 0 \Rightarrow 2k_1\sigma_1^2 + 2\rho(1-k_1)\sigma_1\sigma_2 + 2\rho k_1(-1)\sigma_1\sigma_2 + 2(1-k_1)(-1)\sigma_2^2 = 0 \, .$$

or dividing by 2

$$k_1\sigma_1^2 + \rho(1-k_1)\sigma_1\sigma_2 - \rho k_1\sigma_1\sigma_2 - (1-k_1)\sigma_2^2 = 0 \, .$$

On solving for k_1 in this expression we find

$$k_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2} \, , \tag{3}$$

as claimed. From symmetry $k_2 = 1 - k_1$ is given by

$$k_2 = 1 - k_1 = \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 - \sigma_2^2 + \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} \, . \tag{4}$$

With these values for k_1 and k_2 and introducing

$$D \equiv \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \, ,$$

to simplify notation the minimum mean square error given by Equation 2 becomes

$$
\begin{aligned}
E[\tilde{x}^2] &= \frac{1}{D^2}\left[(\sigma_2^2 - \rho\sigma_1\sigma_2)^2\sigma_1^2 + 2\rho(\sigma_2^2 - \rho\sigma_1\sigma_2)(\sigma_1^2 - \rho\sigma_1\sigma_2)\sigma_1\sigma_2 + (\sigma_1^2 - \rho\sigma_1\sigma_2)^2\sigma_2^2\right] \\
&= \frac{1}{D^2}\left[\sigma_1^2(\sigma_2^4 - 2\rho\sigma_1\sigma_2\sigma_2^2 + \rho^2\sigma_1^2\sigma_2^2)\right. \\
&+ 2\rho\sigma_1\sigma_2(\sigma_1^2\sigma_2^2 - \rho\sigma_1^3\sigma_2 - \rho\sigma_1\sigma_2^3 + \rho^2\sigma_1^2\sigma_2^2) \\
&+ \left.\sigma_2^2(\sigma_1^4 - 2\rho\sigma_1^3\sigma_2 + \rho^2\sigma_1^2\sigma_2^2)\right] \\
&= \frac{1}{D^2}\left[\sigma_1^2\sigma_2^4 - 2\rho\sigma_1^3\sigma_2^3 + \rho^2\sigma_1^4\sigma_2^2\right. \\
&+ 2\rho\sigma_1^3\sigma_2^3 - 2\rho^2\sigma_1^4\sigma_2^2 - 2\rho^2\sigma_1^2\sigma_2^4 + 2\rho^3\sigma_1^3\sigma_2^3 \\
&+ \left.\sigma_1^4\sigma_2^2 - 2\rho\sigma_1^3\sigma_2^3 + \rho^2\sigma_1^2\sigma_2^4\right] \\
&= \frac{1}{D^2}\left[\sigma_1^2\sigma_2^4(1 - 2\rho^2 + \rho^2) + \sigma_1^4\sigma_2^2(\rho^2 - 2\rho^2 + 1) + \sigma_1^3\sigma_2^3(2\rho^3 - 2\rho)\right] \\
&= \frac{\sigma_1^2\sigma_2^2}{D^2}\left[\sigma_2^2(1-\rho^2) + \sigma_1^2(1-\rho^2) + \sigma_1\sigma_2(2\rho)(-1)(1-\rho^2)\right] \\
&= \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{D^2}\left[\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2\right] \\
&= \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\,.
\end{aligned}
$$

Note that this last expression is zero when $\rho = \pm 1$. Our estimate $\hat{x}$ is then given by

$$
\hat{x} = \left(\frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_2^2 - 2\rho\sigma_1\sigma_2 + \sigma_1^2}\right) z_1 + \left(\frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\right) z_2\,. \tag{5}
$$

As before we now consider some special cases. If $\rho = +1$ then the errors are totally positively correlated and we see that

$$
k_1 = \frac{\sigma_2^2 - \sigma_1\sigma_2}{\sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2} = \frac{\sigma_2(\sigma_2 - \sigma_1)}{(\sigma_1 - \sigma_2)^2} = \frac{\sigma_2}{\sigma_2 - \sigma_1}\,,
$$

with k_2 is given by

$$
k_2 = 1 - k_1 = \frac{-\sigma_1}{\sigma_2 - \sigma_1}\,,
$$

so that $\hat{x}$ is given by

$$
\hat{x} = \left(\frac{\sigma_2}{\sigma_2 - \sigma_1}\right) z_1 + \left(\frac{-\sigma_1}{\sigma_2 - \sigma_1}\right) z_2 = \frac{\sigma_2 z_1 - \sigma_1 z_2}{\sigma_2 - \sigma_1}\,.
$$

If $\rho = -1$ the errors are totally negatively correlated and we have

$$
k_1 = \frac{\sigma_2^2 + \sigma_1\sigma_2}{\sigma_1^2 + 2\sigma_1\sigma_2 + \sigma_2^2} = \frac{\sigma_2}{\sigma_2 + \sigma_1}\,.
$$

with k_2 is given by

$$
k_2 = 1 - k_1 = \frac{\sigma_1}{\sigma_2 + \sigma_1}\,,
$$

so that $\hat{x}$ is given by

$$
\hat{x} = \left(\frac{\sigma_2}{\sigma_2 + \sigma_1}\right) z_1 + \left(\frac{\sigma_1}{\sigma_2 + \sigma_1}\right) z_2 = \frac{\sigma_2 z_1 + \sigma_1 z_2}{\sigma_2 + \sigma_1}\,.
$$

Problem 1-2 ($E[\tilde{x}^2]$ without the requirement that $E[\tilde{x}] = 0$)

We are told that our measurements z_1 and z_2 are given as noised measurements of a constant as $z_1 = x + v_1$ and $z_2 = x + v_2$, while our estimate of x or $\hat{x}$ is to be constructed as a linear combination of z_i as $\hat{x} = k_1 z_1 + k_2 z_2$. Now defining $\tilde{x}$ as before we have in this case that

$$\tilde{x} = \hat{x} - x = k_1(x + v_1) + k_2(x + v_2) - x = (k_1 + k_2 - 1)x + k_1 v_1 + k_2 v_2 \,.$$

So that $\tilde{x}^2$ is given by

$$\begin{aligned}
\tilde{x}^2 &= (k_1 + k_2 - 1)^2 x^2 + 2x(k_1 + k_2 - 1)(k_1 v_1 + k_2 v_2) + (k_1 v_1 + k_2 v_2)^2 \\
&= (k_1 + k_2 - 1)^2 x^2 + 2x k_1 (k_1 + k_2 - 1) v_1 + 2x k_2 (k_1 + k_2 - 1) v_2 + (k_1^2 v_1^2 + 2 k_1 k_2 v_1 v_2 + k_2^2 v_2^2) \,.
\end{aligned}$$

Taking the expectation of this expression and using the facts that the mean of the noise is zero so $E[v_i] = 0$ and x is a constant gives

$$E[\tilde{x}^2] = (k_1 + k_2 - 1)^2 x^2 + k_1^2 \sigma_1^2 + 2 k_1 k_2 E[v_1 v_2] + k_2^2 \sigma_2^2 \,.$$

For simplicity lets assume that the two noise sources are uncorrelated i.e. $E[v_1 v_2] = 0$. Then to find the minimum of this expression we take derivatives with respect to k_1 and k_2 set each expression equal to zero and solve for k_1 and k_2. We find the derivatives given by

$$\begin{aligned}
\frac{\partial E[\tilde{x}^2]}{\partial k_1} &= 2(k_1 + k_2 - 1)x^2 + 2 k_1 \sigma_1^2 = 0 \\
\frac{\partial E[\tilde{x}^2]}{\partial k_2} &= 2(k_1 + k_2 - 1)x^2 + 2 k_2 \sigma_2^2 = 0 \,.
\end{aligned}$$

When we group terms by the coefficients k_1 and k_2 we get the following system

$$\begin{aligned}
(x^2 + \sigma_1^2) k_1 + x^2 k_2 &= x^2 \\
x^2 k_1 + (x^2 + \sigma_2^2) k_2 &= x^2 \,.
\end{aligned}$$

To solve this system for k_1 and k_2 we can use Cramer's rule. We find

$$\begin{aligned}
k_1 &= \frac{\begin{vmatrix} x^2 & x^2 \\ x^2 & x^2 + \sigma_2^2 \end{vmatrix}}{\begin{vmatrix} x^2 + \sigma_1^2 & x^2 \\ x^2 & x^2 + \sigma_2^2 \end{vmatrix}} = \frac{x^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2) x^2 + \sigma_1^2 \sigma_2^2} \\
k_2 &= \frac{\begin{vmatrix} x^2 + \sigma_1^2 & x^2 \\ x^2 & x^2 \end{vmatrix}}{(\sigma_1^2 + \sigma_2^2) x^2 + \sigma_1^2 \sigma_2^2} = \frac{x^2 \sigma_1^2}{(\sigma_1^2 + \sigma_2^2) x^2 + \sigma_1^2 \sigma_2^2} \,,
\end{aligned}$$

both of which are functions of the unknown variable x. An interesting idea would be to consider the iterative algorithm where we initially *estimate* x above using an unbiased estimator and then replace the x above with this estimate obtaining values for k_1 and k_2. One could then use these to estimate x again and put this value into the above expressions for k_1 and k_2. Doing this several times one gets an iterative *algorithm* as the estimation procedure.

Problem 1-3 (estimating a constant with three measurements)

For this problem our three measurements are related to the unknown value of x from as $z_1 = x + v_1$, $z_2 = x + v_2$, and $z_3 = x + v_3$, and our estimate will be a linear combination of them as $\hat{x} = k_1 z_1 + k_2 z_2 + k_3 z_3$. To have an unbiased estimate compute the expectation of $\tilde{x} = \hat{x} - x$ which we find to be

$$\begin{aligned}
\tilde{x} &= \hat{x} - x \\
&= k_1 z_1 + k_2 z_2 + k_3 z_3 - x \\
&= k_1(x + v_1) + k_2(x + v_2) + k_3(x + v_3) - x \\
&= (k_1 + k_2 + k_3 - 1)x + k_1 v_1 + k_1 v_1 + k_2 v_2 + k_3 v_3 \,. \qquad (6)
\end{aligned}$$

To make $\hat{x}$ an unbiased estimate of x we require that $E[\tilde{x}] = 0$. This in turn requires $k_1 + k_2 + k_3 - 1 = 0$ or

$$k_3 = 1 - k_1 - k_2 \qquad (7)$$

Thus our unbiased estimate of x now takes the form

$$\hat{x} = k_1 z_1 + k_2 z_2 + (1 - k_1 - k_2) z_3 \,.$$

We will now pick k_1 and k_2 such that the mean square error $E[\tilde{x}^2]$ is a minimum. With this functional form for $\hat{x}$ we have using Equation 6 that

$$\begin{aligned}
\tilde{x}^2 &= (k_1 v_1 + k_2 v_2 + k_3 v_3)^2 \\
&= k_1^2 v_1^2 + k_2^2 v_2^2 + k_3^2 v_3^2 + 2k_1 k_2 v_1 v_2 + 2k_1 k_3 v_1 v_3 + 2k_2 k_3 v_2 v_3 \,.
\end{aligned}$$

Taking the expectation of the above expression, assuming uncorrelated measurements $E[v_i v_j] = 0$ when $i \neq j$ and recalling Equation 7 we have

$$E[\tilde{x}^2] = k_1^2 \sigma_1^2 + k_2^2 \sigma_2^2 + (1 - k_1 - k_2)^2 \sigma_3^2 \,. \qquad (8)$$

to minimize this expression we take the partial derivatives with respect to k_1 and k_2 and set the resulting expressions equal to zero. This gives

$$\begin{aligned}
\frac{\partial E[\tilde{x}^2]}{\partial k_1} &= 2k_1 \sigma_1^2 + 2(1 - k_1 - k_2)(-1)\sigma_3^2 = 0 \\
\frac{\partial E[\tilde{x}^2]}{\partial k_2} &= 2k_2 \sigma_2^2 + 2(1 - k_1 - k_2)(-1)\sigma_3^2 = 0 \,.
\end{aligned}$$

Now solving these two equations for k_1 and k_2 we find

$$\begin{aligned}
k_1 &= \frac{\sigma_2^2 \sigma_3^2}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2} = \frac{1}{\left(\frac{\sigma_1}{\sigma_3}\right)^2 + \left(\frac{\sigma_1}{\sigma_2}\right)^2 + 1} \\
k_2 &= \frac{\sigma_1^2 \sigma_3^2}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2} = \frac{1}{\left(\frac{\sigma_2}{\sigma_3}\right)^2 + 1 + \left(\frac{\sigma_2}{\sigma_1}\right)^2} \,.
\end{aligned}$$

From these we can compute $k_3 = 1 - k_1 - k_2$ to find

$$\begin{aligned} k_3 &= 1 - k_1 - k_2 = 1 - \left(\frac{\sigma_2^2 \sigma_3^2}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2} \right) - \left(\frac{\sigma_1^2 \sigma_3^2}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2} \right) \\ &= \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2} = \frac{1}{1 + \left(\frac{\sigma_3}{\sigma_2}\right)^2 + \left(\frac{\sigma_3}{\sigma_1}\right)^2} . \end{aligned}$$

Then by defining $D \equiv \sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_2^2 \sigma_3^2$ and using Equation 8 we see that

$$\begin{aligned} E[\tilde{x}^2] &= \frac{\sigma_2^4 \sigma_3^4 \sigma_1^2}{D^2} + \frac{\sigma_3^4 \sigma_1^4 \sigma_2^2}{D^2} + \frac{\sigma_1^4 \sigma_2^4 \sigma_3^2}{D^2} = \left(\frac{\sigma_1^2 \sigma_2^2 \sigma_3^2}{D^2} \right) \left(\sigma_2^2 \sigma_3^2 + \sigma_1^2 \sigma_3^2 + \sigma_1^2 \sigma_2^2 \right) = \frac{\sigma_1^2 \sigma_2^2 \sigma_3^2}{D} \\ &= \frac{\sigma_1^2 \sigma_2^2 \sigma_3^3}{\sigma_1^2 \sigma_2^2 + \sigma_1^2 \sigma_3^2 + \sigma_3^2 \sigma_2^2} = \frac{1}{\left(\frac{1}{\sigma_3^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2} \right)} , \end{aligned}$$

as we were to show.

Problem 1-4 (estimating the initial concentration)

We are told that our estimate of the concentration, z_i are noisy measurements of the time-decayed initial concentration x_0 and so have the form

$$z_i = x_0 e^{-a t_i} + v_i , \tag{9}$$

for $i = 1, 2$. The book *provides* us with a functional form of an estimator $\hat{x}_0$ we could use to estimate x_0, and asks us to show that it is unbiased. We could begin by attempting to estimate the initial concentration x_0 using a expression that is linear in the two measurements. That is we might consider

$$\hat{x}_0 = k_1 z_1 + k_2 z_2 ,$$

as has been done else where in the book. From the given form of the measurements in Equation 9 it might be better however to estimate x_0 using the following

$$\hat{x}_0 = k_1 e^{a t_1} z_1 + k_2 e^{a t_2} z_2 ,$$

with k_1 and k_2 unknown. Since in that case the exponential parts $e^{a t_i}$, multiplied by z_i will "remove" the corresponding factor found in Equation 9 and provide a more direct estimate of x_0. We next define our estimation error $\tilde{x}_0$ as $\tilde{x}_0 = \hat{x}_0 - x_0$. To have an unbiased estimator requires that $E[\tilde{x}_0] = 0$. Using this last form form $\hat{x}_0$ this later expectation is given by

$$E[\tilde{x}_0] = E[k_1 e^{a t_1} (x_0 e^{-a t_1} + v_1) + k_2 e^{a t_2} (x_0 e^{-a t_2} + v_2) - x_0] = 0 .$$

Since $E[v_i] = 0$ the above gives $k_1 x_0 + k_2 x_0 - x_0 = 0$ so that $k_2 = 1 - k_1$. Thus our estimator $\hat{x}_0$ looks like

$$\hat{x}_0 = k_1 e^{a t_1} z_1 + (1 - k_1) e^{a t_2} z_2 ,$$

and is in the form suggested in the book. To have the optimal estimator we next select k_1 such that our expected square error is the smallest. To do this we compute our expected square error or $E[\tilde{x}^2]$ and find

$$
\begin{aligned}
E[\tilde{x}_0^2] &= E[(k_1 e^{at_1}(e^{-at_1}x_0 + v_1) + k_2 e^{at_2}(e^{-at_2}x_0 + v_2) - x_0)^2] \\
&= E[(k_1 x_0 + k_1 e^{at_1} v_1 + k_2 x_0 + k_2 e^{at_2} v_2 - x_0)^2] \\
&= E[(k_1 e^{at_1} v_1 + k_2 e^{at_2} v_2)^2] \\
&= E[k_1^2 e^{2at_1} v_1^2 + 2k_1 k_2 e^{at_1} e^{at_2} v_1 v_2 + k_2^2 e^{2at_2} v_2^2] \\
&= k_1^2 e^{2at_1} \sigma_1^2 + k_2^2 e^{2at_2} \sigma_2^2 \,, \qquad (10)
\end{aligned}
$$

assuming uncorrelated measurements $E[v_1 v_2] = 0$. Taking the derivative of this expression with respect to k_1 (while recalling that $k_2 = 1 - k_1$ and setting this derivative equal to zero we get

$$
2k_1 e^{2at_1} \sigma_1^2 + 2(1 - k_1)(-1) e^{2at_2} \sigma_2^2 = 0 \,.
$$

Solving for k_1 we find

$$
k_1 = \frac{(e^{at_2}\sigma_2)^2}{(e^{at_1}\sigma_1)^2 + (e^{at_2}\sigma_2)^2} = \frac{\sigma_2^2}{\sigma_2^2 + \sigma_1^2 e^{-2a(t_2 - t_1)}} \,.
$$

Using this then k_2 becomes

$$
k_2 = 1 - k_1 = \frac{(e^{at_1}\sigma_1)^2}{(e^{at_1}\sigma_1)^2 + (e^{at_2}\sigma_2)^2} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2 e^{2a(t_2 - t_1)}} \,.
$$

To simplify the notation of the algebra that follows we define $A_1 = e^{2at_1}\sigma_1^2$ and $A_2 = e^{2at_2}\sigma_2^2$ so that the variables k_i in terms of A_i are given as $k_1 = \frac{A_2}{A_1 + A_2}$ and $k_2 = \frac{A_1}{A_1 + A_2}$. Then we have that Equation 10 becomes

$$
\begin{aligned}
E[(\hat{x}_0 - x_0)^2] &= \frac{A_2^2}{(A_1 + A_2)^2} A_1 + \frac{A_1^2}{(A_1 + A_2)^2} A_2 = \frac{A_1 A_2}{(A_1 + A_2)^2}(A_1 + A_2) = \frac{A_1 A_2}{A_1 + A_2} \\
&= \frac{1}{\frac{1}{A_2} + \frac{1}{A_1}} = \left(\frac{e^{-2t_1 a}}{\sigma_1^2} + \frac{e^{-2t_2 a}}{\sigma_2^2} \right)^{-1} \,,
\end{aligned}
$$

as we were to show.

Chapter 2: Underlying Mathematical Techniques

Notes On The Text

Least-Squares Techniques

The objective function, J, for least squares is given by

$$J = (z - Hx)^T (z - Hx) \,, \tag{11}$$

which we can expand to write as follows

$$J = z^T z - 2z^T Hx + x^T H^T Hx \,.$$

Taking the first derivative of this expression with respect to the unknown vector x using Equations 311 and 312 gives

$$\frac{\partial J}{\partial x} = -2H^T z + (H^T H + H^T H)x = -2H^T z + 2H^T Hx \,.$$

The second derivative of J with respect to x is given by

$$\frac{\partial^2 J}{\partial x^2} = 2H^T H \,. \tag{12}$$

This matrix is positive semi-definite since if we let ξ be a arbitrary non-zero vector and compute the inner product $\xi^T \frac{\partial^2 J}{\partial x^2} \xi$ we see that this can can be written as a quadratic sum as

$$2(H\xi)^T (H\xi) = 2 \sum_i (H\xi)_i^2 \geq 0 \,,$$

for all possible vectors ξ. Thus $2H^T H$ is positive semi-definite and the solution to the first order optimality condition $\frac{\partial J}{\partial x} = 0$ gives a minimum.

Problem Solutions

Problem 2-1 (the derivative of the matrix inverse)

Since $P(t)P(t)^{-1} = I$, taking the derivative of both sides of this expression and using the product rule gives

$$\dot{P}P^{-1} + P\frac{dP^{-1}}{dt} = 0 \,.$$

Solving for $\frac{dP^{-1}}{dt}$ we find

$$\frac{dP^{-1}}{dt} = -P^{-1}\dot{P}P^{-1} \,, \tag{13}$$

as we were to show.

Problem 2-3 (eigenvalues of positive definite matrices)

We will prove this by showing the equivalence of between two quadratic forms. If we consider the quadratic form $x^T Ax$ then as discussed in the book there exists an orthogonal matrix Q such that $A' = Q^T AQ = Q^{-1}AQ$, is a diagonal matrix. Since A and A' are related by a similarity transformation they have the *same* eigenvalues which are equal to the elements on the diagonal of A'. Thus if we define $x' = Qx$ then $x^T Ax$ can be written as

$$\lambda_1 {x'_1}^2 + \lambda_2 {x'_2}^2 + \cdots + \lambda_n {x'_n}^2 ,$$

where λ_i is the eigenvalue of A (equivalently A'). Now if we are told that A is positive definite then we know that $x^T Ax > 0$ for all x's. If we take $x = q_i$, where q_i is the ith column vector of Q then in that case by the orthogonality of the matrix Q we have $x' = Qq_i = e_i$, a vector of all zeros with a single 1 in the ith spot. For that value of x then $x^T Ax = \lambda_i$. Since $x^T Ax > 0$ for all x we see that $\lambda_i > 0$. On the other hand if we are told that the eigenvalues of A are all positive we know that $\lambda_i > 0$ for all i then from the above decomposition we have that $x^T Ax = \sum_{i=1}^n \lambda_i {x'_i}^2 > 0$ showing that A is positive definite.

Problem 2-4 ($S(t) = \frac{dR(t)}{dt} R^T(t)$ is skew symmetric)

Taking the transpose of the expression for $S(t)$ and we find

$$\begin{aligned} S(t)^T &= R(t)\left(\frac{dR(t)}{dt}\right)^T = R(t)\frac{d}{dt}R(t)^T \\ &= \frac{d}{dt}\left(R(t)R^T(t)\right) - \frac{d}{dt}R(t)R^T(t) . \end{aligned}$$

Since $R(t)$ is orthogonal $R(t)R^T(t) = I$ which has a zero derivative. Since the right-hand-side of the above after this equals $-S(t)$ we have shown

$$S(t)^T = -S(t) ,$$

or that $S(t)$ is skew-symmetric.

Problem 2-5 (uses for the Cayley-Hamilton theorem)

Part (a): The Cayley-Hamilton theorem requires that a matrix A satisfy its own characteristic polynomial. The given matrix has a characteristic polynomial given by $|A - \lambda I| = 0$ or

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 ,$$

or after expanding some

$$(1-\lambda)(4-\lambda) - 6 = 0 ,$$

or finally $\lambda^2 - 5\lambda - 2 = 0$ as we were to show. The eigenvalues of this matrix are then given by the quadratic formula

$$\lambda = \frac{5 \pm \sqrt{33}}{2}. \tag{14}$$

Part (b): Since one definition of e^{At} is

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \frac{t^4}{24}A^4 + \cdots, \tag{15}$$

to evaluate this we need to compute powers of A. Powers of A can be computed using the fact that A satisfies its own characteristic polynomial (the Cayley-Hamilton theorem). We find

$$\begin{aligned}
A^2 &= 2I + 5A \\
A^3 &= (5A + 2I)A = 2A + 5A^2 = 2A + 5(5A + 2I) = 10I + 27A \\
A^4 &= A(A^3) = 10A + 27A^2 = 10A + 27(5A + 2I) = 54I + 145A .
\end{aligned}$$

Using these we can write e^{At} as

$$e^{At} = I + tA + \frac{t^2}{2}(2I + 5A) + \frac{t^3}{6}(10I + 27A) + \frac{t^4}{24}(54I + 145A) + \cdots$$

If we group terms that are multiples of I together and terms that are multiples of A together, we find that the above expression for e^{At} is equal to

$$\begin{aligned}
e^{At} &= I\left[1 + t^2 + \frac{5}{3}t^3 + \frac{9}{4}t^4 + \cdots\right] + A\left[t + \frac{5}{2}t^2 + \frac{9}{2}t^3 + \frac{145}{24}t^4 + \cdots\right] \\
&= a_1(t)I + a_2(t)A ,
\end{aligned}$$

with $a_i(t)$ defined by the respective terms in brackets above.

Part (c): Using the expression derived above

$$e^{At} = a_1(t)I + a_2(t)A ,$$

If we take $A = \lambda_1$ and then $A = \lambda_2$ we get the following system

$$\begin{aligned}
e^{\lambda_1 t} &= a_1(t)I + \lambda_1 a_2(t) \\
e^{\lambda_2 t} &= a_1(t)I + \lambda_2 a_2(t) .
\end{aligned}$$

Solving for the functions $a_1(t)$ and $a_2(t)$ we get

$$\begin{aligned}
a_1(t) &= -\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \\
a_2(t) &= \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} .
\end{aligned}$$

Note that the first expression is the *negative* of the books expression[1]. To verify that these exponential functions are equivalent to the expressions for $a_1(t)$ and $a_2(t)$ given in the book

[1] I think there is a sign mistake in the expression for $a_1(t)$ given in the book.

(and above) we can Taylor expand each of the $a_i(t)$ expressions about $t = 0$ with λ_i given by Equation 14. We do this using Mathematica in the file `chap_2_prob_5.nb`. Where we find

$$
\begin{aligned}
-\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_1 - \lambda_2} &= 1 + t^2 + \frac{5}{3}t^3 + \frac{9}{4}t^4 + \frac{779}{360}t^6 + \cdots \\
\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} &= t + \frac{5}{2}t^2 + \frac{9}{2}t^3 + \frac{145}{24}t^4 + \frac{779}{120}t^5 + \frac{93}{16}t^6 + \cdots,
\end{aligned}
$$

which are the same as the above expressions in brackets, proving the equivalence.

Problem 2-6 (evaluating an integral over the points r such that $r^T E^{-1} r < 1$)

For this problem we want to evaluate the integral $\int_{r^T E^{-1} r < 1} dr$. To do this lets introduce a change of coordinates that decouples the variables in r. Since E is a positive definite matrix so is its inverse E^{-1}, and thus E^{-1} has a Cholesky factorization given by $E^{-1} = GG^T$, where G is an lower triangular matrix. Introduce the vector $v = G^T r$ then the set of possible r values $r^T E^{-1} r < 1$ becomes

$$
r^T G G^T r < 1 \quad \text{or} \quad v^T v < 1 \,.
$$

Our integral under this change of coordinates then becomes

$$
\int_{v^T v < 1} \left| \frac{\partial r}{\partial v} \right| dv \,,
$$

where $\left|\frac{\partial r}{\partial v}\right|$ is the determinant of the Jacobian of the transformation from the v coordinates to the r coordinates. Since $r = G^{-T} v$ we see that

$$
\frac{\partial r}{\partial v} = G^{-T} \,,
$$

and so

$$
\left| \frac{\partial r}{\partial v} \right| = |G^{-T}| = |G^{-1}| = \frac{1}{|G|} \,.
$$

Note that since G is related to E we can express $\left|\frac{\partial r}{\partial v}\right|$ in terms of E by noting that

$$
|E^{-1}| = |G| \cdot |G^T| = |G|^2 \,.
$$

Thus we can replace $\frac{1}{|G|}$ with $\sqrt{|E|}$ to find that $\left|\frac{\partial r}{\partial v}\right| = \sqrt{|E|}$, and that our integral becomes

$$
\sqrt{|E|} \int_{v^T v < 1} dv = \frac{4}{3}\pi^3 \sqrt{|E|} \,,
$$

since we recognized that $\int_{v^T v < 1} dv$ represents the volume of a sphere with radius 1.

Problem 2-7 (weighted least squares)

The objective function, J, for weighted least squares is given by

$$J = (z - Hx)^T W (z - Hx) , \tag{16}$$

which we can expand to write as follows

$$J = z^T W z - 2z^T W H x + x^T H^T W H x = z^T W z - 2(H^T W z)^T x + x^T H^T W H x .$$

Taking the first derivative of this expression with respect to the unknown vector x using Equations 311 and 312 gives

$$\frac{\partial J}{\partial x} = -2H^T W z + (H^T W H + H^T W H)x = -2H^T W z + 2H^T W H x .$$

Setting this derivative equal to zero and solving for x (which we denote as $\hat{x}$) gives

$$\hat{x} = (H^T W H)^{-1} H^T W z , \tag{17}$$

the result quoted in the book. The second derivative of J with respect to x is given by

$$\frac{\partial^2 J}{\partial x^2} = 2H^T W H . \tag{18}$$

This matrix is positive semi-definite if the elements on the diagonal of W are non-negative and the solution given in Equation 17 to the first order optimality condition $\frac{\partial J}{\partial x} = 0$ gives a minimum.

Problem 2-9 (the distribution of the sum of three uniform random variables)

If X is a uniform random variable over $(-1, +1)$ then it has a p.d.f. given by

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} ,$$

while the random variable $Y = \frac{X}{3}$ is another uniform random variable with a p.d.f. given by

$$p_Y(y) = \begin{cases} \frac{3}{2} & -\frac{1}{3} \le x \le \frac{1}{3} \\ 0 & \text{otherwise} \end{cases} .$$

Since the three random variables $X/3$, $Y/3$, and $Z/3$ are independent the characteristic function of the sum of them is the product of the characteristic function of each one of them. For a uniform random variable over the domain (α, β) on can show that the characteristic function $\zeta(t)$ is given by Equation 21 or

$$\zeta(t) = \int_\alpha^\beta \frac{1}{\beta - \alpha} e^{itx} dx = \frac{e^{it\beta} - e^{it\alpha}}{it(\beta - \alpha)} ,$$

note this is a slightly different than the normal definition of the Fourier transform [9], which has e^{-itx} as the exponential argument. Thus for each of the random variables $X/3$, $Y/3$, and $Z/3$ the characteristic function since $\beta = \frac{1}{3}$ and $\alpha = -\frac{1}{3}$ looks like

$$\zeta(t) = \frac{3(e^{it(1/3)} - e^{-it(1/3)})}{2it}.$$

Thus the sum of two uniform random variables like $X/3$ and $Y/3$ has a characteristic function given by

$$\zeta^2(t) = -\frac{9}{4t^2}(e^{it(2/3)} - 2 + e^{-it(2/3)}),$$

and adding in a third random variable say $Z/3$ to the sum of the previous two will give a characteristic function that looks like

$$\zeta^3(t) = -\frac{27}{8i}\left(\frac{e^{it}}{t^3} - \frac{3e^{it(1/3)}}{t^3} + \frac{3e^{-it(1/3)}}{t^3} - \frac{e^{-it}}{t^3}\right).$$

Given the characteristic function of a random variable to compute its probability density function from it we need to evaluate the *inverse Fourier transform* of this function. That is we need to evaluate

$$p_W(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(t)^3 e^{-itw} dt.$$

Note that this later integral is equivalent to $\frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(t)^3 e^{+itw} dt$ (the standard definition of the inverse Fourier transform) since $\zeta(t)^3$ is an even function. To evaluate this integral then it will be helpful to convert the complex exponentials in $\zeta(t)^3$ into trigonometric functions by writing $\zeta(t)^3$ as

$$\zeta(t)^3 = \frac{27}{4}\left(\frac{3\sin\left(\frac{t}{3}\right)}{t^3} - \frac{\sin(t)}{t^3}\right). \tag{19}$$

Thus to solve this problem we need to be able to compute the inverse Fourier transform of two expressions like

$$\frac{\sin(\alpha t)}{t^3}.$$

To do that we will write it as a product with two factors as

$$\frac{\sin(\alpha t)}{t^3} = \frac{\sin(\alpha t)}{t} \cdot \frac{1}{t^2}.$$

This is helpful since we (might) now recognize as the *product* of two functions each of which we know the Fourier transform of. For example one can show [9] that if we define the step function $h_1(w)$ as

$$h_1(w) \equiv \left\{ \begin{array}{cc} \frac{1}{2} & |w| < \alpha \\ 0 & |w| > \alpha \end{array} \right.,$$

then the Fourier transform of this step function $h_1(w)$ is the first function in the product above or $\frac{\sin(\alpha t)}{t}$. Notationally, we can write this as

$$\mathcal{F}\left[\left\{ \begin{array}{cc} \frac{1}{2} & |w| < \alpha \\ 0 & |w| > \alpha \end{array} \right.\right] = \frac{\sin(\alpha t)}{t}.$$

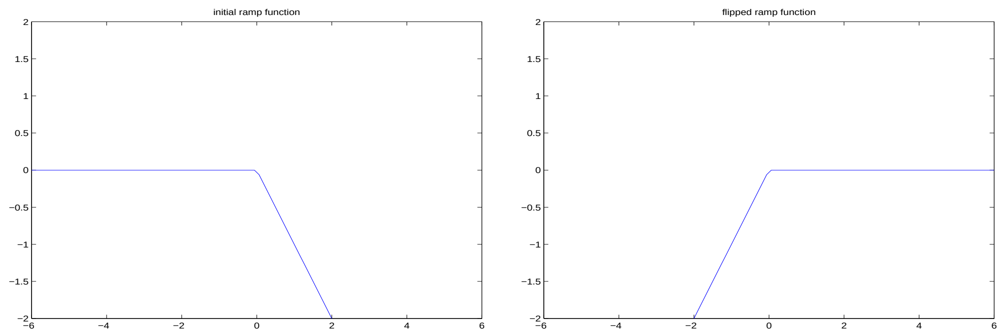

Figure 1: **Left:** The initial function $h_2(x)$ (a ramp function). **Right:** The ramp function flipped or $h_2(-x)$.

In the same way if we define the ramp function $h_2(w)$ as

$$h_2(w) = -w\,u(w)\,,$$

where $u(w)$ is the unit step function

$$u(w) = \begin{cases} 0 & w < 0 \\ 1 & w > 0 \end{cases}\,,$$

then the Fourier transform of $h_2(w)$ is given by $\frac{1}{t^2}$. Notationally in this case we then have

$$\mathcal{F}\left[-wu(w)\right] = \frac{1}{t^2}\,.$$

Since the inverse of a function that is the product of two functions for which we know the individual inverse Fourier transform of is the *convolution* integral of the two inverse Fourier transforms we have that

$$\mathcal{F}^{-1}\left[\frac{\sin(\alpha t)}{t^3}\right] = \int_{-\infty}^{\infty} h_1(x)h_2(w-x)dx\,,$$

the other ordering of the integrands

$$\int_{-\infty}^{\infty} h_1(w-x)h_2(x)dx\,,$$

can be shown to be an equivalent representation. To evaluate the above convolution integral and finally obtain the p.d.f for the sum of three uniform random variables we might as well select a formulation that is simple to evaluate. I'll pick the *first* formulation since it is easy to flip and shift to the ramp function $h_2(\cdot)$ distribution to produce $h_2(w-x)$. Now since $h_2(x)$ looks like the plot given in Figure 1 (left) we see that $h_2(-x)$ then looks like Figure 1 (right). Inserting a right shift by the value w we have $h_2(-(x-w)) = h_2(w-x)$, and this function looks like that shown in Figure 2 (left). The shifted factor $h_2(w-x)$ and our step function $h_1(x)$ are plotted together in Figure 2 (right). These considerations give a functional form

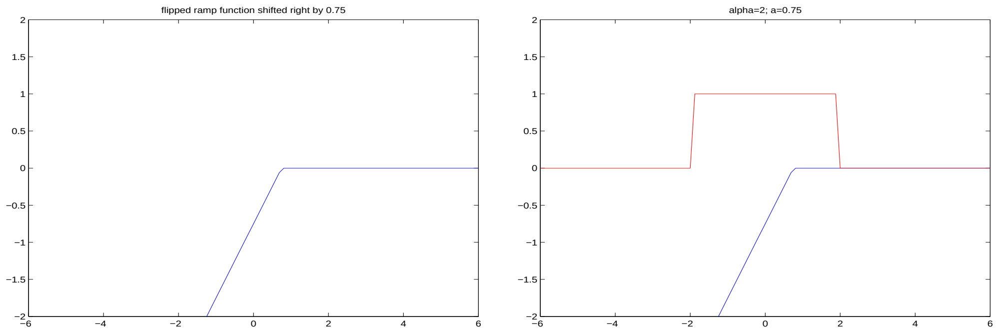

Figure 2: **Left:** The function $h_2(x)$, flipped and shifted by $w = 3/4$ to the right or $h_2(-(x-w))$. **Right:** The flipped and shifted function plotted together with $h_1(x)$ allowing visualizations of function overlap as w is varied.

for the p.d.f of $g_\alpha(w)$ given by

$$
\begin{aligned}
g_\alpha(w) &= \left\{ \begin{array}{cc} 0 & w < -\alpha \\ \int_{-\alpha}^{w} \frac{1}{2}(x-w)dx & -\alpha < w < +\alpha \\ \int_{-\alpha}^{+\alpha} \frac{1}{2}(x-w)dx & w > \alpha \end{array} \right. \\
&= \left\{ \begin{array}{cc} 0 & w < -\alpha \\ -\frac{1}{4}(\alpha+w)^2 & -\alpha < w < +\alpha \\ -\alpha w & w > \alpha \end{array} \right. ,
\end{aligned}
$$

when we evaluate each of the integrals. Using this and Equation 19 we see that

$$
\begin{aligned}
\mathcal{F}^{-1}[\zeta^3(t)] &= \frac{27}{4}(3g_{1/3}(w) - g_1(w)) \\
&= -\frac{81}{4} \left\{ \begin{array}{cc} 0 & w < -\frac{1}{3} \\ \frac{1}{4}(\frac{1}{3}+w)^2 & -\frac{1}{3} < w < +\frac{1}{3} \\ \frac{1}{3}w & w > \frac{1}{3} \end{array} \right. + \frac{27}{4} \left\{ \begin{array}{cc} 0 & w < -1 \\ \frac{1}{4}(1+w)^2 & -1 < w < +1 \\ w & w > 1 \end{array} \right. \\
&= \left\{ \begin{array}{cc} 0 & w < -1 \\ \frac{27}{16}(1+w)^2 & -1 < w < -\frac{1}{3} \\ -\frac{9}{8}(-1+3w^2) & -\frac{1}{3} < w < +\frac{1}{3} \\ \frac{27}{16}(-1+w)^2 & \frac{1}{3} < w < 1 \\ 0 & w > 1 \end{array} \right. ,
\end{aligned}
$$

which is equivalent to what we were to show. In the Mathematical file `chap_2_prob_9.nb` some of the algebra for this problem is worked.

Problem 2-10 (the Poisson probability density)

The distribution function for a Poisson random variable when the mean number of events we expect to observe is μ is given by

$$
F(x) = \sum_{i=0}^{x} \frac{e^{-\mu}\mu^i}{i!} = e^{-\mu} \sum_{i=0}^{x} \frac{\mu^i}{i!} .
$$

When are arrival rate is 0.4 arrivals per minute since in 10 minutes we would have a mean number of arrivals given by $\mu = 10(0.4) = 4$. Thus the probability of exactly four arrivals in 10 min is given by

$$f(x = 4|\mu = 4) = e^{-4}\frac{4^4}{4!} = 0.1954\,,$$

and the probability of no more than four arrivals in 10 minutes is given by

$$F(4) = e^{-4}\sum_{i=0}^{4}\frac{4^i}{i!} = 0.62884\,.$$

See the Matlab file `chap_2_prob_10.m` for calls to the `poisspdf` and `poisscdf` Matlab functions used in evaluating these two probabilities.

Problem 2-11 (a Rayleigh process)

For the first part of this problem lets define the random variable $Z = \sqrt{X^2 + Y^2}$ and attempt to compute the distribution function for the random variable Z. We have

$$\begin{aligned}
F_Z(z) &= \Pr\{Z \leq z\} = \Pr\left\{\sqrt{X^2+Y^2} \leq z\right\} \\
&= \int_{\sqrt{X^2+Y^2}\leq z} p(x,y)dxdy \\
&= \int_{\sqrt{X^2+Y^2}\leq z} p(x)p(y)dxdy \\
&= \int_{\sqrt{X^2+Y^2}\leq z} \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{x^2}{\sigma^2}\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{y^2}{\sigma^2}\right\}dxdy \\
&= \int_{\sqrt{X^2+Y^2}\leq z} \frac{1}{2\pi}\exp\left\{-\frac{1}{2}\frac{(x^2+y^2)}{\sigma^2}\right\}dxdy\,.
\end{aligned}$$

To evaluate this last integral we will change from Cartesian coordinates to polar coordinates. Let $r^2 = x^2 + y^2$ and the integral above becomes

$$\begin{aligned}
F_Z(z) &= \frac{1}{2\pi}\int_{r=0}^{z} e^{-\frac{1}{2}\frac{r^2}{\sigma^2}}2\pi r dr = \int_{r=0}^{z} r e^{-\frac{1}{2}\frac{r^2}{\sigma^2}}dr \\
&= \int_0^{\frac{1}{2}\frac{z^2}{\sigma^2}} e^{-v}dv = 1 - e^{-\frac{1}{2}\frac{z^2}{\sigma^2}}\,.
\end{aligned}$$

We will take the derivative of $F_z(z)$ to get the p.d.f for Z. We find

$$f_Z(z) = F_Z'(z) = \frac{1}{2}\frac{z}{\sigma^2}e^{-\frac{1}{2}\frac{z^2}{\sigma^2}}\,,$$

which is the desired expression.

Next we will compute the expectation of Z and Z^2 directly from the definition of the given Rayleigh density function. We have that

$$E(Z) = \int_{z=0}^{\infty}\frac{z^2}{\sigma^2}e^{-\frac{z^2}{2\sigma^2}}dz\,.$$

To evaluate this integral let $v = \frac{z^2}{2\sigma^2}$ so that $z = \sqrt{2}\sigma\sqrt{v}$ and $dz = \frac{\sigma}{\sqrt{2}}v^{-1/2}dv$ to get

$$\begin{aligned}
E(Z) &= \frac{1}{\sigma^2}\int_{v=0}^{\infty}(2\sigma^2 v)e^{-v}\frac{\sigma}{\sqrt{2}}v^{-1/2}dv \\
&= \sqrt{2}\sigma\int_0^{\infty} v^{\frac{3}{2}-1}e^{-v}dv \\
&= \sqrt{2}\sigma\Gamma(\frac{3}{2}) = \sqrt{2}\sigma\left(\frac{1}{2}\right)\Gamma(\frac{1}{2}) \\
&= \sqrt{\frac{\pi}{2}}\sigma\,.
\end{aligned}$$

Next we calculate $E(Z^2)$. We find

$$E(Z^2) = \frac{1}{\sigma^2}\int_{v=0}^{\infty} z^3 e^{-\frac{z^2}{2\sigma^2}}dz\,.$$

Using the same transformations as was used to evaluate $E(Z)$ we get

$$\begin{aligned}
E(Z^2) &= \frac{1}{\sigma^2}\int_{v=0}^{\infty} 2^{3/2}\sigma^3 v^{3/2}e^{-v}\frac{\sigma}{\sqrt{2}}v^{-1/2}dv \\
&= \sigma^2 2\int_{v=0}^{\infty} v^{0+1}v^{-v}dv = 2\sigma^2\Gamma(0) = 2\sigma^2\,.
\end{aligned}$$

Thus the variance of Z is given by

$$\mathrm{Var}(Z) = E(Z^2) - E(Z)^2 = 2\sigma^2 - \sigma^2\frac{\pi}{2} = \sigma^2\left(2 - \frac{\pi}{2}\right)\,.$$

Problem 2-12 (a maneuvering vehicle)

From the given description the probability of various accelerations A is given by

$$\mathrm{Pr}(a) = \begin{cases} P_{\mathrm{max}} & a = -A_{\mathrm{max}} \\ P_0 & a = 0 \\ P_{\mathrm{max}} & a = +A_{\mathrm{max}} \\ b & -A_{\mathrm{max}} < a < +A_{\mathrm{max}} \end{cases}$$

To be a normalized probability density we must have the value of b satisfy

$$2P_{\mathrm{max}} + P_0 + b(2A_{\mathrm{max}}) = 1\,,$$

or solving for b we find

$$b = \frac{1 - (P_0 - 2P_{\mathrm{max}})}{2A_{\mathrm{max}}}\,.$$

Using this density the expectation of a is then given by

$$E(a) = -A_{\mathrm{max}}P_{\mathrm{max}} + A_{\mathrm{max}}P_{\mathrm{max}} + 0P_0 + \int_{-A_{\mathrm{max}}}^{A_{\mathrm{max}}} a\,b\,da = 0\,.$$

and the expectation of a^2 is given by

$$
\begin{aligned}
E(a^2) &= +A_{\max}^2 P_{\max} + A_{\max}^2 P_{\max} + 0^2 P_0 + \int_{-A_{\max}}^{A_{\max}} a^2\, b\, da \\
&= 2A_{\max}^2 P_{\max} + b\, \frac{a^3}{3}\Big|_{-A_{\max}}^{A_{\max}} \\
&= \frac{A_{\max}^2}{3}\left[1 + 4P_{\max} - P_0\right] ,
\end{aligned}
$$

when we evaluate. Since $E(a) = 0$ the value of the variance is given by $E(a^2)$.

Problem 2-13 (statistics for the uniform distribution)

The uniform distribution has a characteristic function that can be computed directly

$$
\begin{aligned}
\zeta(t) &= E(e^{itX}) = \int_a^b e^{itx} \frac{1}{b-a} dx \qquad (20) \\
&= \frac{1}{b-a}\left(\frac{e^{itb} - e^{ita}}{it}\right) . \qquad (21)
\end{aligned}
$$

We could compute $E(X)$ using the characteristic function $\zeta(t)$ for a uniform random variable. Beginning this calculation we have

$$
\begin{aligned}
E(X) &= \frac{1}{i} \frac{\partial \zeta(t)}{\partial t}\Big|_{t=0} \\
&= \frac{1}{i}\frac{1}{b-a}\left[\frac{1}{it}(ibe^{itb} - iae^{ita}) - \frac{1}{it^2}(e^{itb} - e^{ita})\right]\Bigg|_{t=0} \\
&= -\frac{1}{b-a}\left[\frac{t(ibe^{itb} - iae^{ita}) - (e^{itb} - e^{ita})}{t^2}\right]\Bigg|_{t=0} .
\end{aligned}
$$

To evaluate this expression requires the use of L'Hopital's rule, and seems a somewhat complicated route to compute $E(X)$. The evaluation of $E(X^2)$ would probably be even more work when computed from the characteristic function. For this distribution, it is much easier to compute the expectations directly. We have

$$
E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b = \frac{1}{2}(a+b) .
$$

In the same way we find $E(X^2)$ to be given by

$$
\begin{aligned}
E(X^2) &= \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{b^3 - a^3}{3}\right) \\
&= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{1}{3}(b^2 + ab + a^2) .
\end{aligned}
$$

Using these two results we thus have that the variance of a uniform random variable is

$$\begin{aligned}
\text{Var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{1}{3}(b^2 + ab + a^2) - \frac{1}{4}(a^2 + b^2 + 2ab) \\
&= \frac{(b-a)^2}{12}.
\end{aligned}$$

Problem 2-14 (the distribution of X_1+X_2 when X_1 and X_2 are correlated normals)

The joint p.d.f of X_1 and X_2 is given by

$$f_2(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-\frac{x_1^2}{\sigma_1^2} - 2\rho\frac{x_1}{\sigma_1}\frac{x_2}{\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right]\right\}, \tag{22}$$

and we want to determine what the probability density function of $Z = X_1 + X_2$ is. To do that consider the distribution function for the random variable Z. From the definition of the distribution function we have

$$\begin{aligned}
F_Z(l) &= \Pr\{Z \le l\} = \Pr\{X_1 + X_2 \le l\} \\
&= \int_{x_2=-\infty}^{\infty}\int_{x_1=-\infty}^{l-x_2} f_2(x_1, x_2)dx_1dx_2 .
\end{aligned}$$

It would be nice to be able to evaluate this expression directly but it might be simpler to determine the functional form of $f_Z(l)$ by taking the derivative of the above with respect to l and then evaluating the resulting integral. We find

$$\begin{aligned}
F_Z'(l) &= \int_{x_2=-\infty}^{\infty} f_2(l - x_2, x_2)dx_2 \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{x_2=-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[-\frac{(l-x_2)^2}{\sigma_1^2} - 2\rho\frac{(l-x_2)}{\sigma_1}\frac{x_2}{\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right\}\right\}.
\end{aligned}$$

In the argument in the exponent we can expand everything in terms of x_2, complete the square and write it as

$$-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2} + \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right)\left[x_2 - \left(\frac{l\sigma_2(\rho\sigma_1+\sigma_2)}{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}\right)\right]^2 - \frac{l^2}{2(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)}.$$

Using this we see that the value of $F_Z'(l)$ is the integral of the exponential of this expression over the entire real line. Since x_2 goes from $-\infty$ to $+\infty$ the "shift" amount of $\left(\frac{l\sigma_2(\rho\sigma_1+\sigma_2)}{\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2}\right)$ in the quadratic above can translated away and we get

$$F_Z'(l) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{l^2}{2(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)}} \int_{x_2=-\infty}^{\infty} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{1}{\sigma_1^2} + \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right)x_2^2\right\}dx_2 .$$

To evaluate this expression recall that because of the normalization of the Gaussian probability density that $\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}dx = \sqrt{2\pi}\sigma$ and the above becomes

$$F_Z'(l) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2}} e^{-\frac{l^2}{2(\sigma_1^2+2\rho\sigma_1\sigma_2+\sigma_2^2)}} .$$

Note that this expression is the probability density function of a normal random variable with a mean value of zero and a variance given by $\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2$. In the Mathematical file `chap_2_prob_14.nb` some of the algebra for this problem is worked.

Chapter 3 (Linear Dynamic Systems)

Notes on the text

Notes on Example 3.1.2 (verification of the derivation of the differential system)

Working through the block diagram presented in the text in figure 3.1-4 for this example we find that the various state variables must be related as follows

$$
\begin{aligned}
\varepsilon_a - \phi g &= \delta \dot{v} \\
\delta \dot{p} &= \delta v \\
\frac{\delta v}{R} + \varepsilon_g &= \dot{\phi}\,.
\end{aligned}
$$

If we solve for the derivative variable and assume a state vector given by $\begin{bmatrix} \phi \\ \delta v \\ \delta p \end{bmatrix}$ we find

$$
\begin{aligned}
\dot{\phi} &= \begin{bmatrix} 0 & 1/R & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \delta v \\ \delta p \end{bmatrix} + \varepsilon_g \\
\delta \dot{p} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \delta v \\ \delta p \end{bmatrix} \\
\delta \dot{v} &= \begin{bmatrix} -g & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \delta v \\ \delta p \end{bmatrix} + \varepsilon_a\,.
\end{aligned}
$$

which when written as a first order matrix system is given by the books equation 3.1-13.

Verification of the analytic solution to the continuous linear system

We are told that a solution to the continuous linear system with a time dependent companion matrix $F(t)$ or

$$\dot{x}(t) = F(t)x(t) + L(t)u(t)\,, \tag{23}$$

is given by

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)u(\tau)d\tau\,. \tag{24}$$

To verify this take the derivative of $x(t)$ with respect to time. We find

$$\begin{aligned}
x'(t) &= \Phi'(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi'(t,\tau)L(\tau)u(\tau)d\tau + \Phi(t,t)L(t)u(t) \\
&= F(t)\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} F(t)\Phi(t,\tau)L(\tau)u(\tau)d\tau + L(t)u(t) \\
&= F(t)\left[\Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau)L(\tau)u(\tau)d\tau\right] + L(t)u(t) \\
&= F(t)x(t) + L(t)u(t)\,.
\end{aligned}$$

showing that the expression given in Equation 24 is indeed a solution. Note that in the above we have used the fact that for a fundamental solution $\Phi(t,t_0)$ we have $\Phi'(t,t_0) = F(t)\Phi(t,t_0)$.

Notes on the derivation of the matrix superposition integral (Example 3.3-1)

We will seek a solution $x(t)$ of the form

$$x(t) = \Phi(t,t_0)\xi(t) \tag{25}$$

to our differential equation given by

$$\frac{dx(t)}{dt} = F(t)x(t) + L(t)u(t)\,.$$

When we put our hypothesized expression for $x(t)$ given by Equation 25 into the above equation we get

$$\frac{d}{dt}[\Phi(t,t_0)\xi(t)] = F(t)\Phi(t,t_0)\xi(t) + L(t)u(t)\,,$$

or expanding the time derivative on the left-hand-side we get

$$F(t)\Phi(t,t_0)\xi(t) + \Phi(t,t_0)\frac{d\xi}{dt} = F(t)\Phi(t,t_0)\xi(t) + L(t)u(t)\,.$$

Where we have used the fact that

$$\frac{d}{dt}\Phi(t,t_0) = F(t)\Phi(t,t_0)\,. \tag{26}$$

Canceling the common terms on both sides of this expression we get

$$\Phi(t,t_0)\frac{d\xi}{dt} = L(t)u(t)\,.$$

When we solve this for $\frac{d\xi}{dt}$ we find

$$\frac{d\xi}{dt} = \Phi(t,t_0)^{-1}L(t)u(t) = \Phi(t_0,t)L(t)u(t)\,,$$

since

$$\Phi(t,t_0)^{-1} = \Phi(t_0,t)\,. \tag{27}$$

When we integrate the above expression we find that $\xi(t)$ is given by

$$\xi(t) = \xi(t_0) + \int_{t_0}^{t} \Phi(t_0, \tau) L(\tau) u(\tau) d\tau \,.$$

Putting this expression into Equation 25 we get for $x(t)$ the following

$$x(t) = \Phi(t, t_0)\xi(t_0) + \int_{t_0}^{t} \Phi(t, t_0)\Phi(t_0, \tau) L(\tau) u(\tau) d\tau \,.$$

Since the product of the two Φ functions inside the integral simplifies as

$$\Phi(t, t_0)\Phi(t_0, \tau) = \Phi(t, \tau) \,, \tag{28}$$

and $\xi(t_0) = x(t_0)$ the above expression for $x(t)$ becomes

$$x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, \tau) L(\tau) u(\tau) d\tau \,, \tag{29}$$

or the matrix superposition integral as we were trying to show.

Notes on state vector augmentation: some common correlated noise models

The **random ramp** disturbance can be modeled with the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \,. \end{aligned}$$

From the equation for $x_2(t)$ by integrating we have that $x_2(t) = x_2(0)$ where $x_2(0)$ is the *random* constant initial condition. It is worth repeating the point about the randomness of $x_2(0)$. The value of $x_2(0)$ is not known beforehand but is assumed to be generated from a distribution. Once the random value is generated and observed, the value of $x_2(t)$ is specified for all later time. Then using the first equation we have that $\dot{x}_1 = x_2(0)$ so that $x_1(t) = x_2(0)t + x_1(0)$, where $x_1(0)$ is *another* random initial condition. Thus if we consider $x_1(0)$ to be the "mean value" of $x_1(t)$ then

$$E[(x_1(t) - x_1(0))^2] = E[x_2(0)]t^2 \,,$$

showing the quadratic growth of the variance expected with a random ramp noise model.

For the **exponentially correlated** random variables the state differential equation is given by

$$\dot{x} = -\beta x + w \,,$$

then from this representation the system function F is $-\beta$ and if we assume $w(t)$ is uncorrelated white noise so that $E[w(t)w(\tau)] = q(t)\delta(t - \tau)$ then the **linear variance equation**

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + G(t)Q(t)G(t)^T \,, \tag{30}$$

in this scalar case becomes

$$\dot{p}(t) = -2\beta p(t) + 1^2 q(t)\,.$$

In steady state $\dot{p}(t) = 0$ and taking $q(t) = q$ (a constant) and then solving for p the steady-state error variance with exponentially correlated random variables we have

$$p = E[x^2] = \frac{q}{2\beta}\,,$$

since the *definition* of p is $p = E[x^2]$. If we want to consider the case of exponentially correlated random variables in the *discrete* setting the discrete system equation in that case is given by

$$x_{k+1} = e^{-\beta(t_{k+1}-t_k)} x_k + w_k\,.$$

To evaluate the various terms in the **discrete error covariance extrapolation equation**

$$P_{k+1} = \Phi_k P_k \Phi_k^T + \Gamma_k Q_k \Gamma_k^T\,, \tag{31}$$

we will use the results from the book that translate from the continuous time model to the discrete time model. Recall that the continuous noise produces a discrete noise term $\Gamma_k Q_k \Gamma_k^T$ that is given by

$$\Gamma_k Q_k \Gamma_k^T = \int_{t_k}^{t_{k+1}} \Phi(t_{k+1},\tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t_{k+1},\tau)^T d\tau\,. \tag{32}$$

For the continuous problem where the fundamental solution is given by $\Phi(t, t_0) = e^{-\beta(t-t_0)}$ and $G = 1$ so we can evaluate Equation 32 taking $Q(t) = q$ a constant as

$$\begin{aligned}
\Gamma_k Q_k \Gamma_k^T &= \int_{t_k}^{t_{k+1}} e^{-\beta(t_{k+1}-\tau)} q e^{-\beta(t_{k+1}-\tau)} d\tau \\
&= \frac{q}{2\beta}(1 - e^{-2\beta(t_{k+1}-t_k)})\,,
\end{aligned}$$

or the books equation 3.8-20.

Notes on time series analysis

In the discussion on time series analysis given in the text the focus is on ARMA(p,q) models for the output process z_k given an input process r_k. This means that we assume that our output, z_k, can be expressed as a sum of p values of its past realizations (termed the autoregressive part) and q values of the innovative input process r_k (called the moving average part). Mathematically this is expressed as

$$z_k = \sum_{i=1}^{p} b_i z_{k-i} + r_k - \sum_{i=1}^{q} c_i r_{k-i}\,. \tag{33}$$

for some coefficients b_i and c_i. We can cast this formulation into a state-space representation in several ways. The book recommends the following

$$\mathbf{x}_k = \left[\begin{array}{c} r_{k-q} \\ r_{k-q+1} \\ \vdots \\ r_{k-2} \\ r_{k-1} \\ \hline z_{k-p} \\ z_{k-p+1} \\ \vdots \\ z_{k-2} \\ z_{k-1} \\ \hline \tilde{z}_k(-) \end{array}\right] . \tag{34}$$

The first block of $\mathbf{x}$ is the moving average MA(q) part, the second block of $\mathbf{x}$ is the AR(p) part and the third block (the single element $\tilde{z}(-)$) is discussed below. This third element in the book is written as $z_k(-)$ but with an ∞ symbol above it. Since we *observe* the system output z_k which is determined from the p previous values z_{k-i} for $i = 1, 2, \cdots p$ and the *observed* zero mean random q previous system inputs r_{k-i} for $i = 1, 2, \cdots q$ the state representation above uses those previously observed values. The last element $\tilde{z}_k(-)$ is the best estimate of the *prediction* of z_k given the information thus far. Since we have *not* observed r_k at this point our prediction is given by the sum of the terms we have observed

$$\tilde{z}_k = \sum_{i=1}^{p} b_i z_{k-i} - \sum_{i=1}^{q} c_i r_{k-i} \, . \tag{35}$$

Note that from Equation 33 this is also equal to $z_k - r_k$. To derive the discrete time propagation equation $\mathbf{x}_{k+1} = \Phi_k \mathbf{x}_k$ we note that since

$$\mathbf{x}_{k+1} = \left[\begin{array}{c} r_{k-q+1} \\ r_{k-q+2} \\ \vdots \\ r_{k-1} \\ r_k \\ \hline z_{k-p+1} \\ z_{k-p+2} \\ \vdots \\ z_{k-1} \\ z_k \\ \hline \tilde{z}_{k+1}(-) \end{array}\right] ,$$

most of the variables in $\mathbf{x}_{k+1}$ are "shifted up" and can be directly found in $\mathbf{x}_k$. The ones that are not are r_k, z_k, and $\tilde{z}_{k+1}(-)$. The first, r_k, we treat as a source of process noise. The second, z_k, we can obtain from $\tilde{z}_k(-) + r_k$ the sum of a term in the state $\mathbf{x}_k$ and the process noise r_k. The third we express as follows

$$
\begin{aligned}
\tilde{z}_{k+1}(-) &= b_1 z_k - c_1 r_k + \sum_{i=2}^{p} b_i z_{k+1-i} - \sum_{i=2}^{q} c_i r_{k+1-i} \\
&= b_1(z_k - r_k) + b_1 r_k - c_1 r_k + \sum_{i=2}^{p} b_i z_{k+1-i} - \sum_{i=2}^{q} c_i r_{k+1-i} \\
&= b_1 \tilde{z}_k(-) + (b_1 - c_1) r_k + \sum_{i=2}^{p} b_i z_{k+1-i} - \sum_{i=2}^{q} c_i r_{k+1-i} \,.
\end{aligned}
$$

Taken together all of these considerations given the books equation 3.9-16.

Problem Solutions

Problem 3-1 (proving the solution to the linear variance equation)

For this problem we want to show that $P(t)$ given by

$$
P(t) = \Phi(t, t_0) P(t_0) \Phi(t, t_0)^T + \int_{t_0}^{t} \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T d\tau \,. \tag{36}
$$

is a solution to the linear variance equation. We can do this by first taking the derivative of the given expression for $P(t)$ with respect to t. We find

$$
\begin{aligned}
\frac{dP}{dt} &= \frac{d\Phi(t, t_0)}{dt} P(t_0) \Phi(t, t_0)^T + \Phi(t, t_0) P(t_0) \frac{d\Phi(t, t_0)^T}{dt} \\
&+ \Phi(t, t) G(t) Q(t) G^T(t) \Phi(t, t)^T \\
&+ \int_{t_0}^{t} \frac{d\Phi(t, \tau)}{dt} G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T d\tau + \int_{t_0}^{t} \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \frac{d\Phi(t, \tau)^T}{dt} d\tau \,.
\end{aligned}
$$

Recall that the fundamental solution $\Phi(t, t_0)$ satisfies the following $\frac{d\Phi(t,t_0)}{dt} = F(t)\Phi(t, t_0)$ and that $\Phi(t, t) = I$ with I the identity matrix. With these expressions the right-hand-side of $\frac{dP}{dt}$ then becomes

$$
\begin{aligned}
\frac{dP}{dt} &= F(t) \Phi(t, t_0) P(t_0) \Phi(t, t_0)^T + \Phi(t, t_0) P(t_0) \Phi(t, t_0)^T F^T(t) + G(t) Q(t) G(t)^T \\
&+ \int_{t_0}^{t} F(t) \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T d\tau + \int_{t_0}^{t} \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T F(t)^T d\tau \\
&= F(t) \left[\Phi(t, t_0) P(t_0) \Phi(t, t_0)^T + \int_{t_0}^{t} \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T d\tau \right] \\
&+ \left[\Phi(t, t_0) P(t_0) \Phi(t, t_0)^T + \int_{t_0}^{t} \Phi(t, \tau) G(\tau) Q(\tau) G(\tau)^T \Phi(t, \tau)^T d\tau \right] F(t)^T + G(t) Q(t) G(t)^T \qquad (37) \\
&= F(t) P(t) + P(t) F(t)^T + G(t) Q(t) G(t)^T \,,
\end{aligned}
$$

as a differential equation for $P(t)$.

Problem 3-2 (the steady-state solution to the linear variance equation)

Consider the linear variance equation $\dot{P}(t) = FP + PF^T + Q$, then the solution $P(t)$, to this equation is given in problem 3.1 above in Equation 36. Since our system is time-invariant we have $\Phi(t, \tau) = e^{F(t-\tau)}$ and the expression for $P(t)$ in this case becomes

$$\begin{aligned} P(t) &= e^{F(t-t_0)} P(t_0) e^{F^T(t-t_0)} + \int_{t_0}^{t} e^{F(t-\tau)} Q e^{F^T(t-\tau)} d\tau \\ &= e^{F(t-t_0)} P(t_0) e^{F^T(t-t_0)} + \int_{0}^{t-t_0} e^{Fv} Q e^{F^T v} dv \,. \end{aligned}$$

Where in the second line above we make the substitution $v = t - \tau$ in the integral. To make our above expression for P solve the desired equation from the problem or $FP + PF^T = -Q$, we will consider the *steady-state* solution to the linear variance equation by taking $t \to \infty$ in the above expression. In that case $P(t)$ is a constant so $\dot{P}(t) = 0$ and the linear variance equation reduces to the desired equation $FP + PF^T = -Q$. If our initial state $x(t_0)$ has no uncertainty ($P(t_0) = 0$) or if our linear system is stable we can assume that

$$\Phi(t, t_0) P(t_0) \Phi^T(t, t_0) \to 0 \,,$$

as $t \to \infty$, and the expression for $P(t)$ becomes

$$P = \lim_{t \to +\infty} P(t) = \int_0^{\infty} e^{Fv} Q e^{F^T v} dv \,.$$

the desired expression.

Problem 3-3 (analysis of an autocorrelation function)

Warning: I'm not entirely sure that I've worked this problem correctly since the answer I propose seems too simple to structure a problem around. If anyone sees an error in my solution or can offer verification that this is a correct result please email me.

To get an autocorrelation function of the functional form specified in $\hat{\phi}(\tau)$ we note that we can view it as the sum of three parts: $\sigma^2\alpha_1^2$, $\sigma^2\sigma_2^2\cos(\omega\tau)$, and $\sigma^2\alpha_3^2 e^{-\beta|\tau|}$. We next consider what type of random process give rise to each of these three autocorrelation functional forms.

From figure 3.8-3 in the book the system of a random constant or $\dot{x}_1 = 0$ has a constant autocorrelation function and we can take $\hat{\phi}_1(\tau) = \sigma^2\alpha_1^2$.

From example 2.2-2 in the book the periodic signal

$$x_2(t) = A\sin(\omega t + \theta) \,,$$

with θ a uniform random variable with density $f_\Theta(\theta) = \frac{1}{2\pi}$ on $0 \le \theta \le 2\pi$ has an autocorrelation given by $\frac{A^2}{2}\cos(\omega\tau)$. If we take $\frac{A^2}{2} = \sigma^2\alpha_2^2$ or $A = \sqrt{2}\sigma\alpha_2$ then the signal $x_2(t)$ has a autocorrelation function $\hat{\phi}_2(\tau) = \sigma^2\alpha_2^2\cos(\omega\tau)$

Finally the system

$$\dot{x}_3 = -\beta x_3 + w\,,$$

where $w(t)$ is white noise signal with $E[w(t)w(\tau)] = \sigma^2\alpha_3^2\delta(t-\tau)$ has an autocorrelation function given by $\hat{\phi}_3(\tau) = \sigma^2\alpha_3^2 e^{-\beta|\tau|}$.

Thus if we consider the total system

$$\begin{aligned}
\dot{x}_1 &= 0 \\
\ddot{x}_2 &= -x_2 \\
\dot{x}_3 &= -\beta x_3 + w(t)\,,
\end{aligned}$$

with $E[w^2] = \sigma^2\alpha_3^2$, then $x(t)$ defined as the sum of the three terms

$$x(t) = x_1(t) + x_2(t) + x_3(t)\,,$$

then $x(t)$ will have the given autocorrelation function. Note that the differential equation for $x_2(t)$ is of second order. From the above decomposition we see that $x(t)$ is the sum of three parts, a constant term, an oscillatory term, and an exponentially decaying term.

Problem 3-4 (a simple integrator)

Warning: For this problem I was unable to get the result quoted in the book and was unable to find an error in my work or assumptions below. If anyone sees anything wrong with what I have done please email me, I would be interested in determining what the problem is. Perhaps it is a typo in the books expression for $P(t)$?

The diagram in Figure 3.1 gives the following system for the variables $x_1(t)$ and $x_2(t)$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 + w\,,
\end{aligned}$$

with $E[w(t)w(\tau)] = \sigma^2\delta(t-\tau)$. We have been able to write down the differential equation for $x_2(t)$ from the given expression for its autocorrelation $\phi_{x_2 x_2}(\tau) = \sigma^2 e^{-\beta|\tau|}$ using the discussion in the book on exponentially correlated random variables. If we introduce the state vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then from the above we have a linear system for $\mathbf{x}$ given by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ w \end{bmatrix}.$$

From which we see that our system matrix F is given by $F = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}$. With F defined in this way the linear variance equation given by 30 for this problem them becomes

$$\begin{aligned}
\begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix} \\
&= \begin{bmatrix} 2p_{12} & p_{22} - \beta p_{12} \\ -\beta p_{12} + p_{22} & -2\beta p_{22} + \sigma^2 \end{bmatrix}.
\end{aligned}$$

This gives the following system for $p_{11}(t)$, $p_{12}(t)$, and $p_{22}(t)$

$$\begin{aligned}\dot{p}_{11} &= 2p_{12}\\ \dot{p}_{12} &= p_{22} - \beta p_{12}\\ \dot{p}_{22} &= -2\beta p_{22} + \sigma^2\,.\end{aligned}$$

Here we take initial conditions of $p_{11}(0) = 0$, $p_{12}(0) = 0$, and $p_{22}(0) = \sigma^2$, meaning that initially we have uncertainty only in the component x_2. Then we find a solution to $p_{22}(t)$ given by

$$p_{22}(t) = \frac{\sigma^2}{2\beta}(1 + (2\beta - 1)e^{-2\beta t})\,.$$

which is *not* the same as the expression for $p_{22}(t)$ given in the book which is simply σ^2. In the Mathematical file `chap_3_prob_4.nb` this and the differential equations for $p_{11}(t)$ and $p_{12}(t)$ are solved. I find it strange that the $(2,2)$ component of $P(t)$ is constant independent of t while the other elements are not.

Note that since in this problem the system matrix F is time invariant the fundamental solution is given by $\Phi(t) = e^{Ft}$. Since the matrix F in this case is $\left[\begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array}\right]$, we can compute powers of F directly. We find

$$\begin{aligned}
F^2 &= \left[\begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array}\right]\left[\begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array}\right] = \left[\begin{array}{cc} 0 & -\beta \\ 0 & \beta^2 \end{array}\right]\\
F^3 &= \left[\begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array}\right]\left[\begin{array}{cc} 0 & -\beta \\ 0 & \beta^2 \end{array}\right] = \left[\begin{array}{cc} 0 & \beta^2 \\ 0 & -\beta^3 \end{array}\right]\\
F^4 &= \left[\begin{array}{cc} 0 & 1 \\ 0 & -\beta \end{array}\right]\left[\begin{array}{cc} 0 & \beta^2 \\ 0 & -\beta^3 \end{array}\right] = \left[\begin{array}{cc} 0 & -\beta^3 \\ 0 & \beta^4 \end{array}\right]\\
&\vdots\\
F^{2n} &= \left[\begin{array}{cc} 0 & -\beta^{2n-1} \\ 0 & \beta^{2n} \end{array}\right]\\
F^{2n+1} &= \left[\begin{array}{cc} 0 & \beta^{2n} \\ 0 & -\beta^{2n+1} \end{array}\right].
\end{aligned}$$

Using these we find that

$$\begin{aligned}
\Phi(t) &= e^{Ft} = I + Ft + \frac{1}{2}F^2t^2 + \frac{1}{6}F^3t^3 + \cdots = \sum_{k=0}^{\infty}\frac{F^{2k}t^{2k}}{(2k!)} + \sum_{k=0}^{\infty}\frac{F^{2k+1}t^{2k+1}}{(2k+1)!}\\
&= I + \sum_{k=1}^{\infty}\frac{t^{2k}}{2k!}\left[\begin{array}{cc} 0 & -\beta^{2k-1} \\ 0 & \beta^{2k} \end{array}\right]\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}\left[\begin{array}{cc} 0 & \beta^{2k} \\ 0 & -\beta^{2k+1} \end{array}\right]\\
&= I + \left[\begin{array}{cc} 0 & -\frac{1}{\beta}\sum_{k=1}^{\infty}\frac{t^{2k}\beta^{2k}}{2k!} \\ 0 & \sum_{k=1}^{\infty}\frac{t^{2k}\beta^{2k}}{2k!} \end{array}\right] + \left[\begin{array}{cc} 0 & \frac{1}{\beta}\sum_{k=0}^{\infty}\frac{t^{2k+1}\beta^{2k+1}}{(2k+1)!} \\ 0 & -\sum_{k=0}^{\infty}\frac{t^{2k+1}\beta^{2k+1}}{(2k+1)!} \end{array}\right]\\
&= I + \left[\begin{array}{cc} 0 & -\frac{1}{\beta}(\cosh(t\beta) - 1) \\ 0 & \cosh(t\beta) - 1 \end{array}\right] + \left[\begin{array}{cc} 0 & \frac{1}{\beta}\sinh(t\beta) \\ 0 & -\sinh(t\beta) \end{array}\right]\\
&= \left[\begin{array}{cc} 1 & -\frac{1}{\beta}\cosh(\beta t) + \frac{1}{\beta}\sinh(\beta t) \\ 0 & \cosh(\beta t) - \sinh(\beta t) \end{array}\right].
\end{aligned}$$

Problem 3-5 (is this system observable)

To begin with we express the given diagram figure 3-2 in terms of mathematical equations. We then study the observability of these equations. To begin with from the given diagram we see that the gyro vertical defection (ξ) error e_ξ has two terms, a bias term $e_{\xi b}$ and a random term $e_{\xi r}$ and can be expressed as the sum of these two as

$$e_\xi = e_{\xi b} + e_{\xi r} \,.$$

Following the flow diagram from left to right we next see that the variable δv is given by

$$\int -g(e_\xi + \delta p) = \delta v \,,$$

and that δp in terms of δv is given by

$$\int \frac{1}{R}\delta v = \delta p \,.$$

We expect that the gyro vertical deflection and position bias are driven by random initial constants (which we don't know) and thus have differential equations given by

$$\begin{aligned} \dot{e}_{\xi b} &= 0 \\ \dot{e}_{pb} &= 0 \,. \end{aligned}$$

Finally the velocity and position measurements z_v and z_p are related to the state variables as

$$\begin{aligned} z_v &= \delta v + e_v \\ z_p &= e_{pb} + e_p + \delta p \,. \end{aligned}$$

Thus if we take our state to be $\mathbf{x}^T = \begin{bmatrix} e_{pb} & \delta p & \delta v & e_{\xi b} \end{bmatrix}$ then our dynamical system in companion form is given by

$$\frac{d}{dt}\begin{bmatrix} e_{pb} \\ \delta p \\ \delta v \\ e_{\xi b} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{R}\delta v \\ -ge_{\xi b} - ge_{\xi r} - g\delta p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1/R & 0 \\ 0 & -g & 0 & -g \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{pb} \\ \delta p \\ \delta v \\ e_{\xi b} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -ge_{\xi r} \\ 0 \end{bmatrix} \,. \quad (38)$$

Part (a): If our measurement is z_p and expressed in terms of the state vector $\mathbf{x}$ as

$$z_p = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_{pb} \\ \delta p \\ \delta v \\ e_{\xi b} \end{bmatrix} + e_p \,,$$

so the measurement sensitivity matrix H in this case is $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$. Since our state vector is four dimensional the requirement that the state be observable requires that the block matrix

$$\begin{bmatrix} H^T \,|\, F^T H^T \,|\, (F^T)^2 H^T \,|\, (F^T)^3 H^T \end{bmatrix} \,, \quad (39)$$

have rank equal to four. When we compute the above matrix using the matrices F and H for this problem we find this matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -\frac{g}{R} & 0 \\ 0 & \frac{1}{R} & 0 & -\frac{g}{R^2} \\ 0 & 0 & -\frac{g}{R} & 0 \end{bmatrix} .$$

This matrix has rank 3 and thus our system with only a position measurement is *not* observable.

Part (b): If we have both position and velocity measurements then our measurement vector $\mathbf{z}$ is given by

$$\mathbf{z} = \begin{bmatrix} z_p \\ z_v \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_{pb} \\ \delta p \\ \delta v \\ e_{\xi b} \end{bmatrix} + \begin{bmatrix} e_p \\ e_v \end{bmatrix} .$$

So the measurement sensitivity matrix H in this case is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. When we compute the observability matrix in Equation 39 above we find that is is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -g & -\frac{g}{R} & 0 & 0 & \frac{g^2}{R} \\ 0 & 1 & \frac{1}{R} & 0 & 0 & -\frac{g}{R} & -\frac{g}{R^2} & 0 \\ 0 & 0 & 0 & -g & -\frac{g}{R} & 0 & 0 & \frac{g^2}{R} \end{bmatrix} .$$

For observability this system this matrix must have a rank of 4. Since the first and second row can be combined to yield the fourth row it can have rank at most three. It in fact has a rank of 3 indicating that even with two measurements the given state is still unobservable.

Part (c): Tor this part if we are told that $e_{pb} = 0$, that is the position measurement has no bias, our state is now of dimension three i.e. has the representation given by $\mathbf{x}^T = \begin{bmatrix} \delta p & \delta v & e_{\xi b} \end{bmatrix}$ and for observability of this state we need to consider the matrix

$$\left[\, H^T \, \middle| \, F^T H^T \, \middle| \, (F^T)^2 H^T \, \right] . \tag{40}$$

To be observable this matrix must be of rank 3. This matrix is easy to compute since it is the same observability matrix as in Part (b) above but without the last two columns or

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -g & -\frac{g}{R} & 0 \\ 0 & 1 & \frac{1}{R} & 0 & 0 & -\frac{g}{R} \\ 0 & 0 & 0 & -g & -\frac{g}{R} & 0 \end{bmatrix} .$$

This later matrix does have a rank of three and the resulting system *is* observable. To prevent error in algebraic manipulations the matrix multiplications required above are performed in the Mathematical file `chap_3_prob_5.nb`.

Problem 3-6 (an approximate solution)

We see that the matrix F in this case is $F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix}$. From this matrix we can compute powers of F. We find

$$
\begin{aligned}
F^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} \\
F^3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \\ 0 & 0 & -\alpha^2 \end{bmatrix} \\
F^4 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \\ 0 & 0 & -\alpha^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \alpha^2 \\ 0 & 0 & -\alpha^3 \\ 0 & 0 & \alpha^4 \end{bmatrix} \\
&\vdots \\
F^n &= \begin{bmatrix} 0 & 0 & (-1)^{n-2}\alpha^{n-2} \\ 0 & 0 & (-1)^{n-1}\alpha^{n-1} \\ 0 & 0 & (-1)^n\alpha^n \end{bmatrix}.
\end{aligned}
$$

Recall that the fundamental solution $\Phi(t, t_0)$ for a linear time invariant system is given by $\Phi(t, t_0) = e^{F(t-t_0)}$, which when we use the definition of the matrix exponential to evaluate this expression we find

$$
\begin{aligned}
\Phi(t, t_0) &= \Phi(t - t_0) = e^{F(t-t_0)} \\
&= I + F(t - t_0) + \frac{1}{2}F^2(t - t_0)^2 + \frac{1}{6}F^3(t - t_0)^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} (t - t_0) + \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \end{bmatrix} (t - t_0)^2 \\
&+ \frac{1}{6} \begin{bmatrix} 0 & 0 & -\alpha \\ 0 & 0 & \alpha^2 \\ 0 & 0 & -\alpha^2 \end{bmatrix} (t - t_0)^3 + \cdots
\end{aligned}
$$

Lets take $T = t - t_0$ and sum the components of these matrices. We find that

$$
\Phi(T) = \begin{bmatrix} 1 & T & \frac{T^2}{2} - \alpha\frac{T^3}{6} + \cdots \\ 0 & 1 & T - \alpha\frac{T^2}{2} + \frac{\alpha^2}{6}T^3 + \cdots \\ 0 & 0 & 1 - \alpha T + \frac{\alpha^2}{2}T^2 - \frac{\alpha^3}{6}T^3 + \cdots \end{bmatrix}
$$

Note that we could explicitly evaluate each of these sums directly in terms of the exponential function $e^{\cdot}$, if needed. For example, the $(1, 3)$ element of $\Phi(T)$ above can be written as

$$
\frac{T^2}{2} - \alpha\frac{T^3}{6} + \cdots = \frac{e^{-\alpha T} - 1 + \alpha T}{\alpha^2}.
$$

If we take only the most significant term in each sum above we find that $\Phi(T)$ is approximately equal to

$$\Phi(T) = \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix},$$

as we were to show.

Problem 3-7 (deriving the controllability criterion)

The discrete system given in this problem is

$$x_{k+1} = \Phi x_k + \lambda u_k,$$

where λ is a constant vector. The books discussion on controllability, when specified explicitly for this system gives exactly the requirement stated. That is, the matrix Θ given by

$$\Theta = \left[\; \lambda \,\middle|\, \Phi\lambda \,\middle|\, \Phi^2\lambda \,\middle|\, \cdots \,\middle|\, \Phi^{n-2}\lambda \,\middle|\, \Phi^{n-1}\lambda \; \right], \tag{41}$$

must have rank n for this system to be controllable. As a *direct* way obtain this result, we recall that the definition of controllability is that given an arbitrary input x_0 we can specify a set of controls u_i such that the state x_n after n stages takes any desired value. To build up an intuition for Equation 41 we find that on the first stage after one control u_0, has been specified that we arrive at the state x_1 via

$$x_1 = \Phi x_0 + \lambda u_0.$$

On the second stage after the two controls (u_0 and u_1) have been specified we have the state x_2 via

$$x_2 = \Phi x_1 + \lambda u_1 = \Phi(\Phi x_0 + \lambda u_0) + \lambda u_1 = \Phi^2 x_0 + \Phi\lambda u_0 + \lambda u_1.$$

In the same way, on the third stage after three controls u_0, u_1, and u_2 we have the state x_3 via

$$x_3 = \Phi^3 x_0 + \Phi^2\lambda u_0 + \Phi\lambda u_1 + \lambda u_2.$$

Generalizing the above, at the nth stage we have used n controls and have the state x_n in terms of these controls given by

$$x_n = \Phi^n x_0 + \Phi^{n-1}\lambda u_0 + \Phi^{n-2}\lambda u_1 + \cdots + \Phi^2\lambda u_{n-3} + \Phi\lambda u_{n-2} + \lambda u_{n-1}.$$

We can write the above equation as a vector equation as

$$x_n = \left[\; \lambda \,\middle|\, \Phi\lambda \,\middle|\, \Phi^2\lambda \,\middle|\, \cdots \,\middle|\, \Phi^{n-2}\lambda \,\middle|\, \Phi^{n-1}\lambda \; \right] \begin{bmatrix} u_{n-1} \\ u_{n-2} \\ u_{n-3} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix} + \Phi^n x_0.$$

From the above we see that if the matrix

$$\Theta = \left[\begin{array}{c|c|c|c|c|c} \lambda & \Phi\lambda & \Phi^2\lambda & \cdots & \Phi^{n-2}\lambda & \Phi^{n-1}\lambda \end{array} \right] ,$$

is invertible, then we can specify the n control values u_i to get any state x_n and vice versa. As another way of stating this result is the following. Given an arbitrary initial state x_0 and a target state x_n we can compute a vector $\mathbf{u}$ of controls $\mathbf{u} = \left[\begin{array}{cccccc} u_0 & u_1 & u_2 & \cdots & u_{n-2} & u_{n-1} \end{array} \right]^T$ such that we arrive at the target state x_n in n steps by solving the system

$$\left[\begin{array}{c|c|c|c|c|c} \lambda & \Phi\lambda & \Phi^2\lambda & \cdots & \Phi^{n-2}\lambda & \Phi^{n-1}\lambda \end{array} \right] \left[\begin{array}{c} u_{n-1} \\ u_{n-2} \\ u_{n-3} \\ \vdots \\ u_1 \\ u_0 \end{array} \right] = x_n - \Phi^n x_0 .$$

for the vector $\mathbf{u}$. This requires the invertibility of Θ, or equivalently that Θ must have rank n, which is what we wanted to show.

Problem 3-8 (the discrete state transition matrix)

For this problem I have denoted the value of the signal on the first "loopback" line $x_2(t)$ (since this signal will get multiplied by $\frac{1}{T_2}$) and the value of the signal on the second "loopback" line as $x_1(t)$ (since this signal will get multiplied by $\frac{1}{T_1}$). Under that convention, the differential equation for the system given by figure 3-3 is then given by

$$\begin{aligned} \dot{x}_1(t) &= -\frac{1}{T_1} x_1(t) + \frac{1}{T_1} x_2(t) \\ \dot{x}_2(t) &= -\frac{1}{T_2} x_2(t) + \frac{1}{T_2} w(t) . \end{aligned}$$

If our system state is $\left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right]$ then the system above can be written in terms of matrices as

$$\frac{d}{dt} \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right] = \left[\begin{array}{cc} -\frac{1}{T_1} & \frac{1}{T_1} \\ 0 & -\frac{1}{T_2} \end{array} \right] \left[\begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right] + \left[\begin{array}{c} 0 \\ \frac{1}{T_2} w(t) \end{array} \right] .$$

From this expression we see that the F matrix for this problem is given by

$$F = \left[\begin{array}{cc} -\frac{1}{T_1} & \frac{1}{T_1} \\ 0 & -\frac{1}{T_2} \end{array} \right] .$$

Since this is independent of time the fundamental solution $\Phi(t, t_0) = e^{F(t-t_0)}$ and thus to determine $\Phi(t, t_0)$ we need to evaluate $e^{F(t-t_0)}$. Since this problem is time invariant without loss of generality we can take $t_0 = 0$. To compute $\Phi(t) = e^{Ft}$ we will solve two initial values problems. The first will have initial conditions given by

$$\left[\begin{array}{c} x_1(0) \\ x_2(0) \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \end{array} \right] ,$$

and the second will have initial conditions given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

The solutions to the first initial value problem become the first column of the matrix e^{Ft} and the solution to the second initial value problem will become the second column of e^{Ft}. When we do this we find that

$$e^{Ft} = \begin{bmatrix} e^{-\frac{t}{T_1}} & \frac{T_2}{T_1 - T_2}\left(e^{-\frac{t}{T_1}} - e^{-\frac{t}{T_2}}\right) \\ 0 & e^{-\frac{t}{T_2}} \end{bmatrix}$$

From the above expression we see that $\Phi(\Delta t) = e^{F\Delta t}$ is the same as the expression we are asked to derive in the book. In the Mathematical file `chap_3_prob_8.nb` some of the algebra for this problem is done.

Problem 3-9 (a cascading additive noise integration system)

From figure 3-4 in the book we see that as a system of differential equations we obtain

$$\begin{aligned} \dot{x}_n &= x_{n-1}(t) + w_n(t) \\ \dot{x}_{n-1} &= x_{n-2}(t) + w_{n-1}(t) \\ \dot{x}_{n-2} &= x_{n-3}(t) + w_{n-2}(t) \\ &\vdots \\ \dot{x}_3 &= x_2(t) + w_3(t) \\ \dot{x}_2 &= x_1(t) + w_2(t) \\ \dot{x}_1 &= w_1(t) , \end{aligned}$$

with each white noise term $w_i(t)$ has a spectral density given by $q_i\delta(t)$. If we define the system state vector as $\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$ then our system above in matrix notation is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & & & & \\ 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_{n-2} \\ w_{n-1} \\ w_n \end{bmatrix} .$$

Thus the system matrix F in this case is the zero matrix with ones on the first sub-diagonal. With the above F the linear variance equation $\dot{P} = FP + PF^T + Q$ has a somewhat special form. The product FP is a block row matrix composed of an initial row of zeros followed by the first $n-1$ rows of P. The product PF^T is a block column matrix with the first block a column of zeros and the second block the first $n-1$ columns of the matrix P. With these observations when we write out the linear variance equation for this problem with $\dot{P}(t)$ given by

$$\dot{P}(t) = \begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} & \dot{p}_{13} & \cdots & \dot{p}_{1n} \\ \dot{p}_{21} & \dot{p}_{22} & \dot{p}_{23} & \cdots & \dot{p}_{2n} \\ \dot{p}_{31} & \dot{p}_{32} & \dot{p}_{33} & \cdots & \dot{p}_{3n} \\ \vdots & & & & \vdots \\ \dot{p}_{n1} & \dot{p}_{n2} & \dot{p}_{n3} & \cdots & \dot{p}_{nn} \end{bmatrix},$$

we get the following system

$$\begin{aligned}
\dot{P}(t) &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ \vdots & & & & \vdots \\ p_{n-1,1} & p_{n-1,2} & p_{n-1,3} & \cdots & p_{n-1,n} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & p_{11} & p_{12} & \cdots & p_{1,n-1} \\ 0 & p_{21} & p_{22} & \cdots & p_{2,n-1} \\ 0 & p_{31} & p_{32} & \cdots & p_{3,n-1} \\ \vdots & & & & \vdots \\ 0 & p_{n1} & p_{n2} & \cdots & p_{n,n-1} \end{bmatrix} + \begin{bmatrix} q_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & q_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & q_n \end{bmatrix} \\
&= \begin{bmatrix} q_1 & p_{11} & p_{12} & \cdots & p_{1,n-1} \\ p_{11} & p_{12} + p_{21} + q_2 & p_{13} + p_{22} & \cdots & p_{1n} + p_{2,n-1} \\ p_{21} & p_{22} + p_{31} & p_{23} + p_{32} + q_3 & \cdots & p_{2n} + p_{3,n-1} \\ \vdots & & & & \vdots \\ p_{n-1,1} & p_{n-1,2} + p_{n,1} & p_{n-1,3} + p_{n2} & \cdots & p_{n-1,n} + p_{n,n-1} + q_n \end{bmatrix}.
\end{aligned}$$

Looking at the above expressions we see that in component form we have that the (i, j)th component of the product FP is

$$(FP)_{ij} = p_{i-1,j}(t).$$

for $i \geq 2$ and that the (i, j)th component of the product PF^T is given by

$$(PF^T)_{ij} = p_{i,j-1}(t),$$

for $j \geq 2$. The differential equation for the function $p_{ij}(t)$ is thus given by

$$\dot{p}_{ij}(t) = p_{i-1,j} + p_{i,j-1} + q_i \delta_{ij}, \tag{42}$$

for $2 \leq i \leq n$ and $2 \leq j \leq n$.

Can we solve for $p_{ii}(t)$? This equation would be

$$\dot{p}_{ii} = p_{i-1,i} + p_{i,i-1} + q_i,$$

since P is a symmetric matrix $p_{i-1,i} = p_{i,i-1}$ so the above differential equation becomes

$$\dot{p}_{ii} = 2p_{i-1,i} + q_i \quad \text{for} \quad 2 \leq i \leq n \,,$$

Thus we need to compute $p_{i-1,i}(t)$ to evaluate $p_{ii}(t)$.

If we look at the first row of these equations we have for the $p_{11}(t)$ the following

$$\dot{p}_{11} = q_1 \quad \Rightarrow \quad p_{11}(t) = q_1 t \,.$$

The equation for p_{12} is given by

$$\dot{p}_{12} = p_{11} = q_1 t \quad \Rightarrow \quad p_{12} = \frac{q_1 t^2}{2} \,.$$

The equation for p_{13} next gives

$$\dot{p}_{13} = p_{12} = \frac{q_1 t^2}{2} \quad \Rightarrow \quad p_{13} = \frac{q_1 t^3}{6} \,.$$

In general for the first row we have

$$p_{1j} = \frac{q_1 t^j}{j!} \quad \text{for} \quad 1 \leq j \leq n \tag{43}$$

When we recall that $p_{12}(t) = p_{21}(t)$ by the symmetry of $P(t)$ the equation for $p_{22}(t)$ is

$$\dot{p}_{22} = 2p_{12} + q_2 = q_1 t^2 + q_2 \,.$$

When we integrate this gives

$$p_{22}(t) = \frac{q_1 t^3}{3} + q_2 t \,.$$

Thus we have been able to verify the first two expectations given. This would be a good starting points for an inductive proof of the general case $p_{nn} = E[x_n(t)^2]$. Instead of providing an inductive proof it can be shown in [7] that the fundamental solution $\Phi(t)$ to the given system can be written as

$$\Phi(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ t & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}t^2 & t & 1 & 0 & \cdots & 0 \\ \frac{1}{3!}t^3 & \frac{1}{2}t^2 & t & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(n-1)!}t^{n-1} & \frac{1}{(n-2)!}t^{n-2} & \frac{1}{(n-3)!}t^{n-3} & \frac{1}{(n-4)!}t^{n-4} & \cdots & 1 \end{bmatrix} .$$

We will now use this expression in Equation 36 to derive an expression for $p_{ii}(t)$. In this problem here we have $P(t_0) = 0$, $G(t) = I$, and $Q(t) = Q$ where Q is a diagonal matrix. Then we have

$$P(t) = \int_{t_0}^{t} \Phi(t-\tau) Q \Phi^T(t-\tau) d\tau = \int_0^{t-t_0} \Phi(\tau) Q \Phi(\tau)^T d\tau \,.$$

Since Q is diagonal the product $\Phi(t)Q$ is easy to compute since it is a scalar multiplier of each column of $\Phi(t)$. That is we have

$$\Phi(t)Q = \begin{bmatrix} q_1 & 0 & 0 & 0 & \cdots & 0 \\ q_1 t & q_2 & 0 & 0 & \cdots & 0 \\ \frac{q_1}{2}t^2 & q_2 t & q_3 & 0 & \cdots & 0 \\ \frac{q_1}{3!}t^3 & \frac{q_2}{2}t^2 & q_3 t & q_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{q_1}{(n-1)!}t^{n-1} & \frac{q_2}{(n-2)!}t^{n-2} & \frac{q_3}{(n-3)!}t^{n-3} & \frac{q_4}{(n-4)!}t^{n-4} & \cdots & q_n \end{bmatrix}.$$

From this we see that the elements of the nth row of $\Phi(t)Q$ are given by

$$\frac{q_1}{(n-1)!}t^{n-1}, \frac{q_2}{(n-2)!}t^{n-2}, \cdots, \frac{q_{n-2}}{2!}t^2, q_{n-1}t, q_n.$$

The nth column of $\Phi(t)^T$ is given by the nth row of $\Phi(t)$ and has elements given by

$$\frac{1}{(n-1)!}t^{n-1}, \frac{1}{(n-2)!}t^{n-2}, \cdots, \frac{1}{2!}t^2, t, 1.$$

When we take the dot product of these two vector we see that the the (n,n)th component of $P(t)$ is given by when we take $t_0 = 0$

$$p_{nn}(t) = E[x_{nn}(t)^2] = \int_0^t \sum_{i=1}^n \frac{q_{n+1-i}}{(i-1)!^2}\tau^{2i-2}d\tau = \sum_{i=1}^n \frac{q_{n+1-i}t^{2i-1}}{(i-1)!^2(2i-1)},$$

as we were to show.

Problem 3-10 (steady-state error for a given system)

The system associated with the given diagram figure 3-5 is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 - 2\xi\omega x_2 + w, \end{aligned}$$

where $w(t)$ is a white noise input with spectral density $q\delta(t)$. If we define the state of this system to be $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then our system in terms of $\mathbf{x}$ becomes

$$\frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ w(t) \end{bmatrix}.$$

Thus we see that our system F matrix is given by $F = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}$. With this the linear variance Equation 30 becomes

$$\begin{aligned} \begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & -\omega^2 \\ 1 & -2\xi\omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix} \\ &= \begin{bmatrix} 2p_{12} & p_{22} - \omega^2 p_{11} - 2\xi\omega p_{12} \\ p_{22} - \omega^2 p_{11} - 2\xi\omega p_{12} & -2\omega^2 p_{12} - 4\xi\omega p_{22} + q \end{bmatrix}. \end{aligned}$$

As a system for the functions $p_{ij}(t)$ this is given by

$$
\begin{aligned}
\dot{p}_{11} &= 2p_{12} \\
\dot{p}_{12} &= -\omega^2 p_{11} - 2\xi\omega p_{12} + p_{22} \\
\dot{p}_{22} &= -2\omega^2 p_{12} - 4\xi\omega p_{22} + q \,,
\end{aligned}
$$

In steady-state all time derivatives above are zero. In this case we see that that $p_{12}(t) = 0$, and the other functions must satisfy

$$
\begin{aligned}
0 &= -\omega^2 p_{11} + p_{22} \\
0 &= -4\xi\omega p_{22} + q \,.
\end{aligned}
$$

When we solve for p_{11} and p_{22} using the above system we find

$$
p_{22} = \frac{q}{4\xi\omega} \quad \text{and} \quad p_{11} = \frac{q}{4\xi\omega^3} \,,
$$

as we were to show.

Problem 3-11 (the optimal first-order system)

Note: I think there is an error in this problem. The error has to do with the additive noise function $n(t)$. The book states that the autocorrelation of $n(t)$ is proportional to a delta function, specifically $\phi_{nn}(\tau) = N\delta(t)$. I think what they meant to say was that $E[n(t)n(\tau)] = N^2\delta(t-\tau)$ (note the square on N). In this later case I can show the stated claim: that $k = 1.0$ when $\beta = \sigma^2 = 1.0$ and $N = \frac{1}{2}$.

From the given diagram in figure 3-6 for the unknowns $c(t)$ and $r(t)$ we find the following system of differential equations

$$
\begin{aligned}
\dot{c}(t) &= kr(t) - kc(t) - kn(t) \\
\dot{r}(t) &= -\beta r(t) + w(t) \,.
\end{aligned}
$$

Note in deriving the given differential equation for $r(t)$ we have used the discussion on exponentially correlated random variables, since we are told its autocorrelation function is $\phi_{rr}(\tau) = \sigma^2 e^{-\beta|\tau|}$. In matrix from we find this system is given by

$$
\frac{d}{dt}\begin{bmatrix} c(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} -k & k \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} c(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} n \\ w \end{bmatrix} .
$$

From this expression we see that our system matrix F is given by $F = \begin{bmatrix} -k & k \\ 0 & -\beta \end{bmatrix}$ and using the linear variance equation 30 we have

$$
\begin{aligned}
\begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{bmatrix} &= \begin{bmatrix} -k & k \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} -k & 0 \\ k & -\beta \end{bmatrix} \\
&+ \begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} N^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\begin{bmatrix} -k & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -2kp_{11} + 2kp_{12} + k^2N^2 & -(k+\beta)p_{12} + kp_{22} \\ -(k+\beta)p_{12} + kp_{22} & -2\beta p_{22} + \sigma^2 \end{bmatrix} .
\end{aligned}
$$

If we next restrict to the steady-state version of this, where take all time derivatives equal to zero and solve for $p_{ij}(t)$ to find

$$p_{22} = \frac{\sigma^2}{2\beta}\,, \quad \text{and} \quad p_{12} = \frac{k\sigma^2}{2\beta(k+\beta)}\,, \quad \text{and} \quad p_{11} = \frac{k\sigma^2}{2\beta(k+\beta)} + \frac{kN^2}{2}\,.$$

With these expressions as the steady-state values for a matrix P_{SS} we can compute the value of the error variance, where our error function $e(t)$ is defined as $e(t) = c(t) - r(t)$. Writing this error $e(t)$ as the vector inner product

$$e(t) = \left[\begin{array}{cc} 1 & -1 \end{array}\right] \left[\begin{array}{c} c(t) \\ r(t) \end{array}\right],$$

so that the variance of $e(t)$ as a function of k can be computed using the matrix P_{SS} as

$$\begin{aligned}
\sigma_e(k)^2 &= \left[\begin{array}{cc} 1 & -1 \end{array}\right] P_{SS} \left[\begin{array}{c} 1 \\ -1 \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \end{array}\right] \left[\begin{array}{cc} \frac{k\sigma^2}{2\beta(k+\beta)} + \frac{kN^2}{2} & \frac{k\sigma^2}{2\beta(k+\beta)} \\ \frac{k\sigma^2}{2\beta(k+\beta)} & \frac{\sigma^2}{2\beta} \end{array}\right] \left[\begin{array}{c} 1 \\ -1 \end{array}\right] \\
&= \frac{kN^2}{2} - \frac{k\sigma^2}{2\beta(k+\beta)} + \frac{\sigma^2}{2\beta}\,.
\end{aligned}$$

Since the above expression is a function of k, then to pick k such that this expression is a minimum we take the derivative with respect to k, set the resulting expression equal to zero, and solve for k. When we do this we find that k is given by

$$k = -\beta \pm \frac{\sigma}{\sqrt{N^2}}\,. \tag{44}$$

If we take $\beta = \sigma^2 = 1.0$, and $N = \frac{1}{2}$ then from the above we see that k is given by

$$k = -1 \pm 2 = \left\{ \begin{array}{c} -3 \\ 1 \end{array} \right. .$$

When we put the value of $k = -3$ into the second derivative of $\sigma_e^2(k)$ we see that the value of the second derivative is $-\frac{1}{8}$, which is negative indicating that this value of k gives a maximum of $\sigma_e^2(k)$. When we put the value of $k = +1$ into the second derivative of $\sigma_e(k)^2$ we get a value of $\frac{1}{8}$ which is positive indicating that this value of k is a minimum as we were asked to show. In the Mathematical file `chap_3_prob_11.nb` some of the algebra for this problem is done.

Chapter 4 (Optimal Linear Filtering)

Notes on the text

Recursive filters: estimating a scalar x

Here we explain how to evaluate the books equation 4.0-3 if we have k measurements z_i of the *same* quantity x. As k scalar equations we have $z_i = x + v_i$ for $i = 1, 2, \cdots, k$. This same situation can be viewed as a vector of measurements $\mathbf{z}$ by introducing the measurement sensitivity matrix H for this problem as

$$\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{k-1} \\ v_k \end{bmatrix}.$$

Thus the matrix H in this case is in fact a column vector. The least squares estimate of x given $\mathbf{z}$ is given by equation 4.0-3 or

$$\hat{x} = (H^T H)^{-1} H^T \mathbf{z}.$$

For the H given above we have $H^T H = k$ and $H^T \mathbf{z} = \sum_{i=1}^{k} z_i$ so that our least squares estimate $\hat{x}$ is given by

$$\hat{x} = \frac{1}{k} \sum_{i=1}^{k} z_i ,$$

which is the books equation 4.1-1.

State estimators in Linear Form: the discrete Kalman filter

For this chapter we will consider a certain specific forms for the estimator of the unknown state x at the k-th time step after the k measurement z_k has been observed. We denote this estimate of x as $\hat{x}_k(+)$, and the previous estimate of the state x before the measurement as $\hat{x}_k(-)$. With this notation in this section we want to study estimators that linearly combine these two pieces of information in the following form

$$\hat{x}_k(+) = K'_k \hat{x}_k(-) + K_k z_k . \tag{45}$$

We have yet to determine the optimal choice for the yet undetermined coefficients K'_k and K_k. Since our kth measurement z_k in terms of the true state x_k and measurement noise v_k is given by

$$z_k = H_k x_k + v_k , \tag{46}$$

the above expression for $\hat{x}_k(+)$ can be written as

$$\hat{x}_k(+) = K'_k \hat{x}_k(-) + K_k H_k x_k - K_k v_k \,.$$

Thus we have replaced the measurement z_k with an expression in terms of the state x_k. To replace the value of $\hat{x}_k(-)$ with something in terms of the state x_k we introduce the the error in our a priori estimate $\hat{x}_k(-)$ as $\tilde{x}_k(-)$ defined as

$$\tilde{x}_k(-) = x_k - \hat{x}_k(-) \,. \tag{47}$$

Using this we get for $\hat{x}_k(+)$

$$\begin{aligned}
\hat{x}_k(+) &= K'_k(x_k + \tilde{x}_k(-)) + K_k H_k x_k + K_k v_k \\
&= [K'_k + K_k H_k] x_k + K'_k \tilde{x}_k(-) + K_k v_k
\end{aligned} \tag{48}$$

Introducing the a posteriori error $\tilde{x}_k(+) = x_k + \tilde{x}_k(+)$ into the left-hand-side of Equation 48 gives the following

$$\tilde{x}_k(+) = [K'_k + K_k H_k - I] x_k + K'_k \tilde{x}_k(-) + K_k v_k \,, \tag{49}$$

which is the books equation 4.2-2. If we assume that the a priori estimate $\hat{x}_k(-)$ is unbiased meaning that $E[\hat{x}_k(-)] = x_k$ or equivalently $E[\tilde{x}_k(-)] = 0$ then to have our a posteriori estimate, $\hat{x}_k(+)$, also be unbiased requires that we take

$$K'_k = I - K_k H_k \,, \tag{50}$$

which is the books equation 4.2-3. Using this expression in Equation 45 gives

$$\begin{aligned}
\hat{x}_k(+) &= (I - K_k H_k)\hat{x}_k(-) + K_k z_k \\
&= \hat{x}_k(-) + K_k(z_k - \hat{x}_k(-)) \,.
\end{aligned} \tag{51}$$

In addition, using this expression with Equation 48 gives

$$\tilde{x}_k(+) = [I - K_k H_k]\tilde{x}_k(-) + K_k v_k \,, \tag{52}$$

which is the books equation 4.2-6.

We will now determine K_k by minimizing an appropriate measure of the error in our new estimate $\hat{x}_k(+)$. If we define the value of $P_k(-)$ to be the prior covariance $P_k(-) \equiv E[\tilde{x}_k(-)\tilde{x}_k(-)^T]$ and a posterior covariance error $P_k(+)$ defined in a similar manner namely

$$P_k(+) = E[\tilde{x}_k(+)\tilde{x}_k(+)] \,,$$

then with the value of K'_k given above by $K'_k = I - K_k H_k$ we can use Equation 52 to derive our posterior state estimate as

$$\begin{aligned}
P_k(+) &= E[\tilde{x}_k(+)\tilde{x}_k^T] \\
&= E[((I - K_k H_k)\tilde{x}_k(-) + K_k v_k)(\tilde{x}_k^T(-)(I - K_k H_k)^T + v_k^T K_k^T)] \,.
\end{aligned}$$

By expanding the terms on the right hand side of this expression and remembering that $E[v_k \tilde{x}_k^T(-)] = 0$ gives

$$P_k(+) = (I - K_k H_k) P_k(-) (I - K_k H_k)^T + K_k R_k K_k^T \,, \tag{53}$$

or the so called **Joseph form** of the covariance update equation and is also the books equation 4.2-12. We now introduce the post measurement quadratic objective function $J_k = \text{trace}[P_k(+)]$, for which we want to select the matrix K_k such that J_k is a minimum. Then to find the value of K_k that minimizes this expression we take the derivative of our objective function with respect to our unknown matrix K_k, set the result equal to zero, and solve for K_k. To do this this we will expand the quadratic in Equation 53, the Joseph form of the covariance update equation to write $P_k(+)$ as

$$P_k(+) = P_k(-) - K_k H_k P_k(-) - P_k(-) H_k^T K_k^T + K_k H_k P_k(-) H_k^T K_k^T + K_k R_k K_k^T \,. \tag{54}$$

To evaluate the trace of $P_k(+)$ we will use the **quadratic outer product trace derivative**

$$\frac{\partial}{\partial A}\text{trace}[ABA^T] = 2AB \,, \tag{55}$$

and the **sandwich product trace derivative** identity

$$\frac{\partial}{\partial A}\text{trace}[BAC] = B^T C^T \,. \tag{56}$$

Then to use these two identities when we rotate K_k to be in the middle of the matrix products[2] so that we can use the sandwich product trace derivative we have that $\text{trace}[P_k(+)]$ is given by

$$\begin{aligned} \text{trace}[P_k(+)] &= \text{trace}[P_k(-)] - \text{trace}[P_k(-) K_k H_k] - \text{trace}[P_k(-) K_k H_k] \\ &+ \text{trace}[K_k H_k P_k(-) H_k^T K_k^T] + \text{trace}[K_k R_k K_k^T] \,. \end{aligned}$$

With this our derivative is given by

$$\begin{aligned} \frac{\partial \text{trace}[P_k(+)]}{\partial K_k} &= -2P_k(-) H_k^T + 2K_k H_k P_k(-) H_k^T + 2K_k R_k \\ &= -2P_k(-) H_k^T + K_k (2H_k P_k(-) H_k^T + 2R_k) \,. \end{aligned} \tag{57}$$

Setting this expression equal to zero and solving for K_k we get

$$K_k = P_k(-) H_k^T (H_k P_k(-) H_k^T + R_k)^{-1} \,. \tag{58}$$

which is the books equation 4.2-15.

Now that we have an expression for K_k, alternative forms for the error covariance extrapolation $P_k(+)$ can be obtained through algebraic manipulations. Because every matrix depends explicitly on k, in the following derivations we can drop the k subscript index from the given $P(\pm)$, H and R matrices. The subscript will be added again to the equations that are the most significant. To derive alternative form for $P(+)$ we expand the product on the right-hand-side of Equation 53 to get

$$P(+) = P(-) - KHP(-) - P(-)H^T K^T + KHP(-)H^T K^T + KRK^T \,.$$

[2]We are using two additional facts about traces here namely cyclic permutability $\text{trace}(ABC) = \text{trace}(BCA)$, and transpose invariance $\text{trace}(A) = \text{trace}(A^T)$

Putting in $K = P(-)H^T M^{-1}$ with $M = HP(-)H^T + R$ we find

$$
\begin{aligned}
P(+) &= P(-) - P(-)H^T M^{-1} HP(-) - P(-)H^T M^{-1} HP(-) \\
&+ P(-)H^T M^{-1} HP(-)H^T M^{-1} HP(-) + P(-)H^T M^{-1} R M^{-1} HP(-) \\
&= P(-) - 2P(-)H^T M^{-1} HP(-) + P(-)H^T M^{-1} [HP(-)H^T + R] M^{-1} HP(-) \\
&= P(-) - 2P(-)H^T M^{-1} HP(-) + P(-)H^T M^{-1} HP(-) \\
&= P(-) - P(-)H^T M^{-1} HP(-) \\
&= P_k(-) - P_k(-)H_k^T [H_k P_k(-) H_k^T + R_k]^{-1} H_k P_k(-) \,,
\end{aligned} \tag{59}
$$

which is the books equation 4.2-16 a. When we recall the expression we found for K_k in Equation 58 or $K = P(-)H^T[HP(-)H^T + R]^{-1}$ by using the last line above we get

$$
\begin{aligned}
P_k(+) &= P_k(-) - K_k H_k P_k(-) \\
&= (I - K_k H_k) P_k(-) \,,
\end{aligned} \tag{60}
$$

which is the books equation 4.2-16 b. This later form is most often used in computation.

Some simpler forms for $P_k(+)^{-1}$ and K_k

To begin this subsection we want to show that *inverses* of the state covariance matrices are "easy" to update after obtaining a measurement z_k. Namely we want to show that

$$
P_k(+)^{-1} = P_k(-)^{-1} + H_k^T R_k^{-1} H_k \,, \tag{61}
$$

is true. To do this consider the product $P_k(+)P_k(+)^{-1}$, where $P_k(+)$ is given by Equation 60 and K_k is given by Equation 58. Dropping the subscripts k to ease algebraic manipulation we find

$$
\begin{aligned}
P(+)P(+)^{-1} &= (P(-) + KHP(-))(P(-)^{-1} + H^T R^{-1} H) \\
&= I + P(-)H^T R^{-1} H - KH - KHP(-)H^T R^{-1} H \\
&= I + P(-)H^T R^{-1} H - P(-)H^T [HP(-)H^T + R]^{-1} H \\
&- P(-)H^T [HP(-)H^T + R]^{-1} [HP(-)H^T + R - R] R^{-1} H \\
&= I + P(-)H^T R^{-1} H - P(-)H^T [HP(-)H^T + R]^{-1} H \\
&- P(-)H^T (I - [HP(-)H^T + R]^{-1} R) R^{-1} H \\
&= I + P(-)H^T R^{-1} H - P(-)H^T [HP(-)H^T + R]^{-1} H \\
&- P(-)H^T R^{-1} H + P(-)H^T [HP(-)H^T + R]^{-1} H \\
&= I \,,
\end{aligned}
$$

as we were to show.

To derive another form for K_k we can introduce the product $P_k(+)^{-1}P_k(+) = I$ into the expression for K_k provided in Equation 58 as

$$\begin{aligned}
K_k &= P(-)H^T[HP(-)H^T + R]^{-1} \\
&= [P(+)P(+)^{-1}]P(-)H^T[HP(-)H^T + R]^{-1} \\
&= P(+)[P(-)^{-1} + H^T R^{-1} H]P(-)H^T[HP(-)H^T + R]^{-1} \\
&= P(+)[H^T + H^T R^{-1} HP(-)H^T][HP(-)H^T + R]^{-1} \\
&= P(+)H^T[I + R^{-1}HP(-)H^T][HP(-)H^T + R]^{-1} \\
&= P(+)H^T R^{-1}[R + HP(-)H^T][HP(-)H^T + R]^{-1} \\
&= P_k(+)H_k^T R_k^{-1} ,
\end{aligned} \tag{62}$$

which is the books equation 4.2-20.

Kalman filtering the constant dynamics $x_{k+1} = x_k$ with measurements $z_k = x_k + v_k$

In this subsection we present the algebra and further discussion on the Kalman filtering examples presented in the book. We begin with the estimation of a *constant* x from a series of uncorrelated corrupted noisy measurements. For this example, because there is no dynamics the variance propagation equation is simple $p_k(-) = p_{k-1}(+)$ and with $H_k = 1$ the error covariance update equation due to the measurement z_k is

$$p_k(+) = (1 - k_k)p_k(-) .$$

For this scalar problem we then have $k_k = p_k(-)(p_k(-) + r_0)^{-1}$ and with the above we find $p_k(+)$ given by

$$p_k(+) = p_k(-) - p_k(-)[p_k(-) + r_0]^{-1}p_k(-) = \frac{r_0 p_k(-)}{p_k(-) + r_0} = \frac{p_k(-)}{1 + \frac{p_k(-)}{r_0}} .$$

The iterative equation for $p_k(+)$ is then given by replacing $p_k(-)$ with $p_{k-1}(+)$ in the above expression to get

$$p_k(+) = \frac{p_{k-1}(+)}{1 + \frac{p_{k-1}(+)}{r_0}} .$$

The above expression can be iterated to find the general solution with $p_0(+) = p_0$. We have

$$\begin{aligned}
p_1(+) &= \frac{p_0}{1 + \frac{p_0}{r_0}} \\
p_2(+) &= \frac{p_1(+)}{1 + \frac{p_1(+)}{r_0}} = \frac{\frac{p_0}{1+\frac{p_0}{r_0}}}{1 + \frac{\frac{p_0}{r_0}}{1+\frac{p_0}{r_0}}} = \frac{p_0}{1 + \frac{2p_0}{r_0}} \\
p_3(+) &= \frac{p_2(+)}{1 + \frac{p_2(+)}{r_0}} = \frac{\frac{p_0}{1+\frac{2p_0}{r_0}}}{1 + \frac{\frac{p_0}{r_0}}{1+\frac{2p_0}{r_0}}} = \frac{p_0}{1 + \frac{3p_0}{r_0}} \\
&\vdots \\
p_k(+) &= \frac{p_0}{1 + \frac{kp_0}{r_0}} .
\end{aligned} \tag{63}$$

Given this analytic form for $p_k(+)$ we can write the Kalman gain K_k with Equation 62 as

$$K_k = P_k(+) H_k^T R_k^{-1} = \frac{p_k(+)}{r_0} = \frac{\frac{p_0}{r}}{1 + \frac{kp_0}{r_0}} .$$

Thus our optimal state estimate $\hat{x}_k(+)$ is given by

$$\begin{aligned} \hat{x}_k(+) &= \hat{x}_k(-) + K_k[z_k - \hat{x}_k(-)] \\ &= \hat{x}_k(-) + \left(\frac{\frac{p_0}{r_0}}{1 + \frac{kp_0}{r_0}} \right) (z_k - \hat{x}_k(-)) . \end{aligned} \tag{64}$$

There is no process dynamics in this problem so when we need to propagate the state to the time t_{k+1} and before the next measurement we have $\hat{x}_{k+1}(-) = \hat{x}_k(+)$.

Kalman filtering correlated measurements (Example 4-2.2)

From the given state vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and measurement sensitivity matrix H we seek to determine how a single measurement $\mathbf{z}$ modifies our uncertainty in the state. To do this we will use the a posterior covariance update equation

$$P(+) = P(-) - P(-)H^T[HP(-)H^T + R]^{-1} HP(-) .$$

To evaluate the right-hand-side of the above from the problem description we see that

$$HP(-) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11}(-) & p_{12}(-) \\ p_{12}(-) & p_{22}(-) \end{bmatrix} = \begin{bmatrix} p_{12}(-) & p_{22}(-) \end{bmatrix} = \begin{bmatrix} \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} .$$

Then the matrix $P(-)H^T$ is the transpose of this or

$$P(-)H^T = \begin{bmatrix} p_{12}(-) \\ p_{22}(-) \end{bmatrix} = \begin{bmatrix} \sigma_{12}^2 \\ \sigma_2^2 \end{bmatrix} .$$

Using $P(-)H^T$ just computed the inner product like term $HP(-)H^T$ is given by

$$HP(-)H^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_{12}(-) \\ p_{22}(-) \end{bmatrix} = p_{22}(-) = \sigma_2^2 .$$

Thus using all of these components we find that $P(+)$ is given by

$$P(+) = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} - \frac{1}{\sigma_2^2 + r_2} \begin{bmatrix} \sigma_{12}^2 \\ \sigma_2^2 \end{bmatrix} \begin{bmatrix} \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} .$$

We multiply the two matrices on the right-hand-side and introduce the correlation ρ with

$$\sigma_{12}^2 = \sigma_1 \sigma_2 \rho \quad \text{so that} \quad \frac{\sigma_{12}^4}{\sigma_1^2} = \sigma_2^2 \rho^2 .$$

We then get that $P(+)$ equals

$$P(+) = \left[\begin{array}{cc} \sigma_1^2\left(\frac{\sigma_2^2(1-\rho^2)+r_2}{\sigma_2^2+r_2}\right) & \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \\ \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) & \sigma_2^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \end{array} \right],$$

which is the expression in the book. Some special cases of this result are worth considering. When the measurement z is perfect meaning that there is no estimation error we have $r_2 = 0$ and $P(+)$ becomes

$$P(+) = \left[\begin{array}{cc} \sigma_1^2(1-\rho^2) & 0 \\ 0 & 0 \end{array} \right].$$

Thus we have no uncertainty in the value of x_2 and we have maximally reduced our uncertainly in x_1. Next if the measurement z gives *no* information about x_1 their correlation is zero. When we take $\rho = 0$ in the above we have

$$P(+) = \left[\begin{array}{cc} \sigma_1^2 & \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \\ \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) & \sigma_2^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \end{array} \right],$$

Thus the measurement z provides no information about x_1 and using it does not reduce the initial uncertainty in x_1 so we have $p_{11}(+) = \sigma_1^2$. If the unknowns x_1 and x_2 are perfectly correlated $\rho = \pm 1$ we have

$$P(+) = \left[\begin{array}{cc} \sigma_1^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) & \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \\ \sigma_{12}^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) & \sigma_2^2\left(\frac{r_2}{\sigma_2^2+r_2}\right) \end{array} \right],$$

thus the measurement z provides the same amount of information for both x_1 and x_2 and reduces their initial uncertainty by the same amount (by the fraction $\frac{r_2}{\sigma_2^2+r_2}$).

Kalman filtering the navigation system Omega (Example 4.2-3)

If our a priori estimate of the state is zero $\hat{x}(-) = 0$ then from the posteriori state update equation $\hat{x}(+) = \hat{x}(-) + K_k(z - H\hat{x}_k(-))$ we have $\hat{x}(+) = K_k z$. We compute K_k it the normal way

$$K_k = P_k(-)H_k^T(H_kP_k(-)H_k^T + R_k)^{-1} = P(0)H^T(HP(0)H^T)^{-1},$$

where we have assumed that the measurement noise is zero "very small". Given the pieces from this problem we will now compute K_k. Using the given expression for $P(0)$ and H we find

$$HP(0)H^T = \sigma_\phi^2 \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} 1 & e^{-r_{12}/d} & e^{-r_{13}/d} \\ e^{-r_{12}/d} & 1 & e^{-r_{23}/d} \\ e^{-r_{13}/d} & e^{-r_{23}/d} & 1 \end{array} \right] \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{array} \right] = \sigma_\phi^2 \left[\begin{array}{cc} 1 & e^{-r_{23}/d} \\ e^{-r_{23}/d} & 1 \end{array} \right].$$

The inverse of this matrix is given by

$$(HP(0)H^T)^{-1} = \frac{1}{\sigma_\phi^2(1 - e^{-2r_{23}/d})} \left[\begin{array}{cc} 1 & -e^{-r_{23}/d} \\ -e^{-r_{23}/d} & 1 \end{array} \right].$$

Using this as a factor we next find that the product $K = P(0)H^T(HP(0)H^T)^{-1}$ given by

$$\frac{1}{1-e^{-2r_{23}/d}}\begin{bmatrix} e^{-r_{12}/d}-e^{-r_{13}/d-r_{23}/d} & e^{-r_{13}/d}-e^{-r_{12}/d-r_{23}/d} \\ 1-e^{-2r_{23}/d} & 0 \\ 0 & 1-e^{-2r_{23}/d} \end{bmatrix}.$$

From this matrix we can compute $\hat{x}(+)$. We find since $\hat{x}(+) = \begin{bmatrix} \hat{\phi}_1 \\ \hat{\phi}_2 \\ \hat{\phi}_3 \end{bmatrix} = Kz$ that

$$\begin{aligned} \hat{x}(+) &= \frac{1}{1-e^{-2r_{23}/d}}\begin{bmatrix} e^{-r_{12}/d}-e^{-r_{13}/d-r_{23}/d} & e^{-r_{13}/d}-e^{-r_{12}/d-r_{23}/d} \\ 1-e^{-2r_{23}/d} & 0 \\ 0 & 1-e^{-2r_{23}/d} \end{bmatrix}\begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{1-e^{-2r_{23}/d}}\left((e^{-r_{12}/d}-e^{-(r_{13}+r_{23})/d})\phi_2+(e^{-r_{13}/d}-e^{-(r_{12}+r_{23})/d})\phi_3\right) \\ \phi_2 \\ \phi_3 \end{bmatrix}, \end{aligned}$$

which duplicates the results given in the book. In the Mathematical file `chap_4_2_3.nb` we perform some of the algebra not displayed in the above derivation.

Kalman filtering an inertial navigation system (Example 4.2-4)

The propagation from $t = 0$ to $t = T$ the time of the first fix is done using the state error covariance extrapolation equation or $P(T^-) = \Phi(T,0)P(0)\Phi(T,0)^T$. Using the given matrix $\Phi(T,0)$ for this problem we can compute $P(T^-)$ to find

$$\begin{aligned} P(T^-) &= \begin{bmatrix} 1 & T & \frac{T^2}{2} \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sigma_p^2 & 0 & 0 \\ 0 & \sigma_v^2 & 0 \\ 0 & 0 & \sigma_a^2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ \frac{T^2}{2} & T & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_p^2 & T\sigma_v^2 & \frac{T^2}{2}\sigma_a^2 \\ 0 & \sigma_v^2 & T\sigma_a^2 \\ 0 & 0 & \sigma_a^2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ T & 1 & 0 \\ \frac{T^2}{2} & T & 1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_p^2+T^2\sigma_v^2+\frac{T^4}{4}\sigma_a^2 & T\sigma_v^2+\frac{T^3}{2}\sigma_a^2 & \frac{T^2}{2}\sigma_a^2 \\ T\sigma_v^2+\frac{T^3}{2}\sigma_a^2 & \sigma_v^2+T^2\sigma_a^2 & T\sigma_a^2 \\ \frac{T^2}{2}\sigma_a^2 & T\sigma_a^2 & \sigma_a^2 \end{bmatrix}, \end{aligned} \tag{65}$$

which is the expression in the book. Note I have used the notation $\sigma_p^2 = \overline{\delta p^2(0)}$, $\sigma_v^2 = \overline{\delta v^2(0)}$, and $\sigma_a^2 = \overline{\delta a^2(0)}$ since it is easier to type. After the measurement the new uncertainty $P(T^+)$ is reduced from $P(T^-)$ with

$$P(T^+) = P(T^-) - P(T^-)H^T(HP(T^-)H^T+R)^{-1}HP(T^-). \tag{66}$$

With a measurement sensitivity matrix H of $H = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$ we find

$$H^TP(T^-)H + R = p_{11}(T^-) + \sigma_p^2.$$

and

$$HP(T^-) = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} P(T^-) = -\begin{bmatrix} p_{11}(T^-) & p_{12}(T^-) & p_{13}(T^-) \end{bmatrix},$$

so that $P(T^-)H^T = -\begin{bmatrix} p_{11}(T^-) \\ p_{12}(T^-) \\ p_{13}(T^-) \end{bmatrix}$. With these we find the matrix product given about

$$\begin{aligned} M &= P(T^-)H^T(HP(T^-)H^T + R)^{-1}HP(T^-) \\ &= \frac{1}{p_{11}(T^-) + \sigma_p^2} \begin{bmatrix} p_{11}(T^-) \\ p_{12}(T^-) \\ p_{13}(T^-) \end{bmatrix} \begin{bmatrix} p_{11}(T^-) & p_{12}(T^-) & p_{13}(T^-) \end{bmatrix} \\ &= \frac{1}{p_{11}(T^-) + \sigma_p^2} \begin{bmatrix} p_{11}(T^-)^2 & p_{11}(T^-)p_{12}(T^-) & p_{11}(T^-)p_{13}(T^-) \\ p_{11}(T^-)p_{12}(T^-) & p_{12}(T^-)^2 & p_{12}(T^-)p_{13}(T^-) \\ p_{11}(T^-)p_{13}(T^-) & p_{12}(T^-)p_{13}(T^-) & p_{13}(T^-)^2 \end{bmatrix}. \end{aligned} \quad (67)$$

Since the *total* uncertainty after the fix $P(T^+)$ is given by $P(T^-) - M$, with M computed above we see that the uncertainty of the $(1,1)$ component becomes

$$p_{11}(T^+) = p_{11}(T^-) - \frac{p_{11}(T^-)^2}{p_{11}(T^-) + \sigma_p^2} = \frac{p_{11}(T^-)\sigma_p^2}{p_{11}(T^-) + \sigma_p^2}.$$

If $p_{11}(T^-) \gg \sigma_p^2$ then the above becomes

$$\frac{p_{11}(T^-)\sigma_p^2}{p_{11}(T^-) + \sigma_p^2} = \frac{p_{11}(T^-)\sigma_p^2}{p_{11}(T^-)} = \sigma_p^2,$$

and thus the first fix reduces the error in the position measurement to that of the sensor.

Notes on continuous propagation of covariance

In this section using the discrete results we derive how the continuous covariance matrix $P(t)$ propagates due to the process dynamics and the continuous measurement stream. When we use the approximations $\Phi_k \to I + F\Delta t$, and $Q_k \to GQG^T\Delta t$ in

$$P_{k+1}(+) = \Phi_k P_k(+)\Phi_k^T + Q_k,$$

we get

$$P_{k+1}(-) = P_k(+) + [FP_k(+) + P_k(+)F^T + GQG^T]\Delta t + O(\Delta t^2). \quad (68)$$

Recalling that after a measurement our state uncertainty is updated with

$$P_k(+) = (I - K_kH_k)P_k(-),$$

we can put this expression for $P_k(+)$ into the right-hand-side of Equation 68

$$P_{k+1}(-) = (I-K_kH_k)P_k(-)+[F(I-K_kH_k)P_k(-)+(I-K_kH_k)P_k(-)F^T+GQG^T]\Delta t+O(\Delta t^2).$$

We can manipulate this into a first order difference as

$$\begin{aligned}\frac{P_{k+1}(-) - P_k(-)}{\Delta t} &= FP_k(-) + P_k(-)F^T + GQG^T - \frac{1}{\Delta t}K_k H_k P_k(-) \\ &- FK_k H_k P_k(-) - K_k H_k P_k(-)F^T + O(\Delta t)\,. \end{aligned} \tag{69}$$

To further evaluate this we need to consider the expression $\frac{1}{\Delta t}K_k$. We have

$$\begin{aligned}\frac{1}{\Delta t}K_k &= \frac{1}{\Delta t}P_k(-)H_k^T(H_k P_k(-)H_k^T + R_k)^{-1} \\ &= P_k(-)H_k^T(H_k P_k(-)H_k^T \Delta t + R_k \Delta t)^{-1}\,.\end{aligned}$$

With the discrete covariance matrix R_k converging to the spectral density matrix $R(t)$ when $\Delta t \to 0$ we have $R_k \Delta t \to R$ as $\Delta t \to 0$ and the term $H_k P_k(-)H_k^T \Delta t \to 0$ as $\Delta t \to 0$ and so this term then has the following limit

$$\frac{1}{\Delta t}K_k \to PH^T R^{-1}\,. \tag{70}$$

In the same way the Kalman gain K_k by itself limits to zero since

$$\frac{1}{\Delta t}K_k \to PH^T R^{-1} \quad \text{then} \quad K_k \to \Delta t PH^T R^{-1} = 0\,,$$

as $\Delta t \to 0$. Because of this in Equation 69 the two terms $-FK_k H_k P_k(-)$ and $-K_k H_k P_k(-)F^T$ vanish when we take the limit of Δt tending to zero. Collecting all of these results we are finally left with

$$\dot{P}(t) = FP + PF^T + GQG^T - PH^T R^{-1} HP\,, \tag{71}$$

which is known as the **matrix Riccati equation**, and is the books equation 4.3-8.

Notes on the continuous Kalman filter

Having just developed the matrix Riccati equation which governs how the state covariance matrix $P(t)$ evolves we now perform the same procedure to determine the equation that governs how the continuous state $\hat{x}(t)$ evolves. As before we begin with the corresponding discrete state update equation

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - H_k \hat{x}_k(-)]\,,$$

where we put $\hat{x}_k(-) = \Phi_{k-1}\hat{x}_{k-1}(+)$ into the above to get

$$\hat{x}_k(+) = \Phi_{k-1}\hat{x}_{k-1}(+) + K_k(z_k - H_k \Phi_{k-1}\hat{x}_{k-1}(+))\,,$$

which is the books equation 4.3-10. Next we use the discrete to continuous approximations

$$\begin{aligned}\Phi_{k-1} &= I + F\Delta t \\ K_k &= PH^T R^{-1} \Delta t\,,\end{aligned}$$

to get

$$\hat{x}_k(+) = \hat{x}_{k-1}(+) + F\hat{x}_{k-1}(+)\Delta t + PH^T R^{-1}(z_k - H_k(I + F\Delta t)\hat{x}_{k-1}(+))\Delta t\,,$$

or

$$\frac{\hat{x}_k(+) - \hat{x}_{k-1}(+)}{\Delta t} = F\hat{x}_{k-1}(+) + PH^T R^{-1}(z_k - H_k\hat{x}_{k-1}(+)) + O(\Delta t)\,.$$

In the limit $\Delta t \to 0$ this becomes

$$\dot{\hat{x}}(t) = F\hat{x}(t) + PH^T R^{-1}(z - H\hat{x}(t))\,, \tag{72}$$

which is the **continuous Kalman filter** equation. Note that in the above the expression $P(t)$ given by solving the matrix Riccati Equation 71 for $P(t)$.

Often it is helpful to have a dynamical expression for the *error* in the continuous state estimate $\hat{x}(t)$. To derive the differential equation for this error $\tilde{x}(t) \equiv \hat{x}(t) - x(t)$ we subtract the equation governing the true system dynamics

$$\frac{dx}{dt} = Fx + Gw\,,$$

from the continuous Kalman filter Equation 72 to get

$$\begin{aligned} \frac{\tilde{x}(t)}{dt} &= F\tilde{x}(t) - Gw + PH^T R^{-1}(z - H(\tilde{x}(t) + x(t))) \\ &= F\tilde{x}(t) - Gw - PH^T R^{-1}H\tilde{x}(t) + PH^T R^{-1}v(t)\,. \end{aligned}$$

Where we have used $z(t) - Hx(t) = v(t)$. When we group terms we have

$$\frac{\tilde{x}(t)}{dt} = (F - PH^T R^{-1}H)\tilde{x}(t) - Gw + PH^T R^{-1}v\,.$$

Recalling that $K(t)$ can be expressed as $P(t)H(t)^T R(t)^{-1}$ this later expression becomes

$$\frac{d\tilde{x}}{dt} = (F - KH)\tilde{x} - Gw + Kv\,, \tag{73}$$

which is the books equation 4.3-13.

Notes on correlated process and measurement noise: Y. C. Ho's method

If our process $w(t)$ and measurement $v(t)$ noise are correlated, meaning that $E[w(t)v^T(\tau)] = C(t)\delta(t-\tau)$, then we can transform this problem into one where the new process noise term is uncorrelated with the measurement noise. The algebra to do this are discussed here. Since our measurement $z(t)$ is given in terms of our state via $z = Hx + v$ we can add a multiple (say D) of the expression $z - Hx - v = 0$ to the system dynamics equation giving

$$\frac{dx(t)}{dt} = Fx + Gw + D(z - Hx - v) = (F - DH)x + Dz + Gw - Dv\,. \tag{74}$$

If we take D to be given by the special value of $D = GCR^{-1}$ then we claim that this new process noise term $Gw - Dv$ will be uncorrelated with the measurement noise v and results in a system of the type we have previously been studying. To prove this, we compute the cross-correlation of the new process noise term $Gw - Dv$ with the old measurement noise term v as

$$\begin{aligned}
E[(Gw - Dv)v^T] &= GE[wv^T] - DE[vv^T] \\
&= GC - DR \\
&= GC - GCR^{-1}R = 0\,,
\end{aligned}$$

as we desired to show. We next derive the continuous Kalman filter and the matrix Riccati equation for the system given by Equation 74. To derive the continuous Kalman filter in this case we will use the form given by Equation 72 but with a few modifications. The first modification is that with a deterministic forcing in the system dynamics (as we have here in the form of the Dz term) this forcing must also show up as a term on the right-hand-side of Equation 72. The second modification is that the "F" matrix in Equation 72 is now given by $F - DH$. We thus obtain

$$\begin{aligned}
\dot{\hat{x}}(t) &= (F - GCR^{-1}H)\hat{x}(t) + PH^TR^{-1}(z - H\hat{x}(t)) + GCR^{-1}z \\
&= F\hat{x}(t) - (GCR^{-1}H + PH^TR^{-1}H)\hat{x}(t) + (PH^TR^{-1} + GCR^{-1})z \\
&= F\hat{x} - (PH^T + GC)R^{-1}H\hat{x} + (PH^T + GC)R^{-1}z \\
&= F\hat{x} + (PH^T + GC)R^{-1}(z - H\hat{x})\,. \qquad (75)
\end{aligned}$$

Next we consider the matrix Riccati Equation 71 for this system. As before we need to modify this slightly for the given system. The first modification is again that "F" matrix in Equation 71 becomes $F - DH = F - GCR^{-1}H$. The second modification is that the Q matrix (representing the process noise covariance matrix) needs to correspond to the form of the process noise we have here which has a form given by

$$Gw - Dv = Gw - GCR^{-1}v = G(w - CR^{-1}v)\,.$$

A noise vector of this form will have a covariance matrix given by

$$\begin{aligned}
\mathrm{Cov}(G(w - CR^{-1}v)) &= G\mathrm{Cov}((w - CR^{-1}v))G^T \\
&= G(\mathrm{Cov}(w) + \mathrm{Cov}(CR^{-1}v) - 2\mathrm{Cov}(wv^T)R^{-1}C)G^T \\
&= G(Q + CR^{-1}RR^{-1}C - 2CR^{-1}C)G^T \\
&= G(Q - CR^{-1}C)Q^T\,.
\end{aligned}$$

This later expression will replace the expression GQG^T in Equation 71. When we make these two substitutions into the matrix Riccati equation and perform some manipulations. We find

$$\begin{aligned}
\dot{P}(t) &= (F - DH)P + P(F - DH)^T + G(Q - CR^{-1}C)G^T - PH^TR^{-1}HP \\
&= FP + PF^T + GQG^T - GCR^{-1}HP - PH^TR^{-1}CG^T - GCR^{-1}CG^T - PH^TR^{-1}HP \\
&= FP + PF^T + GQG^T - GCR^{-1}(HP + CG^T) - PH^TR^{-1}(HP + CG^T) \\
&= FP + PF^T + GQG^T - (GCR^{-1} + PH^TR^{-1})(HP + CG^T) \\
&= FP + PF^T + GQG^T - (GC + PH^T)R^{-1}(CG + PH^T)^T \\
&= FP + PF^T + GQG^T - (GC + PH^T)R^{-1}RR^{-1}(CG + PH^T)^T\,. \qquad (76)
\end{aligned}$$

If we define

$$K(t) \equiv (PH^T + GC)R^{-1}\,, \tag{77}$$

then we see that Equation 75 and 76 become

$$\begin{aligned}
\dot{\hat{x}} &= F\hat{x} + K(z - H\hat{x}) \\
\dot{P} &= FP + PF^T + GQG^T - KRK^T\,.
\end{aligned}$$

This result agrees with the ones presented in the book when given a system with correlated process and measurement noises.

A system model that contains deterministic inputs: stochastic observability

Given the continuous system matrix Riccati equation

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP \quad \text{with} \quad P(0) \approx +\infty\,, \tag{78}$$

where $P(0) \approx +\infty$ can be taken to mean that we have no a priori information. We will transform this expression into a differential equation for $P(t)^{-1}$. To do this recall that since $\dot{P}^{-1} = -P^{-1}\dot{P}P^{-1}$ by solving for $\dot{P}(t)$ we get that $\dot{P} = -P\dot{P}^{-1}P$ and using this expression in the left-hand-side of Equation 78 we get

$$-P\dot{P}^{-1}P = FP + PF^T + GQG^T - PH^TR^{-1}HP\,.$$

or by multiplying by P^{-1} once on the left and once on the right and then negating we get

$$\begin{aligned}
\dot{P}^{-1} &= -P^{-1}F - F^TP^{-1} - P^{-1}GQG^TP^{-1} + H^TR^{-1}H \\
&= -F^TP^{-1} - P^{-1}F - P^{-1}GQG^TP^{-1} + H^TR^{-1}H\,,
\end{aligned} \tag{79}$$

where the last equation simple changes the order of the terms in the equation above it. The initial condition $P(0) \approx +\infty$ transforms into the initial condition that $P^{-1}(0) = 0$. If we assume our system has no process noise then the term GQG^T vanishes and this is the books equation 4.4-10. We can solve this equation as in Problem 3.1 on Page 28. Since the fundamental solution to the system with a transition matrix $-F^T$ is given by $e^{-F^T(t-\tau)}$ we see that the solution to $P(t)^{-1}$ is given by

$$\begin{aligned}
P(t)^{-1} &= \int_0^t e^{-F^T(t-\tau)}H(\tau)^TR^{-1}(\tau)H(\tau)e^{-F(t-\tau)}d\tau \\
&= \int_0^t e^{F^T(\tau-t)}H(\tau)^TR^{-1}(\tau)H(\tau)e^{F(\tau-t)}d\tau \\
&= \int_0^t \Phi(\tau,t)^TH(\tau)^TR^{-1}(\tau)H(\tau)\Phi(\tau,t)d\tau\,,
\end{aligned}$$

which is the books equation 4.4-11. In the above $\Phi(t,\tau)$ is the transition matrix corresponding to F.

Notes on correlated measurement errors: continuous time when R is singular

If we define the derived measurement z_1 as

$$z_1 = \dot{z} - Ez \,, \tag{80}$$

then we see that we can write z_1 in terms of our original state x, the original process noise w, and the unexplained measurement noise w_1 as

$$\begin{aligned} z_1 &= \dot{z} - Ez \\ &= \frac{d}{dt}(Hx + v) - E(Hx + v) \\ &= \dot{H}x + H\dot{x} + \dot{v} - EHx - Ev \\ &= \dot{H}x + H(Fx + Gw) + (Ev + w_1) - EHx - Ev \\ &= (\dot{H} + HF - EH)x + HGw + w_1 \\ &= H_1 x + v_1 \,. \end{aligned}$$

For this measurement equation for z_1 we can now calculate its measurement covariance matrix R_1 as $E[v_1 v_1^T]$. Since w and w_1 are uncorrelated $E[ww_1^T] = 0$ and we find

$$R_1 = E[(HGw + w_1)(HGw + w_1)^T] = HGQG^T H^T + Q_1 \,, \tag{81}$$

which is the books equation 4.5.7. The cross correlation matrix C_1 is then computed as

$$\begin{aligned} C_1 &= E[w(t)v_1^T(\tau)] = E[w(t)(HGw(\tau) + w_1(\tau))^T] \\ &= QG^T H^T \,, \end{aligned} \tag{82}$$

which is the books equation 4.5.8. The equivalent problem which we have just formulated is then expressed as

$$\begin{aligned} \dot{x} &= Fx + Gw \quad \text{with} \quad w \sim N(0, Q) \\ z_1 &= H_1 x + v_1 \,, \end{aligned}$$

with the matrices R_1 and C_1 given by Equations 81 and 82 respectively. For this continuous problem, since we have correlated process and measurement noise using Equation 77 we find K_1 given by

$$\begin{aligned} K_1 &= [PH_1^T + GC_1]R_1^{-1} \\ &= [P(\dot{H} + HF - EH)^T + GQG^T H^T](HGQG^T H^T + Q_1)^{-1} \,, \end{aligned} \tag{83}$$

which is the books equation 4.5-9. Then the continuous Kalman filter is given by

$$\begin{aligned} \dot{\hat{x}} &= F\hat{x} + K_1(z_1 - H_1\hat{x}) \\ &= F\hat{x} + K_1(\dot{z} - Ez - H_1\hat{x}) \,, \end{aligned} \tag{84}$$

which is the books equation 4.5-10, and the covariance equation is

$$\dot{P} = FP + PF^T + GQG^T - K_1 R_1 K_1^T \,,$$

which is the books equation 4.5-11. We can avoid having to differentiate our measurement z which is seemingly required by the $\dot{z}$ term on the right-hand-side in Equation 84 by instead taking our state to be $x(t) - K_1(t)z(t)$. Using this expression when we put Equation 84 into the derivative $\frac{d}{dt}(\hat{x}(t) - K_1(t)z(t))$ we get

$$\begin{aligned}
\frac{d}{dt}(\hat{x}(t) - K_1(t)z(t)) &= \dot{\hat{x}} - \dot{K}_1 z - K_1 \dot{z} \\
&= (F\hat{x} + K_1(\dot{z} - Ez - H_1\hat{x})) - \dot{K}_1 z - K_1 \dot{z} \\
&= F\hat{x} - K_1 E z - K_1 H_1 \hat{x} - \dot{K}_1 z \\
&= (F - K_1 H_1)\hat{x} - K_1 E z - \dot{K}_1 z\,.
\end{aligned}$$

Notes on correlated measurement errors: discrete time when R_k is singular

When we have $R_k \equiv 0$, the update equation for the error covariance matrix is given by

$$P_k(+) = P_k(-) - P_k(-)H_k^T[H_k P_k(-)H_k^T]^{-1} H_k P_k(-)\,.$$

If we multiply this by H_k on the left and H_k^T on the right we find that

$$H_k P_k(+) H_k^T = H_k P_k(-) H_k^T - H_k P_k(-) H_k^T [H_k P_k(-) H_k^T]^{-1} H_k P_k(-) H_k^T = 0\,,$$

showing that a linear combination of elements from $P_k(+)$ is zero so a linear combination of states is known exactly.

Notes on the solution of the Riccati equation

In this section we will demonstrate an algebraic transformation that will allow the solution of the Riccati equation in the case where it has *constant* coefficients

$$\dot{P} = FP + PF^T + GQG^T - PH^T R^{-1} HP\,,$$

with $P(t_0)$ given. To show the transformation we will use to solve the equation above we let

$$\lambda = Py\,, \tag{85}$$

and let y satisfy the following differential equation

$$\dot{y} = -F^T y + H^T R^{-1} HPy\,. \tag{86}$$

Then the first derivative of λ is given by

$$\begin{aligned}
\dot{\lambda} &= \dot{P}y + P\dot{y} \\
&= (FP + PF^T + GQG^T - PH^T R^{-1} HP)y + P(-F^T y + H^T R^{-1} HPy) \\
&= FPy + GQG^T y \\
&= F\lambda + GQG^T y\,, \qquad (87)
\end{aligned}$$

which is the books equation 4.6-5. As a matrix system with a vector of unknowns given by $\left[\begin{array}{c} y \\ \lambda \end{array}\right]$ Equations 86 and 87 combine to give

$$\left[\begin{array}{c} \dot{y} \\ \dot{\lambda} \end{array}\right] = \left[\begin{array}{cc} -F^T & H^T R^{-1} H \\ GQG^T & F \end{array}\right] \left[\begin{array}{c} y \\ \lambda \end{array}\right], \tag{88}$$

which is the books equation 4.6-6. Since this is a time-invariant linear dynamical system for the vector of unknowns $\left[\begin{array}{c} y \\ \lambda \end{array}\right]$, let $\Phi = \Phi(t_0 + \tau, t_0)$ be its transition matrix, such that when written in block form

$$\left[\begin{array}{c} y(t_0+\tau) \\ \lambda(t_0+\tau) \end{array}\right] = \left[\begin{array}{cc} \Phi_{yy}(\tau) & \Phi_{y\lambda}(\tau) \\ \Phi_{\lambda y}(\tau) & \Phi_{\lambda\lambda}(\tau) \end{array}\right] \left[\begin{array}{c} y(t_0) \\ \lambda(t_0) \end{array}\right].$$

If we compute components of the product above we find

$$\begin{aligned} y(t_0+\tau) &= \Phi_{yy}(\tau) y(t_0) + \Phi_{y\lambda}(\tau)\lambda(t_0) = \Phi_{yy}(\tau) y(t_0) + \Phi_{y\lambda}(\tau) P(t_0) y(t_0) & (89) \\ \lambda(t_0+\tau) &= \Phi_{\lambda y}(\tau) y(t_0) + \Phi_{\lambda\lambda}(\tau)\lambda(t_0) = \Phi_{\lambda y}(\tau) y(t_0) + \Phi_{\lambda\lambda}(\tau) P(t_0) y(t_0)\,. & (90) \end{aligned}$$

We can replace the left-hand-side of Equation 90 with $\lambda(t_0+\tau) = P(t_0+\tau) y(t_0+\tau)$ and then use Equation 89 to evaluate $y(t_0+\tau)$ to get

$$\begin{aligned} [\Phi_{\lambda y}(\tau) + \Phi_{\lambda\lambda}(\tau) P(t_0)] y(t_0) &= \lambda(t_0+\tau) = P(t_0+\tau) y(t_0+\tau) \\ &= P(t_0+\tau)[\Phi_{yy}(\tau) + \Phi_{y\lambda}(\tau) P(t_0)] y(t_0)\,. \end{aligned}$$

If we "cancel" $y(t_0)$ from both side of this expression and solve for $P(t_0+\tau)$ we get

$$P(t_0+\tau) = [\Phi_{\lambda y}(\tau) + \Phi_{\lambda\lambda}(\tau) P(t_0)][\Phi_{yy}(\tau) + \Phi_{y\lambda}(\tau) P(t_0)]^{-1}, \tag{91}$$

which is the books equation 4.6-8.

As a special case we can use the above result to solve the linear variance equation

$$\dot{P} = FP + PF^T + GQG^T \quad \text{with} \quad P(t_0) \quad \text{given}\,.$$

Since the linear variance equation has $H^T R^{-1} H = 0$ the system in Equation 88 is given by

$$\left[\begin{array}{c} \dot{y} \\ \dot{\lambda} \end{array}\right] = \left[\begin{array}{cc} -F^T & 0 \\ GQG^T & F \end{array}\right] \left[\begin{array}{c} y \\ \lambda \end{array}\right]. \tag{92}$$

In the above the equation for y decouples from that of λ and we have $\dot{y} = -F^T y$ so that the fundamental solution for y is $\Phi_{yy}(\tau) = e^{-F^T\tau}$ and $y(t)$ at any time is then given using that as $y(t) = \Phi_{yy}(t) y_0$. The differential equation for λ now has the known function $y(t)$ as a forcing term and is given by

$$\dot{\lambda} = GQG^T y + F\lambda = F\lambda + GQG^T \Phi(t) y_0\,.$$

As forcing functions like $GQG^T\Phi(t)y_0$ are not important in determining fundamental solutions, the fundamental solution for $\lambda(t)$ is e^{Ft}. Next, to see that $\Phi_{y\lambda}(\tau) = 0$ we can note that for the matrix given in Equation 92 the block matrix fundamental solution $\Phi(\tau)$ is given by

$$\Phi(\tau) = e^{\left[\begin{array}{cc} -F^T & 0 \\ GQG^T & F \end{array}\right]\tau} = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} \left[\begin{array}{cc} -F^T & 0 \\ GQG^T & F \end{array}\right]^k.$$

Each term in the above sum is of the form $\left[\begin{array}{cc} -F^T & 0 \\ GQG^T & F \end{array}\right]^k$, which is the k-th power of a block lower triangular matrix and thus is also block lower triangular. Thus the block $(1, 2)$ term in each component of the sum is 0. Since each component in the sum has a zero $(1, 2)$ term the $(1, 2)$ term for the block fundamental solution $\Phi(\tau)$ will also be zero. Thus we conclude that $\Phi_{y\lambda}(\tau) = 0$. Using this fact, Equation 91 then gives

$$P(t_0 + \tau) = (\Phi_{\lambda y}(\tau) + \Phi_{\lambda\lambda}(\tau)P(t_0))\Phi_{yy}(\tau)^{-1} .$$

Which can further be evaluated by noting that

$$\Phi_{yy}(\tau)^{-1} = (e^{-F^T\tau})^{-1} = e^{F^T\tau} = \Phi_{\lambda\lambda}(\tau)^T ,$$

so

$$P(t_0 + \tau) = \Phi_{\lambda y}(\tau)\Phi_{\lambda\lambda}(\tau)^T + \Phi_{\lambda\lambda}(\tau)P(t_0)\Phi_{\lambda\lambda}(\tau)^T , \tag{93}$$

which is the books equation 4.6-10 and represents a way to solve the linear variance equation.

Problem Solutions

Problem 4-1 (two measurements treated sequentially/simultaneously)

Part (a): If the two measurements are sequential we first observe z_1 and then observe z_2. Assuming no prior information is equivalent to the maximum likelihood estimation method which for Gaussian densities is given by

$$\hat{x}_1(+) = (H^T H)^{-1} H^T z_1 .$$

when there is only one measurement $z_1 = x + v_1$ we see that $H_1 = 1$ and $R_1 = \sigma_1^2$ so the above gives $\hat{x}_1(+) = z_1$. To update the new uncertainty we use

$$P_1^{-1}(+) = P_1^{-1}(-) + H_1^T R_1^{-1} H_1 .$$

If we have no a priori information $P_1^{-1}(-) = 0$ and the above gives

$$P_1^{-1}(+) = \frac{1}{\sigma_1^2} \Rightarrow P_1(+) = \sigma_1^2 .$$

Next, since we are estimating a constant the system dynamics propagate $\hat{x}_1(+)$ to $\hat{x}_2(-)$ as

$$\begin{aligned} \hat{x}_2(-) &= 1\hat{x}_1(+) = z_1 \\ P_2(-) &= P_1(+) = \sigma_1^2 . \end{aligned}$$

The second measurement z_2 is again of the form $z_2 = x + v_2$ so we have $H_2 = 1$, and $R_2 = \sigma_2^2$ so that

$$K_2 = P_2(-)H_2^T[H_2 P_2(-)H_2^T + R_2]^{-1} = \sigma_1^2(\sigma_1^2 + \sigma_2^2)^{-1} = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} .$$

Using this Kalman gain K_2 we have

$$\begin{aligned}
\hat{x}_2(+) &= \hat{x}_2(-) + K_2(z_2 - H_2\hat{x}_2(-)) \\
&= z_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(z_2 - z_1) \\
&= \frac{\sigma_1^2 z_2 + \sigma_2^2 z_1}{\sigma_1^2+\sigma_2^2},
\end{aligned}$$

the same as the books equation 1.0-7. Next we have for $P_2(+)$ the following

$$\begin{aligned}
P_2(+) &= (1 - K_2H_2)P_2(-) = \left(1 - \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\sigma_1^2 \\
&= \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1},
\end{aligned}$$

the books equation 1.0-6.

Part (b): When the two measurements are taken sequentially the each are of the form $z_i = x + v_i$ for $i = 1, 2$ and our measurement vector $\mathbf{z}$ is given by

$$\mathbf{z}_1 = \left[\begin{array}{c} z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] x + \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right],$$

so $H_1 = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$ and the probability density for the measurement error vector $\mathbf{v}_1$ given by $p(\mathbf{v}_1) = N\left(0, \left[\begin{array}{cc} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{array}\right]\right)$. Since we have no a priori information we are required to use *weighted-least squares* which has a update given by

$$\hat{x}_1(+) = (H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1} \mathbf{z}_1,$$

with the matrix $R_1^{-1} = \left[\begin{array}{cc} 1/\sigma_1^2 & 0 \\ 0 & 1/\sigma_2^2 \end{array}\right]$. With the form of H_1 above we compute

$$H_1^T R_1^{-1} H_1 = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}.$$

Using this we can compute the new uncertainty matrix $P_1(+)$ as

$$P_1(+)^{-1} = P_1^{-1}(-) + H_1^T R_1^{-1} H_1 = 0 + \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}.$$

Thus $P_1(+)$ is given by

$$P_1(+) = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1},$$

the same as the books equation 1.0-6. Finally we have $\hat{x}_1(+)$ after this combined measurement $\mathbf{z}_1$ given by

$$\begin{aligned}
\hat{x}_1(+) &= \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1}\left(\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2}\right) \\
&= \frac{1}{\sigma_1^2+\sigma_2^2}\left(\sigma_2^2 z_1 + \sigma_1^2 z_2\right),
\end{aligned}$$

the same as the books equation 1.0-7.

Problem 4-2 (additional Kalman filtering examples)

For this problem we want to rework Problems 1-1 and 1-3 using the Kalman filtering framework developed in this chapter. Problem 1-1 has to do with two measurements z_i of a constant x that are correlated with a correlation coefficient ρ. Problem 1-3 has to with *three* independent measurements.

Problem 1-1: If we assume that our measurements of the constant x of the form $z_i = x + v_i$ for $i = 1, 2$ are correlated, then the noise vector $\mathbf{v}$ takes the form $\mathbf{v} \sim N(0, R)$ with $R = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$. Thus our measurement vector $\mathbf{z}_1$ is given by

$$
\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x + \mathbf{v}_1 \,,
$$

thus $H_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $R_1 = R$ the matrix above. If we assume we have no a priori information on the value of x then our estimate of our state x after the measurement $\mathbf{z}_1$ is given by the weighted least squares estimate

$$
\hat{x}_1(+) = (H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1} \mathbf{z}_1 \,, \tag{94}
$$

and the new uncertainty, $P_1(+)$, can be computed as

$$
P_1(+)^{-1} = P_1(-)^{-1} + H_1^T R_1^{-1} H_1 = H_1^T R_1^{-1} H_1 \,,
$$

since $P_1(-)^{-1} = 0$. From the given form for R_1 we have that its inverse R^{-1} is given by

$$
R_1^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} .
$$

So that the product $R_1^{-1} H_1$ is given by

$$
R_1^{-1} H_1 = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 - \rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 + \sigma_1^2 \end{bmatrix} ,
$$

and the product $H_1^T R_1^{-1} H_1$ is given by

$$
H_1^T R_1^{-1} H_1 = \frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2(1-\rho^2)} .
$$

Thus since this product $H_1^T R_1^{-1} H_1$ equals $P_1(+)^{-1}$ we have that

$$
P_1(+) = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} ,
$$

which is the same result given in the book for the uncertainty of this system. Next using these subresults in Equation 94 we compute $\hat{x}_1(+)$ as

$$
\begin{aligned}
\hat{x}_1(+) &= \left(\frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \right) \cdot \left(\frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \right) \left((\sigma_2^2 - \rho\sigma_1\sigma_2) z_1 + (-\rho\sigma_1\sigma_2 + \sigma_1^2) z_2 \right) \\
&= \left(\frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \right) z_1 + \left(\frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \right) z_2 \,,
\end{aligned}
$$

which also agrees with the solution found in Problem 1.1.

Problem 1-3: In the case when we have three independent measurements, z_i, of an unknown scalar x, our measurement vector $\mathbf{z}_1$ is given by $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x + \mathbf{v}_1$ with $\mathbf{v}_1 \sim N(0, R_1)$ and $R_1 = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$. From this formulation we see that $R^{-1} = \text{diag}(1/\sigma_1^2, 1/\sigma_2^2, 1/\sigma_3^2)$ and the measurement sensitivity matrix $H_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Again assuming no a priori information we have

$$P_1(+)^{-1} = P_1(-)^{-1} + H_1^T R_1^{-1} H_1 = H_1^T R_1^{-1} H_1 \,,$$

and

$$\hat{x}_1(+) = (H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1} \mathbf{z}_1 \,.$$

With the above matrices we have $H_1^T R_1^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{1}{\sigma_2^2} & \frac{1}{\sigma_3^2} \end{bmatrix}$, and thus $H_1^T R_1^{-1} H_1 = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right)$ so that

$$P_1(+) = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right)^{-1} ,$$

and

$$\hat{x}_1(+) = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right)^{-1} \left(\frac{z_1}{\sigma_1^2} + \frac{z_2}{\sigma_2^2} + \frac{z_3}{\sigma_3^2} \right) ,$$

which is the same as the results found in problem 1-3.

Problem 4-3 (Kalman filtering a decaying concentration)

Part (a): For this part of the problem our measurements are $z_i = x_0 e^{-t_i} + v_i$ with $v_i \sim N(0, \sigma_i^2)$ to be taken simultaneously. Since we are told that a priori we have no prior information on the initial concentration x_0 we will take $P(-)^{-1} = 0$ and the initial estimate $\hat{x}(+)$ is the maximum likelihood estimate, which in this case because the two measurements have different uncertainties is given by the *weighted-least-squares* estimate

$$\hat{x}(+) = (H^T R^{-1} H)^{-1} H^T R^{-1} \mathbf{z} \,. \tag{95}$$

In this problem we map our measurement $\mathbf{z}$ to our state via

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{-t_1} \\ e^{-t_2} \end{bmatrix} x_0 + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} ,$$

we see that the matrices H and R are explicitly given by $H = \begin{bmatrix} e^{-t_1} \\ e^{-t_2} \end{bmatrix}$ and $R = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$. Using these we can compute the matrix products needed to evaluate Equation 95 as

$$H^T R^{-1} H = \frac{e^{-2t_1}}{\sigma_1^2} + \frac{e^{-2t_2}}{\sigma_2^2} ,$$

and

$$H^T R^{-1} \mathbf{z} = \frac{e^{-t_1}}{\sigma_1^2} z_1 + \frac{e^{-t_2}}{\sigma_2^2} z_2 \,.$$

Thus $\hat{x}(+)$ is given by

$$\begin{aligned}
\hat{x}(+) &= \left(\frac{e^{-2t_1}}{\sigma_1^2} + \frac{e^{-2t_2}}{\sigma_2^2} \right)^{-1} \left(\frac{e^{-t_1}}{\sigma_1^2} z_1 + \frac{e^{-t_2}}{\sigma_2^2} z_2 \right) = \left(\frac{\sigma_1^2 \sigma_2^2}{\sigma_2^2 e^{-2t_1} + \sigma_1^2 e^{-2t_2}} \right) \left(\frac{e^{-t_1}}{\sigma_1^2} z_1 + \frac{e^{-t_2}}{\sigma_2^2} z_2 \right) \\
&= \frac{\sigma_2^2 e^{-t_1} z_1 + \sigma_1^2 e^{-t_2} z_2}{\sigma_2^2 e^{-2t_1} + \sigma_1^2 e^{-2t_2}} \,,
\end{aligned}$$

and the covariance update equation gives

$$P(+)^{-1} = P(-)^{-1} + H^T R^{-1} H = \frac{e^{-2t_1}}{\sigma_1^2} + \frac{e^{-2t_2}}{\sigma_2^2} \,,$$

so $P(+)$ is given by

$$P(+) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_2^2 e^{-2t_1} + \sigma_1^2 e^{-2t_2}} = \left(\frac{1}{\sigma_1^2} e^{-2t_1} + \frac{1}{\sigma_2^2} e^{-2t_2} \right)^{-1} \,,$$

the same results found in problem 1.3 earlier.

Part (b): If the measurements are now assumed to be obtained sequentially then since $z_1 = e^{-t_1} x_0 + v_1$ is the first one we have $H_1 = e^{-t_1}$ and $R_1 = \sigma_1^2$. Since we have no a priori information on x_0 the state update equation is still the maximum likelihood equation, an applying the information from just this one measurement gives as our new estimate of x_0

$$\hat{x}_1(+) = (H_1^T R_1^{-1} H_1)^{-1} H_1 R_1^{-1} z_1 = \left(\frac{e^{-2t_1}}{\sigma_1^2} \right)^{-1} \frac{e^{-t_1}}{\sigma_1^2} z_1 = e^{t_1} z_1 \,,$$

and

$$P_1(+)^{-1} = P_1(-)^{-1} + H_1^T R_1^{-1} H_1 = \frac{e^{-2t_1}}{\sigma_1^2} \quad \Rightarrow \quad P_1(+) = \sigma_1^2 e^{2t_1} \,.$$

Now before we can incorporate the second equation we must perform state and covariance extrapolation

$$\hat{x}_2(-) = \Phi_1 \hat{x}_1(+) \quad \text{and} \quad P_2(-) = \Phi_1 P_1(+) \Phi_1^T + Q_1 = \Phi_1 P_1(+) \Phi_1^T \,,$$

since $Q_1 = 0$. As the underlying initial state, x_0, we are trying to estimate is a constant we have $\Phi = 1$ (here Φ denotes how the state changes with *time*, not the measurement). Thus

$$\hat{x}_2(-) = \hat{x}_1(+) = e^{t_1} z_1 \quad \text{and} \quad P_2(-) = P_1(+) = \sigma_1^2 e^{2t_1} \,.$$

The second z_2 is related to the initial concentration as $z_2 = e^{-t_2} x_0 + v_2$ we have $H_2 = e^{-t_2}$ and $R_2 = \sigma_2^2$. Next we use the Kalman update equations to obtain the posteriori state and covariance $\hat{x}_2(+)$ and $P_2(+)$ after the second measurement. We find

$$\begin{aligned}
K_2 &= P_2(-) H_2^T [H_2 P_2(-) H_2^T + R_2]^{-1} = \sigma_1^2 e^{2t_1} \cdot e^{-t_2} [e^{-2t_2} \sigma_1^2 e^{2t_1} + \sigma_2^2]^{-1} \\
&= \sigma_1^2 e^{2t_1 - t_2} [\sigma_1^2 e^{-2t_2 + 2t_1} + \sigma_2^2]^{-1} \,.
\end{aligned}$$

Then

$$\begin{aligned}
\hat{x}_2(+) &= \hat{x}_2(-) + K_2(z_2 + H_2\hat{x}_2(-)) \\
&= e^{t_1}z_1 + \sigma_1^2 e^{2t_1-t_2}(\sigma_1^2 e^{-2t_2+2t_1} + \sigma_2^2)^{-1}(z_2 - e^{-t_2}e^{t_1}z_1) \\
&= \frac{\sigma_2^2 e^{-t_1}z_1 + \sigma_1^2 e^{-t_2}z_2}{\sigma_2^2 e^{-2t_1} + \sigma_1^2 e^{-2t_2}},
\end{aligned}$$

and

$$\begin{aligned}
P_2(+) &= (I - K_2H_2)P_2(-) = \left(1 - \frac{\sigma_1^2 e^{2t_1-2t_2}}{\sigma_1^2 e^{-2t_2+2t_1} + \sigma_2^2}\right)\sigma_1^2 e^{2t_1} \\
&= \frac{\sigma_1^2\sigma_2^2}{\sigma_2^2 e^{-2t_1} + \sigma_1^2 e^{-2t_2}},
\end{aligned}$$

both of which agree with what we computed earlier.

Problem 4-4 (weighted least squares and adding an additional measurement)

After having appended a second measurement the same weighted least squares solution for $\hat{x}$ will hold, but with the larger matrices H_1, R_1, and z_1. That is we have

$$\hat{x}(+) = (H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1} z_1 . \tag{96}$$

Since the new measurement is uncorrelated with the others R_1 is block diagonal so its inverse is also block diagonal

$$R_1^{-1} = \begin{bmatrix} R_0^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix},$$

and the measurement sensitivity matrix H_1 also has a block form given by

$$H_1^T = \begin{bmatrix} H_0^T & H^T \end{bmatrix}.$$

Using these two we see that

$$\begin{aligned}
H_1^T R_1^{-1} H_1 &= \begin{bmatrix} H_0^T & H^T \end{bmatrix} \begin{bmatrix} R_0^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} H_0 \\ H \end{bmatrix} = \begin{bmatrix} H_0^T & H^T \end{bmatrix} \begin{bmatrix} R_0^{-1}H_0 \\ R^{-1}H \end{bmatrix} \\
&= H_0^T R_0^{-1} H_0 + H^T R^{-1} H .
\end{aligned} \tag{97}$$

The problem states that we should define $P(-)^{-1}$ as $H_0^T R_0^{-1} H_0$ so if we define $P(+)^{-1}$ in the same way as $H_1^T R_1^{-1} H_1$ then from Equation 97 we have shown that

$$P(+)^{-1} = P(-)^{-1} + H^T R^{-1} H .$$

Next lets compute $\hat{x}(+)$ using Equation 96. We first see that

$$H_1^T R_1^{-1} z_1 = \begin{bmatrix} H_0^T & H^T \end{bmatrix} \begin{bmatrix} R_0^{-1} & 0 \\ 0 & R^{-1} \end{bmatrix} \begin{bmatrix} z_0 \\ z \end{bmatrix} = H_0^T R_0^{-1} z_0 + H^T R^{-1} z ,$$

so that

$$\begin{aligned}\hat{x}(+) &= (H_1^T R_1^{-1} H_1)^{-1}[H_0^T R_0^{-1} z_0 + H^T R^{-1} z] = P(+)[H_0^T R_0^{-1} z_0 + H^T R^{-1} z] \\ &= P(+) H_0^T R_0^{-1} z_0 + P(+) H^T R^{-1} z\,, \end{aligned} \tag{98}$$

using the definition that $(H_1^T R_1^{-1} H_1)^{-1} = P(+)$. Now $P(+)$ is given in terms of $P(-)$ as

$$P(+) = [P(-)^{-1} + H^T R^{-1} H]^{-1}\,.$$

To evaluate this we will use the **matrix inversion identity**

$$B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}\,. \tag{99}$$

with

$$\begin{aligned} B &= P(-)^{-1} + H^T R^{-1} H = P(+)^{-1} \quad \text{and} \\ A &= P(-)^{-1}\,. \end{aligned}$$

For which we find

$$P(+) = P(-) - P(+) H^T R^{-1} H P(-)\,. \tag{100}$$

When we put this expression for $P(+)$ into the first term in Equation 98 we find

$$\begin{aligned}\hat{x}(+) &= P(-) H_0^T R_0^{-1} z_0 - P(+) H^T R^{-1} H P(-) H_0^T R_0^{-1} z_0 + P(+) H^T R^{-1} z \\ &= \hat{x}(-) - P(+) H^T R^{-1} H \hat{x}(-) + P(+) H^T R^{-1} z \\ &= \hat{x}(-) + P(+) H^T R^{-1}(z - H\hat{x}(-))\,, \end{aligned}$$

which is the desired expression. In the above simplifications we have used the fact that

$$\hat{x}(-) = (H_0^T R_0^{-1} H_0)^{-1} H_0^T R_0^{-1} z_0 = P(-) H_0^T R_0^{-1} z_0\,.$$

Problem 4-5 (minimizing the scalar loss functional $J(\hat{x})$)

The given objective function $J(\hat{x})$ can be expanded and written as

$$\begin{aligned} J(\hat{x}) &= [\hat{x} - x(-)]^T P(-)^{-1}[\hat{x} - x(-)] + (z - H\hat{x})^T R^{-1}(z - H\hat{x}) \\ &= \hat{x}^T P(-)^{-1}\hat{x} - \hat{x}^T P(-)^{-1} x(-) - x(-)^T P(-)^{-1}\hat{x} + x(-)^T P(-)^{-1} x(-) \\ &+ z^T R^{-1} z - z^T R^{-1} H\hat{x} - \hat{x}^T H^T R^{-1} z + \hat{x}^T H^T R^{-1} H\hat{x}\,. \end{aligned}$$

Then to find the value of $\hat{x}$ that minimizes this expression we take the derivative of J with respect to $\hat{x}$, set the result equal to zero and then solve for $\hat{x}$. This derivative is given by

$$\begin{aligned} \frac{\partial J}{\partial \hat{x}} &= 2P(-)^{-1}\hat{x} - P(-)^{-1}x(-) - P(-)^{-1}x(-) \\ &- H^T R^{-1} z - H^T R^{-1} z + 2H^T R^{-1} H\hat{x} \\ &= 2[P(-)^{-1} + H^T R^{-1} H]\hat{x} - 2P(-)^{-1}x(-) - 2H^T R^{-1} z\,. \end{aligned}$$

Where to take the derivative above we have used Equations 311 and 312

$$\frac{\partial \mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} = \frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}},$$

and the quadratic derivative Equation 312,

$$\frac{\partial \mathbf{x}^T \mathbf{A} \mathbf{x}}{\partial \mathbf{x}} = (\mathbf{A} + \mathbf{A}^T)\mathbf{x}. \tag{101}$$

Setting the expression $\frac{\partial J}{\partial \hat{x}}$ equal to zero and solving for $\hat{x}$ which we denote $\hat{x}(+)$ we get

$$\hat{x}(+) = (P(-)^{-1} + H^T R^{-1} H)^{-1} (P(-)^{-1} x(-) + H^T R^{-1} z),$$

as the solution to the expressed minimization problem. Motivated by the expression above if we define $P(+)$ as

$$P(+) = (P(-)^{-1} + H^T R^{-1} H)^{-1},$$

then the inverse of $P(+)$ is given directly

$$P(+)^{-1} = P(-)^{-1} + H^T R^{-1} H.$$

Using this definition the above expression for $\hat{x}(+)$ is given as

$$\hat{x}(+) = P(+) P(-)^{-1} x(-) + P(+) H^T R^{-1} z,$$

and for the *first* term in the above we can use the matrix inversion lemma as in the previous problem to write $P(+)$ as given by Equation 100 to obtain

$$\begin{aligned}
\hat{x}(+) &= [P(-) - P(+) H^T R^{-1} H P(-)] P(-)^{-} \hat{x}(-) + P(+) H^T R^{-1} z \\
&= \hat{x}(-) - P(+) H^T R^{-1} H \hat{x}(-) + P(+) H^T R^{-1} z \\
&= \hat{x}(-) + P(+) H^T R^{-1} (z - H \hat{x}(-)), \qquad (102)
\end{aligned}$$

as we were to show.

As an *alternative* way to show the desired expressions for $\hat{x}(+)$ and $P(+)$ that does not use the matrix inversion lemma, we can take the expression for J and write everything in terms of the estimated vs. prior difference or $\tilde{x} = \hat{x} - x(-)$. We find that

$$\begin{aligned}
J &= (\hat{x} - x(-))^T P(-)^{-1} (\hat{x} - x(-)) \\
&+ (z - H(\hat{x} - x(-) + x(-)))^T R^{-1} (z - H(\hat{x} - x(-) + x(-))) \\
&= (\hat{x} - x(-))^T P(-)^{-1} (\hat{x} - x(-)) \\
&+ (z - H(\hat{x} - x(-)))^T R^{-1} (z - H(\hat{x} - x(-))) \\
&- (z - H(\hat{x} - x(-)))^T R^{-1} H x(-) - x(-)^T H^T R^{-1} (z - H(\hat{x} - x(-))) \\
&+ x(-)^T H^T R^{-1} H x(-).
\end{aligned}$$

As before we will want to take the derivative of J with respect to $\hat{x}$, set the result equal to zero and solve for $\hat{x}$. With the above expression since $x(-)$ is a constant, the derivative with respect to $\hat{x}$ is equal to the derivative with respect to the expression $\hat{x} - x(-)$. If we define this expression as $\tilde{x}$, we see that J in terms of $\tilde{x}$ can be written as

$$\begin{aligned}
J &= \tilde{x}^T P(-)^{-1}\tilde{x} + (z - H\tilde{x})^T R^{-1}(z - H\tilde{x}) \\
&- (z - H\tilde{x})^T R^{-1} H x(-) - x(-)^T H^T R^{-1}(z - H\tilde{x}) \\
&+ x(-)^T H^T R^{-1} H x(-) \\
&= \tilde{x}^T P(-)^{-1}\tilde{x} \\
&+ z^T R^{-1} z - z^T R^{-1} H\tilde{x} - \tilde{x}^T H^T R^{-1} z + \tilde{x}^T H^T R^{-1} H\tilde{x} \\
&- z^T R^{-1} H x(-) + \tilde{x}^T H^T R^{-1} H x(-) - x(-)^T H^T R^{-1} z + x(-)^T H^T R^{-1} H\tilde{x} \\
&+ x(-)^T H^T R^{-1} H x(-) \,.
\end{aligned}$$

Taking the $\tilde{x}$ derivative of this expression gives

$$\begin{aligned}
\frac{\partial J}{\partial \tilde{x}} &= 2P(-)^{-1}\tilde{x} \\
&- H^T R^{-1} z - H^T R^{-1} z + 2H^T R^{-1} H\tilde{x} \\
&+ H^T R^{-1} H x(-) + H^T R^{-1} H x(-) \\
&= 2(P(-)^{-1} + H^T R^{-1} H)\tilde{x} - 2H^T R^{-1} z + 2H^T R^{-1} H x(-) \,.
\end{aligned}$$

Seeing this derivative equal to zero ans solving for $\tilde{x}$ we find

$$\begin{aligned}
\tilde{x} &= (P(-)^{-1} + H^T R^{-1} H)^{-1}[-H^T R^{-1} H x(-) + H^T R^{-1} z] \\
&= (P(-)^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1}(z - Hx(-)) \,.
\end{aligned}$$

Thus converting the minimum we just found for $\tilde{x}$ into the variable $\hat{x}$ with $\tilde{x} = \hat{x} - x(-)$ we have that

$$\hat{x} = x(-) + (P(-)^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1}(z - Hx(-)) \,,$$

the same expression as in Equation 102.

Problem 4-6 (the derivation of the maximum likelihood expression)

Using the definition of conditional probability that

$$p(z|x) = \frac{p(x,z)}{p(x)} = \frac{p(x)p(z)}{p(x)} = p(z) \,,$$

since the variables x and v are independent. Let pick the estimate $\hat{x}$ so that it maximizes $p(z|x)$, this is known as the maximum likelihood estimate. The probability density function of the random variable v is said to be a multidimensional normal and is given by

$$p(v) = \frac{1}{(2\pi)^{l/2}|R|^{1/2}} \exp\left\{-\frac{1}{2} v^T R^{-1} v\right\} \,,$$

where l is the dimension of the measurement noise. Then as a function of x since $v = z - Hx$ is given by

$$p(z|x) = \frac{1}{(2\pi)^{l/2}|R|^{1/2}} \exp\left\{-\frac{1}{2}(z - Hx)^T R^{-1}(z - Hx)\right\} \,, \tag{103}$$

so to maximize $p(z|x)$ is equivalent to minimize the product

$$(z - Hx)^T R^{-1}(z - Hx) = z^T R^{-1} z - z^T R^{-1} Hx - x^T H^T R^{-1} z + x^T H^T R^{-1} Hx\,,$$

as a function of x. When we take the derivative of this expression and set the result equal to zero we find that

$$\frac{\partial J}{\partial x} = -H^T R^{-1} z - H^T R^{-1} z + 2H^T R^{-1} Hx = 0\,.$$

Solving for x we find that

$$x = (H^T R^{-1} H)^{-1}(H^T R^{-1} z)\,, \tag{104}$$

for the maximal likelihood solution. This is the same expression we found in Problem 4.4 above and thus the analysis from that problem is valid here. Namely, if we receive another measurement z_2, with a measurement sensitivity matrix H_2, and measurement covariance matrix R_2 the recursive update of our state estimate $\hat{x}$ is given by

$$\begin{aligned} x_2 &= x_1 + P(+) H_2^T R_2^{-1}(z_2 - H_2 x_1) \\ P(+)^{-1} &= H_1^T R_1^{-1} H_1 + H_2^T R_2^{-1} H_2\,, \end{aligned}$$

where x_1 is the estimate of x before receiving the measurement z_2 given by Equation 104 with $H = H_1$, $R = R_1$, $z = z_1$, and $x = x_1$.

Problem 4-7 (the recursive maximum a posteriori estimate)

Part (a): As x is a Gaussian random variable and a linear transformation of Gaussian random variables produces another Gaussian random variable, we see that Hx is another Gaussian random variable. Since v is independent of x and Gaussian and since sums of independent Gaussian random variables are also Gaussian the random variable $Hx + v$ is Gaussian. To determine the full distribution of $Hx + v$, it is sufficient to compute the mean of the we have for the mean and covariance of $z = Hx + v$. For the mean of z we have

$$E[z] = HE[x] + E[v] = H\hat{x}(-) + 0 = H\hat{x}(-)\,.$$

For the $\text{Cov}(z)$ using independence we find

$$\begin{aligned} \text{Cov}(z) &= \text{Cov}(Hx) + \text{Cov}(v) \\ &= H\text{Cov}(x)H^T + R = HP(-)H^T + R\,. \end{aligned}$$

Thus $z \sim N(Hx(-), HP(-)H^T + R)$ as we were to show.

Part (b): Using the definition of conditional probability we find

$$p(x|z) = \frac{p(x,z)}{p(z)} = \frac{p(z|x)p(x)}{p(z)} = \frac{p(v)p(x)}{p(z)}\,,$$

where we have used the fact that $p(z|x) = p(z - Hx|x) = p(v)$.

Part (c): Note that from the problem statement we have that $x \sim N(\hat{x}(-), P(-))$, from Part (a) of this problem we have that $z \sim N(Hx(-), HP(-)H^T + R)$, and from Problem 4-6 above that $p(z|x)$ can be expressed using Equation 103. Thus we can compute $p(x|z)$ using each of these components and obtain the functional form presented in the book.

$$
\begin{aligned}
p(x|z) &= c\exp\{-\frac{1}{2}[(x-\hat{x}(-))^T P(-)^{-1}(x-\hat{x}(-)) + (z-Hx)^T R^{-1}(z-Hx) \\
&- (z-H\hat{x}(-))[HP(-)H^T + R]^{-1}(z-H\hat{x}(-))]\}\,.
\end{aligned}
$$

In the above exponential one can see the three major terms that come from $p(x)$, $p(z|x)$, and $p(z)$ respectively.

Part (d): Sine $p(x|z)$ is another Gaussian density, but with an as yet undetermined mean and covariance, lets denote this unknown mean and covariance by $\hat{x}(+)$ and $P(+)$, and emphasize this by setting the term in the exponential above equal to

$$
-\frac{1}{2}(x-\hat{x}(+))^T P(+)^{-1}(x-\hat{x}(+))\,.
$$

This gives the equation (after we multiply by -2 on both sides)

$$
\begin{aligned}
(x-\hat{x}(+))^T P(+)^{-1}(x-\hat{x}(+)) &= (x-\hat{x}(-))^T P(-)^{-1}(x-\hat{x}(-)) + (z-Hx)^T R^{-1}(z-Hx) \\
&- (z-H\hat{x}(-))[HP(-)H^T + R]^{-1}(z-H\hat{x}(-))\,.
\end{aligned}
$$

Expanding the quadratics on both sides of the above expression gives

$$
\begin{aligned}
x^T P(+)^{-1}x &- 2\hat{x}(+)^T P(+)^{-1}x + \hat{x}(+)^T P(+)^{-1}\hat{x}(+) \\
&= x^T P(-)^{-1}x - 2\hat{x}(-)^T P(-)^{-1}x + \hat{x}(-)^T P(-)^{-1}\hat{x}(-) \\
&+ z^T R^{-1}z - 2z^T R^{-1}Hx + x^T H^T R^{-1}Hx \\
&- (z-H\hat{x}(-))[HP(-)H^T + R]^{-1}(z-H\hat{x}(-))\,.
\end{aligned}
$$

Equating quadratic and terms in x above we see that $P(+)^{-1}$ must be given by

$$
P(+)^{-1} = P(-)^{-1} + H^T R^{-1} H\,. \tag{105}
$$

Equating the linear terms in x above we get that

$$
-2\hat{x}(+)^T P(+)^{-1}x = -2(\hat{x}(-)^T P(-)^{-1} + z^T R^{-1}H)x\,,
$$

or "canceling x" from both sides and taking the transpose we have

$$
P(+)^{-1}\hat{x}(+) = P(-)^{-1}\hat{x}(-) + H^T R^{-1}z\,.
$$

Now if we multiply by $P(+)$ on the right-hand-side of the above we end with

$$
\hat{x}(+) = P(+)P(-)^{-1}\hat{x}(-) + P(+)H^T R^{-1}z\,, \tag{106}
$$

From Equation 105 we see that

$$
\begin{aligned}
P(+) &= (P(-)^{-1} + H^T R^{-1}H)^{-1} \\
&= P(-) - P(+)H^T R^{-1}HP(-)\,,
\end{aligned}
$$

when we use the matrix inversion lemma given in Equation 99. With this expression we can write the product of $P(+)P(-)^{-1}$ as

$$P(+)P(-)^{-1} = I - P(+)H^T R^{-1} H \,, \tag{107}$$

from which we can conclude that $\hat{x}(+)$ is given by

$$\begin{aligned} \hat{x}(+) &= (I - P(+)H^T R^{-1} H)\hat{x}(-) + P(+)H^T R^{-1} z \\ &= \hat{x}(-) + P(+)H^T R^{-1}(z - H\hat{x}(-)) \,. \end{aligned} \tag{108}$$

Proving the results summarized in Equations 105 and 108.

Problem 4-8 (the uncertainty in an estimator of Kalman like form)

The given linear filter we seek is of the form

$$\dot{\hat{x}}(t) = K'\hat{x} + Kz \,,$$

where K' and K chosen such that $\hat{x}$ is unbiased and to have the smallest variance among all estimators of this form. Lets consider the error $\tilde{x}$ defined as $\tilde{x} = \hat{x} - x$. This function has a differential equation given by

$$\begin{aligned} \frac{d\tilde{x}}{dt} &= \frac{d\hat{x}}{dt} - \frac{dx}{dt} \\ &= K'\hat{x} + Kz - Fx - Gw \\ &= K'\hat{x} + K(Hx + v) - Fx - Gw \\ &= K'\hat{x} + KHx + Kv - Fx - Gw \,. \end{aligned}$$

Since $\hat{x} = \tilde{x} + x$ we have that $\frac{d\tilde{x}}{dt}$ in terms of $\tilde{x}$ and x is given by

$$\frac{d\tilde{x}}{dt} = K'\tilde{x} + (K' + KH - F)x + Kv - Gw \,.$$

Then to be unbiased for all x we must pick K' and K such that

$$K' + KH - F = 0 \quad \text{or} \quad K' = F - KH \,. \tag{109}$$

With this expression for K' our estimator is then given by solving the following

$$\begin{aligned} \dot{\hat{x}} &= (F - KH)\hat{x} + Kz \\ &= F\hat{x} + K(z - H\hat{x}) \,, \end{aligned} \tag{110}$$

for $\hat{x}$. With this choice for K' the expression for $\frac{d\tilde{x}}{dt}$ has no terms involving the unknown x and is given by

$$\frac{d\tilde{x}}{dt} = K'\tilde{x} + Kv - Gw \,.$$

If we define $P(t)$ to be $P(t) = E[\tilde{x}\tilde{x}^T]$ from the above we see that

$$\begin{aligned} \dot{P} &= K'P + PK'^T + \mathrm{Cov}(Kv - Gw) \\ &= K'P + PK'^T + K\mathrm{Cov}(v)K^T + G\mathrm{Cov}(w)G^T \\ &= K'P + PK'^T + KRK^T + GQG^T \,. \end{aligned}$$

When we put in the expression for K' found above we obtain

$$\dot{P} = (F - KH)P + P(F - KH)^T + KRK^T + GQG^T \,. \tag{111}$$

Now we want to find the value of K such that our objective function $J = \text{trace}(\dot{P})$ is a minimum. To find this value of K lets first compute the expression for trace$(\dot{P})$. Using Equation 111 we find

$$\begin{aligned} J &= \text{trace}(\dot{P}) \\ &= \text{trace}(FP) + \text{trace}(PF^T) + \text{trace}(GQG^T) \\ &- \text{trace}(KHP) - \text{trace}(PH^TK^T) + \text{trace}(KRK^T) \,. \end{aligned}$$

Next we need to evaluate $\frac{\partial J}{\partial K}$. To do this we will recall the following matrix derivative facts

$$\frac{\partial}{\partial A}\text{trace}(BAC) = B^TC^T \quad \text{so that} \tag{112}$$

$$\frac{\partial}{\partial A}\text{trace}(AC) = I^TC^T = C^T$$

$$\frac{\partial}{\partial A}\text{trace}(CA^T) = \frac{\partial}{\partial A}\text{trace}(AC^T) = I^TC = C \quad \text{and}$$

$$\frac{\partial}{\partial A}\text{trace}(ABA^T) = 2AB \,. \tag{113}$$

Using these results we find that $\frac{\partial J}{\partial K}$ is given by

$$\frac{\partial J}{\partial K} = -PH^T - PH^T + 2KR \,.$$

Setting this derivative equal to zero and solving for K gives

$$K = PH^TR^{-1} \,, \tag{114}$$

as we were to show.

Problem 4-9 (questions about Kalman filters)

Warning: I'm not sure exactly what this problem was asking or how to answer it. If anyone has an idea of the type of solution requested please contact me.

Problem 4-10 (recursive scalar estimation)

That the estimator $\hat{m}_k$ is unbiased can be seen by taking the expectation of its expression

$$E[\hat{m}_k] = \frac{1}{k}\sum_{i=1}^{k} E[x_i] = \frac{1}{k}\sum_{i=1}^{k} m = m \,,$$

where we have used the fact that the expectation of any given sample is the same as the population mean or $E[x_i] = m$.

To show that the estimate of σ^2 is an unbiased estimator of the population variance we will assume that the samples x_i are drawn from a *Gaussian* distribution with a population mean m and variance σ^2. Then it can be shown that $\hat{\sigma}_k^2$ as defined in this problem is related to a chi-squared distribution in that the random variable

$$\frac{(k-1)\hat{\sigma}_k^2}{\sigma^2}\,,$$

is distributed as a χ^2 random variable with $k-1$ degrees of freedom [2, 3]. Recalling that if the random variable, say X, is χ^2 with $k-1$ degrees of freedom then the expectation of X is

$$E[X] = k - 1\,, \tag{115}$$

so that since $\frac{(k-1)\hat{\sigma}_k^2}{\sigma^2}$ is also χ^2 with $k-1$ degrees of freedom

$$E\left[\frac{(k-1)\hat{\sigma}_k^2}{\sigma^2}\right] = k - 1\,.$$

but at the same time

$$E\left[\frac{(k-1)\hat{\sigma}_k^2}{\sigma^2}\right] = \frac{(k-1)}{\sigma^2}E[\hat{\sigma}_k^2]\,.$$

Setting these two expressions equal to each other and solving for $E[\hat{\sigma}_k^2]$ gives

$$E[\hat{\sigma}_k^2] = \sigma^2\,,$$

showing that the estimator $\hat{\sigma}_k^2$ is unbiased.

To derive a recursive form for an estimator for the mean m note that from the given expression for $\hat{m}_k$ note that we have

$$\begin{aligned}
\hat{m}_k &= \frac{1}{k}\sum_{i=1}^{k} x_i = \frac{k-1}{k}\left(\frac{1}{k-1}\sum_{i=1}^{k-1} x_i + \frac{1}{k-1}x_k\right) \\
&= \frac{k-1}{k}\hat{m}_{k-1} + \frac{1}{k}x_k\,,
\end{aligned} \tag{116}$$

showing how given $\hat{m}_{k-1}$ and x_k we can obtain the estimate $\hat{m}_k$.

To derive a recursive form for an estimator for the standard deviation σ^2 we follow much of the same manipulations we did for the mean. We find

$$\begin{aligned}
\hat{\sigma}_k^2 &= \frac{1}{k-1}\sum_{i=1}^{k}(x_i^2 - \hat{m}_k)^2 \\
&= \frac{1}{k-1}\sum_{i=1}^{k}(x_i^2 - 2x_i\hat{m}_k + \hat{m}_k^2) \\
&= \frac{1}{k-1}\sum_{i=1}^{k}x_i^2 - \frac{2}{k-1}\hat{m}_k\sum_{i=1}^{k}x_i + \frac{k}{k-1}\hat{m}_k^2 \\
&= \frac{1}{k-1}\sum_{i=1}^{k}x_i^2 - \frac{k}{k-1}\hat{m}_k^2 \qquad (117) \\
&= \frac{1}{k-1}\left(\sum_{i=1}^{k-1}x_i^2 + x_k^2\right) - \frac{k}{k-1}\hat{m}_k^2 \,. \qquad (118)
\end{aligned}$$

Lets now decrease the index k in Equation 117 so that we can derive an expression for $\sum_{i=1}^{k-1} x_i^2$ (note the upper limit on this summation of $k-1$). We find

$$\hat{\sigma}_{k-1}^2 = \frac{1}{k-2}\sum_{i=1}^{k-1}x_i^2 - \frac{k-1}{k-2}\hat{m}_{k-1}^2 \,,$$

so that the sum $\sum_{i=1}^{k-1} x_i^2$ is given by

$$\sum_{i=1}^{k-1}x_i^2 = (k-2)\hat{\sigma}_{k-1}^2 + (k-1)\hat{m}_{k-1}^2 \,.$$

When we put this expression into Equation 118 we get

$$\begin{aligned}
\hat{\sigma}_k^2 &= \frac{1}{k-1}\left((k-2)\hat{\sigma}_{k-1}^2 + (k-1)\hat{m}_{k-1}^2 + x_k^2\right) - \frac{k}{k-1}\hat{m}_k^2 \\
&= \left(\frac{k-2}{k-1}\right)\hat{\sigma}_{k-1}^2 + \hat{m}_{k-1}^2 - \frac{k}{k-1}\hat{m}_k^2 + \frac{1}{k-1}x_k^2 \,. \qquad (119)
\end{aligned}$$

The above expression is a recursive representation for $\hat{\sigma}_k$ that requires storing and computing the last and most recent estimate of the mean $\hat{m}_k$. Since we can express $\hat{m}_k$ recursively in terms of $\hat{m}_{k-1}$ via Equation 116 if desired we could put this expression into the above and derive an alternative recursive expression for $\hat{\sigma}_k^2$, that only involves the "new" measurement x_k and the old estimates $\hat{\sigma}_{k-1}^2$, $\hat{m}_{k-1}$, that is it does not depend on $\hat{m}_k$.

Problem 4-11 (the system $\dot{x} = ax + w$ with measurements $z = bx + v$)

For this problem everything is a scalar and we have $F = a$, $H = b$, $G = 1$, $Q = q$, and $R = r$. Since the process and measurement noise are uncorrelated the Kalman gain is given by $K = PH^T R^{-1} = \frac{p(t)b}{r}$. The error covariance propagation equation thus given by

$$\begin{aligned} \dot{p}(t) &= 2ap(t) + q - \left(\frac{p(t)b}{r}\right) r \left(\frac{p(t)b}{r}\right) \\ &= 2ap(t) - \frac{b^2}{r}p(t)^2 + q \,, \end{aligned} \tag{120}$$

with an initial condition $p(0) = p_0$. Thus to determine $p(t)$ as a function of t we need to solve the above differential equation. This type of equation is known as a *Riccati* equation and can be transformed into a second order linear equation which can possibly be solved more easily. Note if $q = 0$ this non-linear equation is known as a *Bernoulli* equation. Next we outline the solution to this equation. See [8] for more specific details. The general Riccati equation is given by

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2 \,, \tag{121}$$

for arbitrary functions $P(x)$, $Q(x)$, and $R(x)$. To solve this equation we begin by finding an initial solution y_1 to this equation. Once we have an initial solution if we defined $z(x)$ as

$$z(x) = \frac{1}{y(x) - y_1} \,, \tag{122}$$

or

$$y(x) = y_1 + \frac{1}{z(x)} \,,$$

then when we put the above expression for $y(x)$ into Equation 121 we get the following differential equation for $z(x)$

$$\frac{dz}{dx} = -(Q(x) + 2y_1 R(x))z(x) - R(x) \,.$$

The later, is a first order equation for $z(x)$ which we can solve by quadrature. For the specific problem given here, the initial solution y_1 needed to proceed will be the steady-state or a constant solution. When we take $\dot{p} = 0$ and denote the solution by p_∞ in Equation 120 we have

$$-\frac{b^2}{r}p_\infty^2 + 2ap_\infty + q = 0 \,.$$

When we solve for p_∞ in the above quadratic we find

$$p_\infty = \frac{ar}{b^2}\left(1 \pm \sqrt{1 + \frac{b^2 q}{a^2 r}}\right) \,. \tag{123}$$

Since $p_\infty > 0$ we must take the positive sign in the above expression. Next we let $z(t) = \frac{1}{p(t) - p_\infty}$ and since $P(t) = q$, $Q(t) = 2a$, and $R(t) = -\frac{b^2}{r}$ in the general Riccati solution formulation find the equation for $z(t)$ given by

$$\begin{aligned} z'(t) &= -\left(2a + 2p_\infty\left(-\frac{b^2}{r}\right)\right) z - \left(-\frac{b^2}{r}\right) \\ &= -\left(2a - \frac{2b^2}{r}p_\infty\right) z + \frac{b^2}{r} = \frac{2\sqrt{b^2 q + a^2 r}}{\sqrt{r}} z + \frac{b^2}{r} \,, \end{aligned}$$

when we put in p_∞ and simplify. Consider the coefficient of $z(t)$ in the above equation

$$2\sqrt{\frac{b^2q}{r}+a^2} = 2\sqrt{a^2\left(1+\frac{b^2q}{a^2r}\right)} = 2|a|\sqrt{1+\frac{b^2q}{a^2r}} = 2\beta\,,$$

where we have defined β in the last equality. Thus for $z(t)$ we need to solve

$$z'(t) = 2\beta z(t) + \frac{b^2}{r}\,.$$

When we do this for $z(0) = z_0$ we find

$$z(t) = \frac{-b^2 + b^2e^{2\beta t} + 2r\beta z_0 e^{2\beta t}}{2\beta r}\,.$$

Thus

$$p(t) = p_\infty + \frac{1}{z(t)} = p_\infty + \frac{2\beta r}{b^2(-1+e^{2\beta t}) + 2r\beta z_0 e^{2\beta t}}\,.$$

From this later expression we see that as $t \to \infty$ that $p(t) \to p_\infty$ as it should. Since $p(0) = p_0$ when we let $t = 0$ we find that $p_0 = p_\infty + \frac{1}{z_0}$ or $z_0 = \frac{1}{p_0 - p_\infty}$. Thus

$$p(t) = p_\infty + \frac{2\beta r(p_0 - p_\infty)}{b^2(p_0-p_\infty)(-1+e^{2\beta t}) + 2r\beta e^{2\beta t}}\,. \tag{124}$$

Now note that from the definition of β we have

$$\beta = a\sqrt{1+\frac{b^2q}{a^2r}} = \frac{b^2}{r}p_\infty - a\,,$$

so p_∞ in terms of β is given by

$$p_\infty = \frac{r}{b^2}(\beta + a)\,.$$

When we convert the exponentials above into the hyperbolic functions $\sinh(\cdot)$ and $\cosh(\cdot)$ and replace p_∞ with the above expression for β we find that we can represent $p(t)$ as

$$p(t) = \frac{r\left[(ap_0 - \frac{r}{b^2}(a^2-\beta^2))\sinh(\beta t) + \beta p_0\cosh(\beta t)\right]}{(b^2p_0 - ar)\sinh(\beta t) + \beta r\cosh(\beta t)}\,.$$

Dividing by r on the top and the bottom of this expression and noting that

$$\frac{r}{b^2}\left(a^2-\beta^2\right) = \frac{r}{b^2}\left(a^2 - a^2\left(1+\frac{b^2q}{a^2r}\right)\right) = \frac{r}{b^2}\left(-\frac{b^2q}{r}\right) = -q\,,$$

the above becomes

$$p(t) = \frac{(ap_0+q)\sinh(\beta t) + \beta p_0\cosh(\beta t)}{\left(\frac{b^2p_0}{r} - a\right)\sinh(\beta t) + \beta\cosh(\beta t)}\,, \tag{125}$$

as we were to show. In the Mathematical file `chap_4_prob_11.nb` we perform much of the algebra not displayed in the above derivation.

Problem 4-12 (Kalman filtering a second order system)

The given diagram from the book for this problem implies that $\dot{x}_1 = w$ and

$$x_2 = \int (x_1 - \beta x_2) d\tau \,.$$

Thus as a system of differential equations our system is given by

$$\begin{aligned} \dot{x}_1 &= w \\ \dot{x}_2 &= x_1 - \beta x_2 \,, \end{aligned}$$

or in matrix form the above is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} ,$$

from which we recognize that $F = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}$ and $GQG^T = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$.

The measurement we observe $z(t)$ is related to the state as $z = \alpha x_2 + v$, and so the measurement sensitivity matrix H is given by $H = \begin{bmatrix} 0 & \alpha \end{bmatrix}$ and $R = r$. Using these pieces the matrix Riccati differential equation given by

$$\dot{P} = FP + PF^T + GQG^T - PH^T R^{-1} HP \,.$$

then becomes in steady-state ($\dot{P} = 0$) the following system

$$0 = \begin{bmatrix} q - \frac{\alpha^2 p_{12}^2}{r} & p_{11} - \beta p_{12} - \frac{\alpha^2 p_{12} p_{22}}{r} \\ p_{11} - \beta p_{12} - \frac{\alpha^2 p_{12} p_{22}}{r} & 2p_{12} - 2\beta p_{22} - \frac{\alpha^2 p_{22}^2}{r} \end{bmatrix} .$$

Solving for p_{12} using the $(1,1)$ component above gives

$$p_{12} = \pm \frac{\sqrt{rq}}{\alpha} \,. \tag{126}$$

When we put that value into the $(2,2)$ component of the above expression we get

$$\pm 2 \frac{\sqrt{qr}}{\alpha} - 2\beta p_{22} - \frac{\alpha^2}{r} p_{22}^2 = 0 \,,$$

or as a quadratic equation in standard form

$$\frac{\alpha^2}{r} p_{22}^2 + 2\beta p_{22} \mp \frac{2\sqrt{qr}}{\alpha} = 0 \,.$$

Since we have two signs in the above expression and we have two solutions for each individual quadratic we have four possible solutions for p_{22}. Solving these using the quadratic equation gives

$$p_{22} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\left(\frac{\alpha^2}{r}\right)\left(\mp \frac{2\sqrt{qr}}{\alpha}\right)}}{2\left(\frac{\alpha^2}{r}\right)} = \frac{r}{\alpha^2}\left(-\beta \pm \sqrt{\beta^2 \pm 2\alpha\sqrt{\frac{q}{r}}}\right) \,.$$

Since $p_{22} > 0$ we must take signs such that the resulting expression is positive. Since we are not explicitly told the signs of the variables β and α, lets assume that $\beta > 0$. In that case to guarantee that $p_{22} > 0$ we must take both signs above to be positive. Thus we have

$$p_{22} = \frac{r}{\alpha^2}\left(-\beta + \sqrt{\beta^2 + 2\alpha\sqrt{\frac{q}{r}}}\right) .$$

Now using this expression in the $(1, 2)$ component gives for p_{11}

$$\begin{aligned} p_{11} &= \beta p_{12} + \frac{\alpha^2}{r} p_{12} p_{22} \\ &= \pm\beta\frac{\sqrt{qr}}{\alpha} + \frac{\alpha^2}{r}\left(\pm\frac{\sqrt{qr}}{\alpha}\right)\left(\frac{r}{\alpha^2}\right)\left(-\beta + \sqrt{\beta^2 + 2\alpha\sqrt{\frac{q}{r}}}\right) \\ &= \pm\frac{\sqrt{qr}}{\alpha}\sqrt{\beta^2 + 2\alpha\sqrt{\frac{q}{r}}} . \end{aligned}$$

As $p_{11} > 0$ we must take the positive sign in the above expression. Which means that we know the complete expression for p_{12} given by Equation 126. Now to compute $K(\infty)$ we note that

$$\begin{aligned} K(\infty) &= P(\infty) H^T R^{-1} \\ &= \frac{1}{r}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \frac{\alpha}{r}\begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \\ &= \frac{\alpha}{r}\begin{bmatrix} \frac{\sqrt{qr}}{\alpha} \\ \frac{r}{\alpha^2}\left(-\beta + \sqrt{\beta^2 + 2\alpha\sqrt{\frac{q}{r}}}\right) \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{\frac{q}{r}} \\ \frac{\beta}{\alpha}\left(-1 + \sqrt{1 + 2\frac{\alpha}{\beta^2}\sqrt{\frac{q}{r}}}\right) \end{bmatrix} , \end{aligned}$$

as we were to show. In the Mathematical file `chap_4_prob_12.nb` we perform some of the algebra not displayed in the above derivation.

Problem 4-13 (the optimal filter for detecting a sine wave in white noise)

Warning: I was not able to solve this problem. If anyone has an attempted solution I would be interested in seeing it.

Problem 4-14 (an integrator driven by white noise)

As a continuous system from the problem description the output $x(t)$ of our integrator would satisfy

$$\dot{x} = w ,$$

where $w(t)$ is a white noise process. If we discretize this process we get the discrete system of

$$x_{k+1} = x_k + w_k \,,$$

where now we have that $w_k \sim N(0, q\Delta)$. We are told that the observation equation is given by

$$v_k = x_k + v_k \,.$$

With no a priori information measure we have $P_0(+) = +\infty$, and to compute the a posteriori covariance matrix after each measurement in this problem we will use

$$P_k(+)^{-1} = P_k(-)^{-1} + H_k^T R_k^{-1} H_k \,.$$

From the equations above we can make the association to the standard problem that $\Phi_k = I$, $G_k = I$, $Q_k = q\Delta$, $H_k = 1$, and $R_k = r_0$.

Part (a): In this case we told to assume that $q\Delta \gg r_0$. Now we have $P_0(+) = +\infty$, since there is no a priori information and we get $P_1(-)$ from

$$P_1(-) = P_0(+) + q\Delta = +\infty \,.$$

Then $P_1(+)$ after the first measurement is given by

$$P_1(+)^{-1} = P_1(-)^{-1} + \frac{1}{r_0} = \frac{1}{r_0} \quad \Rightarrow \quad P_1(+) = r_0 \,.$$

For the variance before the second measurement or $P_2(-)$ we get

$$P_2(-) = P_1(+) + q\Delta = r_0 + q\Delta \,.$$

For the updated variance after the second measurement $P_2(+)$ we get

$$P_2(+)^{-1} = P_2(-)^{-1} + \frac{1}{r_0} = \frac{1}{r_0 + q\Delta} + \frac{1}{r_0} \approx \frac{1}{r_0} \quad \Rightarrow \quad P_2(+) = r_0 \,,$$

since $q\Delta \gg r_0$. Now $P_3(-)$ is given by

$$P_3(-) = P_2(+) + q\Delta = r_0 + q\Delta \,,$$

and $P_3(+)$ is given by

$$P_3(+)^{-1} = P_3(-)^{-1} + \frac{1}{r_0} = \frac{1}{r_0 + q\Delta} + \frac{1}{r_0} \quad \Rightarrow \quad P_3(+) = r_0 \,.$$

Continuing the pattern above we conclude that

$$P_k(+) = r_0 \,,$$

and

$$P_{k+1}(-) = r_0 + q\Delta \approx q\Delta \,,$$

when $q\Delta \gg r_0$. This corresponds to the case where the object we are filtering has very large process noise, so that at each timestep when we propagate between measurements we effectively "loose" the object. The measurements are considerably more accurate so when we take a measurement we have a much tighter uncertainty around the tracked object.

Part (b): For this part we assume that $r_0 \gg q\Delta$ and follow the outline as in the previous part. Again we start with $P_0(+) = +\infty$, since there is no a priori information. Then we get $P_1(-)$ from

$$P_1(-) = P_0(+) + q\Delta = +\infty\,.$$

Then $P_1(+)$ is given by

$$P_1(+)^{-1} = P_1(-)^{-1} + \frac{1}{r_0} = \frac{1}{r_0} \quad \Rightarrow \quad P_1(+) = r_0\,.$$

Then for $P_2(-)$ we get

$$P_2(-) = P_1(+) + q\Delta = r_0 + q\Delta \approx r_0\,.$$

Then $P_2(+)$ is given by

$$P_2(+)^{-1} = P_2(-)^{-1} + \frac{1}{r_0} = \frac{1}{r_0 + q\Delta} + \frac{1}{r_0} = \frac{2r_0 + q\Delta}{r_0(r_0 + q\Delta)}\,,$$

so

$$P_2(+) = \frac{r_0(r_0 + q\Delta)}{2r_0 + q\Delta} = \frac{r_0\left(1 + \frac{q\Delta}{r_0}\right)}{2\left(1 + \frac{q\Delta}{2r_0}\right)} \approx \frac{r_0}{2}\,,$$

since $r_0 \gg q\Delta$. Now $P_3(-)$ is given by

$$P_3(-) = P_2(+) + q\Delta = \frac{r_0}{2} + q\Delta \approx \frac{r_0}{2}\,,$$

and $P_3(+)$ is given by

$$P_3(+)^{-1} = P_3(-)^{-1} + \frac{1}{r_0} = \frac{2}{r_0} + \frac{1}{r_0} = \frac{3}{r_0} \quad \Rightarrow \quad P_3(+) = \frac{r_0}{3}\,.$$

Doing one more iteration for completeness we find $P_4(-)$ given by

$$P_4(-) = P_3(+) + q\Delta = \frac{r_0}{3} + q\Delta \approx \frac{r_0}{3}\,,$$

and $P_4(+)$ given by

$$P_4(+)^{-1} = P_4(-)^{-1} + \frac{1}{r_0} = \frac{3}{r_0} + \frac{1}{r_0} = \frac{4}{r_0} \quad \Rightarrow \quad P_4(+) = \frac{r_0}{4}\,.$$

Continuing the pattern above we conclude that

$$P_k(+) = \frac{r_0}{k} = P_{k+1}(-)\,, \tag{127}$$

for $k > 0$ when $q\Delta \gg r_0$. This case corresponds to the situation where the dynamics has very little process noise so once we have "found" the object we are able to keep hold of it relatively easily. As the initial uncertainty is infinite each measurement reduces the uncertainty in a algebraic manner while propagation introduces no additional uncertainty see Equation 127.

Problem 4-15 (an expression for $P_a(T^+)$)

For this problem we are told to take as our state the vector $\mathbf{x} = \begin{bmatrix} \delta p(0) \\ \delta v(0) \\ \delta a(0) \end{bmatrix}$. This is *different* from the state vector specified in example 4.2-4 in that this state is a *constant* vector of initial conditions, while example 4.2-4 in the book used the time dependent state given by $\mathbf{x}(t) = \begin{bmatrix} \delta p(t) \\ \delta v(t) \\ \delta a(t) \end{bmatrix}$, where each function in the state is the appropriate integral of the one below it. The constant state for this problem then satisfies the null dynamics given by $\frac{d\mathbf{x}}{dt} = 0$, which has the fundamental solution $\Phi = I$. We assume that our initial uncertainty in these constants before the measurement at time T is given by

$$P(0) = \begin{bmatrix} p_{11}(0) & 0 & 0 \\ 0 & p_{22}(0) & 0 \\ 0 & 0 & p_{33}(0) \end{bmatrix} = \begin{bmatrix} E[\delta p^2(0)] & 0 & 0 \\ 0 & E[\delta v^2(0)] & 0 \\ 0 & 0 & E[\delta a^2(0)] \end{bmatrix}.$$

The discrete state and covariance extrapolation equations from the time 0 to T^- the time just before the first measurement fix gives

$$\hat{x}(T^-) = I\hat{x}(0) = \begin{bmatrix} \delta p(0) \\ \delta v(0) \\ \delta a(0) \end{bmatrix},$$

and $P(T^-) = P(0)$. Because our state $\mathbf{x}$ is independent of time the given measurement $z(t)$ requires that the measurement sensitivity matrix H now be a function of time because

$$z(t) = -\delta p(t) + e_p = -\begin{bmatrix} 1 & t & \frac{t^2}{2} \end{bmatrix} \begin{bmatrix} \delta p(0) \\ \delta v(0) \\ \delta a(0) \end{bmatrix} + e_p \,,$$

so the measurement sensitivity matrix is given by

$$H(t) = -\begin{bmatrix} 1 & t & \frac{t^2}{2} \end{bmatrix}.$$

With this definition of H we next compute some of the factors needed in computing the a posteriori state and covariance update equations. One expression we require is

$$H(T)P(T^-)H(T)^T = p_{11}(0) + T^2 p_{22}(0) + \frac{T^4}{4} p_{33}(0)\,.$$

From this point on to simplify the notation we will write $p_{11}(0)$ as p_{11} dropping the argument of zero (we follow the same convention for the other expressions). To evaluate $P(T^+)$ we could use Equation 66 with $R = \sigma_p^2$ or we can use the inverse update formulation given by Equation 61 which gives

$$\begin{aligned} P(T^+)^{-1} &= P(T^-)^{-1} + H(T)^T R(T)^{-1} H(T) \\ &= P(T^-)^{-1} + \begin{bmatrix} 1 \\ T \\ \frac{T^2}{2} \end{bmatrix} \left(\frac{1}{\sigma_p^2}\right) \begin{bmatrix} 1 & T & \frac{T^2}{2} \end{bmatrix}. \end{aligned}$$

The book suggest that we invert the right-hand-side of this using the **Sherman-Morrison-Woodbury** formula

$$(A+UV^T)^{-1} = A^{-1} - A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}\,. \tag{128}$$

Using this expression we can compute $P(T^+)$, with the associations $A = P(T^-)^{-1}$ and

$$U = V = \frac{1}{\sigma_p}\begin{bmatrix} 1 \\ T \\ \frac{T^2}{2} \end{bmatrix}.$$

The following algebra, required to derive the expression quoted in the text, is rather tedious and can be skipped if desired. First we evaluate the factor $I + V^TA^{-1}U$ and find

$$\begin{aligned}
I + V^TA^{-1}U &= I + V^TP(T^-)U \\
&= 1 + \frac{1}{\sigma_p^2}\begin{bmatrix} 1 & T & \frac{T^2}{2} \end{bmatrix}\begin{bmatrix} 1 \\ T \\ \frac{T^2}{2} \end{bmatrix} \\
&= 1 + \frac{1}{\sigma_p^2}\left(p_{11} + p_{22}T + p_{33}\frac{T^4}{4}\right).
\end{aligned}$$

Next the expression $\mathbf{M} = A^{-1}U(I+V^TA^{-1}U)^{-1}V^TA^{-1}$ is given by

$$\begin{aligned}
\mathbf{M} &= \frac{P(T^-)}{\sigma_p^2}\begin{bmatrix} 1 \\ T \\ \frac{T^2}{2} \end{bmatrix}\left(\frac{1}{1+\frac{1}{\sigma_p^2}\left(p_{11}+p_{22}T+p_{33}\frac{T^4}{4}\right)}\right)\begin{bmatrix} 1 & T & \frac{T^2}{2} \end{bmatrix}P(T^-) \\
&= \left(\frac{1}{\sigma_p^2 + p_{11} + p_{22}T + p_{33}\frac{T^4}{4}}\right)P(T^-)\begin{bmatrix} 1 & T & \frac{T^2}{2} \\ T & T^2 & \frac{T^3}{2} \\ \frac{T^2}{2} & \frac{T^3}{2} & \frac{T^4}{4} \end{bmatrix}P(T^-)\,.
\end{aligned}$$

Note that from the definition of $\Delta_a(T)$ given we can simplify the denominator above as

$$\sigma_p^2 + p_{11} + p_{22}T + p_{33}\frac{T^4}{4} = p_{11}p_{22}p_{33}\Delta_a(T)\,. \tag{129}$$

When we use this in $\mathbf{M}$ we get

$$\mathbf{M} = \frac{1}{p_{11}p_{22}p_{33}\Delta_a(T)}\begin{bmatrix} p_{11}^2 & p_{11}p_{22}T & p_{11}p_{33}\frac{T^2}{2} \\ p_{22}p_{11}T & p_{22}^2T^2 & p_{22}p_{33}\frac{T^3}{2} \\ p_{33}p_{11}\frac{T^2}{2} & p_{33}p_{22}\frac{T^3}{2} & p_{33}^2\frac{T^4}{4} \end{bmatrix}.$$

Then the expression for $P(T^+)$ then looks like

$$\begin{aligned}
P(T^+) &= P(T^-) - \mathbf{M} \\
&= \frac{1}{\Delta_a(T)}\begin{bmatrix} \Delta_a(T)p_{11} & 0 & 0 \\ 0 & \Delta_a(T)p_{22} & 0 \\ 0 & 0 & \Delta_a(T)p_{33} \end{bmatrix} \\
&- \frac{1}{\Delta_a(T)}\begin{bmatrix} \frac{p_{11}}{p_{22}p_{33}} & \frac{T}{p_{33}} & \frac{T^2}{2p_{22}} \\ \frac{T}{p_{33}} & \frac{p_{22}T^2}{p_{11}p_{33}} & \frac{T^3}{2p_{11}} \\ \frac{T^2}{2p_{22}} & \frac{T^3}{2p_{11}} & \frac{p_{33}T^4}{4p_{11}p_{22}} \end{bmatrix}.
\end{aligned}$$

So $\Delta_a(T)P(T^+)$ then looks like

$$\left[\begin{array}{ccc} \Delta_a(T)p_{11} - \frac{p_{11}}{p_{22}p_{33}} & -\frac{T}{p_{33}} & -\frac{T^2}{2p_{22}} \\ -\frac{T}{p_{33}} & \Delta_a(T)p_{22} - \frac{p_{22}T^2}{p_{11}p_{33}} & -\frac{T^3}{2p_{11}} \\ -\frac{T^2}{2p_{22}} & -\frac{T^3}{2p_{11}} & \Delta_a(T)p_{33} - \frac{p_{33}T^4}{4p_{11}p_{22}} \end{array}\right].$$

We have one more simplification (that we don't fully document) and we have shown the requested result. If we take each of the diagonal elements in the expression for $P(T^+)$ and simplify using the definition of $\Delta_a(T)$ given in Equation 129. For example the $(1,1)$ element becomes

$$\frac{1}{p_{22}p_{33}}\left(\sigma_p^2 + p_{11} + p_{22}T + p_{33}\frac{T^4}{4}\right) - \frac{p_{11}}{p_{22}p_{33}} = \frac{\sigma_p^2}{p_{22}p_{33}} + \frac{T^2}{p_{33}} + \frac{T^4}{4p_{22}},$$

which is the quoted expression in the book. Simplifying the other diagonal terms gives rise to the desired expression for $P(T^+)$.

Problem 4-16 (single-star vs. two-star fixes)

The single-star fix: We are told that our first measurement gives us an estimate of θ_1 and θ_2. Lets assume (for this part and the next) that there is no dynamics in this problem and we just want to observe how the single star and double star fixes change our state uncertainty estimates. For the single star fix the measurement vector $\mathbf{z}$ is related to the state by

$$\mathbf{z} = \left[\begin{array}{c} z_1 \\ z_2 \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]\left[\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array}\right] + \left[\begin{array}{c} v_1 \\ v_2 \end{array}\right],$$

with the measurement noise vector $\left[\begin{array}{c} v_1 \\ v_2 \end{array}\right] \sim N\left(0, \left[\begin{array}{cc} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{array}\right]\right)$. Then we update the a priori covariance to account for this measurement using the standard a posteriori update equation

$$P(+) = P(-) - P(-)H^T(HP(-)H^T + R)^{-1}HP(-)\,. \tag{130}$$

To evaluate this we find that the product $HP(-)$ is given by

$$HP(-) = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]\left[\begin{array}{ccc} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{array}\right] = \left[\begin{array}{ccc} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & 0 \end{array}\right].$$

The matrix $P(-)H^T$ is the transpose of this or $\left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & \sigma^2 \\ 0 & 0 \end{array}\right]$. Next we compute $HP(-)H^T$ and find

$$HP(-)H^T = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]\left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & \sigma^2 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} \sigma^2 & 0 \\ 0 & \sigma^2 \end{array}\right].$$

With this we have

$$(HP(-)H^T + R)^{-1} = \begin{bmatrix} \sigma^2 + \sigma_1^2 & 0 \\ 0 & \sigma^2 + \sigma_1^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\sigma^2+\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma^2+\sigma_2^2} \end{bmatrix},$$

and $P(+)$ is then given by

$$P(-) - P(-)H^T(HP(-)H^T + R)^{-1}HP(-) = \begin{bmatrix} \sigma^2 - \frac{\sigma^4}{\sigma^2+\sigma_1^2} & 0 & 0 \\ 0 & \sigma^2 - \frac{\sigma^4}{\sigma^2+\sigma_2^2} & 0 \\ 0 & 0 & \sigma^2 \end{bmatrix}. \tag{131}$$

From this we see that the trace of this expression is

$$\begin{aligned} \text{trace}(P(+)) &= 3\sigma^2 - \frac{\sigma^4}{\sigma^2 + \sigma_1^2} - \frac{\sigma^4}{\sigma^2 + \sigma_2^2} \\ &\approx 3\sigma^2 - \sigma^2 - \sigma^2 = \sigma^2 , \end{aligned}$$

when we assume that $\sigma^2 \gg \sigma_i^2$.

The two-star fix: For the two-star fix we follow the one-star fix with *another* pair of measurements of the angles θ_1 and θ_3. In this case the second measurement vector has the form

$$\mathbf{z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_3 \end{bmatrix},$$

with

$$\begin{bmatrix} v_1 \\ v_3 \end{bmatrix} \sim N\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix}\right).$$

Thus in this case we have that $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_3^2 \end{bmatrix}$. Performing the same manipulations as above but with these different H and R matrices and using the value computed for $P(+)$ in Equation 131 for the value of $P(-)$ in Equation 130 (the second measurement directly follows the first) we find that $P(+)$ after *both* measurements is given by

$$P(+) = \begin{bmatrix} \frac{\sigma^2\sigma_1^2}{2\sigma^2+\sigma_1^2} & 0 & 0 \\ 0 & \frac{\sigma^2\sigma_2^2}{\sigma^2+\sigma_2^2} & 0 \\ 0 & 0 & \frac{\sigma^2\sigma_3^2}{\sigma^2+\sigma_3^2} \end{bmatrix} \approx \begin{bmatrix} \frac{\sigma_1^2}{2} & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix},$$

when use $\sigma^2 \gg \sigma_i^2$ to simplify terms like

$$\frac{\sigma^2\sigma_i^2}{n\sigma^2 + \sigma_i^2} \approx \frac{\sigma^2\sigma_i^2}{n\sigma^2} = \frac{\sigma_i^2}{n}.$$

From the above we find trace$(P(+))$ to be given by

$$\text{trace}(P(+)) = \frac{\sigma_1^2}{2} + \sigma_2^2 + \sigma_3^2 ,$$

as we were to show.

In the Mathematical file `chap_4_prob_16.nb` we perform some of the algebra not displayed in the above derivation.

Problem 4-17 (a polynomial tracking filter)

The zero forcing dynamic equation $\ddot{x} = 0$ when we introduce the state $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ defined by $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$ has components that satisfy

$$\begin{aligned} \dot{x}_1(t) &= \dot{x}(t) = x_2(t) \\ \dot{x}_2(t) &= \ddot{x}(t) = 0 \,. \end{aligned}$$

so that our equation $\ddot{x} = 0$ has the following companion form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

The measurements for this problem are given by $z = x_1 + v = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$, so the matrices H and R are given by $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $R = r$. The fundamental solution, Φ, to the above companion form representation can be computed as

$$\Phi(t, 0) = e^{Ft} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} .$$

To derive the requested expression for $P_{k+1}(+)$ we sequentially perform error covariance extrapolation followed by error covariance updates until we get to the discrete time $t_{k+1} = (k+1)\tau$. The error covariance extrapolation equation is explicitly given by

$$P_{k+1}(-) = \Phi(\tau, 0) P_k(+) \Phi(\tau, 0)^T \,, \tag{132}$$

and is subsequently followed by an error covariance update step which can be written as

$$\begin{aligned} P_{k+1}(+)^{-1} &= P_{k+1}(-)^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1} \\ &= P_{k+1}(-)^{-1} + \frac{1}{r} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \end{aligned} \tag{133}$$

Once we have computed the matrix $P_{k+1}(+)$ we can compute K_{k+1} via Equation 62 which in this case becomes

$$K_{k+1} = P_{k+1}(+) H_{k+1}^T R_{k+1}^{-1} = \frac{1}{r} P_{k+1}(+) \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{134}$$

While we have not derived the quoted expression for $P_{k+1}(+)$ if we assume that it is correct and compute K_{k+1} with the above formula we get

$$\begin{aligned} K_{k+1} &= \frac{1}{r} P_{k+1}(+) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{r} \frac{2r}{(k+1)(k+2)} \begin{bmatrix} 2k+1 \\ \frac{3}{\tau} \end{bmatrix} \\ &= \frac{2}{(k+1)(k+2)} \begin{bmatrix} 2k+1 \\ \frac{3}{\tau} \end{bmatrix} , \end{aligned}$$

which is the expression given. Thus to finish this problem it remains to derive the expression for $P_{k+1}(+)$. From Equations 132 and 133 we can combine these two expressions into one to get

$$\begin{aligned} P_{k+1}(+)^{-1} &= (\Phi(\tau,0)P_k(+)\Phi(\tau,0)^T)^{-1} + \frac{1}{r}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} P_k(+) \begin{bmatrix} 1 & 0 \\ \tau & 1 \end{bmatrix}\right)^{-1} + \frac{1}{r}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (135)$$

Following the hint in the book if we begin these iterations with $P_0(+) = \frac{1}{\epsilon}I$ we find that

$$P_1(+) = \frac{1}{1+\epsilon r+\tau^2}\begin{bmatrix} r(1+\tau^2) & r\tau \\ r\tau & r+\frac{1}{\epsilon} \end{bmatrix}.$$

We cannot take the limit of this as $\epsilon \to 0$ so we iterate Equation 135 another time to get an expression for $P_2(+)$. When we do this we find that we *can* set $\epsilon = 0$ and get a well defined expression. The resulting expression is

$$P_2(+) = \begin{bmatrix} r & \frac{r}{\tau} \\ \frac{r}{\tau} & \frac{2r}{\tau^2} \end{bmatrix}.$$

Iterating Equation 135 a third time with on the above matrix gives

$$P_3(+) = \begin{bmatrix} \frac{5r}{6} & \frac{r}{2\tau} \\ \frac{r}{2\tau} & \frac{2r}{\tau^2} \end{bmatrix}.$$

Both of these expressions agree with the stated result for $P_{k+1}(+)$ when we take $k = 1$ and $k = 2$. If we hypothesis that

$$P_{k+1}(+) = \frac{2r}{(k+1)(k+2)}\begin{bmatrix} 2k+1 & \frac{3}{\tau} \\ \frac{3}{\tau} & \frac{6}{k\tau^2} \end{bmatrix},$$

we can then use Equation 135 to show by induction that the above expression for $P_{k+1}(+)$ is valid for all k.

Note that in the Mathematical file `chap_4_prob_17.nb` we perform some of the algebra not displayed in the above derivation.

Problem 4-18 (the optimal differentiator)

If we define $y(t)$ by $y(t) = M(t)x(t)$ then $y(t)$ satisfies the system

$$\begin{aligned} \dot{y} &= \dot{M}x + M\dot{x} \\ &= \dot{M}x + M(Fx + Gw) \\ &= (\dot{M} + MF)x + MGw\,. \end{aligned}$$

As this is a linear transformation of $x(t)$ which is itself a Gaussian random process $y(t)$ will also be a Gaussian random process and the estimate of its *mean* will be the optimal a posteriori estimate. Since $E[w] = 0$ we have that the mean of $y(t)$ has dynamics given by

$$\dot{\hat{y}} = E[\dot{y}] = (\dot{M} + MF)E[x] = (\dot{M} + MF)\hat{x}\,,$$

the claimed equation.

Problem 4-19 (the determinant of the posteriori covariance matrix $P_k(+)$)

The discrete covariance matrix update equation is given by

$$P_k(+) = (I - K_k H_k) P_k(-)\,, \tag{136}$$

where K_k is the Kalman gain given by $K_k = P_k(-)H_k^T(H_k P_k(-) H_k^T + R_k)^{-1}$. To derive the requested determinant first consider the following manipulations of the product $H_k K_k$. We have

$$\begin{aligned} H_k K_k &= H_k P_k(-) H_k^T (H_k P_k(-) H_k^T + R_k)^{-1} \\ &= (H_k P_k(-) H_k^T + R_k - R_k)(H_k P_k(-) H_k^T + R_k)^{-1} \\ &= I - R_k (H_k P_k(-) H_k^T + R_k)^{-1}\,. \end{aligned}$$

Thus if we multiply Equation 136 on the left by H_k we get

$$H_k P_k(+) = H_k P_k(-) - H_k K_k H_k P_k(-)\,.$$

When we put in the expression just derived for $H_k K_k$ into the above we get

$$\begin{aligned} H_k P_k(+) &= H_k P_k(-) - (I - R_k (H_k P_k(-) H_k^T + R_k)^{-1}) H_k P_k(-) \\ &= R_k (H_k P_k(-) H_k^T + R_k)^{-1} H_k P_k(-)\,, \end{aligned}$$

the initial expression requested. Taking the determinant of both sides of this then gives

$$|H_k||P_k(+)| = |R_k||H_k P_k(-) H_k^T + R_k|^{-1} |H_k||P_k(-)|\,.$$

We can divide both sides of this equation by $|H_k|$ since H_k is invertible to get

$$|P_k(+)| = \frac{|R_k||P_k(-)|}{|H_k P_k(-) H_k^T + R_k|}\,,$$

the expression we desired.

Problem 4-20 (filtering with a uniform distribution)

Lets look for an optimal linear estimator of the following form for processing the kth measurement z_k

$$\hat{x}_k(+) = k'_k \hat{x}_k(-) + k_k z_k\,.$$

Introducing the a priori and a posteriori estimation errors $\tilde{x}_k(\pm) = \hat{x}_k(\pm) - x_k$, and the measurement equation $z_k = x_k + v_k$ in the above equation we have an recursive update of $\tilde{x}_k(+)$ given by

$$\tilde{x}_k(+) = [k'_k + k_k - 1] x_k + k'_k \tilde{x}(-) + k_k v_k\,.$$

To be an unbiased requires that since $E[v_k] = 0$ that $k'_k = 1 - k_k$ and we have an estimator of

$$\hat{x}_k(+) = (1 - k_k)\hat{x}_k(-) + k_k z_k\,.$$

To determine the value of k_k consider

$$\begin{aligned}
p_k(+) &= E\{\tilde{x}_k(+)\tilde{x}_k(+)^T\} \\
&= E\{(1-k_k)\tilde{x}_k(\tilde{x}_k(1-k_k)+k_k v_k)+k_k v_k(\tilde{x}_k(-)(1-k_k)+k_k v_k)\} \\
&= (1-k_k)^2 E\{\tilde{x}_k(-)^2\} + 2(1-k_k)k_k E\{\tilde{x}_k(-)v_k\} + k_k^2 E\{v_k^2\} \\
&= (1-k_k)^2 p_k(-) + \frac{q^2}{12}k_k^2 .
\end{aligned}$$

Where we have used

$$\begin{aligned}
E[v_k^2] &= \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} x^2 dx = \frac{2}{q}\int_0^{\frac{q}{2}} x^2 dx \\
&= \frac{2}{q}\left.\frac{x^3}{3}\right|_0^{q/2} = \frac{q^2}{12} .
\end{aligned}$$

To find the value of k_k that makes $p_k(+)$ a minimum we take the derivative and set the results equal to zero and solve for k_k. We find for the derivative

$$2(1-k_k)(-1)p_k(-) + \frac{q^2}{6}k_k = 0 .$$

or

$$k_k = \frac{p_k(-)}{p_k(-)+\frac{q^2}{12}} , \tag{137}$$

so

$$1 - k_k = \frac{\frac{q^2}{12}}{p_k(-)+\frac{q^2}{12}} .$$

With this value of k_k the covariance $p_k(+)$ becomes

$$\begin{aligned}
p_k(+) &= \frac{(q^2/12)^2}{(p_k(-)+q^2/12)} p_k(-) + \frac{(q^2/12)^2 p_k(-)^2}{(p_k(-)+q^2/12)^2} \\
&= \frac{(q^2/12)^2 p_k(-)^2}{(p_k(-)+q^2/12)} .
\end{aligned}$$

Since we are estimating a constant with no dynamics we have that $\hat{x}_k(-) = \hat{x}_{k-1}(+)$ and $p_k(-) = p_{k-1}(+)$. In summary then the recursive form of our estimator for the unknown constant starts with

$$\hat{x}_0(+) = m \quad \text{with} \quad p_0(+) = \sigma^2 ,$$

and then iterates for each measurement z_k for $k \geq 1$ the following

$$\begin{aligned}
\hat{x}_k(-) &= \hat{x}_{k-1}(+) \quad \text{and} \quad p_k(-) = p_{k-1}(-) \\
\hat{x}_k(+) &= (1-k_k)\hat{x}_{k-1}(-) + k_k z_k \\
&= \frac{(q^2/12)}{p_{k-1}(-)+(q^2/12)}\hat{x}_{k-1} + \left(\frac{p_{k-1}(-)}{p_{k-1}(-)+q^2/12}\right) z_k \\
p_k(+) &= \frac{(q^2/12)^2 p_k(-)^2}{(p_k(-)+q^2/12)} ,
\end{aligned}$$

It seems that we only needed an expression for $E[v^2]$ but the explicit form of the distribution did not seem to matter.

Problem 4-21 (filtering with multiplicative noise)

Our estimator for this problem will be constructed as $\hat{x} = kz$ for some as of yet unspecified value for the multiplier k. The error using this estimator is computed as

$$\begin{aligned}
\tilde{x} &= \hat{x} - x \\
&= kz - x \\
&= k(1+\eta)x - x \\
&= (k(1+\eta) - 1)x \,.
\end{aligned} \tag{138}$$

For $\hat{x}$ to be an unbiased estimate of x means that $E[\tilde{x}] = 0$. From Equation 138 we see that this requires

$$E[\tilde{x}] = kE[x] + kE[\eta x] - E[x] = 0 \,,$$

since all three expectations are zero. Thus the estimator as defined is unbiased. Next we will pick the value of k so that the variance in the error is as small as possible. The variance in the error is

$$\begin{aligned}
E[\tilde{x}^2] &= E[(\eta k + k - 1)^2 x^2] \\
&= E[(\eta^2 k^2 + 2\eta k(k-1) + (k-1)^2)x^2] \\
&= k^2 E[\eta^2]E[x^2] + 0 + (k-1)^2 E[x^2] \\
&= k^2 \sigma_\eta^2 \sigma_x^2 + (k-1)^2 \sigma_x^2 \,.
\end{aligned}$$

In the above I have assumed that $E[\eta^2 x^2] = E[\eta^2]E[x^2]$, which would be true if x and η are independent random variables. Then we want to minimize the expression $E[\tilde{x}^2]$ when viewed as a function of k. When we take the derivative, of this expression, set the result equal to zero and solve for k we find

$$k = \frac{1}{1+\sigma_\eta^2} \,.$$

We can check that the value above is indeed a minimum by taking the second derivative

$$\frac{d^2 E[\tilde{x}]}{dk^2} = 2\sigma_\eta^2 \sigma_x^2 + 2\sigma_x^2 > 0 \,.$$

Now since

$$k - 1 = \frac{1}{1+\sigma_\eta^2} - 1 = \frac{-\sigma_\eta^2}{1+\sigma_\eta^2} \,,$$

the minimum variance $E[\tilde{x}^2]$ is given by

$$E[\tilde{x}^2] = \frac{\sigma_\eta^2 \sigma_x^2}{(1+\sigma_\eta^2)^2} + \frac{\sigma_\eta^4 \sigma_x^2}{(1+\sigma_\eta^2)^2} = \frac{\sigma_\eta^2 \sigma_x^2}{1+\sigma_\eta^2} \,.$$

Problem 4-22 (filtering with spectral densities)

Warning: I'm not sure exactly what this problem was asking or how to answer it. If anyone has an idea of the type of solution requested please contact me.

Problem 4-23 (filtering a constant angular rate)

If we define the state variables x_1 and x_2 for this problem to be $x_1 = \theta$ and $x_2 = \dot{\theta}$ then as a differential system we have

$$\frac{d}{dt}\mathbf{x} = \left[\begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right] = \left[\begin{array}{c} x_2 \\ 0 \end{array} \right] = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right] .$$

Then using the power series definition for the fundamental solution we have

$$\Phi(t+T,t) = e^{FT} = I + FT + \frac{1}{2}F^2T^2 + \cdots .$$

For the F given above $F^2 = 0$ and so the above sum explicitly stops after two terms. Evaluating this two term sum we find that $\Phi(t+T,t)$ given by

$$\Phi(t+T,t) = \left[\begin{array}{cc} 1 & T \\ 0 & 1 \end{array} \right] .$$

Also since $z_k = \theta_k + v_k = x_1(kT) + v_k$ the measurement sensitivity matrix H is independent of time and given by $H = \left[\begin{array}{cc} 1 & 0 \end{array} \right]$, and $R = 5^2$. We are told to take initial state estimate and uncertainty for this problem given by

$$\hat{x}_0(+) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \quad \text{and} \quad P_0(+) = \left[\begin{array}{cc} 20^2 & 0 \\ 0 & 20^2 \end{array} \right] = 20^2 I .$$

The filtering equations that will produce the optimal estimates of position and velocity are given by the Kalman equations. We will do the first of these updates "by hand" and then one could write a simple program to generate the rest. We first need to propagate the initial state and uncertainty to the first measurement time

$$\begin{aligned}
\hat{x}_1(-) &= \Phi_0 \hat{x}_0(+) = \left[\begin{array}{cc} 1 & T \\ 0 & 1 \end{array} \right] \left[\begin{array}{c} 0 \\ 0 \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \\
P_1(-) &= \Phi_0 P_0(+) \Phi_0^T = \left[\begin{array}{cc} 1 & T \\ 0 & 1 \end{array} \right] 20^2 \, I \left[\begin{array}{cc} 1 & 0 \\ T & 1 \end{array} \right] = 20^2 \left[\begin{array}{cc} 1+T^2 & T \\ T & 1 \end{array} \right] .
\end{aligned}$$

Next we observe the first measurement z_1 and update the state and covariance matrix with with Equations 51, 58, and 59. We begin with Equation 58 or

$$K_1 = P_1(-) H_1^T [H_1 P_1(-) H_1^T + R_1]^{-1} .$$

Now to compute this we need to add

$$H_1 P_1(-) H_1^T = 20^2 \left[\begin{array}{cc} 1 & 0 \end{array} \right] \left[\begin{array}{cc} 1+T^2 & T \\ T & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = 20^2(1+T^2) .$$

to $R_1 = 5^2$, giving $H_1 P_1(-) H_1^T + R_1 = 20^2(1+T^2) + 5^2$. Next we compute

$$P_1(-) H_1^T = \left[\begin{array}{cc} 1+T^2 & T \\ T & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 0 \end{array} \right] = \left[\begin{array}{c} 1+T^2 \\ T \end{array} \right] ,$$

so K_1 is explicitly given by

$$K_1 = \frac{1}{(20^2(1+T^2)+5^2)} \begin{bmatrix} 1+T^2 \\ T \end{bmatrix} .$$

Then the application of Equation 51 and 59 then give

$$\begin{aligned}
\hat{x}_1(+) &= \hat{x}_1(-) + K_1(z_1 - H\hat{x}_1(-)) = K_1 z_1 \\
&= \frac{1}{(20^2(1+T^2)+5^2)} \begin{bmatrix} 1+T^2 \\ T \end{bmatrix} z_1 \\
P_1(+) &= (I - K_1 H_1) P_1(-) .
\end{aligned}$$

Since Φ and H do not depend on the index k the steps in this process are summarized as follows. Given an initial starting values of $\hat{x}(+)$ and $P(+)$ as each measurement z comes in compute

$$\begin{aligned}
\hat{x}(-) &= \Phi \hat{x}(+) \\
P(-) &= \Phi P(+) \Phi^T \\
K &= P(-) H^T (H P(-) H^T + R)^{-1} \\
\hat{x}(+) &= \hat{x}(-) + K(z - H\hat{x}(-)) \\
P(+) &= (I - KH) P(-) .
\end{aligned}$$

Problem 4-24 (Kalman filtering with discrete measurement noise)

For this problem we are told that $E[x_0] = 1$ and $E[x_0^2] = 2$. From this we can conclude that the variance of the initial state x_0 is given by

$$p_0(+) = \mathrm{Var}(x_0) = E[x_0^2] - E[x_0]^2 = 2 - 1^2 = 1 .$$

Our dynamic system model for this problem is

$$x_{k+1} = e^{-T/\tau} x_k + w_k \quad \text{for} \quad k \geq 0 ,$$

where since $T = \tau$ the value of the exponential is above is actually e^{-1}. Our fundamental solution matrix is then $\Phi_k = e^{-1}$ with a process noise variance of $q_k = 2$. With measurements of this process given by

$$z_k = x_k + v_k ,$$

we have $h_k = 1$. To derive statistics of the measurement noise process v_k recall that the density of the measurement noise v_k is discrete and specifically given by

$$P(v_k = -2) = P(v_k = +2) = \frac{1}{2} ,$$

so that $E[v_k] = 0$. The variance of noise distributed like this is given by

$$r_k = E[v_k^2] = \frac{1}{2} 4 + \frac{1}{2} 4 = 4 ,$$

With all of the above information we can apply the Kalman filtering framework to this problem.

Part (a-b): With initial conditions for this problem are given by $\hat{x}_0(+) = 1$ with $p_0(+) = 1$, our estimate for $\hat{x}_1(-)$ and $p_1(-)$ is given by

$$\hat{x}_1(-) = \Phi_0 \hat{x}_0(+) = e^{-1} ,$$

and

$$p_1(-) = \Phi_0 p_0(+) \Phi_0^T + Q_0 = e^{-2} + 2 .$$

Then we observe the measurement z_1, which we can incorporate using the Kalman measurement update Equations 51, 58, and 59. Rather than document these in detail again, please see the `python` file `chap_4_prob_24.py` for some numerical code where we do these calculations for the two measurements z_1 and z_2. When we implement these equations and execute the above script we find

$$\begin{aligned} \hat{x}_1(+) &= 0.7619 \quad p_1(+) = 1.3921 \quad \text{and} \\ \hat{x}_1(+) &= 1.2420 \quad p_1(+) = 1.4145 . \end{aligned}$$

Problem 4-25 (Kalman filtering the motion of a one-dimensional ship)

Warning: I was not sure about this problem. If anyone has any ideas please contact me.

Problem 4-26 (an airplane autopilot)

Warning: I was not sure how to deal with the derivative of the expression $h_C(t)$ in the noise term on the right-hand-side of the differential equation for $h(t)$. If anyone has any ideas please contact me.

Problem 4-27 (measuring the voltage in the black box)

Denote by $i_1(t)$ and $i_2(t)$ the currents in the left most and right most cell in Figure 4-4 respectively. We assume that the currents are running in a clockwise direction. Then Kirchhoff's voltage law (KVL) [5] around the left most cell gives

$$u(t) - R_1 i_1 - v_1 = 0 , \tag{139}$$

while Kirchhoff's voltage law around the right most cell gives

$$v_1 - R_2 i_2 - v_2 = 0 , \tag{140}$$

where v_i is the voltage of the capacitor C_i. Also the current flowing from top down through the capacitor C_1 gives rise to a change in voltage as

$$i_1 - i_2 = C_1 \frac{dv_1}{dt} . \tag{141}$$

The same consideration for the current flowing from top down through the capacitor C_2 gives $i_2 = C_2 \frac{dv_2}{dt}$ so that with this we can write i_1 in terms of v_i. From Equation 141 we have

$$i_1 = i_2 + C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} + C_1 \frac{dv_1}{dt} \,.$$

With these expressions for i_1 and i_2, using Equations 139 and 140 our system differential equation in terms of the variables v_1 and v_2 is

$$\begin{aligned} u(t) - R_1 \left(C_1 \frac{dv_1}{dt} + C_2 \frac{dv_2}{dt} \right) - v_1 &= 0 && (142) \\ v_1 - R_2 C_2 \frac{dv_2}{dt} - v_2 &= 0 \,. && (143) \end{aligned}$$

Solving this second equation for $\frac{dv_2}{dt}$ gives

$$\frac{dv_2}{dt} = \frac{1}{R_2 C_2}(v_1 - v_2) \,.$$

When we put that expression into Equation 142 and solving for $\frac{dv_1}{dt}$ we find

$$\frac{dv_1}{dt} = -\frac{1}{C_1}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] v_1 + \frac{1}{R_2 C_1} v_2 + \frac{1}{R_1 C_1} u(t) \,.$$

When we view these two equations as a matrix system with a state $\mathbf{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ we find

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left[\frac{1}{R_1} + \frac{1}{R_2}\right] & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & -\frac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{1}{R_1 C_1} \begin{bmatrix} u(t) \\ 0 \end{bmatrix} \,.$$

If we next simplify the system above to the case where $R_1 = R_2 = 1$ and $C_1 = C_2 = 1$ the above system becomes

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u(t) \\ 0 \end{bmatrix} \,.$$

Thus for this problem we see that our system matrix $F = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$. We are told that the measurement for this system is of $v_2(t)$ and is exact or

$$z(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \,.$$

Since numerically having no measurement noise can be harder to we will simulate this by taking R to be a very small number say 10^{-6}.

This problem, as specified, is continuous but we want to compute our estimates the discrete times so we will discretize it and apply the discrete Kalman filtering equations. To do that we need the discrete transition matrix Φ_k given by

$$\Phi_k = \Phi((k+1)\Delta t, k\Delta t) = e^{F\Delta t} \approx I + F\Delta t + \frac{1}{2} F^2 \Delta t^2 \,.$$

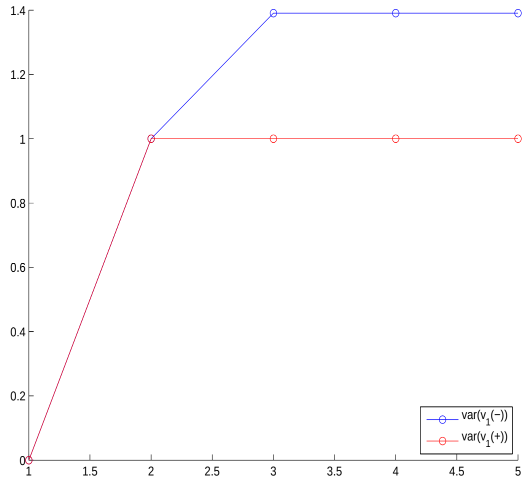

Figure 3: Plots of the a priori (in blue) and a posteriori (in red) covariance for the voltage across the capacitor C_1 as a function of the index in the discrete Kalman filtering algorithm. The "index" 1 corresponds to the time 0.

The discrete process noise Q_k is given by

$$Q_k = \Delta t Q = \left[\begin{array}{cc} 2 & 0 \\ 0 & 0 \end{array} \right] \Delta t \,,$$

since $u \sim N(0, 2)$. Then the optimal estimate of the voltage across C_1 is given by the discrete Kalman filter. For this problem statement we have $\Delta t = 0.5$ seconds, and to reach the time $T = 2$ seconds we need four iterations. We will take the initial conditions for this system as

$$\hat{\mathbf{x}}_0(+) = \left[\begin{array}{c} 0 \\ 0 \end{array} \right] \quad \text{and} \quad P_0(+) = 0 \,,$$

since we assume that the initial conditions are known exactly. Then to finish this problem we need to iterate the discrete Kalman filtering covariance equations

$$\begin{aligned} P_k(-) &= \Phi_{k-1} P_{k-1}(+) \Phi_{k-1}^T + Q_{k-1} \\ K_k &= P_k(-) H_k^T [H_k P_k(-) H_k^T + R_k]^{-1} = P_k(-) H_k^T [H_k P_K(-) H_k^T]^{-1} \\ P_k(+) &= [I - K_k H_k] P_k(-) \,, \end{aligned}$$

and then plot the $(1,1)$th element of the matrices $P_k(\pm)$ after each iteration. In the MATLAB/Octave file `chap_4_prob_27.m` we perform the Kalman filtering iterations needed to produce the plot above. We see that the value of the variance of v_1 after the first measurement goes to 1 and stays there for all further iterations.

Problem 4-28 (Kalman filtering the inverse square law)

To begin, first consider the given equations under the conditions that $u_1 = u_2 = 0$, which are given by

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \frac{G_0}{r^2} \\ \ddot{\theta} &= -2\dot{\theta}\frac{\dot{r}}{r} \,. \end{aligned}$$

Then if $r = R$ is a constant we see that $\dot{r} = \ddot{r} = 0$ and the above becomes

$$\begin{aligned} 0 &= R\dot{\theta}^2 - \frac{G_0}{R^2} \\ \ddot{\theta} &= 0 \,. \end{aligned}$$

The first equation above gives $\dot{\theta}^2 = \frac{G_0}{R^3}$ or

$$\dot{\theta} = \frac{\sqrt{G_0}}{R^{3/2}} \,,$$

so that as a function of t when we integrate we find

$$\theta(t) = \frac{\sqrt{G_0}}{R^{3/2}} t + \theta_0 \,,$$

where θ_0 is an arbitrary constant. Note that this solution also satisfies $\ddot{\theta} = 0$. To get the circular orbit solution quoted in the book we take $\theta_0 = 0$ and then $\theta(t) = \omega t$ with ω given by

$$\omega = \frac{\sqrt{G_0}}{R^{3/2}} \,,$$

or equivalently $R^3 \omega^2 = G_0$.

We are told to introduce state variables x_1, x_2, x_3, and x_4 to be given by

$$\begin{aligned} x_1 &= r - R & (144) \\ x_2 &= \dot{r} & (145) \\ x_3 &= R(\theta - \omega t) & (146) \\ x_4 &= R(\dot{\theta} - \omega) \,. & (147) \end{aligned}$$

Note that with the above definitions of x_i when we evaluate the state vector $\mathbf{x}$ at the equilibrium solution $r(t) = R$ and $\theta(t) = \omega t$ we have $x_i = 0$ for $i = 1, 2, 3, 4$. Our next step will to derive the differential equations satisfied by the variables x_1, x_2, x_3, and x_4. To begin note that

$$\dot{x}_1 = \dot{r} = x_2 \,. \qquad (148)$$

Then

$$\begin{aligned} \dot{x}_2 &= \ddot{r} = r\dot{\theta}^2 - \frac{G_0}{r^2} + u_1 \\ &= (x_1 + R)\left[\frac{x_4}{R} + \omega\right]^2 - \frac{G_0}{(x_1 + R)^2} + u_1 \,. \end{aligned}$$

Next

$$\begin{aligned}\dot{x}_3 &= R(\dot{\theta} - \omega) = R\left(\frac{x_4}{R} + \omega - \omega\right) \\ &= x_4 \,.\end{aligned}$$

Finally

$$\begin{aligned}\dot{x}_4 &= R\ddot{\theta} = R\left(-2\dot{\theta}\frac{\dot{r}}{r} + \frac{u_2}{r}\right) \\ &= -2R\left[\frac{x_4}{R} + \omega\right]\left(\frac{x_2}{x_1 + R}\right) + \frac{Ru_2}{x_1 + R} \,.\end{aligned}$$

Each of the expressions above shows how the first derivative of x_i can be expressed purely in terms of the function values x_i. Thus as a matrix system we have

$$\frac{d}{dt}\left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] = \left[\begin{array}{c} x_2 \\ (x_1 + R)\left[\frac{x_4}{R} + \omega\right]^2 - \frac{G_0}{(x_1+R)^2} + u_1 \\ x_4 \\ -2R\left[\frac{x_4}{R} + \omega\right]\left(\frac{x_2}{x_1+R}\right) + \frac{Ru_2}{x_1+R} \end{array}\right] \,. \tag{149}$$

We will take this nonlinear system and split it into two parts to write it as

$$\left[\begin{array}{c} x_2 \\ (x_1 + R)\left[\frac{x_4}{R} + \omega\right]^2 - \frac{G_0}{(x_1+R)^2} + u_1 \\ x_4 \\ -2R\left[\frac{x_4}{R} + \omega\right]\left(\frac{x_2}{x_1+R}\right) + \frac{Ru_2}{x_1+R} \end{array}\right] = \left[\begin{array}{c} x_2 \\ (x_1 + R)\left[\frac{x_4}{R} + \omega\right]^2 - \frac{G_0}{(x_1+R)^2} \\ x_4 \\ -2R\left[\frac{x_4}{R} + \omega\right]\left(\frac{x_2}{x_1+R}\right) \end{array}\right] + \left[\begin{array}{c} 0 \\ u_1 \\ 0 \\ \frac{Ru_2}{x_1+R} \end{array}\right] \,. \tag{150}$$

This writes the right-hand-side as the sum of two vectors each that are nonlinear in the state $\mathbf{x}$ and the components of the noise vector $\mathbf{u} = \left[\begin{array}{c} u_1 \\ u_2 \end{array}\right]$. If we denote the first vector as $\mathbf{f}(\mathbf{x})$ (since it does not depend on the noise vector $\mathbf{u}$) then we will linearized it about the state $\mathbf{x}_0$. We do this as

$$\left[\begin{array}{c} x_2 \\ (x_1 + R)\left[\frac{x_4}{R} + \omega\right]^2 - \frac{G_0}{(x_1+R)^2} \\ x_4 \\ -2R\left[\frac{x_4}{R} + \omega\right]\left(\frac{x_2}{x_1+R}\right) \end{array}\right] \approx \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\right] \tag{151}$$

The point $\mathbf{x}_0$ is the equilibrium point for circular orbits and corresponds to $\mathbf{x}_0 = 0$. Using the fact that $\omega^2 = \frac{G_0}{R^3}$ we have that $f(\mathbf{x}_0) = 0$. To complete this derivation recall the definition of $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ which is given by

$$\begin{aligned}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} &= \left[\begin{array}{cccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{array}\right] \\ &= \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ \left[\frac{x_4}{R} + \omega\right]^2 + \frac{2G_0}{(x_1+R)^3} & 0 & 0 & \frac{2}{R}(x_1 + R)\left[\frac{x_4}{R} + \omega\right] \\ 0 & 0 & 0 & 1 \\ 2R\left(\frac{x_4}{R} + \omega\right)\left(\frac{x_2}{(x_1+R)^2}\right) & -2R\left(\frac{x_4}{R} + \omega\right)\left(\frac{1}{x_1+R}\right) & 0 & -2R\left(\frac{1}{R}\right)\left(\frac{x_2}{x_1+R}\right) \end{array}\right] \,.\end{aligned}$$

We now evaluate this at the point $\mathbf{x}_0$. We find that when we use the fact that $\omega^2 = \frac{G_0}{R^3}$ we get

$$\left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 + \frac{2G_0}{R^3} & 0 & 0 & \frac{2}{R}R\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2R\frac{\omega}{R} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}. \tag{152}$$

The second term in the sum in Equation 150 is the non-linear forcing function given by $\begin{bmatrix} 0 \\ u_1 \\ 0 \\ \frac{Ru_2}{x_1+R} \end{bmatrix}$. To expand this vector about the joint point $(\mathbf{x}_0, \mathbf{u}_0) = (\mathbf{0}, \mathbf{0}) = \mathbf{0}$ we have

$$\begin{aligned}
\begin{bmatrix} 0 \\ u_1 \\ 0 \\ \frac{Ru_2}{x_1+R} \end{bmatrix} &\approx \mathbf{g}(\mathbf{0}) + \frac{\partial}{\partial x_1}\left.\begin{bmatrix} 0 \\ u_1 \\ 0 \\ \frac{Ru_2}{x_1+R} \end{bmatrix}\right|_{\mathbf{0}} x_1 + \frac{\partial}{\partial \mathbf{u}}\left.\begin{bmatrix} 0 \\ u_1 \\ 0 \\ \frac{Ru_2}{x_1+R} \end{bmatrix}\right|_{\mathbf{0}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{Ru_2}{(x_1+R)^2} \end{bmatrix}\right|_{\mathbf{0}} x_1 + \left.\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{R}{x_1+R} \end{bmatrix}\right|_{\mathbf{0}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\end{aligned} \tag{153}$$

When we combine Equations 149 151, 152, and 153 we have the equation we wanted to show.

In the two parts below it seemed strange that the measurement noise had a variance that was the same symbol q as the process noise symbol. Thus I've changed the notation below to use the notation r_i for the variance of the measurement z_i.

Part (a): In this case $z(t) = x_3(t) + v_3(t)$ with $v_3 \sim N(0, r_3)$ so we have a measurement sensitivity matrix H given by

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},$$

with a measurement noise variance given by $R = r_3$.

Part (b): In this case $z(t) = x_1(t) + v_1(t)$ with $v_1 \sim N(0, r_1)$ so we have a measurement sensitivity matrix H given by

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

with a measurement noise variance given by $R = r_1$.

In comparing the prescriptions from Part (a) and Part (b) the better estimator will be the come with the smaller value of trace(P_∞), so we need to solve the steady-state for the Riccati equation

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP,$$

when $\dot{P} = 0$ and with F given by the above, $Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$, $G = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, and H and R given by the different parts as above.

In the Mathematical file `chap_4_prob_28.nb` we perform some of the algebra in attempting to solving for the steady state error covariance matrix $P(\infty)$.

Warning: I ran into trouble in that Mathematica could not solve the above nonlinear system for the components p_{ij} in the time I gave it. I then tried to solve the matrix Riccati equation using the methods discussed on Page 57 above. Unfortunately the eigenvalues of the system matrix F do not have a negative real parts since they are zero or entirely imaginary and this method cannot be used. Thus algebraically in the time I had to work on this I was unable to determine which of the two methods is better. If we specify numerical values for the above variances one could easy do a numerical simulation and make some headway. If anyone has any insight into this problem I would be interested in hearing your comments.

Chapter 5 (Optimal Linear Smoothing)

Notes on the text

Notes on the matrix inverse of a sum

Many of the results from the initial section use the following simple matrix inverse identity which we now derive. Since we can write the sum $P + P_b$ as

$$P + P_b = P_b(P_b^{-1} + P^{-1})P\,,$$

when we take the inverse of this sum $P + P_b$ we find that this inverse is given by

$$(P + P_b)^{-1} = P^{-1}(P_b^{-1} + P^{-1})^{-1}P_b^{-1} = P_b^{-1}(P^{-1} + P_b^{-1})^{-1}P^{-1}\,. \tag{154}$$

Notes on the derivation of the backward filter covariance matrix

To fully specify the backwards smoothing equations

$$\frac{d\hat{x}_b}{d\tau} = -F\hat{x}_b + P_bH^TR^{-1}[z - H\hat{x}_b] \tag{155}$$

$$\frac{dP_b}{d\tau} = -FP_b - P_bF^T + GQG^T - P_bH^TR^{-1}HP_b\,, \tag{156}$$

we must specify the initial condition on $\hat{x}_b(\tau)$. Now we don't know the value of $\hat{x}_b(t = T)$ but we know that it must be finite. Since we know that $P_b^{-1}(t = T) = 0$ we can try to derive an alternative differential equation for the product $P_b^{-1}(\tau)\hat{x}_b(\tau)$, since we know the value of this expression when $\tau = 0$ $(t = T)$ is 0. We start by recalling the matrix derivative of an inverse given by

$$\frac{d}{d\tau}P(\tau)^{-1} = -P(\tau)^{-1}\left(\frac{dP(\tau)}{d\tau}\right)P(\tau)^{-1}\,.$$

If we take the backwards covariance propagation equation

$$\frac{P_b(\tau)}{d\tau} = -FP_b - P_bF^T + GQG^T - P_bH^TR^{-1}HP_b\,,$$

and multiply on the left by P_b^{-1} and on the right by P_b^{-1} (and then negate the entire expression) we get

$$-P_b^{-1}\left(\frac{d}{d\tau}P_b^{-1}\right)P_b^{-1} = P_b^{-1}F + F^TP_b^{-1} - P_b^{-1}GQG^TP_b^{-1} + H^TR^{-1}H\,.$$

As the expression on the left-hand-side is $\frac{d}{d\tau}P_b(\tau)^{-1}$ this is the books equation 5.2-12. Using this we can now derive the differential equation for the variable $s(t) = P_b^{-1}(t)\hat{x}_b(t)$. Taking this derivative and using the product rule (and dropping the b subscript) we have

$$\begin{aligned}\frac{ds}{d\tau} &= \left(\frac{dP^{-1}(\tau)}{d\tau}\right)\hat{x}(\tau) + P^{-1}(\tau)\frac{d\hat{x}(\tau)}{d\tau} \\ &= (P^{-1}F + F^TP^{-1} - P^{-1}GQG^TP^{-1} + H^TR^{-1}H)\hat{x} + P^{-1}(-F\hat{x} + PH^TR^{-1}(z - H\hat{x})) \\ &= (F^T - P^{-1}GQG^T + H^TR^{-1}HP)P^{-1}\hat{x} + H^TR^{-1}(z - HPP^{-1}\hat{x}) \\ &= (F^T - P^{-1}GQG^T)s + H^TR^{-1}z\,,\end{aligned}$$

which is the books equation 5.2-13 and the expression we wanted to show.

Notes on the forward-backwards filter formulation of the smoother Table 5.2-1

In this subsubsection we derive the expression for the optimal smoother expressed in Table 5.2-1 and which is based on combining the forward filtering equations with the backwards filtering equations. In that table the forward filter and the backwards filter are the same as given in the text in many places. What is not directly obvious is the given expression for the optimal fixed-interval smoother $\hat{x}(t|T)$ and $P(t|T)$. To derive these equations we will use the matrix identity

$$B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}\,,$$

to evaluate $[P^{-1} + P_b^{-1}]^{-1}$ in the expression for $P(t|T)$. By taking $B = P^{-1} + P_b^{-1}$ and $A = P^{-1}$ we have

$$\begin{aligned}(P^{-1} + P_b^{-1})^{-1} &= P - (P^{-1} + P_b^{-1})^{-1}P_b^{-1}P \\ &= P - (P_b(P^{-1} + P_b^{-1}))^{-1}P \\ &= P - (I + P_bP^{-1})^{-1}P \\ &= P - (PP^{-1} + P_bP^{-1})^{-1}P \\ &= P - P(P + P_b)^{-1}P \\ &= P - PP_b^{-1}(I + PP_b^{-1})^{-1}P\,,\end{aligned}$$

which is the books equation for $P(t|T)$ found in table 5.2-1. Next we compute $\hat{x}(t|T)$ using the definition of $s(t)$ as

$$\begin{aligned}\hat{x}(t|T) &= P(t|T)[P^{-1}(t)\hat{x}(t) + P_b^{-1}\hat{x}_b(t)] \\ &= P(t|T)[P^{-1}(t)\hat{x}(t) + s(t)] = P(t|T)P^{-1}(t)\hat{x}(t) + P(t|T)s(t)\,.\end{aligned}$$

We next write this expression as

$$\begin{aligned}\hat{x}(t|T) &= (P^{-1} + P_b^{-1})^{-1}P^{-1}\hat{x}(t) + P(t|T)s(t) \\ &= (I + PP_b^{-1})^{-1}\hat{x}(t) + P(t|T)s(t)\,.\end{aligned}$$

Warning: This is *different* from the expression in the book for $\hat{x}(t|T)$ found in table 5.2-1 in that the books expression does not have an inverse on the factor $I + PP_b^{-1}$. If anyone finds anything wrong with the above expression or derivation please contact me.

The derivation of the Rauch-Tung-Striebel smoother equations

To derive the Rauch-Tung-Striebel smoother equations we begin by taking the t derivative of the books equation 5.1-11

$$P^{-1}(t|T) = P^{-1}(t) + P_b^{-1}(t)\,, \tag{157}$$

which expresses the smoothed covariance $P(t|T)$ in terms of the forward and backwards covariances. To do this we will applying the matrix inverse derivative identity

$$\frac{d}{dt}A^{-1} = -A^{-1}\left(\frac{dA}{dt}\right)A^{-1}\,,$$

to the left-hand-side of the above equation (but not to the right-hand-side) giving

$$\begin{aligned}
\frac{d}{dt}P(t|T)^{-1} &= -P^{-1}(t|T)\left(\frac{dP(t|T)}{dt}\right)P^{-1}(t|T) &(158)\\
&= \frac{d}{dt}P(t)^{-1} + \frac{d}{dt}P_b(t)^{-1} \\
&= \frac{d}{dt}P(t)^{-1} - \frac{d}{d\tau}P_b(\tau)^{-1}\,, &(159)
\end{aligned}$$

where we have converted the t derivative into a $\tau \equiv T - t$ derivative in the derivative of P_b^{-1} in the last term above. Now recall that from Equation 79 that the time derivative of P^{-1} is given by

$$\frac{d}{dt}P^{-1} = -F^TP^{-1} - P^{-1}F - P^{-1}GQG^TP^{-1} + H^TR^{-1}H\,,$$

and using the books equation 5.2-12 that the τ derivative of P_b^{-1} is given by

$$\frac{d}{d\tau}P_b^{-1} = P_b^{-1}F + F^TP_b^{-1} - P_b^{-1}GQG^TP_b^{-1} + H^TR^{-1}H\,. \tag{160}$$

If we use these two expression in Equation 159 we find

$$\begin{aligned}
\frac{d}{dt}P(t|T)^{-1} &= -F^TP^{-1} - P^{-1}F - P^{-1}GQG^TP^{-1} + H^TR^{-1}H \\
&- F^TP_b^{-1} - P_b^{-1}F + P_b^{-1}GQG^TP_b^{-1} - H^TR^{-1}H \\
&= -F^T(P^{-1} + P_b^{-1}) - (P^{-1} + P_b^{-1})F - P^{-1}GQG^TP^{-1} + P_b^{-1}GQG^TP_b^{-1} \\
&= -F^TP(t|T)^{-1} - P(t|T)^{-1}F - P^{-1}GQG^TP^{-1} + P_b^{-1}GQG^TP_b^{-1}\,.
\end{aligned}$$

To solve for $\frac{dP(t|T)}{dt}$ we use Equation 158 by premultiplying and postmultiplying by $P(t|T)$ and then negating the resulting expression. This procedure gives

$$\begin{aligned}
\frac{dP(t|T)}{dt} &= P(t|T)F^T + FP(t|T) \\
&+ P(t|T)P^{-1}GQG^TP^{-1}P(t|T) - P(t|T)P_b^{-1}GQG^TP_b^{-1}P(t|T)\,. &(161)
\end{aligned}$$

Lets now try to "remove" the terms with P_b from this expression. To do that recall if we premultiply by $P(t|T)$ in Equation 157, we get

$$I = P(t|T)P^{-1} + P(t|T)P_b^{-1}\,, \tag{162}$$

or solving for $P(t|T)P_b^{-1}$
$$P^{-1}(t|T)P_b^{-1} = I - P(t|T)P^{-1}\,. \tag{163}$$
Next we postmultiply by $P(t|T)$ in Equation 157, to get
$$I = P^{-1}P(t|T) + P_b^{-1}P(t|T)\,,$$
or solving for $P_b^{-1}P(t|T)$
$$P_b^{-1}P(t|T) = I - P^{-1}P(t|T)\,. \tag{164}$$
Then using these two expressions 163 and 164 in Equation 161 we obtain
$$\begin{aligned}
\frac{dP(t|T)}{dt} &= P(t|T)F^T + FP(t|T) + P(t|T)P^{-1}GQG^TP(t|T) \\
&- (I - P(t|T)P^{-1})GQG^T(I - P^{-1}P(t|T)) \\
&= P(t|T)F^T + FP(t|T) - GQG^T + P(t|T)P^{-1}GQG^T + GQG^TP^{-1}P(t|T) \\
&= (F + GQG^TP^{-1})P(t|T) + P(t|T)(F + GQG^TP^{-1})^T - GQG^T\,, \qquad (165)
\end{aligned}$$
or the books equation 5.2-15.

We next derive the differential expression satisfied by the smoothed estimate $\hat{x}(t|T)$. To begin recall the books equation 5.1-12,
$$\hat{x}(t|T) = P(t|T)[P^{-1}\hat{x}(t) + P_b^{-1}\hat{x}_b]\,, \tag{166}$$
from which we see that the time derivative of this expression is given by
$$\begin{aligned}
\frac{d\hat{x}(t|T)}{dt} &= \frac{dP(t|T)}{dt}[P^{-1}\hat{x} + P_b^{-1}\hat{x}_b] + P(t|T)[\frac{d}{dt}(P^{-1}\hat{x}) + \frac{d}{dt}(P_b^{-1}\hat{x}_b)] \\
&= [(F + GQG^TP^{-1})P(t|T) + P(t|T)(F + GQG^TP^{-1})^T - GQG^T]P^{-1}(t|T)\hat{x}(t|T) \\
&+ P(t|T)\left[\frac{dP^{-1}}{dt}\hat{x} + P^{-1}\frac{d\hat{x}}{dt} + \frac{dP_b^{-1}}{dt}\hat{x}_b + P_b^{-1}\frac{d\hat{x}_b}{dt}\right]\,.
\end{aligned}$$
Since the forward and backward state estimates must satisfy
$$\begin{aligned}
\frac{d\hat{x}}{dt} &= F\hat{x} + PH^TR^{-1}(z - H\hat{x}) \\
\frac{d\hat{x}_b}{dt} &= -(-F\hat{x}_b + P_bH^TR^{-1}(z - H\hat{x}_b)) = F\hat{x}_b - P_bH^TR^{-1}(z - H\hat{x}_b)\,,
\end{aligned}$$
when we put these into the above expression we find that
$$\begin{aligned}
\frac{d\hat{x}(t|T)}{dt} &= (F + GQG^TP^{-1})\hat{x}(t|T) \\
&+ [P(t|T)(F + GQG^TP^{-1})^T - GQG^T]P^{-1}(t|T)\hat{x}(t|T) \\
&+ P(t|T)\left[-F^TP^{-1}\hat{x} - P^{-1}F\hat{x} - P^{-1}GQG^TP^{-1}\hat{x} + H^TR^{-1}H\hat{x}\right] \\
&+ P(t|T)\left[P^{-1}F\hat{x} + H^TR^{-1}(z - H\hat{x})\right] \\
&+ P(t|T)\left[-P_b^{-1}F\hat{x}_b - F^TP_b^{-1}\hat{x}_b + P_b^{-1}GQG^TP_b^{-1}\hat{x}_b - H^TR^{-1}H\hat{x}_b\right] \\
&+ P(t|T)\left[P_b^{-1}F\hat{x}_b - H^TR^{-1}(z - H\hat{x}_b)\right]\,.
\end{aligned}$$

Many terms cancel in this expression and we are left with

$$\begin{aligned}
\frac{d\hat{x}(t|T)}{dt} &= (F + GQG^T P^{-1})\hat{x}(t|T) && (167)\\
&+ \left[P(t|T)(F + GQG^T P^{-1})^T - GQG^T\right] P^{-1}(t|T)\hat{x}(t|T) && (168)\\
&+ P(t|T)\left[-F^T P^{-1}\hat{x} - P^{-1}GQG^T P^{-1}\hat{x}\right] && (169)\\
&+ P(t|T)\left[-F^T P_b^{-1}\hat{x}_b + P_b^{-1}GQG^T P_b^{-1}\hat{x}_b\right] . && (170)
\end{aligned}$$

Notice that the terms $-P(t|T)F^T P^{-1}\hat{x}$ and $-P(t|T)F^T P_b^{-1}\hat{x}_b$ on the lines 169 and 170 combine using Equation 166 to give $-P(t|T)F^T P^{-1}(t|T)\hat{x}(t|T)$, which cancels the the first term on line 168 above to give

$$\begin{aligned}
\frac{d\hat{x}(t|T)}{dt} &= (F + GQG^T P^{-1})\hat{x}(t|T)\\
&+ P(t|T)P^{-1}GQG^T P^{-1}(t|T)\hat{x}(t|T) - GQG^T P^{-1}(t|T)\hat{x}(t|T)\\
&- P(t|T)P^{-1}GQG^T P^{-1}\hat{x} + P(t|T)P_b^{-1}GQG^T P_b^{-1}\hat{x}_b . && (171)
\end{aligned}$$

Again trying to "remove" the terms that contain $\hat{x}_b$ or P_b we note that from Equation 166 we get

$$P_b^{-1}\hat{x}_b = P^{-1}(t|T)\hat{x}(t|T) - P^{-1}\hat{x}(t) ,$$

and from Equation 157 we have $P_b^{-1} = P(t|T)^{-1} - P^{-1}$ so when we use these two expression in the last term in line 171 we find it is equal to

$$\begin{aligned}
P(t|T)P_b^{-1}GQG^T P_b^{-1}\hat{x}_b &= P(t|T)(P(t|T)^{-1} - P^{-1})GQG^T(P^{-1}(t|T)\hat{x}(t|T) - P^{-1}\hat{x})\\
&= GQG^T P^{-1}(t|T)\hat{x}(t|T) - GQG^T P^{-1}\hat{x}\\
&- P(t|T)P^{-1}GQG^T P^{-1}(t|T)\hat{x}(t|T) + P(t|T)P^{-1}GQG^T P^{-1}\hat{x} .
\end{aligned}$$

After this expansion when we use it in Equation 171 many terms cancel to give

$$\begin{aligned}
\frac{d\hat{x}(t|T)}{dt} &= (F + GQG^T P^{-1})\hat{x}(t|T) - GQG^T P^{-1}\hat{x}\\
&= F\hat{x}(t|T) + GQG^T P^{-1}(\hat{x}(t|T) - \hat{x}) , && (172)
\end{aligned}$$

the equation we were to show. Recall that $\hat{x}$ is the forward filtering solution and thus is a function of time even thought we don't explicitly denote it as such in the above expression.

Notes on the smoothabilty

When $Q = 0$ the Rauch-Tung-Striebel equations for the smoothed covariance $P(t|T)$ is

$$\dot{P}(t|T) = FP(t|T) + P(t|T)F^T .$$

Lets prove that the claimed expression for $P(t|T)$ or $\Phi(t, T)P(T)\Phi(t, T)^T$ is indeed a solution to this equation. From $P(t|T) = \Phi(t, T)P(T)\Phi(t, T)^T$ using the product rule to take the time derivative we have that

$$\dot{P} = \frac{d}{dt}\Phi(t, T)P(T)\Phi^T(t, T) + \Phi(t, T)P(T)\frac{d}{dt}\Phi(t, T)^T .$$

Since Φ is a fundamental solution we have $\frac{d}{dt}\Phi(t,T) = F(t)\Phi(t,T)$ and we can conclude that

$$\frac{d}{dt}\Phi(t,T)^T = (F\Phi)^T = \Phi^T F^T\,,$$

so the above first derivative of $P(t|T)$ becomes

$$\begin{aligned}\dot{P} &= F\Phi(t,T)P(T)\Phi^T(t,T) + \Phi(t,T)P(T)\Phi^T F^T \\ &= FP + PF^T\,,\end{aligned}$$

as we were to show.

Notes on the Books Example 5.2-1

In part one of this example we perform fixed-interval smoothing using the forward-backwards optimal filters. Thus to begin with we need to solve the continuous forward filtering Riccati equation. To do that note that for this problem we have $f = 0$, $g = h = 1$ so that Equation 71 in this case becomes

$$\dot{p} = q - \frac{p^2}{r}\,.$$

In steady-state $\dot{p} = 0$ so $p^2 = rq$ or $p = +\sqrt{rq} \equiv \alpha$. The backwards error covariance from Equation 156 is given by

$$\frac{dp_b}{d\tau} = q - \frac{p_b^2}{r}\,.$$

In steady-state $\frac{dp_b}{d\tau} = 0$ so $p_b^2 = rq$ or $q_b = +\sqrt{rq} = \alpha$. Thus in steady-state the smoothed state has the following error covariance

$$p^{-1}(t|T) = p^{-1}(t) + p_b^{-1}(t) = \frac{1}{\alpha} + \frac{1}{\alpha} = \frac{2}{\alpha}\,,$$

and so

$$p(t|T) = \frac{\alpha}{2}\,.$$

Next the smoothed state estimate is given by

$$\hat{x}(t|T) = p(t|T)\left(\frac{\hat{x}(t)}{p(t)} + \frac{\hat{x}_b(t)}{p_b(t)}\right) = \frac{\alpha}{2}\left(\frac{\hat{x}}{\alpha} + \frac{\hat{x}_b}{\alpha}\right) = \frac{1}{2}\left(\hat{x} + \hat{x}_b\right)\,. \tag{173}$$

For part 2 of this example we want to perform fixed-interval smoothing using the Rauch-Tung-Striebel equations, which in general are given by Equations 165 and 172. Specifying these to the problem at hand we find Equation 165 becomes

$$\begin{aligned}\dot{p}(t|T) &= \left(\frac{q}{\alpha}\right)p(t|T) + p(t|T)\left(\frac{q}{\alpha}\right) - q \\ &= \frac{2q}{\alpha}p(t|T) - q\,,\end{aligned}$$

as our differential equation to solve for $p(t|T)$. This equation has the final condition given by $p(T|T) = p(T)$, where $p(T)$ the forward smoother's error covariance value at the time $t = T$. Define β to be $\beta = \frac{1}{\alpha}$ then solving this differential equation is done as follows

$$\begin{aligned}
\dot{p}(t|T) - 2\beta p(t|T) &= -q \quad \text{or} \\
\frac{d}{dt}\left(e^{-2\beta t} p(t|T)\right) &= -q e^{-2\beta t} \quad \text{integrating both sides gives} \\
e^{-2\beta t} p(t|T) &= \frac{q}{2\beta} e^{-2\beta t} + C_0 \quad \text{for some constant } C_0 \text{ thus} \\
p(t|T) &= \frac{q}{2\beta} + C_0 e^{2\beta t}.
\end{aligned}$$

Note that $p(T) = \alpha$ since we assume that T is large enough so that the forward filtering equation is in steady-state. With this to satisfy the final condition on $p(t|T)$ of $p(T|T) = p(T) = \alpha$ requires C_0 satisfy

$$\frac{q}{2\beta} + C_0 e^{2\beta T} = \alpha \Rightarrow C_0 = \left(\alpha - \frac{q}{2\beta}\right) e^{-2\beta T}.$$

Thus we have for $p(t|T)$ the following

$$\begin{aligned}
p(t|T) &= \frac{q}{2\beta} + \left(\alpha - \frac{q}{2\beta}\right) e^{-2\beta T} e^{2\beta t} \\
&= \frac{\alpha}{2}\left(1 + e^{-2\beta(T-t)}\right) \quad \text{for} \quad t < T.
\end{aligned}$$

The differential equation for the smoothed state derived from Equation 172 is

$$\dot{\hat{x}}(t|T) = \frac{q}{\alpha}(\hat{x}(t|T) - \hat{x}(t)) = \beta(\hat{x}(t|T) - \hat{x}(t)).$$

This can be shown to be equivalent to Equation 173 by taking the time derivative of that equation which gives us

$$\dot{\hat{x}}(t|T) = \frac{1}{2}(\dot{\hat{x}} + \dot{\hat{x}}_b).$$

Using the differential equations for $\hat{x}$ and $\hat{x}_b$ which in this case are given by

$$\begin{aligned}
\dot{\hat{x}} &= \frac{p}{r}(z - \hat{x}) = \sqrt{\frac{q}{r}}(z - \hat{x}) \\
\dot{\hat{x}}_b &= -\frac{p_b}{r}(z - \hat{x}) = -\sqrt{\frac{q}{r}}(z - \hat{x}_b).
\end{aligned}$$

When we sum these two expressions (as required by $\dot{\hat{x}}(t|T)$) we find

$$\begin{aligned}
\dot{\hat{x}}(t|T) &= \frac{1}{2}\sqrt{\frac{q}{r}}(\hat{x}_b - \hat{x}) = \frac{1}{2}\sqrt{\frac{q}{r}}(2\hat{x}(t|T) - \hat{x} - \hat{x}) \\
&= \sqrt{\frac{q}{r}}(\hat{x}(t|T) - \hat{x}),
\end{aligned}$$

where we have expressed $\hat{x}_b$ in terms of $\hat{x}$ and $\hat{x}(t|T)$ using Equation 173.

Notes on a steady-state, fixed-interval smoother solution

In this subsection we show an alternative method to solve for the fixed-interval linear smoother covariance equation for $P(t|T)$ governed by the differential Equation 165. We start by defining an unknown λ in terms of the variable y as

$$\lambda = P(t|T)y \,, \tag{174}$$

where y is chosen to satisfy the following differential equation

$$\frac{dy}{dt} = -[F + GQG^T P^{-1}]^T y \,. \tag{175}$$

With such a definition taking the time derivative of λ above and using the product rule followed by replacing $\dot{P}(t|T)$ with the right-hand-side of Equation 165 we find

$$\begin{aligned}
\dot{\lambda} &= \dot{P}(t|T)y - P(t|T)(F + GQG^T P^{-1})^T y \\
&= (F + GQG^T P^{-1})P(t|T)y + P(t|T)(F + GQG^T P^{-1})^T y - GQG^T y \\
&- P(t|T)(F + GQG^T P^{-1})^T y \\
&= (F + GQG^T P^{-1})P(t|T)y - GQG^T y \\
&= (F + GQG^T P^{-1})\lambda - GQG^T y \,.
\end{aligned} \tag{176}$$

Then as a system in terms of the vector unknown $\begin{bmatrix} y \\ \lambda \end{bmatrix}$ we have

$$\frac{d}{dt}\begin{bmatrix} y \\ \lambda \end{bmatrix} = \begin{bmatrix} -(F + GQG^T P^{-1})^T & 0 \\ -GQG^T & F + GQG^T P^{-1} \end{bmatrix} \begin{bmatrix} y \\ \lambda \end{bmatrix} ,$$

which is the books equation 5.2-14.

Derivations of the equations for optimal fixed-point smoothers

In this subsection we provide somewhat more complete derivations of many of the stated fixed-point smoother equations. While the algebra for some of these can be tedious and I include most of it, the hope is that someone could simple "read" these derivations and observe their correctness. In other-words I don't want to have any of the steps that lead up to a result be mysterious. By cataloging these derivations and results in one place I won't have to revisit this work again in the future.

The first statement of this section is that we can write the explicit solution to the fixed-interval smoother differential Equation 172 in terms of a smoothing fundamental solution $\Phi_s(t, T)$. The claimed functional form for $\hat{x}(t|T)$ is given by

$$\hat{x}(t|T) = \Phi_s(t, T)\hat{x}(T) - \int_T^t \Phi_s(t, \tau)GQG^T P^{-1}(\tau)\hat{x}(\tau)d\tau \,, \tag{177}$$

where $\Phi_s(t, T)$ is the fundamental solution for Equation 172 and thus satisfies

$$\dot{\Phi}_s(t, T) = (F + GQG^T P^{-1}(t))\Phi_s(t, T) \quad \text{with} \quad \Phi_s(t, t) = I \,. \tag{178}$$

As a note on our notation, when dealing with multiple matrix products as in GQG^TP^{-1} if all factors in the product are to evaluated at the same argument we will present that argument only on the last factor. Thus the expression $GQG^TP^{-1}(\tau)$ is really a short-hand for $G(\tau)Q(\tau)G(\tau)^TP^{-1}(\tau)$. In the same way, the addition of another matrix to a product expression will be evaluated at the same argument as the product expression. Thus the expression $F + GQG^TP^{-1}(\tau)$ is really a short-hand for $F(\tau) + G(\tau)Q(\tau)G^T(\tau)P^{-1}(\tau)$.

Now we will show that Equation 177 is a solution to Equation 172 by explicitly evaluating its time derivative. Using Leibniz's rule and Equation 177 itself to replace any resulting integrals with simpler expressions it then follows that

$$
\begin{aligned}
\dot{\hat{x}}(t|T) &= (F + GQG^TP^{-1}(t))\Phi_s(t,T)\hat{x}(T) - \Phi_s(t,t)GQG^TP^{-1}(t)\hat{x}(t) \\
&- \int_T^t \dot{\Phi}_s(t,\tau)GQG^TP^{-1}(\tau)\hat{x}(\tau)d\tau \\
&= (F + GQG^TP^{-1}(t))\Phi_s(t,T)\hat{x}(T) - GQG^TP^{-1}(t)\hat{x}(t) \\
&- (F + GQG^TP^{-1}(t))\int_T^t \Phi_s(t,\tau)GQG^TP^{-1}(\tau)\hat{x}(\tau)d\tau \\
&= (F + GQG^TP^{-1}(t))\Phi_s(t,T)\hat{x}(T) - GQG^TP^{-1}(t)\hat{x}(t) \\
&- (F + GQG^TP^{-1}(t))[-\hat{x}(t|T) + \Phi_s(t,T)\hat{x}(T)] \\
&= (F + GQG^TP^{-1}(t))\hat{x}(t|T) - GQG^TP^{-1}(t)\hat{x}(t)\,,
\end{aligned}
$$

or an expression equivalent to Equation 172 proving that Equation 177 is a representation of its solution.

The next steps in the derivation are to derive expressions for the T evolution of $\hat{x}(t|T)$ and $P(t|T)$ or explicit equations for $\frac{d\hat{x}(t|T)}{dT}$ and $\frac{dP(t|T)}{dT}$. To derive an expression for $\frac{d\hat{x}(t|T)}{dT}$ we will need to be able to evaluate the expression $\frac{d\Phi_s(t,T)}{dT}$ which the book claims is given by

$$
\frac{d\Phi_s(t,T)}{dT} = -\Phi_s(t,T)(F + GQG^TP^{-1}(T))\,, \tag{179}
$$

where the expression $F+GQG^TP^{-1}(T)$ means that every matrix has its argument evaluated at T. To show this is true, consider the t derivative of the identity $\Phi_s(t,T)\Phi_s(T,t) = I$, which by the product rule is given by

$$
\frac{d\Phi_s(t,T)}{dt}\Phi_s(T,t) + \Phi_s(t,T)\frac{d\Phi_s(T,t)}{dt} = 0\,.
$$

Solving for $\frac{d\Phi_s(T,t)}{dt}$ and using the expression for $\frac{d\Phi_s(t,T)}{dt}$ given by Equation 178 we get

$$
\begin{aligned}
\frac{d\Phi_s(T,t)}{dt} &= -\Phi_s(t,T)^{-1}\frac{d\Phi_s(t,T)}{dt}\Phi_s(T,t) \\
&= -\Phi_s(T,t)\frac{d\Phi_s(t,T)}{dt}\Phi_s(T,t) \\
&= -\Phi_s(T,t)(F + GQG^TP^{-1}(t))\Phi_s(t,T)\Phi_s(T,t) \\
&= -\Phi_s(T,t)(F + GQG^TP^{-1}(t))\,.
\end{aligned} \tag{180}
$$

Then to get the desired expression for $\frac{d}{dT}\Phi_s(t,T)$ we exchange T and t in Equation 180 to get Equation 179 or the books equation 5.3-5. Once the expression for $\frac{d\Phi_s(t,T)}{dT}$ has been established the equation for $\frac{d\hat{x}(t|T)}{dT}$ is give by using Leibniz' rule on Equation 177 in a straightforward manner.

Having just derived an expression for $\frac{d\hat{x}(t|T)}{dT}$, we proceed do the same thing for $\frac{dP(t|T)}{dT}$. To do this we start with an explicit solution for $P(t|T)$ in terms of the fundamental smoothing matrix $\Phi_s(t,T)$, and proceed to take the T derivative of that solution using Equation 179 to simplify the resulting expressions. Now we claim that a solution to the differential equation for $\frac{dP(t|T)}{dt}$ given by Equation 165 can be expressed as

$$P(t|T) = \Phi_s(t,T)P(T)\Phi_s^T(t,T) - \int_T^t \Phi_s(t,\tau)GQG^T(\tau)\Phi_s(t,\tau)^T d\tau\,. \tag{181}$$

To verify this expression is indeed a solution we can take its t derivative to get

$$\begin{aligned}
\dot{P}(t|T) &= (F + GQG^T P^{-1}(t))\Phi_s(t,T)P(T)\Phi_s^T(t,T) \\
&+ \Phi_s(t,T)P(T)\Phi_s(t,T)^T(F + GQG^T P^{-1}(t))^T \\
&- \Phi_s(t,t)GQG^T(t)\Phi_s^T(t,t) \\
&- \int_T^t \left[\dot{\Phi}_s(t,\tau)GQG^T(\tau)\Phi_s^T(t,\tau) + \Phi_s(t,\tau)GQG^T(\tau)\dot{\Phi}_s^T(t,\tau)\right] d\tau \\
&= (F + GQG^T P^{-1}(t))\Phi_s(t,T)P(T)\Phi_s^T(t,T) \\
&+ \Phi_s(t,T)P(T)\Phi_s^T(t,T)(F + GQG^T P^{-1}(t))^T \\
&- GQG^T(t) \\
&- \int_T^t (F + GQG^T P^{-1}(t))\Phi_s(t,\tau)GQG^T(\tau)\Phi_s^T(t,\tau)d\tau \\
&- \int_T^t \Phi_s(t,\tau)GQG^T(\tau)\Phi_s^T(t,\tau)(F + GQG^T P^{-1}(t))^T d\tau\,.
\end{aligned}$$

From the claimed solution for $P(t|T)$ given by Equation 181 we have

$$\int_T^t \Phi_s(t,\tau)GQG^T(\tau)\Phi_s^T(t,\tau)d\tau = \Phi_s(t,T)P(T)\Phi_s^T(t,T) - P(t|T)\,,$$

so using this the above expression for $\dot{P}(t|T)$ becomes

$$\begin{aligned}
\dot{P}(t|T) &= (F + GQG^T P^{-1}(t))\Phi_s(t,T)P(T)\Phi_s^T(t,T) \\
&+ \Phi_s(t,T)P(T)\Phi_s(t,T)^T(F + GQG^T P^{-1}(t))^T \\
&- GQG^T(t) \\
&- (F + GQG^T P^{-1}(t))[\Phi_s(t,T)P(T)\Phi_s^T(t,T) - P(t|T)] \\
&- [\Phi_s(t,T)P(T)\Phi_s^T(t,T) - P(t|T)](F + GQG^T P^{-1}(t))^T \\
&= [F + GQG^T P^{-1}(t)]P(t|T) + P(t|T)[F + GQG^T P^{-1}(t)]^T - GQG^T(t)\,,
\end{aligned}$$

when we simplify. This is the books equation 5.2-15 showing that Equation 181 is indeed a solution to Equation 165 as claimed.

With the explicit representation for $P(t|T)$ given by Equation 181 we next take the T

derivative of this expression. The product rule and Leibniz' rule gives

$$\begin{aligned}
\frac{dP(t|T)}{dT} &= \frac{d\Phi_s(t,T)}{dT}P(T)\Phi_s^T(t,T) + \Phi_s(t,T)\frac{dP(T)}{dT}\Phi_s^T(t,T) \\
&+ \Phi_s(t,T)P(T)\frac{d\Phi_s^T(t,T)}{dT} + \Phi_s(t,T)GQG^T(T)\Phi_s(t,T)^T .
\end{aligned}$$

Now using Equation 71 and 179 into the above we have

$$\begin{aligned}
\frac{dP(t|T)}{dT} &= -\Phi_s(t,T)(F + GQG^T P^{-1}(T))P(T)\Phi_s^T(t,T) \\
&+ \Phi_s(t,T)\left[FP + PF^T + GQG^T - PH^TR^{-1}HP(T)\right]\Phi_s^T(t,T) \\
&- \Phi_s(t,T)P(T)(F + GQG^T P^{-1}(T))^T\Phi_s^T(t,T) + \Phi_s(t,T)GQG^T(T)\Phi_s(t,T)^T \\
&= -\Phi_s(t,T)PH^TR^{-1}HP(T)\Phi_s^T(t,T) , \qquad (182)
\end{aligned}$$

which is the book's equation 5.3-8.

Derivations of the equations for optimal fixed-lag smoothing

In this subsection of we present notes and derivations on the equations fixed-lag smoothers must satisfy. Starting with Equation 177 but by taking $t = T - \Delta$ gives the equation

$$\hat{x}(T-\Delta|T) = \Phi_s(T-\Delta,T)\hat{x}(T) - \int_T^{T-\Delta}\Phi_s(T-\Delta,\tau)GQG^TP^{-1}(\tau)\hat{x}(\tau)d\tau . \qquad (183)$$

To derive the ordinary differential equation that the optimal fixed-lag state estimate or $\hat{x}(T-\Delta|T)$ must satisfy we will take the T derivative of the above expression. To take the T derivative of the above requires us to evaluate $\frac{d\Phi_s(T-\Delta,T)}{dT}$. This derivative can be evaluated by writing $\Phi_s(T-\Delta,T) = \Phi_s(T-\Delta,t)\Phi_s(t,T)$, using the product rule followed by Equations 178 and 179. We find

$$\begin{aligned}
\frac{d\Phi_s(T-\Delta,T)}{dT} &= \frac{d\Phi_s(T-\Delta,t)}{dT}\Phi_s(t,T) + \Phi_s(T-\Delta,t)\frac{d\Phi_s(t,T)}{dT} \\
&= (F + GQG^TP^{-1}(T-\Delta))\Phi_s(T-\Delta,t)\Phi_s(t,T) \\
&- \Phi_s(T-\Delta,t)\Phi_s(t,T)(F + GQG^TP^{-1}(T)) \\
&= (F + GQG^TP^{-1}(T-\Delta))\Phi_s(T-\Delta,T) \\
&- \Phi_s(T-\Delta,T)(F + GQG^TP^{-1}(T)) . \qquad (184)
\end{aligned}$$

which is the books equation 5.4-3.

With this result we are ready to evaluate $\frac{d\hat{x}(T-\Delta|T)}{dT}$ using Equation 183. We find

$$\begin{aligned}
\frac{d\hat{x}(T-\Delta|T)}{dT} &= \frac{d\Phi_s(T-\Delta,T)}{dT}\hat{x}(T) + \Phi_s(T-\Delta,T)\frac{d\hat{x}(T)}{dT} \\
&- \Phi_s(T-\Delta,T-\Delta)GQG^TP^{-1}(T-\Delta)\hat{x}(T-\Delta) \\
&+ \Phi_s(T-\Delta,T)GQG^TP^{-1}(T)\hat{x}(T) \\
&- \int_T^{T-\Delta}\frac{d\Phi_s(T-\Delta,\tau)}{dT}GQG^TP^{-1}(\tau)\hat{x}(\tau)d\tau .
\end{aligned}$$

Using Equation 178 to evaluate $\frac{d\Phi_s(T-\Delta,\tau)}{dT}$ the integral term above becomes

$$(F + GQG^T P^{-1}(T-\Delta)) \int_T^{T-\Delta} \Phi_s(T-\Delta,\tau) GQG^T P^{-1}(\tau)\hat{x}(\tau) d\tau \,.$$

In terms of $\hat{x}(T-\Delta|T)$ from Equation 183 this is

$$(F + GQG^T P^{-1}(T-\Delta)) \left[\Phi_s(T-\Delta,T)\hat{x}(T) - \hat{x}(T-\Delta|T)\right] \,.$$

Thus we find our derivative of $\hat{x}(T-\Delta|T)$ given by

$$\begin{aligned}
\frac{d\hat{x}(T-\Delta|T)}{dT} &= (F + GQG^T P^{-1}(T-\Delta))\Phi_s(T-\Delta,T)\hat{x}(T) \\
&- \Phi_s(T-\Delta,T)(F + GQG^T P^{-1}(T))\hat{x}(T) \\
&+ \Phi_s(T-\Delta,T)\left[F(T)\hat{x}(T) + K(T)(z(T) - H(T)\hat{x}(T))\right] \\
&- \Phi_s(T-\Delta,T-\Delta) GQG^T P^{-1}(T-\Delta)\hat{x}(T-\Delta) \\
&+ \Phi_s(T-\Delta,T) GQG^T P^{-1}(T)\hat{x}(T) \\
&- (F + GQG^T P^{-1}(T-\Delta)) \left[\Phi_s(T-\Delta,T)\hat{x}(T) - \hat{x}(T-\Delta|T)\right] \\
&= (F + GQG^T P^{-1}(T-\Delta))\hat{x}(T-\Delta|T) \\
&- GQG^T P^{-1}(T-\Delta)\hat{x}(T-\Delta) \\
&+ \Phi_s(T-\Delta,T)K(T)(z(T) - H(T)\hat{x}(T)) \,, \qquad (185)
\end{aligned}$$

which is the books equation 5.4-3 and is the desired differential equation for $\hat{x}(T-\Delta|T)$.

Next we derive the differential equation for $P(T-\Delta|T)$ under optimal fixed-lag smoothing. To do this we set $t = T-\Delta$ in Equation 181 and get

$$P(T-\Delta|T) = \Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T) - \int_T^{T-\Delta} \Phi_s(T-\Delta,\tau) GQG^T(\tau)\Phi_s^T(T-\Delta,\tau) d\tau \,.$$

We follow the same procedure to derive the corresponding differential equation we have been performing above. The algebra for this seems quite involved and can be skipped at first reading. Taking the T derivative of this expression we find

$$\begin{aligned}
\frac{dP(T-\Delta|T)}{dT} &= \frac{d\Phi_s(T-\Delta,T)}{dT} P(T)\Phi_s^T(T-\Delta,T) + \Phi_s(T-\Delta,T)\frac{dP(T)}{dT}\Phi_s^T(T-\Delta,T) \\
&+ \Phi_s(T-\Delta,T)P(T)\frac{d\Phi_s^T(T-\Delta,T)}{dT} - GQG^T(T-\Delta) \\
&+ \Phi_s(T-\Delta,T) GQG^T(T)\Phi_s^T(T-\Delta,T) \\
&- \int_T^{T-\Delta} \frac{d\Phi_s(T-\Delta,\tau)}{dT} GQG^T(\tau)\Phi_s^T(T-\Delta,\tau) d\tau \\
&- \int_T^{T-\Delta} \Phi_s(T-\Delta,\tau) GQG^T(\tau)\frac{d\Phi_s^T(T-\Delta,\tau)}{dT} d\tau \,.
\end{aligned}$$

Again we will use Equation 178 to evaluate $\frac{d\Phi_s(T-\Delta,\tau)}{dT}$ in the above integrals and then write them in terms in terms of $P(T-\Delta|T)$ and $\Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T)$ using the proposed integral solution for $P(T-\Delta|T)$. When we do this along with other simplifications of derivatives that appear we obtain

$$
\begin{aligned}
\frac{dP(T-\Delta|T)}{dT} &= (F+GQG^TP^{-1}(T-\Delta))\Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T) \\
&- \Phi_s(T-\Delta,T)(F+GQG^TP^{-1}(T))P(T)\Phi_s^T(T-\Delta,T) \\
&+ \Phi_s(T-\Delta,T)[FP+PF^T+GQG^T-PH^TR^{-1}HP(T)]\Phi_s^T(T-\Delta,T) \\
&+ \Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T)(F+GQG^TP^{-1}(T-\Delta))^T \\
&- \Phi_s(T-\Delta,T)P(T)(F-GQG^TP^{-1}(T))^T\Phi_s^T(T-\Delta,T) \\
&- GQG^T(T-\Delta) \\
&+ \Phi_s(T-\Delta,T)GQG^T(T)\Phi_s^T(T-\Delta,T) \\
&- (F+GQG^TP^{-1}(T-\Delta))[\Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T)-P(T-\Delta|T)] \\
&- [\Phi_s(T-\Delta,T)P(T)\Phi_s^T(T-\Delta,T)-P(T-\Delta|T)](F+GQG^TP^{-1}(T-\Delta))^T\,.
\end{aligned}
$$

As expected, many terms cancel in the above expression and when the smoke clears we find we are left with

$$
\begin{aligned}
\frac{dP(T-\Delta|T)}{dT} &= (F+GQG^TP^{-1}(T-\Delta))P(T-\Delta|T) \\
&+ P(T-\Delta|T)(F+GQG^TP^{-1}(T-\Delta))^T \\
&- \Phi_s(T-\Delta,T)PH^TR^{-1}HP\Phi_s^T(T-\Delta,T) \\
&- GQG^T(T-\Delta)\,,
\end{aligned}
\tag{186}
$$

which is the books equation 5.4-4.

Problem Solutions

Problem 5-1 (the smoothing equation via minimization)

To solve this problem lets begin by expanding the given objective function as

$$
\begin{aligned}
J &= (x-\hat{x})^TP^{-1}(x-\hat{x}) + (x-\hat{x}_b)^TP_b^{-1}(x-\hat{x}_b) \\
&= x^TP^{-1}x - 2\hat{x}^TP^{-1}x + \hat{x}^TP^{-1}\hat{x} \\
&+ x^TP_b^{-1}x - 2\hat{x}_b^TP_b^{-1}x + \hat{x}_b^TP_b^{-1}\hat{x}_b\,.
\end{aligned}
$$

Then using Equations 311 and 312 we can compute the derivative of J with respect to x. We find

$$\frac{\partial J}{\partial x} = 2P^{-1}x + 2P_b^{-1}x - 2(P^{-1}\hat{x}) - 2(P_b^{-1}\hat{x}_b)\,.$$

Setting this result equal to zero we have

$$(P^{-1}+P_b^{-1})x = P^{-1}\hat{x} + P_b^{-1}\hat{x}_b\,,$$

or solving for x and calling the result $\hat{x}(t|T)$ we have

$$\hat{x}(t|T) = (P^{-1}+P_b^{-1})^{-1}(P^{-1}\hat{x} + P_b^{-1}\hat{x}_b)\,. \tag{187}$$

If we define $P(t|T)$ to be

$$P(t|T) = (P^{-1} + P_b^{-1})^{-1} ,$$

then the above is

$$\hat{x}(t|T) = P(t|T)(P^{-1}\hat{x} + P_b^{-1}\hat{x}_b) ,$$

as we were to show.

Problem 5-2 (deriving the Rauch-Tung-Striebel smoother equations)

This exercise is worked beginning on Page 100 in this text.

Problem 5-3 (deriving the Bryson-Frazier fixed-interval smoother equations)

Warning: I was unable to derive the given expression for $\dot{\Lambda}(t)$ or to show the identity

$$P(t|T) = P(t) - P(t)\Lambda(t)P(t) ,$$

as requested in this problem. Below I present the algebraic steps I took and where I got stuck. If anyone sees what to do next or an alternative solution please contact me.

If we consider the estimate $\hat{x}(t|T)$ decomposed as

$$\hat{x}(t|T) = \hat{x}(t) - P(t)\lambda(t) , \tag{188}$$

then taking the derivative of $\hat{x}(t|T)$ using the product rule gives

$$\frac{\hat{x}(t|T)}{dt} = \frac{d\hat{x}}{dt} - \frac{dP}{dt}\lambda(t) - P(t)\frac{\lambda(t)}{dt} . \tag{189}$$

Now using equation 5.2-14 from the book to evaluate $\frac{\hat{x}(t|T)}{dt}$ on the left-hand-side followed by the forward filtering equations given by

$$\frac{d\hat{x}}{dt} = F\hat{x} + PH^T R^{-1}(z - H(t)\hat{x}) ,$$

and a similar equation for $\frac{dP}{dt}$ in the right-hand-side of Equation 189 we find

$$\begin{aligned} F\hat{x}(t|T) + GQG^T P^{-1}(\hat{x}(t|T) - \hat{x}) &= F\hat{x} + PH^T R^{-1}(z - H\hat{x}) \\ &- (FP + PF^T + GQG^T - PH^T R^{-1} HP)\lambda - P\frac{d\lambda}{dt} . \end{aligned}$$

Putting the expression for $\hat{x}(t|T)$ given by Equation 188 into the left-hand-side the above expression and then canceling the common terms we obtain

$$0 = PH^T R^{-1}(z - H(t)\hat{x}) - PF^T\lambda + PH^T R^{-1} HP\lambda - P\frac{d\lambda}{dt} .$$

Solving for $\frac{d\lambda}{dt}$ we obtain

$$
\begin{aligned}
\frac{d\lambda}{dt} &= -F^T\lambda + H^T R^{-1} H P \lambda + H^T R^{-1}(z - H\hat{x}) \\
&= -[F^T - P H^T R^{-1} H]^T \lambda + H^T R^{-1}(z - H\hat{x}) \,,
\end{aligned}
$$

as we were to show. Note that since $\hat{x}(t|T)$ when $t = T$ is given by $\hat{x}(T|T) = \hat{x}(T)$, we see that this translates into an initial condition on $\lambda(t)$ of the following

$$
\hat{x}(T|T) = \hat{x}(T) - P(T)\lambda(T) \quad \text{so} \quad \lambda(T) = 0 \,.
$$

Using the definition of $\Lambda(t)$ as $E[\lambda(t)\lambda(t)^T]$ we have that the first derivative of this expression (when we use the results from above) is

$$
\begin{aligned}
\frac{d}{dt}\Lambda(t) &= E\left[\frac{d\lambda}{dt}\lambda^T\right] + E\left[\lambda\frac{d\lambda^T}{dt}\right] \\
&= -(F - PH^T R^{-1} H)^T E[\lambda\lambda^T] + H^T R^{-1} E[(z - H\hat{x})\lambda^T] \\
&- E[\lambda\lambda^T](F - PH^T R^{-1} H) + E[\lambda(z - H\hat{x})^T] R^{-1} H \\
&= -(F - PH^T R^{-1} H)^T \Lambda(t) - \Lambda(t)(F - PH^T R^{-1} H) \qquad (190) \\
&+ H^T R^{-1} E[(z - H\hat{x})\lambda^T] + E[\lambda(z - H\hat{x})^T] R^{-1} H \,. \qquad (191)
\end{aligned}
$$

This result is similar to the expression we are attempting to derive for $\dot{\Lambda}$. To make the two expressions the same we need to evaluate the last two terms above. Since the two terms on line 191 are transposes of each other we will evaluate only the first one and get the second one by transposition. From the definition of λ we have

$$
\lambda = P^{-1}(\hat{x} - \hat{x}(t|T)) \,,
$$

thus we see that

$$
E[(z - H\hat{x})\lambda^T] = E[(z - H\hat{x})(\hat{x} - \hat{x}(t|T))^T] P^{-1} \,. \qquad (192)
$$

Now since by assumption our measurement z is related to the state via $z = Hx + v$ where x is our true system state and our estimate $\hat{x}$ is the true system state plus an error as $\hat{x} = x + \tilde{x}$. Using these two relationships we can write the first factor in the product above as

$$
z - H\hat{x} = Hx + v - H(x + \tilde{x}) = v - H\tilde{x} \,. \qquad (193)
$$

Next lets consider the second factor in the product above. From the definition of $\hat{x}(t|T)$ in Equation 166 and using Equation 162 we see that

$$
\begin{aligned}
\hat{x} - \hat{x}(t|T) &= \hat{x} - P(t|T)P^{-1}\hat{x} - P(t|T)P_b^{-1}\hat{x}_b \\
&= (I - P(t|T)P^{-1})\hat{x} - P(t|T)P_b^{-1}\hat{x}_b \\
&= P(t|T)P_b^{-1}\hat{x} - P(t|T)P_b^{-1}\hat{x}_b \\
&= P(t|T)P_b^{-1}(\hat{x} - \hat{x}_b) = P(t|T)P_b^{-1}(\tilde{x} - \tilde{x}_b) \,.
\end{aligned}
$$

With this result we can now compute the inner product needed in Equation 192. We find

$$
(z - H\hat{x})(\hat{x} - \hat{x}(t|T))^T = (v - H\tilde{x})(\tilde{x} - \tilde{x}_b)^T P_b^{-1} P(t|T) \,.
$$

Now taking expectations and using the facts that

$$\begin{aligned} E[v\tilde{x}^T] &= 0 \\ E[v\tilde{x}_b^T] &= 0\,, \end{aligned}$$

and the fact that the backwards filter is independent of the forward filter so that

$$E[\tilde{x}\tilde{x}_b^T] = 0\,,$$

we find the needed expectation given by

$$\begin{aligned} E[(z - H\hat{x})(\hat{x} - \hat{x}(t|T))^T] &= -HE[\tilde{x}\tilde{x}^T]P_b^{-1}P(t|T) \\ &= -HPP_b^{-1}P(t|T)\,. \end{aligned}$$

Putting everything back together we find the term $H^T R^{-1} E[(z - H\hat{x})\lambda^T]$ given by

$$\begin{aligned} H^T R^{-1} E[(z - H\hat{x})\lambda^T] &= H^T R^{-1} E[(z - H\hat{x})(\hat{x} - \hat{x}(t|T))^T]P^{-1} \\ &= -H^T R^{-1} HPP_b^{-1}P(t|T)P^{-1}\,. \end{aligned}$$

Now we have two terms like this to add together on line 191 where the second is the transpose of the first we need to simplify

$$-H^T R^{-1} HPP_b^{-1}P(t|T)P^{-1} - P^{-1}P(t|T)P_b^{-1}PH^T R^{-1}H\,.$$

Warning: I don't see how to turn this remaining expression into $H^T R^{-1} H$. If anyone sees how to proceed with this derivation please contact me.

Problem 5-4 (an example of the reduction in uncertainty with smoothing)

Part (a): See the problem 4-11 on Page 73 where we do this calculation in detail.

Part (b): We will consider the Rauch-Tung-Striebel (RTS) covariance Equation 165 in steady-state where $\dot{P}(t|T) = 0$ but specified for this problem where all system matrices are scalars and constant. Specifically we have $F = a$, $G = 1$, $Q = q$, $H = b$, and $R = r$ so the RTS equation becomes

$$0 = 2\left(a + \frac{q}{p_\infty}\right)p_\infty(t|T) - q\,.$$

When we solve this for $p_\infty(t|T)$ we get

$$p_\infty(t|T) = \frac{q}{2\left(a + \frac{q}{p_\infty}\right)} = \frac{p_\infty}{2\left(1 + \frac{a}{q}p_\infty\right)}\,.$$

To solve this problem another way one could consider the backwards covariance filtering equation given by

$$\begin{aligned} \frac{d}{d\tau}P_b^{-1}(T-\tau) &= P_b^{-1}(T-\tau)F(T-\tau) + F^T(T-\tau)P_b^{-1}(T-\tau) \\ &- P_b^{-1}(T-\tau)G(T-\tau)Q(T-\tau)G^T(T-\tau)P_b^{-1}(T-\tau) \\ &+ H^T(T-\tau)R^{-1}(T-\tau)H(T-\tau)\,. \end{aligned}$$

Set $\frac{dP_b^{-1}}{d\tau} = 0$ and solve for $P_b(\infty)$. For this problem the above becomes

$$0 = \frac{2a}{p_b(\infty)} - \frac{q}{p_b(\infty)^2} + \frac{b^2}{r}\,.$$

which we would need to solve for $p_b(\infty)$. Given this value we can compute the desired expression $P_\infty(t|T)$ using $P_\infty(t|T)^{-1} = P^{-1}(\infty) + P_b^{-1}(\infty)$.

Part (c): Using the above two results we find that

$$\frac{p_\infty(t|T)}{p_\infty} = \frac{1}{2\left(1 + \frac{a}{q}p_\infty\right)} = \frac{1}{2\left(1 + \frac{a}{q}\cdot\frac{ar}{b^2}\left(1 + \sqrt{1 + \frac{b^2 q}{a^2 r}}\right)\right)}\,.$$

Defining γ^2 as $\gamma^2 = \frac{b^2 q}{a^2 r}$ the above becomes

$$\frac{1}{2\left(1 + \frac{1}{\gamma^2}\left(1 + \sqrt{1+\gamma^2}\right)\right)}\,,$$

which if we multiply by γ^2 on the top and bottom of this expression gives the desired result.

Problem 5-6 (smoothing an integrator)

For this problem we desire to apply fixed interval smoothing to a discrete system which looks like

$$\begin{aligned} x_k &= x_{k-1} + w_{k-1} \quad \text{for} \quad w_{k-1} \sim N(0, q\Delta) \\ z_k &= x_k + v_k \quad \text{for} \quad v_k \sim N(0, r_0)\,. \end{aligned}$$

Thus we have that $\Phi_k = 1$, $Q_k = q\Delta$, $H_k = 1$, and $R_k = r_0$. Note that the forward filtering part of this problem is the same as that of Problem 4-14 on page 77.

Part (a): For this part we want to use fixed-interval smoothing to compute $p_{0|2}$ and $p_{1|2}$, so $N = 2$ and to solve this problem using the Rauch-Tung-Striebel algorithm we first need to compute the forward smoothed solution $p_k(\pm)$.

Since we are told to assume no a-priori information on the knowledge of the state we must take $p_0(+) \approx +\infty$. If we do this directly it seems that we run into problems when we perform backwards filtering (in that we obtain the indefinite ratio of ∞/∞) with the above forward filtered results. Thus I'll take our initial condition on $p_0(+)$ to be

$$p_0(+) = \frac{1}{\epsilon}\,,$$

where ϵ is a small number. Just as in Problem 4-14 we iterate the discrete Kalman filter equations for $k = 0, 1, 2$ to find when we take $\epsilon = 0$ we get

$$
\begin{aligned}
p_0(+) &= +\infty \\
p_1(-) &= +\infty \\
p_1(+) &= r_0 \\
p_2(-) &= r_0(1+\gamma) \\
p_2(+) &= r_0 \frac{1+\gamma}{2+\gamma} .
\end{aligned}
$$

When we keep $\epsilon \neq 0$ we can then perform the discrete RTS filtering equations backwards. Starting with $p_{N|N} = p_{2|2} = p_2(+)$ we compute for $k = 1$ and then $k = 0$ the following

$$
\begin{aligned}
A_k &= P_k(+)\Phi_k^T P_{k+1}^{-1}(-) \\
P_{k|N} &= P_k(+) + A_k[P_{k+1|N} - P_{k+1}(-)]A_k^T .
\end{aligned}
$$

The calculations when $p_0(+) = \frac{1}{\epsilon}$ and the subsequent limit as $\epsilon \to 0$ are rather tedious and are done in the Mathematica file `chap_5_prob_6.nb`. Performing the above iterations we obtain

$$
\begin{aligned}
p_{2|2} &= p_2(+) = r_0 \frac{1+\gamma}{2+\gamma} \\
a_1 &= p_1(+)\Phi_1^T p_2^{-1}(-) = \frac{1}{1+\gamma} \\
p_{1|2} &= p_1(+) + a_1[p_{2|2} - p_2(-)]a_1^T \\
&= p_1(+) + a_1^2[p_2(+) - p_2(-)] = r_0 \left(\frac{1+\gamma}{2+\gamma} \right) \\
a_0 &= p_0(+)\Phi_k^T p_1^{-1}(-) = 1 \\
p_{0|2} &= p_0(+) + a_0[p_{1|2} - p_1(-)]a_0^T \\
&= r_0 \left(\frac{1+3\gamma+\gamma^2}{2+\gamma} \right) .
\end{aligned}
$$

Warning: Note that these expressions are somewhat different than the ones presented for this problem. If anyone sees an error in what I've done or can verify that these are correct please contact me.

Part (c): In fixed-point smoothing we desire a smoothed estimate of the state at a particular point of interest while the "end point" of the interval grows. Specifically, in fixed-point optimal smoothing we will *fix* the index k and then let the index N increase. For this problem since we want to compute $p_{0|1}$ and $p_{0|2}$ that means we take $k = 0$ and let $N = 1$ and $N = 2$. Once k is fixed and using the a priori and a posteriori covariance estimate $P_i(\pm)$ for $i \geq k$ computed from forward filtering we will compute the desired fixed-point smoothed solutions $P_{k|N}$ for $N = k+1, k+2, \cdots$ by using

$$
\begin{aligned}
B_N &= \prod_{i=k}^{N-1} P_i(+)\Phi_i^T P_{i+1}^{-1}(-) \\
P_{k|N} &= P_{k|N-1} + B_N[P_k(+) - P_k(-)]B_N^T ,
\end{aligned}
$$

with $P_{k|k} = P_k(+)$.

To iterate these equations when $N = 1$ we have

$$
\begin{aligned}
B_1 &= \prod_{i=0}^{0} P_i(+)\Phi_i^T P_{i+1}^{-1}(-) = P_0(+)P_1^{-1}(-) = \frac{P_0(+)}{P_1(-)} = 1 \\
P_{0|1} &= P_{0|0} + B_1(P_0(+) - P_0(-))B_1^T \\
&= 2P_0(+) - P_0(-)\,.
\end{aligned}
$$

Warning: I don't see how to evaluate the term $P_0(-)$ since our initial a posteriori uncertainty was to be infinite $P_0(+) = \infty$. This might mean that $P_0(-) = \infty$. In any case these results don't agree with what the book claims this expression should be.

Chapter 6 (Nonlinear Estimation)

Notes on the text

Notes on the extended Kalman filter

If we perform a power series expansion of our nonlinear function $f(x,t)$ in terms of the current estimate (the conditional mean $\hat{x}(t)$) then we have

$$f(x,t) \approx f(\hat{x},t) + \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}} (x-\hat{x}) + \cdots = f(\hat{x},t) + F(x-\hat{x}) + \cdots ,$$

where F is a function of the state, $\hat{x}$, we linearize about and time t i.e. $F = F(\hat{x},t)$. Then the state estimate $\hat{x}$ satisfies

$$\dot{\hat{x}}(t) = \hat{f}(x(t),t) \,. \tag{194}$$

Next using the books equation 6.1-5 or

$$\dot{P}(t) = \widehat{xf^T} - \hat{x}\hat{f}^T + \widehat{fx^T} - \hat{f}\hat{x}^T + Q \,, \tag{195}$$

we will evaluate the right-hand-side using the above power series expansion for $f(x,t)$. For the term $\widehat{xf^T} = E[xf^T]$ we find

$$\begin{aligned} E[xf^T] &= E[xf(\hat{x},t)^T] + E[x(x-\hat{x})^T F^T] \\ &= E[x]f(\hat{x},t)^T + E[(x-\hat{x})(x-\hat{x})^T]F^T + E[\hat{x}(x-\hat{x})^T]F^T \\ &= \hat{x}f(\hat{x},t)^T + PF^T \,. \end{aligned}$$

To evaluate $\widehat{fx^T} = E[fx^T]$ we simply take the transpose of the above result. To evaluate the expression $\hat{f}$ we have

$$\hat{f} = E[f(x,t)] \approx f(\hat{x},t) + E[(F(x-\hat{x}))^T] = f(\hat{x},t) \,.$$

Using these two expressions in Equation 195 we have for $\dot{P}$

$$\begin{aligned} \dot{P}(t) &= \hat{x}f(\hat{x},t)^T + PF^T \\ &- \hat{x}f(\hat{x},t)^T \\ &+ f(\hat{x},t)\hat{x}^T + FP \\ &- f(\hat{x},t)\hat{x}^T + Q \\ &= PF^T + FP + Q \,, \end{aligned} \tag{196}$$

which is the book's equation 6.1-8.

Notes on the extended Kalman filter: incorporating measurements

We will estimate the state at time t_k or x_k after the measurement z_k using a formula like

$$\hat{x}_k(+) = a_k + K_k z_k \,. \tag{197}$$

Then introducing the definition of the a priori and a posteriori state error $\tilde{x}_k(\pm)$ as

$$\tilde{x}_k(\pm) = \hat{x}_k(\pm) - x_k \,, \tag{198}$$

and first using $\tilde{x}_k(+)$ on the left-hand-side of the proposed estimator Equation 197 above we get

$$\tilde{x}_k(+) + x_k = a_k + K_k(h_k(x_k) + v_k) \,.$$

Next using $\tilde{x}_k(-)$ to replace x_k on the left-hand-side of this expression we get

$$\tilde{x}_k(+) = a_k + K_k h_k(x_k) + K_k v_k + \tilde{x}_k(-) - \hat{x}_k(-) \,, \tag{199}$$

which is the books equation 6.1-11. Now taking the expectation of both sides of this expression and assuming that our earlier estimate of x_k was unbiased that is $E[\tilde{x}_k(-)] = 0$ then to make our a posteriori estimate of x_k unbiased we require the following

$$E[\tilde{x}_k(+)] = a_k + K_k E[h_k(x_k)] - E[\hat{x}_k(-)] = 0 \,.$$

Since $E[\hat{x}_k(-)] = \hat{x}_k(-)$ when we solve for a_k we find

$$a_k = \hat{x}_k(-) - K_k E[h_k(x_k)] \,,$$

and the a posteriori estimate $\hat{x}_k(+)$ in Equation 197 then takes the form

$$\begin{aligned} \hat{x}_k(+) &= a_k + K_k z_k \\ &= \hat{x}_k(-) + K_k(z_k - E[h_k(x_k)]) \,, \end{aligned} \tag{200}$$

which is the books equation 6.1-13. Using this expression for a_k we can go back to the expression above for the a posteriori estimate error $\tilde{x}_k(+)$ or Equation 199 where we find

$$\begin{aligned} \tilde{x}_k(+) &= \hat{x}_k(-) - K_k E[h_k(x_k)] + K_k h_k(x_k) + K_k v_k + \tilde{x}_k(-) - \hat{x}_k(-) \\ &= \tilde{x}_k(-) + K_k(h_k(x_k) - E[h_k(x_k)]) + K_k v_k \,, \end{aligned} \tag{201}$$

or the books equation 6.1-14. This expression makes it easy to compute $P_k(+)$ since it is the expectation of the above expression "squared". Specifically $P_k(+) = E[\tilde{x}_k(+)\tilde{x}_k(+)^T]$ and this quadratic product is given by

$$\begin{aligned} \tilde{x}_k(+)\tilde{x}_k(+)^T &= \tilde{x}_k(-)\tilde{x}_k(-)^T + \tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T K_k^T + \tilde{x}_k(-) v_k^T K_k^T \\ &+ K_k(h_k(x_k) - E[h_k(x_k)])\tilde{x}_k(-)^T \\ &+ K_k(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T K_k^T \\ &+ K_k(h_k(x_k) - E[h_k(x_k)]) v_k^T K_k^T \\ &+ K_k v_k \tilde{x}_k(-)^T + K_k v_k (h_k(x_k) - E[h_k(x_k)])^T K_k^T + K_k v_k v_k^T K_k^T \,. \end{aligned}$$

When we take the expectation of the above many terms simplify. Specifically using

$$\begin{aligned} E[\tilde{x}_k(\pm)\tilde{x}_k(\pm)^T] &= P_k(\pm) \\ E[\tilde{x}_k(-) v_k^T] &= 0 \\ E[(h_k(x_k) - E[h_k(x_k)]) v_k^T] &= 0 \\ E[v_k v_k^T] &= R_k \,, \end{aligned}$$

the above simplifies to

$$\begin{aligned}
P_k(+) &= P_k(-) + E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T]K_k^T \\
&+ K_k E[(h_k(x_k) - E[h_k(x_k)])\tilde{x}_k(-)^T] \\
&+ K_k E[(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T]K_k^T + K_k R_k K_k^T\,. \quad (202)
\end{aligned}$$

which is the books equation 6.1-15.

As we have done before we will select K_k so that $P_k(+)$ has a minimum trace. Defining $J_k = \text{trace}(P_k(+))$, we then seek to minimize J_k as a function of K_k by taking the K_k derivative of J_k, setting the result equal to zero and then solving for K_k. From the Equation 202 we have several types of derivatives to take. Using Equation 112 with either B or C equal to the identity matrix we can take the derivative of the second and third terms in Equation 202, while using Equation 113 we can take the derivative of the fourth and fifth terms. When we use these expressions we find we need to solve

$$\begin{aligned}
\frac{\partial J_k}{\partial K_k} &= E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T] \\
&+ E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T] \\
&+ 2K_k E[(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T] + 2K_k R_k = 0\,,
\end{aligned}$$

for K_k. Doing this gives

$$\begin{aligned}
K_k &= -E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T] \\
&\times \left\{E[(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T] + R_k\right\}^{-1}\,, \quad (203)
\end{aligned}$$

or the books equation 6.1-17. We next want to put this expression into Equation 202 to evaluate what the minimum value of the objective function J_k is. To do this we will briefly introduce some short-hand notation so that the manipulations are more manageable. We define the symbol "xh^T" as

$$xh^T = E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T]\,.$$

and the symbol "hh^T" as

$$hh^T = E[(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T]\,.$$

With this short-hand we have $K_k = -xh^T(hh^T + R_k)^{-1}$ and find that $P_k(+)$ becomes

$$\begin{aligned}
P_k(+) &= P_k(-) + xh^T(hh^T + R_k)^{-1}(hh^T)(hh^T + R_k)^{-1}hx^T \\
&- xh^T(hh^T + R_k)^{-1}hx^T - xh^T(hh^T + R_k)^{-1}hx^T \\
&+ xh^T(hh^T + R_k)^{-1}R_k(hh^T + R_k)^{-1}hx^T\,.
\end{aligned}$$

Combining the second and fifth terms gives

$$xh^T(hh^T + R_k)^{-1}(hh^T + R_k)(hh^T + R_k)^{-1}hx^T = xh^T(hh^T + R_k)^{-1}hx^T\,,$$

which cancels with the third term. Thus we get (expressed in terms of the expressions with expectations and not the short-hand notation)

$$P_k(+) = P_k(-) + K_k E[(h_k(x_k) - E[h_k(x_k)])\tilde{x}_k(-)^T]\,, \quad (204)$$

or the books equation 6.1-18.

If, as the book suggests, we Taylor expand the nonlinear function $h_k(x_k)$ about the a priori state estimate $\hat{x}_k(-)$ as

$$h_k(x_k) = h_k(\hat{x}_k(-)) + H_k(\hat{x}_k(-))(x_k - \hat{x}_k(-)), \tag{205}$$

then using this we observe that the expectation of $h_k(x_k)$ denoted by $E[h_k(x_k)]$ is equal to $h_k(\hat{x}_k(-))$ and thus

$$h_k(x_k) - E[h_k(x_k)] = H_k(x_k - \hat{x}_k(-)) = -H_k\tilde{x}_k(-).$$

Thus some of the expectations in the formulas for K_k and $P_k(+)$ simplify as

$$E[(h_k(x_k) - E[h_k(x_k)])(h_k(x_k) - E[h_k(x_k)])^T] = H_k P_k(-) H_k^T,$$

and

$$E[\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T] = -P_k(-) H_k^T.$$

Using both of these observations we see that Equation 204 becomes

$$P_k(+) = P_k(-) - K_k H_k(\hat{x}_k(-)) P_k(-),$$

for the a posteriori covariance update equation for the extended Kalman filter and the books equation 6.1-21.

Notes on Higher-Order Filters

In this section we will attempt to derive many of the expressions for a second-order filter presented in the book. To begin we will perform a *second-order* Taylor expansion of $f(x(t), t)$ and $h_k(x_k)$ about $\hat{x}(t)$ and $\hat{x}_k(-)$ respectively as follows

$$\begin{aligned} f(x(t), t) &= f(\hat{x}(t), t) - F(\hat{x}(t), t)\tilde{x}(t) + \frac{1}{2}\partial^2(f, \tilde{x}(t)\tilde{x}(t)^T) + \cdots && (206) \\ h_k(x_k) &= h_k(\hat{x}_k(-)) - H(\hat{x}_k(-))\tilde{x}_k(-) + \frac{1}{2}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T) + \cdots, && (207) \end{aligned}$$

where for any matrix B the expression $\partial^2(f, B)$ is a *vector* with an ith component defined as

$$\partial_i^2(f, B) \equiv \text{trace}\left\{ \left[\frac{\partial^2 f_i}{\partial x_p \partial x_q} \right] B \right\}. \tag{208}$$

When these expressions are put into the state dynamic Equation 194 or $\dot{\hat{x}}(t) = \hat{f}(x(t), t)$ we get

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{f}(x(t), t) = E[f(x(t), t)] \\ &= E[f(\hat{x}(t), t) - F(\hat{x}, t)\tilde{x}(t) + \frac{1}{2}\partial^2(f, \tilde{x}\tilde{x}^T)] \\ &= f(\hat{x}(t), t) + \frac{1}{2}\partial^2(f, P(t)), \end{aligned}$$

since $E[F(\hat{x},t)\tilde{x}(t)] = F(\hat{x},t)E[\tilde{x}(t)] = 0$.

Next we want to put the second-order Taylor expansions above into Equation 195 or

$$\dot{P}(t) = \widehat{xf^T} - \hat{x}\hat{f}^T + \widehat{fx^T} - \hat{f}\hat{x}^T + Q\,.$$

Since we know how to evaluate $\hat{f}$, the expectation of f, lets first consider the term $\widehat{xf^T}$. Before we take the expectation, under the second order Taylor expansion of $f(x,t)$ we find xf^T is given by

$$xf^T = x\left(f(\hat{x})^T - \tilde{x}^T F(\hat{x})^T + \frac{1}{2}\partial^2(f, \tilde{x}\tilde{x}^T)^T\right)\,.$$

When we take expectations of this using the fact that $x = \hat{x} - \tilde{x}$ we get

$$\begin{aligned}
\widehat{xf^T} &= E[xf^T] = \hat{x}f(\hat{x})^T - E[(\hat{x}-\tilde{x})\tilde{x}^T]F(\hat{x})^T + \frac{1}{2}E[(\hat{x}-\tilde{x})\partial^2(f,\tilde{x}\tilde{x}^T)^T] \\
&= \hat{x}f(\hat{x})^T + P(t)F(\hat{x})^T + \frac{1}{2}\hat{x}\partial^2(f,P(t))^T - \frac{1}{2}E[\tilde{x}\partial^2(f,\tilde{x}\tilde{x}^T)^T]\,.
\end{aligned}$$

From which we see that we now need to evaluate the expectation of the matrix $\tilde{x}\partial^2\left(f,\tilde{x}\tilde{x}^T\right)^T$ which has an ijth component given by

$$(\tilde{x}\partial^2\left(f,\tilde{x}\tilde{x}^T\right)^T)_{ij} = \tilde{x}_i \text{trace}\left\{\left[\frac{\partial^2 f_j}{\partial x_p \partial x_q}\right]\tilde{x}\tilde{x}^T\right\}\,.$$

Now consider the matrix product

$$\left[\frac{\partial^2 f_j}{\partial x_p \partial x_q}\right]\tilde{x}\tilde{x}^T\,,$$

which has a pnth component given by

$$\sum_{q=1}^{n} \frac{\partial^2 f_j}{\partial x_p \partial x_q}\tilde{x}_q\tilde{x}_n\,,$$

thus the trace in the above expression becomes

$$\text{trace}\left\{\left[\frac{\partial^2 f_j}{\partial x_p \partial x_q}\right]\tilde{x}\tilde{x}^T\right\} = \sum_{p=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 f_j}{\partial x_p \partial x_q}\tilde{x}_q\tilde{x}_p\,. \tag{209}$$

When we multiply this by by $\tilde{x}_i$ we finally find

$$(\tilde{x}\partial^2\left(f,\tilde{x}\tilde{x}^T\right)^T)_{ij} = \sum_{p=1}^{n}\sum_{q=1}^{n} \frac{\partial^2 f_j}{\partial x_p \partial x_q}\tilde{x}_i\tilde{x}_q\tilde{x}_p\,. \tag{210}$$

When we take the expectation of this we get zero, assuming that $\tilde{x}_i$ are independent Gaussian random variables with zero expectation because then $E[\tilde{x}_i\tilde{x}_q\tilde{x}_p] = 0$. After all of this we finally arrive at

$$\widehat{xf^T} = \hat{x}f(\hat{x})^T + P(t)F(\hat{x})^T + \frac{1}{2}\hat{x}\partial^2(f,P(t))^T\,.$$

Now the expectation of f is given by $\hat{f} = f(\hat{x}) + \frac{1}{2}\partial^2(f, P(t))$ so we can now evaluate $\dot{P}(t)$ using Equation 195. We find

$$\begin{aligned}
\dot{P}(t) &= \hat{x}f(\hat{x})^T + P(t)F(\hat{x})^T + \frac{1}{2}\hat{x}\partial^2(f, P(t))^T - \hat{x}f(\hat{x})^T - \frac{1}{2}\hat{x}\partial^2(f, P(t))^T \\
&+ f(\hat{x})\hat{x}^T + F(\hat{x})P(t) + \frac{1}{2}\partial^2(f, P(t))\hat{x}^T - f(\hat{x})\hat{x}^T - \frac{1}{2}\partial^2(f, P(t))\hat{x}^T + Q \\
&= P(t)F(\hat{x})^T + F(\hat{x})P(t) + Q\,,
\end{aligned}$$

the desired expression in 6.1-26.

Next we evaluate Equation 200 or

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - \hat{h}_k(x_k)]\,.$$

From the given second-order Taylor series expansion for $h_k(x_k)$ we have the expectation of $h_k(x_k)$ denoted by $\hat{h}_k(x_k)$ given by

$$\hat{h}_k(x_k) = E[h_k(x_k)] = h_k(\hat{x}_k(-)) + \frac{1}{2}\partial^2(h_k, P_k(-))\,.$$

Thus we see that Equation 200 becomes

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - \hat{h}_k(\hat{x}_k(-)) - \frac{1}{2}\partial^2(h_k, P_k(-))]\,,$$

the desired equation in 6.1-26.

Next we simplify Equation 203 to derive the equation for K_k under the second-order Taylor series approximation. To do this we first evaluate

$$\begin{aligned}
h_k(x_k) - \hat{h}_k(x_k) &= h_k(\hat{x}_k(-)) - H(\hat{x}_k(-))\tilde{x}_k(-) + \frac{1}{2}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T) \\
&- h_k(\hat{x}_k(-)) - \frac{1}{2}\partial^2(h_k, P_k(-)) \\
&= -H(\hat{x}_k(-))\tilde{x}_k(-) + \frac{1}{2}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T) - \frac{1}{2}\partial^2(h_k, P_k(-))\,.
\end{aligned}$$

Using this expression we see that the product $\tilde{x}_k(-)(h_k(x_k) - \hat{h}_k(x_k))^T$ is then

$$-\tilde{x}_k(-)\tilde{x}_k(-)^T H(\hat{x}_k(-))^T + \frac{1}{2}\tilde{x}_k(-)\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)^T - \frac{1}{2}\tilde{x}_k(-)\partial^2(h_k, P_k(-))^T\,.$$

Taking expectation of this the third term vanishes and by using using Equation 210 the second term also vanishes. Thus we are left with

$$E\left\{\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T\right\} = -P_k(-)H(\hat{x}_k(-))^T\,. \tag{211}$$

Next we can now compute the inner product required in the expression for the matrix inverse portion of K_k or

$$[h_k(x_k) - \hat{h}_k(x_k)][h_k(x_k) - \hat{h}_k(x_k)]^T\,.$$

To do this lets define this product as T, and use the shorthand that $H \equiv H(\hat{x}_k(-))$. Then this product has nine terms and is given by

$$
\begin{aligned}
T &= H\tilde{x}_k(-)\tilde{x}_k(-)^T H^T - \frac{1}{2}H\tilde{x}_k(-)\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T) + \frac{1}{2}H\tilde{x}_k(-)\partial^2(h_k, P_k(-)) \\
&- \frac{1}{2}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)x_k(-)^T H^T \\
&+ \frac{1}{4}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)^T \\
&- \frac{1}{4}\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)\partial^2(h_k, P_k(-))^T \\
&- \frac{1}{2}\partial^2(h_k, P_k(-))\tilde{x}_k(-)^T H^T \\
&- \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)^T \\
&+ \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, P_k(-))^T .
\end{aligned}
$$

Taking the required expectation of this expression and recalling Equation 210 we see that the second, third, fourth, and seventh terms vanish and we get

$$
\begin{aligned}
E[T] &= HP_k(-)H^T + \frac{1}{4}E[\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)^T] \\
&- \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, P_k(-))^T - \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, P_k(-))^T \\
&+ \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, P_k(-))^T ,
\end{aligned}
$$

or canceling terms that

$$
\begin{aligned}
E[T] &= HP_k(-)H^T + \frac{1}{4}E[\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)\partial^2(h_k, \tilde{x}_k(-)\tilde{x}_k(-)^T)^T] \\
&- \frac{1}{4}\partial^2(h_k, P_k(-))\partial^2(h_k, P_k(-))^T . \qquad (212)
\end{aligned}
$$

In the above expression notice that the last two terms are equal to the definition of the matrix A_k in the book. Next lets evaluate the above expression for A_k. To begin with, for notational simplicity, we will drop the k subscripts and the $(-)$ notation by considering the second term in the above expression or

$$\partial^2(h, \tilde{x}\tilde{x}^T)\partial^2(h, \tilde{x}\tilde{x}^T)^T .$$

This matrix since it is an outer product has an ijth element given by

$$
\begin{aligned}
\partial^2(h, \tilde{x}\tilde{x}^T)_i\partial^2(h, \tilde{x}\tilde{x}^T)_j &= \left(\sum_p\sum_q \frac{\partial^2 h_i}{\partial x_p \partial x_q}\tilde{x}_q\tilde{x}_p\right)\left(\sum_m\sum_n \frac{\partial^2 h_j}{\partial x_m \partial x_n}\tilde{x}_n\tilde{x}_m\right) \\
&= \sum_{p,q,m,n} \frac{\partial^2 h_i}{\partial x_p \partial x_q}\frac{\partial^2 h_j}{\partial x_m \partial x_n}\tilde{x}_p\tilde{x}_q\tilde{x}_m\tilde{x}_n .
\end{aligned}
$$

Taking the expectation of this expression and using the fact that for Gaussian random variables we have

$$E[\tilde{x}_p\tilde{x}_q\tilde{x}_m\tilde{x}_n] = p_{pq}p_{mn} + p_{pm}p_{qn} + p_{pn}p_{qm} , \qquad (213)$$

we can write the above as

$$\sum_{p,q,m,n} \frac{\partial^2 h_i}{\partial x_p \partial x_q} \frac{\partial^2 h_j}{\partial x_m \partial x_n} \left[p_{pq} p_{mn} + p_{pm} p_{qn} + p_{pn} p_{qm} \right] .$$

At the same time the ijth element of the other term in the definition of A_k is

$$\partial^2(h,P)_i \partial^2(h,P)_j = \sum_{p,q,m,n} \frac{\partial^2 h_i}{\partial x_p \partial x_q} \frac{\partial^2 h_j}{\partial x_m \partial x_n} p_{pq} p_{mn} .$$

Thus these terms cancel and we are left with

$$A_{ij} = \sum_{p,q,m,n} \frac{\partial^2 h_i}{\partial x_p \partial x_q} \frac{\partial^2 h_j}{\partial x_m \partial x_n} \left[p_{pm} p_{qn} + p_{pn} p_{qm} \right] . \tag{214}$$

This combined with the other term in Equation 212 gives the books equation 6.1.28. Combining all of the expressions obtained thus far we finally end with

$$K_k = P_k(-) H_k(\hat{x}_k(-))^T \left[H_k(\hat{x}_k(-)) P_k(-) H_k(\hat{x}_k(-))^T + R_k + A_k \right]^{-1} ,$$

as we were to show.

In this section we have computed all of the needed expectations required to evaluate Equation 204. Using everything from earlier we find that

$$P_k(+) = P_k(-) - K_k H_k(\hat{x}_k(-)) P_k(-) ,$$

the same as in the book.

Notes on Statistical Linearization

In this section we see to approximate the nonlinear vector function $f(x)$ with the linear form

$$f(x) \approx a + N_f x , \tag{215}$$

where the vector a and the matrix N_f are determined by statistical linearization. To determine the specific form for a and N_f introduce the approximation error e as

$$e = f(x) - a - N_f x ,$$

and seek to minimized an objective function of a and N_f defined by

$$\begin{aligned} J &= E[e^T A e] \\ &= E[(f(x) - a - N_f x)^T A (f(x) - a - N_f x)] , \end{aligned} \tag{216}$$

where A is some symmetric positive semidefinite matrix. To find the minimum of J with respect to a, e take the derivative with respect to a, set the resulting expression equal to zero and then solve for a. Using Equation 312 to take the derivative we find

$$\frac{\partial J}{\partial a} = E[-2A(f(x) - a - N_f x)] = 0 .$$

When we solve for a we get

$$a = E[f(x)] - N_f E[x] = \hat{f} - N_f \hat{x} \,, \tag{217}$$

or the books equation 6.2-7. When we put this expression for a back into our approximate expression for $f(x)$ given by Equation 215 we get

$$f(x) \approx \hat{f} + N_f(x - \hat{x}) \,, \tag{218}$$

and for J given by Equation 216 we find that

$$J = E[(f - \hat{f} - N_f(x - \hat{x}))^T A (f - \hat{f} - N_f(x - \hat{x}))] \,,$$

and we need to find the minimum of the above expression as a function of N_f. Taking the N_f derivative of the above expression is made easier if we write J as

$$\begin{aligned} J &= E[(f - \hat{f})^T A (f - \hat{f})] - E[(A(f - \hat{f}))^T N_f (x - \hat{x})] \\ &- E[(x - \hat{x})^T N_f^T A (f - \hat{f})] + E[(x - \hat{x})^T N_f^T A N_f (f - \hat{f})] \,. \end{aligned}$$

Then using the product rule for the fourth term and Equations 319 and 320 to evaluate the matrix derivatives we see that

$$\begin{aligned} \frac{\partial J}{\partial N_f} &= -E[A(f - \hat{f})(x - \hat{x})^T] - E[A(f - \hat{f})(x - \hat{x})^T] \\ &+ E[A N_f (x - \hat{x})(x - \hat{x})^T] + E[A N_f (x - \hat{x})(x - \hat{x})^T] \\ &= -2AE[(f - \hat{f})(x - \hat{x})^T] + 2AN_f E[(x - \hat{x})(x - \hat{x})^T] \,. \end{aligned}$$

When we set this last expression equal to zero and solve for N_f we find

$$N_f = E[(f - \hat{f})(x - \hat{x})^T] E[(x - \hat{x})(x - \hat{x})^T]^{-1} \,. \tag{219}$$

Introducing $\tilde{x}$ as $\tilde{x} = \hat{x} - x$ we have $E[(x - \hat{x})(x - \hat{x})^T] = P$ and

$$\begin{aligned} E[(f - \hat{f})(x - \hat{x})^T] &= E[f x^T] - E[f \hat{x}^T] - E[\hat{f} x^T] + E[\hat{f} \hat{x}^T] \\ &= E[f x^T] - \hat{f} \hat{x}^T - \hat{f} \hat{x}^T + \hat{f} \hat{x}^T \\ &= \widehat{f x^T} - \hat{f} \hat{x}^T \,, \end{aligned}$$

so

$$N_f = (\widehat{f x^T} - \hat{f} \hat{x}^T) P^{-1} \,, \tag{220}$$

or the book's equation 6.2-9.

Notes on Computational Considerations

We now consider the evaluation of ξ_{ij} or

$$\xi_{ij} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_j f_i(x) p(x) dx \,, \tag{221}$$

when $f_i(x)$ only depends on a limited number, say x_s of state elements. To envision this case think of this this way. Given all of the $f_i(\cdot)$ functions, if they are functions of only a limited number of state variables then we can find an index (say j) such that our ith nonlinearity, $f_i(x)$, does not depend on the state x_j. Then if we let x_s be the state variables without the variable x_j then we can write the joint density $p(x)$ as x_j conditional on x_s as

$$p(x) = p(x_s, x_j) = p(x_j|x_s)p(x_s)\,.$$

With this we can then write the expression for ξ_{ij} as

$$\begin{aligned}
\xi_{ij} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_j f_i(x_s) p(x_j|x_s) p(x_s) dx_s dx_i \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i(x_s) p(x_s) \int_{-\infty}^{\infty} x_j p(x_j|x_s) dx_j dx_s\,.
\end{aligned} \tag{222}$$

We now need to evaluate $\int_{-\infty}^{\infty} x_j p(x_j|x_s)dx_j$. Since x_s and x_j are jointly normal this integral is in fact the conditional *mean* of x_j *given* the vector x_s. Since x_j and x_s are jointly Gaussian this expression has the form given by

$$E[x_j|x_s] = \int_{-\infty}^{\infty} x_j p(x_j|x_s) dx_j = \hat{x}_j + p_{js}^T \Sigma_{ss}^{-1}(x_s - \hat{x}_s)\,.$$

see [3]. Where we have defined

$$\begin{aligned}
p_{js} &= E[(x_j - \hat{x}_j)(x_s - \hat{x}_s)] \\
\Sigma_{ss} &= E[(x_s - \hat{x}_s)(x_s - \hat{x}_s)^T]\,.
\end{aligned}$$

Note that p_{js} is a column vector and contains the elements in the jth column/row of $P = E[\tilde{x}\tilde{x}^T]$ excluding the jth diagonal element, while Σ_{ss} is a matrix. When we put these two into Equation 222 we get

$$\begin{aligned}
\xi_{ij} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i(x_s) p(x_s) \left\{\hat{x}_j + p_{js}^T \Sigma_{ss}^{-1}(x_s - \hat{x}_s)\right\} dx_s \\
&= E\left[\left\{\hat{x}_j + p_{js}^T \Sigma_{ss}^{-1}(x_s - \hat{x}_s)\right\} f_i(x_s)\right] \\
&= \hat{x}_j E\left[f_i(x_s)\right] + p_{js}^T \Sigma_{ss}^{-1} E\left[(x_s - \hat{x}_s) f_i(x_s)\right]\,,
\end{aligned}$$

since $\hat{x}_j$, p_{js} and Σ_{ss} are all constants with respect to the expectation over x_s. Now since the expression $p_{js}^T \Sigma_{ss}^{-1} E[(x_s - \hat{x}_s) f_i(x_s)]$ is a scalar we can take its transpose and not change its value. Doing this gives

$$\xi_{ij} = \hat{f}_i \hat{x}_j + n_{si}^T p_{js}\,, \tag{223}$$

where we have defined n_{si} as

$$\begin{aligned}
n_{si}^T &= E[f_i(x_s)(x_s - \hat{x}_s)^T][\Sigma_{ss}^{-1}]^T \\
&= E[f_i(x_s)(x_s - \hat{x}_s)^T] E[(x_s - \hat{x}_s)(x_s - \hat{x}_s)^T]^{-1}\,,
\end{aligned} \tag{224}$$

which is the books equation 6.2-36. Note that I think the book is missing a transpose on its definition of n_{si}.

Notes on Direct Statistical Analysis of Nonlinear Systems (CADET)

We approximate $f(x)$ with

$$f(x) \approx f_a(x) = N_m m + N_r r \,, \tag{225}$$

thus the error e is given by

$$e = f(x) - f_a(x) = f(x) - N_m m - N_r r \,.$$

Thus ee^T is given by

$$\begin{aligned} ee^T &= (f(x) - N_m m - N_r r)(f(x) - N_m m - N_r r)^T \\ &= f f^T - f m^T N_m^T - f r^T N_r^T \\ &- N_m m f^T + N_m m m^T N_m^T + N_m m r^T N_r^T \\ &- N_r r f^T + N_r r m^T N_m^T + N_r r r^T N_r^T \,. \end{aligned}$$

Taking the expectation of this and using the fact that $E[rm^T] = E[mr^T] = 0$ and that m is a constant gives

$$\begin{aligned} E[ee^T] &= E[f f^T] - E[f] m^T N_m^T - E[f r^T] N_r^T \\ &- N_m m E[f^T] + N_m m m^T N_m^T \\ &- N_r E[r f^T] + N_r E[r r^T] N_r^T \,. \end{aligned}$$

We next want to take the trace of this expression and use it to evaluate the N_m and N_r derivatives needed to find a minimum of the objective function $J = \text{trace}(E[ee^T])$. The derivative expressions we need are

$$\frac{\partial}{\partial N_m} \text{trace}(E[ee^T]) = 0 \quad \text{and} \quad \frac{\partial}{\partial N_r} \text{trace}(E[ee^T]) = 0 \,.$$

To evaluate these derivatives we will use Equations 313, 314, 315, 316, 317, and 318. For the derivative of N_m we find

$$\frac{\partial}{\partial N_m} \text{trace}(E[ee^T]) = -E[f] m^T - E[f] m^T + 2 N_m m m^T = 0 \,,$$

or that N_m must satisfy

$$N_m m m^T = E[f] m^T \,, \tag{226}$$

which is the books equation 6.4-4. For the derivative of N_r we find

$$\frac{\partial}{\partial N_r} \text{trace}(E[ee^T]) = -E[f r^T] - E[f r^T] + N_r (2 E[r r^T]) = 0 \,,$$

or that N_r must satisfy

$$N_r E[r r^T] = E[f r^T] \,, \tag{227}$$

which is the books equation 6.4-5.

Now our dynamic equation is given by $\dot{x} = f(x,t) + w$ which under the assumption that $x = m + r$ and Equation 225 becomes

$$\dot{m} + \dot{r} = N_m m + N_r r + w\,.$$

When $w \sim N(b, Q)$ we can introduce the variable u as $u = w - b$ and get

$$\dot{m} + \dot{r} = N_m m + N_r r + b + u\,.$$

If we assume that we can decouple into two equations the expressions for the mean from the residual expressions we get the following

$$\dot{m} = N_m m + b \tag{228}$$

$$\dot{r} = N_r r + u\,, \tag{229}$$

which are the book's equations 6.4-9. From Equation 229 we can derive the differential equation for $S \equiv E[rr^T]$ to find

$$\dot{S} = N_r(m,S)S + SN_r^T(m,S) + Q\,, \tag{230}$$

since $w \sim N(b, Q)$ so $u \equiv w - b \sim N(0, Q)$.

If our system is linear $f(x) = Fx = Fm + Fr$ we can evaluate the Equations 226 and 227. For Equation 227 we find that $E[fr^T]$ is given by

$$E[fr^T] = E[Fmr^T + Frr^T] = FE[rr^T] = FS(t)\,.$$

So N_r becomes

$$N_r = E[fr^T]S^{-1} = FSS^{-1} = F\,.$$

In the same way we find that Equation 226 becomes

$$N_m m = E[f(x)] = Fm \quad \text{so} \quad N_m = F\,,$$

also.

Next we want to prove that when r is Gaussian we have the identity

$$N_r(m,S) = \frac{d}{dm}E[f(x)]\,. \tag{231}$$

To do this we note that if r is Gaussian with a covariance matrix S then $E[f(x)]$ can be written

$$E[f(x)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(m+r)N(r;0,S)dr\,,$$

where to simplify notation we have introduced the notation

$$N(r;\mu,\Sigma) \equiv \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(r-\mu)^T\Sigma^{-1}(r-\mu)\right\}\,, \tag{232}$$

to represent the probability density function of a n dimensional Gaussian random variable. Using the above expression for $E[f(x)]$ we see that the m derivative of this is given by

$$\frac{\partial E[f(x)]}{\partial m} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial f(m+r)}{\partial m} N(r;0,S)dr\,.$$

Note that this m derivative is really also an r derivative as

$$\frac{\partial f(m+r)}{\partial m} = \frac{\partial f(m+r)}{\partial r},$$

and thus we need to evaluate

$$\frac{\partial E[f(x)]}{\partial m} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial f(m+r)}{\partial r} N(r;0,S) dr\,.$$

In the above integration the expression $\frac{\partial f(m+r)}{\partial r}$ is a *matrix* with an ijth component given by $\frac{\partial f_i(m+r)}{\partial r_j}$. If we then consider just the integral over r_j (denoted by I_j) in the above expression then by integration by parts we have

$$I_j = \int_{r_j=-\infty}^{\infty} \frac{\partial f_i(m+r)}{\partial r_j} N(r;0,S) dr_j = 0 - \int_{r_j=-\infty}^{\infty} f_i(m+r) \frac{\partial}{\partial r_j} N(r;0,S) dr_j\,.$$

Evaluating the r_j derivative above we see that

$$\begin{aligned}
\frac{\partial}{\partial r_j} N(r;0,S) &= \frac{\partial}{\partial r_j}\left(\frac{1}{(2\pi)^{n/2}|S|^{1/2}} \exp\left\{-\frac{1}{2} r^T S^{-1} r\right\}\right) \\
&= \frac{1}{(2\pi)^{n/2}|S|^{1/2}} \exp\left\{-\frac{1}{2} r^T S^{-1} r\right\} \left(-\frac{1}{2}(e_j^T S^{-1} r + r^T S^{-1} e_j)\right) \\
&= -(r^T S^{-1} e_j) N(r;0,S)\,.
\end{aligned}$$

Thus the vector derivative of $N(r;0,S)$ is given by

$$\frac{\partial}{\partial r} N(r;0,S) = -N(r;0,S) S^{-1} r \tag{233}$$

Using these results we see that

$$\begin{aligned}
\frac{\partial E[f_i(x)]}{\partial m} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial f_i(m+r)}{\partial r} N(r;0,S) dr \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_i(m+r) r N(r;0,S) dr S^{-1}\,.
\end{aligned}$$

When we then consider this expression for all values of i we see that

$$\frac{\partial E[f(x)]}{\partial m} = E[f(x) r^T] S^{-1} = N_r(m,S)\,,$$

as we were to show.

In the special case where f is a scalar function and we assume that the random perturbation r is Gaussian than taking $n_r \equiv N_r(m,S)$ we find

$$\begin{aligned}
n_r &= E[f(x) r^T] S^{-1} = \frac{1}{\sigma^2}\left[\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} f(m+r) r e^{-r^2/2\sigma^2} dr\right] \\
&= \frac{1}{\sqrt{2\pi}\sigma^3} \int_{-\infty}^{\infty} f(m+r) r e^{-r^2/2\sigma^2} dr\,.
\end{aligned} \tag{234}$$

At the same time we find the scalar version of the equation for N_m or $N_m(m,S)m = E[f]$ becomes

$$n_m = \frac{1}{\sqrt{2\pi}\sigma m} \int_{-\infty}^{\infty} f(m+r) e^{-r^2/2\sigma^2} dr\,. \tag{235}$$

Problem Solutions

Problem 6-1 (a density for x)

The given expression for the probability density function (p.d.f) for x is a special case of distribution known as the *gamma* distribution. If X is given by a gamma distribution then it has a p.d.f given by

$$f(x|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}\,. \tag{236}$$

From which we see that the books expression can be obtained by taking $\alpha = 2$ and $\beta = \lambda$. We now derive several properties of the gamma distribution and then answer the requested questions by making the substitution $\alpha = 2$ and $\beta = \lambda$ in the resulting expressions.

The characteristic function for a gamma random variable is given by

$$\begin{aligned}\zeta(t) &= E(e^{itX}) = \int_{x=0}^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{itx} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{x=0}^\infty x^{\alpha-1} e^{-(\beta-it)x} dx\,.\end{aligned}$$

To evaluate this integral let $v = (\beta - it)x$ so that $x = \frac{v}{\beta - it}$ and $dv = (\beta - it)dx$ and we get

$$\zeta(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{(\beta-it)^{\alpha-1}} \int_{v=0}^\infty v^{\alpha-1} e^{-v} \frac{dv}{\beta - it}\,.$$

If we recall the definition of the Gamma function

$$\Gamma(\alpha) \equiv \int_{v=0}^\infty v^{\alpha-1} e^{-v} dv\,, \tag{237}$$

we see that the above integral becomes

$$\begin{aligned}\zeta(t) &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\beta-it)^\alpha} = \left(\frac{\beta}{\beta-it}\right)^\alpha \\ &= \left(1 - \frac{it}{\beta}\right)^{-\alpha}\,.\end{aligned} \tag{238}$$

Using this expression we could compute $E(X)$ and $E(X^2)$ via derivatives. Alternatively we could compute these expectations directly as follows

$$\begin{aligned}E(X) &= \int_{x=0}^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{v=0}^\infty \frac{v^\alpha}{\beta^\alpha} e^{-v} \frac{dv}{\beta} \\ &= \frac{1}{\beta\Gamma(\alpha)} \int_{v=0}^\infty v^\alpha e^{-v} dv = \frac{\Gamma(\alpha+1)}{\beta\Gamma(\alpha)} = \frac{\alpha}{\beta}\,,\end{aligned} \tag{239}$$

when we make the substitution $v = \beta x$. Next we find $E(X^2)$ given by

$$\begin{aligned}E(X^2) &= \int_{x=0}^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\beta^{\alpha+1}} \frac{1}{\beta} \int_{v=0}^\infty v^{\alpha+1} e^{-v} dv \\ &= \frac{1}{\beta^2 \Gamma(\alpha)} \Gamma(\alpha+2) = \frac{(\alpha+1)\alpha}{\beta^2}\,.\end{aligned} \tag{240}$$

Thus the variance of a gamma random variable is given by

$$\mathrm{Var}(X) = \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}\,. \tag{241}$$

Part (a): If we take $\alpha = 2$ and $\beta = \lambda$ in the expression from Equation 239 we get

$$E(X) = \frac{2}{\lambda}\,.$$

Part (b): For this part to find the maximum value of $f(x|\alpha,\beta)$ when X is a gamma random variable we take the x derivative of f, set the result equal to zero, and then solve for x. We find

$$\frac{df(x|\alpha,\beta)}{dx} = \frac{\beta^\alpha}{\Gamma(\alpha)}\left((\alpha-1)x^{\alpha-2}e^{-\beta x} - \beta x^{\alpha-1}e^{-\beta x}\right) = 0\,.$$

When we solve for x we find

$$x = \frac{\alpha-1}{\beta}\,.$$

If we take $\alpha = 2$ and $\beta = \lambda$ in the above expression we get

$$x = \frac{1}{\lambda}\,.$$

Part (c): The expectation of y is given by $E[Y] = \int y p(y) dy$, while the value of y that maximizes $p(y)$ is given by the solution to $p'(y) = 0$. If these two points are are the same then we must have

$$p'(E[y]) = 0\,.$$

Problem 6-2 (the non-linear expectation reduces to the linear)

For this problem we use the "E" notation for expectation rather than the books "hat" notation. In symbols $E[X] \equiv \widehat{X}$. The books equation 6.1-5 is

$$\dot{P}(t) = E[xf^T] - E[x]E[f]^T + E[fx^T] - E[f]E[x]^T + Q\,. \tag{242}$$

If our function f is in fact a linear function $f(x) = Fx$ then $E[f] = FE[x]$ where we are assuming that F is not state dependent. Next $xf^T = xx^TF^T$ under this linear assumption, so taking expectations we have

$$E[xf^T] = E[xx^T]F^T\,.$$

Since $P(t)$ by definition can be written as

$$P(t) = E[(x-E[x])(x-E[x])^T] = E[xx^T] - E[x]E[x]^T\,,$$

we have that

$$E[xx^T] = P(t) + E[x]E[x]^T\,.$$

and we see that $E[xf^T]$ is given by

$$E[xf^T] = FP(t) + E[x]E[x]^T F^T .$$

In the same way since fx^T is just the transpose of xf^T we see that

$$E[fx^T] = P(t)F^T + FE[x]E[x]^T .$$

Thus the differential equation for $P(t)$ becomes

$$\begin{aligned}\dot{P}(t) &= PF^T + E[x]E[x]^T F^T - E[x]E[x]^T F^T \\ &+ FP + FE[x]E[x]^T - FE[x]E[x]^T + Q \\ &= PF^T + FP + Q ,\end{aligned}$$

the expression we were to show.

Problem 6-3 (filtering x_k using a quadratic expression for the measurements)

Part (a): We will estimate $x(t_k)$ after observing the measurement z_k using and expression quadratic in z_k or

$$\hat{x}_k(+) = a_k + b_k z_k + c_k z_k^2 .$$

Since $z_k = h(x_k) + v_k$ in terms of $h(\cdot)$ the above becomes

$$\hat{x}_k(+) = a_k + b_k h(x_k) + b_k v_k + c_k h(x_k)^2 + 2c_k h(x_k) v_k + c_k v_k^2 . \tag{243}$$

To have the above expression for $\hat{x}_k(+)$ be an unbiased estimator of x_k we require that $E[\hat{x}_k(+)] = E[x_k] = \hat{x}_k(-)$. Using this with $E[v_k] = 0$ and $E[v_k^2] = r$ when we take the expectation of Equation 243 we get

$$a_k + b_k E[h(x_k)] + c_k E[h(x_k)^2] + c_k r = \hat{x}_k(-) . \tag{244}$$

This is the same expression we were asked to derive.

Problem 6-4 (deriving the linearized Kalman filter)

For this problem we derive the expressions for a *linearized* Kalman filter that are summarized in the book. To begin we consider a first-order Taylor expansion of $f(x(t), t)$ and $h_k(x_k)$ about a *known* trajectory $\bar{x}(t)$ as follows

$$f(x(t), t) = f(\bar{x}(t), t) + \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}(t)} (x - \bar{x}) + \cdots \tag{245}$$

$$h_k(x_k) = h_k(\bar{x}(t_k)) + \left.\frac{\partial h_k}{\partial x}\right|_{x=\bar{x}(t_k)} (x_k - \bar{x}(t_k)) + \ . \tag{246}$$

To simplify notation we will define the matrices F and H_k to be

$$\begin{aligned} F &= F(\bar{x}(t), t) = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}(t)} \\ H_k &= H_k(\bar{x}(t_k), t_k) = \left.\frac{\partial h}{\partial x}\right|_{x=\bar{x}(t_k)} . \end{aligned}$$

When the expression for $f(x(t), t)$ above it put into the state dynamic Equation 194 or $\dot{\hat{x}}(t) = \hat{f}(x(t), t)$ we get

$$\dot{\hat{x}}(t) = \hat{f}(x(t), t) = E[f(x(t), t)] = f(\bar{x}(t), t) + F(\bar{x}(t), t)(\hat{x} - \bar{x}) .$$

Next we want to put our Taylor expansions above into Equation 195 or

$$\dot{P}(t) = \widehat{xf^T} - \hat{x}\hat{f}^T + \widehat{fx^T} - \hat{f}\hat{x}^T + Q .$$

Since we know how to evaluate $\hat{f}$, the expectation of f, lets first consider the term $\widehat{xf^T}$. Before we take the expectation, under the Taylor expansion above $f(x, t)$ we find xf^T is given by

$$xf^T = xf(\bar{x}(t), t)^T + x(x - \bar{x})^T F(\bar{x}(t), t)^T .$$

When we use the fact that $x = \hat{x} - \tilde{x}$ we get xf^T equal to

$$\begin{aligned} xf^T &= (\hat{x} + \tilde{x})f(\bar{x}(t), t)^T + (\hat{x} + \tilde{x})(\hat{x} + \tilde{x} - \bar{x})^T F(\bar{x}(t), t)^T \\ &= (\hat{x} + \tilde{x})f(\bar{x}(t), t)^T + \hat{x}(\hat{x} + \tilde{x} - \bar{x})^T F(\bar{x}(t), t)^T + \tilde{x}(\hat{x} + \tilde{x} - \bar{x})^T F(\bar{x}(t), t)^T \end{aligned}$$

From which we can now take the expectation to find that

$$\widehat{xf^T} = \hat{x}f(\bar{x}(t), t)^T + \hat{x}(\hat{x} - \bar{x})^T F(\bar{x}(t), t)^T + P(t)F(\bar{x}(t), t)^T .$$

We can now evaluate $\dot{P}(t)$ using Equation 195. We find

$$\begin{aligned} \dot{P}(t) &= \hat{x}f(\bar{x}, t)^T + \hat{x}(\hat{x} - \bar{x})^T + P(t)F(\bar{x}, t)^T \\ &- \bar{x}f(\bar{x}, t)^T - \hat{x}(\hat{x} - \bar{x})^T F(\bar{x}, t)^T \\ &+ f(\bar{x}, t)\bar{x}^T + F(\bar{x}, t)(\hat{x} - \bar{x})\hat{x}^T + F(\bar{x}, t)P(t) \\ &- f(\bar{x}, t)\bar{x}^T - F(\bar{x}, t)(\hat{x} - \bar{x})\hat{x}^T + Q \\ &= P(t)F(\bar{x}, t)^T + F(\bar{x}, t)P(t) + Q , \end{aligned}$$

the desired expression.

Next we evaluate Equation 200 or

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - \hat{h}_k(x_k)] .$$

From the given Taylor series expansion for $h_k(x_k)$ we have the expectation of $h_k(x_k)$ denoted by $\hat{h}_k(x_k)$ given by

$$\hat{h}_k(x_k) = E[h_k(x_k)] = h_k(\bar{x}(t_k)) + H(\bar{x}(t_k), t_k)(\hat{x}_k(-) - \bar{x}(t_k)) .$$

Thus we see that Equation 200 becomes

$$\hat{x}_k(+) = \hat{x}_k(-) + K_k[z_k - h_k(\bar{x}(t_k)) - H(\bar{x}(t_k), t_k)(\hat{x}_k(-) - \bar{x}(t_k))] ,$$

the desired equation.

Next we simplify Equation 203 to derive the equation for K_k under the linearization above. To do this we first evaluate

$$\begin{aligned} h_k(x_k) - \hat{h}_k(x_k) &= h_k(\bar{x}(t_k)) + H(\bar{x}(t_k))(x_k - \bar{x}_k) \\ &- h_k(\bar{x}(t_k)) - H(\bar{x}(t_k))(\hat{x}_k - \bar{x}_k) \\ &= H(\bar{x}(t_k))(x_k - \hat{x}_k) \\ &= -H(\bar{x}(t_k))\tilde{x}_k(-) . \end{aligned}$$

Using this expression we see that the product $\tilde{x}_k(-)(h_k(x_k) - \hat{h}_k(x_k))^T$ is then

$$-\tilde{x}_k(-)\tilde{x}_k(-)^T H(\bar{x}(t_k))^T .$$

Taking expectation of this we are left with

$$E\left\{\tilde{x}_k(-)(h_k(x_k) - E[h_k(x_k)])^T\right\} = -P_k(-)H(\bar{x}(t_k))^T .$$

Next we can now compute the inner product required in the expression for the matrix inverse portion of K_k or $[h_k(x_k) - \hat{h}_k(x_k)][h_k(x_k) - \hat{h}_k(x_k)]^T$. From the above expression for $h_k(x_k) - \hat{h}_k(x_k)$ we see that this is given by

$$E[[h_k(x_k) - \hat{h}_k(x_k)][h_k(x_k) - \hat{h}_k(x_k)]^T] = H(\bar{x}(t_k))P_k(-)H(\bar{x}(t_k))^T .$$

Combining all of the expressions obtained thus far we finally end with

$$K_k = P_k(-)H(\bar{x}(t_k))^T \left[H(\bar{x}(t_k))P_k(-)H(\bar{x}(t_k))^T + R_k\right]^{-1} ,$$

as we were to show.

In this section we have computed all of the needed expectations required to evaluate Equation 204. Using everything from earlier we find that

$$P_k(+) = P_k(-) - K_k H(\bar{x}(t_k))P_k(-) ,$$

the same as in the book.

Problem 6-6 (a density for x)

To begin we will square the given expression to get several terms. We find

$$\begin{aligned} (f(x) - a - bx - cx^2)^2 &= a^2 - 2af(x) + f(x)^2 \\ &+ 2abx - 2bf(x)x + b^2x^2 \\ &+ 2acx^2 - 2cf(x)x^2 + 2bcx^3 + c^2x^4 . \end{aligned}$$

We now take the expectation of the above expression. We find

$$
\begin{aligned}
E[(f(x) - a - bx - cx^2)^2] &= a^2 - 2aE[f(x)] + E[f(x)^2] \\
&+ 2abE[x] - 2bE[f(x)x] + b^2 E[x^2] \\
&+ 2acE[x^2] - 2cE[f(x)x^2] + 2bcE[x^3] + c^2 E[x^4] \,.
\end{aligned}
$$

To find the values of a, b, and c such that the above expression is a minimum we take the derivative of $E[(f(x) - a - bx - cx^2)^2]$ with respect to each of these values, set the resulting expressions equal to zero and solve for them. We find

$$
\begin{aligned}
a &= \hat{f} - b\hat{x} - c\widehat{x^2} \\
b &= \frac{(-\hat{f}\widehat{x^2}\widehat{x^3} + \widehat{fx^2}(-\hat{x}\widehat{x^2} + \widehat{x^3}) + \widehat{fx}(\widehat{x^2}^2 - \widehat{x^4}) + \hat{f}\hat{x}\widehat{x^4})}{(\widehat{x^2}^3 + \widehat{x^3}^2 + \hat{x}^2\widehat{x^4} - \widehat{x^2}(2\hat{x}\widehat{x^3} + \widehat{x^4}))} \\
c &= \frac{\widehat{fx^2}(\hat{x}^2 - \widehat{x^2}) + \widehat{fx}(-\hat{x}\widehat{x^2} + \widehat{x^3}) + \hat{f}(\widehat{x^2}^2 - \hat{x}\widehat{x^3})}{(\widehat{x^2}^3 + \widehat{x^3}^2 + \hat{x}^2\widehat{x^4} - \widehat{x^2}(2\hat{x}\widehat{x^3} + \widehat{x^4}))} \,.
\end{aligned}
$$

These calculations are done in the Mathematica file `chap_6_prob_6.nb`. Now to try to make this expressions look more like the ones in the book we could transform these "raw moments" i.e. the expressions $E[x^i]$ into central moments m_i defined by $m_i = E[(x - \hat{x})^i]$. This can be done with the "inverse binomial transform" (see [10]) or

$$
E[x^n] = \sum_{k=0}^{n} \binom{n}{k} m_k \hat{x}^{n-k} \,.
$$

Using this expression we have

$$
\begin{aligned}
E[x] &= \hat{x} \\
E[x^2] &= m_2 + \hat{x}^2 \\
E[x^3] &= m_3 + 3m_2\hat{x} + \hat{x}^3 \\
E[x^4] &= m_4 + 4m_3\hat{x} + 6m_2\hat{x}^2 + \hat{x}^4 \,.
\end{aligned}
$$

Warning: When we do this however the results for a, b, and c don't seem to match the book's results. If anyone sees an error with what I've done please contact me.

Problem 6-7 (evaluating $E[x^n]$)

Warning: While this seems like a simple problem, I was unable to show the desired result for n even. If anyone sees anything wrong with what I've done or has an alternative way to solve this problem please contact me.

Recall (see [3]) that if a given random variable X has a characteristic function $\zeta(t)$ and the expectation $E[x^n]$ exists for some positive integer n then it can be evaluated from

$$
E[X^n] = i^{-n}\zeta^{(n)}(0) \,. \tag{247}
$$

When X is a Gaussian random variable with mean μ and variance σ^2 then it has a p.d.f given by $f(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$. In what follows we will try to evaluate $\zeta(t)$ directly. We have

$$\zeta(t) = E(e^{itX}) = \frac{1}{(2\pi)^{1/2}\sigma} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx\,.$$

The argument of the exponential in the above expression is given by

$$\begin{aligned}
& -\frac{1}{2\sigma^2}\left[x^2 - 2\mu x + \mu^2 - 2i\sigma^2 tx\right] \\
= & -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu + i\sigma^2 t)x + \mu^2\right] \\
= & -\frac{1}{2\sigma^2}\left[x^2 - 2(\mu + i\sigma^2 t)x + (\mu + i\sigma^2 t)^2 - (\mu + i\sigma^2 t)^2 + \mu^2\right] \\
= & -\frac{1}{2\sigma^2}(x - (\mu + i\sigma^2 t))^2 + \frac{(\mu + i\sigma^2 t)^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} \\
= & -\frac{1}{2\sigma^2}(x - (\mu + i\sigma^2 t))^2 + \frac{\mu^2 + 2\mu\sigma^2 ti - \sigma^4 t^2 - \mu^2}{2\sigma^2} \\
= & -\frac{1}{2\sigma^2}(x - (\mu + i\sigma^2 t))^2 + \frac{2\mu\sigma^2 ti - \sigma^4 t^2}{2\sigma^2}\,.
\end{aligned}$$

Thus the integral expression we seek to evaluate looks like

$$\zeta(t) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{it\mu - \frac{\sigma^2 t^2}{2}\right\} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+i\sigma^2 t))^2} dx\,.$$

To evaluate this let $v = x - (\mu + i\sigma^2 t)$ so that $dx = dv$ and the integral above becomes

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}v^2} dv = (2\pi)^{1/2}\sigma\,.$$

Thus the characteristic function for a Gaussian random variable is given by

$$\zeta(t) = \exp\left\{it\mu - \frac{\sigma^2 t^2}{2}\right\}\,, \tag{248}$$

as we were to show. If our Gaussian has zero mean $\mu = 0$ and unit variance then $\sigma^2 = 1$ and the above expression simplifies to

$$\zeta(t) = \exp\left\{-\frac{t^2}{2}\right\}\,,$$

We can use this result to compute the expectation of X^n when X has a unit variance. If X does not have a unit variance then derivation below changes slightly but is effectively the same. Thus we will evaluate $E[X^n]$ in the case where X has unit variance. To determine this expectations requires that we evaluate derivatives of $\zeta(t)$. We find

$$
\begin{aligned}
\zeta^{(0)}(t) &= e^{-\frac{t^2}{2}} \\
\zeta^{(1)}(t) &= e^{-\frac{t^2}{2}}(-t) = -te^{-\frac{t^2}{2}} \\
\zeta^{(2)}(t) &= -e^{-\frac{t^2}{2}} + t^2 e^{-\frac{t^2}{2}} = (-1+t^2)e^{-\frac{t^2}{2}} \\
\zeta^{(3)}(t) &= 2te^{-\frac{t^2}{2}} + (-1+t^2)(-t)e^{-\frac{t^2}{2}} = (3t-t^3)e^{-\frac{t^2}{2}} \\
\zeta^{(4)}(t) &= (3-6t^2+t^4)e^{-\frac{t^2}{2}} \\
\zeta^{(5)}(t) &= (-15t+10t^3-t^5)e^{-\frac{t^2}{2}} \\
\zeta^{(6)}(t) &= (-15+45t^2-15t^4+t^6)e^{-\frac{t^2}{2}} .
\end{aligned}
$$

Some of these calculations are done in the Mathematica file `chap_6_prob_7.nb`. By performing these derivatives we see that the form of $\zeta^{(n)}(t)$ looks like it takes the form

$$
\zeta^{(n)}(t) = \phi_n(t) e^{-\frac{t^2}{2}} , \tag{249}
$$

where $\phi_n(t)$ is a nth degree polynomial. In fact for n odd the polynomial $\phi_n(t)$ has only odd powers of t (with no intercept term) and for n even it looks like $\phi_n(t)$ has only even powers of t. Thus with the above expression for $\zeta^{(n)}(t)$ we see that to evaluate expectations of powers of X we have

$$
E[X^n] = i^{-n}\zeta^{(n)}(0) = i^{-n}\phi_n(0) ,
$$

thus we need to be able to evaluate the polynomial $\phi_n(t)$ at $t=0$.

From the above expression for $\zeta^{(n)}(t)$ in Equation 249 we see that using the product rule $\zeta^{(n+1)}(t)$ is given by

$$
\zeta^{(n+1)}(t) = \phi_n'(t)e^{-\frac{t^2}{2}} - \phi_n(t)te^{-\frac{t^2}{2}} = (\phi_n'(t) - t\phi_n(t))e^{-\frac{t^2}{2}} ,
$$

and that $\zeta^{(n+2)}(t)$ is given by

$$
\begin{aligned}
\zeta^{(n+2)}(t) &= [\phi_n''(t) - \phi_n(t) - t\phi_n'(t) - t(\phi_n'(t) - t\phi_n(t))]\, e^{-\frac{t^2}{2}} \\
&= \left[\phi_n''(t) - 2t\phi_n'(t) + (-1+t^2)\phi_n(t)\right] e^{-\frac{t^2}{2}} .
\end{aligned}
$$

Thus the recursive relationship between the coefficient polynomials $\phi_{n+2}(t)$ and the one two previous $\phi_n(t)$ is

$$
\phi_{n+2}(t) = \phi_n''(t) - 2t\phi_n'(t) + (-1+t^2)\phi_n(t) . \tag{250}
$$

Given the examples of $\phi_1(t)$, $\phi_3(t)$, and $\phi_5(t)$ presented at the beginning of this problem lets form the induction hypothesis that when n is odd then $\phi_n(t)$ is an odd polynomial that is

$$
\phi_{2n+1}(t) = \sum_{k=0}^{n} a_{2k+1} t^{2k+1} . \tag{251}
$$

This statement is true for the polynomials $\phi_1(t)$, $\phi_3(t)$, and $\phi_5(t)$ above. If we assume that $\phi_{2n+1}(t)$ has the form given by Equation 251 then we see from Equation 250 that $\phi_{2n+3}(t)$ must also have a form given by Equation 251. This is because each of the terms in

Equation 251 is odd polynomial and so the sum is another odd polynomial. In this case we see that $\phi^{2n+1}(t) = 0$ and by Equation 247 all odd powers of X have zero expectation.

Given the examples of $\phi_2(t)$, $\phi_4(t)$, and $\phi_6(t)$ presented at the beginning of this problem lets form the induction hypothesis that when n is even then $\phi_n(t)$ is an even polynomial that is

$$\phi_{2n}(t) = \sum_{k=0}^{n} a_{2k} t^{2k} . \tag{252}$$

Again using Equation 250 we see that if $\phi_{2n}(t)$ has this form then $\phi_{2n+2}(t)$ will also have this form.

At this point I would like to derive a recursive expression for $\phi_{2n}(0)$ since that would enable me to evaluate the desired expectations. I was unable to do this however. If anyone sees a method to do this please let me know.

Problem 6-8 (deriving the expression ξ_{ij})

See the notes on Page 125 for this derivation.

Problem 6-9 (deriving the expressions for $N_m(m, S)$ and $N_r(m, S)$)

See the notes on Page 127 for the requested derivation.

Problem 6-10 (show that $E[ex^T] = 0$)

Since $e = f(x) - f_a(x) = f(x) - N_m m - N_r r$ we find

$$\begin{aligned} E[ex^T] &= E[(f - N_m m - N_r r)x^T] \\ &= E[fx^T] - N_m m E[x^T] - N_r E[rx^T] . \end{aligned}$$

Since $x = m + r$ and m is a constant this becomes

$$E[f]m^T + E[fr^T] - N_m m m^T - N_r E[rr^T] .$$

Using Equations 226 and 227 we see that all terms cancel and we end with $E[ex^T] = 0$ as we were to show.

Problem 6-11 (evaluating some multiple-input describing function gains)

For this problem $f(x) = x(1 + x^2)$ so that using Equation 234 to compute n_r we get

$$\begin{aligned} n_r(m, \sigma_r^2) &= \frac{1}{\sqrt{2\pi}\sigma_r^3} \int_{-\infty}^{\infty} (m + r)(1 + (m + r)^2) r e^{-r^2/2\sigma_r^2} dr \\ &= 1 + 3m^2 + 3\sigma_r^2 . \end{aligned}$$

Using Equation 235 to compute n_m we get

$$\begin{aligned} n_m(m, \sigma_r^2) &= \frac{1}{\sqrt{2\pi} m \sigma_r} \int_{-\infty}^{\infty} (m + r)(1 + (m + r)^2) e^{-r^2/2\sigma_r^2} dr \\ &= 1 + m^2 + 3\sigma_r^2 . \end{aligned}$$

Where we have evaluated the above integrals in the Mathematica file `chap_6_prob_11.nb`. Note that to evaluate these integrals "by hand" we would expand the polynomial argument, for example

$$(m + r)(1 + (m + r)^2) ,$$

in the case of n_m into a polynomial in r and then use the results on the expectation of powers of a zero mean Gaussian random variable with variance σ^2. This later result is that if X is such a random variables then $E[X^n] = 0$ when n is odd and

$$E[X^n] = 1 \cdot 3 \cdot 5 \cdots (n - 1)\sigma^n ,$$

when n is even.

Problem 6-12 (describing function gains for some simple probability densities)

For the ideal relay nonlinearity $f(x)$ defined by

$$f(x) = \begin{cases} D & x > 0 \\ 0 & x = 0 \\ -D & x < 0 \end{cases} ,$$

we want to evaluate n_r under several different assumptions on the probability density for r (the residual). Since $f(\cdot)$ is a scalar function we have

$$n_r = \frac{1}{\sigma^2} E[f(x) r] = \frac{1}{\sigma^2} \int r f(m + r) p(r) dr = \frac{1}{\sigma^2} \int r f(r) p(r) dr ,$$

when we assume that m is zero.

For a **Gaussian density** recall that $p(r) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}}$ and we compute

$$\begin{aligned}
n_r &= \frac{1}{\sigma^2} \int r f(r) p(r) dr \\
&= -\frac{D}{\sigma^2} \int_{-\infty}^{0} r \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}} dr + \frac{D}{\sigma^2} \int_{0}^{\infty} r \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}} dr \\
&= -\frac{D}{\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} (-\sigma^2) \left. e^{-\frac{r^2}{2\sigma^2}} \right|_{-\infty}^{0} + \frac{D}{\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} (-\sigma^2) \left. e^{-\frac{r^2}{2\sigma^2}} \right|_{0}^{\infty} \\
&= \frac{D}{\sigma\sqrt{2\pi}} (1 - 0) - \frac{D}{\sigma\sqrt{2\pi}} (0 - 1) = \sqrt{\frac{2}{\pi}} \frac{D}{\sigma} .
\end{aligned}$$

Next recall that a **Triangular density** between $-b$ and $+b$ had an analytic representation of its density of

$$p(r) = \left\{ \begin{array}{cc} 0 & r < -b \\ \frac{1}{b^2}(r+b) & -b < r < 0 \\ -\frac{1}{b^2}(r-b) & 0 < r < b \\ 0 & r > b \end{array} \right. .$$

To use this, we first compute the expression for the variance σ^2 of this density in terms of the parameter b. We find

$$\begin{aligned}
\sigma^2 &= \int r^2 p(r) dr \\
&= \int_{-b}^{0} r^2 \frac{1}{b^2}(r+b) dr - \int_{0}^{b} r^2 \frac{1}{b^2}(r-b) dr \\
&= \frac{1}{b^2} \left[\int_{-b}^{0} r^3 dr + b \int_{-b}^{0} r^2 dr \right] - \frac{1}{b^2} \left[\int_{0}^{b} r^3 dr - b \int_{0}^{b} r^2 dr \right] \\
&= \frac{1}{b^2} \left[\left. \frac{r^4}{4} \right|_{-b}^{0} + b \left. \frac{r^3}{3} \right|_{-b}^{0} \right] - \frac{1}{b^2} \left[\left. \frac{r^4}{4} \right|_{0}^{b} - b \left. \frac{r^3}{3} \right|_{0}^{b} \right] \\
&= \frac{1}{b^2} \left[-\frac{b^4}{4} + \frac{b^4}{3} \right] - \frac{1}{b^2} \left[\frac{b^4}{4} - \frac{b^4}{3} \right] = \frac{1}{6} b^2 .
\end{aligned}$$

Next we calculate n_r. We find

$$\begin{aligned}
n_r &= \frac{1}{\sigma^2} \int r f(r) p(r) dr \\
&= \frac{1}{\sigma^2} \int_{-b}^{0} r(-D) \frac{1}{b^2}(r+b) dr + \frac{1}{\sigma^2} \int_{0}^{b} r(D) \left(-\frac{1}{b^2}(r-b) \right) dr \\
&= -\frac{D}{\sigma^2 b^2} \int_{-b}^{0} (r^2 + br) dr - \frac{D}{\sigma^2 b^2} \int_{0}^{b} (r^2 - br) dr \\
&= -\frac{D}{\sigma^2 b^2} \left[\frac{r^3}{3} + \left. \frac{br^2}{2} \right|_{-b}^{0} \right. - \frac{D}{\sigma^2 b^2} \left[\frac{r^3}{3} - \left. \frac{br^2}{2} \right|_{0}^{b} \right. \\
&= -\frac{D}{\sigma^2 b^2} \left[\frac{b^3}{3} - \frac{b^3}{2} \right] - \frac{D}{\sigma^2 b^2} \left[\frac{b^3}{3} - \frac{b^3}{2} \right] \\
&= \frac{Db}{3\sigma^2} .
\end{aligned}$$

When we use the fact that $b = \sqrt{6}\sigma$ we get

$$n_r = \sqrt{\frac{2}{3}}\frac{D}{\sigma}\,.$$

For a **uniform density** between $-\frac{a}{2}$ and $\frac{a}{2}$ we start by recalling that the variance is related to the end points of the density by

$$\sigma^2 = \int r^2 p(r) dr = \frac{a^2}{12}\,.$$

Next we calculate n_r as

$$\begin{aligned}
n_r &= \frac{1}{\sigma^2}\int_{-\frac{a}{2}}^{\frac{a}{2}} r f(r)\frac{1}{a} drn = \frac{1}{a\sigma^2}\left[\int_{-\frac{a}{2}}^{0} -Drdr + \int_{0}^{\frac{a}{2}} Drdr\right] \\
&= \frac{D}{a\sigma^2}\left[-\frac{r^2}{2}\bigg|_{-\frac{a}{2}}^{0} + \frac{r^2}{2}\bigg|_{0}^{\frac{a}{2}} rdr\right] = \frac{D}{a\sigma^2}\left[\frac{1}{2}\left(\frac{a^2}{4}\right) + \frac{1}{2}\left(\frac{a^2}{4}\right)\right] = \frac{aD}{4\sigma^2}\,.
\end{aligned}$$

Since $\sigma^2 = \frac{a^2}{12}$ or $a = \sqrt{12}\sigma$, so in terms of σ only n_r is given by

$$n_r = \sqrt{\frac{3}{4}}\frac{D}{\sigma}\,.$$

Problem 6-13 (CADET applied to a nonlinear differential equation)

We want to approximate $f(x) = a_1 x + a_2 x^2$ under the CADET philosophy. That is we take $f(x) \approx f_a(x) = n_m m + n_r r$ for two functions n_m and n_r. For n_r assuming a Gaussian density for r i.e. $r \sim N(0, \sigma^2)$ we find

$$\begin{aligned}
n_r &= \frac{1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} f(r+m) r e^{-r^2/2\sigma^2} dr \\
&= \frac{1}{\sqrt{2\pi}\sigma^3}\int_{-\infty}^{\infty} (a_1(r+m) + a_2(r+m)^2) r e^{-r^2/2\sigma^2} dr \\
&= a_1 + 2a_2 m\,.
\end{aligned}$$

While for n_m we find

$$\begin{aligned}
n_m &= \frac{1}{\sqrt{2\pi} m\sigma}\int_{-\infty}^{\infty} f(r+m) e^{-r^2/2\sigma^2} dr \\
&= \frac{1}{\sqrt{2\pi} m\sigma}\int_{-\infty}^{\infty} (a_1(r+m) + a_2(r+m)^2) e^{-r^2/2\sigma^2} dr \\
&= \frac{1}{m}\left(a_1 m + a_2(m^2 + \sigma^2)\right)\,.
\end{aligned}$$

Where we have evaluated the above integrals in the Mathematica file `chap_6_prob_13.nb`. Then using Equations 228 and 229 with $\sigma^2 = p$ we have

$$\begin{aligned}
\dot{m} &= m_m m + b = a_1 m + a_2(m^2 + \sigma^2) + b = a_1 m + a_2(m^2 + p) + b \\
\dot{r} &= n_r r + u = (a_1 + 2a_2 m) r + u\,.
\end{aligned}$$

Then using Equation 230 we get for the evolution of p

$$\dot{p} = 2n_r p + q = 2(a_1 + 2a_2 m)p + q\,.$$

Chapter 7 (Suboptimal Filter Design and Sensitivity Analysis)

Notes on the text

Notes on Example 7.1-2

We can derive the state covariance update equations by noting that figure 7.1-5 is the same system as that given in Example 7.1-1 but with the value of γ taken to be 1, with $q_{22} = 0$, and with the matrix $PH^TR^{-1}HP$ taken to be zero. This last fact is because we are not getting the reduction in state uncertainty from any measurements. Using these facts and the results from Exercise 7-3 on Page 155 the state covariance equation

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP\,,$$

becomes the set of scalar equations

$$\begin{aligned}
\dot{p}_{11} &= -2\alpha_1 p_{11} + q_1 \\
\dot{p}_{12} &= -(\alpha_1 + \alpha_2)p_{12} + p_{11} \\
\dot{p}_{22} &= -2\alpha_2 p_{22} + 2p_{12}\,,
\end{aligned}$$

which are the same ones given in the book. If we want to get the steady-state values for the covariance errors under the system above we set $\dot{P} = 0$ and then solve for the elements of P. When we do this and by solving these equations from top to bottom we find that the steady-state values for each element of P are

$$\begin{aligned}
p_{11} &= \frac{q}{2\alpha_1} \\
p_{12} &= \frac{p_{11}}{\alpha_1 + \alpha_2} = \frac{q}{2\alpha_1(\alpha_1 + \alpha_2)} \\
p_{22} &= \frac{p_{12}}{\alpha_2} = \frac{q}{2\alpha_1\alpha_2(\alpha_1 + \alpha_2)}\,,
\end{aligned}$$

which is the book's equation 7.1-14.

Notes on Example 7.1-4

From the given system

$$\begin{aligned}
\dot{x} &= ax + w \quad \text{with} \quad w \sim N(0, q) && (253) \\
z &= bx + v \quad \text{with} \quad v \sim N(0, r)\,, && (254)
\end{aligned}$$

we have $F = a$, $G = 1$, $Q = q$, $H = b$, and $R = r$, and thus $k = \frac{bp(t)}{r}$ and the Kalman filter error covariance is

$$\dot{p} = 2ap + q - \frac{b^2p^2}{r}\,,$$

with $p(0) = p_0$. As shown in Chapter 4 Problem 4-11 that $p_\infty = \frac{ar}{b^2}\left(1 + \frac{\beta}{a}\right)$ and so we find k_∞ given by

$$k_\infty = p_\infty H^T R^{-1} = \frac{p_\infty b}{r} = \frac{a}{b}\left(1 + \frac{\beta}{a}\right) = \frac{a+\beta}{b}\,.$$

When we define the estimation error between the Wiener and Kalman filters as $\delta p(t) = p_w(t) - p_k(t)$, we can use the bound presented in the book to show

$$||\delta p(t)|| \le \frac{||p_0 - p_\infty||^2 ||H^T R^{-1} H||}{8|\alpha_{\max}|} = \frac{(p_0 - p_\infty)^2\left(\frac{b^2}{r}\right)}{8|\alpha_{\max}|}\,,$$

where $\alpha_{\max}$ is the maximum real part of the eigenvalues of the matrix $F - K_\infty H$. In this case everything is a scalar and we find

$$F - K_\infty H = a - \left(\frac{a+\beta}{b}\right) b = \beta\,.$$

So the above error discrepancy between the Kalman and Wiener filters becomes

$$||\delta p(t)|| \le \frac{(p_0 - p_\infty)^2 b^2}{8r\beta}\,,$$

the same result given in the book.

Notes on Example 7.1-5

For this example we assume that we are filtering the given system using

$$\dot{\hat{x}} = -\hat{x} + K(z - \hat{x})\,,$$

with K a constant as of yet unspecified. Now K is not totally unconstrained, since we note that the above is equivalent to

$$\dot{\hat{x}} = -(1+K)\hat{x} + Kz\,,$$

and the condition of stability of this differential equation is that the coefficient of $\hat{x}$ be negative or that $1 + K > 0$ or $K > -1$.

We will take our filtering performance measure given by $J = p_\infty$, where p_∞ is the steady-state state error covariance for this problem. Since we assume that we will operate the filter with a constant gain (rather than the optimal time varying Kalman gain) and the correct system dynamics the covariance propagation for this filter will follow the *Wiener* filtering equations

$$\dot{P} = (F - KH)P + P(F - KH)^T + GQG^T + KRK^T\,, \tag{255}$$

where K is a *constant*. In this example, we have $F = -1$, $G = 1$, $Q = q$, $H = 1$, and $R = r$, and so the expression for $\dot{P}$ becomes

$$\begin{aligned}\dot{p} &= (-1-k)p + p(-1-k) + q + k^2 r \\ &= -2(k+1)p + q + k^2 r\,.\end{aligned}$$

To compute the steady-state solution p_∞ we can solve this differential equation for $p(0) = p_0$ and take the limit $t \to \infty$ or by taking $\dot{p} = 0$ and solving for p. When we do the later we find

$$p_\infty = \frac{q + k^2 r}{2(k+1)} .$$

The actual real world parameters α that are unknown to us are the two values of q and r or the variances of the process noise and measurement noise. Because of this, in the minimum sensitivity design formulation we take $\alpha = (q, r)$ and now need to compute the function $J_0(\alpha) = J_0(q, r)$. For *given* values of $\alpha = (q, r)$ the actual parameters, the *minimum* value of J over k is denoted $J_0(\alpha)$. To find this function $J_0(\alpha)$ we can take the derivative with respect to k of the function $J = p_\infty$, computed above, set the result equal to zero, solve for k, and then put this value of k back into the given expression for J. Taking the k derivative of J we find

$$\frac{dJ}{dk} = \frac{2kr}{2(1+k)} - \frac{k^2 r + q}{2(1+k)^2} = \frac{2kr + k^2 r - q}{2(k+1)^2} .$$

Setting this expression equal to zero, then solving for k with the quadratic formula we find

$$k = \frac{-r \pm \sqrt{r^2 + rq}}{r} = -1 \pm \sqrt{1 + \frac{q}{r}} ,$$

If we put this expression into $J(\cdot)$ we find $p_\infty = -r \pm \sqrt{r(q+r)}$. Since p_∞ must be positive we need to take the plus sign. If we do this and denote the resulting expression by $J_0(q, r)$ we find

$$J_0(q, r) = -r + r\sqrt{q + r} ,$$

which is the books equation 7.1-17.

We next proceed to evaluate the S_1, S_2, and S_3 criterion expressions for this example.

Given the above expressions for the objective S_1 we have

$$\begin{aligned} S_1 &= \min_{\beta \in B} \max_{\alpha \in A} J(\alpha, \beta) = \min_{k} \max_{(q,r)} J(k; q, r) \\ &= \min_{k} \max_{(q,r)} \frac{q + k^2 r}{2(k+1)} = \min_{k} \frac{k^2 + 1}{2(1+k)} . \end{aligned}$$

Where have used the fact that since $0 < q < 1$ and $0 < r < 1$ the maximum of $J(k; q, r)$ over (q, r) is obtained when $r = 1$ and $q = 1$. To perform the next minimization we take the derivative with respect to k, set the result equal to zero, and solve for k. We find the derivative given by

$$\frac{2k}{2(1+k)} + \frac{k^2 + 1}{2(1+k)^2} = \frac{k^2 + 2k - 1}{2(1+k)^2} .$$

When we set that equal to zero and solve the resulting quadratic equation for k we find

$$k = \frac{-2 \pm \sqrt{4 + 4}}{2} = -1 \pm \sqrt{2} .$$

To have stability requires that $k > -1$ and we must take the positive solution found above giving $k = \sqrt{2} - 1$ in agreement with the book. Thus using this criterion function we should filter our signal with $k = \sqrt{2} - 1$.

For S_2 criterion we have

$$\begin{aligned} S_2 &= \min_{\beta \in B} \max_{\alpha \in A} (J(\alpha, \beta) - J_0(\alpha)) \\ &= \min_k \max_{q,r} \left(\frac{k^2 r + q}{2(1+k)} - (r^2 + rq)^{1/2} + r \right) \end{aligned}$$

To get an understanding of the the inner maximization in the above min-max problem we simply *plot* the above expression as a function of $0 \le q \le 1$ and $0 \le r \le 1$ for several values of $k > -1$. This is done in the Mathematical file `example_7_1_5.nb` where it is found that for all k the maximization above happen at the end values of r namely either when $r = 0$ from which we see

$$\left. \left(\frac{k^2 r + q}{2(1+k)} - (r^2 + rq)^{1/2} + r \right) \right|_{r=0} = \frac{q}{2(1+k)},$$

when $r = 0$ and when $r = 1$ we find

$$\left. \left(\frac{k^2 r + q}{2(1+k)} - (r^2 + rq)^{1/2} + r \right) \right|_{r=1} = 1 - \sqrt{1+q} + \frac{k^2 + q}{2(1+k)}.$$

For all k the first expression is maximized as a function of q when $0 \le q \le 1$ when $q = 1$ and gives $\frac{1}{2(1+k)}$, for its maximum value. To find the minimum or maximum of the second expression as a function of q we take the q derivative, set the result equal to zero, and solve for q. Before performing all of this work we note that the *second* derivative with respect to q of the expression under discussion is given by

$$\frac{1}{4(1+q)^{3/2}},$$

which is always positive indicating that the value of q that makes first derivative zero is a minimum and not a maximum. Thus the maximum in the case when $r = 1$ must occur at the end points of the domain i.e. $q = 0$ or $q = 1$. The expression above at these points has the values of

$$\frac{k^2}{2(1+k)} \quad \text{and} \quad 1 - \sqrt{2} + \frac{1+k^2}{2(1+k)},$$

respectively. In summary then, when we fix the value of k the value we could get from the expression $\max_{\alpha \in A} (J(\alpha, \beta) - J_0(\alpha))$ is given by

$$\max \left\{ \frac{1}{2(1+k)}, \frac{k^2}{2(1+k)}, 1 - \sqrt{2} + \frac{1+k^2}{2(1+k)} \right\}.$$

We plot the three functions that make up this maximization in Figure 4. To evaluate the S_2 criterion and design our filter we have to pick k such that S_2 is minimized since

$$S_2 = \min_k \max \left\{ \frac{1}{2(1+k)}, \frac{k^2}{2(1+k)}, 1 - \sqrt{2} + \frac{1+k^2}{2(1+k)} \right\}.$$

From Figure 4 we see that this minimum occurs at $k = 1$ and this would correspond to our filtering design parameter to use. This result agrees with that discussed in the book but no detail as to how the book got its results was provided. In the Mathematical file `example_7_1_5.nb` much of the algebra for this problem is worked.

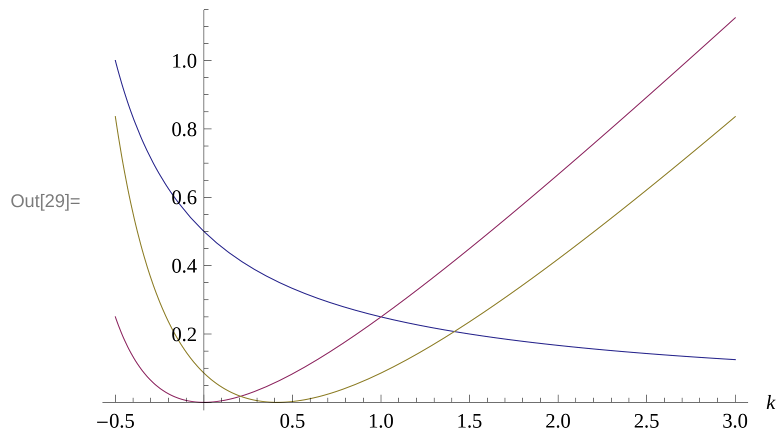

Figure 4: A plot of the three functions $\frac{1}{2(1+k)}$ (in blue) $\frac{k^2}{2(1+k)}$ (in red) and $1-\sqrt{2}+\frac{1+k^2}{2(1+k)}$ (in brown). For each value of k the maximum of these three functions is the result of the maximization over α of $J(\alpha,\beta)-J_0(\alpha)$ and is a function of $\beta=k$.

Notes on Example 7.1-6

For this example the true physical system has parameters given by $F=-\beta$, $G=1$, $Q=q$, $H=1$, and $R=r$, while we choose to filter our system with possibly non optimal parameters $F^*=-\beta_f$, $K^*=k$, and $H^*=1$. Because of this we are filtering with an incorrect implementation of dynamics and measurements and need to use the results derived below to obtain the steady-state error covariance expression p_∞.

To derive an equation for p_∞ we use the books equations 7.2-14, 7.2-15, and 7.2-16 or Equations 263, 264, and 265 below with the values for F, F^* etc given above. In this case we first note that $\Delta F = F^* - F = -\beta_f + \beta$ and $\Delta H = 0$, so that in steady-state when $\dot{P}=\dot{V}=\dot{U}=0$ the given system becomes

$$
\begin{aligned}
0 &= 2(-\beta_f - k)p_\infty + 2(-\beta_f+\beta)v_\infty + q + k^2 r \\
0 &= -\beta v_\infty + (-\beta_f - k)v_\infty + (-\beta_f+\beta)u_\infty - q \\
0 &= -2\beta u_\infty + q\,.
\end{aligned}
$$

When we solve this system for p_∞, v_∞ and u_∞ we find

$$
\begin{aligned}
p_\infty &= \frac{\beta_f^2 q + \beta^2 k^2 r + \beta(\beta_f+k)(q+k^2 r)}{2\beta(\beta_f+k)(\beta+\beta_f+k)} \qquad (256) \\
v_\infty &= -\frac{q(\beta+\beta_f)}{2\beta(\beta+\beta_f+k)} \\
u_\infty &= \frac{q}{2\beta}\,.
\end{aligned}
$$

A couple equivalent expressions for p_∞ are given by simple manipulations of the above expression. We find

$$
\begin{aligned}
p_\infty &= \frac{k^2 r}{2(\beta_f + k)} + \frac{(\beta_f^2 + \beta(\beta_f + k))q}{2\beta(\beta_f + k)(\beta + \beta_f + k)} \\
&= \frac{k^2 r}{2(\beta_f + k)} + \frac{(\beta_f^2 + \beta(\beta + \beta_f + k - \beta))q}{2\beta(\beta_f + k)(\beta + \beta_f + k)} \\
&= \frac{k^2 r + q}{2(\beta_f + k)} + \frac{(\beta_f^2 - \beta^2)q}{2\beta(\beta_f + k)(\beta + \beta_f + k)} \,.
\end{aligned}
$$

The first expression for p_∞ is equation 7 in the original reference for this section see [1], while the last equation is the result presented in the book.

Given the above expression for p_∞ the vector "α" in this case or the actual physical parameters is given by the three unknown scalar values (β, q, r) and the design parameter vector "β", is given by the two parameters (β_f, k).

Next as the design criterion S_2 and S_3 requires we need to compute the expression for $J_0(\cdot)$ defined as a minimum in terms of the unknowns for this problem by

$$
\begin{aligned}
J_0(\beta, q, r) &\equiv \min_{\beta_f, k} p_\infty(\beta, q, r; \beta_f, k) \\
&= \min_{\beta_f, k} \left(\frac{k^2 r + q}{2(\beta_f + k)} + \frac{(\beta_f^2 - \beta^2)q}{2\beta(\beta_f + k)(\beta + \beta_f + k)} \right) .
\end{aligned}
$$

We could find this minimum analytically, numerically, or more easily by recognizing that its *value* is equal to the *optimal* Kalman steady-state value of p_∞ when we filter with the true system value. This in turn is given by the steady-state value of

$$
\dot{P} = FP + PF^T + GQG^T - PH^T R^{-1} HP \,.
$$

With $F = -\beta$, $G = 1$, $Q = q$, $H = 1$, and $R = r$ When we specify the above equation to the given expression we get

$$
0 = -2\beta p_\infty + q - \frac{p_\infty^2}{r} \,, \tag{257}
$$

or

$$
p_\infty^2 + 2\beta r p_\infty - qr = 0 \,.
$$

Solving this with the quadratic equation gives

$$
p_\infty = \frac{-2\beta r \pm \sqrt{4\beta^2 r^2 - 4(-qr)}}{2} = -\beta \pm \sqrt{\beta^2 r^2 + qr} \,.
$$

Since $p_\infty > 0$ we need to take the positive sign in the above and we have

$$
J_0(\beta, q, r) \equiv p_\infty = -\beta r + \sqrt{\beta^2 r^2 + qr} \,. \tag{258}
$$

With this background discussion we now proceed to determine the performance of optimal, S_1, S_2, S_3 and $\beta = 0.5$ filters when $q = 10$, $r = 1$ and $0.1 \leq \beta \leq 1$ as documented in this example.

- **The Kalman Optimal Filter:**

  To design and plot the optimal filtering performance result for this problem note that this system is a special case of that in example 7.1-4, with $a = -\beta$ and $b = +1$. In that example we found that p_∞ is given by

$$
\begin{aligned}
p_\infty &= \frac{ar}{b^2}\left(1 + \frac{\tilde{\beta}}{a}\right) = -\beta r \left(1 + \frac{\tilde{\beta}}{\beta}\right) \quad \text{with} \\
\tilde{\beta} &= a\sqrt{1 + \frac{b^2 q}{a^2 r}} = -\beta\sqrt{1 + \frac{q}{\beta^2 r}}
\end{aligned}
$$

  Thus we let $q = 10$, $r = 1$ and $0.1 \leq \beta \leq 1$ and plot p_∞ as a function of β.

- **The S_1 Filter:**

  The design of the S_1 filter is defined by

$$
\begin{aligned}
S_1 &= \min_{\beta \in B} \max_{\alpha \in A} J(\alpha, \beta) \\
&= \min_{(\beta_f, k)} \max_{(\beta, q, r)} \left(\frac{k^2 r + q}{2(\beta_f + k)} + \frac{(\beta_f^2 - \beta^2) q}{2\beta(\beta_f + k)(\beta + \beta_f + k)} \right).
\end{aligned}
$$

  There are probably many ways to implement such a filter. For this example we will do this in a brute force way. What this means is that we will create a grid that samples from the possible values for β_f and k. Then for each value of β_f and k we have as a candidate pair to filter with we then need to compute the inner maximization over (β, q, r). Again we do this by simply sampling the provided function on a discretized grid of points. Having evaluated the above function at each of these points we return the maximum. We then move to the next candidate pair for (β_f, k) and repeat this procedure. The filter designer would then pick the values of β_f and k that gave the minimum over all tested pairs.

- **The S_2 and S_3 Filter:**

  We design these two filters in the same way we do the S_1 filter except that the objective function is slightly different than in the S_1 case.

  The $\beta = 0.5$ case:

  In this case we assume that we think know the dynamics exactly $\beta = 0.5$ and that we design a optimal Kalman filter under that assumption. Thus we have $\beta_f = 0.5$ and k would be given by the optimal Kalman gain under the assumption that $\beta = 0.5$.

  We can find this value of k by using the value of p_∞

  and then take $k = K_\infty = P_\infty H^T R^{-1}$.

  From this we have that k is given by

$$
k = -\beta + \sqrt{\beta^2 + \frac{q}{r}}.
$$

  Since we assume that we know that the value of β is $1/2$ the value of k that we will filter is given by taking $\beta = 1/2$ in the above expression.

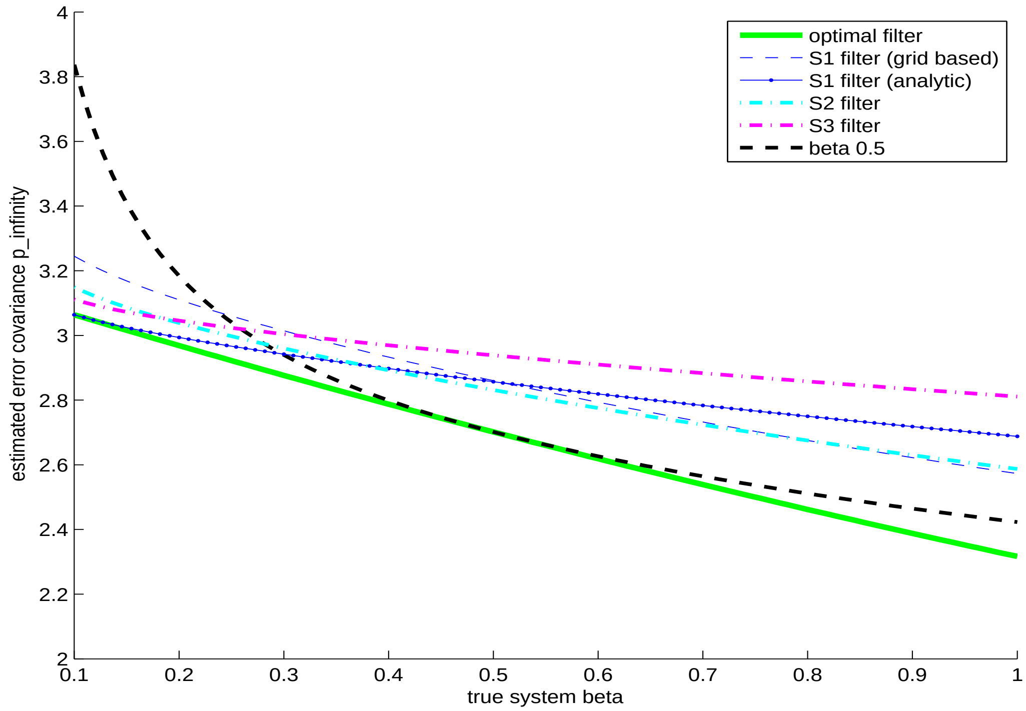

Figure 5: Attempted duplication of the results found in figure 7.1-11 from the book. This is qualitatively very similar to the corresponding figure from the text. See the main text for more discussion on this plot.

As discussed in [1] the S_1 criterion can be determine exactly given the functional form for $J(\alpha, \beta)$. We have

$$
\begin{aligned}
S_1 &= \min_{\beta} \max_{\alpha} p(\alpha, \beta) \\
&= \min_{(\beta_f, k)} \max_{(\beta, q, r)} p(\beta, q, r; \beta_f, k) \\
&= \min_{(\beta_f, k)} \max_{\beta} p(\beta, q_{\max}, r_{\max}; \beta_f, k) \\
&= \min_{(\beta_f, k)} p(\beta_{\min}, q_{\max}, r_{\max}; \beta_f, k) \\
&= J_0(\beta_{\min}, q_{\max}, r_{\max}) \,,
\end{aligned}
$$

where J_0 is given by Equation 258. This means that we can exactly analyze the S_1 criterion. Unfortunately I was not able to numerically duplicate these expected analytic results. In the Figure 5 one will see a numerical duplication of the filters discussed above. For the S_1 filter I present *both* the analytic and the grid based numeric result. These plots are produced in the Matlab file `example_7_1_6_plots_brute_force_optimization.m`, and if anyone sees anything wrong with what I have done please contact me. These results, as they stand, are very similar to the ones presented in the books figure 7.1-11. In addition, qualitatively Figure 5 shows the statements given in the text on min/max filters. Namely that they enable a filter design that is very close to the optimal result (the green line). From the plot it looks like the S_2 filter is the closest to the optimal result. Finally, we mention that the algebra to derive some of these expressions is can be found in the Mathematical file `example_7_1_6.nb`.

Notes on incorrect implementation of dynamics and measurement

When we filter with *incorrect* Kalman gain K^*, measurement sensitivity H^*, and dynamics F^*, the differential equation for the error $\tilde{x} \equiv \hat{x} - x$ in the continuous case can be derived using the implemented equation for $\hat{x}$ and *true* state dynamic equation for x as follows

$$\begin{aligned}
\frac{d}{dt}\tilde{x} &= \frac{d}{dt}(\hat{x} - x) \\
&= F^*\hat{x} + K^*(z - H^*\hat{x}) - Fx - Gw \\
&= (F^* - K^*H^*)\hat{x} - Fx + K^*z - Gw \\
&= (F^* - K^*H^*)\hat{x} - Fx + K^*(Hx + v) - Gw \\
&= (F^* - K^*H^*)\hat{x} - (F - K^*H)x + K^*v - Gw\,. \qquad (259)
\end{aligned}$$

or the books equation 7.2-8. Let $\Delta F = F^* - F$ and $\Delta H = H^* - H$ so that $F = F^* - \Delta F$ and $H = H^* - \Delta H$ and then Equation 259 in terms of ΔF and ΔH becomes

$$\begin{aligned}
\frac{d}{dt}\tilde{x} &= (F^* - K^*H^*)\hat{x} - (F^* - \Delta F - K^*(H^* - \Delta H))x + K^*v - Gw \\
&= (F^* - K^*H^*)\hat{x} - (F^* - K^*H^*)x + (\Delta F - K^*\Delta H)x + K^*v - Gw \\
&= (F^* - K^*H^*)\tilde{x} + (\Delta F - K^*\Delta H)x + K^*v - Gw\,, \qquad (260)
\end{aligned}$$

or the books equation 7.2-9. Since this equation involves $\tilde{x}$ *and* x on the right-hand-side, let x' be denoted as the vector of $\tilde{x}$ and x as $x' = \begin{bmatrix} \tilde{x} \\ x \end{bmatrix}$, then since the dynamics of x is governed by $\frac{dx}{dt} = Fx + Gw$ the system for x' is given by

$$\frac{dx'}{dt} = \begin{bmatrix} F^* - K^*H^* & \Delta F - k^*\Delta H \\ 0 & F \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x \end{bmatrix} + \begin{bmatrix} K^*v - Gw \\ Gw \end{bmatrix} \equiv F'x' + w'\,, \qquad (261)$$

which is the books equation 7.2-10 and in which we have implicitly defined the variables F' and w'. Now using the system theory from earlier we have that the covariance matrix for the variable x' satisfies the following differential equation

$$\frac{dE[x'x'^T]}{dt} = F'E[x'x'^T] + E[x'x'^T]F'^T + E[w'w'^T]\,. \qquad (262)$$

What we really want to study however is the behavior of the covariance matrix for $\tilde{x}$ only since this represents the difference between the true state x and our estimate $\tilde{x}$. To obtain this lets block partition the covariance of the vector x' by introducing the matrices P, V, and U as

$$E[x'x'^T] \equiv \begin{bmatrix} P & V^T \\ V & U \end{bmatrix}.$$

Now from the definition of w' we can compute $E[w'w'^T]$ as

$$\begin{aligned}
E[w'w'^T] &= E\left[\begin{bmatrix} K^*v - Gw \\ Gw \end{bmatrix} \begin{bmatrix} v^TK^{*T} - w^TG^T & w^TG^T \end{bmatrix}\right] \\
&= E\left[\begin{bmatrix} K^*vv^TK^{*T} - K^*vw^TG^T - Gwv^TK^{*T} + Gww^TG^T & K^*vw^TG^T - Gww^TG^T \\ Gwv^TK^{*T} - Gww^TG^T & Gww^TG^T \end{bmatrix}\right] \\
&= \begin{bmatrix} K^*RK^{*T} + GQG^T & -GQG^T \\ -GQG^T & GQG^T \end{bmatrix},
\end{aligned}$$

which is the expression for $E[w'w'^T]$ presented in the books equation 7.2-13. Using this expression and the definition of F' we can construct the right-hand-side of Equation 262. Setting this equal to the block partitioned form for $\frac{dE[x'x'^T]}{dt}$, we obtain a dynamical system for the components. When we do this we obtain the following system

$$
\begin{aligned}
\dot{P} &= (F^* - K^*H^*)P + P(F^* - K^*H^*)^T + (\Delta F - K^*\Delta H)V \\
&+ V^T(\Delta F - K^*\Delta H)^T + GQG^T + K^*RK^{*T} \qquad (263) \\
\dot{V} &= FV + V(F^* - K^*H^*)^T + U(\Delta F - K^*\Delta H)^T - GQG^T \qquad (264) \\
\dot{U} &= FU + UF^T + GQG^T \,, \qquad (265)
\end{aligned}
$$

with as before $\Delta F \equiv F^* - F$ and $\Delta H \equiv H^* - H$.

If we are *deleting* states from the true state in order to derive the filter we will process measurements with, then the filter equations for this case can be obtained from the ones above if we make the substitutions

$$
\begin{aligned}
F^* &\to W^T F^* W \\
H^* &\to H^* W \\
K^* &\to W^T K^* \,.
\end{aligned}
$$

In addition, we now define ΔF and ΔH as $\Delta F = W^T F^* W - F$ and $\Delta H = H^* W - H$. Using this, we can derive the following for the equation for $\dot{P}$ from Equation 263 as follows

$$
\begin{aligned}
\dot{P} &= (W^T F^* W - W^T K^* H^* W)P + P(W^T F^* W - W^T K^* H^* W)^T + (\Delta F - W^T K^* \Delta H)V \\
&+ V^T(\Delta F - W^T K^* \Delta H)^T + GQG^T + W^T K^* R K^{*T} W \\
&= W^T(F^* - K^* H^*)WP + PW^T(F^* - K^*H^*)^T W + (\Delta F - W^T K^* \Delta H)V \\
&+ V^T(\Delta F - W^T K^* \Delta H)^T + GQG^T + W^T K^* R K^{*T} W \,, \qquad (266)
\end{aligned}
$$

which duplicates the books equation 7.2-19. The other equations would be done in a similar manner.

Problem Solutions

Problem 7-1 (the fixed gain k_∞ gives the same steady-state error covariance)

Equation 7.2-3 from the book is

$$\dot{P} = (F - KH)P + P(F - KH)^T + GQG^T + KRK^T \,,$$

and is the equation that our covariance satisfies if we *don't* use the optimal Kalman gain but instead filter with another value say k. In example 7.1-3 we have $F = 0$, $G = 1$, $Q = q$, $H = 1$, and $R = r$ so this equation becomes

$$\dot{p} = -kp - kp + q + k^2 r = -2kp + q + k^2 r \,.$$

If in particular we filter with the value $k = \sqrt{\frac{q}{r}}$, the above equation becomes

$$\dot{p} = -2\sqrt{\frac{q}{r}}p + q + \frac{q}{r} \cdot r = -2\sqrt{\frac{q}{r}}p + 2q\,.$$

To find the steady-state solution to this equation we could solve it for $p(t)$ and then take the limit as $t \to \infty$ or simply recall that in steady-state $\dot{p} = 0$ and then solve for $p = p_\infty$ in the above equation. When we do that we find

$$p_\infty = \sqrt{rq}\,,$$

the same steady-state value result we obtain if we had in fact done *optimal* Kalman filtering.

Problem 7-2 (discrete equations under errors in measurements and dynamics)

Recalling our definition of the a priori state error of $\tilde{x}_k(-) = \hat{x}_k(-) - x_k$, when we increment k by one we get

$$\begin{aligned}
\tilde{x}_{k+1}(-) &= x_{k+1} - \hat{x}_{k+1}(-) \\
&= \Phi_k^* \hat{x}_k(+) - \Phi_k x_k - w_k\,.
\end{aligned}$$

Now Introduce the notation $\Delta\Phi_k \equiv \Phi_k^* - \Phi_k$ so that $\Phi_k = \Phi_k^* - \Delta\Phi_k$ so the above becomes

$$\tilde{x}_{k+1}(-) = \Phi_k^* \hat{x}_k(+) - (\Phi_k^* x_k - \Delta\Phi_k x_k) - w_k = \Phi_k^* \tilde{x}_k(+) + \Delta\Phi_k x_k - w_k\,.$$

This last result expresses $\tilde{x}_{k+1}(-)$ in terms of $\tilde{x}_k(+)$ and x_k. Next introduce the stacked vector $x'_k(-)$ defined as as $x'_k(-) = \left[\begin{array}{c} \tilde{x}_k(-) \\ x_k \end{array}\right]$, then we have that $x'_k(-)$ satisfies

$$\begin{aligned}
x'_{k+1}(-) &= \left[\begin{array}{c} \tilde{x}_{k+1}(-) \\ x_k \end{array}\right] = \left[\begin{array}{c} \Phi_k^* \tilde{x}_k(+) + \Delta\Phi_k x_k \\ \Phi_k x_k \end{array}\right] + \left[\begin{array}{c} -w_k \\ w_k \end{array}\right] \\
&= \left[\begin{array}{cc} \Phi_k^* & \Delta\Phi_k \\ 0 & \Phi_k \end{array}\right] \left[\begin{array}{c} \tilde{x}_k(+) \\ x_k \end{array}\right] + \left[\begin{array}{c} -w_k \\ w_k \end{array}\right]. \qquad (267)
\end{aligned}$$

Then defining the blocks of the covariance of $x'_{k+1}(-)$ as

$$E[x'_{k+1}(-) x'_{k+1}(-)^T] \equiv \left[\begin{array}{cc} P_{k+1}(-) & V_{k+1}(-)^T \\ V_{k+1}(-) & U_{k+1}(-) \end{array}\right],$$

by using Equation 267 we find the block matrix expression for $\mathcal{P} \equiv E[x'_{k+1}(-) x'_{k+1}(-)^T]$ as

$$\mathcal{P} = \left[\begin{array}{cc} \Phi_k^* & \Delta\Phi_k \\ 0 & \Phi_k \end{array}\right] \left[\begin{array}{cc} P_k(+) & V_k(+)^T \\ V_k(+) & U_k(+) \end{array}\right] \left[\begin{array}{cc} \Phi_k^{*T} & 0 \\ \Delta\Phi_k^T & \Phi_k^T \end{array}\right] + \left[\begin{array}{cc} Q_k & -Q_k \\ -Q_k & Q_k \end{array}\right].$$

To evaluate this first we multiply the rightmost matrices together as

$$\left[\begin{array}{cc} P_k(+) & V_k(+)^T \\ V_k(+) & U_k(+) \end{array}\right] \left[\begin{array}{cc} \Phi_k^{*T} & 0 \\ \Delta\Phi_k^T & \Phi_k^T \end{array}\right] = \left[\begin{array}{cc} P_k(+)\Phi_k^{*T} + V_k(+)^T \Delta\Phi_k^T & V_k(+)^T \Phi_k{}^T \\ V_k(+)\Phi_k^{*T} + U_k(+)\Delta\Phi_k^T & U_k(+)\Phi_k^T \end{array}\right],$$

multiply the resulting matrix on the left by the matrix $\begin{bmatrix} \Phi_k^* & \Delta\Phi_k \\ 0 & \Phi_k \end{bmatrix}$ and then add the matrix $\begin{bmatrix} Q_k & -Q_k \\ -Q_k & Q_k \end{bmatrix}$. When we do this and equate the result to $\begin{bmatrix} P_{k+1}(-) & V_{k+1}(-)^T \\ V_{k+1}(-) & U_{k+1}(-) \end{bmatrix}$ as components we get a system of equations given by

$$\begin{aligned}
P_{k+1}(-) &= \Phi_k^* P_k(+)\Phi_k^{*T} + \Phi_k^* V_k(+)^T \Delta\Phi_k^T + \Delta\Phi_k V_k(+)\Phi_k^{*T} + \Delta\Phi_k U_k(+)\Delta\Phi_k^T + Q_k \\
V_{k+1}(-) &= \Phi_k V_k(+)\Phi_k^{*T} + \Phi_k U_k(+)\Delta\Phi_k^T - Q_k \\
U_{k+1}(-) &= \Phi_k U_k(+)\Phi_k^T + Q_k \,.
\end{aligned}$$

These correspond to the book's equations 7.2-20.

To derive a recursive relationship *across* a measurement note that we can write $\tilde{x}_k(+)$ as

$$\begin{aligned}
\tilde{x}_k(+) &= x_k - \hat{x}_k(+) = x_k - (\hat{x}_k(-) + K_k^*(z_k - H_k^*\hat{x}_k(-))) \\
&= \tilde{x}_k(-) - K_k^*(H_k x_k + v_k - H_k^*\hat{x}_k(-)) \,.
\end{aligned}$$

Again using that $\Delta H \equiv H^* - H$ or putting $H = \Delta H + H^*$ in the above we get

$$\begin{aligned}
\tilde{x}_k(+) &= \tilde{x}_k(-) - K_k^*(H_k^*\tilde{x}_k(-) + \Delta H_k x_k + v_k) \\
&= (I - K_k^* H_k^*)\tilde{x}_k(-) - K_k^*\Delta H_k x_k - K_k^* v_k \,.
\end{aligned}$$

If we introduce the vector $x_k'(+)$ as $\begin{bmatrix} \tilde{x}_k(+) \\ x_k \end{bmatrix}$ then we see that in terms of $\tilde{x}_k(-)$ and x_k by using the above we have

$$\begin{aligned}
x_k'(+) &= \begin{bmatrix} (I - K_k^* H_k^*)\tilde{x}_k(-) - K_k^*\Delta H_k x_k - K_k^* v_k \\ x_k \end{bmatrix} \\
&= \begin{bmatrix} I - K_k^* H_k^* & -K_k^*\Delta H_k \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{x}_k(-) \\ x_k \end{bmatrix} + \begin{bmatrix} -K_k^* v_k \\ 0 \end{bmatrix} .
\end{aligned}$$

Using the above block definitions we find the matrix expression for $\mathcal{P} \equiv E[x_k'(+)x_k'(+)^T]$ as

$$\mathcal{P} = \begin{bmatrix} I - K_k^* H_k^* & -K_k^*\Delta H_k \\ 0 & I \end{bmatrix} \begin{bmatrix} P_k(-) & V_k(-)^T \\ V_k(-) & U_k(-) \end{bmatrix} \begin{bmatrix} (I - K_k^* H_k^*)^T & 0 \\ -\Delta H_k{}^T K_k^{*T} & I \end{bmatrix} + \begin{bmatrix} K_k^* R_k K_k^{*T} & 0 \\ 0 & 0 \end{bmatrix} .$$

Performing the matrix products, adding the matrix $\begin{bmatrix} K_k^* R_k K_k^{*T} & 0 \\ 0 & 0 \end{bmatrix}$ and equating the result to $\begin{bmatrix} P_k(+) & V_k(+)^T \\ V_k(+) & U_k(+) \end{bmatrix}$ gives the following system

$$\begin{aligned}
P_k(+) &= (I - K_k^* H_k^*)P_k(-)(I - K_k^* H_k^*)^T - (I - K_k^* H_k^*)V_k(-)^T \Delta H_k^T K_k^{*T} \\
&- K_k^*\Delta H_k V_k(-)(I - K_k^* H_k^*)^T + K_k^*\Delta H_k U_k(-)\Delta H_k^T K_k^{*T} + K_k^* R_k K_k^{*T} \\
V_k(+) &= V_k(-)(I - K_k^* H_k^*)^T - U_k(-)\Delta H_k K_k^{*T} \\
U_k(+) &= U_k(-) \,,
\end{aligned}$$

these results agree with the book's equations 7.2-21.

Problem 7-3 (The fully coupled system $\gamma \neq 0$)

For the given system we have $F = \begin{bmatrix} -\alpha_1 & 0 \\ \gamma & -\alpha_2 \end{bmatrix}$, (so that $F^T = \begin{bmatrix} -\alpha_1 & \gamma \\ 0 & -\alpha_2 \end{bmatrix}$), $Q = \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $R = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$. Then the matrix Riccati Equation 71 for this problem becomes

$$\begin{aligned} \begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{bmatrix} &= \begin{bmatrix} -\alpha_1 p_{11} & \alpha_1 p_{12} \\ \gamma p_{11} - \alpha_2 p_{12} & \gamma p_{12} - \alpha_2 p_{22} \end{bmatrix} + \begin{bmatrix} -\alpha_1 p_{11} & \gamma p_{11} - \alpha_2 p_{12} \\ -\alpha_1 p_{12} & \gamma p_{12} - \alpha_2 p_{22} \end{bmatrix} \\ &+ \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{r_{11}} p_{11}^2 + \frac{1}{r_{22}} p_{12}^2 & \frac{1}{r_{11}} p_{11} p_{12} + \frac{1}{r_{22}} p_{12} p_{22} \\ \frac{1}{r_{11}} p_{11} p_{12} + \frac{1}{r_{22}} p_{12} p_{22} & \frac{1}{r_{11}} p_{12}^2 + \frac{1}{r_{22}} p_{22}^2 \end{bmatrix} . \end{aligned}$$

As a system of scalar equations this becomes

$$\begin{aligned} \dot{p}_{11} &= -2\alpha_1 p_{11} + q_{11} - \frac{1}{r_{11}} p_{11}^2 - \frac{1}{r_{22}} p_{12}^2 \\ \dot{p}_{12} &= -\alpha_1 p_{11} + \gamma p_{11} - \alpha_2 p_{12} - \frac{1}{r_{11}} p_{11} p_{12} - \frac{1}{r_{22}} p_{12} p_{22} \\ &= -\left(\alpha_1 + \alpha_2 + \frac{p_{11}}{r_{11}} + \frac{p_{12}}{r_{22}}\right) p_{12} + \gamma p_{11} \\ \dot{p}_{22} &= 2\gamma p_{12} - 2\alpha_2 p_{22} + q_{22} - \frac{1}{r_{11}} p_{12}^2 - \frac{1}{r_{22}} p_{22}^2 \\ &= -2\alpha_2 p_{22} + q_{22} - \frac{p_{22}^2}{r_{22}} - \frac{p_{12}^2}{r_{11}} + 2\gamma p_{12} , \end{aligned}$$

the same equations given by the book in Example 7.1-1.

Problem 7-6 (the expression for $P_k(+)$ under non-optimal filtering)

The requested expression for $P_k(+)$ is a specialization of the discussion given in the section on filtering with incorrect dynamics and measurement found on Page 151, in that the result we desire to show here can be obtained if we take $\Delta H = 0$ and $\Delta F = 0$. This means that we are filtering with the correct dynamics and measurement sensitivity matrix but with a potentially incorrect Kalman gain K^*. In this case Equation 263 becomes

$$P_k(+) = (I - K_k^* H_k) P_k(-) (I - K_k^* H_k) + K_k^* R_k K_k^{*T} ,$$

which is the result requested.

Problem 7-7 (the error covariance differential equation for a Kalman like filter)

Part (a): From Table 4.3-1 a Kalman like filter means that we should derive an estimate of our state $\hat{x}(t)$ by integrating

$$\frac{d\hat{x}}{dt} = F\hat{x}(t) + K(t)(z(t) - H(t)\hat{x}(t)) .$$

For this problem we assume that $K(t)$ is general and not necessarily given by the optimal expression PH^TR^{-1}. Since the true state x follows the dynamics given by

$$\frac{dx}{dt} = Fx + Gw\,,$$

the error vector $\tilde{x} = \hat{x} - x$ has a differential equation given by

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= \frac{d\hat{x}}{dt} - \frac{dx}{dt} \\ &= F\hat{x} + K(z - H\hat{x}) - Fx - Gw\,.\end{aligned}$$

Now the measurement z in terms of the true state x is given by $z = Hx + v$ so we can write the above as

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= F(\hat{x} - x) - Gw + K(H(x - \hat{x}) + v) \\ &= (F - KH)\tilde{x} - Gw + Kv\,,\end{aligned}$$

or the desired error differential equation.

Part (b): Given this differential equation for $\tilde{x}$ and following by example the results from Chapter 4 we have that

$$\begin{aligned}\dot{P} &= \frac{dE[\tilde{x}\tilde{x}^T]}{dt} \\ &= (F - KH)E[\tilde{x}\tilde{x}^T] + E[\tilde{x}\tilde{x}^T](F - KH)^T + E[(Gw - Kv)(Gw - Kv)^T] \\ &= (F - KH)P + P(F - KH)^T + GQG^T + KRK^T\,.\end{aligned}$$

As we were to show. Note that if K equals the Kalman optimal value of PH^TR^{-1} then we see that the equation for $\dot{P}$ becomes

$$\begin{aligned}\dot{P} &= (F - PH^TR^{-1}H)P + P(F - PH^TR^{-1}H)^T + GQG^T + PH^TR^{-1}RR^{-1}HP \\ &= FP + PF^T + GQG^T - PH^TR^{-1}HP - PH^TR^{-1}HP + PH^TR^{-1}HP \\ &= FP + PF^T + GQG^T - PH^TR^{-1}HP\,,\end{aligned}$$

or the matrix Riccati equation as it should.

Problem 7-8 (estimating a random ramp function)

For the system given for this problem with no process noise we have

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ z &= x_2 + v \quad \text{with} \quad v \sim N(0, r)\,,\end{aligned}$$

With that representation we have $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $G = 1$, $Q = 0$, $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and $R = r$. The matrix Riccati equation of

$$\dot{P} = FP + PF^T + GQG^T - PH^TR^{-1}HP\,,$$

in component form is given by

$$
\begin{aligned}
\begin{bmatrix} \dot{p}_{11} & \dot{p}_{12} \\ \dot{p}_{12} & \dot{p}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} + 0 - \frac{1}{r} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & p_{11} \\ p_{11} & 2p_{12} \end{bmatrix} - \frac{1}{r} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ p_{12} & p_{22} \end{bmatrix} \\
&= \begin{bmatrix} 0 & p_{11} \\ p_{11} & 2p_{12} \end{bmatrix} - \frac{1}{r} \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{r}p_{12}^2 & p_{11} - \frac{1}{r}p_{12}p_{22} \\ p_{11} - \frac{1}{r}p_{12}p_{22} & 2p_{12} - \frac{1}{r}p_{22}^2 \end{bmatrix} .
\end{aligned}
$$

From this we find the following system of scalar equations

$$
\begin{aligned}
\dot{p}_{11} &= -\frac{1}{r}p_{12}^2 \\
\dot{p}_{12} &= p_{11} - \frac{1}{r}p_{12}p_{22} \\
\dot{p}_{22} &= 2p_{12} - \frac{1}{r}p_{22}^2 ,
\end{aligned}
$$

as we were to show. If we seek the steady state solution were $\dot{p}_{11} = \dot{p}_{12} = \dot{p}_{22} = 0$, from the above we see that $p_{12} = 0$, $p_{11} = 0$, and $p_{22} = 0$. Then since $K = P_\infty H^T R^{-1}$ we see that $K = 0$ also.

To consider the case where the true system *has* process noise, but we in fact performed a filter design *without* it recall that this is an example where we are using the correct dynamics and measurement matrices in the implementation of the filter, but an incorrect process noise vector $q^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ rather than the true value of $q = \begin{bmatrix} w \\ 0 \end{bmatrix}$. To determine the effect that this error has on our filtering equations we recall the section entitled "Exact Implementation of Dynamics and Measurements" since in this case we are correctly modeling the F and H matrices. In that section a procedure is outlined for assessing the true filters performance under modeling errors. The procedure to follow is

- Assume all filter design values are *correct* i.e. take $q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and calculate the optimal Kalman gain K in that situation.
- Use the value of K found above to solve for P in

$$
\dot{P} = (F - KH)P + P(F - KH)^T + GQG^T + KRK^T , \tag{268}
$$

  where in the above expression all variables (except K) are their *true* values.

The first part of the above procedure where we take $q = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ done earlier and where we have shown that in steady-state we get $P = 0$ and thus $K = 0$. When we put this value into Equation 268 we get

$$
\dot{P} = FP + PF^T + Q = \begin{bmatrix} 0 & p_{11} \\ p_{11} & 2p_{12} \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} q & p_{11} \\ p_{11} & 2p_{12} \end{bmatrix} ,
$$

which when written as a system of scalar equations gives the desired expression.

Chapter 8 (Implementation Considerations)

Notes on the text

Notes on the ϵ technique

In this section of the notes we derive the expression for $\hat{x}_{k+1}(+)$ expressed via the books equation 8.1-10 when we use the ϵ technique. From equation 8.1-5, the introduced expression for ϵ', and the expression for $\Delta\hat{x}_{k+1}(+)$ we have

$$
\begin{aligned}
\hat{x}_{k+1}(+) - \Phi_k\hat{x}_k(+) &= K_{k+1}\left[z_{k+1} - H_{k+1}\Phi_k\hat{x}_k(+)\right] \\
&+ \left(\frac{\epsilon r_{k+1}}{H_{k+1}P_{k+1}(-)H_{k+1}^T + r_{k+1}}\right) H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}[z_{k+1} - H_{k+1}\Phi_k\hat{x}_k(+)] \\
&= \left[K_{k+1} + \frac{\epsilon r_{k+1}H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}}{H_{k+1}P_{k+1}(-)H_{k+1}^T + r_{k+1}}\right](z_{k+1} - H_{k+1}\Phi_k\hat{x}_k(+)) .
\end{aligned}
$$

When we take K_{k+1} to be the optimal Kalman gain given by

$$
K_{k+1} = P_{k+1}(-)H_{k+1}^T(H_{k+1}P_{k+1}(-)H_{k+1}^T + r_{k+1})^{-1} ,
$$

in the above we get for the leading coefficient of $(z_{k+1} - H_{k+1}\Phi_k\hat{x}_k(+))$ the following

$$
K = \frac{P_{k+1}(-)H_{k+1}^T + \epsilon\frac{r_{k+1}H_{k+1}^T}{H_{k+1}H_{k+1}^T}}{H_{k+1}P_{k+1}(-)H_{k+1}^T + r_{k+1}} .
$$

This is the book's equation 8.1-10. Recall that H_{k+1} is a *row* matrix and r_k is a *scalar*, so K is a *column* matrix.

From the above expression we see that in this case the Kalman gain K we are using to filter with is composed of two parts $K = K_{\text{reg}} + K_{\text{ow}}$. To determine how this non optimal gain performs i.e. what the error covariance matrix $P_k(+)$ will be for such a filter we need to use the results from chapter 7. Namely the results under the section "Exact Implementation of Dynamics and Measurements". There the true a posterior error covariance matrix $P_k(+)$ when filtering with a Kalman gain K_k is given by

$$
P_k(+) = (I - K_kH_k)P_k(-)(I - K_kH_k)^T + K_kR_kK_k^T .
$$

Using the expression in this section for K we find that $P_k(+)$ is given by (dropping the k subscript for notational simplicity)

$$
\begin{aligned}
P(+) &= (I - KH)P(-)(I - KH)^T + KRK^T \\
&= (I - K_{\text{reg}}H - K_{\text{ow}}H)P(-)(I - K_{\text{reg}}H - K_{\text{ow}}H)^T + (K_{\text{reg}} + K_{\text{ow}})R(K_{\text{reg}} + K_{\text{ow}})^T \\
&= (I - K_{\text{reg}}H)P(-)(I - K_{\text{reg}}H)^T + K_{\text{reg}}RK_{\text{reg}}{}^T \\
&- (I - K_{\text{reg}}H)P(-)H^TK_{\text{ow}}{}^T - K_{\text{ow}}HP(-)(I - K_{\text{reg}}H)^T + K_{\text{ow}}HP(-)H^TK_{\text{ow}}{}^T \\
&+ K_{\text{reg}}RK_{\text{ow}}{}^T + K_{\text{ow}}RK_{\text{reg}}{}^T + K_{\text{ow}}RK_{\text{ow}}{}^T \\
&= [P(+)]_{\text{reg}} \\
&- P(-)H^TK_{\text{ow}}{}^T + K_{\text{reg}}HP(-)H^TK_{\text{ow}}{}^T - K_{\text{ow}}HP(-) + K_{\text{ow}}HP(-)H^TK_{\text{reg}}{}^T \\
&+ K_{\text{ow}}HP(-)H^TK_{\text{ow}}{}^T + rK_{\text{reg}}K_{\text{ow}}{}^T + rK_{\text{ow}}K_{\text{reg}}{}^T + rK_{\text{ow}}K_{\text{ow}}{}^T .
\end{aligned}
$$

To further evaluate this expression we will need to simplify the terms after $[P(+)]_{\text{reg}}$. To do that we will first replace K_{reg} with $P(-)H^T(HP(-)H^T + r)^{-1}$ to get

$$
\begin{aligned}
P(+) &= [P(+)]_{\text{reg}} \\
&- P(-)H^T {K_{\text{ow}}}^T + \frac{P(-)H^T HP(-)H^T {K_{\text{ow}}}^T}{HP(-)H^T + r} - K_{\text{ow}} HP(-) \\
&+ \frac{K_{\text{ow}} HP(-)H^T HP(-)}{HP(-)H^T + r} + K_{\text{ow}} HP(-)H^T {K_{\text{ow}}}^T \\
&+ \frac{rP(-)H^T {K_{\text{ow}}}^T}{HP(-)H^T + r} + \frac{rK_{\text{ow}} HP(-)}{HP(-)H^T + r} + rK_{\text{ow}} {K_{\text{ow}}}^T .
\end{aligned}
$$

Counting terms after the $[P(+)]_{\text{reg}}$ expression starting at one let us combine the first and sixth term, the third and sevenths terms, and the fifth and eighth terms to get

$$
\begin{aligned}
P(+) &= [P(+)]_{\text{reg}} \\
&+ \frac{(-HP(-)H^T)P(-)H^T {K_{\text{ow}}}^T}{HP(-)H^T + r} + \frac{P(-)H^T HP(-)H^T {K_{\text{ow}}}^T}{HP(-)H^T + r} \\
&+ \frac{(-HP(-)H^T)P(-)K_{\text{ow}} HP(-)}{HP(-)H^T + r} + \frac{K_{\text{ow}} HP(-)H^T HP(-)}{HP(-)H^T + r} \\
&+ (HP(-)H^T + r)K_{\text{ow}} {K_{\text{ow}}}^T .
\end{aligned}
$$

Now to simplify this we note that since H is a row vector the expression $HP(-)H^T$ is a scalar and can be factored out if needed. This cancels four terms and we get

$$
\begin{aligned}
P(+) &= [P(+)]_{\text{reg}} + (HP(-)H^T + r)K_{\text{ow}} {K_{\text{ow}}}^T \\
&= [P(+)]_{\text{reg}} + (HP(-)H^T + r)\frac{\epsilon^2 r H^T H r}{(HP(-)H^T + r)^2} \\
&= [P(+)]_{\text{reg}} + \frac{\epsilon^2 r H^T H r}{HP(-)H^T + r} .
\end{aligned}
$$

as we were to show.

Notes on fading memory filters and age-weighting: Example 8.1-3

In Example 8.1-3 The continuous system specified by Figure 8.1-8 is given by

$$
\begin{aligned}
\dot{x} &= 0 \quad \text{with} \quad x(0) = x_0 \\
z &= x + v \quad \text{with} \quad v \sim N(0, \sigma^2) .
\end{aligned}
$$

As a discrete system this is then given by

$$
\begin{aligned}
x_k &= x_{k-1} \\
z_k &= x_k + v_k ,
\end{aligned}
$$

with $x_0 = x_0$ and $v_k \sim N(0, \sigma^2)$. From this the variables defined in the discrete Kalman filtering case are $\phi = 1$, $q = 0$, $h = 1$ and $r = \sigma^2$. Using these the recursive Kalman filter given by equation 8.1-16 is

$$\begin{aligned} p'_k(-) &= s p'_{k-1}(+) \\ p'_k(+) &= p'_k(-) - \frac{p'_k(-)^2}{p'_k(-) + \sigma^2} \\ &= s p'_{k-1}(+) - \frac{s^2 p'_{k-1}(+)^2}{s p'_{k-1}(+) + \sigma^2} = \frac{\sigma^2 s p'_{k-1}(+)}{\sigma^2 + s p'_{k-1}(+)} . \end{aligned} \tag{269}$$

In the steady state we have $p'_k(+) = p'_{k-1}(+) = p'_\infty$ and using the above expression we see that p'_∞ is given by

$$p'_\infty = \frac{\sigma^2 s p'_\infty}{\sigma^2 + s p'_\infty} ,$$

or

$$s {p'_\infty}^2 + \sigma^2 p'_\infty - \sigma^2 s p'_\infty = 0 .$$

Solving for p'_∞ in the above gives

$$p'_\infty = \frac{\sigma^2(s-1)}{s} = \sigma^2 \left(1 - \frac{1}{s}\right) .$$

To determine k_∞ note that it is given by

$$k_\infty = p'_\infty(-) H_\infty^T (H_\infty p'_\infty(-) H_\infty^T + \sigma^2)^{-1} = \frac{p'_\infty(-)}{p'_\infty(-) + \sigma^2} ,$$

with $p'_\infty(-)$ related to $p'_\infty(+)$ (which we know) by Equation 269 or

$$p'_\infty(-) = s p'_\infty(+) = \sigma^2(s-1) .$$

Thus we get

$$k_\infty = \sigma^2(s-1)(\sigma^2(s-1) + \sigma^2)^{-1} = 1 - \frac{1}{s} .$$

To calculate the true error covariances $p_k(+)$ (note no prime) see Problem 8-2.

Notes on prefiltering

For the simple example given we find that we can evaluate the expectation of the additional noise due to smoothing the signal as

$$\begin{aligned} E\left[\left(\frac{1}{2}\sum_{i=1}^{2} x_i - x_2\right)^2\right] &= \frac{1}{4} E\left[\left(\sum_{i=1}^{2} x_i - 2x_2\right)^2\right] \\ &= \frac{1}{4} E\left[(x_1 - x_2)^2\right] = \frac{1}{4} E[x_1^2 - 2x_1x_2 + x_2^2] \\ &= \frac{1}{4}(\sigma_x^2 - 2\sigma_x^2 e^{-\Delta t} + \sigma_x^2) = \frac{\sigma_x^2}{2}(1 - e^{-\Delta t}) , \end{aligned}$$

which is the books equation 8.2-8.

Notes on algorithms and integration rules

From the definition of Q_k in terms of the continuous system recall that we have

$$Q_k = \int_{t_k}^{t} \Phi(t,\tau)Q(\tau)\Phi^T(t,\tau)d\tau \,, \tag{270}$$

so that we can take the t derivative of Q_k using Leibniz' rule as

$$\begin{aligned} \frac{dQ_k}{dt} &= Q(t) + \int_{t_k}^{t} F(t)\Phi(t,\tau)Q(\tau)\Phi^T(t,\tau)d\tau + \int_{t_k}^{t} \Phi(t,\tau)Q(\tau)\Phi^T(t,\tau)F^T(t)d\tau \\ &= Q(t) + F(t)Q_k + Q_kF^T(t)\,, \end{aligned}$$

gives equation 8.3-6. We have used the facts that $\frac{d\Phi(t,\tau)}{dt} = F(t)\Phi(t,\tau)$ and $\Phi(t,t) = I$.

Notes on integration algorithms

Take a second order approximation of the Taylor expansion of $x(t)$ about t_k as

$$x_{k+1} = x_k + \dot{x}_k\Delta t_k + \frac{1}{2}\ddot{x}_k{\Delta t_k}^2\,,$$

where the approximation we will use for $\ddot{x}_k$ is

$$\ddot{x}_k = \frac{\dot{x}_{k+1} - \dot{x}_k}{\Delta t_k}\,.$$

When we put this approximation for $\ddot{x}_k$ in the above we get

$$x_{k+1} = x_k + \dot{x}_k\Delta t_k + \frac{1}{2}{\Delta t_k}^2\left(\frac{\dot{x}_{k+1} - \dot{x}_k}{\Delta t_k}\right) = x_k + \frac{\Delta t_k}{2}(\dot{x}_{k+1} + \dot{x}_k)\,.$$

To finish this evaluation we need to find an expression for $\dot{x}_{k+1}$. From our differential equation $\dot{x} = f(x,t)$, if we take $\dot{x}_{k+1} \approx f(x_k + \dot{x}_k\Delta t_k, t_k)$ then the above becomes

$$x_{k+1} = x_k + \frac{\Delta t_k}{2}(f(x_k + \dot{x}_k\Delta t_k, t_k) + \dot{x}_k)\,. \tag{271}$$

or the books equation 8.3-16. This method is known as the **modified Euler** method. Note that since our differential equation is given by $\dot{x} = f(x,t)$ one might also evaluate $\dot{x}_{k+1}$ as $f(x_k + \dot{x}_k\Delta t_k, t_{k+1})$ where we have evaluated f at the time t_{k+1} rather than t_k. Since we are assuming that $\Delta t_k = t_{k+1} - t_k \ll 1$ there is not much difference between either approximation.

Notes on the mathematical form of equations

In this subsection of these notes we argue that different forms for the a priori to a posteriori equation (i.e. computing $P(+)$ from $P(-)$) have different computational properties and that the so called Joseph's form for computing $P(+)$ from $P(-)$ is to be preferred all other things being equal. Dropping subscripts for notational simplicity, to begin we consider computing $P(+)$ via $P(+) = (I - KH)P(-)$ under a perturbation in the Kalman gain K. To do this we take $K \to K + \delta K$ and see that the new $P(+)$ then becomes

$$P(+) = (I - KH)P(-) \to (I - KH - \delta KH)P(-) = (I - KH)P(-) - \delta KHP(-)\,,$$

thus the change in $P(+)$ or $\delta P(+)$ is given by

$$\delta P(+) = -\delta KHP(-)\,.$$

When we compute $P(+)$ using the Joseph form and the same perturbation in K we have

$$\begin{aligned}
P(+) &= (I - KH)P(-)(I - KH)^T + KRK^T \\
&\to (I - KH - \delta KH)P(-)(I - KH - \delta KH)^T + (K + \delta K)R(K + \delta K)^T \\
&= (I - KH)P(-)(I - KH)^T - (I - KH)P(-)(\delta KH)^T - \delta KHP(-)(I - KH)^T \\
&+ \delta KHP(-)(\delta KH)^T + KRK^T + \delta KRK^T + KR\delta K^T + \delta KR\delta K^T\,.
\end{aligned}$$

Therefore the change in $P(+)$ is given by

$$\begin{aligned}
\delta P(+) &= -(I - KH)P(-)H^T\delta K^T - \delta KHP(-)(I - KH)^T + \delta KHP(-)H^T\delta K^T \\
&+ \delta KRK^T + KR\delta K^T + \delta KR\delta K^T \\
&= \delta K[RK^T - HP(-)(I - KH)^T] + [KR - (I - KH)P(-)H^T]\delta K^T \\
&+ \delta K[HP(-)H^T + R]\delta K^T\,.
\end{aligned}$$

To simplify this, consider the expression for $RK^T - HP(-)(I - KH)^T$ when we put in the optimal Kalman gain $K = P(-)H^T(HP(-)H^T + R)^{-1}$. We see that we get

$$\begin{aligned}
RK^T - HP(-)(I - KH)^T &= RK^T - HP(-) + HP(-)H^TK^T \\
&= (R + HP(-)H^T)K^T - HP(-) = 0\,.
\end{aligned}$$

Thus in this case we see that

$$\delta P(+) = \delta K[HP(-)H^T + R]\delta K^T\,,$$

or is zero to first order.

Problem Solutions

Problem 8-1 (the covariance matrix $P_k(+)$ when using the ϵ technique)

This result is verified in these notes in the section on the ϵ technique. See Page 158 where it is derived.

Problem 8-2 (the expression for p_∞ for Example 8.1-3)

If we assume when working this example that we will be filtering with the *correct* dynamic and measurement model i.e. with correct F and H matrices but with the the non-optimal Kalman gain k_∞ given by

$$k_\infty = 1 - \frac{1}{s},$$

then from Chapter 7 in the section entitled "Exact Implementation of Dynamics and Measurements" the true error covariance is given by $P_k(+)$ obtained by solving the following

$$\begin{aligned}
P_k(+) &= (I - K_k H_k) P_k(-) (I - K_k H_k)^T + K_k R_k K_k^T \\
P_{k+1}(-) &= \Phi_k P_k(+) \Phi_k^T + Q_k .
\end{aligned}$$

In the case considered here these become

$$\begin{aligned}
p_k(+) &= \left(1 - \left(1 - \frac{1}{s}\right)\right) p_k(-) \left(1 - \left(1 - \frac{1}{s}\right)\right) + \left(1 - \frac{1}{s}\right)^2 \sigma^2 \\
p_{k+1}(-) &= p_k(+) .
\end{aligned}$$

So we have

$$p_k(+) = \left(1 - \frac{1}{s}\right)^2 \sigma^2 + \frac{1}{s^2} p_k(+) + \frac{1}{s^2} p_k(+) .$$

When we let $k \to \infty$ where we get

$$\left(1 - \frac{1}{s^2}\right) p_\infty(+) = \left(1 - \frac{1}{s}\right)^2 \sigma^2 ,$$

or when we solve for $p_\infty(+)$ and simplify some we get

$$p_\infty(+) = \frac{(s-1)^2}{s^2 - 1} \sigma^2 = \frac{s-1}{s+1} \sigma^2 ,$$

the result we were to show.

Problem 8-3 (verification of sequential observations)

For the second measurement we have $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $R = [1]$ so this measurement updates the $P_i(+)$ covariance matrix (i for intermediate) as follows

$$\begin{aligned}
P(+) &= P_i(+) - P_i(+) H^T [H P_i(+) H^T + R]^{-1} H P_i(+) \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{8} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{7}{8} + 1\right)^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{8} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{8} \end{bmatrix} - \frac{8}{15} \begin{bmatrix} \frac{1}{4} \\ \frac{7}{8} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{7}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{7}{8} \end{bmatrix} - \frac{8}{15} \begin{bmatrix} \frac{1}{16} & \frac{7}{32} \\ \frac{7}{32} & \frac{49}{64} \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{15} & \frac{2}{15} \\ \frac{2}{15} & \frac{7}{15} \end{bmatrix} ,
\end{aligned}$$

which provides the result we want to show.

Problem 8-6 (the exponential series)

The books equation 8.3-18 is given by

$$\Phi(t_2, t_1) = \sum_{n=0}^{\infty} \frac{\Delta t^n F(t_1)^n}{n!} = I + \Delta t F(t_1) + \frac{\Delta t^2}{2} F(t_1)^2 + \cdots , \tag{272}$$

where $\Delta t = t_2 - t_1$. Since the function Φ has translational invariance with respect to time, that is $\Phi(t_2, t_1) = \Phi(t_2 - t_1, 0)$, we can simplify the problem by considering only a single variable by taking $t_1 = 0$ and $t_2 = t$. In addition, since we are interested only in small times from the time t_k we can consider methods for approximating $\Phi(\Delta t, 0) = \Phi(t_{k+1}, t_k)$, where $\Delta t = t_{k+1} - t_k$.

To show the equivalence of this expression with various integration methods, we first recall that $\Phi(t, 0)$ is the solution to $\frac{d\Phi(t,0)}{dt} = F(t)\Phi(t, 0)$ with initial condition given by $\Phi(0, 0) = I$. Then note that if we approximate the solution to this differential equation at Δt using Euler's method

$$x_{k+1} = x_k + f(x_k, t_k)\Delta t_k , \tag{273}$$

so that the state $x(t)$ is $\Phi(t, 0)$, the initial time t_k is 0, the final time $t_{k+1} = \Delta t$, and $f(x, t) = F(t)x$ so that $f(x_k, t_k) = F(0)\Phi(0, 0)$ to get

$$\Phi(\Delta t, 0) = \Phi(0, 0) + \Delta t F(0)\Phi(0, 0) = I + \Delta t F(0) ,$$

or the first two terms in Equation 272. Alternatively if we approximate this differential equation with the modified Euler method given by Equation 271 we get

$$\begin{aligned}
\Phi(\Delta t, 0) &= \Phi(0, 0) + \frac{\Delta t}{2} \left[F(0)(\Phi(0, 0) + F(0)\Delta t) + F(0)\right] \\
&= I + \frac{\Delta t}{2}[2F(0) + \Delta t F(0)^2] = I + \Delta t F(0) + \frac{\Delta t^2}{2} F(0)^2 ,
\end{aligned}$$

or the first three terms in Equation 272.

Problem 8-7 (a WW^T decomposition)

We let the matrix W be $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we find

$$WW^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} .$$

Setting this expression equal to $P = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ gives the scalar equations

$$\begin{aligned}
a^2 + b^2 &= 2 \\
ac + bd &= 1 \\
c^2 + d^2 &= 2 .
\end{aligned}$$

From this we see that we have three equations and four unknowns and therefore no unique solution. If we take W to be lower triangular then $b = 0$ and the equations above simplify to

$$\begin{aligned} a^2 &= 2 \\ ac &= 1 \\ c^2 + d^2 &= 2\,. \end{aligned}$$

One solution to these is to take $a = \sqrt{2}$, and then $c = \frac{1}{\sqrt{2}}$ and $d^2 = 2 - \frac{1}{2} = \frac{3}{2}$ so $d = \sqrt{\frac{3}{2}}$.

Chapter 9 (Additional Topics)

Notes on the text

Notes on adaptive Kalman filtering

Recall that the state error $\tilde{x}$ is defined as $\tilde{x} = \hat{x} - x$ and using that we can write the innovation ν as $\nu = -H\tilde{x} + v$. Consider two times t_1 and t_2 where $t_2 > t_1$ and lets compute $E[\nu(t_2)\nu(t_1)^T]$. In terms of $\tilde{x}$ and v this is given by

$$\begin{aligned}
E[\nu(t_2)\nu(t_1)^T] &= E[(H(t_2)\tilde{x}(t_2) - v(t_2))(H(t_1)\tilde{x}(t_1) - v(t_1))^T] \\
&= H(t_2)E[\tilde{x}(t_2)\tilde{x}(t_1)^T]H(t_1)^T - H(t_2)E[\tilde{x}(t_2)v(t_1)^T] \\
&- E[v(t_2)\tilde{x}(t_1)^T]H(t_1)^T + E[v(t_2)v(t_1)^T] .
\end{aligned}$$

Now to evaluate this expression we note that the measurement errors observed at the time t_1 i.e. $v(t_1)$ *can* and will affect our estimate error at the later time t_2 i.e. $\tilde{x}(t_2)$, thus we can't conclude that $E[\tilde{x}(t_2)v(t_1)^T] = 0$ since $\tilde{x}(t_2)$ depends in on what $v(t_1)$ was. On the other hand the measurement errors observed at the later time t_2 i.e. $v(t_2)$ will not affect or modify our estimation error made earlier i.e. $\tilde{x}(t_1)$, thus $E[v(t_2)\tilde{x}(t_1)^T] = 0$. Using the known correlation of v i.e. $E[v(t_2)v(t_1)^T] = R(t_1)\delta(t_1 - t_2)$ we have

$$E[\nu(t_2)\nu(t_1)^T] = H(t_2)E[\tilde{x}(t_2)\tilde{x}(t_1)^T]H(t_1)^T - H(t_2)E[\tilde{x}(t_2)v(t_1)^T] + R(t_1)\delta(t_2 - t_1) , \quad (274)$$

or the book's equation 9.1-7.

Recall that we have derived the differential equation that $\tilde{x}$ satisfies in Equation 73. From this equation we see that the solution for $\tilde{x}(t_2)$ given by

$$\tilde{x}(t_2) = \Phi(t_2, t_1)\tilde{x}(t_1) - \int_{t_1}^{t_2} \Phi(t_2, \tau)[G(\tau)w(\tau) - K(\tau)v(\tau)]d\tau , \quad (275)$$

where $\Phi(t_2, t_1)$ is the transition matrix corresponding to $F - KH$. Using this expression for $\tilde{x}(t_2)$ we can now compute terms needed to evaluate Equation 274. To begin we compute $E[\tilde{x}(t_2)\tilde{x}^T(t_1)]$ as

$$\begin{aligned}
&= E\left[\Phi(t_2, t_1)\tilde{x}(t_1)\tilde{x}^T(t_1) - \int_{t_1}^{t_2} \Phi(t_2, \tau)[G(\tau)w(\tau)\tilde{x}^T(t_1) - K(\tau)v(\tau)\tilde{x}^T(t_1)]d\tau\right] \\
&= \Phi(t_2, t_1)P(t_2) ,
\end{aligned}$$

where we have used the facts that $E[w(\tau)\tilde{x}^T(t_1)] = 0$ and $E[v(\tau)\tilde{x}^T(t_1)] = 0$ when $\tau > t_1$. Next we evaluate $E[\tilde{x}(t_2)v^T(t_1)]$ and find

$$\begin{aligned}
&= E\left[\Phi(t_2, t_1)\tilde{x}(t_1)v^T(t_1) - \int_{t_1}^{t_2} \Phi(t_2, \tau)[G(\tau)w(\tau)v^T(t_1) - K(\tau)v(\tau)v^T(t_1)]d\tau\right] \\
&= 0 + \int_{t_1}^{t_2} \Phi(t_2, \tau)K(\tau)E[v(\tau)v^T(t_1)]d\tau = \Phi(t_2, t_1)K(t_1)R(t_1) .
\end{aligned}$$

Thus using these two expressions in Equation 274 we find that

$$\begin{aligned} E[\nu(t_2)\nu(t_1)^T] &= H(t_2)\Phi(t_2,t_1)P(t_1)H^T(t_1) - H(t_2)\Phi(t_2,t_1)K(t_1)R(t_1) + R(t_1)\delta(t_2-t_1) \\ &= H(t_2)\Phi(t_2,t_1)[P(t_1)H^T(t_1) - K(t_1)R(t_1)] + R(t_1)\delta(t_2-t_1)\,, \end{aligned} \tag{276}$$

this is the books equation 9.1-12. If our filter is optimal the the optimal expression for K is given by $K(t_1) = P(t_1)H^T(t_1)R^{-1}(t_1)$ so that the above then becomes

$$E[\nu(t_2)\nu(t_1)^T] = R(t_1)\delta(t_2-t_1)\,.$$

If our dynamics $\dot{x} = Fx + Gw$ is linear and time-invariant then the transition matrix $\Phi(t_2, t_1)$ is a function of only $\tau = t_2 - t_1$ as

$$\Phi(t_2,t_1) = \Phi(t_1+\tau,t_1) = e^{(F-KH)|\tau|}\,,$$

and Equation 276 becomes

$$E[\nu(t_1+\tau)\nu(t_1)^T] = He^{(F-KH)|\tau|}[PH^T - KR] + R\delta(\tau)\,, \tag{277}$$

or the books equation 9.1-14. In the above all matrices F, K, etc are evaluated at $t = t_1$.

Notes on Example 9.1-1

For this example our system and measurement equations are given by

$$\begin{aligned} \dot{x} &= w \quad \text{with} \quad w \sim N(0,q) \\ z &= x + v \quad \text{with} \quad v \sim N(0,r)\,, \end{aligned}$$

So the system and measurement matrices are scalars with $f = 0$ and $h = 1$. We will filter our signal z using $\dot{\hat{x}} = k(z - \hat{x})$ where k is *non optimal* i.e. derived from erroneous values of q and r. As we filter z with this value of k, we will be observing the innovations ν at each time defined as $\nu = z - h\hat{x} = z - \hat{x}$. Using Equation 277 for this system we find that it becomes

$$E[\nu(t)\nu(t-\tau)] = e^{-k|\tau|}(p_\infty - kr) + r\delta(\tau)\,. \tag{278}$$

Note that in the above expression we can compute the left-hand-side based on the *realized* observations of the innovation function $\nu(t)$ and call this empirically computed function $\phi_{\nu\nu}(\tau)$. By performing a least squares fit or (using some other method) we fit the empirically obtained $\phi_{\nu\nu}(\tau)$ function to a autocorrelation model of the form $Ae^{-k|\tau|} + B$, for some unknown coefficients A and B. Once we have empirical estimates of the coefficients A and B using Equation 278 from the above model we see that these are estimates of the expressions $p_\infty - kr$ and r. Since we know the value of k using in filtering this means we have an estimate of p_∞. The expression p_∞ is the steady state solution to the linear variance equation for *Wiener* filtering, since we are not filtering with the optimal gain value k (but are instead using its steady-state value). Thus we need to find the steady state solution p_∞ to

$$\dot{P} = (F-KH)P + P(F-KH)^T + GQG^T + KRK^T\,,$$

or

$$0 = -2kp_\infty + q + k^2 r\,,$$

which means that

$$p_\infty = \frac{q + k^2 r}{2k}\,.$$

Since we have estimates of r and p_∞ and we know the value of k we now have an estimate of q. Showing that for this simple example we can identify the initially unknown values of q and r.

Notes on observers for deterministic systems

In this subsection and the next we introduce and discuss the notation of an *observer*. Basically an observer is another transformation of the state $x(t)$ (in addition to the measurement $z(t) = H(t)x(t)$) that will estimate and that will allow us to determine a complete specification of our state $x(t)$. We begin by requiring that the relationship between our observer $\xi(t)$ and state $x(t)$ should be

$$\xi(t) = T(t)x(t)\,.$$

In addition we would like our observe to have the property that if we know $\xi(t)$ and $z(t)$ then we can construct an estimate of $x(t)$ by inverting the combined measurement observer system

$$\left[\frac{\xi(t)}{z(t)}\right] = \left[\frac{T(t)}{H(t)}\right] x(t)\,,$$

as

$$x(t) = \left[\frac{T(t)}{H(t)}\right]^{-1} \left[\frac{\xi(t)}{z(t)}\right]\,.$$

Once we have specified the expression we will use for $T(t)$ we can actually compute the inverse $\left[\frac{T(t)}{H(t)}\right]^{-1}$ above. Since this inverse then multiplies the stacked vector $\left[\frac{\xi(t)}{z(t)}\right]$, we will *define* it in terms of two more unknowns $A(t)$ and $B(t)$ as the matrix $\left[\begin{array}{c|c} A(t) & B(t) \end{array}\right]$. These unknowns makes the state $x(t)$ from the observer $\xi(t)$ and measurement $z(t)$ equation simple

$$x(t) = A(t)\xi(t) + B(t)z(t)\,. \tag{279}$$

Thus one way to state what we are doing is to observe that if we can obtain an expression for $T(t)$ then we can form the stacked matrix $\left[\frac{T(t)}{H(t)}\right]$, invert it, and obtain the block matrices $A(t)$ and $B(t)$. With these we can construct $x(t)$ using Equation 279.

We next derive some relationships between the block matrices introduced thus far. From the definition that $\left[\begin{array}{c|c} A & B \end{array}\right]$ is the inverse of $\left[\frac{T}{H}\right]$ we have

$$\left[\begin{array}{c|c} A & B \end{array}\right] \left[\frac{T}{H}\right] = AT + BH = I\,. \tag{280}$$

On taking the product in the other order we have

$$\left[\begin{array}{c} T \\ \hline H \end{array}\right] \left[\begin{array}{c|c} A & B \end{array}\right] = I \,, \tag{281}$$

which by evaluating the matrix product on the left-hand-side we have the block identity

$$\begin{bmatrix} TA & TB \\ HA & HB \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} . \tag{282}$$

This in turn gives the four equations $TA = I$, $TB = 0$, $HA = 0$, and $HB = I$. Taking the time derivative of the above block matrix identity while using the product rule gives another set of four constraints

$$\begin{bmatrix} \dot{T}A + T\dot{A} & \dot{T}B + T\dot{B} \\ \dot{H}A + H\dot{A} & \dot{H}B + H\dot{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} . \tag{283}$$

The result of this expression is that they allow us to move the time derivative on one factor in a product to the other factor in the product while we introduce a negative sign. For example, the $(1,1)$ and $(1,2)$ components imply the relationships $\dot{T}A = -T\dot{A}$ and $\dot{T}B = -T\dot{B}$.

From how we have defined the observer $\xi(t)$ its differential equation can be computed using the relationships introduced above and the true state dynamics of $x(t)$ as

$$\begin{aligned} \dot{\xi} &= \dot{T}x + T\dot{x} \\ &= \dot{T}(A\xi + Bz) + T(F(A\xi + Bz) + Lu) \\ &= (\dot{T}A + TFA)\xi + (\dot{T}B + TFB)z + TLu \,. \end{aligned} \tag{284}$$

which is the books equation 9.2-10. If we use two expressions $\dot{T}A = -T\dot{A}$ and $\dot{T}B = -T\dot{B}$ in Equation 284 we get

$$\dot{\xi} = (TFA - T\dot{A})\xi + (TFB - T\dot{B})z + TLu \,, \tag{285}$$

which is the books equation 9.2-11. Then assuming we had a T matrix (and thus the A and B matrices) we would use Equation 285 to propagate an estimate of $\xi(t)$ namely $\hat{\xi}(t)$ and then use this estimate in Equation 279 to derive an estimate of x. As a next step we must make sure that whatever choice we make for T any initial error in our estimate of ξ and x will exponentially propagate to zero. Thus we need to study the properties of the *error* in our estimates of ξ and x.

To do this we begin with the error in ξ as $\tilde{\xi}$ defined in the normal way as $\tilde{\xi} = \hat{\xi} - \xi$ with ξ satisfying Equation 285 and our estimate $\hat{\xi}$ satisfying the same functional form as the differential equation that ξ satisfies. That is we propagate $\hat{\xi}$ using

$$\dot{\hat{\xi}} = (TFA - T\dot{A})\hat{\xi} + (TFB - T\dot{B})z + TLu \,.$$

From these two equation we see that our error $\tilde{\xi}$ satisfies

$$\dot{\tilde{\xi}} = (TFA - T\dot{A})\tilde{\xi} \,. \tag{286}$$

Thus how the error in ξ behaves is determined by the eigenvalues of the matrix $TFA - T\dot{A}$. This observation guides the specification of the T matrix in that we would like this matrix to have small eigenvalues and thus convergence of $\hat{\xi}$ to ξ to be "fast".

Now to study the error in x or $\tilde{x} = \hat{x} - x$. Using the facts that $x = A\xi + Bz$ and $\hat{x} = A\hat{\xi} + Bz$ we see that $\tilde{x}$ can be written as

$$\tilde{x} = \hat{x} - x = A\hat{\xi} + Bz - A\xi - Bz = A(\hat{\xi} - \xi) = A\tilde{\xi}\,,$$

or the simple relationship

$$\tilde{x} = A\tilde{\xi}\,, \tag{287}$$

which is the book's equation 9.2-16. If we premultiply this by T and use the fact that $TA = I$ we get

$$\tilde{\xi} = T\tilde{x}\,, \tag{288}$$

which is the book's equation 9.2-17. Now we can get the differential equation for the error in x from the corresponding differential equation for the error in ξ as

$$\dot{\tilde{x}} = \frac{d}{dt}(A\tilde{\xi}) = \dot{A}\tilde{\xi} + A\dot{\tilde{\xi}}\,,$$

using Equation 286 we have

$$\dot{\tilde{x}} = (\dot{A} + A(TFA - T\dot{A}))\tilde{\xi}\,,$$

but $\tilde{\xi} = T\tilde{x}$ so we

$$\begin{aligned}
\dot{\tilde{x}} &= (\dot{A} + ATFA - AT\dot{A})T\tilde{x} \\
&= (\dot{A}T + ATFAT - AT\dot{A}T)\tilde{x}\,. \qquad (289)
\end{aligned}$$

Note that we can further simplify this by noting that if we premultiply Equation 288 by A to get $A\tilde{\xi} = AT\tilde{x}$ and then use Equation 287 to replace $A\tilde{\xi}$ with $\tilde{x}$ we end with

$$\tilde{x} = AT\tilde{x}\,. \tag{290}$$

Thus replacing AT in the second term on the right-hand-side of Equation 289 we have

$$\dot{\tilde{x}} = (\dot{A}T + ATF - AT\dot{A}T)\tilde{x}\,, \tag{291}$$

which is the books equation 9.2-18. From Equation 280 or $AT + BH = I$ we can write AT as $AT = I - BH$ and then get for $\dot{\tilde{x}}$ the following

$$\dot{\tilde{x}} = (\dot{A}T + (I - BH)F - (I - BH)\dot{A}T)\tilde{x} = (F - BHF + BH\dot{A}T)\tilde{x}\,.$$

We next replace the $H\dot{A}$ in the third term in the above with $-\dot{H}A$ from Equation 283 to get a third term that looks like

$$BH\dot{A}T\tilde{x} = -B\dot{H}AT\tilde{x} = -B\dot{H}\tilde{x}\,,$$

where we used $AT\tilde{x} = \tilde{x}$, to simplify. Using this for the third term for $\dot{\tilde{x}}$ we finally get

$$\dot{\tilde{x}} = (F - BHF - B\dot{H})\tilde{x}\,, \tag{292}$$

which is the books equation 9.2-19.

Notes on Example 9.2-1

For this given example we have the noiseless measurement

$$z = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so that $H = \begin{bmatrix} 1 & 0 \end{bmatrix}$. It is helpful to consider the dimensions of the matrices involved in this problem. Now our state dimension n is 2 and $H \in \mathbb{R}^{1\times 2}$ gives us one noiseless measurement $(m = 1)$ thus to derive $n - m = 1$ more with an observer we have $T \in \mathbb{R}^{1\times 2}$ to give a second observation via our observer $\xi \in \mathbb{R}$. Then given the measurement z and an estimate of our observer, $\hat{\xi}$, we use matrices $A(t)$ and $B(t)$ as Kalman like gains to construct an estimate of x from

$$\hat{x} = A(t)\hat{\xi}(t) + B(t)z(t)\,.$$

From which we see that the dimensions of A and B are $A \in \mathbb{R}^{2\times 1}$ and $B \in \mathbb{R}^{2\times 1}$. From the $(2, 2)$ component of Equation 282 in terms of these vectors gives

$$HB = I = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1 = 1\,,$$

and thus b_2 is currently unspecified. The differential equation for $\tilde{x}$ or $\dot{\tilde{x}} = (F - BHF)\tilde{x}$ for this problem has the matrix $F - BHF$ given by

$$\begin{aligned} F - BHF &= \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} - \begin{bmatrix} 1 \\ b_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ b_2 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ -b_2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -b_2 - \beta \end{bmatrix}. \end{aligned}$$

A nice property would be to have $\tilde{x}$ converge to zero faster than the system response time which is β. To achieve this we would like to make the $m = 1$ eigenvalue of $F - BHF$ which is $\lambda = -(\beta + b_2)$ "significantly" smaller than β. One way to do this is to take $\lambda = -5\beta$ so that $b_2 = 4\beta$ and we now have that $B = \begin{bmatrix} 1 \\ 4\beta \end{bmatrix}$.

From the matrix dimensions discussed above, for the most general A and T, we can take $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $T = \begin{bmatrix} t_1 & t_2 \end{bmatrix}$. Using these general expressions, the three additional requirements from Equation 282 become

$$\begin{aligned} TA &= t_1 a_1 + t_2 a_2 = 1 \\ TB &= \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 1 \\ 4\beta \end{bmatrix} = t_1 + 4\beta t_2 = 0 \\ HA &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 = 0\,. \end{aligned}$$

Since $a_1 = 0$ the one requirement from Equation 280 is

$$AT + BH = \begin{bmatrix} 0 \\ a_2 \end{bmatrix} \begin{bmatrix} t_1 & t_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4\beta \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 a_2 + 4\beta & t_2 a_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we end with the set of equations

$$\begin{aligned} t_2 a_2 &= 1 \\ t_1 + 4\beta t_2 &= 0 \\ t_1 a_2 + 4\beta &= 0\,. \end{aligned}$$

Since the last equation can be obtained by multiplying the second equation by a_2 and using the first equation we have two equations and three unknowns. One solution can be found by taking $a_2 = t_2 = 1$, and then $t_1 = -4\beta$.

To finish this example we would solve Equation 285 (with ξ replaced with $\hat{\xi}$) and then estimate x using $\hat{x} = A(t)\hat{\xi}(t) + B(t)z(t)$. Equation 285 for $\hat{\xi}$ in this case is

$$\begin{aligned} \dot{\hat{\xi}} &= \left(\begin{bmatrix} -4\beta & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \hat{\xi} + \left(\begin{bmatrix} -4\beta & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} 1 \\ 4\beta \end{bmatrix} \right) z \\ &+ \begin{bmatrix} -4\beta & 1 \end{bmatrix} \begin{bmatrix} 0 \\ l \end{bmatrix} u \\ &= -5\beta\hat{\xi} - (16\beta^2 + \beta) z + lu = -5\hat{\xi} - 17z - 1\,. \end{aligned}$$

With an initial condition on $\hat{\xi}$ given by

$$\hat{\xi}(0) = T(0)\hat{x}(0) = \begin{bmatrix} -4\beta & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = -4\beta x_1(0) + x_2(0) = -4(1) + 0 = -4\,.$$

The true system evolves as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

with initial condition $x_1(0) = 1$, $x_2(0) = 1$, and we solve the above differential equation for $0 \le t \le \infty$. Then since our measurement $z = x_1$ solving these three equations is equivalent to solving the coupled set system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 17 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix},$$

with initial condition of $\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}$. Once we have $\hat{\xi}$ as a function of time, x is reconstructed via

$$\hat{x} = A\hat{\xi} + Bz = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{\xi} + \begin{bmatrix} 1 \\ 4\beta \end{bmatrix} x_1(t) = \begin{bmatrix} x_1(t) \\ \hat{\xi} + 4x_1(t) \end{bmatrix}.$$

Notes on observers for stochastic systems

In this section of these notes we provide further details on observers, but in this case we consider the situation where in addition to exact measurements (considered above) we have noisy measurements. In this case the measurements are a combination of noisy and noise-free as

$$z = \left[\frac{z_1}{z_2}\right] = \left[\frac{H_1}{H_2}\right] x + \left[\frac{v_1}{0}\right] .$$

Here z is a vector of dimension m and we consider the case where there are m_1 noise measurements and m_2 noise-free measurements where m_2 must equal $m - m_1$.

Using the standard definition of the error in ξ as $\tilde{\xi} = \hat{\xi} - \xi$ we can derive the differential equation for $\tilde{\xi}$ by take the time derivative of this difference by using the postulated expressions for $\dot{\xi}$ and $\dot{\hat{\xi}}$. When we do this we find

$$\dot{\tilde{\xi}} = (TFA - T\dot{A})\tilde{\xi} + TB_1(z_1 - H\hat{x}) - TGw . \tag{293}$$

We next would like to derive the expression for the differential equation for the error in our state $\tilde{x}$. To do this we need to derive a few axillary results. The first is to note that that $\xi = Tx$ and $\hat{\xi} = T\hat{x}$, so that $\tilde{\xi} = T\tilde{x}$. The second is to note that that we can write the error correction term above as

$$z_1 - H_1\hat{x} = H_1 x + v_1 - H_1\hat{x} = -H_1\tilde{x} + v_1 .$$

Next we show that $\tilde{x} = A\tilde{\xi}$ which can be done as follows

$$\begin{aligned} \tilde{x} &= \hat{x} - x \\ &= A\hat{\xi} + B_2 z_2 - (A\xi + B_2 z_2) \\ &= A\tilde{\xi} . \end{aligned} \tag{294}$$

Starting with this last expression, $\tilde{x} = A\tilde{\xi}$, by taking the time derivative as $\dot{\tilde{x}} = \dot{A}\tilde{\xi} + A\dot{\tilde{\xi}}$, when we use $\dot{\tilde{\xi}}$ given by Equation 293 we get

$$\dot{\tilde{x}} = \dot{A}\tilde{\xi} + A(TFA - T\dot{A})\tilde{\xi} + ATB_1(z_1 - H_1\hat{x}) - ATGw .$$

Since $\tilde{\xi} = T\tilde{x}$ and $z_1 - H_1\hat{x} = -H_1\tilde{x} + v_1$ the above becomes

$$\begin{aligned} \dot{\tilde{x}} &= \dot{A}T\tilde{x} + ATFAT\tilde{x} - AT\dot{A}T\tilde{x} - ATB_1H_1\tilde{x} + ATB_1v_1 - ATGw \\ &= (\dot{A}T + ATFAT - AT\dot{A}T - ATB_1H_1)\tilde{x} + ATB_1v_1 - ATGw . \end{aligned}$$

Now we will simplify this by showing that $AT\tilde{x} = \tilde{x}$. By premultipling $\tilde{\xi} = T\tilde{x}$ by A we have $A\tilde{\xi} = AT\tilde{x}$ and since $A\tilde{\xi} = \tilde{x}$ by Equation 294 we have shown that

$$\dot{\tilde{x}} = (\dot{A}T + ATF - AT\dot{A}T - ATB_1H_1)\tilde{x} + ATB_1v_1 - ATGw , \tag{295}$$

which is the books equation 9.2-32. To further simplify this recall that from 9.2-18 we derived Equation 292 an equivalent express for the first three terms in the above or

$$\dot{A}T + ATF - AT\dot{A}T = F - BHF - B\dot{H} .$$

To modify this expression for the case of noiseless and noisy measurements considered here we take $B \to B_2$ and $H \to H_2$ since the subscript 2 represents the noiseless measurements. Using this expression in the first three terms and $AT = I - B_2 H_2$ in the last term the differential equation for $\tilde{x}$ becomes

$$\dot{\tilde{x}} = (F - B_2 H_2 F - B_2 \dot{H}_2 - ATB_1 H_1)\tilde{x} + ATB_1 v_1 + (I - B_2 H_2) G w \,, \tag{296}$$

which is the books equation 9.2-33.

Notice that if we replace B_1 in the above with ATB_1 we see that the expression ATB_1 becomes $AT(ATB_1) = ATATB_1 = ATB_1$, since $TA = I$. Thus the transformation given by $B_1 \to ATB_1$ leave the right-hand-side of the above unmodified. The book argues that this means that we can *also* perform the transformation $ATB_1 \to B_1$.

Warning: I don't really see the logic in the books argument. If anyone knows of a better argument for making this substitution please let me know.

If we can do this transformation however we get for $\tilde{x}$ the following

$$\dot{\tilde{x}} = (F - B_2 H_2 F - B_2 \dot{H}_2 - B_1 H_1)\tilde{x} + B_1 v_1 + (I - B_2 H_2) G w \,, \tag{297}$$

or the books equation 9.2-34.

We now verify that in special cases these results duplicate known results. If we consider the case where there is no noisy measurements ($v_1 = 0$ and $B_1 = 0$) and no process noise $G = 0$ we then get

$$\dot{\tilde{x}} = (F - B_2 H_2 F - B_2 \dot{H}_2)\tilde{x} \,,$$

or Equation 291, which is the expected result for observers of deterministic systems. In the case where there are no noise free measurements $B_2 = H_2 = 0$ (and only noisy measurements) we get

$$\dot{\tilde{x}} = (F - B_1 H_1)\tilde{x} + B_1 v_1 - G w \,.$$

which is the standard Kalman filter error dynamics when B_1 is the Kalman gain.

Notes on the optimal choice for B_1 and B_2

Using Equation 297 we can write down the differential equation satisfied by $P = E[\tilde{x}\tilde{x}^T]$, where we find

$$\begin{aligned}
\dot{P} &= (F - B_2 H_2 F - B_2 \dot{H}_2 - B_1 H_1) P + P (F - B_2 H_2 F - B_2 \dot{H}_2 - B_1 H_1)^T \\
&+ B_1 R_1 B_1^T + (I - B_2 H_2) G Q G^T (I - B_2 H_2)^T \,.
\end{aligned}$$

As in other parts of this text we seek expressions for B_1 and B_2 that make trace($\dot{P}$) as small as possible. This requires taking the B_1 and B_2 derivatives, setting the results equal to zero and solving for B_1 and B_2. To take these derivatives we will use Equations 313, 315, and 317. Performing this procedure to determine the optimal value for B_1 first to evaluate $\frac{\partial}{\partial B_1}\text{trace}(\dot{P})$ we find the three derivatives we need to evaluate given by

$$
\begin{aligned}
\frac{\partial}{\partial B_1}\text{trace}((F - B_2H_2F - B_2\dot{H}_2 - B_1H_1)P) &= -\frac{\partial}{\partial B_1}\text{trace}(B_1H_1P) \\
&= -(H_1P)^T = -PH_1^T \\
\frac{\partial}{\partial B_1}\text{trace}(P(F - B_2H_2F - B_2\dot{H}_2 - B_1H_1)^T) &= -\frac{\partial}{\partial B_1}\text{trace}(P(B_1H_1)^T) \\
&= -\frac{\partial}{\partial B_1}\text{trace}(PH_1^TB_1^T) \\
&= -\frac{\partial}{\partial B_1}\text{trace}(B_1H_1P) = -PH_1^T \\
\frac{\partial}{\partial B_1}\text{trace}(B_1R_1B_1^T) &= 2B_1R_1\,.
\end{aligned}
$$

Thus $\frac{\partial}{\partial B_1}\text{trace}(\dot{P}) = 0$ becomes

$$
-2PH_1^T + 2B_1R_1 = 0\,,
$$

or when we solve for B_1 we find

$$
B_1^{\text{opt}} = PH_1^TR_1^{-1}\,. \tag{298}
$$

When we use the optimal value for B_1 found above we find that $\dot{P}$ is given by

$$
\begin{aligned}
\dot{P} &= (F - B_2H_2F - B_2\dot{H}_2)P + P(F - B_2H_2F - B_2\dot{H}_2)^T + (I - B_2H_2)GQG^T(I - B_2H_2)^T \\
&- PH_1^TR_1^{-1}H_1P - PH_1^TR_1^{-1}H_1P + PH_1^TR_1^{-1}R_1R_1^{-1}H_1P \\
&= (F - B_2H_2F - B_2\dot{H}_2)P + P(F - B_2H_2F - B_2\dot{H}_2)^T + (I - B_2H_2)GQG^T(I - B_2H_2)^T \\
&- PH_1^TR_1^{-1}H_1P\,,
\end{aligned} \tag{299}
$$

since several terms cancel. This is the books equation 9.2-37. Now to minimize the trace of $\dot{P}$ in Equation 299 with respect to B_2 we need to take the derivative of the above expression with respect to B_2. The various derivatives we need in this calculation are given by

$$
\begin{aligned}
\frac{\partial}{\partial B_2}\text{trace}((F - B_2H_2F - B_2\dot{H}_2)P) &= -\frac{\partial}{\partial B_2}\text{trace}(B_2H_2FP) - \frac{\partial}{\partial B_2}\text{trace}(B_2\dot{H}_2P) \\
&= -(H_2FP)^T - (\dot{H}_2P)^T \\
&= -PF^TH_2^T - P\dot{H}_2^T\,.
\end{aligned}
$$

The trace of the second term on the right-hand-side of Equation 299 has the same derivative since it is the transpose of the first. Next we evaluate

$$
\begin{aligned}
\frac{\partial}{\partial B_2}\text{trace}((I - B_2H_2)GQG^T(I - B_2H_2)^T) &= -\frac{\partial}{\partial B_2}\text{trace}(GQG^TH_2^TB_2^T) \\
&- \frac{\partial}{\partial B_2}\text{trace}(B_2H_2GQG^T) \\
&+ \frac{\partial}{\partial B_2}\text{trace}(B_2H_2GQG^TH_2^TB_2^T)\,.
\end{aligned}
$$

Note that the first term and second term are equal since the arguments of the traces are transposes of each other. Thus we get for this part of the total derivative

$$-2(H_2 G Q G^T)^T + 2B_2 H_2 G Q G^T H_2^T .$$

The total derivative of trace$(\dot{P})$ is then given by adding up all of the parts seen thus far to get

$$\frac{\partial}{\partial B_2}\text{trace}(\dot{P}) = -2PF^T H_2^T - 2P\dot{H}_2^T - 2GQG^T H_2^T + 2B_2 H_2 G Q G^T H_2^T = 0 .$$

Thus solving for B_2 we see that B_2 is given by

$$B_2^{\text{opt}} = (PF^T H_2^T + GQG^T H_2^T + P\dot{H}_2^T)(H_2 G Q G^T H_2^T)^{-1} , \tag{300}$$

or the books equation 9.2.38.

Notes on specialization to correlated measurement errors

We will solve the problem of correlated measurement errors by incorporating the correlated dynamics of the measurement noise v

$$\dot{v} = Ev + w_1 ,$$

into the state by forming an $n+m$th order augmented "prime" system, where the new state x' is is the old state x plus the measurement noise v defined as $x'^T = \left[\begin{array}{c|c} x^T & v^T \end{array}\right]$. Such an augmented system has new system matrices F', G', H_2', and Q' as given in the book. We now show that the state estimation error $\tilde{x}'$ is orthogonal to the noise-free measurements represented by H_2' or

$$H_2'\tilde{x}' = \left[\begin{array}{cc} H & I \end{array}\right] \left[\begin{array}{c} \tilde{x} \\ \tilde{v} \end{array}\right] = H\tilde{x} + \tilde{v} = 0 . \tag{301}$$

To show this recall that $\tilde{x}' = A\tilde{\xi}$ and premultiply this relationship by H_2' to get

$$H_2'\tilde{x}' = H_2' A\tilde{\xi} ,$$

and by Equation 281 for the augmented system we have that

$$\left[\begin{array}{c} T \\ \hline H_2' \end{array}\right] \left[\begin{array}{c|c} A & B \end{array}\right] = \left[\begin{array}{cc} I & 0 \\ 0 & I \end{array}\right] ,$$

or $H_2' A = 0$ meaning that $H_2'\tilde{x}' = 0$ showing the claimed orthogonalization in Equation 301. Using this expression we can derive expressions for the augmented state error covariance matrix $P' = E[\tilde{x}'\tilde{x}'^T]$ as

$$P' = E[\tilde{x}'\tilde{x}'^T] = E\left[\left[\begin{array}{c} \tilde{x} \\ \hline \tilde{v} \end{array}\right] \left[\begin{array}{c|c} \tilde{x}^T & \tilde{v}^T \end{array}\right]\right] = \left[\begin{array}{cc} P & E[\tilde{x}\tilde{v}^T] \\ E[\tilde{v}\tilde{x}^T] & E[\tilde{v}\tilde{v}^T] \end{array}\right] .$$

By post-multiplying the relationship $H\tilde{x} + \tilde{v} = 0$ by $\tilde{x}^T$ we have $H\tilde{x}\tilde{x}^T + \tilde{v}\tilde{x}^T = 0$ so taking expectations we get

$$HP + E[\tilde{v}\tilde{x}^T] = 0 ,$$

or

$$E[\tilde{v}\tilde{x}^T] = -HP\,.$$

The transpose of this is $E[\tilde{x}\tilde{v}^T] = -PH^T$ and $E[\tilde{v}\tilde{v}^T$ is computed as

$$E[\tilde{v}\tilde{v}^T] = E[(-H\tilde{x})(-H\tilde{x})^T] = HPH^T\,.$$

Thus using all of these parts we find

$$P' = \begin{bmatrix} P & -PH^T \\ -HP & HPH^T \end{bmatrix} \tag{302}$$

which is the books equation 9.2-47.

With this augmented system we are now in a situation where we can apply the results of the previous section. That is we will put the primed system, and Equation 302 into Equation 300. To do this we first need to evaluate various products. To begin we find $G'Q'G'^T = \begin{bmatrix} GQG^T & 0 \\ 0 & Q_1 \end{bmatrix}$ so that

$$G'Q'G'^T H_2'^T = \begin{bmatrix} GQG^T & 0 \\ 0 & Q_1 \end{bmatrix} \begin{bmatrix} H^T \\ I \end{bmatrix} = \begin{bmatrix} GQG^T H^T \\ Q_1 \end{bmatrix},$$

and

$$H_2' G'Q'G'^T H_2'^T = HGQG^T H^T + Q_1\,.$$

Next we find

$$\begin{aligned} P'F'^T H_2'^T &= \begin{bmatrix} P & -PH^T \\ -HP & HPH^T \end{bmatrix} \begin{bmatrix} F^T & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} H^T \\ I \end{bmatrix} = \begin{bmatrix} P & -PH^T \\ -HP & HPH^T \end{bmatrix} \begin{bmatrix} F^T H^T \\ E^T \end{bmatrix} \\ &= \begin{bmatrix} PF^T H^T - PH^T E^T \\ -HPF^T H^T + HPH^T E^T \end{bmatrix}, \end{aligned}$$

and

$$P\dot{H}_2'^T = \begin{bmatrix} P & -PH^T \\ -HP & HPH^T \end{bmatrix} \begin{bmatrix} \dot{H}^T \\ 0 \end{bmatrix} = \begin{bmatrix} P\dot{H}^T \\ -HP\dot{H}^T \end{bmatrix}.$$

Thus the sum of the three needed terms in B_2^{opt} is given by

$$P'F'^T H_2'^T + G'Q'G'^T H_2'^T + P'\dot{H}_2'^T = \begin{bmatrix} PF^T H^T - PH^T E^T + GQG^T H^T + P\dot{H}^T \\ -HPF^T H^T + HPH^T E^T + Q_1 - HP\dot{H}^T \end{bmatrix}.$$

When we group terms then for the matrix B_2^{opt} we have

$$\begin{bmatrix} B_{21}^{\text{opt}} \\ B_{22}^{\text{opt}} \end{bmatrix} = \begin{bmatrix} P(\dot{H} + HF - EH)^T + GQG^T H^T \\ -HP(\dot{H} + HF - EH)^T + Q_1 \end{bmatrix} (HGQG^T H^T + Q_1)^{-1},$$

or the books equation 9.2-51.

Notes on stochastic approximation: estimating x_0 from $z_k = x_0 + v_k$

If our measurements are noised versions of the constant x_0 or $z_k = x_0 + v_k$ then our stochastic estimation algorithm is

$$\hat{x}_{k+1} = \hat{x}_k + k_k(z_k - \hat{x}_k)\,.$$

In this case $g(x) = x_0 - x$, and so $g'(x) = -1$. Thus the required convergence condition on the sign of k_k of $\operatorname{sgn}(k_k) = -\operatorname{sgn}(g'(x)) = -(-1) = +1$ thus we must have $k_k > 0$ for convergence.

Notes on stochastic approximation: estimating x_0 from $z_j = h_j x_0 + v_j$

When $z_j = h_j x_0 + v_j$ for $j = 1, 2, \cdots k-1$ then by tabulating these equations for each value of j gives

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{k-1} \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{k-1} \end{bmatrix} x_0 + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{k-1} \end{bmatrix}.$$

This is an over determined system and to solve for x_0 using the least-squares methodology we multiply both sides by the transpose of the coefficient in front of x_0 or $\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{k-1} \end{bmatrix}^T$ to get

$$\begin{aligned}
\hat{x}_k &= \left(\begin{bmatrix} h_1 & h_2 & \cdots & h_{k-1} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{k-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} h_1 & h_2 & \cdots & h_{k-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{k-1} \end{bmatrix} \\
&= \frac{\sum_{j=1}^{k-1} h_j z_j}{\sum_{j=1}^{k-1} h_j^2}\,. \qquad (303)
\end{aligned}$$

This is the books equation 9.3-29. We denote this estimate $\hat{x}_k$ since it is the best predictor "going into" the kth measurement. In other words it is the prior estimate of the value of x_0 before we obtain the kth measurement. From the above expression for $\hat{x}_k$ a recursive estimate of $\hat{x}_{k+1}$ can be derived as follows

$$\begin{aligned}
\hat{x}_{k+1} &= \frac{z_k h_k + \sum_{j=1}^{k-1} z_j h_j}{\sum_{j=1}^{k} h_j^2} = \frac{z_k h_k + \hat{x}_k \sum_{j=1}^{k-1} h_j^2}{\sum_{j=1}^{k} h_j^2} = \frac{z_k h_k + \hat{x}_k (\sum_{j=1}^{k} h_j^2 - h_k^2)}{\sum_{j=1}^{k} h_j^2} \\
&= \hat{x}_k + \frac{1}{\sum_{j=1}^{k} h_j^2}(z_k h_k - h_k^2 \hat{x}_k) = \hat{x}_k + \frac{h_k}{\sum_{j=1}^{k} h_j^2}(z_k - h_k \hat{x}_k)\,,
\end{aligned}$$

which is the books equation 9.3-30.

Notes on Example 9.3-1

As an example of these techniques we will use stochastic approximation methods to estimate the value of a Gaussian random variable x_0 (with mean μ_0 and variance σ_0^2) from noised measurements like $z_k = x_0 + v_k$, where $v_k \sim N(0, \sigma^2)$. If we take $k_k = \frac{1}{k}$ then since $g(x) = x_0 - x$ we find our stochastic approximation algorithm given by

$$\hat{x}_{k+1} = \hat{x}_k + k_k m_k = \hat{x}_k + \frac{1}{k}(z_k - \hat{x}_k) .$$

Note that if we start with an initial guess at x_0 denoted by $\hat{x}_1$ (since it is our guess before the measurement z_1 is obtained) taken to be μ_0 and we receive the measurements z_k we see our estimates of x_0 become

$$\begin{aligned}
\hat{x}_2 &= \hat{x}_1 + \frac{1}{1}(z_1 - \hat{x}_1) = z_1 \\
\hat{x}_3 &= \hat{x}_2 + \frac{1}{2}(z_2 - \hat{x}_2) = z_1 + \frac{1}{2}(z_2 - z_1) = \frac{1}{2}(z_1 + z_2) \\
\hat{x}_4 &= \hat{x}_3 + \frac{1}{3}(z_3 - \hat{x}_3) = \frac{1}{2}(z_1 + z_2) + \frac{1}{3}(z_3 - \frac{1}{2}(z_1 + z_2)) = \frac{1}{3}(z_1 + z_2 + z_3) .
\end{aligned}$$

From this sequence it look like in general than we have that

$$\hat{x}_{k+1} = \frac{1}{k}\sum_{j=1}^{k} z_j ,$$

or the *average* of the k data points. Then the statement $E[(\hat{x}_{k+1} - x_0)^2] = \frac{\sigma^2}{k}$ is the well known result on the variance in the estimate of the mean. We can prove its correctness simply as

$$\begin{aligned}
E\left[\left(\frac{1}{k}\sum_{j=1}^{k} z_j - x_0\right)^2\right] &= E\left[\left(\frac{1}{k}\sum_{j=1}^{k}(x_0 + v_j) - x_0\right)\right] \\
&= E\left[\left(\frac{1}{k}\sum_{j=1}^{k} v_j\right)^2\right] = \frac{1}{k^2}\sum_{j=1}^{k} E[v_j^2] = \frac{1}{k^2}\sigma^2 k = \frac{\sigma^2}{k} ,
\end{aligned}$$

where we have used the fact that the sequence of measurement noise v_j are independent i.e. $E[v_i v_j] = \delta_{ij}\sigma^2$.

Now in the present case, where $x_0 \sim N(\mu_0, \sigma_0^2)$ and when taking measurements $z_j = x_0 + v_j$ with $v_j \sim N(0, \sigma^2)$ in terms of a Kalman filter framework by taking our initial guess at the state, x_0, and its uncertainty as $\hat{x}_0 = \mu_0$ and $p_0(-) = \sigma_0^2$, we see that this example is exactly like Example 4.2-1 discussed on Page 47. To make the notation from that example match this example we need to take $r_0 \to \sigma^2$ and $p_0 \to \sigma_0^2$. Under this similarity using Equation 63 we have that our state uncertainty changes with measurements as

$$p_k(+) = \frac{p_0}{1 + \frac{p_0}{r_0}k} = \frac{r_0}{\frac{r_0}{p_0} + k} \to \frac{\sigma^2}{k + \frac{\sigma^2}{\sigma_0^2}} ,$$

which is the books equation 9.3-39. The state update Equation 64 from that example and using the above transformations gives the books equation 9.3-38.

Notes on deterministic optimal linear systems – duality

In this section of these notes we will simply derive and verify many of the book's equations. Given the quadratic performance index J specified in the book we seek to transform it using a time-varying symmetric matrix $S(t)$ with certain properties. Since $S(t)$ is a function of time we have

$$\frac{d}{dt}(x^T Sx) = \dot{x}^T Sx + x^T \dot{S}x + x^T S\dot{x}\,.$$

Using the fact that our system state satisfies $\dot{x} = F(t)x(t) + L(t)u(t)$ this becomes

$$\begin{aligned}
\frac{d}{dt}x^T Sx &= u^T L^T Sx + x^T F^T Sx + x^T \dot{S}x + x^T SFx + x^T SLu \\
&= s^T(F^T S + SF + \dot{S})x + u^T L^T Sx + x^T SLu\,.
\end{aligned}$$

We next add and subtract $x^T Vx + u^T Uu$ to this expression to get that $\frac{d}{dt}x^T Sx$ equals

$$x^T(F^T S + SF + \dot{S} + V)x + u^T L^T Sx + x^T SLu + u^T Uu - x^T V x^T - u^T Uu\,. \tag{304}$$

or the books equation 9.5-8. We claim that we can write this as

$$\frac{d}{dt}x^T Sx = (x^T SL + u^T U)U^{-1}(L^T Sx + Uu) - x^T Vx - u^T Uu\,,$$

if we impose some restrictions on S. To show this expand out the first term to get

$$x^T SLU^{-1}L^T Sx + x^T SLu + u^T L^T Sx + u^T Uu\,.$$

This will be equal to Equation 304 if

$$F^T S + SF + \dot{S} + V = SLU^{-1}L^T S\,, \tag{305}$$

or the books equation 9.5-10. Thus since we have just argued that

$$x^T Vx + u^T Uu = (x^T SL + u^T U)U^{-1}(L^T Sx + Uu) - \frac{d}{dt}(x^T Sx)\,,$$

and requiring that at t_f the matrix S equals V_f or

$$x(t_f)^T S(t_f)x(t_f) = x(t_f)^T V_f x(t_f)\,,$$

we can write our quadratic performance index J as

$$\begin{aligned}
J &= x(t_f)^T V_f x(t_f) + \int_{t_0}^{t_f}(x^T Vx + u^T Uu)dt \\
&= x(t_f)^T S(t_f)x(t_f) + \int_{t_0}^{t_f}(x^T SL + u^T U)U^{-1}(L^T Sx + Uu)dt \\
&- (x(t_f)^T S(t_f)x(t_f) - x(t_0)^T S(t_0)x(t_0)) \\
&= x(t_0)^T S(t_0)x(t_0) + \int_{t_0}^{t_f}(x^T SL + u^T U)U^{-1}(L^T Sx + Uu)dt\,, \qquad (306)
\end{aligned}$$

which is the book's equation 9.5-12. From this we see that we can minimize J if we require

$$L^T Sx + Uu = 0\,, \tag{307}$$

or that the control u should be given by

$$u(t) = -U^{-1}(t)L(t)^T S(t)x(t)\,. \tag{308}$$

Notes on optimal linear stochastic control systems – separation principles

From the discussion in the book we arrive at a minimization problem for u of the form

$$\bar{J}_u = \int_{t_0}^{t_f} E[(x^T SL + u^T U)U^{-1}(L^T Sx + Uu)]\,,$$

which we desire to minimize as a function of u. The u derivative of this expression is

$$\begin{aligned}
\frac{\partial}{\partial u}E[(x^T SL + u^T U)U^{-1}(L^T Sx + Uu)] &= \frac{\partial}{\partial u}E[x^T SLU^{-1}L^T Sx + x^T SLu + u^T L^T Sx + u^T Uu] \\
&= \frac{\partial}{\partial u}(\hat{x}^T SLU^{-1}L^T S\hat{x} + \hat{x}^T SLu + u^T L^T S\hat{x} + u^T Uu) \\
&= (\hat{x}SL)^T + L^T S\hat{x} + (U + U^T)u\,.
\end{aligned}$$

When we simplify and set this equal to zero we get

$$2L^T S\hat{x} + 2Uu = 0\,.$$

Solving for u we find

$$u = -U^{-1}L^T S\hat{x}\,,$$

the same solution as in Equation 308 but evaluated at the mean state vector $\hat{x}$.

Problem Solutions

Problem 1 (an adaptive filtering example)

Note this is a linear-time invariant system and so the innovations are generated by Equation 277, which in this case becomes

$$E[\nu(t_1 + \tau)\nu(t_1)] = e^{(-\beta-k)|\tau|}(p_\infty - kr) + r\delta(\tau)\,,$$

As discussed in the example 9.1-1 on Page 167 we empirically compute the left-hand-side of the above (we call this $\phi_{\nu\nu}(\tau)$) and then fit the empirical values to a function of the form $Ae^{-(\beta+k)|\tau|}(p_\infty - kr) + B\delta(\tau)$. Once we have done this we have estimate of $p_\infty - kr$ and r. Next we look for the steady-state solution to

$$\dot{P} = (F - KH)P + P(F - KH)^T + QGQ^T + KRK^T\,,$$

Which for this system is given by

$$0 = 2(-\beta - k)p_\infty + q + k^2 r\,,$$

or

$$p_\infty = \frac{q + k^2 r}{2(\beta + k)}\,.$$

Thus the adaptive filtering procedure for this problem then is as follows

1. Measure the autocorrelation of the innovations $\nu(t)$ and denote this $\phi_{\nu\nu}(\tau)$.

2. Fit a model of the form $Ae^{-(\beta+k)|\tau|} + B\delta(\tau)$ to the measured function $\phi_{\nu\nu}(\tau)$, obtaining estimates of A and B.

3. From the earlier discussion these two values of A and B should satisfy

$$A = p_\infty - kr = \frac{q + k^2 r}{2(\beta + k)} - kr \quad \text{and} \quad B = r\,.$$

   Thus we can use these estimates to solve for q and r with k fixed. These two values of q and r should be better estimates of q and r than we previously had and could be used to modify the value of k using in filtering.

Problem 3 (relationships between the covariance of ξ and x)

Since $\tilde{x}$ and $\tilde{\xi}$ are related via $\tilde{\xi} = T\tilde{x}$ see Equation 287 and since $AT = I$ when we premultiply by A this means that $A\tilde{\xi} = \tilde{x}$. From these two expressions we see that the error covariances for $\tilde{x}$ and $\tilde{\xi}$ are related via

$$\Pi = E[\tilde{\xi}\tilde{\xi}^T] = TE[\tilde{x}\tilde{x}^T]T^T = TPT^T\,, \tag{309}$$

and

$$P = E[\tilde{x}\tilde{x}^T] = AE[\tilde{\xi}\tilde{\xi}^T]A^T = A\Pi A^T\,, \tag{310}$$

as we were to show.

Problem 5 (convergence of the modified Newton's algorithm)

For the iterations of the modified Newton's algorithm

$$\hat{x}_{k+1} = \hat{x}_k - k_0 \frac{g(\hat{x}_k)}{g'(\hat{x}_k)}\,,$$

introduce the error, e_k, defined to be $e_k = \hat{x}_k - x_0$. Then

$$\begin{aligned} e_{k+1} &= \hat{x}_{k+1} - x_0 = \hat{x}_k - x_0 - \frac{k_0 g(\hat{x}_k)}{g'(\hat{x}_k)} \\ &= e_k - \frac{k_0 g(x_0 + e_k)}{g'(x_0 + e_k)}\,. \end{aligned}$$

Taylor expand $g(x_0 + e_k)$ and $g'(x_0 + e_k)$ about x_0 to get

$$g(x_0 + e_k) = g'(x_0)e_k + \frac{1}{2}g''(x_0){e_k}^2 + \cdots$$

Since x_0 is a root of $g(\cdot)$ so that $g(x_0) = 0$. Next Taylor expand $g'(x_0 + e_k)$ about x_0 to get

$$g'(x_0 + e_k) = g'(x_0) + g''(x_0)e_k + \cdots\,.$$

With these two expressions the iterations of e_k satisfy

$$e_{k+1} = e_k - k_0 \left(\frac{g'(x_0)e_k + \frac{1}{2}g''(x_0)e_k^2 + \cdots}{g'(x_0) + g''(x_0)e_k + \cdots} \right) .$$

When we keep only the highest order terms in e_k on the top and the bottom we obtain

$$e_{k+1} = e_k - k_0 e_k = (1 - k_0)e_k \, .$$

When we iterate this over k we get

$$e_k = (1 - k_0)^{k-1} e_1 \quad \text{for} \quad k \geq 2 \, .$$

Thus we see that if $|1 - k_0| < 1$ then this method converges since $e_k \to 0$ in that case. This means that convergence is guaranteed when $-1 < 1 - k_0 < 1$ or $0 < k_0 < 2$. We are told that $g(x)$ satisfies $0 \leq a \leq |g(x)| \leq b < \infty$, from which we conclude that $0 < \frac{a}{b} < 1$ so when we impose the requirement that k_0 be such that $0 < k_0 < \frac{a}{b}$ this requires that $0 < k_0 < 1$, which is stricter than was is truly required for convergence (which is $k_0 < 2$).

Problem 7 (requirements for stochastic convergence)

All the examples given for the gain sequence, k_k, are examples that can be shown similar to that of the classic divergent series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Problem 8 (some derivations)

See the notes that accompany Example 9.3-1 on Page 179.

A Appendix

A.1 Matrix and Vector Derivatives

In this section of the appendix we enumerate several matrix and vector derivatives that are used in the previous document. We begin with some derivatives of scalar forms

$$\begin{aligned} \frac{\partial \mathbf{x}^T \mathbf{a}}{\partial \mathbf{x}} &= \frac{\mathbf{a}^T \mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} && (311) \\ \frac{\partial \mathbf{x}^T \mathbf{B} \mathbf{x}}{\partial \mathbf{x}} &= (\mathbf{B} + \mathbf{B}^T)\mathbf{x}\,. && (312) \end{aligned}$$

Next we present some derivatives involving traces. We have

$$\begin{aligned} \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{A}\mathbf{X}) &= \mathbf{A}^T && (313) \\ \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}\mathbf{A}) &= \mathbf{A}^T && (314) \\ \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{A}\mathbf{X}^T) &= \mathbf{A} && (315) \\ \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}^T\mathbf{A}) &= \mathbf{A} && (316) \\ \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}^T\mathbf{A}\mathbf{X}) &= (\mathbf{A} + \mathbf{A}^T)\mathbf{X} && (317) \\ \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}\mathbf{A}\mathbf{X}^T) &= \mathbf{X}(\mathbf{A} + \mathbf{A}^T)\,. && (318) \end{aligned}$$

Note that we can derive Equations 317 and 318 given the previous trace derivative identities using the "product rule". To do this we assume that one of the terms $\mathbf{X}$ (or $\mathbf{X}^T$) is constant when we take the derivative with respect to the other $\mathbf{X}$ term. For example to derive Equation 318 we have

$$\begin{aligned} \frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}\mathbf{A}\mathbf{X}^T) &= \left.\frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{X}\mathbf{A}\mathbf{V})\right|_{\mathbf{V}=\mathbf{X}^T} + \left.\frac{\partial}{\partial \mathbf{X}} \mathrm{trace}(\mathbf{V}\mathbf{A}\mathbf{X}^T)\right|_{\mathbf{V}=\mathbf{X}} \\ &= \left.(\mathbf{A}\mathbf{V})^T\right|_{\mathbf{V}=\mathbf{X}^T} + \left.(\mathbf{V}\mathbf{A})\right|_{\mathbf{V}=\mathbf{X}} = (\mathbf{A}\mathbf{X}^T)^T + \mathbf{X}\mathbf{A} \\ &= \mathbf{X}(\mathbf{A} + \mathbf{A}^T)\,. \end{aligned}$$

Next we present some matrix derivatives that are helpful to know. We have

$$\begin{aligned} \frac{\partial}{\partial \mathbf{X}} (\mathbf{a}^T \mathbf{X} \mathbf{b}) = \mathbf{a}\mathbf{b}^T && (319) \\ \frac{\partial}{\partial \mathbf{X}} (\mathbf{a}^T \mathbf{X}^T \mathbf{b}) = \mathbf{b}\mathbf{a}^T\,, && (320) \end{aligned}$$

where as before $\mathbf{X}$ is a matrix. Derivations of expressions of this form are derived in [4, 6].

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