Notes and Solutions for the Book: Adaptive Filtering Theory: Third Edition by Symon Haykin

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January 5, 2003

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Chapter 4 (Eigenanalysis)

Problem Solutions

Problem 1 (some eigenvalues)

Part (a): We have

$$\text{trace}(R) = \lambda_1 + \lambda_2 = 2.0\,.$$

Part (b): Let the matrix $Q = \left[\begin{array}{cc} q_{11} & q_{21} \\ q_{12} & q_{22} \end{array} \right]$ be the matrix with the eigenvectors of R as its columns. Then we must have

$$Q^H R Q = \left[\begin{array}{cc} 0.5 & 0 \\ 0 & 1.5 \end{array} \right]\,.$$

Since the eigenvalues for R are unique the eigenvectors are unique under the assumption that $q_i^H q_i = 1$ so the decomposition is unique.

Problem 2 (the eigenvalues of triangular matrix)

Consider the expansion of the characteristic equation $|A - \lambda I| = 0$. From the form of A this is equal to the product of the diagonal elements of $A - \lambda I$ and has zeros given by the diagonal elements of A.

Problem 3 (pairs of eigenvectors)

If $q_r + jq_i$ is an eigenvector of the system specified then

$$(A_r + jA_i)(q_r + jq_i) = \lambda(q_r + jq_i)\,, \tag{1}$$

and we want to know if $q_i - jq_r$ is an eigenvector. We can show that this is true by multiplying Equation 1 by $-j$ for which we get

$$(A_r + jA_i)(q_i - jq_r) = \lambda(q_i - jq_r)\,,$$

showing that the vector $q_i - jq_r$ has the same eigenvalue.

Problem 4 (eigenvalues of correlation matrix)

From the definition of the correlation matrix and the vector u we have

$$R = \begin{bmatrix} \sigma_u^2 & x & x & \cdots \\ x & \sigma_u^2 & x & \cdots \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

That is a matrix with σ_u^2 on the diagonals and the off diagonal elements don't matter. Then from Property 6: the sum of the λ_i equals the trace of R. Thus we have

$$\sum_i \lambda_i = \text{trace}(R) = M\sigma_u^2\,.$$

Problem 5 (the square root of a matrix R)

Part (a): Consider the given expression for $R^{1/2}$

$$R^{1/2} = \sum_{i=1}^{M} \lambda_i^{1/2} q_i q_i^H\,,$$

Then

$$R^{1/2}R^{1/2} = \sum_{i=1}^{M}\sum_{i=1}^{M} \lambda_i^{1/2}\lambda_j^{1/2} q_i q_i^H q_j q_j^H\,.$$

If $q_1, q_2, \cdots, q_M$ correspond to *distinct* eigenvalues of then the eigenvectors are orthogonal and $q_i^H q_j = 1$ if $i = j$ and 0 if $i \neq j$ (we assume normalized of eigenvectors) so the above sum becomes

$$\sum_{i=1}^{M} \lambda_i q_i q_i^H\,,$$

which equals R showing the intended summation expression for $R^{1/2}$ when squared does produce R.

Part (b): Based on Part (a) we compute the unitary similarity transform i.e. we find the eigenvalues λ_i and the eigenvectors q_i of R such that $Q^H R Q = \Lambda$. We then form the sum

$$\sum_{i=1}^{M} \lambda_i^{1/2} q_i q_i^H\,.$$

Then from Part (a) this equals $R^{1/2}$.

Problem 6 (the determinant of R)

Use the similarity transform of R as $Q^H R Q = \Lambda$ and take the determinant of both sides. We have

$$|Q^H R Q| = |\Lambda| = \prod_{i=1}^{M} \lambda_i \,.$$

The left-hand-side is $|Q^H||R||Q|$ as Q is unitary $Q^H = Q^{-1}$ thus we have

$$|Q^H| = |Q^{-1}| = \frac{1}{|Q|} \,,$$

thus the left-hand-side is $|R|$.

Problem 7 (the product of two unitary matrices)

Let U_1 and U_2 be unitary matrices that is $U_i^H = U_i^{-1}$.

Part (a): Then consider $(U_1 U_2)^H = U_2^H U_1^H = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1}$ showing that $U_1 U_2$ is unitary.

Part (b): Let $V = U^{-1}$ with U a unitary matrix. Since

$$V^H = (U^H)^{-1} = (U^{-1})^{-1} = U \,,$$

we have $V V^H = U^{-1} U = I$ showing that V is unitary.

Problem 8 (the Schur decomposition)

Part (a): By the unitary similarity transformation property of positive semidefinite matrices discussed in the book the Schur decomposition $Z^H A Z = T$ of the correlation matrix R takes the form above with $T = \Lambda$ a diagonal matrix.

Part (b): The subspace $\text{span}(z_1, z_2, \cdots, z_k)$ mentioned in the Schur decomposition is equivalent to the subspace spanned by the eigenvectors of R with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$.

Problem 9 (the LQ factorization)

We are told that $A_n - k_n I = Q_n L_n$ thus $L_n = Q_n^H (A_n - k_n I)$ since Q_n is unitary. We now from A_{n+1} to find

$$\begin{aligned}
A_{n+1} &= L_n Q_n + k_n I \\
&= Q_n^H (A_n - k_n I) Q_n + k_n I \\
&= Q_n^H A_n Q_n \,,
\end{aligned}$$

as we were to show.

Part (a): From the given expression for $\mathbf{v}$ we have $\mathbf{v}_i^H \mathbf{v}_i = 1$ and $\mathbf{v}_i^H \mathbf{v}_j = 0$ if $i \neq j$, thus we compute $c_i(n)$ as

$$c_i(n) = \mathbf{v}_i^H \mathbf{u}(n) \quad \text{for} \quad 0 \leq i \leq M-1 \, .$$

Part (b): Consider

$$\begin{aligned} E[c_i(n)c_j^*(n)] &= E[\mathbf{v}_i^H \mathbf{u}(n)(\mathbf{v}_j^H \mathbf{u}(n))^*] = \mathbf{v}_i^H E[\mathbf{u}(n)\mathbf{u}^H(n)]\mathbf{v}_j \\ &= \mathbf{v}_i^H R \mathbf{v}_j \geq 0 \, , \end{aligned}$$

since R is positive semidefinite. We have assumed that $\mathbf{u}(n)$ is zero mean. Thus the Fourier coefficients *are* correlated.

Part (c): From the above we see that $E[|c_i(n)|^2] = \mathbf{v}_i^H R \mathbf{v}_i$ or the power in the ith Fourier mode.

Problem 11 (the condition number of A and UA)

Recall that the condition number of a matrix A is given by $\chi(A) = ||A||||A^{-1}||$. If we use the spectral norm for the definition of the norm $|| \cdot ||$ then one way to express this norm is

$$||A||_s^2 = \max_{x \neq 0} \frac{||Ax||^2}{||x||^2} \, .$$

Consider the matrix A but multiplied by a unitary matrix U or the matrix UA. We will show that $||A||_s = ||UA||_s$ and $||A^{-1}||_s = ||(UA)^{-1}||_s$ from which we can conclude that $\chi(A)$ and $\chi(UA)$ are the same. To begin note that

$$||UAx||^2 = (UAx)^H(UAx) = x^H A^H U^H U A x = x^H A^H A x = ||Ax||^2 \, .$$

Thus

$$||UA||_s^2 = \max_{x \neq 0} \frac{||UAx||^2}{||x||^2} = \max_{x \neq 0} \frac{||Ax||^2}{||x||^2} = ||A||_s^2 \, .$$

Lets now consider $||(UA)^{-1}||_s^2$. We have

$$||(UA)^{-1}||_s^2 = \max_{x \neq 0} \frac{||A^{-1}U^{-1}x||^2}{||x||^2} = \max_{x \neq 0} \frac{||A^{-1}U^H x||^2}{||x||^2} \, .$$

To simplify the above note that

$$||U^H x||^2 = (U^H x)^H (U^H x) = x^H U U^H x = x^H x = ||x||^2 \, ,$$

thus the norm of x is not changed by applying a unitary transformation to it. We can use this to replace the denominator in the expression for $||(UA)^{-1}||_s^2$ above and get

$$\begin{aligned} ||(UA)^{-1}||_s^2 &= \max_{x \neq 0} \frac{||A^{-1}U^H x||^2}{||U^H x||^2} \\ &= \max_{v \neq 0} \frac{||A^{-1}v||^2}{||v||^2} = ||A^{-1}||_s^2 \, , \end{aligned}$$

by replacing the maximization over x with a maximization over $v \equiv U^H x$ since U^H is an invertible transform. Combining these two results gives the desired result.

Problem 14 (Szego's theorem)

If we take the function g to be $g(x) = \ln(x)$ then the left-hand-side of the given expression is

$$\lim_{M\to\infty} \left(\frac{1}{M} \sum_{i=1}^{M} \ln(\lambda_i) \right) = \lim_{M\to\infty} \left(\ln \left(\prod_{i=1}^{M} \lambda_i \right)^{1/M} \right) .$$

Taking exponentials of this expression and the right-hand-side (the integral of $g(S(\omega))$) gives

$$\lim_{M\to\infty} \left(\prod_{i=1}^{M} \lambda_i \right)^{1/M} = \exp \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln(S(\omega)) d\omega \right) .$$

Since the determinant of R is equal to the product of the eigenvalues we have shown the desired expression.

Problem 15 (condition numbers and linear systems)

Part (a): We are told to consider $Rw = p$ and $(R + \delta R)(w + \delta w) = p$. Expanding the left-hand-side of the second equation gives

$$Rw + R\delta w + \delta R(w + \delta w) = p .$$

Using $Rw = p$ in the above gives

$$R\delta w + \delta R(w + \delta w) = 0 .$$

Solving for δw we get

$$\delta w = -R^{-1} \delta R(w + \delta w) .$$

Taking vector norms on both sides we get

$$\begin{aligned}
||\delta w|| &\leq ||R^{-1}|| ||\delta R|| (||w|| + ||\delta w||) \\
&= ||R^{-1}|| ||R|| \left(\frac{||\delta R||}{||R||} \right) (||w|| + ||\delta w||) \\
&= \chi(R) \frac{||\delta R||}{||R||} ||w|| + \chi(R) \frac{||\delta R||}{||R||} ||\delta w|| ,
\end{aligned}$$

using the definition of the condition number $\chi(R) \equiv ||R|| ||R^{-1}||$. When we solve for $||\delta w||$ in the above we get

$$\left(1 - \chi(R) \frac{||\delta R||}{||R||} \right) ||\delta w|| \leq \chi(R) \left(\frac{||\delta R||}{||R||} \right) ||w|| .$$

It can be shown that if our matrix perturbation δR is small enough that the new matrix $R + \delta R$ is still invertible then $\frac{||\delta R||}{||R||} \leq \frac{1}{\chi(R)}$ see [?]. In that case left-hand-side has a leading coefficient that is positive and we can divide by it to get

$$\frac{||\delta w||}{||w||} \leq \frac{\chi(R)}{1 - \chi(R)\frac{||\delta R||}{||R||}} \frac{||\delta R||}{||R||} . \quad (2)$$

If we assume that $\chi(R)\frac{||\delta R||}{||R||} \approx 0$ then we get

$$\frac{||\delta w||}{||w||} \leq \chi(R)\frac{||\delta R||}{||R||} .$$

the desired expression.

Part (b): We are told to consider $Rw = p$ and $R(w + \delta w) = p + \delta p$. Expanding the left-hand-side of the second equation and using $Rw = p$ gives

$$R\delta w = \delta p ,$$

or solving for δw we get

$$\delta w = R^{-1}\delta p .$$

Taking vector norms on both sides we get

$$||\delta w|| \leq ||R^{-1}||||\delta p|| = \chi(R)\frac{||\delta p||}{||R||} ,$$

using the definition of the condition number $\chi(R) \equiv ||R||||R^{-1}||$. Since $Rw = p$ we have $||p|| \leq ||R||||w||$ so $\frac{1}{||R||} \leq \frac{||w||}{||p||}$ and we get

$$||\delta w|| \leq \chi(R)\frac{||\delta p||}{||p||}||w|| ,$$

or

$$\frac{||\delta w||}{||w||} \leq \chi(R)\frac{||\delta p||}{||p||} ,$$

the desired expression.

Problem 17 (doubly symmetric matrices)

An example of the matrix J and its actions on a matrix R can be helpful. For a 3-by-3 matrix we have

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \quad (3)$$

Then for a given matrix R

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} ,$$

we find JR given by

$$JR = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} r_{31} & r_{32} & r_{33} \\ r_{21} & r_{22} & r_{23} \\ r_{11} & r_{12} & r_{13} \end{bmatrix}, \tag{4}$$

i.e. the rows of R are reversed. Next RJ is given by

$$RJ = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_{13} & r_{12} & r_{11} \\ r_{23} & r_{22} & r_{21} \\ r_{33} & r_{32} & r_{13} \end{bmatrix}, \tag{5}$$

i.e. the columns of R are reversed. Finally we find JRJ given by

$$JRJ = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} r_{33} & r_{32} & r_{31} \\ r_{23} & r_{22} & r_{21} \\ r_{13} & r_{12} & r_{11} \end{bmatrix}, \tag{6}$$

i.e. the rows and columns of R are reversed.

Part (a): Note that if $JRJ = R$ then

$$R^{-1} = (JRJ)^{-1} = J^{-1}R^{-1}J^{-1} = JR^{-1}J\,,$$

showing that R^{-1} is doubly symmetric.

Part (b): Let $\mathbf{q}$ be an eigenvector of R so that $R\mathbf{q} = \lambda\mathbf{q}$. Then replacing R with JRJ since R is double symmetric we get

$$JRJ\mathbf{q} = \lambda\mathbf{q}\,,$$

or multiplying by $J^{-1} = J$ on both sides gives

$$RJ\mathbf{q} = \lambda J\mathbf{q}\,, \tag{7}$$

showing that $J\mathbf{q}$ is also an eigenvector of R with the *same* eigenvalue λ. Since we are told that the eigenvalues of R are distinct R has distinct eigenvectors (one for each eigenvalue λ) thus we must have $J\mathbf{q} \propto \mathbf{q}$ since in that case Equation 7 would be satisfied. From the form of J (a matrix with only zeros and ones) we only have to consider $J\mathbf{q} = \pm\mathbf{q}$. Thus I have shown that the matrix R has either symmetric or skew-symmetric eigenvectors.

Note: I'm not sure how to show that the *number* of these symmetric and skew-symmetric eigenvectors equals $\lfloor (M+1)/2 \rfloor$ and $\lfloor M/2 \rfloor$ respectively. If anyone knows how to do this part please contact me.

Problem 19 (more about doubly symmetric matrices)

Part (a): For a 3-by-3 matrix R given we note that

$$R\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1-\rho_2 \\ \rho_1-\rho_2 \\ \rho_2-1 \end{bmatrix} = (1-\rho_2)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = (1-\rho_2)\mathbf{q}_1\,.$$

Thus $\mathbf{q}_1$ is an eigenvector of R with eigenvalue as claimed. Now with J given as in Equation 3 we have

$$J\mathbf{q}_1 = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \left(\frac{1}{\sqrt{2}} \left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right] \right) = \frac{1}{\sqrt{2}} \left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right] = -\mathbf{q}_1 \,,$$

thus this is our $\lfloor M/2 \rfloor = \lfloor 3/2 \rfloor = 1$ skew-symmetric eigenvector. Now for R as given we find $R\mathbf{q}_i$ given by

$$R\mathbf{q}_i = \frac{1}{\sqrt{1+c_i^2}} \left[\begin{array}{c} 1+\rho_1 c_i + \rho_2 \\ \rho_1 + c_i + \rho_1 \\ \rho_2 + \rho_1 c_i + 1 \end{array}\right] = \frac{1}{\sqrt{1+c_i^2}} \left[\begin{array}{c} 1+\rho_1 c_i + \rho_2 \\ 2\rho_1 + c_i \\ 1 + \rho_1 c_i + \rho_2 \end{array}\right] .$$

To have this equal $\lambda_i \mathbf{q}_i$ requires (we can ignore the scaling factor $\sqrt{1+c_i^2}$)

$$\begin{aligned} 1 + \rho_1 c_i + \rho_2 &= \lambda_i \\ 2\rho_1 + c_i &= \lambda_i c_i \,. \end{aligned}$$

The first of these equations gives

$$c_i = \frac{\lambda_i - 1 - \rho_2}{\rho_1} \,,$$

and the second of these equations gives

$$c_i = \frac{2\rho_1}{\lambda_i - 1} \,.$$

Now note that

$$J\mathbf{q}_i = \frac{1}{\sqrt{1+c_i^2}} \left[\begin{array}{c} 1 \\ c_i \\ 1 \end{array}\right] = \mathbf{q}_i \,,$$

which are the two symmetric eigenvectors.

Problem 20 (low-rank modeling)

When we apply low-rank modeling the output vector at the receiver is

$$y_{\text{indirect}} = \sum_{i=1}^{p} c_i(n)\mathbf{q}_i + \sum_{i=1}^{p} v_i(n)\mathbf{q}_i \,.$$

Then since $\mathbf{u}(n) = \sum_{i=1}^{M} c_i(n)\mathbf{q}_i$ we have the error given by

$$\begin{aligned} \epsilon_{\text{indirect}} &= E\left[||y_{\text{indirect}}(n) - \mathbf{u}(n)||^2 \right] \\ &= E\left[\left|\left| - \sum_{i=p+1}^{M} c_i(n)\mathbf{q}_i + \sum_{i=1}^{p} v_i(n)\mathbf{q}_i \right|\right|^2 \right] = E\left[\left|\left| \sum_{i=1}^{M} d_i(n)\mathbf{q}_i \right|\right|^2 \right] \,, \end{aligned}$$

where we have defined $d_i(n)$ as

$$d_i(n) = \begin{cases} v_i(n) & 1 \le i \le p \\ -c_i(n) & p+1 \le i \le M \end{cases} .$$

Thus evaluating $\epsilon_{\text{indirect}}$ we have

$$\epsilon_{\text{indirect}} = E\left[\sum_{i=1}^{M}\sum_{j=1}^{M} d_i^*(n) d_j(n) \mathbf{q}_i^H \mathbf{q}_j\right] .$$

Unless $i = j$ the term in the above summation vanishes due to the orthogonality of $\mathbf{q}_i$ and $\mathbf{q}_j$ and we get

$$\begin{aligned}
\epsilon_{\text{indirect}} &= \sum_{i=1}^{M} E[d_i^*(n) d_i(n)] = \sum_{i=1}^{p} E[v_i^*(n) v_i(n)] + \sum_{i=p+1}^{M} E[c_i^*(n) c_i(n)] \\
&= \sum_{i=1}^{p} \sigma^2 + \sum_{i=p+1}^{M} \lambda_i = \sum_{i=p+1}^{M} \lambda_i + p\sigma^2 ,
\end{aligned}$$

as we were to show.

Problem 21 (a minimum eigenfilter)

We would be minimizing the SNR at the output, since we are minimizing the Rayleigh quotient

$$\min_{w \neq 0} \frac{w^H R w}{w^H w} = \lambda_{\min} .$$

Chapter 5 (Wiener Filters)

Notes on the Text

Notes on the principle of orthogonality

With a cost function J defined as $J = E[e(n)e^*(n)]$ and ∇_k defined in terms of the real components a_k and b_k as

$$\nabla_k J = \frac{\partial J}{\partial a_k} + j\frac{\partial J}{\partial b_k}\,, \tag{8}$$

using the product rule we find $\nabla_k J$ is then given by

$$\begin{aligned}
\nabla_k J &= E\left[\frac{\partial(e(n)e^*(n))}{\partial a_k} + j\frac{\partial(e(n)e^*(n))}{\partial b_k}\right] \\
&= E\left[\frac{\partial e(n)}{\partial a_k}e^*(n) + e(n)\frac{\partial e^*(n)}{\partial a_k} + j\frac{\partial e(n)}{\partial b_k}e^*(n) + je(n)\frac{\partial e^*(n)}{\partial b_k}\right]\,.
\end{aligned} \tag{9}$$

To evaluate this expression recall that the definition of the error sequence $e(n)$ in terms of the complex numbers $w_k = a_k + jb_k$ and the delayed signal $u(n-k)$ is explicitly given by

$$e(n) = d(n) - y(n) = d(n) - \sum_{k=0}^{\infty} w_k^* u(n-k) = d(n) - \sum_{k=0}^{\infty}(a_k - jb_k)u(n-k)\,.$$

From this the complex conjugate of this expression is easily computed

$$e^*(n) = d^*(n) - y^*(n) = d^*(n) - \sum_{k=0}^{\infty} w_k u^*(n-k) = d^*(n) - \sum_{k=0}^{\infty}(a_k + jb_k)u^*(n-k)\,.$$

Using each of these expressions we can directly compute the derivatives needed in Equation 9. We find

$$\begin{aligned}
\frac{\partial e(n)}{\partial a_k} &= -u(n-k) \\
\frac{\partial e(n)}{\partial b_k} &= ju(n-k) \\
\frac{\partial e^*(n)}{\partial a_k} &= -u^*(n-k) \\
\frac{\partial e^*(n)}{\partial b_k} &= -ju^*(n-k)\,.
\end{aligned}$$

When we put these expressions into Equation 9 we find

$$\nabla_k J = -2E\left[u(n-k)e^*(n)\right]\,.$$

Appendix B (Differentiation with Respect to a Vector)

Notes On The Text

Verification of some simple properties of $\frac{\partial}{\partial w_k}$

In this little subsection we validate some of the simple statements made about the complex derivative $\frac{\partial}{\partial w_k}$. Consider first $\frac{\partial w_k}{\partial w_k}$. We find

$$\begin{aligned}
\frac{\partial w_k}{\partial w_k} &= \frac{\partial}{\partial w_k}(x_k + j y_k) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_k} - j\frac{\partial}{\partial y_k}\right)(x_k + j y_k) \\
&= \frac{1}{2}(1+1) = 1\,.
\end{aligned}$$

Next consider $\frac{\partial w_k}{\partial w_k^*}$. We find

$$\begin{aligned}
\frac{\partial w_k}{\partial w_k^*} &= \frac{\partial}{\partial w_k^*}(x_k + j y_k) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_k} + j\frac{\partial}{\partial y_k}\right)(x_k + j y_k) \\
&= \frac{1}{2}(1-1) = 0\,.
\end{aligned}$$

Finally consider $\frac{\partial w_k^*}{\partial w_k}$. We find

$$\begin{aligned}
\frac{\partial w_k^*}{\partial w_k^*} &= \frac{1}{2}\left(\frac{\partial}{\partial x_k} - j\frac{\partial}{\partial y_k}\right)(x_k - j y_k) \\
&= \frac{1}{2}(1-1) = 0\,.
\end{aligned}$$