pg 340 Hirsch Vol I

Prob 8.12

4 schemes 1) first order upwind eq $u_t + u_x = 0$

2) Lax-Friedrich

3) Lax-Wendroff

4) Leap frog

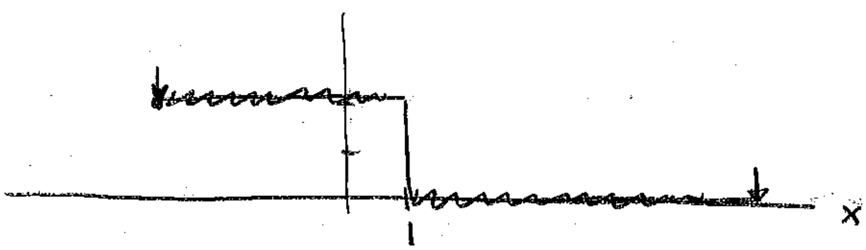
$x \in [2.0, 4.0]$

n time steps = 50 w/ $\Delta x = 0.05$ $\sigma = 0.8 = \frac{a\Delta t}{\Delta x} \Rightarrow \Delta t = .8(.05) = 0.04$

Then time = $50\Delta t = 2.0$ since the exact solution is now located at 3.0

it must have started propagating at $x = 1.0$, so final location will be at 3.0.

Thus initial condition is $U_0(x)$ $U_{exact}(x,t) = U_0(x-at)$



Take initial domain to be [0,4] then to get a cell size of .05

need $\frac{4-0}{.05} = 80$ cells.

Prob 8.13 Advection eq $u_t + a u_x = 0$ w/ $a=0$

I am not exactly sure what the initial conditions are so I will do a modified problem that I hope captures the same essence.

$$U_0(x) = \begin{cases} \sin(2\pi n x) & 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

Now $\Delta x = 0.05$, $\sigma = 0.8$, nTimesteps = 50

$$N = \frac{3.5 - 0}{\Delta x} = 70$$

Prob 8.16

$u_t + u u_x = 0$ w/ leap frog

$\sigma = .8 = \frac{(\pm 1)\Delta t}{\Delta x}$

$\Rightarrow \Delta t = (\pm 1)(.8)\Delta x$

$$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t} = -\frac{U_i^n}{2\Delta x}(U_{i+1}^n - U_{i-1}^n)$$

$$\Rightarrow U_i^{n+1} = U_i^{n-1} - \left(\frac{\Delta t}{\Delta x}\right) U_i^n (U_{i+1}^n - U_{i-1}^n)$$

How is the courant number defined for non linear problems? I will just use

$\sigma \equiv \frac{\Delta t}{\Delta x} \Rightarrow \Delta t = \Delta x \sigma = .05(.8) = .04$

A discontinuity in U propagates at the speed $s = \frac{1}{2}(u_\ell + u_r)$

$= \frac{1}{2}(-1+1) = 0$

& the leap frog scheme can be written as

$$U_i^{n+1} = U_i^{n-1} - \sigma U_i^n U_{i+1}^n + \sigma U_i^n U_{i-1}^n$$

$\Delta x = .05 \Rightarrow N = \frac{2-0}{.05} = 40$

I can't get 80 timesteps w/o the solution diverging.

pg 341 Hirsch Vol I

Prob 8.21

Prob 5.14: $\frac{du}{dt} = f(u)$

$$\frac{1}{6}\left[\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right] = \frac{1}{2\Delta x}(f_{i+1} - f_{i-1})$$

let $f(u) = -au$ generalized trapezoid rule

$$\frac{1}{6}\left[\frac{u_{i-1}^{n+1} - u_{i-1}^{n}}{\Delta t} + \frac{4(u_i^{n+1} - u_i^n)}{\Delta t} + \frac{(u_{i+1}^{n+1} - u_{i+1}^{n})}{\Delta t}\right]$$

$$= \frac{-a}{2\Delta x}\theta\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) - \frac{a}{2\Delta x}(1-\theta)(u_{i+1}^n - u_{i-1}^n)$$

$$\Rightarrow \quad u_{i-1}^{n+1} - u_{i-1}^{n} + 4(u_i^{n+1} - u_i^n) + (u_{i+1}^{n+1} - u_{i+1}^{n})$$

$$+ 3\theta\sigma(u_{i+1}^{n+1} - u_{i-1}^{n+1}) + 3\sigma(1-\theta)(u_{i+1}^n - u_{i-1}^n) = 0$$

Compute amplification factor consider one fourier mode $u_i^n = \hat{u}^n e^{I\phi i}$

$\phi = k\Delta x$

$$\Rightarrow G e^{-I\phi} - e^{-I\phi} + 4(G-1) + G e^{+I\phi} - e^{+I\phi} \qquad \text{w/ } \hat{u}^{n+1} = G\hat{u}^n$$

$$+ 3\theta\sigma G(e^{+I\phi} - e^{-I\phi}) + 3(1-\theta)\sigma(e^{I\phi} - e^{-I\phi}) = 0$$

$$\Rightarrow (G-1)e^{-I\phi} + 4(G-1) + (G-1)e^{I\phi} + 3\theta\sigma G\, 2I\sin\phi$$

$$+ 3(1-\theta)\sigma\, 2I\sin\phi = 0$$

$$\Rightarrow e^{-I\phi} + 4 + e^{+I\phi} + 6I\sigma\sin\phi\frac{[G\theta + (1-\theta)]}{G-1} = 0$$

08-02-01 Z

$$2\cos\phi + 4 + 6I\sigma\sin\phi\left(\frac{\theta(G-1)+1}{G-1}\right) = 0$$

$$\Rightarrow \cancel{2}(2+\cos\phi) + \overset{3}{\cancel{6}}I\sigma\sin\phi\,\theta + \frac{\overset{3}{\cancel{6}}I\sigma\sin\phi}{G-1} = 0$$

$$\frac{3I\sigma\sin\phi}{G-1} = -(2+\cos\phi) - 3I\sigma\sin\phi\,\theta$$

$$\Rightarrow G-1 = \frac{3I\sigma\sin\phi}{-(2+\cos\phi) - 3I\sigma\sin\phi\,\theta} = \frac{-3I\sigma\sin\phi}{3I\sigma\sin\phi\,\theta + 2+\cos\phi}$$

$$G = \frac{3I\sigma\sin\phi\,\theta + 2+\cos\phi - 3I\sigma\sin\phi}{3I\sigma\sin\phi\,\theta + 2+\cos\phi}$$

$$= \frac{2+\cos\phi + 3I\sigma(\theta-1)\sin\phi}{2+\cos\phi + 3I\sigma\theta\sin\phi} = \frac{2+\cos\phi - 3(1-\theta)I\sigma\sin\phi}{2+\cos\phi + 3\theta I\sigma\sin\phi}$$

$$GG^* = \frac{2+\cos\phi - 3(1-\theta)I\sigma\sin\phi}{2+\cos\phi + 3\theta I\sigma\sin\phi}\;\frac{2+\cos\phi + 3(1-\theta)I\sigma\sin\phi}{2+\cos\phi - 3\theta I\sigma\sin\phi}$$

$$= \frac{(2+\cos\phi)^2 + 9(1-\theta)^2\sigma^2\sin^2\phi}{(2+\cos\phi)^2 + 9\theta^2\sigma^2\sin^2\phi} < \frac{3^2 + 9(1-\theta)^2\sigma^2}{4}$$

$$= \frac{9}{4}\left(1+\sigma^2(1-\theta)^2\right)$$

$$GG^* = \frac{\left(\frac{2+\cos\phi}{3}\right)^2 + (1-\theta)^2 \sigma^2 \sin^2\phi}{\left(\frac{2+\cos\phi}{3}\right)^2 + \theta^2 \sigma^2 \sin^2\phi}$$

$$= \frac{Y + (1-\theta)^2 X}{Y + \theta^2 X} \quad \text{w/ } Y = \left(\frac{2+\cos\phi}{3}\right)^2$$

$$\text{\& } X = (\sigma \sin\phi)^2$$

Note $X > 0$ & $Y > 0$

& $GG^*(\theta = 1) = \frac{Y}{Y+X} < 1$

$GG^* = 1$ when $(1-\theta)^2 = \theta^2 \Rightarrow \theta = 1/2$

So $|G|^2 < 1 \quad \forall \quad \frac{1}{2} < \theta < 1$ unconditionally

If $\theta = \frac{1}{2}$ Then $|G|^2 = 1 \Rightarrow$ No dissipation error

let $G = |G| e^{-I\Phi}$

w/ $|G| = e^{\eta \Delta t}$ & $\Phi = \xi \Delta t$

write $G = e^{-I\omega\Delta t}$

For this scheme w/ $\theta = \frac{1}{2}$ $\quad G = \frac{2+\cos\phi - 3(1-\theta) I \sigma \sin\phi}{2+\cos\phi + 3\theta I \sigma \sin\phi}$

$$G(1/2 = \theta) = \frac{2+\cos\phi - 3/2\, I \sigma \sin\phi}{2+\cos\phi + 3/2\, I\sigma\sin\phi}$$

Then since $G \ni |G| = 1$ $\qquad G = \frac{x}{x^*}$

There is no dissipation error: $\qquad |G|^2 = \left(\frac{x}{x^*}\right)\left(\frac{x^*}{x}\right) = 1$

The numerical dispersion relationship is given by $Re\left(\frac{dw}{dt}\right)$

$$G = e^{-Iw\Delta t} \underset{\substack{\uparrow \\ \text{For trapezium} \\ \text{relation}}}{=} \frac{2+\cos\phi - \frac{3}{2} I G \sin\phi}{2+\cos\phi + \frac{3}{2} I G \sin\phi}$$

$$-\frac{I\Delta t}{\Delta x} e^{-Iw\Delta t} \frac{dw}{dk} = \frac{\left(-\sin\phi - \frac{3}{2} I G \cos\phi\right)}{2+\cos\phi+\frac{3}{2}IG\sin\phi} - \frac{\left(2+\cos\phi - \frac{3}{2}IG\sin\phi\right)}{\left(2+\cos\phi+\frac{3}{2}IG\sin\phi\right)^2} \cdot \left(-\sin\phi + \frac{3}{2}IG\cos\phi\right)$$

$$\Rightarrow +I\frac{G}{a} e^{-Iw\Delta t}\frac{dw}{dk} = \frac{\left[\left(-\sin\phi - \frac{3}{2}IG\cos\phi\right)\left(2+\cos\phi+\frac{3}{2}IG\sin\phi\right) - \left(2+\cos\phi-\frac{3}{2}IG\sin\phi\right)\left(-\sin\phi+\frac{3}{2}IG\cos\phi\right)\right]}{\left(2+\cos\phi+\frac{3}{2}IG\sin\phi\right)^2}$$

$$= \frac{AB - \bar{B}\bar{A}}{(\qquad)^2} =$$

$x+iy-(x-iy) = 2iy$
$= 2I \operatorname{img}(AB)$

By MMA

$$= \frac{+3IB(1+2\cos(\phi))}{\left(2+\cos\phi+\frac{3}{2}IG\sin\phi\right)^2}$$

$$\frac{dw}{dk} = \frac{3(1+2\cos(\phi))\,a}{(2+\cos\phi+\frac{3}{2}I\sigma\sin\phi)^2}\; e^{I\omega\Delta t}$$

$$\frac{dw}{dk} = \frac{3a(1+2\cos(\phi))}{(2+\cos\phi+\frac{3}{2}I\sigma\sin\phi)^{\cancel{2}}}\left(\frac{\cancel{2+\cos\phi+\frac{3}{2}I\sigma\sin\phi}}{2+\cos\phi-\frac{3}{2}I\sigma\sin\phi}\right)$$

$$= \frac{3a(1+2\cos(\phi))}{(2+\cos\phi+\frac{3}{2}I\sigma\sin\phi)}\;\frac{1}{(2+\cos\phi-\frac{3}{2}I\sigma\sin\phi)}$$

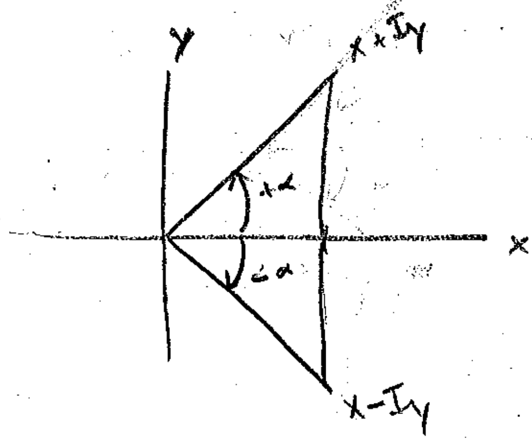
$$= \frac{3a(1+2\cos\phi)}{(2+\cos\phi)^2+\frac{9}{4}\sigma^2\sin^2\phi}$$

But they ask for a more accurate dispersion relation. given

$$e^{-I\omega\Delta t} = \frac{2+\cos\phi-\frac{3}{2}\sigma I\sin\phi}{2+\cos\phi+\frac{3}{2}I\sigma\sin\phi}$$

since RHS is + that $|z|=1$ let $x = 2+\cos\phi$
+ $y = \frac{3}{2}\sigma\sin\phi$

Then RHS eqs $\dfrac{x-Iy}{x+Iy}$



Now If $x+Iy = \sqrt{x^2+y^2}\,e^{I\alpha}$
$x-Iy = \sqrt{}\,e^{-I\alpha}$

so that $\dfrac{x+Iy}{x+Iy} = e^{-2I\alpha}$

Thus $e^{-I\omega\Delta t} = e^{-2I\alpha}$

$$\Rightarrow \quad -\omega\Delta t = -2\alpha \quad \Rightarrow \quad \frac{\omega\Delta t}{2} = \alpha$$

$$\& \quad \alpha = \tan^{-1}\left[\frac{3/2\, G \sin\phi}{2+\cos\phi}\right] = \frac{\omega\Delta t}{2}$$

$$\Rightarrow \quad \tan\left(\frac{\omega\Delta t}{2}\right) = \frac{\frac{3}{2} G \sin\phi}{2+\cos\phi}$$

Prob 8,22

8,3,6

$$\tilde{v}_G(k) = \frac{d\tilde{\omega}}{dk} \qquad V_G(k) = \frac{d\xi}{dk} = Re\left(\frac{d\omega}{dk}\right) G = e^{-I\omega\Delta t}$$

$$\phi = k\Delta x$$

First order upwind: 8.1.19 gives

$$G = 1 - 2\sigma \sin^2 \phi/2 - I\sigma \sin\phi$$

$$\|$$

$$e^{-I\omega\Delta t} = 1 - 2\sigma \sin^2\phi/2 - I\sigma\sin\phi$$

Also from eq 8,1,20

$$\frac{d}{dk} \to$$

$$-I\Delta t \frac{d\omega}{dk} e^{-I\omega\Delta t} = -4\sigma\sin(\phi/2)\cos(\phi/2)\frac{\Delta x}{2} - I\sigma\cos\phi\,\Delta x$$

$$\frac{d\omega}{dk} = \frac{\Delta x e^{I\omega\Delta t}}{\Delta t I}\left(2\sigma\sin(\phi/2)\cos(\phi/2) + I\sigma\cos\phi\right)$$

$$= \frac{a}{\sigma}\left(-2I\sigma\sin(\phi/2)\cos(\phi/2) + \sigma\cos\phi\right)e^{I\omega\Delta t}$$

$$\sigma = \frac{a\Delta t}{\Delta x} \qquad = \frac{a}{\sigma}\frac{\left(\sigma\cos\phi - 2I\sigma\sin(\phi/2)\cos(\phi/2)\right)}{\left(1 - 2\sigma\sin^2(\phi/2) - I\sigma\sin\phi\right)}$$

$$\frac{\Delta x}{\Delta t} = \frac{a}{\sigma}$$

Now I want the Real part of this expression

$$\frac{d\omega}{dt} = \frac{a}{\sigma}\frac{(\sigma\cos\phi - I\sigma\sin\phi)}{(1-2\sigma\sin^2(\phi/2) - I\sigma\sin\phi)}\frac{(1-2\sigma\sin^2(\phi/2)+I\sigma\sin\phi)}{(1-2\sigma\sin^2(\phi/2)+I\sigma\sin\phi)}$$

$$= \frac{a}{\sigma}\frac{\left(\sigma\cos\phi(1-2\sigma\sin^2(\phi/2)) + \sigma^2\sin^2\phi + I(\text{ terms })\right)}{(1-2\sigma\sin^2(\phi/2))^2 + \sigma^2\sin^2\phi}$$

Thus $\mathrm{Re}\left(\frac{d\omega}{dt}\right) = \frac{a}{\sigma}\frac{\left(\sigma\cos\phi(1-2\sigma\sin^2(\phi/2)) + \sigma^2\sin^2\phi\right)}{(1-2\sigma\sin^2(\phi/2))^2 + \sigma^2\sin^2\phi}$

Now $2\sin^2(\phi/2) = ?$

$$\cos(2\phi) = \cos^2\phi - \sin^2\phi$$
$$= 1 - 2\cos^2\phi$$
$$= 1 - 2\sin^2\phi \rightarrow 2\sin^2\phi = 1-\cos(2\phi)$$

$\therefore \quad 2\sin^2(\phi/2) = 1-\cos(\phi)$

So

$$V_\sigma = \mathrm{Re}\left(\frac{d\omega}{dk}\right) = \frac{a}{\sigma}\frac{\left(\sigma\cos\phi(1-\sigma(1-\cos\phi)) + \sigma^2\sin^2\phi\right)}{\left((1+\sigma(\cos\phi-1))^2 + \sigma^2\sin^2\phi\right)}$$

$$= \frac{a}{\sigma}\frac{\left(\sigma\cos\phi - \sigma^2\cos\phi + \sigma^2\cos^2\phi + \sigma^2\sin^2\phi\right)}{\left((1+\sigma(\cos\phi-1))^2 + \sigma^2\sin^2\phi\right)}$$

$$= \frac{a(\cos\phi - \sigma\cos\phi + \sigma)}{''} = \frac{a((1-\sigma)\cos\phi + \sigma)}{''}$$

Lax - Friedrich : From E8.3.2

$$G = e^{-I\omega\Delta t} = \cos\phi - IG\sin\phi$$

$$\frac{d}{dk} \Rightarrow$$

$$\frac{d\omega}{dk} e^{-I\omega\Delta t}(-I\Delta t) = \Delta x\left(-\sin\phi - IG\cos\phi\right) \qquad \frac{\Delta x}{\Delta t} = \frac{a}{G}$$

$$\frac{d\omega}{dk} = \frac{a\left(-\sin\phi - IG\cos\phi\right)}{G\left(\cos\phi - IG\sin\phi\right)(-I)}$$

$$= \frac{a}{G} I \frac{\left(-\sin\phi + IG\cos\phi\right)\left(\cos\phi + IG\sin\phi\right)}{\cos^2\phi + G^2\sin^2\phi}$$

$$= \frac{aI}{G} \frac{\left(-\sin\phi\cos\phi - IG\sin^2\phi - IG\cos^2\phi + G^2\cos\phi\sin\phi\right)}{\cos^2\phi + G^2\sin^2\phi}$$

$$= \frac{a}{G}\left[\frac{(G^2-1)\sin\phi\cos\phi - IG}{\cos^2\phi + G^2\sin^2\phi}\right] I$$

$$\therefore \mathrm{Re}\left(\frac{d\omega}{dk}\right) = \frac{a}{\cos^2\phi + G^2\sin^2\phi}$$

Lax-Wendroff: $G = e^{-I\omega\Delta t} = 1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi)$ ✓

$$-I\Delta t\, e^{-I\omega\Delta t}\frac{d\omega}{dk} = \Delta x\left[-I\sigma\cos\phi - \sigma^2\sin\phi\right]$$ ✓

$$\frac{d\omega}{dk} = \frac{a}{\sigma}\frac{1}{(-I)}\frac{(-I\sigma\cos\phi - \sigma^2\sin\phi)}{(1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi))} = \frac{aI(-I\cos\phi - \sigma\sin\phi)}{(1-\sigma^2(1-\cos\phi) - I\sigma\sin\phi)}$$ ✓

$1-\cos\phi$
$\parallel$

$$\cos 2\phi = \cos^2\phi - \sin^2\phi$$
$$= 1 - 2\sin^2\phi$$
$$= 2\cos^2\phi - 1 = -(1 - 2\cos^2\phi)$$

$$\frac{d\omega}{dk} = \frac{a(\cos\phi - I\sigma\sin\phi)\,(1-\sigma^2(1-\cos\phi)+I\sigma\sin\phi)}{(1-\sigma^2(1-\cos\phi) - I\sigma\sin\phi)(1-\sigma^2(1-\cos\phi)+I\sigma\sin\phi)}$$ ✓

$$= \frac{a\left(\cos\phi(1-\sigma^2(1-\cos\phi)) + \sigma^2\sin^2\phi + I\{\text{terms}\}\right)}{(1-\sigma^2(1-\cos\phi))^2 + \sigma^2\sin^2\phi}$$

Thus

$$Re\left(\frac{d\omega}{dk}\right) = \frac{a(\cos\phi - \sigma^2\cos\phi + \sigma^2\cos^2\phi + \sigma^2\sin^2\phi)}{}$$

Note Above real expression is the exact one given in text

$$2\sin^2(\phi/2) \stackrel{?}{=} 1 - \cos\phi$$

$$2\frac{(e^{I\phi/2} - e^{-I\phi/2})^2}{(-4)} = -\frac{1}{2}(e^{I\phi} - 2 + e^{-I\phi}) = 1 - \frac{1}{2}(e^{I\phi} + e^{-I\phi})$$

Thus

$$2\sin^2(\phi/2) = 1-\cos\phi$$

$\therefore$

$$V_G = \underline{a\left((1-\sigma^2)\cos\phi+1\right)}$$

eq 8.3.16 is

$$u_i^{n+1} = u_i^n + \frac{\alpha\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Then using 9.1.3 we get

$$\cancel{u_i^n} + \Delta t(u_t)_i + \frac{\Delta t^2}{2}(u_{tt})_i + \frac{\Delta t^3}{6}(u_{ttt})_i + O(\Delta t^4)$$

$$= \cancel{u_i^n} + \frac{\alpha\Delta t}{\Delta x^2}\left(\Delta x^2(u_{xx})_i + \frac{2\Delta x^4}{4!}(u_{xxxx})_i + \frac{2}{6!}\Delta x^6(u_{VI})\right)$$

$$\Rightarrow u_t + \frac{\Delta t}{2}(u_{tt})_i + O(\Delta t^2) = \alpha\left((u_{xx})_i + \frac{2}{4!}\alpha\Delta x^2(u_{xxxx})_i + \frac{2\alpha}{6!}\Delta x^4(u_{VI})_i\right)$$

$$u_t - \alpha u_{xx} = -\frac{\Delta t}{2}u_{tt} + \frac{\alpha\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{\alpha\Delta x^4}{3\underbrace{(5!)}}(u_{VI}) + \cdots$$

$$-\frac{\Delta t^2}{6}u_{ttt}$$

$5 \cdot 24 = 120$

$\cdot 3 = 360$

$5! = 5\cdot4\cdot3\cdot2$

6

$5\cdot 24 =$

24
5
120

eq 9.1.5.

$$u_t = \alpha u_{xx}$$

$$u_{tt} = \alpha(u_t)_{xx} = \alpha(\alpha u_{xx})_{xx} = \alpha^2 u_{xxxx}$$

$$u_{ttt} = \alpha^2(u_t)_{xxxx} = \alpha^3\frac{\partial^6 u}{\partial x^6}$$

$$u_t - \alpha u_{xx} = -\frac{\Delta t\,\alpha^2}{2}\frac{\partial^4 u}{\partial x^4} + \alpha\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)$$

$$+ \alpha\frac{\Delta x^4}{360}\frac{\partial^6 u}{\partial x^6} - \frac{\Delta t^2}{6}\alpha^3\frac{\partial^6 u}{\partial x^6} + \ldots$$

$$= -\frac{1}{2}\alpha\,\Delta x^2\left(\frac{\alpha\,\Delta t}{\Delta x^2} - \frac{1}{6}\right)\frac{\partial^4 u}{\partial x^4}$$

$$+ \alpha\,\underline{\Delta x^4}$$

Goal is to get u_t written in terms of only u_x's

u_{tt}

u_{ttt} etc. to a specified order of accuracy.

If I can get $u_t = \mathcal{L}(u, u_x, u_{xx}, \cdots) + O(\Delta x^p)$

$u_{tt} = \mathcal{J}(u_t, u_{tx}, u_{txx}, \cdots) + O(\Delta x^p)$

$u_{tt} = \mathcal{J}(\mathcal{L}(u, u_x, u_{xx}, \cdots) + O(\Delta x^p), \mathcal{L}(u_x, u_{xx}, u_{xxx}, \cdots) + O(\Delta x^p), \cdots)$

$+ O(\Delta x^p)$ etc

Thus since from the known eq 9.1.5

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} - \frac{\Delta t}{2} u_{tt} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6} - \frac{\Delta t^2}{6} u_{ttt} + O(\Delta t^3) + O(\Delta x^6)$$

$$= \underbrace{\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6}}_{\text{independent of } t \text{ derivatives}} - \frac{\Delta t}{2} u_{tt} - \frac{\Delta t^2}{6} u_{ttt} + O(\Delta t^3) + O(\Delta x^6)$$

Then putting this operator into itself gives

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6}$$

$$- \frac{\Delta t}{2} \frac{\partial}{\partial t}\left[\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6} - \frac{\Delta t}{2} u_{tt} - \frac{\Delta t^2}{6} u_{ttt} + O(\Delta t^3) + O(\Delta x^6)\right]$$

$$- \frac{\Delta t^2}{6} \frac{\partial^2}{\partial t^2}\left[\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6} - \frac{\Delta t}{2} u_{tt} - \frac{\Delta t^2}{6} u_{ttt} + O(\Delta t^3) + O(\Delta x^6)\right]$$

$\Rightarrow$

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6}$$

$$- \frac{\Delta t}{2} \alpha \frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t}$$

$$- \alpha \frac{\Delta t \Delta x^2}{24} \frac{\partial^4}{\partial x^4} \frac{\partial u}{\partial t}$$

$$+ O(\Delta t \Delta x^4)$$

$$+ \frac{\Delta t^2}{4} u_{ttt} + O(\Delta t^3)$$

$$- \frac{\Delta t^2}{6} \alpha \frac{\partial^2}{\partial x^2} \frac{\partial^2 u}{\partial t^2}$$

$$+ O(\Delta t^2 \Delta x^2)$$

to the order specified & w/ more x derivatives now than before.

$\Rightarrow$

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6}$$

$$- \frac{\Delta t}{2} \alpha \frac{\partial^2}{\partial x^2} \left[\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6} - \alpha \frac{\Delta t}{2} \frac{\partial^2}{\partial x^2} \frac{\partial u}{\partial t} \right]$$

$\Leftarrow$ Note: this substitution should use the simplification above, & not the 1st one

$u_t = \alpha u_{xx} - \frac{\Delta t}{2} u_{xxt} + \dots$

This is kinda a trick.

$$- \alpha \frac{\Delta t \Delta x^2}{24} \frac{\partial^4}{\partial x^4} \left[\alpha \frac{\partial^2 u}{\partial x^2} \right] + O(\Delta t^3) + O(\Delta t \Delta x^4) + O(\Delta x^6)$$

$$+ \frac{\Delta t^2}{4} \frac{\partial^2}{\partial t^2} \left[\alpha \frac{\partial^2 u}{\partial x^2} \right]$$

$$- \alpha \frac{\Delta t^2}{6} \frac{\partial^2}{\partial x^2} \frac{\partial}{\partial t} \left[\alpha \frac{\partial^2 u}{\partial x^2} \right]$$

$$\Rightarrow u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^4}{360}\frac{\partial^6 u}{\partial x^6}$$

$$- \alpha^2 \frac{\Delta t}{2}\frac{\partial^4 u}{\partial x^4} - \alpha^2 \frac{\Delta t \Delta x^2}{24}\frac{\partial^6 u}{\partial x^6} + \alpha^2 \frac{\Delta t^2}{4}\frac{\partial^4}{\partial x^4}\frac{\partial u}{\partial t}$$

$$- \frac{\alpha^2}{24}\Delta t \Delta x^2 \frac{\partial^6 u}{\partial x^6} + \alpha \frac{\Delta t^2}{4}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}\right] - \alpha^2 \frac{\Delta t^2}{6}\frac{\partial^4}{\partial x^4}\frac{\partial u}{\partial t} + O(\Delta t^3) + O(\Delta t \Delta x^4) + O(\Delta x^6)$$

Thus

$$u_t = \alpha \frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{2}\left(\frac{\Delta x^2}{6} - \alpha \Delta t\right)\frac{\partial^4 u}{\partial x^4}$$

$$+ \left(\frac{\alpha \Delta x^4}{360} - \frac{\alpha^2 \Delta t \Delta x^2}{24} - \frac{\alpha^2 \Delta t \Delta x^2}{24}\right)\frac{\partial^6 u}{\partial x^6} + \alpha^2 \Delta t^2 \left(\frac{1}{4} - \frac{1}{6}\right)\frac{\partial^4}{\partial x^4}\frac{\partial u}{\partial t}$$

$$+ \alpha \frac{\Delta t^2}{4}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}\right]$$

$$\Rightarrow u_t = \alpha \frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{2}\left(\frac{\Delta x^2}{6} - \alpha \Delta t\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^2}{12}\left(\frac{\Delta x^2}{30} - \frac{2\Delta t \alpha}{2}\right)\frac{\partial^6 u}{\partial x^6}$$

$$360 = 6 \cdot 60 = \cancel{6} \cdot \cancel{6} \cdot 10 \qquad + \frac{\alpha^2 \Delta t^2}{12}\frac{\partial^4}{\partial x^4}\frac{\partial u}{\partial t} + \alpha \frac{\Delta t^2}{4}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^2 u}{\partial t^2}\right]$$

$$24 = 2 \cdot 2 \cdot 3 \cdot 2$$

Now again put the above expression into itself keeping only the terms that will give the correct order desired

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{2}\left(\frac{\Delta x^2}{6} - \alpha \Delta t\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^2}{12}\left(\frac{\Delta x^2}{30} - \Delta t \alpha\right)\frac{\partial^6 u}{\partial x^6}$$

$$+ \frac{\alpha^2 \Delta t^2}{12}\frac{\partial^4}{\partial x^4}\left[\alpha \frac{\partial^2 u}{\partial x^2}\right]$$

$$+ \frac{\alpha \Delta t^2}{4}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}\left[\alpha \frac{\partial^2 u}{\partial x^2}\right]$$

$$= \alpha \frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{2}\left(\frac{\Delta x^2}{6} - \alpha \Delta t\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^2}{12}\left(\frac{\Delta x^2}{30} - \Delta t \alpha\right)\frac{\partial^6 u}{\partial x^6}$$

$$+ \alpha^3 \frac{\Delta t^2}{12}\frac{\partial^6 u}{\partial x^6} + \frac{\alpha^2 \Delta t^2}{4}\frac{\partial^4}{\partial x^4}\frac{\partial u}{\partial t}$$

$\parallel$

can only put in $\alpha \frac{\partial^2 u}{\partial x^2}$ to

keep global expression at correct order of magnitude.

Thus in summary

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \frac{\alpha}{2}\left(\frac{\Delta x^2}{6} - \alpha \Delta t\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^2}{12}\left(\frac{\Delta x^2}{30} - \Delta t \alpha\right)\frac{\partial^6 u}{\partial x^6}$$

$$+ \frac{\alpha^3 \Delta t^2}{12}\frac{\partial^6 u}{\partial x^6} + \frac{\alpha^3 \Delta t^2}{4}\frac{\partial^6 u}{\partial x^6}$$

$\frac{1}{4} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$

$$\Rightarrow u_t - \alpha u_{xx} = -\frac{\alpha \Delta x^2}{2}\left(\frac{\alpha \Delta t}{\Delta x^2} - \frac{1}{6}\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^4}{12}\left(\frac{1}{30} - \frac{\Delta t \alpha}{\Delta x^2} + \frac{\alpha^2 \Delta t^2 (12)}{3 \Delta x^4}\right)\frac{\partial^6 u}{\partial x^6}$$

Defining $\beta = \frac{\alpha \Delta t}{\Delta x^2}$

$$\Rightarrow u_t - \alpha u_{xx} = -\alpha \frac{\Delta x^2}{2}\left(\frac{\alpha \Delta t}{\Delta x^2} - \frac{1}{6}\right)\frac{\partial^4 u}{\partial x^4} + \frac{\alpha \Delta x^4}{3}\left(\frac{1}{120} - \frac{\beta}{4} + \beta^2\right)\frac{\partial^6 u}{\partial x^6}$$

eq 9.1.6

9-2-011

pg 344 Hirsh Vol I

$G = e^{-I\omega t}$ $\qquad$ $u_k(t) \doteq e^{-I\omega t} e^{Ikx}$

$$-I\omega \cancel{u_k(t)} = \alpha (Ik)^2 \cancel{u_k(t)} + r_1 (Ik)^4 \cancel{u_k(t)} + r_2 (Ik)^6 \cancel{u_k(t)}$$

$$= -\alpha k^2 + r_1 k^4 - r_2 k^6 \qquad \text{eq } 9.1.11$$

$$\Rightarrow \quad \omega = -(\alpha k^2 - r_1 k^4 + r_2 k^6) I$$

So

$$u_k(t) = e^{+II(\alpha k^2 - r_1 k^4 + r_2 k^6) t} e^{Ikx} = e^{-(\alpha k^2 - r_1 k^4 + r_2 k^6) t} e^{Ikx} \qquad \text{eq } 9.1.12$$

$$= e^{-(\alpha - r_1 k^2 + r_2 k^4) k^2 t} e^{Ikx}$$

Require $\alpha - r_1 k^2 + r_2 k^4 \geq 0$

$\alpha > 0 \qquad r_1 < 0 \qquad r_2 > 0 \qquad$ eq 9.1.14

Thus require that $\alpha > 0$

$B - \frac{1}{6} > 0 \qquad B > \frac{1}{6} \quad \Rightarrow \qquad B <$

Thus doesn't make sense?

pg 344–345 Hirsch Vol I

9-12-01 1

$$u_i^{n+1} - u_i^{n-1} = 2\beta\left(u_{i+1}^{n} - u_i^{n+1} - u_i^{n-1} + u_{i-1}^{n}\right)$$

$$\Delta t\, u_t + \cancel{\frac{\Delta t^2}{2} u_{tt}} + \frac{\Delta t^3}{6} u_{ttt} + \cancel{O(\Delta t^4)}$$

$$-\left(-\Delta t\, u_t + \cancel{\frac{\Delta t^2}{2} u_{tt}} - \frac{\Delta t^3}{6} u_{ttt} + \cancel{O(\Delta t^4)}\right) = 2\beta($$

$$2\Delta t\, u_t + 2\frac{\Delta t^3}{6} u_{ttt} + 2\frac{\Delta t^5}{5!} u_{ttttt} + O(\Delta t^7)$$

$$= 2\beta($$

$$\cancel{\phi} + \cancel{\Delta x\, u_x} + \frac{\Delta x^2}{2} u_{xx} + \cancel{\frac{\Delta x^3}{6} u_{xxx}} + \frac{\Delta x^4}{24} u_{xxxx} + O(\Delta x^5)$$

$$-\left(\cancel{\phi} + \cancel{\Delta t\, u_t} + \frac{\Delta t^2}{2} u_{tt} + \cancel{\frac{\Delta t^3}{6} u_{ttt}} + \frac{\Delta t^4}{24} u_{tttt} + O(\Delta t^5)\right)$$

$$-\left(\cancel{\phi} - \cancel{\Delta t\, u_t} + \frac{\Delta t^2}{2} u_{tt} - \cancel{\frac{\Delta t^3}{6} u_{ttt}} + \frac{\Delta t^4}{24} u_{tttt} + O(\Delta t^5)\right)$$

$$+\left(\cancel{\phi} - \cancel{\Delta x\, u_x} + \frac{\Delta x^2}{2} u_{xx} - \cancel{\frac{\Delta x^3}{6} u_{xxx}} + \frac{\Delta x^4}{24} u_{xxxx} + O(\Delta x^5)\right)$$

$$)$$

$$= 2\beta\left(\frac{2\Delta x^2}{2} u_{xx} + \frac{\Delta x^4}{12} u_{xxxx} - \Delta t^2 u_{tt} - \frac{\Delta t^4}{12} u_{tttt}\right)$$

$$u_t + \frac{\Delta t^2}{6}u_{ttt} + \underbrace{\frac{\Delta t^4}{5!}u_{ttttt}}_{} = \frac{\beta}{\Delta t}\left(\Delta x^2 u_{xx} + \frac{\Delta x^4}{12}u_{xxxx} - \Delta t^2 u_{tt} - \frac{\Delta t^4}{12}u_{tttt} + \frac{2\,\Delta x^6}{6!}u_{xxxxxx}\right)$$

$= O(\Delta x^8)$

$\beta = \frac{\alpha\,\Delta t}{\Delta x^2}$

$\Delta t = O(\Delta x^2)$

$$u_t - \alpha u_{xx} = -\frac{\Delta t^2}{6}u_{ttt} + \frac{\alpha\Delta x^2}{12}u_{xxxx} - \frac{\alpha\Delta t^2}{\Delta x^2}u_{tt} - \alpha\frac{\Delta t^4}{12\Delta x^2}u_{tttt} + \frac{2\alpha\Delta x^4}{6!}u_{xxxxxx} + \cdots$$

$$\frac{2}{6!} = \frac{\cancel{2}}{6\cdot 5\cdot 4\cdot 3\cdot \cancel{2}\cdot 1} =$$

30

120

360

$$u_t - \alpha u_{xx} = -\frac{\Delta t^2}{6}u_{ttt} + \frac{\alpha\Delta x^2}{12}u_{xxxx} + \frac{\alpha}{360}\Delta x^4\frac{\partial^6 u}{\partial x^6} - \alpha\left(\frac{\Delta t}{\Delta x}\right)^2 u_{tt} - \alpha\frac{\Delta t^4}{12\Delta x^2}u_{tttt} + \cdots$$

eq E9.1.2

pg 345 Hirsh Vol I

given E9.1.2

$$U_t - \alpha U_{xx} = -\frac{\Delta t^2}{6} u_{ttt} + \alpha \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right) + \alpha \frac{\Delta x^4}{360}\left(\frac{\partial^6 u}{\partial x^6}\right)$$

$$-\alpha \frac{\Delta t^2}{\Delta x^2} u_{tt} - \frac{\alpha \Delta t^4}{12 \Delta x^2}\left(\frac{\partial^4 u}{\partial t^4}\right) + \cdots$$

let $\Delta x, \Delta t \to 0$

$\Rightarrow \frac{\Delta t}{\Delta x} = C_1$ say

Then

$$u_t - \alpha U_{xx} + \alpha \left(\frac{\Delta t}{\Delta x}\right)^2 u_{tt} = 0$$

$$\Rightarrow \quad u_t + \alpha \left(\frac{\Delta t}{\Delta x}\right)^2 u_{tt} = \alpha U_{xx} \quad \text{eq E9.1.3}$$

Fixing $\beta = \alpha \frac{\Delta t}{\Delta x^2}$ Then Dufort-Frankel scheme becomes

$\Delta t = \frac{\beta \Delta x^2}{\alpha}$ so that E9.1.2 becomes

$$U_t - \alpha U_{xx} = O(\Delta x^4) + O(\Delta x^2) + O(\Delta x^4) + O(\Delta x^2) + O\left(\frac{\Delta x^8}{\Delta x^2}\right) \to 0$$

From E9.1.2

$$U_t = \alpha U_{xx} - \frac{\Delta t^2}{6} u_{ttt} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} + \alpha \frac{\Delta x^4}{360} \frac{\partial^6 u}{\partial x^6} - \alpha \frac{\Delta t^2}{\Delta x^2} u_{tt} - \frac{\alpha \Delta t^4}{12 \Delta x^2} \frac{\partial^4 u}{\partial t^4} + \cdots$$

✓

$$u_{tt} = \alpha (U_t)_{xx} - \frac{\Delta t^2}{6} u_{tttt} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4}{\partial x^4}(U_t) + \frac{\alpha \Delta x^4}{360} \frac{\partial^6}{\partial x^6}(u_t) - \alpha \left(\frac{\Delta t}{\Delta x}\right)^2 u_{ttt}$$

$$-\alpha \frac{\Delta t^4}{12 \Delta x^2} \frac{\partial^5 u}{\partial t^5} + \cdots$$

Now:

$\left(\frac{\Delta t}{\Delta x}\right)^2 \approx C_1^2 \Delta x^2 = O(\Delta x^2)$ thus we desire an $O(\Delta x^2)$ approximation

keeping terms of this order or lower gives for eq. E 9.1.2

$$u_t - \alpha u_{xx} = \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha \frac{\Delta t^2}{\Delta x^2} u_{tt} + \cdots$$

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha \frac{\Delta t^2}{\Delta x^2} u_{tt} + \cdots$$

$$\therefore u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial}{\partial t}\left(\alpha \frac{\partial^2 u}{\partial x^2} + \cdots\right)$$

$$\quad = \quad '' \quad\quad '' \quad - \alpha^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right)$$

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha^2 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^2}{\partial x^2}\left(\alpha \frac{\partial^2 u}{\partial x^2}\right)$$

$\Rightarrow$

$$u_t = \alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^4 u}{\partial x^4} + \cdots$$

Then

$$u_{tt} = \alpha \frac{\partial^2}{\partial x^2}\left[u_t\right] + \alpha \frac{\Delta x^2}{12} \frac{\partial^4}{\partial x^4}\left[u_t\right] - \alpha^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^4}{\partial x^4}\left[u_t\right] + \cdots$$

$$= \alpha \frac{\partial^2}{\partial x^2}\left[\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^4 u}{\partial x^4}\right]$$

$$+ \alpha \frac{\Delta x^2}{12} \frac{\partial^4}{\partial x^4}\left[\alpha \frac{\partial^2 u}{\partial x^2}\right] - \alpha^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^4}{\partial x^4}\left(\alpha \frac{\partial^2 u}{\partial x^2}\right)$$

$$\Rightarrow u_{tt} = \alpha^2 \frac{\partial^4 u}{\partial x^4} + \frac{\alpha^2 \Delta x^2}{12} \frac{\partial^6 u}{\partial x^6} - \alpha^4 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^6 u}{\partial x^6}$$

$$+ \frac{\alpha^2 \Delta x^2}{12} \frac{\partial^6 u}{\partial x^6} - \alpha^4 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^6 u}{\partial x^6}$$

$$= \alpha^2 \frac{\partial^4 u}{\partial x^4} + \frac{\alpha^2 \Delta x^2}{6} \frac{\partial^6 u}{\partial x^6} - 2\alpha^4 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^6 u}{\partial x^6} \qquad *$$

$$u_{ttt} = \alpha^2 \frac{\partial^4}{\partial x^4}\left[\alpha \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\Delta x^2}{12} \frac{\partial^4 u}{\partial x^4} - \alpha^3 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^4 u}{\partial x^4}\right]$$

$$+ \alpha^2 \frac{\Delta x^2}{6}\left(\alpha \frac{\partial^8 u}{\partial x^8}\right) - 2\alpha^4 \left(\frac{\Delta t}{\Delta x}\right)^2 \left(\alpha \frac{\partial^8 u}{\partial x^8}\right)$$

$$= \alpha^3 \frac{\partial^6 u}{\partial x^6} + \alpha^3 \frac{\Delta x^2}{12} \frac{\partial^8 u}{\partial x^8} - \alpha^5 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^8 u}{\partial x^8}$$

$$+ \alpha^3 \frac{\Delta x^2}{6} \frac{\partial^8 u}{\partial x^8} - 2\alpha^5 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^8 u}{\partial x^8}$$

$$= \alpha^3 \frac{\partial^6 u}{\partial x^6} + \alpha^3 \frac{\Delta x^2}{4} \frac{\partial^8 u}{\partial x^8} - 3\alpha^5 \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^8 u}{\partial x^8} \qquad **$$

Then E9.1.2 becomes w/o t derivatives:

$$u_t - \alpha u_{xx} = -\frac{\Delta t^2}{6}\left[\alpha^3 \frac{\partial^6 u}{\partial x^6} + \underset{*}{\frac{\alpha^3 \Delta x^2}{4}\frac{\partial^8 u}{\partial x^8}} - \underset{*}{3\alpha^5\left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^8 u}{\partial x^8}}\right]$$

$$+ \alpha \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \alpha\frac{\Delta x^4}{360}\frac{\partial^6 u}{\partial x^6}$$

$$- \alpha\left(\frac{\Delta t}{\Delta x}\right)^2\left[\alpha^2 \frac{\partial^4 u}{\partial x^4} + \alpha^2 \frac{\Delta x^2}{6}\frac{\partial^6 u}{\partial x^6} - \underset{*}{2\alpha^4\left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^8 u}{\partial x^6}}\right]$$

* Note: terms denoted w/ a * should not have been included because collectively their order is greater than 4th.

$$\Rightarrow u_t - \alpha u_{xx} = -\alpha \Delta x^2\left[-\frac{1}{12} + \frac{\alpha^2}{\Delta x^2}\left(\frac{\Delta t}{\Delta x}\right)^2\right]\frac{\partial^4 u}{\partial x^4}$$

$$+ \alpha \Delta x^4\left[-\frac{\Delta t^2}{6\Delta x^4}\alpha^2 + \frac{1}{360} - \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\alpha^2}{\Delta x^2}\frac{1}{6}\right]\frac{\partial^6 u}{\partial x^6}$$

$$+ O(\Delta x^6)$$

Define $\beta = \frac{\alpha \Delta t}{\Delta x^2}$ +

$$u_t - \alpha u_{xx} = -\alpha \Delta x^2\left[\beta^2 - \frac{1}{12}\right]\frac{\partial^4 u}{\partial x^4} + \alpha \Delta x^4 \underbrace{\left[\frac{1}{360} - \frac{\beta^2}{6} - \frac{\beta^2}{6}\right]}_{\left(\frac{1}{360} - \frac{\beta^2}{3}\right)}\frac{\partial^6 u}{\partial x^6}$$

$$\beta^2 - \frac{1}{12} > 0 \qquad \beta^2 > \frac{1}{12} \qquad \beta > \frac{1}{2\sqrt{3}}$$

Assuming Hirsh is correct.

$$\& \quad \frac{1}{360} - \frac{\beta^2}{4} + \beta^4 \geq 0$$

pg 346 Hrsh Vol I

Given any consistent discritization of $u_t + au_x = 0$ we can taylor expand the discretization for smooth solutions & then by transposing to the R.H.S. all terms besides $u_t + au_x$ one can obtain the taylor expansion of $u_t + au_x$ in terms of Δt & Δx. Setting $\Delta t = \lambda \Delta x$ we get a taylor expansion involving Δx only. Writing it as

$$\sum_{l=1}^{+\infty} a_{2l} \frac{\partial^{2l} u}{\partial x^{2l}} \Delta x^{2l-1} + a_{2l+1} \frac{\partial^{2l+1} u}{\partial x^{2l+1}} \Delta x^{2l}$$

Note that Δx^{2l-1} goes w/ $\frac{\partial^{2l} u}{\partial x^{2l}}$ & Δx^{2l} goes w/ $\frac{\partial^{2l+1} u}{\partial x^{2l+1}}$

Because the consistency of our discretization requires

$u_t + au_x = O(\Delta x)$ ... Don't see in an easy way how to show that $\Delta x^{2l-1} \longleftrightarrow \frac{\partial^{2l} u}{\partial x^{2l}}$ etc. besides the argument that a normal taylor series has term $\Delta x^{2l} \frac{\partial^{2l} u}{\partial x^{2l}}$ & then ÷ by Δx to obtain $u_t + au_x = \cdots$ gives expression shown

If scheme is order p then $\quad u_t + a u_x = O(\Delta x^p)$

So $\quad a_{2\ell} = a_{2\ell+1} = 0 \qquad 2\ell, 2\ell+1 < p$

let $U = e^{i(kx - \omega t)} = e^{-i\omega t} e^{ikx} \qquad \phi = k\Delta x$

9.2.1 is

$$u_t = -a u_x + \sum_{\ell=1}^{\infty} \left[a_{2\ell} \frac{\partial^{2\ell} u}{\partial x^{2\ell}} \Delta x^{2\ell-1} + a_{2\ell+1} \frac{\partial^{2\ell+1} u}{\partial x^{2\ell+1}} \Delta x^{2\ell} \right]$$

Now let $\quad U(x,t) = e^{-I\omega t} e^{Ikx}$

$$-I\omega = -akI + \sum_{\ell=1}^{\infty} \left[a_{2\ell} (Ik)^{2\ell} \Delta x^{2\ell-1} + a_{2\ell+1} (Ik)^{2\ell+1} \Delta x^{2\ell} \right]$$

$$\Rightarrow I\omega = akI - \sum_{\ell=1}^{\infty} \left[a_{2\ell} (-1)^{\ell} \phi^{2\ell} \Delta x^{-1} + a_{2\ell+1} (-1)^{\ell} I \phi^{2\ell+1} \Delta x^{-1} \right]$$

$$I\omega = akI - \frac{1}{\Delta x} \sum_{\ell=1}^{\infty} \left[a_{2\ell} (-1)^{\ell} \phi^{2\ell} + I (-1)^{\ell} a_{2\ell+1} \phi^{2\ell+1} \right] \qquad \text{eq } 9.2.4$$

$$U_t + aU_x = \sum_{l=1}^{\infty} \left[a_{2l} \frac{\partial^{2l} U}{\partial x^{2l}} \Delta x^{2l-1} + a_{2l+1} \frac{\partial^{2l+1} U}{\partial x^{2l+1}} \Delta x^{2l} \right]$$

see previous notes on this form.

see notes on derivation of eq 9.2.4

Nessasary condition for stability is $(-1)^r a_{2r} < 0$

for $2r$ lowest derivative in expression.

$p = 1$ require $\{2l, 2l+1 < 1 \Rightarrow l = -1$

1st order scheme

Thus

$a_1 = 0$ only $a_2 \neq 0$ $a_3 \neq 0$

$\uparrow$ dissapative term $\uparrow$ dispersive term

For $p = 2$ Then $a_2 = 0$ $a_3 \neq 0$ $a_4 \neq 0$

$p = 3$ then $a_2 = 0$ $a_3 = 0$ $a_4 \neq 0$

$(2l)_{min} = 4$
$\Rightarrow l_{min} = 2$
$\Rightarrow r = 2$

$\{[k] = \frac{1}{L}\}$

$G = \underbrace{|G|}\, e^{-I\Phi}$ $c_\phi = \frac{\Phi}{ka\Delta t} = ?$

represents decomposing G into real & imag components

$$I\omega = \frac{-1}{\Delta x} \sum_l a_{2l}(-1)^l \phi^{2l} + Ika - \frac{I}{\Delta x} \sum (-1)^l a_{2l+1} \phi^{2l+1} =$$

$$\omega = ka - \frac{1}{\Delta x} \sum_l (-1)^l a_{2l+1} \phi^{2l+1} + \frac{I}{\Delta x} \sum_l a_{2l}(-1)^l \phi^{2l}$$

Thus since $G = e^{-I\omega \Delta t}$

$$\Rightarrow G = e^{-I\Delta t\left(ka - \frac{1}{\Delta x}\sum (-1)^l a_{2l+1}\phi^{2l+1} + \frac{I}{\Delta x}\sum_l a_{2l}(-1)^l \phi^{2l}\right)}$$

$$= \exp\left\{ -I\Delta t\left(ka - \frac{1}{\Delta x}\sum(-1)^l a_{2l+1}\phi^{2l+1}\right) + \frac{\Delta t}{\Delta x}\sum_l a_{2l}(-1)^l \phi^{2l}\right\}$$

Thus $= |G| e^{-I\Phi}$

we get $\Phi = \Delta t\left(ka - \frac{1}{\Delta x}\sum(-1)^l a_{2l+1}\phi^{2l+1}\right)$ $\quad \phi = k\Delta x$

So that $\varepsilon_\phi = \dfrac{\Phi}{ak\Delta t} = 1 - \dfrac{1}{ak\Delta x}\sum(-1)^l a_{2l+1}\phi^{2l+1}$

$$= 1 - \frac{1}{a}\sum(-1)^l a_{2l+1}\phi^{2l} \qquad \text{eq } 9.2.7$$

$$\& \ |G| = \exp\left\{\frac{\Delta t}{\Delta x}\sum_l a_{2l}(-1)^l \phi^{2l}\right\} \qquad \text{eq } 9.2.8$$

For 1st order schemes $a_2 \neq 0$ (may). Then dissipation term would be

$l = 1$ in $\sum_l a_{2l}(-1)^l\phi^{2l} \Longrightarrow a_2(-1)\phi^2 < 0$ for stability

$$\Rightarrow a_2 > 0$$

If $a_2 = 0$ next stability term comes from a_4 since a_3 contributes a dispersive error to the numerical solution

$l = 2$ in $\sum_l a_{2l}(-1)^l\phi^{2l} \Rightarrow a_4(-1)^2\phi^2 < 0$ $\quad$ w/ $\frac{\Delta t}{\Delta x}$ we obtain eq 9.2.9

$$a_4 < 0$$

$$U_i^{n+1} = U_i^{n-1} - \sigma\left(U_{i+1}^n - U_{i-1}^n\right) \qquad \text{eq E9.21}$$

$\|$

$$\cancel{U_i^n} + U_t\Delta t + U_{tt}\cancel{\frac{\Delta t^2}{2}} + O(\Delta t^3) = \cancel{U_i^n} - \Delta t U_t + \Delta t^2\cancel{\frac{U_{tt}}{2}} + O(\Delta t^3)$$

$$-\sigma\Big(\cancel{U_i^n} + \Delta x U_x + \cancel{\frac{\Delta x^2}{2}}U_{xx} + \frac{\Delta x^3}{6}U_{xxx} + \cancel{\frac{\Delta x^4}{4!}}U_{xxxx} + \frac{\Delta x^5}{5!}\frac{\partial^5 U}{\partial x^5} + \cdots$$

$$- \cancel{U_i^n} + \Delta x U_x - \cancel{\frac{\Delta x^2}{2}}U_{xx} + \frac{\Delta x^3}{6}U_{xxx} - \cancel{\frac{\Delta x^4}{4!}}\frac{\partial^4 U}{\partial x^4} + \frac{\Delta x^5}{5!}\frac{\partial^5 U}{\partial x^5} + \cdots\Big)$$

$2\Delta t U_t + O(\Delta t^3) =$ This is not of a high enough order in $\Delta t \sim \Delta x$.

This is because I only went here to $O(\Delta t^3)$ but for the linear convection problem $\Delta t = O(\Delta x)$. Thus to $O(\Delta t^6) + O(\Delta x^6)$ we have

$$\cancel{U_i^n} + U_t\Delta t + U_{tt}\cancel{\frac{\Delta t^2}{2}} + U_{ttt}\frac{\Delta t^3}{3!} + U_{tttt}\cancel{\frac{\Delta t^4}{4!}} + \frac{\partial^5 U}{\partial t^5}\frac{\Delta t^5}{5!} + \cancel{\frac{\partial^6 U}{\partial t^6}\frac{\Delta t^6}{6!}} + O(\Delta t^7)$$

$$= \cancel{U_i^n} - U_t\Delta t + U_{tt}\cancel{\frac{\Delta t^2}{2}} - U_{ttt}\frac{\Delta t^3}{3!} + \cancel{\frac{\partial^4 U}{\partial t^4}\frac{\Delta t^4}{4!}} - \frac{\partial^5 U}{\partial t^5}\frac{\Delta t^5}{5!} + \cancel{\frac{\partial^6 U}{\partial t^6}\frac{\Delta t^6}{6!}} + O(\Delta t^7)$$

$$-\sigma\Big(\cancel{U_i^n} + \Delta x U_x + \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 U}{\partial x^2}} + \frac{\Delta x^3}{3!}\frac{\partial^3 U}{\partial x^3} + \cancel{\frac{\Delta x^4}{4!}\frac{\partial^4 U}{\partial x^4}} + \frac{\Delta x^5}{5!}\frac{\partial^5 U}{\partial x^5} + \cancel{\frac{\Delta x^6}{6!}\frac{\partial^6 U}{\partial x^6}} + O(\Delta x^7)$$

$$-\cancel{U_i^n} + \Delta x U_x - \cancel{\frac{\Delta x^2}{2}\frac{\partial^2 U}{\partial x^2}} + \frac{\Delta x^3}{3!}\frac{\partial^3 U}{\partial x^3} - \cancel{\frac{\Delta x^4}{4!}\frac{\partial^4 U}{\partial x^4}} + \frac{\Delta x^5}{5!}\frac{\partial^5 U}{\partial x^5} - \cancel{\frac{\Delta x^6}{6!}\frac{\partial^6 U}{\partial x^6}} + O(\Delta x^7)\Big)$$

$$\Rightarrow \Delta t\, u_t + \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^5}{5!}\frac{\partial^5 u}{\partial t^5} + O(\Delta t^7)$$

$$= -\Delta t\, u_t - \frac{\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^5}{5!}\frac{\partial^5 u}{\partial t^5} + O(\Delta t^7)$$

$$-\sigma\left(2\Delta x\, u_x + 2\frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{2\Delta x^5}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^7)\right) \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

$$\Rightarrow 2\Delta t\, u_t + \frac{2\Delta t^3}{3!}\frac{\partial^3 u}{\partial t^3} + \frac{2\Delta t^5}{5!}\frac{\partial^5 u}{\partial t^5} + O(\Delta t^7)$$

$$= -2\Delta x\,\sigma\left(u_x + \frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)\right)$$

$$\Rightarrow \cancel{2}\Delta t\left(u_t + \frac{\Delta t^2}{3!}\frac{\partial^3 u}{\partial t^3} + \frac{\Delta t^4}{5!}\frac{\partial^5 u}{\partial t^5} + O(\Delta t^6)\right)$$

$$= -\cancel{2}\Delta x\,\sigma\left(u_x + \frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)\right)$$

$$u_t + a u_x = -\frac{\Delta t^2}{3!}\frac{\partial^3 u}{\partial t^3} - \frac{\Delta t^4}{5!}\frac{\partial^5 u}{\partial t^5} + O(\Delta t^6)$$

$$-\frac{a\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - \frac{a\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

With the above expression we see that we need $\frac{\partial^3 u}{\partial t^3}$ to $O(\Delta x^2)$ only

+ $\frac{\partial^5 u}{\partial t^5}$ to $O(1)$ only to maintain an $O(\Delta x^4)$

expansion in Δx.

Thus solving for u_t + putting this into the RHS + being sure that we only keep the orders listed w/ $\Delta t \sim \Delta x$ (How know?)

$$u_t + a u_x = -a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - a\frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$- \frac{\Delta t^2}{3!}\frac{\partial^2}{\partial t^2}\left[-a u_x - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - \frac{\Delta t^2}{3!}\frac{\partial^3 u}{\partial t^3}\right]$$

$$- \frac{\Delta t^4}{5!}\frac{\partial^4}{\partial t^4}\left[-a u_x\right]$$

$$= -a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - a\frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$+ \frac{\Delta t^2 a}{3!}\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2 a}{(3!)^2}\frac{\partial^3}{\partial x^3}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^4}{(3!)^2}\frac{\partial^5 u}{\partial t^5}$$

$$+ \frac{a\Delta t^4}{5!}\frac{\partial}{\partial x}\frac{\partial^4 u}{\partial t^4}$$

$$\Rightarrow u_t + a u_x = -a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - a\frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$+ \frac{a\Delta t^2}{3!}\frac{\partial}{\partial x}\frac{\partial}{\partial t}\left[-a u_x - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + a\frac{\Delta t^2}{3!}\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial t^2} + \frac{a\Delta t^2}{(3!)^2}\frac{\partial^3}{\partial x^3}\frac{\partial^2 u}{\partial t^2}\right]$$

$$+ \frac{a\Delta t^2}{(3!)^2}\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial t}\left[-a u_x - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{\Delta t^2}{3!}a\frac{\partial}{\partial x}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2 a}{(3!)^2}\frac{\partial^3}{\partial x^3}\frac{\partial^2 u}{\partial t^2}\right]$$

$$u_t + a u_x = -\frac{a\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - \frac{a\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$-\frac{a^2\Delta t^2}{3!}\frac{\partial^2}{\partial x^2}u_t - \frac{a^2\Delta t^2\Delta x^2}{(3!)^2}\frac{\partial^4}{\partial x^4}u_t + \frac{a^2\Delta t^4}{(3!)^2}\frac{\partial^2}{\partial x^2}\frac{\partial^3 u}{\partial t^3}$$

$$+\frac{a^2\Delta t^4}{(3!)^3}\frac{\partial^4}{\partial x^4}\frac{\partial^3 u}{\partial t^3}$$

$$-\frac{a^2\Delta t^2}{(3!)^2}\frac{\partial^4}{\partial x^4}u_t - \frac{a^2\Delta t^2\Delta x^2}{(3!)^3}\frac{\partial^6}{\partial x^6}u_t$$

$$+\frac{a^2\Delta t^4}{(3!)^3}\frac{\partial^4}{\partial x^4}\frac{\partial^3}{\partial t^3}u + \frac{a^2\Delta t^2}{(3!)^4}\frac{\partial^6}{\partial x^6}\frac{\partial^3}{\partial t^3}u$$

$$= -\frac{a\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - \frac{a\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$-\frac{a^2\Delta t^2}{3!}\frac{\partial^2}{\partial x^2}u_t$$

Can I get this result simpler? How about using the $O(\Delta x^2)$ data somehow? 9-27-01 5

$$u_t + a u_x = -\frac{\Delta t^2}{3!}\frac{\partial^3 u}{\partial t^3} - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$u_t + a u_x = -\frac{\Delta t^2}{3!}\frac{\partial^2}{\partial t^2}\left[-a u_x\right] - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$= \frac{a\Delta t^2}{3!}\frac{\partial}{\partial x}\frac{\partial}{\partial t}(u_t) - a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$= \frac{a\Delta t^2}{3!}\frac{\partial}{\partial x}\frac{\partial}{\partial t}(-a u_x) - \text{''}$$

$$= -\frac{a^2\Delta t^2}{3!}\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}u - \frac{a\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^2} + O(\Delta x^4)$$

$$u_t + a u_x = \frac{a^3\Delta t^2}{3!}\frac{\partial^3}{\partial x^3}u - \frac{a\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$= \frac{\Delta x^2}{3!}\left(\frac{a^2\Delta t^2}{\Delta x^2} - 1\right)\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

$$= \frac{\Delta x^2}{3!}(\sigma^2 - 1)\frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)$$

... This was relatively easy to obtain. Can I use it to get the $O(\Delta x^4)$ correction?

$$u_{tt} = -a u_{xt} + \frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial \partial^3 u}{\partial t \partial x^3} + O(\Delta x^4)$$

$$= -a\frac{\partial}{\partial x}\left(-a u_x + \frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^3 u}{\partial x^3}\right) +$$

$$\frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^3}{\partial x^3}(-a u_x) + O(\Delta x^4)$$

$$= +a^2\frac{\partial^2 u}{\partial x^2} - \frac{a\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^4 u}{\partial x^4} - a\frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^4 u}{\partial x^4} + O(\Delta x^4)$$

$$= a^2\frac{\partial^2 u}{\partial x^2} - \frac{2a\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^4 u}{\partial x^4} + O(\Delta x^4)$$

$$u_{ttt} = a^2\frac{\partial^2}{\partial x^2}\left[-a u_x + a\frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^3 u}{\partial x^3}\right]$$

$$-\frac{2a^2\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^4}{\partial x^4}\left[-a u_x\right] + O(\Delta x^4)$$

$$\Rightarrow u_{ttt} = -a^3\frac{\partial^3 u}{\partial x^3} + a^3\frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^5 u}{\partial x^5} + 2a^3\frac{\Delta x^2}{3!}(\sigma^2-1)\frac{\partial^5 u}{\partial x^5} + O(\Delta x^4)$$

$$= -a^3\frac{\partial^3 u}{\partial x^3} + \frac{3a^3\Delta x^2(\sigma^2-1)}{3!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^4)$$

$$+ \quad u_{tttt} = a^4\frac{\partial^4 u}{\partial x^4} + O(\Delta x^2)$$

$$\frac{\partial^5 u}{\partial t^5} = -a^5\frac{\partial^5 u}{\partial x^5} + O(\Delta x^2)$$

Thus from pg 2

$$u_t + a u_x = -\frac{\Delta t^2}{3!}\left[-a^3 \frac{\partial^3 u}{\partial x^3} + \frac{3a^3 \Delta x^2 (\sigma^2 - 1)}{3!}\frac{\partial^5 u}{\partial x^5}\right] - \frac{\Delta t^4}{5!}\left(-a^5 \frac{\partial^5 u}{\partial x^5}\right) + O(\Delta x^6)$$

$$-a\frac{\Delta x^2}{3!}\frac{\partial^3 u}{\partial x^3} - a\frac{\Delta x^4}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$= \left(a^3\frac{\Delta t^2}{3!} - a\frac{\Delta x^2}{3!}\right)\frac{\partial^3 u}{\partial x^3} + \left(-\frac{3a^3\Delta t^2 \Delta x^2(\sigma^2-1)}{(3!)^2} - a\frac{\Delta x^4}{5!} + a^5\frac{\Delta t^4}{5!}\right)\frac{\partial^5 u}{\partial x^5}$$

$$+ O(\Delta x^6)$$

$$= \frac{a\Delta x^2}{6}\left(\frac{a^2\Delta t^2}{\Delta x^2} - 1\right)\frac{\partial^3 u}{\partial x^3} + \Delta x^4\left(-\frac{3a^3\Delta t^2}{\Delta x^2}\frac{1}{(3!)^2}(\sigma^2-1) - \frac{a}{5!} + \frac{a^5\Delta t^4}{\Delta x^4 5!}\right)\frac{\partial^5 u}{\partial x^5}$$

$$+ O(\Delta x^6)$$

$$= \frac{a\Delta x^2}{6}(\sigma^2-1)\frac{\partial^3 u}{\partial x^3} + \Delta x^4\left(-\frac{3\sigma^2(\sigma^2-1)a}{(3!)^2} - \frac{a}{5!}(1-\sigma^4)\right)\frac{\partial^5 u}{\partial x^5} + O(\Delta x^6)$$

$$= \frac{a\Delta x^2}{6}(\sigma^2-1)\frac{\partial^3 u}{\partial x^3} - a\Delta x^4\left(\frac{\cancel{3}(\sigma^4-\sigma^2)}{\cancel{6}\cdot 6\cdot 2} - \frac{1-\sigma^4}{5\cdot\cancel{4}\cdot 3\cdot 2}\right)$$

$$\frac{a\Delta x^4}{6\cdot 2}\left(\sigma^4 - \sigma^2 - \frac{1-\sigma^4}{10}\right)$$

$$\frac{a\Delta x^4}{120}\left(10\sigma^4 - 10\sigma^2 - 1 + \sigma^4\right)$$

$$9\sigma^4 - 10\sigma^2 - 1 = (9\sigma^2 - 1)(\sigma^2 - 1)$$

pg 347 Hirsch Vol I

eq 9.2.1

$$u_t + a u_x = \sum_{l=1}^{+\infty} \left[a_{2l} \left(\frac{\partial^{2l} u}{\partial x^{2l}} \right) \Delta x^{2l-1} + a_{2l+1} \left(\frac{\partial^{2l+1} u}{\partial x^{2l+1}} \right) \Delta x^{2l} \right]$$

$$= a_2 \frac{\partial^2 u}{\partial x^2} \Delta x^1 + a_3 \frac{\partial^3 u}{\partial x^3} \Delta x^2 + a_4 \frac{\partial^4 u}{\partial x^4} \Delta x^3 + a_5 \frac{\partial^5 u}{\partial x^5} \Delta x^4 + \dots$$

w/ eq E9.2.2

$$u_t + a u_x = \dots$$

gives

$$a_2 = 0, \ a_3 = \frac{a}{6}(\sigma^2 - 1), \ a_4 = 0, \ a_5 = -\frac{a}{120}(9\sigma^2 - 1)(\sigma^2 - 1), \ a_6 = 0.$$

Since $a_2 = 0$ $a_3 \neq 0$ our scheme is second order accurate.

No even derivatives $\Rightarrow$ no diffusion error.

$$u_i^{n+1} = \sum_j b_j u_{j+i}^n$$

upwind scheme:

$$u_i^{n+1} = u_i^n - \frac{a \Delta t}{\Delta x}(u_i^n - u_{i-1}^n) = (1-\sigma) u_i^n + \sigma u_{i-1}^n$$

$b_{-1} = \sigma \quad b_0 = 1 - \sigma \quad b_{+1} = 0$ eq 9.2.11a

Lax-Friedrich scheme: eq

$$u_i^{n+1} = \frac{1}{2}(1-\sigma) u_{i+1}^n + \frac{1}{2}(1+\sigma) u_{i-1}^n$$ eq E8.2.7.

$b_{-1} = \frac{1}{2}(1+\sigma) \quad b_0 = 0 \quad b_{+1} = \frac{1}{2}(1-\sigma)$ eq 9.2.12a

Lax-Wendroff: E 8.3.11

$$U_i^{n+1} = U_i^n - \frac{\beta}{2}\left(U_{i+1}^n - U_{i-1}^n\right) + \frac{\beta^2}{2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

$$= (1-\beta^2)U_i^n + \left(\frac{\beta}{2} + \frac{\beta^2}{2}\right)U_{i-1}^n + \left(-\frac{\beta}{2} + \frac{\beta^2}{2}\right)U_{i+1}^n$$

$$=$$

$$= \frac{\beta}{2}(1+\beta)U_{i-1}^n + (1-\beta^2)U_i^n + \frac{\beta}{2}(-1+\beta)U_{i+1}^n \qquad \text{eq } 9.2.13b$$

$$b_{-1} = \frac{\beta}{2}(1+\beta) \qquad b_0 = 1-\beta^2 \qquad b_{+1} = \frac{\beta}{2}(-1+\beta) \qquad \text{eq } 9.2.13a$$

eq 9.2.10

$$u_i^{n+1} = \sum_j b_j u_{i+j}^n$$

to be pth order

$$u_i^{n+1} - \sum_j b_j u_{i+j}^n = O(\Delta x^p)$$

$\Rightarrow O(1), O(\Delta x), \ldots O(\Delta x^{p-1})$ terms must be set to 0.

$\Rightarrow$ p equations, How get p+1?

Because $u_i^{n+1} = u_i^n + O(\Delta t)$ + ∴ correct expression is

$$\frac{1}{\Delta t}\left(u_i^{n+1} - \sum_j b_j u_{i+j}^n\right) = O(\Delta x^p)$$

Since $O(\Delta t) = O(\Delta x)$

$$\Longleftrightarrow u_i^{n+1} - \sum_j b_j u_{i+j}^n = O(\Delta x^{p+1}) \Rightarrow p+1 \text{ eqs.}$$

$u_i^{n+1} - \sum_j b_j u_{i+j}^n$ let $u_i^n = C$ const

$$\Rightarrow C = \sum_j b_j C \Rightarrow 1 = \sum_j b_j$$

$$u^n_{i+j} = u^n_i + \sum_{m=1}^{\infty} \frac{(j\cdot\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

+ 9.216 into 9.210 gives

$$\cancel{u^n_i} + \sum_{m=1}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = \cancel{\sum_j b_j u^n_i} + \sum_j b_j \sum_{m=1}^{\infty} \frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \sum_{m=2}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = \sum_j b_j j \Delta x \frac{\partial u}{\partial x} + \sum_j b_j \sum_{m=2}^{\infty} \frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

By eq $\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

$$\Rightarrow \underbrace{\Delta t(-a)\frac{\partial u}{\partial x} + \sum_{m=2}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = \Delta x\frac{\partial u}{\partial x}\sum_j j b_j}_{\text{comparing coeff of } \frac{\partial u}{\partial x}} + \sum_j b_j \sum_{m=2}^{\infty} \frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

$$-a\Delta t = \Delta x \sum_j j b_j \Rightarrow \sum_j j b_j = -\sigma \quad \text{eq } 9.218$$

$$\Rightarrow \Delta t \frac{\partial u}{\partial t} + \sigma \Delta x \frac{\partial u}{\partial x} = -\sum_{m=2}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} + \sum_j b_j \sum_{m=2}^{\infty} \frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

$$\frac{\partial u}{\partial t} + \underbrace{\frac{\sigma \Delta x}{\Delta t}}_{a}\frac{\partial u}{\partial x} = -\sum_{m=2}^{\infty} \frac{\Delta t^{m-1}}{m!}\frac{\partial^m u}{\partial t^m} + \frac{1}{\Delta t}\sum_j b_j$$

$$u_t + a u_x = \sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{l+1} u}{\partial x^{l+1}}$$

Idea consider a $\underset{\text{simple}}{\wedge}$ numerical scheme. Derive the equivalent D.E. solve this eq numerically + see that it matches very well w/ the given numerical sol to or original equation.

As an operator (works for lin operators only)

$$\partial_t = \left[-a\,\partial_x + \sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{l+1}}{\partial x^{l+1}} \right]$$

$$\therefore \frac{\partial^m u}{\partial t^m} = \left[-a\frac{\partial}{\partial x} + \sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{l+1}}{\partial x^{l+1}} \right]^m u$$

$$= (-a)^m \frac{\partial^m}{\partial x^m} + m(-a)^{m-1} \sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{m+l}}{\partial x^{m+l}} u$$

$$+ \frac{m(m-1)}{2}(-a)^{m-2} \left(\sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{l+1}}{\partial x^{l}} \right)^2 \frac{\partial^{m-2}}{\partial x^{m-2}} u + O(\Delta x^{3p})$$

lowest Δx power in these terms

$$\therefore \frac{\partial^m u}{\partial t^m} = (-a)^m \frac{\partial^m}{\partial x^m} + m(-a)^{m-1} \sum_{l=p}^{\infty} \Delta x^{l} a_{l+1} \frac{\partial^{m+l}}{\partial x^{m+l}} u$$

$$+ \frac{m(m-1)}{2}(-a)^{m-2} \sum_{l,k=p}^{\text{two}} \Delta x^{l+k} a_{l+1} a_{k+1} \frac{\partial^{l+1+k+1+m-2}}{\partial x^{l+k+m}} + O(\Delta x^{3p})$$

$$u_t + a u_x = + \frac{1}{\Delta t}\sum_j \sum_{m=2}^{\infty} b_j \frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}$$

$$-\sum_{m=2}^{\infty} \frac{\Delta t^{m-1}}{m!}(-a)^m \frac{\partial^m u}{\partial x^m} - \sum_{m=2}^{\infty} \frac{\Delta t^{m-1}}{m!}\, m(-a)^{m-1}\sum_{l=p}^{\infty} \Delta x^l a_{l+1}\frac{\partial^{m+l}u}{\partial x^{m+l}}$$

$$-\sum_{m=2}^{\infty} \frac{\Delta t^{m-1}}{m!}\frac{m(m-1)}{2}(-a)^{m-2}\sum_{l,k=p}^{\infty} \Delta x^{l+k} a_{l+1}a_{k+1}\frac{\partial^{l+k+m}u}{\partial x^{l+k+m}} + O(\ )$$

$$\uparrow \Delta x^{8}$$

$$u_t + a u_x = \sum_{m=2}^{\infty}\left(\frac{1}{\Delta t}\Big(\sum_j b_j j^m\Big)\frac{\Delta x^m}{m!} - \underbrace{\frac{\Delta t^{m-1}}{m!}(-1)^m\frac{\sigma^m\Delta x^m}{\Delta t^m}}_{\Delta t^{-1}}\right)\frac{\partial^m u}{\partial x^m}$$

$$\sigma = a\frac{\Delta t}{\Delta x} \Rightarrow a = \frac{\sigma\Delta x}{\Delta t}$$

$$-\sum_{m=2}^{\infty}\frac{\cancel{\Delta t}^{m-1}}{(m-1)!}(-1)^{m-1}\frac{\sigma^{m-1}\Delta x^{m-1}}{\cancel{\Delta t}^{m-1}}\sum_{l=p}^{\infty}\Delta x^l a_{l+1}\frac{\partial^{m+l}u}{\partial x^{m+l}}$$

$$-\frac{1}{2}\sum_{m=2}^{\infty}\frac{(-1)^{m-2}}{(m-2)!}\frac{\sigma^{m-2}\Delta x^{m-2}}{\Delta t^{-1}}\sum_{l,k=p}^{\infty}\Delta x^{l+k}a_{l+1}a_{l+1}\frac{\partial^{l+k+m}u}{\partial x^{l+k+m}} + O(?)$$

$$\cancel{\Delta t}^{m-1}\,\frac{\sigma^{m-2}\Delta x^{m-2}}{\Delta t^{\cancel{m-2}}\ \Delta t^{-1}}$$

$$u_t + a u_x = \frac{\Delta x}{\Delta t} \sum_{m=2}^{\infty} \left[\sum_j b_j j - (-b)^m \right] \frac{\Delta x^{m-1}}{m!} \frac{\partial^m u}{\partial x^m}$$

$$- \sum_{m=2}^{\infty} \frac{(-b)^{m-1}}{(m-1)!} \sum_{l=p}^{\infty} \Delta x^{l+m-1} \frac{\partial^{m+l} u}{\partial x^{m+l}}$$

$$- \frac{\Delta t}{2\Delta x} \sum_{m=2}^{\infty} \frac{(-b)^{m-2}}{(m-2)!} \sum_{l,k=p}^{\infty} \Delta x^{l+k+m-1} a_{l+1} a_{k+1} \frac{\partial^{l+k+m} u}{\partial x^{l+k+m}} \qquad \text{eq } 9.2.22$$

1st sum $O(\Delta x^{p+1})$

2nd sum $O(\Delta x^{2p+2-1})$ $O(\Delta x^{2p+1})$

If our scheme is order p

$u_t = -a u_x$

$u_{tt} = (-a)^2 u_{xx}$

$u_{ttt} = (-a)^3 u_{xxx}$ etc.

$\therefore$ from eq 9.2.19

$$u_t + a u_x = \underbrace{\frac{1}{\Delta t} \sum_{m=2}^{\infty} b_j j^m \frac{\Delta x^m}{m!} \frac{\partial^m u}{\partial x^m} - \frac{\Delta t^m}{m!} (-a)^m \frac{\partial^m u}{\partial x^m}}_{\text{must vanish up to order } p.}$$

$$u_t + a u_x = \frac{1}{\Delta t}\sum_{m=2}^{+\infty}\left(\sum_j b_j j^m - (-\sigma)^m\right)\frac{\Delta x^m}{m!}\frac{\partial^m u}{\partial x^m} = 0$$

up to order p $\quad = O(\Delta x^p)$

$$\Rightarrow \quad \sum_j b_j j^m = (-\sigma)^m \qquad m = 0, 1, 2, \dots, p-1 \qquad \text{eq } 9.2.23$$

in eq 9.2.22 terms up to $O(\Delta x^p)$ must vanish $\Rightarrow$ eq 9.2.23 up to $p-1$. Then coefficient of Δx^p given by eq 9.2.22 is:

$$\frac{\Delta x}{\Delta t}\left[\sum_j b_j j^{p+1} - (-\sigma)^{p+1}\right]\frac{1}{(p+1)!} = a_{p+1} \qquad \text{eq } 9.2.24$$

$$\equiv \frac{\Delta x}{\Delta t}\alpha_{p+1}$$

Now coeff of Δx^{p+1} $\frac{\partial^{p+2} u}{\partial x^{p+2}}$

$$\frac{\Delta x}{\Delta t}\left[\sum_j b_j j^{p+2} - (-\sigma)^{p+2}\right]\frac{1}{(p+2)!}$$

$$- \frac{(-\sigma)^1}{1!} a_{p+1} = a_{p+2}$$

2nd term $m = 2$
$\ell = p$
$\Rightarrow \Delta x^{p+1} \frac{\partial^{p+2} u}{\partial x^{p+2}}$.
only term that contributes to this order.

3rd term:
$m = 2,\ \ell, k = p$
$\Rightarrow \Delta x^{2p+1}$ to large.

$\Rightarrow$ w/ def of α_{p+2} from 9.2.25

$$\Rightarrow \quad a_{p+2} = \frac{\Delta x}{\Delta t}\alpha_{p+2} + \sigma a_{p+1} \qquad \text{eq } 9.2.26$$

Now coeff of $\Delta x^{p+2} \frac{\partial^{p+3} U}{\partial x^{p+3}}$ would be

1st term $m = p+3$

2nd term $m=2, l=p+1$

$\Rightarrow \Delta x^{p+2} \frac{\partial^{p+3} U}{\partial x^{p+3}}$

$m=3, l=p$

3rd term

$m=2, l,k=p$

Δx^{2p+1} will be a contributing term iff $p=1$

$$\frac{\Delta x}{\Delta t} \alpha_{p+3}$$

$$- \frac{(-6)^1}{1!} a_{p+2} - \frac{(-6)^2}{2!} a_{p+1}$$

$$- \frac{\Delta t}{2\Delta x} \frac{(-6)^0}{0!} a_{p+1} a_{p+1} = a_{p+3}$$

$$\Rightarrow a_{p+3} = \frac{\Delta x}{\Delta t} \alpha_{p+3} - \frac{6^2}{2} a_{p+1} + 6\, a_{p+2}$$

$$- \frac{\Delta t}{2\Delta x} a_{p+1}^2 \qquad \text{eq } 9.2.27$$

If $p=2$ last term gives $O(\Delta x^5)$ & is too small.

$$\therefore \quad a_{p+3} = \frac{a}{6}\left(\alpha_{p+3} - \frac{6^3}{2a} a_{p+1} + \frac{6^2}{a} a_{p+2}\right)$$

$6 = \frac{\Delta t\, a}{\Delta x}$

$\frac{\Delta x}{\Delta t} = \frac{a}{6}$

$$= \cancel{\frac{a}{6} \alpha_{p+3}}$$

Don't see 9.2.28

Addendum:

$$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n)$$

$$= u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\sigma}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad \text{eq E9.2.4}$$

Pg 351 Hirsh Vol I

$\cancel{u_i^{n+1}} = u$

$$u_i^{n+1} = \sigma u_{i-1}^n + (1-\sigma) u_i^n \quad \text{eq } 9.2.11b$$

$$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n) \quad \text{eq } E9.2.3$$

$$\Rightarrow b_{-1} = +\sigma, \; b_0 = 1-\sigma, \; b_{+1} = 0 \quad \text{1st order scheme}$$

$$\sum_j b_j = 1 \quad \checkmark$$

$a_0 = 0$

$a_1 = 0$

$a_2 \neq 0$

$$\sum_j j b_j = -1(+\sigma) + 0 = -\sigma \quad \checkmark$$

$$a_2 = \left[\sum_j b_j j^2 - (-\sigma)^2 \right] \frac{1}{2!} \left(\frac{\Delta x}{\Delta t} \right)$$

$$\sigma = \frac{\Delta t\, a}{\Delta x}$$

$$= \frac{\left((+1)\sigma - (-\sigma)^2\right)}{2} \frac{\Delta x}{\Delta t} = \frac{\sigma(1-\sigma)}{2} \frac{\Delta x}{\Delta t}$$

$$= \frac{a(1-\sigma)}{2} \quad E9.2.5.$$

$$a_3 = \frac{\Delta x}{\Delta t} \alpha_3 + \sigma a_2$$

$$= \frac{\Delta x}{\Delta t} \left[\sum_j b_j j^3 - (-\sigma)^3 \right] \frac{1}{3!} + \sigma^2 \frac{(1-\sigma)}{2} \frac{\Delta x}{\Delta t}$$

$$a_3 = \frac{\Delta x}{\Delta t} \left[-\sigma - (-\sigma)^3 \right] \frac{1}{6} + \frac{a\sigma(1-\sigma)}{2}$$

$$a_3 = \frac{\sigma\Delta x}{\Delta t}\left[-1+\sigma^2\right]\frac{1}{6} + \frac{a(1-\sigma)\sigma}{2}$$

$$= \frac{a}{6}(\sigma^2-1) + \frac{a}{2}\sigma(1-\sigma)$$

$$= \frac{a}{6}(\sigma-1)(\sigma+1) + \frac{a}{2}\sigma(1-\sigma)$$

$$= \frac{a}{6}(\sigma-1)\left[\sigma+1 \mp 3\sigma\right] = \frac{a}{6}(\sigma-1)(1-2\sigma) \quad \text{eq E9.2.6}$$

$$a_4 = \frac{\Delta x}{\Delta t}\alpha_4 - \frac{\sigma^2}{2}a_2 + \sigma a_3 - \frac{\Delta t}{2\Delta x}a_2^2$$

$$= \frac{\Delta x}{\Delta t}\left[\sigma(1) - (-\sigma)^4\right] - \frac{\sigma^2}{2}\frac{a}{2}(1-\sigma) + \frac{\sigma a}{6}(\sigma-1)(1-2\sigma)$$

$$- \frac{\Delta t}{2\Delta x}\frac{a^2}{4}(1-\sigma)^2$$

$$= \frac{\Delta x}{\Delta t}\sigma(1-\sigma^3) - \frac{a\sigma^2}{4}(1-\sigma) + \frac{a}{6}\sigma(\sigma-1)(1-2\sigma)$$

$$- \frac{a}{8}\sigma(1-\sigma)^2$$

$$= \frac{a}{24}(1-\sigma)\left[1+\sigma+\sigma^2 - \sigma^2(6) \mp 4\sigma(1-2\sigma)\right.$$

$$\left. - 3\sigma(1-\sigma)\right]$$

$$\left.\begin{array}{l} 1+\cancel{\sigma}+\cancel{\sigma^2} \\ \cancel{\sigma}-\cancel{\sigma^2}-\sigma^3 \end{array}\right\}$$

$$= \frac{a(1-\sigma)}{24}\left[1+\sigma+\sigma^2-6\sigma^2-4\sigma+8\sigma^2-3\sigma+3\sigma^2\right]$$

$$= \frac{a(1-\sigma)}{24}\left[1-5\sigma+(\underbrace{1-6+8+3}_{6})\sigma^2\right]$$

$$= \frac{a(1-\sigma)}{24}(1-6\sigma+6\sigma^2) \quad \text{eq E.9.2.7}$$

$$\frac{a\Delta x}{2}(1-\sigma) = \frac{1}{2}\left(\frac{\Delta x}{\Delta t}\sigma\right)\Delta x(1-\sigma) \qquad \frac{\Delta t a}{\Delta x} = \sigma$$

$$= \frac{1}{2}\frac{\Delta x^2}{\Delta t}\sigma(1-\sigma)$$

For $\sigma = 1$ scheme becomes

$$u_i^{n+1} = \cancel{u_i^n} - \cancel{u_i^n} + u_{i-1}^n$$

$$u_i^{n+1} = u_{i-1}^n$$

Lax-Fridrichs:

$$u_i^{n+1} = \frac{1}{2}(1+\sigma)u_{i-1}^n + \frac{1}{2}(1-\sigma)u_{i+1}^n \quad \text{from eq 9.2.12b}$$

$$= \cancel{}$$

$$= u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(u_{i-1}^n - 2u_i^n + u_{i+1}^n) \quad \text{eq E9.2.10}$$

Then check

$$b_{-1} = \frac{1}{2}(1+\sigma) \qquad b_{+1} = \frac{1}{2}(1-\sigma)$$

$$\sum_j b_j = 1 \checkmark \qquad \sum_j b_j j = -\frac{1}{2}(1+\sigma) + \frac{1}{2}(1-\sigma)$$

$$= -\sigma \quad \checkmark$$

$$\sum_j b_j j^2 \overset{?}{=} +(-\sigma)^2$$

$$\parallel$$

$$\frac{1}{2}(1+\sigma) + \frac{1}{2}(1-\sigma) = 1 \overset{\text{Not?}}{=} -\sigma^2.$$

Is Lax-Fridrichs 1st order? Yes, must be. $p=1$

Then from pg 357 $a_2 = \frac{\Delta x}{\Delta t}\left[1-\sigma^2\right]\frac{1}{2} = \frac{1}{2}\frac{\Delta x}{\Delta t}(1-\sigma^2)$

$$a_3 = \frac{\Delta x}{\Delta t}\alpha_3 + \frac{\sigma}{2}\frac{\Delta x}{\Delta t}(1-\sigma^2)$$

$$= \frac{1}{3!}\frac{\Delta x}{\Delta t}\left[\sum_j b_j j^3 - (-\sigma)^3\right] + \frac{\Delta x}{\Delta t}\frac{\sigma}{2}(1-\sigma^2)$$

$$a_3 = \frac{1}{6}\frac{\Delta x}{\Delta t}\left[-\sigma+\sigma^3\right] + \frac{\Delta x}{\Delta t}\frac{\sigma}{2}(1-\sigma^2)$$

$$= \frac{1}{6}\frac{\Delta x}{\Delta t}\sigma\left[(-1+\sigma^2)+3(1-\sigma^2)\right]$$

$$= \frac{\sigma}{6}\frac{\Delta x}{\Delta t}\left[-1+\sigma^2+3-3\sigma^2\right]$$

$$= \frac{\sigma}{6}\frac{\Delta x}{\Delta t}(2-2\sigma^2) = \frac{\sigma}{3}\frac{\Delta x}{\Delta t}(1-\sigma^2) = \frac{a}{3}(1-\sigma^2)$$

Then

$$u_t + a u_x = a_2 \Delta x \frac{\partial^2 u}{\partial x^2} + a_3 \Delta x^2 \frac{\partial^3 u}{\partial x^3} + O(\Delta x^3)$$

$$= \frac{\Delta x^2}{2\Delta t}(1-\sigma^2)\frac{\partial^2 u}{\partial x^2} + \frac{a\Delta x^2}{3}(1-\sigma^2)\frac{\partial^3 u}{\partial x^3} + O(\Delta x^3)$$

$$\nu_{num} = \frac{\Delta x^2}{2\Delta t}(1-\sigma^2) = \frac{\Delta x}{2}\left(\frac{a}{\sigma}\right)(1-\sigma^2) \qquad \left\{ \sigma = \frac{a\Delta t}{\Delta x} \right. \qquad \left\{ \frac{\Delta x}{\Delta t} = \frac{a}{\sigma} \right.$$

eq E9.2.12

Lax-Friedrichs dispersion error: From pg 306

$$\varepsilon_\phi = \frac{\Phi}{\sigma\phi} = \frac{\tan^{-1}(\sigma\tan\phi)}{\sigma\phi} \qquad \phi \approx k_x\Delta x \qquad \phi \ll 1$$

→ ~~[illegible]~~ long wave components?

$$= \underset{\text{taylor expand}}{} \; 1 + \frac{(1-\sigma^2)}{3}\phi^2 + O(\Delta x^4) \qquad \text{eq E9.2.13}$$

$\left\{ k = \frac{2\pi}{\lambda} \right\}$

k = wave #

If $|\sigma| < 1$ as required by stability ⇒

$1 - \sigma^2 > 0$

∴ $\varepsilon_\phi > 1$ ⇒ computed waves travel faster than physical waves.

leading phase error.

eq 9.2.7 $\quad \varepsilon_\phi = 1 - \sum_{l=1}^{\infty} (-1)^l \frac{\phi^{2l}}{a} a_{2l+1}$

9.2.8 $\quad \varepsilon_D = |G| = \exp\left[\sum_l a_{2l} (-1)^l \phi^{2l} \frac{\Delta t}{\Delta x} \right]$

let $u_i^n = v^n e^{Ii\phi}$ $\qquad \phi = k\Delta x$

Defining ~~[illegible]~~ $G(\phi) = \frac{v^{n+1}}{v^n}$ $\qquad u_i^{n+1} = \sum_j b_j u_{i+j}^n$

~~$(G(\phi))$~~ = ⇒ $v^{n+1} \cancel{e^{Ii\phi}} = \sum_j b_j v^n \cancel{e^{Ii\phi}} e^{Ij\phi}$

eq 9.2.29

$$\frac{v^{n+1}}{v^n} = \sum_j b_j e^{Ij\phi} = \sum_j b_j \cos(j\phi) + I \sum_j b_j \sin(j\phi)$$

$$\mathcal{E}_D = \exp\left[\sum_{\ell=1}^{\infty} a_{2\ell} (-1)^\ell \phi^{2\ell} \frac{\Delta t}{\Delta x} \right]$$ small ϕ.

coefficients of even derivatives.

$$\cong 1 + \sum_{\ell=1}^{\infty} a_{2\ell} (-1)^\ell \phi^{2\ell} \frac{\Delta t}{\Delta x} + \cdots$$ eq 9.2.30

$$\mathcal{E}_\phi = 1 - \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{a} a_{2\ell+1} \phi^{2\ell}$$ eq 9.2.31

If order of accuracy p is odd $\Rightarrow \exists\, \ell \ni \quad 2\ell = p+1$

$\ell = \frac{p+1}{2}$

+ $a_0 = 0 = a_1 = a_2 = \cdots = a_{p-1}$

$a_p \neq 0.$ where a_i's given by

By eq 9.2.20

$$u_t + a u_x = \sum_{\ell=p}^{\infty} \Delta x^\ell a_{\ell+1} \frac{\partial^{\ell+1} u}{\partial x^{\ell+1}}$$

Since p is odd + only $\mathcal{E}_\phi$ has odd terms ~~$\mathcal{E}_D$~~ The 1st nonzero term in $\mathcal{E}_D$ will be of order $a_{p+1} = a_{2\ell} \ni \underset{\text{n.b.}}{\ell =} \frac{p+1}{2}$

Then $\mathcal{E}_D \cong 1 + (-1)^{\frac{p+1}{2}} \phi^{p+1} \frac{\Delta t}{\Delta x} a_{p+1} + O(\phi^{p+3})$ ~~eq~~

since only odd terms are present.

Now $a_{p+1} = \frac{\Delta x}{\Delta t} \alpha_{p+1}$ by eq 9.2.25

$$\Rightarrow \quad \varepsilon_D \cong 1 + (-1)^{\frac{p+1}{2}} \alpha_{p+1} \phi^{p+1} + O(\phi^{p+3}) \qquad \text{eq } 9.2.32$$

Since p is odd expect 1st nonzero term of a's in 9.2.31 to be

$2l+1 = p$ $\quad a_p \quad$ what happened to the a_p th term?

$l = \frac{p-1}{2}$

$$\varepsilon_\phi = 1 - \underbrace{\frac{(-1)^{\frac{p-1}{2}}}{a} a_p \phi^{p-1}}_{\text{where did this term go?}} - \frac{(-1)^{\frac{p+1}{2}}}{a} a_{p+2}\phi^{p+1} + O(\phi^{p+3})$$

Then $\quad a_{p+2} = \frac{\Delta x}{\Delta t} \alpha_{p+2} + \sigma a_{p+1}$

Then $\quad \frac{a_{p+2}}{a} = \frac{\alpha_{p+2}}{\sigma} + \underbrace{\frac{\sigma}{a} a_{p+1}}_{\alpha_{p+1}} \qquad \left\{ \frac{\sigma}{a} = \frac{\Delta t}{\Delta x} \right\}$

$$\therefore \quad \varepsilon_\phi = 1 - \underbrace{\frac{(-1)^{\frac{p-1}{2}}}{a} a_p \phi^{p-1}}_{\text{where is this term?}} - (-1)^{\frac{p+1}{2}} \left(\frac{\alpha_{p+2}}{\sigma} + \alpha_{p+1} \right) \phi^{p+1} + O(\phi^{p+3})$$

stability requires that $\varepsilon_D < 1$

$$\Rightarrow \quad (-1)^{\frac{p+1}{2}} \alpha_{p+1} \phi^{p+1} < 0$$

$$(-1)^{\frac{p+1}{2}} \alpha_{p+1} < 0 \qquad \text{eq } 9.2.34$$

For even orders of accuracy $2l = p$

$$\mathcal{E}_\phi \cong 1 + \underbrace{q_p (-1)^{p/2} \phi^p \frac{\Delta t}{\Delta x}}_{\text{where is this term?}} + q_{p+2}(-1)^{\frac{p+2}{2}} \phi^{p+2} \frac{\Delta t}{\Delta x} + \cdots$$

$$= 1 - (-1)^{p/2}\left(\alpha_{p+2} + \frac{\Delta t}{\Delta x}\sigma a_{p+1}\right)\phi^{p+2} \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

But $\frac{\Delta t}{\Delta x} a_{p+1} = \alpha_{p+1}$

$$\therefore \quad \mathcal{E}_\phi \cong 1 - (-1)^{p/2}\left(\alpha_{p+2} + \sigma\alpha_{p+1}\right)\phi^{p+2} + O(\phi^{p+4}) \qquad \text{eq } 9.2.35$$

$$\mathcal{E}_\phi = 1 - \frac{(-1)^{p/2}}{a} a_{p+1}\phi^p + O(\phi^{p+2})$$

$$\frac{a_{p+1}}{a} = \frac{\Delta x}{a\Delta t}\alpha_{p+1} = \frac{\alpha_{p+1}}{\sigma}$$

$$\mathcal{E}_\phi = 1 - (-1)^{p/2}\frac{\alpha_{p+1}}{\sigma}\phi^p + O(\phi^{p+2}) \qquad \text{eq } 9.2.36$$

Stability is then the ~~term~~ $\cancel{-\frac{(-1)^{p/2}}{a} a_{p+1}\phi^p < 0}$

$$-(-1)^{p/2}\left(\alpha_{p+2} + \sigma\alpha_{p+1}\right)\phi^{p+2} < 0$$

$$(-1)^{p/2}\left(\alpha_{p+2} + \sigma\alpha_{p+1}\right) > 0 \qquad \text{eq } 9.2.37$$

$$G(\pi) = \sum_j b_j \cos(j\pi) = \sum_j (-1)^j b_j = \sum_{j\,\text{even}} b_j - \sum_{j\,\text{odd}} b_j \qquad \text{eq } 9.2.38$$

$|G(\pi)| < 1$

$$-1 \leq \underbrace{\sum_{j\,\text{even}} b_j - \sum_{j\,\text{odd}} b_j} \leq 1$$

$$\sum_j b_j = 1 \qquad \underbrace{\sum_j b_j} - \sum_{j\,\text{odd}} b_j - \sum_{j\,\text{odd}} b_j$$

$$-1 \leq 1 - 2\sum_{j\,\text{odd}} b_j \leq 1 \rightarrow -2 \leq -2\sum_{j\,\text{odd}} b_j \leq 0$$

$$0 \leq \sum_{j\,\text{odd}} b_j \leq 1$$

or $\sum_{j\,\text{odd}} b_j = 1 - \sum_{j\,\text{even}} b_j$ this gives

$$0 \leq 1 - \sum_{j\,\text{even}} b_j \leq 1$$

$$-1 \leq -\sum_{j\,\text{even}} b_j \leq 0 \rightarrow 0 \leq \sum_{j\,\text{even}} b_j \leq 1 \quad \text{eq } 9.2.40$$

Lax-Friedrichs $\quad b_{-1} = \frac{1}{2}(1+\sigma)\,;\; b_0 = 0\,;\; b_{+1} = \frac{1}{2}(1-\sigma)$

~~$0 \leq 0$~~ $\quad 0 < 0 \quad \rightarrow\leftarrow.$

eq 9.2.29

$$G(\phi) = \sum_j b_j \cos j\phi + I \sum_j b_j \sin(j\phi) = \sum_j b_j e^{Ij\phi}$$

$$|G(\phi)|^2 = \sum_j b_j e^{Ij\phi} \cdot \sum_{j'} b_{j'}^* e^{-Ij'\phi} = \sum_{j,j'} b_j b_{j'}^* e^{I(j-j')\phi} \qquad **$$

From $G(\phi) = \sum_j b_j \cos j\phi + I \sum_j b_j \sin(j\phi)$

$$G^*(\phi)G(\phi) = \left(\sum_j b_j \cos j\phi + I \sum_j b_j \sin(j\phi)\right)\left(\sum_j b_j \cos(j\phi) - I \sum_j b_j \sin(j\phi)\right)$$

$$= \left[\sum_j b_j \cos j\phi\right]^2 + \left[\sum_j b_j \sin(j\phi)\right]^2 \quad \text{+ from eq } **$$

we get $|G(\phi)|^2 = \sum_{jk} b_j b_k^* e^{I(j-k)\phi}$ since $b_k \in \mathbb{R}$

$$= \sum_{jk} b_j b_k \cos(j-k)\phi + \underbrace{I \sum_{j,k} b_j b_k \sin(j-k)\phi}_{=0}$$

+ since $|G(\phi)|^2 \in \mathbb{R}$

$$\therefore \quad |G(\phi)|^2 = \sum_{jk} b_j b_k \cos(j-k)\phi \qquad \text{eq } 9.2.41$$

$\cos(n\phi) = ?$

$\cos(\phi) = 1 + (-1)^1 2^1 \sin^2 \phi/2 = 1 - 2\sin^2 \phi/2$ ~~[illegible]~~. True ✓

$\Leftrightarrow \quad \cos(2\phi) = 1 - 2\sin^2\phi$

$\quad\quad \| $

$\quad\quad \cos^2\phi - \sin^2\phi = 1 - 2\sin^2\phi$ ✓.

$\cos(2\phi) = \cancel{\ldots}\ \cancel{\cos 2\phi} \cancel{\sin^2\phi} =$

$= 1 + (-1) 2 \sin^2(\phi/2) + 2^3 \sin^4 \phi/2$

$= 1 - 2\sin^2(\phi/2) + 8\sin^4\phi/2$

$= 1 - 2(1-\cos^2(\phi/2)) + 8(1-\cos^2(\phi/2))^2$

$= 1 - 2 + 2\cos^2(\phi/2) + 8(1 - 2\cos^2(\phi/2) + \cos^4(\phi/2))$

$= -1 + 8 - 14\cos^2(\phi/2) + 8\cos^4(\phi/2)$

$= 7 - 14\cos^2(\phi/2) + 8\cos^4(\phi/2)$

$= 7 - 14\left(\frac{e^{I\phi/2} + e^{-I\phi/2}}{2}\right)^2 + 8\left(\frac{e^{I\phi/2} + e^{-I\phi/2}}{2}\right)^4$

...

Something is incorrect w/ this expression Eq 9.2.42. The correct expression is

$\cos(nx) = (-1)^n \left\{ 2^{2n-1} \sin^{2n} \frac{x}{2} - \frac{2n}{1!} 2^{2n-3} \sin^{2n-2} \frac{x}{2} + \frac{2n(2n-3)}{2!} \ldots \right.$

$$\cancel{\cos(nx) = 1 + \sum_{l=1}^{n} 2^{2l-1} (-1)^l \sin^{2l}\left(\tfrac{x}{2}\right)}.$$

$$\cos(2nx) = 1 + \sum_{l=1}^{2n} (-1)^l 2^{2l} \binom{2n}{k} \sin^{2l} x$$

$$\binom{2n}{0} = 1 \qquad \binom{2n}{2} = \frac{2n!}{2!\,(2n-2)!} = \frac{2n(2n-1)\cancel{(2n-2)}}{2}$$

$$\binom{2n}{1} = 2n$$

See pg 34 Gradshteyn + Ryzhik.

$$|G(\phi)|^2 = \sum_j \sum_k b_j b_k \cos(j-k)\phi = \sum_j \sum_k b_j b_k \left(1 + \sum_{l=1}^{j-k} C_l \sin^{2l}(\phi/2)\right)$$

$$= \sum_j \sum_k b_j b_k + \sum_j \sum_k b_j b_k \sum_{l=1}^{j-k} C_l \sin^{2l}(\phi/2)$$

(m-terms)(m-terms)
= sum of m terms

$$= 1 + \sum_{l=1}^{m} B_l z^l$$

eq 9.2.8 $\quad \xi = |G| = \exp\left[\sum_l a_{2l} (-1)^l \phi^{2l} \frac{\Delta t}{\Delta x}\right]$

$$\phi^2 \to 0$$

Now from $$|G(\phi)|^2 = \sum_j \sum_k b_j b_k \cos(j-k)\phi \quad **$$

$$= \sum_{j=-m}^{+m} \sum_{k=-m}^{+m} b_j b_k \left(1 + \sum_{l=1}^{j-k} c_l \sin^{2l}(\phi/2)\right)$$

$$= 1 + \sum_{j=-m}^{+m} \sum_{k=-m}^{+m} \sum_{l=1}^{j-k} b_j b_k c_l \, z^l$$

From ** since $\cos(-\phi) = \cos\phi$ the sum can run from 0 to m

$$\therefore \quad |G(\phi)|^2 = \sum_{j=0}^{m} \sum_{k=0}^{m} b_j' b_k' \cos(j-k)\phi = \sum_{j=0}^{m} \sum_{k=0}^{m} b_j' b_k' \left(1 + \sum_{l=1}^{j-k} \frac{(-1)^l 2^{2l-1}}{\cdots} \sin^{2l}\phi/2\right)$$

$$= 1 + \sum_{j=0}^{m} \sum_{k=0}^{m} b_j' b_k' \sum_{l=1}^{j-k} c_l \, z^l$$

$m=2 \quad r = 1$ or 2 + $s =$

Lax-Wendroff $\qquad b_{-1} = \frac{\sigma}{2}(1+\sigma); \; b_0 = 1-\sigma^2; \; b_{+1} = -\frac{\sigma}{2}(1-\sigma)$

$$|G(\phi)|^2 = \sum_{j=-1}^{+1} \sum_{k=-1}^{+1} b_j b_k$$

9.2.23

$$\sum_j j^m b_j = (-\sigma)^m \qquad m=0,1,\cdots,p$$

p+1 eqs

Assume p is correct but have question. Seems to be of order p+1?

If $b_j \neq 0$ for $\exists$ M values of ~~j~~ j. $\Rightarrow$ M unknowns b_j

M − p − 1 coeff can be choosen $\Rightarrow$ total # of unknowns M.

" " " eqs M − p − 1
+ p+1

M closed system

Then w/ p = M−1 total # of unknown coeff (of that would be determined arbitrarilly) would be

$M-(M-1)-1 = 0.$

$j = \pm 1, 0$

$b_{-1} + b_0 + b_1 = 1$ eq 7.3.1

$-b_{-1} + b_1 = -\sigma \quad = \quad b_{-1} - b_1 = \sigma$

$S(b_{-1}, b_0, b_1)$

$S_{LF}($ eq 9.2.12a

S_U eq 9.2.11a

S_C

eq 7.2.5 $\quad u_i^{n+1} - u_i^n = -\frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) \Rightarrow u_i^{n+1} = u_i^n + \frac{\sigma}{2}u_{i-1}^n - \frac{\sigma}{2}u_{i+1}^n$

$b_0 = 1$, $b_{-1} = \frac{\beta}{2}$, $b_{+1} = \frac{-\beta}{2}$

System 9,3,1

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \beta-1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-\beta \\ 0 \end{pmatrix}$$

Since there are more eqs than unknowns system is singular

$\Rightarrow$ Null space is non empty

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1-\beta \\ 0 \end{pmatrix}$$

$b_{-1} + \cancel{2b_0} = 0 \quad b_{-1} = -2b_0$

wrong indices

$b_{-1} + b_0 + b_1 = 0 \Rightarrow b_1 = -b_{-1} - b_0 \Rightarrow b_1 = 2b_0 - b_0$
$= b_0$

$$\therefore \quad \vec{b} = \begin{pmatrix} b_{-1} \\ b_0 \\ b_{+1} \end{pmatrix} = \begin{pmatrix} b_0 \\ b_0 \\ -2b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Why is this different than books result of $\begin{matrix} 1 \\ -2 \\ 1 \end{matrix}$? Because of Algebra mistake!!

Find Null space of original matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_{+1} \end{pmatrix} = \vec{0}$$

$b_{-1} \cancel{=} b_0$ wrong index!!

$b_{-1} + b_0 + b_{+1} = 0 \Rightarrow b_0 = \cdot$

$\Rightarrow \quad b_{+1} = -2b_0$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_{+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$b_{-1} + b_0 + b_{+1} = 0 \Rightarrow b_{-1} = +2b_{+1} - b_{+1} = b_{+1}$$
$$b_0 + 2b_{+1} = 0 \Rightarrow b_0 = -2b_{+1}$$

$$\Rightarrow \vec{b} = \begin{pmatrix} b_{-1} \\ b_0 \\ b_{+1} \end{pmatrix} = \begin{pmatrix} b_{+1} \\ -2b_{+1} \\ b_{+1} \end{pmatrix} = b_{+1}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$r = \frac{\sigma}{2}$$

$$S(b_{-1}, b_0, b_{+1}) = S_c(\sigma/2, 1, -\sigma/2) + \frac{\sigma}{2} H(1, -2, 1)$$
$$= S(\sigma, 1-\sigma, 0) \quad \text{upwind}$$

$$r = 1/2$$

$$S(b_{-1}, b_0, b_{+1}) = S_c\left(\frac{\sigma}{2} + \frac{1}{2}, 0, -\frac{\sigma}{2} + \frac{1}{2}\right)$$
$$= S_c\left(\frac{1+\sigma}{2}, 0, \frac{1-\sigma}{2}\right) \quad \text{Lax-Friedrich}$$

$$u_i^{n+1} = b_{-1} u_{i-1}^n + b_0 u_i^n + b_{+1} u_{i+1}^n$$
$$= \left(\frac{\sigma}{2} + r\right) u_{i-1}^n + (1 - 2r) u_i^n + \left(r - \frac{\sigma}{2}\right) u_{i+1}^n$$
$$= u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + r\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) \quad \text{eq 9.3.3.}$$

9.2.28

$$\alpha_{p+1} = \frac{\Delta x}{\Delta t}\left[\sum_j b_j j^{p+1} - (-\sigma)^{p+1}\right]\frac{1}{(p+1)!}$$

$$(-1)^{\left(\frac{p+1}{2}\right)}\alpha_{p+1} < 0 \qquad \text{w/} \quad p = +1$$

$$\Rightarrow \quad (-1)\,\alpha_2 < 0$$

$$\Rightarrow \quad (-1)\left[\sum_j b_j j^2 - (-\sigma)^2\right]\frac{1}{2} < 0$$

$$\Rightarrow \quad \left(\frac{\sigma}{2}+\gamma\right)(-1)^2 + \left(\gamma - \frac{\sigma}{2}\right)(1)^2 - \sigma^2 > 0$$

$$2\gamma - \sigma^2 > 0 \qquad \text{eq } 9.3.4$$

For a 2nd order scheme $p = 2$ in 9.3.23 gives

$$\sum_j j^m b_j = (-\sigma)^m \qquad m = 2$$

$$b_{-1} + b_{+1} = +\sigma^2 \qquad \text{eq } 9.3.5$$

$\Rightarrow$ 3 eqs + 3 unknows

$$\begin{pmatrix} 1 & 1 & 1 \\ +1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_0 \\ b_{+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \sigma \\ \sigma^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \sigma-1 \\ \sigma^2-1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & -2 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-\sigma^2 \\ \sigma-1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 1-\sigma^2 \\ \sigma-\sigma^2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} \sigma^2 \\ 1-\sigma^2 \\ -\frac{(\sigma-\sigma^2)}{2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} b_{-1} \\ b_1 \\ b_0 \end{pmatrix} = \begin{pmatrix} \sigma^2 + \frac{1}{2}\sigma - \frac{\sigma^2}{2} \\ 1-\sigma^2 \\ -\frac{(\sigma-\sigma^2)}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\sigma^2+\sigma) \\ 1-\sigma^2 \\ \frac{1}{2}(\sigma^2-\sigma) \end{pmatrix}$$

$$\therefore \quad b_{-1} = \frac{\sigma}{2} + \frac{\sigma^2}{2}$$

$$b_0 = 1 - \sigma^2 \quad \Rightarrow \quad r = \frac{\sigma^2}{2}$$

$$b_{+1} = -\frac{\sigma}{2} + \frac{\sigma^2}{2}$$

Then from def of b_i's

$$u_i^{n+1} = \left(\frac{\sigma}{2} + \frac{\sigma^2}{2}\right)u_{i-1}^n + (1-\sigma^2)u_i^n + \left(-\frac{\sigma}{2} + \frac{\sigma^2}{2}\right)u_{i+1}^n$$

$$= \frac{\sigma}{2}(1+\sigma)u_{i-1}^n + (1-\sigma^2)u_i^n - \frac{\sigma}{2}\left(1 - \cancel{\tfrac{\sigma}{2}}\,\sigma\right)u_{i+1}^n \qquad \text{eq 9.3.6}$$

or from eq 9.3.3 w/ $r = \sigma^2/2$ gives eq 9.3.7

$$\approx \Delta x^2 u_{xx} \frac{\sigma^2}{2} \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

$$\Delta x \sigma = a \Delta t$$

$$= \frac{a^2 \Delta t^2}{2} u_{xx}$$

From eq 9.2.22

$$u_t + a u_x = \frac{\Delta x}{\Delta t}\left[\alpha_m \Delta x^{m-1} \frac{\partial^m u}{\partial x^m}\bigg|_{m=2} + \alpha_m \Delta x^{m-1} \frac{\partial^m u}{\partial x^m}\bigg|_{m=3} + \alpha_4 \Delta x^3 \frac{\partial^4 u}{\partial x^4} + O(\Delta x^4) \right]$$

$$+ - \frac{(-\sigma)^1}{1!} \Delta x^3 a_3 \frac{\partial^4 u}{\partial x^4} + O(\Delta x^4)$$

m

$$p + 2 - 1 = p+1$$

$$p + p + 2 - 1 = 2p+1\Big|_{p=2} = 5.$$

$$= \frac{\Delta x}{\Delta t}\left[\alpha_2 \Delta x \frac{\partial^2 u}{\partial x^2} + \alpha_3 \Delta x^2 \frac{\partial^3 u}{\partial x^3} + \alpha_4 \Delta x^3 \frac{\partial^4 u}{\partial x^4} + O(\Delta x^4) \right]$$

$$+ \sigma \Delta x^3 a_3 \frac{\partial^4 u}{\partial x^4} + O(\Delta x^4)$$

$$\text{w/ } a_3 = \left| \sum_j b_j j^3 - (-\sigma)^3 \right| \frac{\Delta x}{\Delta t} \frac{1}{3!}$$

$$= \Big(\underbrace{(-b_{-1} + b_{+1})}_{-\sigma} + \sigma^3 \Big) \frac{\Delta x}{\Delta t} \frac{1}{6} = \frac{(\sigma^3 - \sigma)}{6} \frac{\Delta x}{\Delta t}$$

$$\alpha_2 = \frac{1}{2} \Big[\sum_j b_j j^2 - (-\sigma)^2 \Big] = \frac{1}{2} \Big[\underbrace{(b_{-1} + b_{+1}) - \sigma^2}_{=0.} \Big]$$

$$\alpha_3 = \frac{1}{6} \Big[\underbrace{(-b_{-1} + b_{+1})}_{-\sigma} + \sigma^3 \Big] = + \frac{\sigma(\sigma^2 - 1)}{6}$$

$$\alpha_4 = \frac{1}{24} \Big[\underbrace{b_{-1} + b_{+1}}_{\sigma^2} - \sigma^4 \Big] = \frac{\sigma^2 (1 - \sigma^2)}{24}$$

$$\sigma = \frac{a \Delta t}{\Delta x}$$

Then

$$u_t + a u_x = \frac{\Delta x}{\Delta t} \Big[\frac{\sigma(\sigma - 1)(\sigma + 1)}{6} \Delta x^2 u_{xxx} \cancel{\pm} \underline{\frac{\sigma^2 (1 - \sigma^2)}{24}} \Delta x^3 u_{xxxx} + O(\Delta x^4) \Big]$$

$$+ \underline{\sigma \Delta x^3 \frac{\Delta x}{\Delta t} \frac{\sigma}{6} (\sigma^2 - 1) u_{xxxx}} + \cdots$$

$$= -a \frac{(1 - \sigma^2)}{6} \Delta x^2 u_{xxx} + a \sigma (1 - \sigma^2) \underbrace{\Big[\frac{1}{24} - \frac{1}{6} \Big]}_{-\frac{1}{8}} \Delta x^3 u_{xxxx} + O(\Delta x^4)$$

eq 9.3.8

pg 358 Hirsch Vol I

$$u_i^{n+1} = b_{-2}u_{i-2} + b_{-1}u_{i-1} + b_0 u_i$$

Then eq 9.2.23 imply

$m=0$:

$$b_{-2} + b_{-1} + b_0 = +1$$

$m=1$:

$$-2b_{-2} - b_{-1} + 0 = -\sigma \qquad \text{eq 9.3.10}$$

$m=2$:

$$4b_{-2} + b_{-1} = \sigma^2$$

$\Rightarrow$ ~~$b_{-2} = \frac{\sigma}{2}(1+\sigma)$ $b_{-1} =$~~

$$b_{-2} = \frac{\sigma}{2}(-1+\sigma)\ ;\ b_{-1} = (2-\sigma)\sigma\ ;\ b_0 = \frac{1}{2}(1-\sigma)(2-\sigma)$$

eq 9.3.11.

$$\Rightarrow u_i^{n+1} = \frac{\sigma}{2}(\sigma-1)u_{i-2}^n + (2-\sigma)\sigma u_{i-1}^n + \frac{1}{2}(2-3\sigma+\sigma^2)u_i^n \qquad \text{eq 9.3.12}$$

$$= u_i^n - \frac{3}{2}\sigma u_i^n + \frac{\sigma^2}{2}u_i^n + 2\sigma u_{i-1}^n - \sigma^2 u_{i-1}^n$$

$$+ \frac{\sigma^2}{2}u_{i-2}^n - \frac{\sigma}{2}u_{i-2}^n$$

$$= u_i^n - \sigma(u_i^n - u_{i-1}^n) + \sigma u_i^n - \sigma u_{i-1}^n$$

$$- \frac{3}{2}\sigma u_i^n + 2\sigma u_{i-1}^n$$

$$+ \frac{\sigma^2}{2}u_i^n - \sigma^2 u_{i-1}^n + \frac{\sigma^2}{2}u_{i-2}^n - \frac{\sigma}{2}u_{i-2}^n$$

$$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n) - \frac{\sigma}{2}u_i^n + \sigma u_{i-1}^n$$

$$+\frac{\sigma^2}{2}u_i^n - \sigma^2 u_{i-1}^n + \frac{\sigma^2}{2}u_{i-2}^n - \frac{\sigma}{2}u_{i-2}^n$$

$$= u_i^n - \sigma(u_i^n - u_{i-1}^n) - \frac{\sigma}{2}(u_i^n - 2u_{i-1}^n + u_{i-2}^n)$$

$$+\frac{\sigma^2}{2}(u_i^n - 2u_{i-1}^n + u_{i-2}^n)$$

$$= u_i^n - \sigma(u_i^n - u_{i-1}^n) + \frac{\sigma}{2}(\sigma-1)(u_i^n - 2u_{i-1}^n + u_{i-2}^n) \qquad \text{eq } 9.3.13$$

Dissipative term

$$\frac{\sigma}{2}(\sigma-1)(u_i^n - 2u_{i-1}^n + u_{i-2}^n) = \frac{\sigma\Delta x^2}{2}(\sigma-1)\,u_{xx}(x_{i-1}) + O(\Delta x^3)$$

$$\left\{\sigma = \frac{a\Delta t}{\Delta x}\right\} \qquad = \frac{a\Delta t\Delta x}{2}(\sigma-1)\,u_{xx}(x_{i-1}) + O(\Delta x^3)$$

From eq 9.2.23

$u_t + au_x = \frac{\Delta x}{\Delta t}\Big[\quad \cdots$ starting sum at $m = p+1$ gives

$$= a_{p+1}\Delta x^{p}\frac{\partial^{p+1} u}{\partial x^{p+1}} \qquad \text{from eq } 9.2.24$$

$$=$$

$$a_3 = \frac{\Delta x}{\Delta t}\left[\cancel{}\ \cancel{b_{-2}}\ \cancel{8b_{-1}} - (-\sigma)^3\right]\frac{1}{6}$$

$$-8b_{-2} - 1b_{-1}$$

$$a_3 = \frac{\Delta x}{b\Delta t}\left[-\frac{\cancel{2}8}{2}\sigma(\sigma-1) - \cancel{8}1\,\sigma(2-\sigma) + \sigma^3 \right]$$

$$= \frac{1}{6}\frac{\Delta x}{\Delta t}\sigma(1-\sigma)(2-\sigma) = \frac{a}{6}(1-\sigma)(2-\sigma)$$

Then the next order terms will be from 9.2.22

$\cancel{a_4} \leftarrow \cancel{\frac{\Delta x}{\Delta t}}$

$$\frac{\Delta x}{\Delta t}\left[\sum_j b_j j^4 - \sigma^4\right]\frac{\Delta x^3}{24}\frac{\partial^4 u}{\partial x^4}$$

$$\left.- \frac{(-\sigma)^1}{1!}\Delta x^3 a_3 \frac{\partial^4 u}{\partial x^4} \right\} O(\Delta x^3) \text{ contribution.}$$

$\Delta x^{p+2-1 \leftarrow p+1} = \Delta x$

$2p+2-1 = q-1$

$\therefore$ last term does not contribute

$$= \frac{\Delta x}{\Delta t}\left[+16 b_{-2} + b_{-1} - \sigma^4\right]\frac{\Delta x^3}{24}\frac{\partial^4 u}{\partial x^4} + \frac{a\sigma}{6}(1-\sigma)(2-\sigma)\frac{\partial^4 u}{\partial x^4}\Delta x^3$$

$$= \left[\frac{\Delta x}{\Delta t}\frac{\sigma}{24}(2-\sigma)(\sigma-1)(\sigma+3) + \frac{a\sigma}{6}(1-\sigma)(2-\sigma)\right]\frac{\partial^4 u}{\partial x^4}\Delta x^3$$

$$= \frac{a}{6}(2-\sigma)(1-\sigma)\left[\frac{-(\sigma+3)}{4} + \sigma\right]\frac{\partial^4 u}{\partial x^4}\Delta x^3$$

$$= \frac{a(2-\sigma)(1-\sigma)}{\cancel{6}\,2}\,\frac{\cancel{3}(\sigma-1)}{4}\,u_{xxxx}\Delta x^3$$

$$= \frac{a(2-\sigma)(1-\sigma)(\sigma-1)}{8}u_{xxxx}\Delta x^3$$

$$\therefore\ u_t + a u_x = \frac{a}{6}\Delta x^2(1-\sigma)(2-\sigma)u_{xxx} - \frac{a\Delta x^3}{8}(1-\sigma)^2(2-\sigma)u_{xxxx} + O(\Delta x^4)$$

eq 9.3.14

eq 9.2.5 $\Rightarrow \quad (-1)^r a_{2r} < 0$

$p = 2 \quad 2l = 4 \quad r = 2 \quad \Rightarrow \quad a_4 < 0$

$$\Rightarrow \quad -\frac{a}{8}(1-\sigma)^2(2-\sigma) < 0$$

$0 \le \sigma \le 2$ don't see

why?

Von-Neumann $\Rightarrow$ in eq 9.3.12 decompose u_i^n into Fourier modes

$u_i^n = G^n e^{+Ii\phi} \qquad \phi = k\Delta x \qquad \{ e^{i\theta} = \cos\theta + i\sin\theta \}$

Then

$$G = 1 - \sigma(1 - e^{-I\phi}) + \tfrac{1}{2}\sigma(\sigma-1)(1 - 2e^{-I\phi} + e^{-2I\phi})$$

$$= 1 - \sigma(1 - \cos\phi + I\sin\phi) + \tfrac{1}{2}\sigma(\sigma-1)(1 - 2(\cos\phi - I\sin\phi) + (\cos(2\phi) - I\sin(2\phi)))$$

$$= \cancel{1 - 2\sigma(1 - \ldots)(1-\sigma)}$$

$$= 1 - 2\sigma(1 - (1-\sigma)\cos(\phi))\sin^2(\phi/2) - I\sigma(1 - (1-\sigma)\cos\phi)\sin\phi$$

imaginary term is also:

$$\left\{\cos\phi = \cos^2\phi/2 - \sin^2\phi/2 = 1 - 2\sin^2\phi/2\right.$$

$$= -I\sigma\sin\phi\left(1-(1-\sigma)(1-2\sin^2\phi/2)\right)$$

$$= -I\sigma\sin\phi\left(1-1+\sigma+2(1-\sigma)\sin^2\phi/2\right)$$

$$= -I\sigma\sin\phi\left(\sigma+2(1-\sigma)\sin^2\phi/2\right)$$

↑ got σ here rather than $\underline{1}$.

$$|G|^2 = \left(1-2\sigma(1-(1-\sigma)\cos\phi)\sin^2\phi/2\right)^2$$

$$+\left(\sigma\sin\phi\left(\sigma+2(1-\sigma)\sin^2\phi/2\right)\right)^2$$

$$= 1+\sigma(-2+\sigma)(-1+\sigma)^2\Big[\frac{3}{2} - 2\cos\left(\frac{\phi}{2}\right)^2 + \frac{1}{2}\cos\left(\frac{\phi}{2}\right)^4$$

$$+ 2\sin\left(\frac{\phi}{2}\right)^2 - 3\cos\left(\frac{\phi}{2}\right)^2\sin\left(\frac{\phi}{2}\right)^2 + \frac{1}{2}\sin\left(\frac{\phi}{2}\right)^4\Big]$$

$$= 1+\sigma(-2+\sigma)(-1+\sigma)^2\Big[\frac{3}{2} - 2\left(1-\sin\left(\frac{\phi}{2}\right)^2\right) + \frac{1}{2}\left(1-\sin\left(\frac{\phi}{2}\right)^2\right)^2$$

$$+ 2\sin\left(\frac{\phi}{2}\right)^2 - 3\sin\left(\frac{\phi}{2}\right)^2\left(1-\sin\left(\frac{\phi}{2}\right)^2\right) + \frac{1}{2}\sin\left(\frac{\phi}{2}\right)^4\Big]$$

$$= 1+\sigma(-2+\sigma)(-1+\sigma)^2\left[4\sin\left(\frac{\phi}{2}\right)^4\right]$$

$$= 1-4\sigma(1-\sigma)^2(2-\sigma)\sin^4\left(\frac{\phi}{2}\right)$$

11-26-01 6

$G(\pi) = 1 - 4\sigma(1-\sigma)^2(2-\sigma)$

Req $|G|<1 \Rightarrow |4\sigma(1-\sigma)^2(2-\sigma)\sin^4(\frac{\phi}{2})| < 2$

$\forall \phi \quad 0 \le \phi \le \pi$

$-1 \le 1 - x \le 1$

$-2 \le -x \le 0$

$0 \le x \le 2$

w/ $x = 4\sigma(1-\sigma)^2(2-\sigma)\sin^4(\phi/2)$

$-1 \le x \le 1$

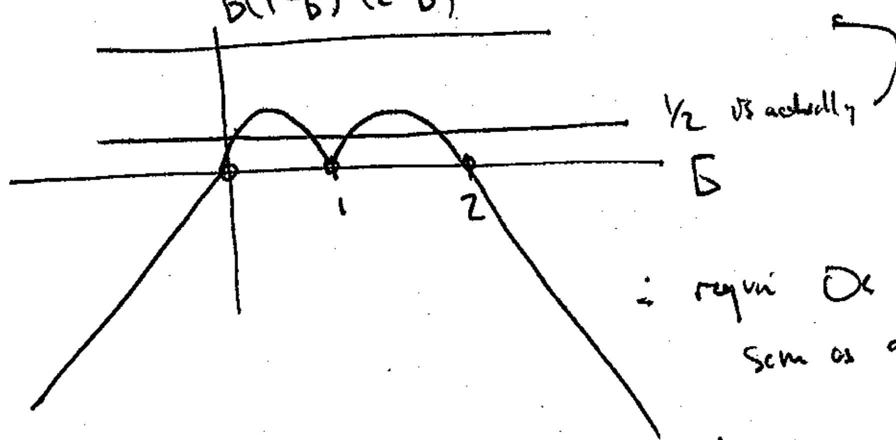
$-1 \le -x \le +1$

$0 \le -x \le 2$

$$0 < 4\sigma(1-\sigma)^2(2-\sigma)\sin^4(\phi/2) < 2$$

$\Rightarrow \sigma > 0 \qquad \sigma(1-\sigma)^2(2-\sigma) < \frac{1}{2}$

$\sigma(1-\sigma)^2(2-\sigma)$



$\therefore$ require $0 < \sigma < 2$

Same as 9.3.15!!

where ½ falls will

enter

$$0 < 4\sigma(1-\sigma)^2(2-\sigma) < \frac{2}{\sin^4(\phi/2)}$$

This inequality will be inforced if we require that

$$0 < 4\sigma(1-\sigma)^2(2-\sigma) < 2$$

eq 9.2.36 gives:

$$\varepsilon_\phi = 1 - (-1)^{p/2} \frac{1}{\sigma} \alpha_{p+1} \phi^p + O(\phi^{p+2})$$ which becomes

$$\varepsilon_\phi = 1 + \frac{1}{\sigma} \alpha_3 \phi^2 + O(\phi^4)$$

$$= 1 + \frac{1}{\sigma} \frac{1}{6} \sigma (1-\sigma)(2-\sigma) \phi^2 + O(\phi^4)$$

$$= 1 + \frac{1}{6}(1-\sigma)(2-\sigma) \phi^2 + O(\phi^4)$$ eq 9.3.19

Now $\frac{1}{6}(1-\sigma)(2-\sigma) > 0 \qquad 0<\sigma<1$

$\frac{1}{6}(1-\sigma)(2-\sigma) < 0 \qquad 1<\sigma<2$

$\therefore \quad \varepsilon_\phi > 1 \qquad 0<\sigma<1 \Rightarrow$ leading phase error.

$\varepsilon_\phi < 1 \qquad 1<\sigma<2 \Rightarrow$ lagging phase error.

eq 9.4.10 ⇒ 9.4.11 by grouping powers of Δx

Then

$$g(u_{i-1},u_i) = g(u_i,u_i) + \frac{\partial g}{\partial u_{i-1}}\Big|_{u_i}\Delta u + \frac{1}{2}\frac{\partial^2 g}{\partial u_{i-1}^2}\Big|_{u_i}\Delta u^2 +$$

$$\frac{1}{6}\frac{\partial^3 g}{\partial u_{i-1}^3}\Big|_{u_i}\cancel{\Delta x}^3 + O(\Delta u^4)$$

Defining as 9.4.9 & $\Delta u = u_{i-1} - u_i$

$$= -u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} - u_{xxx}\frac{\Delta x^3}{6} + O(\Delta x^4)$$

$$= g(u_i,u_i) + g_1\left(-u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} - u_{xxx}\frac{\Delta x^3}{6} + O(\Delta x^4)\right)$$

$$+ \frac{1}{2}g_{11}\left(+u_x^2\Delta x^2 - u_x u_{xx}\Delta x^3 + O(\Delta x^4)\right)$$

$$+ \frac{1}{6}g_{111}\left(-u_x^3\Delta x^3 + O(\Delta x^4)\right)$$

$$= g(u_i,u_i) - g_1 u_x\Delta x +$$

$$+ \frac{1}{2}\left(g_1 u_{xx} + g_{11}u_x^2\right)\Delta x^2$$

$$+ \frac{1}{6}\left(-g_1 u_{xxx} - 3g_{11}u_x u_{xx} - g_{111}u_x^3\right)\Delta x^3 + O(\Delta x^4)$$

Then

$$g(u_i,u_{i+1}) - g(u_{i-1},u_i) = \Delta x\,(g_2 + g_1)\,u_x$$

$$+ \frac{\Delta x^2}{2}\left(g_2 u_{xx} + g_{22} u_x^2 - g_1 u_{xx} - g_{11} u_x^2\right)$$

$$+ \frac{\Delta x^3}{6}\left(g_2 u_{xxx} + 3 g_{22} u_x u_{xx} + g_{222} u_x^3 + g_1 u_{xxx} + 3 g_{11} u_x u_{xx} + g_{111} u_x^3\right) + O(\Delta x^4)$$

$$= \Delta x\,(g_1 + g_2)\,u_x + \frac{\Delta x^2}{2}\left[u_{xx}(g_2 - g_1) + (g_{22} - g_{11})\,u_x^2\right]$$

$$+ \frac{\Delta x^3}{6}\left[(g_1+g_2)\,u_{xxx} + 3(g_{11}+g_{22})\,u_x u_{xx} + \cancel{\;}(g_{111} + g_{222})\,u_x^3\right]$$

$$+ O(\Delta x^4)$$

Now $$\frac{\partial}{\partial x}\left((g_2 - g_1)\,u_x\right) = \cancel{g_{21} u_x + g_{22} u_x - g_{12} u_x - g_{11} u_x}$$

$$= \cancel{g_{21}} u_x + g_{22} u_x - \cancel{g_{12}} u_x - g_{11} u_x$$

$$= (g_{22} - g_{11})\,u_x$$

$$\frac{\partial}{\partial x}\left(f_{xx} - 3 g_{12} u_x^2\right)$$

$$f(u) = g(u,u)$$

$$f_x = g_1 u_x + g_2 u_x$$

$$f_{xx} = g_{11} u_x^2 + \ldots$$

Now w/ $f(u) = g(u,u)$

$$f_x = g_1 u_x + g_2 u_x$$

$$f_{xx} = g_{11} u_x^2 + g_{12} u_x^2 + g_1 u_{xx} + g_{12} u_x^2 + g_{22} u_x^2 + g_2 u_{xx}$$

$$= (g_{11} + 2g_{12} + g_{22}) u_x^2 + (g_1 + g_2) u_{xx}$$

So

$$f_{xx} - 3g_{12} u_x^2$$

$$= (g_{11} - g_{12} + g_{22}) u_x^2 + (g_1 + g_2) u_{xx}$$

So that

$$\frac{\partial}{\partial x}(f_{xx} - 3g_{12} u_x^2)$$

$$= (g_{111} + \cancel{g_{112}} - \cancel{g_{112}} - \cancel{g_{122}} + \cancel{g_{122}} + g_{222}) u_x^3 + 2(g_{11} - \cancel{g_{12}} + g_{22}) u_x u_{xx}$$

$$+ (g_{11} + \cancel{g_{12}} + \cancel{g_{12}} + g_{22}) u_{xx} u_x + (g_1 + g_2) u_{xxx}$$

$$= (g_{111} + g_{222}) u_x^3 + 3(g_{11} + g_{22}) u_x u_{xx}$$

$$+ (g_1 + g_2) u_{xxx} \quad \text{coefficient of } O(\Delta x^3)$$

9.4.2 is

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\right]$$

w/ 9.2.16

$$u_i^{n+1} = u_i^n + \sum_{m=1}^{\infty} \frac{\Delta t^m}{m!} \frac{\partial^m u}{\partial t^m}$$

$$\sum_{m=1}^{\infty} \frac{\Delta t^m}{m!} \frac{\partial^m u}{\partial t^m} = -\frac{\Delta t}{\Delta x}\left[g(u_i, u_{i+1}) - g(u_{i-1}, u_i)\right]$$

$$\frac{\Delta t}{1!} u_t + \frac{\Delta t^2}{2} u_{tt} + \frac{\Delta t^3}{3!} u_{ttt} + O(\Delta t^4)$$

$$= -\frac{\Delta t}{\Delta x}\left[g(u_i, u_{i+1}) - g(u_{i-1}, u_i)\right]$$

$$= -\frac{\Delta t}{\Delta x}\left[\Delta x (g_1 + g_2) u_x + \frac{\Delta x^2}{2} \frac{\partial}{\partial x}\left[(g_2 - g_1) u_x\right] + \frac{\Delta x^3}{6} \frac{\partial}{\partial x}\left[f_{xx} - 3 g_{12} u_x^2\right] + O(\Delta x^4)\right]$$

$$= -\Delta t (g_1 + g_2) u_x - \frac{\Delta t \Delta x}{2} \frac{\partial}{\partial x}\left[(g_2 - g_1) u_x\right] - \frac{\Delta t \Delta x^2}{6} \frac{\partial}{\partial x}\left[f_{xx} - 3 g_{12} u_x^2\right] + O(\Delta x^3 \Delta t)$$

Thus we require ~~Δt~~ $-(g_1 + g_2) u_x = u_t$

$$\Rightarrow u_t + (g_1 + g_2) u_x = 0 \quad \text{or that} \quad g_1 + g_2 = A \quad \text{eq 9.4.15.}$$

$$\therefore \; A(u_i) = \frac{\partial g(u_i, u_{i+1})}{\partial u_i} + \frac{\partial g(u_i, u_{i+1})}{\partial u_{i+1}}$$

~~Then~~ ~~[illegible]~~ ~~Teq~~ If this condition is matched we then have

$$u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} + O(\Delta t^3) = -(g_1 + g_2) u_x - \frac{\Delta x}{2} \frac{\partial}{\partial x}\left[(g_2 - g_1) u_x\right] - \frac{\Delta x^2}{6} \frac{\partial}{\partial x}\left[\cdots\right]$$

$$u_t + f_x = -\frac{\Delta x}{2}\frac{\partial}{\partial x}\left[(g_2 - g_1)u_x\right] - \frac{\Delta x^2}{6}\frac{\partial^2}{\partial x^2}\left[f_{xx} - 3g_{12}u_x^2\right]$$

$$-\frac{\Delta t}{2}u_{tt} - \frac{\Delta t^2}{6}u_{ttt} + O(\Delta x^3)$$

If we assume $u_t + f_x = \Delta x^p Q(u)$ in terms of all x derivatives.

Then higher derivatives in terms of only x-derivatives,

$$u_{tt} = -f_{xt} + \Delta x^p \frac{\partial Q}{\partial t} + O(\Delta x^{p+1})$$

$$= -(f_t)_x + \text{''} + \text{''}$$

$$= -(Au_t)_x + \text{''} + \text{''}$$

$$= -\left(A\left(-f_x + \Delta x^p Q(u)\right)\right)_x + \text{''} + \text{''}$$

$$= -\left(-Af_x + \Delta x^p A Q(u)\right)_x + \Delta x^p \frac{\partial Q}{\partial t} + O(\Delta x^{p+1})$$

$$= (AAu_x)_x - \Delta x^p (AQ(u))_x + \Delta x^p \frac{\partial Q}{\partial t} + O(\Delta x^{p+1})$$

& dropping the term $\Delta x^p (AQ(u))_x$ gives

$$u_{tt} = (A^2 u_x)_x + \Delta x^p \frac{\partial Q}{\partial t} + O(\Delta x^{p+1})$$

Pg 363 Hirsch Vol I

$$u_{ttt} = (A^2 u_x)_{xt} + O(\Delta x^p)$$

$$= (A^2 u_x)_{tx} + O(\Delta x^p)$$

$$= (2AA_u u_t u_x + A^2 u_{xt})_x + O(\Delta x^p)$$

$$= (2AA_u u_x u_t + A^2 (u_t)_x)_x + O(\Delta x^p)$$

$$= (2AA_u u_x(-f_x)) + A^2(-f_x)_x)_x + O(\Delta x^p)$$

$$= -(2AA_u f_x u_x + A^2 f_{xx})_x + O(\Delta x^p)$$

$$= -(2AA_u A u_x^2 + A^2 f_{xx})_x + O(\Delta x^p)$$

Now $f_x = Au_x$

$f_{xx} = A_u u_x^2 + Au_{xx}$ so the above becomes:

$$= -(2AA_u A u_x^2 + A^2 A_u u_x^2 + A^3 u_{xx})_x + O(\Delta x^p)$$

Assuming $AA_uA = A^2 A_u$ (might Not be true)

$$= -(3A^2 A_u u_x^2 + A^3 u_{xx})_x + O(\Delta x^p) \quad \text{then}$$

$$u_{ttt} = -3 \cdot 2 \underline{A} A_u^2 u_x^3 - 3A^2 \underline{A_{uu}} u_x^3 - 3A^2 \underline{\underline{A_u}} 2 u_x u_{xx} \qquad -6-3=-9$$

$$- 3A^2 \underline{\underline{A_u}} u_x u_{xx} - A^3 u_{xxx} + O(\Delta x^p)$$

$$= -3A(AA_{uu} + 2A_u^2)u_x^3 - 9A^2 A_u u_x u_{xx} - A^3 u_{xxx} + O(\Delta x^p) \quad \text{eq } 9.4.20$$

9.4.17 becomes

$$u_t + f_x = -\frac{\Delta x}{2}\frac{\partial}{\partial x}\left[(g_2 - g_1)u_x\right] - \frac{\Delta x^2}{6}\frac{\partial}{\partial x}\left(f_{xx} - 3g_{12}u_x^2\right)$$

$$- \frac{\Delta t}{2}(A^2 u_x)_x + \frac{\Delta t^2}{6}(2A^2 A_u u_x^2 + A^2 f_{xx})_x + O(\Delta x^3)$$

$$= -\frac{\Delta x}{2}\frac{\partial}{\partial x}\left[\left(g_2 - g_1 + A^2\frac{\Delta t}{\Delta x}\right)u_x\right] - \frac{\Delta x^2}{6}\frac{\partial}{\partial x}\left[f_{xx} - 3g_{12}u_x^2 - \frac{\Delta t^2}{\Delta x^2}(2A^2A_u u_x^2 + A^2 f_{xx})\right]$$

$$+ O(\Delta x^3)$$

$$= -\frac{\Delta x}{2}\frac{\partial}{\partial x}\left[(g_2 - g_1 + A^2\tau)u_x\right] - \frac{\Delta x^2}{6}\frac{\partial}{\partial x}\left[(1 - \tau^2A^2)f_{xx} - (3g_{12} + 2\tau^2A^2A_u)u_x^2\right]$$

$$+ O(\Delta x^3) \quad \text{eq } 9.4.21$$

$$g_1 - g_2 = \tau A^2 \quad \text{eq } 9.4.22$$

9.4.3	$g(u,u) = f(u)$	consistency
9.4.15	$g_1 + g_2 = A$	~~...~~ consistency
9.4.22	$g_1 - g_2 = \tau A^2$	2nd order accuracy.

Now the generalization of $\sum_j j^m b_j = (-\sigma)^m$

~~...~~ Alternately we can write 9.4.21 as (the 2nd order term as)

$$\cong -\frac{\Delta x^2}{6}\left[\frac{\partial}{\partial x}(f_{xx} - 3g_{12}u_x^2) + \tau^2 u_{ttt}\right]$$

$$= -\frac{\Delta x^2}{6}\left[f_{xxx} - 3(g_{112} + g_{122})u_x^3 - 6g_{12}u_x u_{xx} + \tau^2 u_{ttt}\right]$$

From 12-01-01 $f_{xx} = A_{u}u_x^2 + Au_{xx}$

$$f_{xxx} = A_{uu}u_x^3 + 2A_u u_x u_{xx} + A_u u_x u_{xx} + Au_{xxx}$$

$$= A_{uu}u_x^3 + 3A_u u_x u_{xx} + Au_{xxx}.$$

Thus w/ expression for u_{ttt} from eq 9.4.20 we have

$$-\frac{\Delta x^2}{6}\Big[A_{uu}u_x^3 + 3A_u u_x u_{xx} + Au_{xxx} - 3(g_{112} + g_{122})u_x^3 - 6g_{12}u_x u_{xx}$$

$$+ \tau^2\left(-3A(AA_{uu} + 2A_u^2)u_x^3 - 9A^2A_u u_x u_{xx} - A^3u_{xxx}\right)\Big]$$

$$= -\frac{\Delta x^2}{6}\Big(A(1-\tau^2A^2)u_{xxx} - \left[(9A^2\tau^2 - 3)A_u + 6g_{12}\right]u_x u_{xx}$$

$$+ \left[(1-3\tau^2A^2)A_{uu} - 6\tau^2AA_u^2 - 3(g_{112} + g_{122})\right]u_x^3 \qquad \text{eq } 9.4.24$$

$$-\frac{\Delta x^2}{6}(-3)\left[(3A^2\tau^2 - 1)A_u + 2g_{12}\right]u_x u_{xx}$$

$$= \frac{\Delta x^2}{2}\left[(3A^2\tau^2 - 1)A_u - 2g_{12}\right]u_x u_{xx} \qquad \text{eq } 9.4.25$$

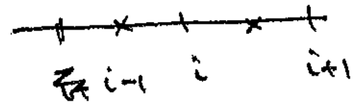
E8.3.8 $\quad u_i^{n+1} = u_i^n + \underbrace{\Delta t (u_t)_i}_{\Delta t(-f_x)} + (u_{tt})_i \frac{\Delta t^2}{2} + O(\Delta t^3)$

$$u_{tt} = -(f_x)_t = -(f_t)_x = -(Au_t)_x = +(Af_x)_x$$

$$\therefore \quad u_i^{n+1} = u_i^n - \Delta t (F_x)_i + \frac{\Delta t^2}{2}\frac{\partial}{\partial x}(A F_x) + O(\Delta t^3)$$ eq 9.4.26

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}(F_{i+1} - F_{i-1}) + \frac{\Delta t^2}{2\Delta x^2}\left[A_{i+1/2}(F_{i+1} - F_i) - A_{i-1/2}(F_i - F_{i-1}) \right] + O(\Delta t^3)$$

eq 9.4.27



Then Numerical flux $u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\left[g(u_{\cancel{x}i+1}, u_i) - g(\cancel{x}, u_{i-1}) \right]$

$\left[g(u_i, u_{i+1}) - g(u_{i-1}, u_i) \right]$

is $g(u_i, u_{i+1}) = \frac{1}{2} f_{i+1}$

$- \frac{1}{2}\frac{\Delta t}{\Delta x} A_{i+1/2}(f_{i+1} - f_i)$? Why won't this work?

Then $g(u_{i-1}, u_i) = \frac{1}{2} f_i$

Also $g(u_i, u_{i+1}) = \frac{1}{2}(f_{i+1} + f_i) - \frac{\tau}{2} A_{i+1/2}(f_{i+1} - f_i)$ eq 9.4.28

$$A_{i+1/2} = A\left(\frac{u_i + u_{i+1}}{2}\right)$$

$\{ f_{i+1} - f_i - f_i + f_{i-1}$
$= f_{i+1} - 2f_i + f_{i-1} \}$

w/ Burgers eq $u_t + (u^2/2)_x = 0$

$F(u) = u^2/2 \qquad A = u$

$$g(u_i, u_{i+1}) = \frac{1}{2}\left(\frac{u_{i+1}^2}{2} + \frac{u_i^2}{2}\right) - \frac{\tau}{2}\left(\frac{u_i + u_{i+1}}{2}\right)\left(\frac{u_{i+1}^2}{2} - \frac{u_i^2}{2}\right)$$

So that

$$g_1 = \frac{\partial g}{\partial u_i} = +\frac{2}{4}u_i - \frac{\tau}{2}\,\frac{1}{2}\left(\frac{u_{i+1}^2}{2} - \frac{u_i^2}{2}\right) - \frac{\tau}{2}\left(\frac{u_i + u_{i+1}}{2}\right)\left(-\frac{2u_i}{2}\right)$$

$$g_1 = +\frac{1}{2}u_i - \frac{\tau}{4}\left(\frac{u_{i+1}^2}{2} - \frac{u_i^2}{2}\right) + \frac{\tau}{2}\left(\frac{u_i + u_{i+1}}{2}\right)u_i$$

$$= +\frac{1}{2}u_i - \frac{\tau}{8}(u_{i+1}^2 - u_i^2) + \frac{\tau}{4}(u_i + u_{i+1})u_i$$

$$g_2 = \frac{\partial g}{\partial u_{i+1}} = \frac{1}{2}u_{i+1} - \frac{\tau}{2}\left(\frac{1}{2}\right)\left(\frac{u_{i+1}^2}{2} - \frac{u_i^2}{2}\right) - \frac{\tau}{2}\left(\frac{u_i + u_{i+1}}{2}\right)(1)\,u_{i+1}$$

$$= \frac{1}{2}u_{i+1} - \frac{\tau}{8}(u_{i+1}^2 - u_i^2) - \frac{\tau}{4}(u_i + u_{i+1})u_{i+1}$$

$$g_{12} = \frac{\partial^2 g}{\partial u_i \partial u_{i+1}} = +\frac{\tau}{4}u_i - \frac{\tau}{4}u_i = 0 \quad \checkmark$$

Then ~~eq~~ checking eq ~~9~~ 9.4.3

$$g(u,u) = f(u) \quad \checkmark$$

checking $g_1 + g_2 = A$ eq 9.4.15 Req for ~~a~~ consistency, gives:

$$= \frac{1}{2}(u_{i+1} + u_i) - \frac{2\tau}{8}(u_{i+1}^2 - u_i^2) - \frac{\tau}{4}(u_i + u_{i+1})^2$$

$$= \frac{1}{2}(u_{i+1} + u_i) - \frac{\tau}{4}\left(u_{i+1}^2 - u_i^2 + u_i^2 + 2u_iu_{i+1} + u_{i+1}^2\right)$$

* $g_1 + g_2 = A$ is eq 9.4.16 evaluated at u_i

$= u_i = A(u_i)$ Yes!!

$f = u^2/2$

$f_u = u$

$f_{uu} = 1$

$\Rightarrow$ ~~u_i^2~~

Checking eq $J_1 - J_2 = \tau A^2 \Big|_{u_{i+1} = u_i}$

$\Rightarrow 2 + \frac{\tau}{4}\left(2u_i^2\right) = \frac{2\tau}{2} u_i^2 = \tau u_i^2$ Yes !!

Then eq 9.4.24 becomes: (or eq 9.4.21 which is easier to work w/)

~~$u_t + au$~~ $u_t + (u^2/2)_x = -\frac{\Delta x^2}{6}\frac{\partial}{\partial x}\left[(1-\tau^2 u^2)\right.$ ... actually 9.4.24

is easier

$$u_t + (u^2/2)_x = -\frac{\Delta x^2}{6}\Big[\left[u(1-\tau^2u^2)u_{xxx} - \left[(9\tau^2u^2 - 3)\right]u_x u_{xx}\right.$$

$$\left. + \left[(1-3\tau^2u^2)\cdot 0 - 6\tau^2 u\, u_x^3\right]\right]$$

$$= -\frac{\Delta x^2}{6} u(1-\sigma^2)u_{xxx} + \frac{\Delta x^2}{2}(3\sigma^2 - 1)u_x u_{xx}$$

$$+ \Delta x^2 \tau(\sigma) u_x^3 \qquad \text{eq } 9.4.30 \sim$$

The term $\frac{\Delta x^2}{2}(3\sigma^2 - 1) < 0$ when $|\sigma| < \frac{1}{\sqrt{3}}$

$|\tau u| < \frac{1}{\sqrt{3}}$

Nonlinear condition

want

~~$u_t + au_x$~~

$u_t + f_x = \kappa u_{xx}$

$\kappa > 0$ ✓

for stability.

pg 367 Hirsh Vol I

Prob 9.1 E9.2.2. has been derived see page 347

E8.3.25 is $\varepsilon_\phi = \pm \dfrac{\sin^{-1}(\sigma \sin\phi)}{\sigma\phi}$ See MMA for a taylor

expansion
$$= \pm\left(1 + \tfrac{1}{6}(\sigma^2-1)\phi^2 + O(\phi^4)\right)$$

Prob 9.2

Thus $|\varepsilon_\phi| < 1$ iff $|\sigma| < 1$

Thus we have a lagging error. From eq E9.2.2 we see that

$$u_t + a u_x = \underbrace{\frac{a\Delta x^2}{6}}(\sigma^2-1) u_{xxx} + O(\Delta x^4)$$

negative ⇒ lagging error (I think)

At least I know that u_{xxx} terms result in dispersion errors not dissipative errors.

Check lagging errors result in a ~~decrease~~ negative value of the coefficient of $\underline{u_{xxx}}$.

Prob 9.2 For upwind scheme:

9.2.2 is $u_t + a u_x = \dfrac{a\Delta x}{2}(1-\sigma) u_{xx} + \dfrac{a\Delta x^2}{6}(2\sigma-1)(1-\sigma) u_{xxx}$

By eq 8.5.14 $\qquad + \dfrac{a\Delta x^3}{24}(1-\sigma)(1+6\sigma^2-6\sigma) u_{xxxx} + O(\Delta x^4)$

$|G| = 1 - \frac{\sigma}{2}(1-\sigma)\phi^2 + O(\phi^4)$ eq E8.3.6 is full expression

$$\varepsilon_\phi = \frac{\tan^{-1}\left[\dfrac{(\sigma \sin\phi)}{1-\sigma+\sigma\cos\phi}\right]}{\sigma\phi} = \frac{1}{1-2\sigma} - \frac{(\sigma-1)\phi^2}{\sigma(2\sigma-1)^3}\phi^2 + O(\phi^4)$$

Why is $\varepsilon_\phi \neq 1 + O(\phi^2)$? There must be an error somewhere or else the scheme is not consistent.

We see from $|G|$ that for $|\sigma| < 1$ the coefficient of ϕ^2 is < 0 $\therefore$ method is stable. At the same time

$|\sigma|<1 \Rightarrow 1-\sigma > 0$ + the coefficient of u_{xx} is positive.

~~Eq~~ which is required for stability.

For the Lax-Friedrichs schemes:

eq 7.2.3 gives:

$$u_t + au_x = \frac{\Delta x^2}{2\Delta t}(1-\sigma^2)u_{xx} + \frac{a\Delta x^2}{3}(1-\sigma^2)u_{xxx} + O(\Delta x^3)$$

+ eqs $\quad$ eq E

$$\varepsilon_\phi \quad \{\phi = k\Delta x\}$$

$$|G| = (\cos^2\phi + \sigma\sin^2\phi)^{1/2} = 1 + \tfrac{1}{2}(\sigma-1)\phi^2 + O(\phi^4) \quad E\,8.3.3$$

$$\varepsilon_\phi = \frac{\tan^{-1}(\sigma\tan\phi)}{\sigma\phi} = 1 + \tfrac{1}{3}(1-\sigma^2)\phi^2 + O(\phi^4) \quad E\,8.3.5$$

Thus $|\sigma|<1$ for stability $\Rightarrow$ coefficient of u_{xx} is positive as required

$\Rightarrow \frac{1}{3}(1-\sigma^2) > 0 \Rightarrow$ ~~lagging~~ leading error + we see that

Coefficient of u_{xxx} $\frac{a\Delta x^2}{3}(1-\sigma^2) > 0 \Rightarrow$ leading error in numerical scheme.

From eq 9.2.1 the definition of a_k is the coefficient of the $\frac{\partial^k u}{\partial x^k}$'th term. & the Δx^{k-1} th power of Δx.

$\therefore$ a_4 is the coefficient of the Δx^3 powered term.

$\Rightarrow$ $m = 4$ in 1st term in 9.2.22 $\qquad$ $p = 1$ for Lax-Friedrich.

2nd term has powers of $\quad l + m - 1 = 3$

$m = 2 \Rightarrow l = 2$

$m = 3 \Rightarrow l = 1$

$m = 4 \Rightarrow l = 0$ not possible.

3rd term has powers of

$l + k + m - 1 = 3$

$m = 2 \qquad l + k = 2$

$l = 1 \quad k = 1$ only poss.

$m = 3 \qquad l + k = 1$

no possibilities.

Thus ~~by~~ expanding the a_4 th term from 9.2.22 gives.

$$u_t + a u_x \underset{\substack{4th\\term}}{\cong} \left(\frac{\Delta x}{\Delta t}\right)\left[\sum_j b_j j^4 - (-\sigma)^4\right] \frac{\Delta x^3}{4!} \frac{\partial^4 u}{\partial x^4}$$

$$- \frac{(-\sigma)^1}{1!} \Delta x^3 a_3 \frac{\partial^4 u}{\partial x^4} - \frac{(-\sigma)^2}{2!} \Delta x^3 a_2 \frac{\partial^4 u}{\partial x^4}$$

$$- \frac{\Delta t}{2\Delta x} \frac{(-\sigma)^0}{0!} \Delta x^3 a_2 a_2 \frac{\partial^4 u}{\partial x^4}$$

$$= \Delta x^3 \left\{ \left(\frac{\Delta x}{\Delta t}\right)\left[\sum_j b_j j^4 - (-\sigma)^4\right] \frac{1}{4!} + \sigma a_3 - \frac{\sigma^2}{2} a_2 - \frac{\Delta t}{2\Delta x} a_2^2 \right\} \frac{\partial^4 u}{\partial x^4}.$$

Now $\sum_j b_j j^4 = b_{-1} + b_1 = 1$ & w/ def of a_2 & a_3 on pg 353

$$a_2 = \frac{a(1-\sigma^2)}{2\sigma} \qquad a_3 = \frac{a(1-\sigma^2)}{3} \qquad \text{we have}$$

$$= \Delta x^3 \left\{ \left(\frac{\Delta x}{\Delta t}\right)\frac{(1-\sigma^4)}{4\cdot 3\cdot 2} + \frac{a\sigma(1-\sigma^2)}{3} - \frac{\sigma^2}{2}\frac{a(1-\sigma^2)}{2\sigma} - \left(\frac{\Delta t}{2\Delta x}\right)\frac{a^2(1-\sigma^2)^2}{4\sigma^2} \right\} \frac{\partial^4 u}{\partial x^4}$$

12-03-a 8

$$= \frac{\Delta x^3}{4} \Big\{ \left(\frac{\Delta x}{\Delta t}\right) \frac{(1-\sigma^4)}{6} + \frac{4a\sigma(1-\sigma^2)}{3} - \frac{\sigma a(1-\sigma^2)}{\cancel{2}}$$

$$- \sigma \ldots \Big\}$$

$$\sigma = \frac{a\Delta t}{\Delta x} \quad \therefore \left(\frac{\Delta t}{\Delta x}\right)^{-1} = \frac{a}{\sigma}$$

form $\underline{a_1}$

pg 367 Hirsh Vol I

12-04-01 1

Prob 9.3

$$u_i^{n+1} = u_i^{n-1} + \sum_j b_j u_{i+j}^n$$ b_j determined by numerical scheme.

Now: $$u_{i+j}^n = u_i^n + \sum_{m=1}^{\infty} \frac{(j\Delta x)^m}{m!}\left(\frac{\partial^m u}{\partial x^m}\right)$$

& $$u_i^{n+1} = u_i^n + \sum_{m=1}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m}$$

$$u_i^{n-1} = u_i^n + \sum_{m=1}^{\infty} \frac{(-1)^m \Delta t^m}{m!}\frac{\partial^m u}{\partial t^m}$$

$\Rightarrow$ we get

$$\cancel{u_i^n} + \sum_{m=1}^{\infty} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = \cancel{u_i^n} + \sum_{m=1}^{\infty} \frac{(-1)^m \Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} + \sum_j b_j \left(u_i^n + \sum_{m=1}^{\infty}\frac{(j\Delta x)^m}{m!}\frac{\partial^m u}{\partial x^m}\right)$$

$$\sum_{m=1}^{\infty} \frac{\Delta t^m}{m!}\left(1-(-1)^m\right)\frac{\partial^m u}{\partial t^m} = u_i^n \sum_j b_j + \sum_{m=1}^{\infty} \frac{\Delta x^m}{m!}\frac{\partial^m u}{\partial x^m}\sum_j j^m b_j$$

when m even $= 2,4,6,8,\dots$, we obtain $1-(-1)^m = 0$.

when m odd $= 1,3,5,7,\dots$ $1-(-1)^m = 2$

$$\Rightarrow 2\sum_{m=1,3,5,7,\dots} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = u_i^n \sum_j b_j + \sum_{m=1}^{\infty} \frac{\left(\sum_j j^m b_j\right)}{m!}\Delta x^m \frac{\partial^m u}{\partial x^m}$$

taking $\Delta x \to 0$ $\Delta t \to 0$ $\Rightarrow$ $\sum_j b_j = 0$ for consistency.

Releasing the 1st term ; Pg 367 Hirsch Vol I

$$\Rightarrow 2\Delta t\, u_t + 2\sum_{m=3,5,7,\ldots} \frac{\Delta t^m}{m!}\frac{\partial^m u}{\partial t^m} = u_i^n \sum_j b_j + \left(\sum_j j b_j\right)\Delta x\, u_x + \sum_{m=2}^{\infty} \frac{\left(\sum_j j^m b_j\right)}{m!}\frac{\partial^m u}{\partial x^m}\Delta x^m$$

$\Rightarrow \div$ by $2\Delta t$

$$\Rightarrow u_t - \frac{\Delta x}{2\Delta t}\sum_j j b_j\, u_x = \frac{u_i^n}{2\Delta t}\sum_j b_j + \sum_{m=2}^{\infty}\frac{\left(\sum_j j^m b_j\right)}{m!}\frac{\Delta x^m}{2\Delta t}\frac{\partial^m u}{\partial x^m} - \sum_{m=3,5,7,\ldots}\frac{\Delta t^{m-1}}{m!}\frac{\partial^m u}{\partial t^m}$$

let $\sigma \equiv \frac{a\Delta t}{\Delta x} \Rightarrow \frac{\Delta x}{\Delta t} = \frac{a}{\sigma}$ $\qquad \Delta t = \frac{\Delta x \sigma}{a}$ ✓

$$\Rightarrow u_t + a\left(\frac{-1}{2\sigma}\sum_j j b_j\right)u_x = \frac{u_i^n}{2\Delta t}\sum_j b_j + \sum_{m=2}^{\infty}\frac{\left(\sum_j j^m b_j\right)}{m!}\frac{\Delta x^m}{2\Delta t}\frac{\partial^m u}{\partial x^m} - \sum_{m=3,5,7,\ldots}\frac{\Delta t^{m-1}}{m!}\frac{\partial^m u}{\partial t^m}$$

Requiring: $\sum_j j b_j = -2\sigma$

Now: $u_t + a u_x = \sum_{l=p}^{\infty} \Delta x^l a_{l+1} \frac{\partial^{l+1} u}{\partial x^{l+1}}$ for an order p-scheme.

So that $\frac{\partial}{\partial t} = -a\frac{\partial}{\partial x} + \sum_{l=p}^{\infty} \Delta x^l a_{l+1}\frac{\partial^{l+1}}{\partial x^{l+1}}$

So as in eq 9.2.21

$$\frac{\partial^m u}{\partial t^m} = \left[-a\frac{\partial}{\partial x} + \sum_{l=p}^{\infty}\Delta x^l a_{l+1}\frac{\partial^{l+1}}{\partial x^{l+1}}\right]^m u$$ + eq 9.2.21 holds.

Now:

$$u_t + a u_x = \sum_{m=2}^{\infty} \frac{(\sum_j j^m b_j)}{m!} \frac{\Delta x^m}{2\Delta t} \frac{\partial^m u}{\partial x^m} - \sum_{m=3,5,7,\dots}^{\infty} \frac{\Delta t^{m-1}}{m!} (-a)^m \frac{\partial^m u}{\partial x^m}$$

$$\left\{ \begin{array}{l} \sigma = \frac{a \Delta t}{\Delta x} \\ a\Delta t = \Delta x \sigma \end{array} \right.$$

$$- \sum_{m=3,5,7,\dots}^{\infty} \frac{\Delta t^{m-1}}{m!} m(-a)^{m-1} \sum_{l=p}^{\infty} \Delta x^l a_{l+1} \frac{\partial^{l+m} u}{\partial x^{l+m}}$$

$$- \sum_{m=3,5,7,\dots}^{\infty} \frac{\Delta t^{m-1}}{m!} \frac{m(m-1)}{2} (-a)^{m-2} \sum_{l,k=p}^{\infty} (\Delta x)^{l+k} a_{l+1} a_{k+1} \frac{\partial^{l+k+m} u}{\partial x^{l+k+m}} + O(\Delta x^{3p})$$

$$= \frac{1}{2} \frac{\Delta x}{\Delta t} \sum_{m=2,4,6,8,\dots}^{\infty} \frac{(\sum_j j^m b_j)}{m!} \Delta x^{m-1} \frac{\partial^m u}{\partial x^m} + \frac{\Delta x}{2\Delta t} \sum_{m=3,5,7,\dots}^{\infty} \frac{(\sum_j j^m b_j)}{m!} \Delta x^{m-1} \frac{\partial^m u}{\partial x^m}$$

$$- \frac{1}{2\Delta t} \sum_{m=3,5,7,\dots}^{\infty} \frac{2}{m!} \Delta x^m \sigma^m (-1)^m \frac{\partial^m u}{\partial x^m}$$

$$- \sum_{m=3,5,7,\dots}^{\infty} \frac{\Delta x^{m-1} (-\sigma)^{m-1}}{(m-1)!} \sum_{l=p}^{\infty} \Delta x^l a_{l+1} \frac{\partial^{l+m} u}{\partial x^{l+m}}$$

$$- \frac{1}{2} \sum_{m=3,5,7,\dots}^{\infty} \frac{\Delta t^{-1} \Delta x^{m-2} (-\sigma)^{m-2}}{(m-2)!} \sum_{l,k=p}^{\infty} (\Delta x)^{l+k} a_{l+1} a_{k+1} \frac{\partial^{l+k+m} u}{\partial x^{l+k+m}} + O(\Delta x^{3p})$$

$$= \frac{1}{2}\frac{\Delta x}{\Delta t} \sum_{m=2,4,6,8,\dots}^{\infty} \frac{\left(\sum_j j^m b_j\right)}{m!} \Delta x^{m-1} \frac{\partial^m U}{\partial x^m}$$

$$+ \frac{\Delta x}{2\Delta t} \sum_{m=3,5,7,\dots}^{\infty} \frac{1}{m!}\left(\sum_j j^m b_j - 2(-\sigma)^m \right) \Delta x^{m-1} \frac{\partial^m U}{\partial x^m}$$

$$- \sum_{m=3,5,7,\dots}^{\infty} \frac{(-\sigma)^{m-1}}{(m-1)!} \sum_{l=p}^{\infty} \Delta x^{l+m-1} a_{l+1} \frac{\partial^{l+m} U}{\partial x^{l+m}}$$

$$- \frac{\Delta x}{2\Delta t} \sum_{m=3,5,7,\dots}^{\infty} \frac{(-\sigma)^{m-2}}{(m-2)!} \sum_{l,k=p}^{\infty} \Delta x^{l+k+m-1} a_{l+1} a_{k+1} \frac{\partial^{l+k+m} U}{\partial x^{l+k+m}} + \cdots$$

Settig R.H.S $= 0$ gives:

$$\sum_j j^m b_j = 0 \qquad m = 2, 4, 6, 8, \dots\ p \qquad m=0 \text{ works also}$$

$$\sum_j j^m b_j = 2(-\sigma)^m \qquad m = 3, 5, 7, \dots,\ p \qquad m=1 \text{ works also}$$

The leap frog scheme is 1st order $\Rightarrow$ $p=1$ From eq 8.3.33

$$U_i^{n+1} = U_i^{n-1} - \frac{a\Delta t}{\Delta x}\left(U_{i+1}^n - U_{i-1}^n\right) = U_i^{n-1} - \sigma U_{i+1}^n + \sigma U_{i-1}^n$$

$$= U_i^{n-1} + \sigma U_{i-1}^n - \sigma U_{i+1}^n$$

Then Check ~~p~~ $m=0$

$$b_{-1} + b_{+1} = \delta - \delta = 0 \quad ✓$$

Check $m=1$

$$-b_{-1} + b_{+1} = -\delta - \delta = -2\delta \stackrel{?}{=} 2(-\delta) \quad \text{Yes} ✓.$$

Then the coefficients of the expansion a_{p+1} are derived as ~~eq~~ ~~T.T.E.~~ 2.1 on pg 351

Thus $u_t + a u_x = \frac{1}{2} \frac{\Delta x}{\Delta t} \frac{\sum_j j^m b_j}{m!} \Delta x^{m-1} \frac{\partial^m u}{\partial x^m}\Big|_{m=2} + O(\Delta x^2)$

etc. expanding sums give full taylor expansion of the modified

D.E.

Prob 9.4 $\quad u_t = \alpha u_{xx} \qquad \alpha > 0$

$$u_i^{n+1} = \sum_j b_j u_{i+j}^n$$ taylor expanding both sides gives

$$\sum_{k=0}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^k u}{\partial t^k} = \sum_j b_j \sum_{k=0}^{\infty} \frac{(\Delta x\, j)^k}{k!}\frac{\partial^k u}{\partial x^k}$$

$$\Rightarrow \qquad = \sum_{k=0}^{\infty} \frac{\left(\sum_j b_j j^k\right)}{k!} \Delta x^k \frac{\partial^k u}{\partial x^k}$$

$$u + \Delta t\, u_t + \sum_{k=2}^{\infty} \frac{\Delta t^k}{k!}\frac{\partial^k u}{\partial t^k} = \left(\sum_j b_j\right) u + \left(\sum_j b_j j\right)\Delta x\, u_x + \left(\sum_j b_j j^2\right)\frac{\Delta x^2}{2} u_{xx} + \sum_{k=3}^{\infty} \frac{\sum_j b_j j^k}{k!}\Delta x^k \frac{\partial^k u}{\partial x^k}$$

For consistency $\quad 1 = \sum_j b_j \quad (1)$

$$0 = \sum_j j\, b_j \quad (2)$$

$$\Rightarrow \quad u_t - \underbrace{\sum_j b_j j^2 \left(\frac{\Delta x^2}{\Delta t}\right)\frac{1}{2} u_{xx}}_{-\alpha u_{xx}} + \sum_{k=2}^{\infty} \frac{\Delta t^{k-1}}{k!}\frac{\partial^k u}{\partial t^k} - \sum_{k=3}^{\infty} \frac{\sum_j b_j j^k}{k!}\frac{\Delta x^k}{\Delta t}\frac{\partial^k u}{\partial x^k} = 0$$

$$\alpha = \left(\frac{\Delta x}{\Delta t}\right)^2 \frac{1}{2} \sum_j b_j j^2 \qquad (3)$$

$$\therefore \quad \sum_j b_j j^2 = 2\alpha \frac{\Delta t}{\Delta x^2} \qquad (3')$$

Now:

$$u_t = \alpha u_{xx} - \sum_{k=2}^{\infty} \frac{\Delta t^{k-1}}{k!} \frac{\partial^k u}{\partial t^k} +$$

Assuming an eq of the form

$$u_t + \alpha u_{xx} = \sum_{l=p}^{\infty} \Delta x^l a_{l+1} \frac{\partial^{l+1} u}{\partial x^{l+1}}$$

we can follow the same ideas through eq 9.2.21

to obtain ...

pg 367 Hirsch Vol I

Prob 9.5 $u_t + a u_x = 0$

Von-Neumann amplification factors are:

1st order upwind:

$$G = 1 - 2\sigma \sin^2 \phi/2 - I\sigma \sin\phi \qquad 8.1.19$$

Lax-Friedrichs:

$$G = \cos\phi - I\sigma \sin\phi \qquad E8.3.2$$

Lax-Wendroff:

$$G = 1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi) \qquad E8.3.12$$

Then $|G|^2$ is for each method:

1st order upwind:

$$|G|^2 = 1 - 4\sigma(1-\sigma)\sin^2\phi/2 \qquad E8.3.6$$

Lax-Friedrichs:

$$|G|^2 = \cos^2\phi + \sigma^2 \sin^2\phi \qquad E8.3.4$$

Lax-Wendroff:

$$|G|^2 = 1 - 4\sigma^2(1-\sigma^2)\sin^4\phi/2 \qquad E8.3.14$$

w/ $z = \sin^2 \phi/2$

Methods ~~____~~ (amplification factor becomes)

1st order

$$|G|^2 = 1 - 4\sigma(1-\sigma)z$$

Lax-Friedrichs:

$$|G|^2 = 1 - \sin^2\phi + \sigma^2 \sin^2\phi$$

Note: $\cos(2\phi) = \cos^2\phi - \sin^2\phi$

$$= 1 - 2\sin^2\phi$$

$$\therefore |G|^2 = \cos^2\phi + \sigma^2(1 - \cos^2\phi)$$

$$= (1 - 2\sin^2\phi/2)^2 + \sigma^2(1 - (1 - 2\sin^2\phi/2)^2)$$

let $z = \sin^2\phi/2$

$$= (1-2z)^2 + \sigma^2(1 - (1-2z)^2)$$

$$= 1 - 4z + 4z^2 + \sigma^2(1 - (1 - 4z + 4z^2))$$

$$= 1 - 4z + 4z^2 + \sigma^2(4z - 4z^2)$$

$$= 1 - 4\left((1-\sigma^2)z - (1-\sigma^2)z^2\right) = 1 - 4(1-\sigma^2)z(1-z)$$

$$=$$

$$= 1 - z\{4(1-\sigma^2)(1-z)\}$$

Lax-Wendroff:

$$|G|^2 = 1 - 4\sigma^2(1-\sigma^2)z^2$$

$\square$

Thus

$$S_{upwind}(z) = 4\sigma(1-\sigma)z$$

$$S_{Lax-Friedrich}(z) = 4(1-\sigma^2)(1-z)$$

$$S_{Lax-Wendroff}(z) = 4\sigma^2(1-\sigma^2)$$ Because we factor z^2 out in def of $S(z)$

Pg 368 Hirsh Vol I

Prob 9.7

2nd order upwind Warming + Beam: $G = |G| e^{-I\Phi}$

$$G = 1 - 2\sigma\left[1-(1-\sigma)\cos\phi\right]\sin^2\phi/2 - I\sigma\sin\phi\left[1+2(1-\sigma)\sin^2\phi/2\right]$$

$$|G|^2 = \left(1-2\sigma(1-(1-\sigma)\cos\phi)\sin^2\phi/2\right)^2 + \sigma^2\sin^2\phi\left(1+2(1-\sigma)\sin^2\phi/2\right)^2 = \varepsilon_D$$

phase error: $\varepsilon_\phi = \dfrac{\Phi}{}$

For linear scheme $u_t + au_x = 0$ Fourier decomposing gives (in space) Assuming a bounded domain $(0,L)$

$$u(x,t) = \sum_{n=-\infty}^{\infty} U_n(t) e^{inx} \text{ on } (0,2\pi) \qquad \left(\frac{2\pi}{L}\right)n$$

$$= \sum_{n=-\infty}^{+\infty} U_n(t) e^{i\left(\frac{2\pi}{L}n\right)x}$$ Thus putting this into (1) gives

$$\sum_{n=-\infty}^{+\infty}\left\{U_n'(t)e^{i\left(\frac{2\pi}{L}n\right)x} + aU_n(t)\,i\left(\frac{2\pi}{L}\right)n\,e^{i\left(\frac{2\pi}{L}n\right)x}\right\} = 0$$

$\div\ e^{i\left(\frac{2\pi}{L}n\right)}$ we obtain

$\Rightarrow$

$$U_n'(t) + ai\left(\frac{2\pi}{L}\right)n\,U_n(t) = 0$$

$$\Rightarrow U_n'(t) = -a\left(\frac{2\pi}{L}\right)ni\,U_n(t) \Rightarrow \frac{dU_n}{U_n} = -a\left(\frac{2\pi}{L}\right)ni\,dt$$

$$V_n(t) = V_n^0 e^{-i\left(\frac{2\pi}{L}\right)ant}$$

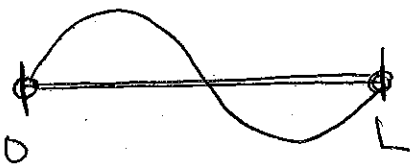
$\therefore$ After 1 timestep of Δt $\quad V_n(t+\Delta t) = V_n^0 e^{-i\left(\frac{2\pi}{L}\right)ant} \cdot e^{-i\left(\frac{2\pi}{L}\right)an\Delta t}$

$$= V_n(t) e^{-i\left(\frac{2\pi}{L}\right)an\Delta t}$$

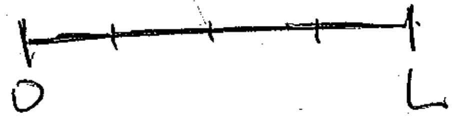
$\therefore \quad \Phi = \left(\frac{2\pi}{L}\right)an\Delta t \qquad$ w/ $\quad \frac{2\pi}{L}n \equiv k_n$ wave #.

$$\Phi = k a \Delta t$$

$k_1 = \frac{2\pi}{L}$



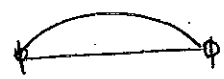
$k_2 = \frac{2\pi}{L} \cdot 2$



$\therefore \quad \varepsilon_\phi \equiv \dfrac{\Phi}{ka\Delta t}$

"$\sin(k_n x)$"

where is



mode?

$[k] = \frac{1}{L}$

Thus $\quad G = |G| e^{-i\Phi}$

w/ $\quad \Phi = -\tan^{-1}\left[\dfrac{-\sigma \sin\phi\left(1+2(1-\sigma)\sin^2\phi/2\right)}{1-2\sigma(1-(1-\sigma)\cos\phi)\sin^2\phi/2}\right]$

$$= \tan^{-1}\left[\dfrac{\sigma \sin\phi\left(1+2(1-\sigma)\sin^2\phi/2\right)}{1-2\sigma(1-(1-\sigma)\cos\phi)\sin^2\phi/2}\right]$$

$\varepsilon_\phi = \dfrac{\Phi}{\sigma k\Delta x} \qquad k\Delta x \equiv \phi \qquad \phi \in (0, 2\pi)$

$$= \frac{\Phi}{\sigma\phi}$$

$1 - \tau = \frac{1}{\tau}$

$\tau - \tau^2 = 1$

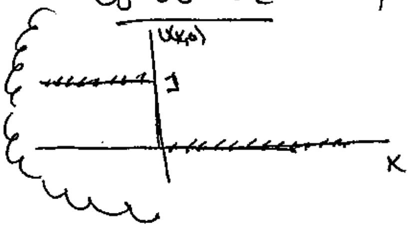
$\tau^2 - \tau + 1 = 0$

pg 368 Hirsch Vol I

08-06-02

1

Prob 9.9 Solve $u_t + a u_x = 0$ w/ $u(x,0) = \begin{cases} 1 & x<0 \\ 0 & x>0 \end{cases}$



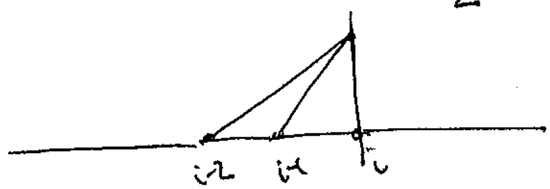
$n = 10, 50, 100$ $\qquad \Delta x = \frac{1}{60}$ & $a = 1$ w/ $\sigma \Rightarrow \Delta t$

$\sigma = .1, .5, .9$ $\qquad \sigma = \dfrac{a\Delta t}{\Delta x}$

upwind: 8.1.19

2nd order upwind:

$$U_i^{n+1} = \frac{\sigma}{2}(\sigma-1)U_{i-2}^n + \sigma(2-\sigma)U_{i-1}^n + \left(1 - \frac{3\sigma}{2} + \frac{\sigma^2}{2}\right)U_i^n$$



w/ (1) Upwind 1st order
(2) Lax-Friedrich
(3) Lax-Wendroff
(4) 2nd order upwind

Problem only discretize out to 3 but plot to 6.

~~Max~~ distance shock travels is $n\,\Delta t$ & ~~$n = n_{max}$~~

$$\cancel{= n_{max}\Delta t = n_{max}\left(\frac{\Delta x}{a}\right)\sigma = n_{max}\left(\frac{1}{60}\right)\frac{1}{(1)}}$$

$$\leq n_{max}\Delta t = n_{max}\,\sigma\frac{\Delta x}{a} \leq n_{max}\,\sigma_{max}\left(\frac{\Delta x}{a}\right) = \frac{n_{max}\,\sigma_{max}}{60}$$

$$= \frac{200(.9)}{60} = \underline{\underline{3}}$$