$u = 0$ eq 7.1.3 becomes

$$\rho \frac{\partial e}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + q_H$$

$$e = c_v T$$

$$c_v \rho \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + q_H$$

$$\frac{\partial T}{\partial t} = \frac{k}{c_v \rho}\frac{\partial^2 T}{\partial x^2} + \frac{q_H}{c_v \rho} = \alpha \frac{\partial^2 T}{\partial x^2} + q$$

Flow incompressible $\nabla \cdot u = 0$

$$\Leftrightarrow \frac{\partial u}{\partial x} = 0 \Rightarrow u = \text{constant}(t)$$

Assume u = constant independent of time thermal conductivity k is constant.

$$\rho\left(\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x}\right) = k \frac{\partial^2 T}{\partial x^2} + q_H$$

$$\Rightarrow c_v \rho \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x}\right) = k \frac{\partial^2 T}{\partial x^2} + q_H$$

μ is dynamic viscosity
κ is thermal diffusion/ component,

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} = \frac{k}{c_v \rho}\frac{\partial^2 T}{\partial x^2} + \frac{q_H}{c_v \rho}$$

ν is kinematic viscosity.

$$\text{Peclet \#} = \frac{LU}{\alpha} = \left(\frac{\nu}{\alpha}\right)\frac{LU}{\nu}$$

$$= \left(\frac{\nu}{\alpha}\right) Re$$

$$\frac{\mu}{\rho} = \nu$$

$\frac{\nu}{\alpha}$ & Prandtl #?

$$\alpha = \frac{k}{c_v \rho}$$

$$Pr \equiv \frac{\nu}{\kappa}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + \cancel{p}\right)^0 = \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right)$$ μ dynamic viscosity.

ρ constant $\qquad \frac{\partial u}{\partial x} = 0$ incompressibility in 1d then

$$\rho_t + \cancel{\rho u_x}^0 + \rho_x u = 0$$

$$\rho_t + u\rho_x = 0$$

$$\rho \frac{\partial u}{\partial t} + \rho 2u \frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial t} + 2u\frac{\partial u}{\partial x} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial x^2}$$

$$= \nu \frac{\partial^2 u}{\partial x^2}$$

Incompressible ρ = constant & $\nabla \cdot \bar{u} = 0 \Rightarrow \bar{u} = \nabla\phi$ $\quad \phi$ scalar

eq 2.9.5 $\qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\nabla\phi) = 0$

$$\Rightarrow \nabla^2\phi = 0 \qquad \text{eq } 7.1.13$$

Energy eq 1.5.11

$$\frac{\partial}{\partial t}(\rho E) + \nabla\cdot\left[\cancel{\rho \bar{v}} H^0 - k\nabla T - \cancel{\bar{\bar{\tau}}\cdot\bar{v}}^0\right] = W_F + q_H$$

w/ No velocity → No shear & No work W_F & no time dependence

$$= -\nabla\cdot\left[k\nabla T\right] = q_H$$

w/ k thermal conductivity constant & in 2D

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{q_H}{k} \qquad \text{eq } 7.1.14$$

General transport eq. U is a general scalar {U per unit mass}

$$\frac{\partial}{\partial t}(\rho U) + \nabla \circ (\rho \bar{V} U) = \nabla \circ (k \rho \nabla U) + Q_v + \nabla \circ Q_s$$

k = diffusivity constant m^2/s

Medium at rest & $\bar{V} = 0$ U = quantity that is conserved

$$\cancel{\frac{\partial}{\partial t}(\rho U)}^{0} = \nabla \circ (k \rho \nabla U) + Q_v + \nabla \circ Q_s$$

$$\Rightarrow \nabla \circ (k \rho \nabla U) = -Q_v - \nabla \circ Q_s$$

pg 270 Hirsh Vol I 05-25-01

From eq 1.5.15.

$$\rho \frac{de}{dt} = -p(\nabla \cdot \vec{v}) + \varepsilon_v + \nabla \cdot (k \nabla T) + q_H \qquad \left\{ \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right\}$$

w ideal gas $e = C_v T$ taking $\varepsilon_v = 0$ & incompressible fluid $\nabla \cdot \vec{v} = 0$

$$\rho C_v \left[\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right] = -p(\cancelto{0}{\nabla \cdot \vec{v}}) + \nabla \cdot (k \nabla T) + q_H$$

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = \frac{k}{\rho C_v} \nabla \cdot (\nabla T) + \frac{q_H}{\rho C_v}$$

For 2d temp dist $T = T(x,y)$ $\quad \alpha = \frac{k}{\rho C_v}$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + q \qquad \text{eq } 7.1.15$$

Eq 2.9.25 is

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} + \phi_{zz} = \frac{1}{a^2} (\phi_{tt} + 2 \phi_x \phi_{xt})$$

For $a = f\infty$ & 2d flow

$$(1 - M_\infty^2) \phi_{xx} + \phi_{yy} = 0$$

$$\frac{\partial^2 \phi}{\partial y^2} - (M_\infty^2 - 1) \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{eq } 7.1.16$$

$$M_\infty = \frac{u_\infty}{c} = \frac{\partial \phi / \partial x (+\infty)}{c} \quad \{\text{think}\}$$

For subsonic flows $M_\infty < 1$ $M_\infty^2 - 1 < 0$

eq is elliptic

As flow at ∞ increase past sonic $M_\infty > 1$ $M_\infty^2 - 1 > 0$

eq is hyperbolic

$u_t + a u_x = 0$ wave traveling to the right.

$$(u_t)_i = -a u_x \cong -\frac{a}{2\Delta x}(u_{i+1} - u_{i-1}) \quad \text{eq } 7.2.4$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{a}{2\Delta x} u_{i+1}^{n+1} + \frac{a}{2\Delta x} u_{i-1}^{n+1}$$

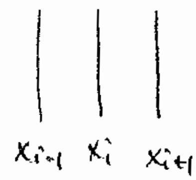
$$\frac{a}{2\Delta x} u_{i+1}^{n+1} + \frac{1}{\Delta t} u_i^{n+1} - \frac{a}{2\Delta x} u_{i-1}^{n+1} = \frac{1}{\Delta t} u_i^n$$

$$\Rightarrow \frac{a\Delta t}{2\Delta x} u_{i+1}^{n+1} + u_i^{n+1} - \frac{a\Delta t}{2\Delta x} u_{i-1}^{n+1} = u_i^n$$

$$\begin{pmatrix} & & \\ +a\frac{\Delta t}{2\Delta x} & 1 & -a\frac{\Delta t}{2\Delta x} \\ & & \end{pmatrix} \begin{pmatrix} u_{i-1}^{n+1} \\ u_i^{n+1} \\ u_{i+1}^{n+1} \end{pmatrix} = \begin{pmatrix} u_{i-1}^n \\ u_i^n \\ u_{i+1}^n \end{pmatrix}$$

Backwards difference for $(u_t)_i \cong \dfrac{u_i^n - u_i^{n-1}}{\Delta t}$ gives same eq as 7.2.6

$(u_t)_i \cong -\frac{a}{\Delta x}(x_i - x_{i-1})$ eq 7.2.7



$$1+\frac{x}{2}\Big|_{x=-\frac{2}{10}} = 1+\frac{1}{2}\left(-\frac{2}{10}\right) = 1-\frac{1}{10} = \frac{9}{10} \neq 0 \quad ?$$

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) \quad \text{eq } 7.2.12$$

$$\sigma \equiv \frac{a\Delta t}{\Delta x}$$

$7.2.8 \Rightarrow$

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) = u_i^n - \sigma\left(u_i^n - u_{i-1}^n\right) \quad 7.2.13$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} - (u_t + a u_x)_i^n$$

$$= \cancel{(u_t)_i^n} + \frac{\Delta t}{2}(u_{tt})_i^n + a\left[\cancel{(u_x)_i^n} + \frac{\frac{\Delta x^3}{3}(u_{xxx})_i^n}{2\Delta x} + \cdots\right] - \cancel{(u_t)_i^n} - \cancel{a(u_x)_i^n}$$

$$= \frac{\Delta t}{2}(u_{tt})_i^n + a\frac{\Delta x^2}{6}(u_{xxx})_i^n + O(\Delta t^2, \Delta x^4) \quad \text{eq } 7.2.15$$

If u_i^n is the exact solution to

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{a(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} \equiv 0$$

Then eq 7.2.15 becomes

$$-(\bar{u}_t + a\bar{u}_x)_i^n = \frac{\Delta t}{2}(u_{tt})_i^n + \frac{\Delta x^2}{6} a(u_{xxx})_i + O(\Delta t^2, \Delta x^4)$$

$$\Rightarrow (\bar{u}_t + a\bar{u}_x)_i^n = -\frac{\Delta t}{2}(u_{tt})_i^n - \frac{\Delta x^2}{6} a(u_{xxx})_i + O(\Delta t^2, \Delta x^4) \quad \text{eq } 7.2.16$$

$$(u_t)_i^n = -a(u_x)_i^n + O(\Delta t, \Delta x^2)$$

$$(u_{tt})_i^n = -a(u_{tx})_i^n + O(\Delta t, \Delta x^2) = -a(-au_x)_{x\,i} + O(\Delta t, \Delta x^2)$$

$$= a^2(u_{xx})_i^n + O(\Delta t, \Delta x^2) \quad \text{eq } 7.2.19$$

$$\varepsilon_T = -\frac{\Delta t}{2}\left[a^2(u_{xx})_i^n\right] + O(\Delta t^2, \Delta x^2) - \frac{\Delta x^2}{6} a(u_{xxx})_i^n + O(\Delta t^2, \Delta x^4)$$

$$\Rightarrow \varepsilon_T = -\frac{\Delta t}{2} a^2(u_{xx})_i^n - a\frac{\Delta x^2}{6}(u_{xxx})_i^n + O(\Delta t^2, \Delta x^2) \quad \text{eq } 7.2.20$$

$$u_t + au_x = -\frac{\Delta t}{2} a^2 u_{xx} + O(\Delta t^2, \Delta x^2)$$

$\uparrow$

viscosity term that is negative!!

$$U_i^{n+1} = +\frac{\sigma}{2}U_{i-1}^n + U_i^n - \frac{\sigma}{2}U_{i+1}^n$$

$$CU^n = \begin{pmatrix} \sigma/2 & 1 & -\sigma/2 & & \\ & +\sigma/2 & 1 & -\sigma/2 & \\ & & \sigma/2 & 1 & -\sigma/2 \end{pmatrix} \begin{pmatrix} U_1^n \\ \vdots \\ U_{i-1}^n \\ U_i^n \\ U_{i+1}^n \\ \vdots \\ U_N^n \end{pmatrix}$$ eq 7.2.28

Tridiagonal matrix w/ lower diagonal

$\sigma/2\,(1,1,\cdots,1)$

$(1,1,\cdots,1)$

$-\sigma/2\,(1,1,\cdots,1)$

$$U_i^{n+1} = +\sigma U_{i-1}^n + (1-\sigma)U_i^n$$

$$CU^n = \begin{pmatrix} 1-\sigma & 0 & & & \\ \sigma & (1-\sigma) & 0 & & \\ & \sigma & 1-\sigma & 0 & \end{pmatrix} \begin{pmatrix} \vdots \\ U_{i-1}^n \\ U_i^n \\ U_{i+1}^n \\ \vdots \end{pmatrix}$$ eq 7.2.29

Tridiagonal w/

$\sigma(1,1,\cdots,1)$

$(1-\sigma)(1,1,1,\cdots,1)$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{-a}{2\Delta x}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right)$$

$$u_i^{n+1} - u_i^n = -a\frac{\Delta t}{2\Delta x}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) = -\frac{\sigma}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right)$$

$$-\frac{\sigma}{2}u_{i-1}^{n+1} + u_i^{n+1} + \frac{\sigma}{2}u_{i+1}^{n+1} = u_i^n$$

$$\begin{pmatrix} -\sigma/2 & 1 & +\sigma/2 & & \\ & -\sigma/2 & 1 & +\sigma/2 & \\ & & -\sigma/2 & 1 & +\sigma/2 \end{pmatrix} \begin{pmatrix} \vdots \\ u_{i-1}^{n+1} \\ u_i^{n+1} \\ u_{i+1}^{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ u_{i-1}^n \\ u_i^n \\ u_{i+1}^n \\ \vdots \end{pmatrix} \quad \text{eq } 7.2.31$$

Tridiagonal matrix + diagonals are

$$-\sigma/2\,(1,1,\cdots,1)$$
$$(1,1,\cdots,1)$$
$$\sigma/2\,(1,1,\cdots,1)$$

$$-\sigma u_{i-1}^{n+1} + \sigma u_i^{n+1} + u_i^{n+1} = u_i^n$$

$$-\sigma u_{i-1}^{n+1} + (1+\sigma)u_i^{n+1} = u_i^n$$

$$\begin{pmatrix} -\sigma & 1+\sigma & & & & \\ & -\sigma & 1+\sigma & & & \\ & & -\sigma & 1+\sigma & & \\ & & & -\sigma & 1+\sigma & \\ & & & & -\sigma & 1+\sigma \end{pmatrix} \begin{pmatrix} u_{i-1}^{n+1} \\ u_i^{n+1} \end{pmatrix} \quad \text{eq } 7.2.33$$

Tridiagonal matrix w/

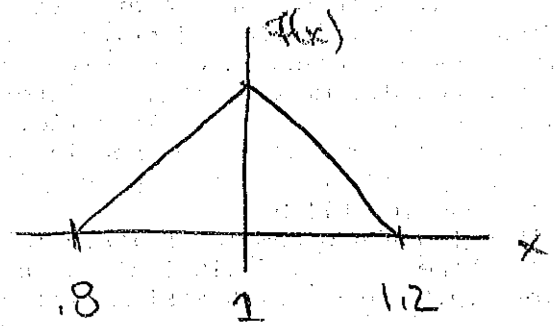
$$-\sigma(1,1,\cdots,1)$$
$$(1+\sigma)(1,1,\cdots 1)$$
$$(0,0,\cdots,0)$$

(Prob 7.1) Initial condition corresponding to picture in Fig 7.2.1 is in correct (7.2.10 is wrong)

want zero at .8 + 1.2 & is 1 at 1.

Then

$$Y_L = \frac{(x-.8)}{.2}$$



$$Y_R = -\frac{(x-1.2)}{.2}$$

Numerical scheme is 7.2.5

$$U_i^{n+1} = U_i^n - \frac{a\Delta t}{2\Delta x}\left(U_{i+1}^n - U_{i-1}^n\right) = U_i^n - \frac{\sigma}{2}\left(U_{i+1}^n - U_{i-1}^n\right).$$

would like to plot exact + initial solution after each time step.

Exact solution $f(x-t)$ Extend calculation up to 5 timesteps.

Figure 7.2.1 has $\sigma = 1/2$ $\sigma = 3/2$

write function that given # of timesteps + σ produces plots like Figure 7.2.1.

Exact solution is $f(x - n\Delta t)$ $\sigma = \frac{1\,\Delta t}{\Delta x} \Rightarrow \Delta t = \Delta x\,\sigma$

so $f(x - n\Delta x\,\sigma)$

making $\sigma \ll 1 \Leftrightarrow \Delta t \ll 1$

(Prob 7.2) Scheme 7.2.13 is upwind scheme

$$U_i^{n+1} = U_i^n - \sigma (U_i^n - U_{i-1}^n)$$

$\sigma = 1/2$ & $\sigma = 3/2$ $\sigma = 1$. into above gives

$$U_i^{n+1} = \cancel{U_i^n} - \cancel{U_i^n} + U_{i-1}^n$$

$U_i^{n+1} = U_{i-1}^n$ translation to the Right at speed 1.

See MMA

(Prob 7.3) Scheme 7.2.13 w/ $\sigma = 1/2$

& $\Delta x = 0.05$ $\Delta x = 0.025$

input Δx $\Rightarrow$ generate x_i $\Rightarrow$ generate initial conditions IC

Run as normal.

(Prob 7.4)

Scheme 7.2.13 is $u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n)$

taylor expanding everything about $i+n$. $u_i^n \equiv \tilde{u}(i\Delta x, n\Delta t)$

$$\cancel{u_i^n} + (u_t)_i^n \Delta t + (u_{tt})_i^n \frac{\Delta t^2}{2} + (u_{ttt})_i^n \frac{\Delta t^3}{6} + O(\Delta t^4)$$

$$= \cancel{u_i^n} - \sigma\left(\cancel{u_i^n} - \left(\cancel{u_i^n} - \Delta x (u_x)_i^n + \frac{\Delta x^2}{2}(u_{xx})_i^n - \frac{\Delta x^3}{6}(u_{xxx})_i^n + O(\Delta x^4)\right)\right) \checkmark$$

$$\Rightarrow \Delta t (u_t)_i^n + \frac{\Delta t^2}{2}(u_{tt})_i^n + O(\Delta t^3) = -\sigma \Delta x (u_x)_i^n + \sigma \frac{\Delta x^2}{2}(u_{xx})_i^n + O(\Delta x^3) \checkmark$$

$$\Rightarrow (u_t)_i^n + \frac{\Delta x \sigma}{\Delta t}(u_x)_i^n = -\frac{\Delta t^1}{2}(u_{tt})_i^n + O(\Delta t^2) + \frac{\sigma \Delta x^2}{2\Delta t}(u_{xx})_i^n + O\left(\frac{\Delta x^3}{\Delta t}\right) \checkmark$$

Now:

$$\sigma = \frac{a\Delta t}{\Delta x}$$

?

$$\Rightarrow (u_t)_i^n + a(u_x)_i^n = -\frac{\Delta t}{2}(u_{tt})_i^n + \frac{a\cancel{\Delta t}}{\cancel{\Delta x}} \cdot \frac{\Delta x^2}{2\cancel{\Delta t}}(u_{xx})_i^n + O(\Delta x^2)$$

$$= -\frac{\Delta t}{2}(u_{tt})^n + \frac{\Delta x a}{2}(u_{xx})_i^n + O(\Delta x^2) \checkmark$$

From PDE $u_t = -a u_x$

$u_{tt} = -a u_{xt} = -a(u_t)_x = -a(-a u_x)_x = +a^2 u_{xx}$

$$\Rightarrow (u_t)_i^n + a(u_x)_i^n = -\frac{\Delta t}{2}(a^2 u_{xx})_i^n + \frac{\Delta x a}{2}(u_{xx})_i^n + O(\Delta x^2)$$

$$(U_t)_i^n + a(U_x)_i^n = \frac{a}{2}(\Delta x - a\Delta t)(U_{xx})_i^n + O(\Delta x^2)$$

$$= \underbrace{\frac{a}{2}\Delta x(1-\sigma)}(U_{xx})_i^n + O(\Delta x^2)$$

if $\sigma = \frac{a\Delta t}{\Delta x}$ positive obtain the diffusion eq iff

$1-\sigma \geq 0 \Rightarrow \sigma \leq 1$ for stability.

Prob 7.5

Generalized trapezoid formula:

$$(u_t)_i = H(u_i)$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \theta H(u_i^{n+1}) + (1-\theta) H(u_i^n) \qquad \theta = \begin{cases} 0 & \text{Forward Euler} \\ 1/2 & \text{Lax-Wendroff} \\ 1 & \text{Backwards Euler} \end{cases}$$

For $u_t + a u_x = 0$

$$(u_t) \approx -a\left[\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right]$$

Generalized trapeziod rule is then

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -a\theta\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right] - a(1-\theta)\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right]$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\theta\left[\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x}\right] + a(1-\theta)\left[\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}\right] = 0$$

Expanding each term in a taylor series gives:

$$(u_t)_i^n + \frac{\Delta t}{2}(u_{tt})_i^n + \frac{\Delta t^2}{6}(u_{ttt})_i^n + O(\Delta t^3)$$

$$+\frac{a\theta}{2\Delta x}\Big[\cancel{u^{n+1}} + \Delta x(u_i^{n+1})_x + \cancel{\frac{\Delta x^2}{2}(u_i^{n+1})_{xx}} + \frac{\Delta x^3}{6}(u_i^{n+1})_{xxx} + \cancel{\frac{\Delta x^4}{24}(u_i^{n+1})_{xxxx}} + O(\Delta x^5)$$

$$-\cancel{u_i^{n+1}} + \Delta x(u_i^{n+1})_x - \cancel{\frac{\Delta x^2}{2}(u_i^{n+1})_{xx}} + \frac{\Delta x^3}{6}(u_i^{n+1})_{xxx} - \cancel{\frac{\Delta x^4}{24}(u_i^{n+1})_{xxxx}} + O(\Delta x^5)\Big]$$

$$+\frac{a(1-\theta)}{2\Delta x}\left[2\Delta x(U_i^n)_x+\frac{2\Delta x^3}{6}(U_i^n)_{xxx}+\frac{2\Delta x^5}{5!}(U_i^n)_{xxxxx}+\mathcal{O}(\Delta x^7)\right]=0$$

$$\Rightarrow (U_t)_i^n+\frac{\Delta t}{2}(U_{tt})_i^n+\frac{\Delta t^2}{6}(U_{ttt})_i^n+\mathcal{O}(\Delta t^3)$$

$$+\frac{a\theta}{2\Delta x}\left[2\Delta x(U_i^{n+1})_x+\frac{2\Delta x^3}{6}(U_i^{n+1})_{xxx}+\frac{2\Delta x^5}{5!}(U_i^{n+1})_{xxxxx}+\mathcal{O}(\Delta x^7)\right]$$

$$+\frac{a(1-\theta)}{2\Delta x}\left[2\Delta x(U_i^n)_x+\frac{2\Delta x^3}{6}(U_i^n)_{xxx}+\frac{2\Delta x^5}{5!}(U_i^n)_{xxxxx}+\mathcal{O}(\Delta x^7)\right]=0$$

$$\Rightarrow (U_t)_i^n+\frac{\Delta t}{2}(U_{tt})_i^n+\frac{\Delta t^2}{6}(U_{ttt})_i^n+\mathcal{O}(\Delta t^3)\;+$$

$$a\theta\left[(U_i^{n+1})_x+\frac{\Delta x^2}{6}(U_i^{n+1})_{xxx}+\frac{\Delta x^4}{5!}(U_i^{n+1})_{xxxxx}+\mathcal{O}(\Delta x^6)\right]$$

$$+a(1-\theta)\left[(U_i^n)_x+\frac{\Delta x^2}{6}(U_i^n)_{xxx}+\frac{\Delta x^4}{5!}(U_i^n)_{xxxxx}+\mathcal{O}(\Delta x^6)\right]=0$$

$$\Rightarrow (U_t)_i^n+\frac{\Delta t}{2}(U_{tt})_i^n+\frac{\Delta t^2}{6}(U_{ttt})_i^n+\mathcal{O}(\Delta t^3)\;+$$

$$a\theta\left[(U_x)_i^n+\Delta t(U_{xt})_i^n+\frac{\Delta t^2}{2}(U_{xtt})_i^n+\mathcal{O}(\Delta t^3)+\frac{\Delta x^2}{6}\left[(U_{xxx})_i^n+(U_{xxxt})_i^n\Delta t+(U_{xxxtt})_i^n\frac{\Delta t^2}{2}+\mathcal{O}(\Delta t^3)\right]\right.$$

$$+ \frac{\Delta x^4}{5!}\left[(u_{xxxxx})_i^n + O(\Delta t)\right] + O(\Delta x^6) \Bigg]$$

$$+ a(1-\theta)\left[(u_x)_i^n + \frac{\Delta x^2}{6}(u_{xxx})_i^n + \frac{\Delta x^4}{5!}(u_{xxxxx})_i^n + O(\Delta x^6)\right] = 0$$

$$\Rightarrow (u_t)_i^n + \frac{\Delta t}{2}(u_{tt})_i^n + \frac{\Delta t^2}{6}(u_{ttt})_i^n + O(\Delta t^3) +$$

$$a(u_x)_i^n + a\theta(u_{xt})_i^n \Delta t + \frac{a\theta}{2}(u_{xtt})_i^n \Delta t^2 + \frac{a\theta}{6}(u_{xxx})_i^n \Delta x^2$$

$$+ a(1-\theta)\frac{\Delta x^2}{6}(u_{xxx})_i^n + O(\Delta t^3) + O(\Delta t, \Delta x^2) = 0$$

$$\Rightarrow (u_t)_i^n + a(u_x)_i^n = -\frac{\Delta t}{2}(u_{tt})_i^n - a\theta\Delta t(u_{xt})_i^n + O(\Delta t^2, \Delta x^2)$$

Now $u_{xt} = (u_t)_x = (-au_x)_x = -au_{xx}$

$u_{tt} = (u_t)_t = (-au_x)_t = -a(u_{xt}) = -a(-a)u_{xx} = a^2 u_{xx}$

$$\Rightarrow (u_t)_i^n + a(u_x)_i^n = -\frac{a^2\Delta t}{2}(u_{xx})_i^n + a\theta\Delta t a(u_{xx})_i^n + O(\Delta t^2, \Delta x^2)$$

$$= \underbrace{a^2\Delta t\left(-\frac{1}{2}+\theta\right)}_{\text{truncation error}}(u_{xx})_i^n + O(\Delta t^2, \Delta x^2)$$

∴ Note $\theta = 1/2$ higher order terms must be kept,

For stability diffusion term must be positive

$$\Rightarrow -\frac{1}{2} + \theta \geq 0 \Rightarrow \theta \geq \frac{1}{2}$$

∴ scheme is unstable if $\theta < 1/2$