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$$u\frac{\partial \rho}{\partial t} + u\nabla\cdot(\rho\bar{v}) = 0$$

1.1.7 $\Rightarrow$ $$u\cancel{\frac{\partial \rho}{\partial t}} + \rho\frac{\partial u}{\partial t} + \cancel{u\nabla\cdot(\rho\bar{v})} + \nabla u\cdot\rho\bar{v}$$

$$= \nabla\cdot(k\rho\nabla u) + Q_v + \nabla\cdot\bar{Q}_s$$

Subtract

$$\rho\frac{\partial u}{\partial t} + \rho\bar{v}\cdot\nabla u = \nabla\cdot(k\rho\nabla u) + Q_v + \nabla\cdot\bar{Q}_s$$

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$$\frac{\partial\bar{\xi}}{\partial t} - \nabla\times(\bar{v}\times\bar{\xi}) = -\nabla\times\left(\frac{1}{\rho}\nabla p\right) + \nabla\times\left(\frac{1}{\rho}\nabla\cdot\bar{\bar{\tau}}\right) + \nabla\times\bar{f}_e$$

$$\|$$

$$-\left[(\bar{\xi}\cdot\nabla)\bar{v} - (\bar{v}\cdot\nabla)\bar{\xi} + \bar{v}\cancelto{0}{(\nabla\cdot\bar{\xi})} - \bar{\xi}(\nabla\cdot\bar{v})\right]$$

$$\Rightarrow \frac{\partial\bar{\xi}}{\partial t} + (\bar{v}\cdot\nabla)\bar{\xi} = (\bar{\xi}\cdot\nabla)\bar{v} - \bar{\xi}(\nabla\cdot\bar{v}) - \nabla\left(\frac{1}{\rho}\right)\times\nabla p - \cancelto{0}{\frac{1}{\rho}\nabla\times\nabla p}$$

$$\|$$

$$+\nabla p\times\nabla\left(\frac{1}{\rho}\right)$$

Now

$\nabla \times \left(\frac{1}{\rho} \nabla \circ \bar{\bar{\tau}}\right) =$

$\tau_{ij} = \mu(\partial_i v_j + \partial_j v_i)$

$\nabla \circ \bar{\bar{\tau}} =$

$\sum_i \partial_i \tau_{ij}$

$\nabla \circ \bar{\bar{\tau}} \stackrel{?}{=} \mu(\partial_i^2 v_j + \partial_{ij}^2 v_i)$

$\frac{1}{\rho} \nabla \circ \bar{\bar{\tau}} = \nu(\partial_i^2 v_j + \partial_{ij}^2 v_i)$

$\nabla \times \left(\frac{1}{\rho} \nabla \circ \bar{\bar{\tau}}\right) = \epsilon_{ijk}\, \nu\left(\partial_l^2 v_k + \partial_{lk}^2 v_l\right)_{,j}$

$\boxed{\nabla \times \bar{A} = \epsilon_{ijk} A_{k,j}}$

$= \epsilon_{ijk}\, \nu\left(\partial_{llj}^3 v_k + \partial_{lkj}^3 v_l\right) = \nu\, \epsilon_{ijk}\left(v_{k,llj} + v_{l,lkj}\right)$

Check

$\cancel{\nabla^2 \xi = \nabla^2(\nabla \times \bar{v})}$

$\cancel{= \nabla^2(\epsilon_{ijk} v_{k,j}) = \epsilon_{ijk}}$

$\partial_{ij}^2 v_i = \partial_j(\cancel{\partial_i v_i})$ 0

$\therefore$

$\frac{1}{\rho} \nabla \circ \bar{\bar{\tau}} = \nu\, \partial_i^2 v_j$

$\therefore \nabla \times \frac{1}{\rho} \nabla \circ \bar{\bar{\tau}} = \nu\, \partial_i^2 (\nabla \times \bar{v})$ same ✓

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1.3.8 $\rightarrow$ $$\rho \frac{d\vec{v}}{dt} = -\nabla p + \nabla \circ \bar{\bar{\tau}} + \rho \vec{f}_e$$

$$\rightarrow \rho \partial_t \vec{v} + \rho(\vec{v} \circ \nabla)\vec{v} = -\nabla p + \nabla \circ \bar{\bar{\tau}} + \rho \vec{f}_e$$

$$\partial_t \vec{v} + \underbrace{(\vec{v} \circ \nabla)\vec{v}}_{} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e.$$

$$\nabla\left(\frac{v^2}{2}\right) - \vec{v} \times (\nabla \times \vec{v})$$

gives

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \times \vec{\xi} = -\nabla\left(\frac{v^2}{2}\right) - \frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e$$

eq 1.3.13 $\nabla \times \rightarrow$

$$\partial_t \vec{\xi} - \nabla \times (\vec{v} \times \vec{\xi}) = -\nabla \times \left(\frac{1}{\rho}\nabla p\right) + \nabla \times \left(\frac{1}{\rho}\nabla \circ \bar{\bar{\tau}}\right) + \nabla \times \vec{f}_e$$

$$\partial_t \vec{\xi} - [$$

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Fm

$$\frac{\partial \rho}{\partial t} + \nabla \circ (\rho \vec{V}) = 0 \qquad \text{get} \quad \frac{\partial \rho}{\partial t} + \nabla \circ (\rho \vec{W}) = 0.$$

$$\vec{V} = \vec{W} + \vec{U} = \vec{W} + \vec{\omega} \times \vec{r}.$$

Then $\nabla \circ (\rho \vec{V}) = \nabla \circ (\rho \vec{W} + \rho \vec{\omega} \times \vec{r})$

$$= \nabla \circ (\rho \vec{W}) + \nabla \rho \circ \vec{\omega} \times \vec{r} +$$

But $w_j \partial_{x_j} w_i$ is $(\vec{W} \cdot \nabla)\vec{w}$ in vector form

$$\Rightarrow \nabla\left(\frac{W^2}{2}\right) - \vec{W} \times (\nabla \times \vec{W})$$

$\therefore$ ith comp $\Rightarrow$ $\partial_{x_i}\left(\frac{W^2}{2}\right) - (\vec{W} \times \nabla \times \vec{W})_i$

$\Rightarrow$ we get

$$W_{i_t} + \left(\underline{\nabla\left(\frac{W^2}{2}\right)} - \vec{W} \times (\nabla \times \vec{W})\right)_i = f_{e_i} - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))_i$$
$$- 2(\vec{\omega} \times \vec{W})_i$$
$$- \underline{\frac{1}{\rho}} \partial_{x_i} p + \underline{\frac{1}{\rho}} \partial_{x_j} \tau_{ij}$$

~~Almost if~~

~~I can show~~

$$\vec{\xi} = \nabla \times (\vec{W} + \vec{\omega} \times \vec{r})$$
$$= \nabla \times \vec{W} + \nabla \times (\vec{\omega} \times \vec{r})$$
$$= \nabla \times \vec{W} +$$

$$\Rightarrow (\vec{W}_t)_i + - (\vec{W} \times \nabla \times \vec{W})_i = f_{e_i}$$

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$$-\frac{1}{\rho}\nabla p - \nabla\left(\frac{w^2}{2} - \frac{u^2}{2}\right) + \frac{1}{\rho}\nabla\circ\bar{\bar{\tau}} + \bar{f}_e$$

$$\frac{\partial\rho}{\partial t}\bar{w} + \rho\frac{\partial\bar{w}}{\partial t} + \cdot$$

$\bar{\xi} = \nabla\times\bar{V}$

$\bar{V} = \bar{w} + \bar{\underline{\underline{u}}}$

$\bar{w} + \bar{\omega}\times\bar{r}$

ith comp.

$$\frac{\partial\rho}{\partial t}w_i + \rho\frac{\partial w_i}{\partial t} + (\rho w_i w_j)_{,j}$$

$$= \rho\bar{f}_e - \rho\left(-\nabla\left(\tfrac{1}{2}(\bar{\omega}\times\bar{r})^2\right)\right) - 2\rho(\bar{\omega}\times\bar{w}) - \nabla p + \nabla\circ\bar{\bar{\tau}}$$

$$\Rightarrow \cancel{\frac{\partial\rho}{\partial t}w_i} + \rho\frac{\partial w}{\partial t} + \cancel{(\rho w_j)_{,j}}\, w_i + w_{i,j}\rho w_j = \cdots$$

$$= \cdots$$

$$\Rightarrow \frac{\partial w_i}{\partial t} + w_{i,j}w_j = f_{e_i} - \nabla\left(-\frac{u^2}{2}\right)\circ\hat{e}_i - \frac{\nabla p}{\rho}\circ\hat{e}_i + \frac{1}{\rho}\nabla\circ\bar{\bar{\tau}}\circ\hat{e}_i - 2(\bar{\omega}\times\bar{w})$$

All terms correct except ~~$w_{i,j}w_j$~~

$$w_{i,j}w_j = -2\epsilon_{ijk}\omega_j w_k$$

$\xi_i \epsilon$

$\bar{w} \times \bar{\xi} = \epsilon_{ijk} w_j \xi_k$

$\vec{\xi} = \nabla \times \bar{v}$

$\xi_k = \epsilon_{k\ell n} v_{n,\ell}$

$\epsilon_{ijk}\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$

$\therefore \quad \bar{w} \times \bar{\xi} = \epsilon_{ijk}\epsilon_{k\ell n} w_j v_{n,\ell}$

$= -\epsilon_{kij}\epsilon_{k\ell n} w_j v_{n,\ell}$

$= -[\delta_{i\ell}\delta_{jn} - \delta_{in}\delta_{j\ell}] w_j v_{n,\ell}$

$= \underline{-w_n v_{n,i} + v_{i,\ell} w_\ell}$

Add this term to the LHS

$\therefore$ ~~$\epsilon \cdot$ $\frac{\partial w}{}$~~ Right hand side is correct ~~except~~ if ξ I

can make

$-w_n v_{n,i} + v_{i,\ell} w_\ell + -2\epsilon_{ijk} \omega_j w_k$

$\bar{\omega} \to \bar{\Omega}$

$$\frac{\partial}{\partial t}(\rho\bar{\omega}) + \nabla\circ(\rho\bar{\omega}\otimes\bar{\omega}) = \rho\bar{f}_e - \rho(\bar{\Omega}\times(\bar{\Omega}\times\bar{r}))$$

$$-2\rho(\bar{\Omega}\times\bar{\omega}) - \nabla p + \nabla\circ\bar{\bar{\tau}}$$

+ cont

$$\frac{\partial\rho}{\partial t} + \nabla\circ(\rho\bar{\omega}) = 0$$

$$\rho_t + (\rho\omega_i)_{,i} \qquad \bar{\Omega}\times(\bar{\Omega}\times\bar{r}) = -\nabla\left(\tfrac{1}{2}(\bar{\Omega}\times\bar{r})^2\right)$$

$$\cancel{\frac{\partial\rho}{\partial t}}\omega_i + \rho\frac{\partial\omega_i}{\partial t} + (\rho\omega_i\omega_j)_{,j} = \rho f_{ei} + \rho\nabla\left(\tfrac{1}{2}(\bar{\Omega}\times\bar{r})^2\right) - 2\rho(\bar{\Omega}\times\bar{\omega})$$

$$\| \qquad -\nabla p + \nabla\circ\bar{\bar{\tau}}$$

$$\cancel{\rho\omega_{j,j}}\omega_i$$

$$+\rho\omega_j\omega_{i,j}$$

$\cancel{\Omega\times\omega\times\Omega\times r}$

$\div\rho$

$$\frac{\partial\omega_i}{\partial t} + \omega_j\omega_{i,j} = f_{ei} + \nabla\left(\tfrac{1}{2}\bar{v}^2\right) - 2(\bar{\Omega}\times\bar{\omega}) - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\circ\bar{\bar{\tau}}$$

$$\bar{\xi} = \nabla\times\bar{v} = \epsilon_{ijk}v_{k,j}$$

$$\bar{\omega}\times\bar{\xi} = \epsilon_{ikl}\omega_k\epsilon_{lmn}v_{n,m} = \epsilon_{ikl}\epsilon_{lmn}\omega_k v_{n,m}$$

$$\epsilon_{ikl}\epsilon_{lmn} = -\epsilon_{lik}\epsilon_{lmn} = -\delta_{im}\delta_{kn} + \delta_{in}\delta_{km}$$

$$\text{“}\epsilon_{\underline{i}jk}\epsilon_{\underline{i}rs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}\text{”}$$

$$\therefore \quad \bar{\omega} \times \bar{\xi} = -\delta_{im}\delta_{kn}\omega_k V_{n,m} + \delta_{in}\delta_{km}\omega_k V_{n,m}$$

$$= -\omega_n V_{n,i} + \omega_m V_{i,m}$$

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$$\gamma_p \kappa \nabla e = k \nabla T$$

$$c_p \rho \kappa \nabla\left(\frac{e}{c_v}\right) = k \nabla T$$

$$\frac{\mu}{\rho} = \nu$$

if $T = \frac{e}{c_v}$ $\quad$ $\underline{k = \rho c_p \kappa}$

$$\rho = \frac{\mu}{\nu}$$

$$k = \frac{\mu c_p}{(\nu/\kappa)} = \frac{\mu c_p}{Pr} \quad \text{if } Pr = \nu/\kappa$$

i.e. $\underline{e = T c_v}$ ✓.

$$Q_v = \rho \bar{f}_e \circ \bar{v} + q_H$$

$$\bar{Q}_s = \bar{\bar{\sigma}} \circ \bar{v} = -p\bar{v} + \bar{\bar{\tau}} \circ \bar{v}$$

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$\nabla \circ (\bar{\bar{\sigma}} \circ \bar{v})$ $\qquad \bar{\bar{\sigma}} = -p\bar{\bar{I}} + \bar{\bar{\tau}}$

$= \nabla \circ (-p\bar{v} + \bar{\bar{\tau}} \circ \bar{v})$

$\Rightarrow$ 1.5.9 becomes

$\cancel{\frac{\partial}{\partial t}(\rho E)}$ $\quad \frac{\partial}{\partial t}(\rho E) + \nabla \circ (\rho \bar{v} E + p\bar{v} - \bar{\bar{\tau}} \circ \bar{v} - k\nabla T)$

$= W_f + q_H$

$\frac{\partial}{\partial t}(\rho E) + \nabla \circ (\bar{v} (\rho E +$

$\nabla \circ \left(\rho \bar{v}\left(E + \frac{p}{\rho}\right) - k\nabla T - \bar{\bar{\tau}} \circ \bar{v}\right)$

$H =$

Pg 19 High

$$\frac{\partial}{\partial t}\int_{\Omega} \rho E \, d\Omega + \underbrace{\int_{S} \rho E \bar{v} \circ d\bar{S}}_{\text{convection flux}} = \underbrace{\oint_{S} k \nabla T \circ d\bar{S}}_{\text{diff. flux}} + \int_{\Omega} (\rho \bar{f}_e \circ \bar{v} + q_H)$$

$$+ \int_{S} (\bar{\bar{\sigma}} \circ \bar{v}) \circ d\bar{S}$$

1.5.9

$$\Rightarrow \frac{\partial}{\partial t}(\rho E) + \nabla \circ (\rho \bar{v} E + p \bar{v} - \bar{\bar{\tau}} \circ \bar{v} - k \nabla T) = W_f + q_H$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho \underset{\substack{\parallel \\ e + \frac{\bar{v}^2}{2}}}{E}) + \nabla \circ (\rho \bar{v} \underbrace{(E + \frac{p}{\rho})}_{H = h + \frac{\bar{v}^2}{2}} - k \nabla T - \bar{\bar{\tau}} \circ \bar{v}) = W_f + q_H$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho e + \rho \frac{\bar{v}^2}{2}) + \nabla \circ (\rho \bar{v} h + \rho \bar{v} \frac{\bar{v}^2}{2})$$

$$= \nabla \circ (k \nabla T) + q_H + W_f + \nabla \circ (\bar{\bar{\tau}} \circ \bar{v})$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho e) + \nabla \circ (\rho \bar{v} h) = \nabla \circ (k \nabla T) + q_H + W_f + \nabla \circ (\bar{\bar{\tau}} \circ \bar{v})$$

$$- \frac{\partial}{\partial t}(\rho \frac{v^2}{2}) - \nabla \circ (\rho \bar{v} \cancel{\frac{v^2}{2}})$$

<u>RHS</u>: w/o $\nabla\circ(k\nabla T)$ + q_H terms

mmt eq $\bar{v}\circ \rightarrow$

$$\bar{v}\circ\frac{\partial}{\partial t}(\rho\bar{v}) + \bar{v}\circ\nabla\circ(\rho\bar{v}\otimes\bar{v} + p\bar{\bar{I}} - \bar{\bar{\tau}}) = \underbrace{\rho\bar{f}_e\circ\bar{v}}_{W_f}$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\,\bar{v}\circ\bar{v}) - \rho\bar{v}\circ\frac{\partial\bar{v}}{\partial t}$$

$$+ \bar{v}\circ\nabla\circ(\rho v_i v_j + p\delta_{ij} - \mu(\partial_i v_j + \partial_j v_i)) = W_f$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho v_i^2) - \rho v_i\frac{\partial v_i}{\partial t} + v_i\partial_j(\rho v_i v_j + p\delta_{ij} - \mu(\partial_i v_j + \partial_j v_i))$$

$$= W_f$$

Prob 1.1 get 1.4.7

$$\frac{\partial \vec{w}}{\partial t} - (\vec{w} \times \vec{\xi}) = -\frac{1}{\rho}\nabla p - \nabla\left(\frac{w^2}{2} - \frac{u^2}{2}\right) + \frac{1}{\rho}\nabla \circ \vec{\vec{\tau}} + \vec{f}_e$$

write conservation mont in non conservation form

$$\frac{\partial}{\partial t}(\rho\vec{w}) + \nabla \circ (\rho\vec{w} \otimes \vec{w}) = \rho\vec{f}_e - \rho\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$-2\rho(\vec{\omega} \times \vec{w}) - \nabla p + \nabla \circ \vec{\vec{\tau}}$$ is eq 1.4.6

$\Rightarrow$ ~~$\rho_t \vec{w} + \rho \vec{w}_t$~~ ith component.

$$\rho_t w_i + \rho w_{i_t} + \partial_{x_j}(\rho w_j w_i) = \rho f_{ei} - \rho(\vec{\omega} \times (\vec{\omega} \times \vec{r}))_i$$

$$-2\rho(\vec{\omega} \times \vec{w})_i - \partial_{x_i} p + (\nabla \circ \vec{\vec{\tau}})_i$$

$$\Rightarrow \rho_t w_i + \rho w_{i_t} + \cancel{\partial_{x_j}(\ldots)}$$

$$+ \rho w_j \partial_{x_j} w_i + w_i \partial_{x_j}(\rho w_j)$$

$$= \rho f_{ei} - \rho(\vec{\omega} \times (\vec{\omega} \times \vec{r}))_i - 2\rho(\vec{\omega} \times \vec{w})_i - \partial_{x_i} p + \partial_{x_j} \tau_{ji}$$

But $w_i(\rho_t + \partial_{x_j}(\rho w_j)) = 0$

$\div$ by ρ.

$$\Rightarrow w_{i_t} + w_j \partial_{x_j} w_i = f_{ei} - (\vec{\omega} \times (\vec{\omega} \times \vec{r}))_i - 2(\vec{\omega} \times \vec{w})_i - \frac{1}{\rho}\partial_{x_i} p + \frac{1}{\rho}\partial_{x_j}\tau_{ji}$$

Prob 1.3

energy eq 1.5.14 is w/ incompressible fluid ⇒ ρ constant

$$\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho\bar{v}h) = (\bar{v}\cdot\nabla)p + \epsilon_v + q_H + \nabla\cdot(k\nabla T)$$

Assuming $e = c_v T$ {when is this true? Ideal gases only correct?}

$$\Rightarrow \rho c_v \frac{\partial T}{\partial t} + \nabla\cdot(\bar{v}(\rho e + P)) = (\bar{v}\cdot\nabla)P + q_H + \nabla\cdot(k\nabla T)$$

↑

$\{h = e + \frac{P}{\rho} \Rightarrow \rho h = \rho e + P$

+ Dropping ϵ_v (why?)

For fluid at rest $\bar{v} = 0$ ⇒

$$\rho c_v \frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T) + q_H$$

Problem 1.7

$$T\nabla S = \nabla h - \frac{\nabla p}{\rho}$$

(1.3.13) $\Rightarrow$

$$\frac{\partial \bar{v}}{\partial t} - (\bar{v} \times \bar{\xi}) = \underbrace{-\frac{1}{\rho}\nabla p}_{T\nabla S - \nabla h} - \nabla\left(\frac{v^2}{2}\right) + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e$$

Defining $H = e + \frac{p}{\rho} + \frac{v^2}{2} = h + \frac{v^2}{2} = E + \frac{p}{\rho}$ As total enthalpy.

$$\Rightarrow \frac{\partial \bar{v}}{\partial t} - (\bar{v} \times \bar{\xi}) = T\nabla S - \underbrace{\nabla\left(h + \frac{v^2}{2}\right)}_{\nabla H} + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \bar{f}_e$$

+ replacing $\frac{\nabla P}{\rho}$ in the rel form of the momentum.

from (1.4.7)

$$\frac{\partial \bar{w}}{\partial t} - (\bar{w} \times \bar{\xi}) = T\nabla S - \nabla h - \nabla\left(\frac{w^2}{2} - \frac{u^2}{2}\right) + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e$$

$$= T\nabla S - \underbrace{\nabla\left(h - \frac{u^2}{2}\right)}_{} - \nabla \frac{w^2}{2} + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \bar{f}_e$$

$$\frac{\partial \vec{w}}{\partial t} - (\vec{w} \times \vec{\xi}) = T\nabla S - \nabla\underbrace{\left(h + \frac{w^2}{2} - \frac{U^2}{2}\right)}_{I} + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e$$

Define $I = h + \dfrac{w^2}{2} - \dfrac{U^2}{2} = H - \vec{U} \circ \vec{V}$

$$\Rightarrow \quad \frac{\partial \vec{w}}{\partial t} - (\vec{w} \times \vec{\xi}) = T\nabla S - \nabla I + \frac{1}{\rho}\nabla \circ \bar{\bar{\tau}} + \vec{f}_e$$

Prob 2.1 obtain (1.5.14) ⇒

$$\frac{\partial}{\partial t}(\rho e) + \nabla\circ(\rho\vec{v}h) = (\vec{v}\circ\nabla)p + \varepsilon_v + q_H + \nabla\circ(k\nabla T)$$

w/ $\varepsilon_v = \tau_{ij}\frac{\partial v_i}{\partial x_j} = \frac{1}{2\mu}(\bar{\bar{\tau}} \otimes \bar{\bar{\tau}}^T) = (\bar{\bar{\tau}}\circ\nabla)\circ\vec{v}$

considering ~~(1.3.8) multiplied by~~ $\vec{v}\circ$

$$\cancel{\rho(\vec{v}_t + (\vec{v}\circ\nabla)\vec{v}) = -\nabla p + \nabla\circ\bar{\bar{\tau}} + \rho\vec{f}_e}$$

$\vec{v}\circ$

$$\cancel{\rho(\vec{v}\circ\vec{v}_t + \vec{v}\circ(\vec{v}\circ\nabla)\vec{v}) = -\vec{v}\circ\nabla p + \vec{v}\circ\nabla\circ\bar{\bar{\tau}} + \rho\vec{v}\circ\vec{f}_e}$$

(1.5.9) is $\frac{\partial}{\partial t}\Big(\rho\underbrace{\Big(e + \frac{v^2}{2}\Big)}_{\text{take } E = e + \frac{v^2}{2}}\Big) + \nabla\circ\Big(\rho\vec{v}\underbrace{\Big(h + \frac{v^2}{2} - \frac{p}{\rho}\Big)}_{\text{take } E = h + \frac{v^2}{2} - \frac{p}{\rho}}\Big)$

$$= \nabla\circ(k\nabla T) + \nabla\circ(-p\bar{\bar{I}} + \bar{\bar{\tau}}) + \rho\vec{f}_e\circ\vec{v} + q_H$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho e) + \nabla\circ(\rho\vec{v}h) = -\frac{\partial}{\partial t}\Big(\frac{v^2}{2}\rho\Big) - \nabla\circ\Big(\rho\vec{v}\Big(\frac{v^2}{2} - \frac{p}{\rho}\Big)\Big)$$

$$+ \nabla\circ(k\nabla T) - \nabla p + \nabla\circ\bar{\bar{\tau}} + \rho\vec{f}_e\circ\vec{v} + q_H$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho e) + \nabla\circ(\rho\vec{v}h) - q_H - \nabla\circ(k\nabla T) = -\frac{\partial}{\partial t}\Big(\frac{v^2}{2}\rho\Big) - \nabla\circ\Big(\rho\vec{v}\frac{v^2}{2}\Big)$$

$$+ \nabla\circ(\vec{v}p) - \nabla p + \nabla\circ\bar{\bar{\tau}} + \rho\vec{f}_e\circ\vec{v}$$

~~Now - Mmt eq is~~

$$\cancel{\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + \rho \vec{v} \circ \vec{f_e} =}$$

considering 1.3.7 multiplied by $\vec{v}$ we get

$$\vec{v} \circ \frac{\partial}{\partial t}(\rho \vec{v}) + \vec{v} \circ \nabla \left(\rho \vec{v} \otimes \vec{v} + p\bar{\bar{I}} - \bar{\bar{\tau}}\right) = \rho \vec{v} \circ \vec{f_e}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho \frac{v^2}{2}\right) -$$

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2} \vec{v} \circ \vec{v}\right) = \frac{1}{2} \frac{\partial}{\partial t}\left(\rho \vec{v} \circ \vec{v}\right) = \frac{1}{2} \frac{\partial}{\partial t}(\rho v) \circ \vec{v} + \frac{1}{2} \rho \vec{v} \circ \frac{\partial \vec{v}}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho \vec{v} \circ \vec{v}) = \partial_t (\rho \vec{v}) \circ \vec{v} + \rho \vec{v} \circ \partial_t \vec{v}$$

$$\Rightarrow \vec{v} \circ \partial_t (\rho \vec{v}) = \partial_t (\rho \vec{v}^2) - \rho \vec{v} \circ \partial_t \vec{v}.$$

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Ideal gas:

$$e = C_v T = \frac{C_v}{r}\frac{P}{\rho}$$

From 2.1.12 + 2.1.13

$$\frac{C_p}{r} = \frac{\gamma}{\gamma-1}$$

$$\frac{C_v}{C_p}\cdot\frac{C_p}{r} = \frac{\gamma}{\gamma-1}\cdot\frac{1}{\gamma} = \frac{1}{\gamma-1}$$

$$\therefore e = \frac{1}{\gamma-1}\frac{P}{\rho}$$

$h = C_p T$ def on enthapy per unit mass.

$$= \frac{C_p}{r}\frac{P}{\rho} = \frac{\gamma}{\gamma-1}\frac{P}{\rho}$$

eq 1.5.16 is:

$$T ds = de + p\,d(1/\rho) = dh - \frac{dp}{\rho}$$

Using $ds = \frac{dh}{T} - \frac{dp}{T\rho}$ + EOS $P = r\rho T$

+ $h = C_p T$

$$\Rightarrow ds = C_p\frac{dT}{T} - r\frac{dp}{P}$$

integrating

$$S - S_A = C_p \ln(T/T_A) - r\ln(P/P_A)$$

$$S - S_A = -r\left[-\frac{C_p}{r}\ln(T/T_A) + \ln(P/P_A)\right]$$

But $\frac{C_p}{r} = \frac{\gamma}{\gamma-1}$ Thus $S - S_A = -r\left[\ln\left(\frac{T_A}{T}\right)^{\frac{\gamma}{\gamma-1}} + \ln\left(\frac{P}{P_A}\right)\right]$

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$$\Rightarrow \quad S - S_A = -r\left[\ln\left(\frac{P}{P_A}\cdot\left(\frac{T_A}{T}\right)^{\frac{r}{r-1}}\right)\right]$$

$$= -r\ln\frac{(P/P_A)}{(T/T_A)^{\frac{r}{r-1}}}$$

Use E.O.S. to remove T dependence

$$\frac{P}{P_A} = \frac{\rho T}{\rho_A T_A} \quad \Rightarrow \quad \frac{T}{T_A} = \frac{P}{P_A}\left(\frac{\rho_A}{\rho}\right)$$

$$\therefore \; S - S_A = -r\ln\frac{P/P_A}{\left(\frac{P}{P_A}\right)^{\frac{r}{r-1}}\left(\frac{\rho_A}{\rho}\right)^{\frac{r}{r-1}}}$$

$$1 - \frac{r}{r-1} = \frac{r-1-r}{r-1} = -\frac{1}{r-1}$$

$$= -r\ln\frac{\left(\frac{P}{P_A}\right)^{1-\frac{r}{r-1}}}{\left(\frac{\rho}{\rho_A}\right)^{-\frac{r}{r-1}}}$$

$$= -\frac{r(-1)}{r-1}\ln\left(\frac{(P/P_A)}{(\rho/\rho_A)^{r}}\right)$$

Now $\frac{r}{r-1} = \frac{C_p}{r}$ eq 2.1.12

From 2.1.13 $\frac{C_p}{r} = C_v$

$$= C_v\ln\left(\frac{(P/P_A)}{(\rho/\rho_A)^{r}}\right)$$

$$H \equiv \underset{\text{defined}}{E} + \frac{P}{\rho} = e + \frac{V^2}{2} + \frac{P}{\rho} = \underbrace{e + \frac{P}{\rho}}_{\equiv h} + \frac{V^2}{2}$$

evaluate at inlet/outlet conditions.

$$h + \frac{\vec{V}^2}{2} = c_p T + \frac{V^2}{2} = c_p \underbrace{\left(T + \frac{V^2}{2c_p}\right)}_{\equiv T_0 \text{ define}}$$

$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s = ?$ Taking $\left.\frac{\partial}{\partial \rho}\right|_s$ of 2.1.17 $\Rightarrow$

$$0 = c_v \frac{(P/P_A)^{\gamma}}{(P/P_A)} \cdot \frac{\partial}{\partial \rho}\left(\frac{P/P_A}{(\rho/\rho_A)^{\gamma}}\right)$$

$$\Rightarrow \quad 0 = \frac{\partial}{\partial \rho}\left(\frac{P/P_A}{(\rho/\rho_A)^{\gamma}}\right) = \left.\frac{\partial P}{\partial \rho}\right|_s \frac{1}{P_A} \frac{1}{(\rho/\rho_A)^{\gamma}} + \left(\frac{P}{P_A}\right)(-\gamma)\left(\frac{\rho}{\rho_A}\right)^{-\gamma-1} \cdot \frac{1}{\rho_A}$$

$$\Rightarrow \quad 0 = \frac{1}{P_A} \left.\frac{\partial P}{\partial \rho}\right|_s + \frac{P}{P_A}(-\gamma)\left(\frac{\cancel{\rho_A}}{\rho}\right) \frac{1}{\cancel{\rho_A}}$$

$$\Rightarrow \quad \left.\frac{\partial P}{\partial \rho}\right|_s = \gamma \frac{P}{\rho} = \gamma r T = c^2$$

Then $T = \dfrac{c^2}{\gamma r}$

So $$T_0 = T + \frac{\vec{V}^2}{2c_p} = T\left(1 + \frac{\vec{V}^2}{2Tc_p}\right) = T\left(1 + \frac{\vec{V}^2}{2c_p\left(\frac{c^2}{\gamma r}\right)}\right)$$

$$= T\left(1 + \frac{\gamma}{2}\frac{\vec{V}^2}{c^2}\left(\frac{r}{c_p}\right)\right)$$ from 2.1.12 $\frac{r}{c_p} = \frac{\gamma - 1}{\gamma}$

$$\therefore \quad T_0 = T\left(1 + \frac{\gamma-1}{2}\frac{\vec{V}^2}{c^2}\right) \qquad \text{let } M = \frac{|\vec{V}|}{c}$$

$$T_0 = T\left(1 + \frac{\gamma-1}{2}M^2\right)$$

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$$\rho \frac{\partial u}{\partial t} + \rho(\vec{v}\circ\nabla)u = -\frac{\partial p}{\partial x} + \mu\Delta u$$

$x = Lx'$

$t = Tt'$

$\vec{v} = V\vec{v}'$

$p = \rho V^2 p'$

$$[[p]] = \frac{F}{A} = \frac{kg\cdot m/s^2}{m^2} = \frac{M}{LT^2}$$

$= \rho\cdot V^2$ works $\qquad [\rho] = \frac{M}{L^3}$

$=$

Then putting in & dropping primes

$$\frac{\rho V}{T}\frac{\partial u}{\partial t} + \rho\frac{V^2}{L}(\vec{v}\circ\nabla)u = \underbrace{-\frac{\rho V^2}{L}}_{\div}\frac{\partial p}{\partial x} + \frac{\mu V}{L^2}\Delta u$$

$$\frac{\cancel{\rho}\cancel{V}L}{\rho V^2 T}\frac{\partial u}{\partial t} + (\vec{v}\circ\nabla)u = -\frac{\partial p}{\partial x} + \frac{\mu}{L\rho V}\Delta u$$

$$\Rightarrow \frac{L}{VT}\frac{\partial u}{\partial t} + (\vec{v}\circ\nabla)u = -\frac{\partial p}{\partial x} + \underbrace{\frac{\mu}{\rho V L}}_{\frac{1}{Re}}\Delta u$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'}\frac{\partial t'}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t'}$$

$$Re = \frac{\rho VL}{\mu} = \frac{VL}{\mu/\rho} = \frac{VL}{\nu}$$

$$Re \frac{VT}{L} \frac{\partial u}{\partial t} = -Re \frac{\partial p}{\partial x} + \Delta u$$

$$\frac{V\cancel{L}}{\nu} \cdot \frac{VT}{\cancel{L}} = \frac{V^2 T}{\nu}$$

My eq gives

$$\frac{L}{VT} \cdot Re \frac{\partial u}{\partial t} = -Re \frac{\partial p}{\partial x} + \Delta u$$

$$\frac{L}{\cancel{V}T} \cdot \frac{\cancel{V}L}{\nu} \frac{\partial u}{\partial t} =$$

$$\frac{L^2}{T\nu} \frac{\partial u}{\partial t}$$

--- going w/ Hirsh's result

$$-\frac{V^2 T}{L} + \Delta u = Re \frac{\partial p}{\partial x}$$

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$$a\frac{\partial^2\phi}{\partial x^2} + 2b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} = 0$$

let $u = \frac{\partial\phi}{\partial x}$ & $v = \frac{\partial\phi}{\partial y}$ then the above becomes

$$a\frac{\partial u}{\partial x} + 2b\frac{\partial u}{\partial y} + c\frac{\partial v}{\partial y} = 0$$

$$-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \begin{pmatrix} a & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 2b & c \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y = \vec{0}$$

let $U = \hat{U}e^{i(\vec{n}\circ\vec{x})} = \hat{U}e^{i(n_x x + n_y y)}$

$U_x = n_x i \hat{U} e^{i(\vec{n}\circ\vec{x})}$ $\quad U_y = n_y i \hat{U} e^{i(\vec{n}\circ\vec{x})}$

put in

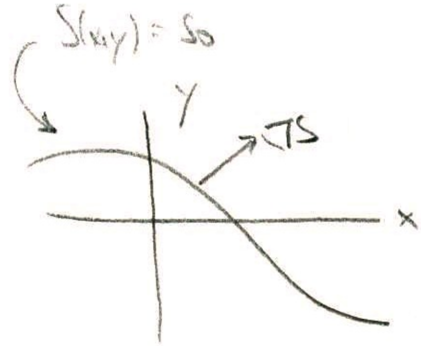
$$(A^1 n_x + A^2 n_y)\hat{U} = 0$$

$$|A^1 n_x + A^2 n_y| = 0$$

$$\Rightarrow \begin{vmatrix} a n_x + 2b n_y & c n_y \\ -n_y & n_x \end{vmatrix} = 0 \quad \Rightarrow$$

$$a n_x^2 + 2b n_y n_x + c n_y^2 = 0$$

either n_x or $n_y \neq 0$ Assume $n_y \neq 0$ What about $n_y = 0$?



Then

$$a\left(\frac{n_x}{n_y}\right)^2 + 2b\left(\frac{n_x}{n_y}\right) + c = 0$$

$$"b"^2 - 4"a""c" = 4b^2 - 4ac = b^2 - ac$$

$$\frac{n_x}{n_y} = \frac{-2b \pm \sqrt{}}{2a} = \frac{-b}{a}$$

$$\frac{dy}{dx} = \frac{-S_x}{S_y}$$

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$$b^2 - ac = \frac{u^2 v^2}{c_0^4} - \left(1 - \frac{u^2}{c_0^2}\right)\left(1 - \frac{v^2}{c_0^2}\right)$$

$$= \cancel{\frac{u^2 v^2}{c_0^4}} - \left(1 - \frac{1}{c_0^2}(u^2+v^2) + \cancel{\frac{u^2 v^2}{c_0^4}}\right)$$

$$= \frac{u^2+v^2}{c_0^2} - 1 = M^2 - 1 \qquad \text{eq E3,2,3}$$

$$(1 - M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$a = 1 - M_\infty^2, \quad b = 0, \quad c = 1$$

Obtained from transforming this small potential equation into a 1st order equation.

{ May not be unique }

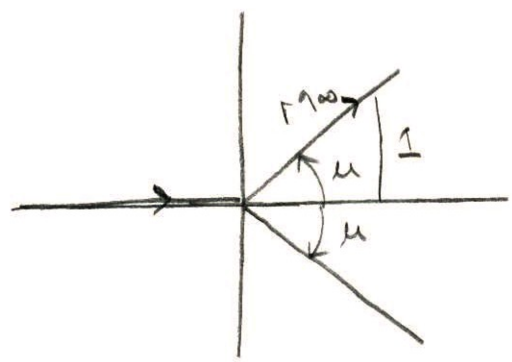
Then $b^2 - ac = -(1 - M_\infty^2)$

3.2.10 $\Rightarrow$ $a\left(\frac{n_x}{n_y}\right)^2 + 2b\left(\frac{n_x}{n_y}\right) + c = 0$

$$\frac{n_x}{n_y} = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$= \frac{\pm\sqrt{M_\infty^2 - 1}}{M_\infty^2 - 1} = \pm \frac{1}{\sqrt{M_\infty^2 - 1}}$$

$$\frac{dy}{dx} = -\frac{n_x}{n_y} = \pm \frac{1}{\sqrt{M_\infty^2 - 1}}$$



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$$\underbrace{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}}_{A^1} \begin{pmatrix} u \\ v \end{pmatrix}_x + \underbrace{\begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}}_{A^2} \begin{pmatrix} u \\ v \end{pmatrix}_y = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\det | A^k n_k | = 0$$

$$\Rightarrow \begin{vmatrix} a n_x & c n_y \\ d n_y & b n_x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a\frac{n_x}{n_y} & c \\ d & b\frac{n_x}{n_y} \end{vmatrix} = 0$$

$$ab\left(\frac{n_x}{n_y}\right)^2 = cd$$

$$\left(\frac{n_x}{n_y}\right)^2 = \frac{cd}{ab}$$

$a=b=c=d=1$, $f_1=f_2=0$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}+\underbrace{\frac{\partial^2 v}{\partial x\partial y}}=0$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0 \qquad -\frac{\partial^2 u}{\partial y^2}=0$$

If $a=b=1$, $c=-d=-1$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 v}{\partial x\partial y}=0$$

$$\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}=0 \qquad -\left(-\frac{\partial^2 u}{\partial y^2}\right)=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$b=0$ Then E3.2.11 gives

$$\begin{vmatrix} \frac{a n_x}{n_y} & c \\ d & 0 \end{vmatrix} = 0 \qquad ?$$

If $a=1$, $b=0$, $c=-d=-1$ $f_1=0$

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0 \quad \text{from 2nd eq} \quad \frac{\partial v}{\partial y}=\frac{\partial^2 u}{\partial y^2} \quad \text{put in 1st eq}$$

$$\frac{\partial u}{\partial y}=v \qquad \frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial y^2}$$

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$$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}\begin{pmatrix} h \\ u \\ v \end{pmatrix}_x + \begin{pmatrix} v & 0 & h \\ 0 & v & 0 \\ g & 0 & v \end{pmatrix}\begin{pmatrix} h \\ u \\ v \end{pmatrix}_y = \vec{0}$$

$$\det | A^k n_k | = 0$$

$$\Rightarrow \begin{vmatrix} un_x + vn_y & hn_x & hn_y \\ gn_x & un_x + vn_y & 0 \\ gn_y & 0 & un_x + vn_y \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{1}{n_y}\right)^3 \begin{vmatrix} u\lambda + v & h\lambda & h \\ g\lambda & u\lambda + v & 0 \\ g & 0 & u\lambda + v \end{vmatrix} = 0$$

$$\Rightarrow (u\lambda+v)\left[(u\lambda+v)^2\right] - g\lambda\left[h\lambda(u\lambda+v)\right] + g\left[-h(u\lambda+v)\right] = 0$$

$$(u\lambda+v)\left[(u\lambda+v)^2 - gh\lambda^2 - gh\right] = 0$$

one sol: $\lambda = -\frac{v}{u}$ & $u^2\lambda^2 + 2uv\lambda + v^2 - gh\lambda^2 - gh = 0$

$$(u^2 - gh)\lambda^2 + 2uv\lambda + (v^2 - gh) = 0$$

$$\lambda^{(2),(3)} = \frac{-2uv \pm \sqrt{4u^2v^2 - 4(u^2-gh)(v^2-gh)}}{2(u^2-gh)}$$

$$= \frac{-uv \pm \sqrt{\cancel{u^2v^2} - (\cancel{u^2v^2} - ghu^2 - ghv^2 + g^2h^2)}}{u^2-gh}$$

$$= \frac{-uv \pm \sqrt{gh(u^2+v^2-gh)}}{u^2-gh}$$

$\vec{n}^{(1)} = (n_x, n_y) \ni \frac{n_x}{n_y} = \frac{-v}{u}$

$\vec{n}^{(1)} \cdot \vec{V} = n_x u + n_y v = -vu + uv = 0$

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$$(u^2 - gh)\lambda^2 + 2\lambda uv + (v^2 - gh) = 0 \qquad \text{eq E3,2.21}$$

$$\lambda = \frac{-\cancel{2}uv \pm \sqrt{\cancel{4}u^2v^2 - \cancel{4}(u^2 - gh)(v^2 - gh)}}{2(u^2 - gh)}$$

$$= \frac{-uv \pm \sqrt{\cancel{u^2v^2} - (\cancel{u^2v^2} - ghu^2 - ghv^2 + g^2h^2)}}{(u^2 - gh)}$$

$$= \frac{-uv \pm \sqrt{ghu^2 + ghv^2 - g^2h^2}}{u^2 - gh}$$

$$= -uv \pm \sqrt{gh(u^2 + v^2 - gh)}$$

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$$S(x^1, x^2, \dots, x^M) = 0$$

$$dS = \frac{\partial S}{\partial x^k} dx^k = 0$$

$$= \frac{\partial S}{\partial x^1} dx^1 + \frac{\partial S}{\partial x^2} dx^2 + \dots \frac{\partial S}{\partial x^M} dx^M = 0$$

$$\frac{\partial S}{\partial x^k} + \frac{\partial S}{\partial x^1}\frac{dx^1}{dx^k} + \dots + \frac{\partial S}{\partial x^{k-1}}\frac{dx^{k-1}}{dx^k} + \frac{\partial S}{\partial x^k} + \frac{\partial S}{\partial x^{k+1}}\frac{\partial x^{k+1}}{\partial x^k} + \dots + \frac{\partial S}{\partial x^m}\frac{dx^m}{dx^k} = 0$$

Now Assuming x^M can be solved for in terms of $x^1, x^2, \dots, x^{M-1}$ (the other variables) Thus $\frac{\partial x^i}{\partial x^j} = \delta_{ij}$ $\quad 1 \le i, j \le m-1$ $\cancel{i \neq j}$

Then All $$\frac{\partial S}{\partial x^M}\frac{dx^m}{dx^k} = -\left(\frac{\partial S}{\partial x^1}\frac{dx^1}{dx^k} + \dots \frac{\partial S}{\partial x^{k-1}}\frac{dx^{k-1}}{dx^k} + \frac{\partial S}{\partial x^{k+1}}\frac{\partial x^{k+1}}{\partial x^k} + \dots + \frac{\partial S}{\partial x^{m-1}}\frac{\partial x^{m-1}}{\partial x^k}\right)$$

$$- \frac{\partial S}{\partial x^k}$$

→ Assuming $x^1, \dots, x^{m-1}$ are all independent, thus All terms in the parenthesis vanish

$$\Rightarrow \frac{\partial S}{\partial x^M}\frac{dx^M}{dx^k} = -\frac{\partial S}{\partial x^k}$$

$$\frac{\partial x^M}{\partial x^k} = \frac{-\partial S/\partial x^k}{\partial S/\partial x^M} = \frac{-n_k}{n_m}$$

Assuming $x^m = x^m(x^1, x^2, \dots, x^{m-1})$

The

$$\frac{\bar{\partial}}{\partial x^k} = \frac{\partial}{\partial x^k} + \frac{\partial x^m}{\partial x^k}\frac{\partial}{\partial x^m} = \frac{\partial}{\partial x^k} + -\left(\frac{n_k}{n_m}\right)\frac{\partial}{\partial x^m}$$

$$l^i A^k_{ij}\left[\frac{\bar{\partial}}{\partial x^k} + \frac{n_k}{n_m}\frac{\partial}{\partial x^m}\right]u^j = l^i Q_i$$

$$\frac{\partial}{\partial x^k} = \frac{\bar{\partial}}{\partial x^k} + \frac{n_k}{n_m}\frac{\partial}{\partial x^m}$$ from eq 3.3.4.

Then

$$l^i A^k_{ij}\frac{\bar{\partial} u^j}{\partial x^k} + \underbrace{l^i A^k_{ij}\frac{n_k}{n_m}\frac{\partial u^j}{\partial x^m}}_{=0} = l^i Q_i$$

$$\forall j \quad l^i A^k_{ij} n_k = 0$$

$$\rightarrow \quad l^i A^k_{ij}\frac{\partial S}{\partial x^k} = 0$$

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Small disturbance potential eq:

$$(1-M_\infty^2)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

let $u=\frac{\partial\phi}{\partial x}$ $\quad v=\frac{\partial\phi}{\partial y}$

$$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

$$+(1-M_\infty^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \quad +\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

$$(1-M_\infty^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\Rightarrow \begin{pmatrix}0 & 1\\ 1-M_\infty^2 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_x+\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_y=\vec{0}$$

The

$$\begin{pmatrix}0 & 1\\ (1-M_\infty^2) & 0\end{pmatrix}n_x+\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}n_y=\begin{pmatrix}-n_y & n_x\\ (1-M_\infty^2)n_x & n_y\end{pmatrix}$$

Then the left eigenvectors for this matrix $\Rightarrow$

$$\vec{l}^T\begin{pmatrix}-n_y & n_x\\ (1-M_\infty^2)n_x & n_y\end{pmatrix}=\vec{0}$$

taking transpose:

$$\begin{pmatrix} -n_y & (1-M_\infty^2)\, n_x \\ n_x & n_y \end{pmatrix} \vec{l} = \vec{0} \qquad \checkmark d$$

Now $\dfrac{n_y}{n_x} = \dfrac{\pm 1}{\sqrt{M_\infty^2 - 1}}$ (~~Direction $\perp$ to the characteristic surface S~~)
Direction normal to characteristic surface S

$\Rightarrow n_y = \dfrac{\pm n_x}{\sqrt{M_\infty^2 - 1}}$ & The above becomes (Not correct?)

$$\begin{pmatrix} \mp \dfrac{n_x}{\sqrt{M_\infty^2-1}} & (1-M_\infty^2)\, n_x \\ n_x & \pm \dfrac{n_x}{\sqrt{M_\infty^2-1}} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 0$$

$\cdot \mp \dfrac{\sqrt{M_\infty^2-1}}{n_x}$ top

$$\Rightarrow \begin{pmatrix} 1 & \pm (M_\infty^2 - 1)^{3/2} \\ n_x & \dfrac{\pm n_x}{\sqrt{M_\infty^2-1}} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \\ 0 & \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

Now This det must be zero

$$\Rightarrow \quad -n_y^2 - (1-M_\infty^2) n_x^2 = 0$$

$$\left(\frac{n_y}{n_x}\right)^2 = M_\infty^2 - 1$$

$$\frac{n_y}{n_x} = \pm\sqrt{M_\infty^2 - 1}$$ Thus eigenvectors l_1, l_2 become

$$\begin{pmatrix} \pm\sqrt{M_\infty^2-1}\, n_x & (1-M_\infty^2) n_x \\ n_x & \pm\sqrt{M_\infty^2-1}\, n_x \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \mp\sqrt{M_\infty^2-1} \\ n_x & \pm\sqrt{M_\infty^2-1}\, n_x \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \mp\sqrt{M_\infty^2-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

Take ~~$l_1 = 1$~~

$$l_1 = \pm\sqrt{M_\infty^2-1}\; l_2$$

$$\vec{l} = \begin{pmatrix} \pm\sqrt{M_\infty^2-1}\; l_2 \\ l_2 \end{pmatrix} = l_2 \begin{pmatrix} \pm\sqrt{M_\infty^2-1} \\ 1 \end{pmatrix} \simeq \tilde{l}_2 \Big(\;1$$

The matrix

$$\begin{pmatrix} (1-M_\infty^2)n_x & n_y \\ -n_y & n_x \end{pmatrix} = \begin{pmatrix} 1-M_\infty^2 & 0 \\ 0 & 1 \end{pmatrix} n_x + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} n_y$$

$$\Rightarrow \quad (1-M_\infty^2)\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} + -\frac{\partial u}{\partial y} = 0$$

$$\Rightarrow v = \frac{\partial \phi}{\partial y} \quad \& \quad u = \frac{\partial \phi}{\partial x}$$

check 1st eq becomes

$$(1-M_\infty^2)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Thus $\quad (l^1, l^2)\begin{pmatrix} (1-M_\infty^2)n_x & n_y \\ -n_y & n_x \end{pmatrix} = \vec{0}$

taking transpose

$$\begin{pmatrix} (1-M_\infty^2)n_x & -n_y \\ n_y & n_x \end{pmatrix}\begin{pmatrix} l^1 \\ l^2 \end{pmatrix} = \vec{0}$$

This matrix must be singular $\Rightarrow$

$$(1-M_\infty^2)\, n_x^2 + n_y^2 = 0.$$

Thus $\left(\frac{n_y}{n_x}\right)^2 = -(1-M_\infty^2)$

$$\frac{n_y}{n_x} = \pm\sqrt{M_\infty^2 - 1}$$

Then; the matrix becomes:

$$\begin{pmatrix} (1-M_\infty^2)n_x & \mp\sqrt{M_\infty^2-1}\,n_x \\ \pm\sqrt{M_\infty^2-1}\,n_x & n_x \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \pm\frac{1}{\sqrt{M_\infty^2-1}} \\ \pm\sqrt{M_\infty^2-1}\,n_x & n_x \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\mp\sqrt{M_\infty^2-1}\,n_x$$

$$\Rightarrow \begin{pmatrix} 1 & \pm\frac{1}{\sqrt{M_\infty^2-1}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

Then $l_1 = \mp\frac{l_2}{\sqrt{M_\infty^2-1}}$

$$\vec{l} = \begin{pmatrix} \mp\frac{l_2}{\sqrt{M_\infty^2-1}} \\ l_2 \end{pmatrix} = \mp\frac{l_2}{\sqrt{M_\infty^2-1}}\begin{pmatrix} 1 \\ \mp\sqrt{M_\infty^2-1} \end{pmatrix}$$

taking $l_1 = 1$ & $l_2 = +\sqrt{M_\infty^2-1}$

$$Z_j^k = \ell^i A^k_{ij} \qquad A^1 = \begin{pmatrix} (1-M_\infty^2) & 0 \\ 0 & 1 \end{pmatrix}$$

$$Z_j^k = \qquad A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left(1, \sqrt{M_\infty^2 - 1}\right)\begin{pmatrix} (1-M_\infty^2) & 0 \\ 0 & 1 \end{pmatrix} = \left((1-M_\infty^2), \underbrace{\sqrt{M_\infty^2-1}}_{\lambda}\right)$$

$$= \left(1-M_\infty^2, \lambda\right)$$

$$\vec{Z}_2 = \left(1, \sqrt{M_\infty^2-1}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left(-\sqrt{M_\infty^2-1}, 1\right) = (-\lambda, 1)$$

$$\vec{Z}_1 = \left(-\lambda^2, \lambda\right) = \lambda(-\lambda, 1)$$

$$\vec{n} = (1, \lambda)$$

$$Z_1 \circ \vec{n} = (1-M_\infty^2) + \lambda^2 = 0$$

$$Z_2 \circ \vec{n} = -\lambda + \lambda = 0$$

3.3.9 becomes $Z_j^k \frac{\partial u^j}{\partial x^k} = l^i Q_i$

$Z_1 = (1 - M_\infty^2, \lambda)$

$Z_2 = (-\lambda, 1)$

$Z_j^k =$ jth vector, kth component.

$$Z_1^1 \frac{\partial u^1}{\partial x^1} + Z_2^1 \frac{\partial u^2}{\partial x^1} +$$

$$Z_1^1 \frac{\partial u^1}{\partial x^1} + Z_1^2 \frac{\partial u^1}{\partial x^2} + Z_2^1 \frac{\partial u^2}{\partial x^1} + Z_2^2 \frac{\partial u^2}{\partial x^2} = 0$$

$$\left((1-M_\infty^2)\frac{\partial}{\partial x} + \lambda\frac{\partial}{\partial y}\right)u^1 + \left(-\lambda\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u^2 = 0$$

Now $\sin\mu = \frac{1}{M_\infty}$ & $\tan\mu = \frac{1}{\sqrt{M_\infty^2 - 1}} = \frac{1}{\lambda}$

$$\cot\mu = \sqrt{M_\infty^2 - 1}$$

$\therefore$

$$\sqrt{M_\infty^2 - 1}\left(-\sqrt{M_\infty^2 - 1}\,\frac{\partial}{\partial x} + \frac{\lambda}{\sqrt{M_\infty^2 - 1}}\frac{\partial}{\partial y}\right)u$$

$$\pm\left(\mp\lambda\frac{\partial}{\partial x} \pm \frac{\partial}{\partial y}\right)v = 0$$

Now if $\tan\mu = \dfrac{1}{\sqrt{M_\infty^2 - 1}} \Big\{ = \dfrac{\sin\mu}{\cos\mu}$

$$\sin\mu = \frac{1}{M_\infty} \quad \Rightarrow \quad \cos\mu = \frac{\sin\mu}{\tan\mu}$$

Then $\qquad \Rightarrow \cos\mu = \dfrac{1/M_\infty}{\dfrac{1}{\sqrt{M_\infty^2-1}}} = \dfrac{\sqrt{M_\infty^2-1}}{M_\infty}$

$\lambda = +\sqrt{M_\infty^2 - 1}$

$\Rightarrow$ last eq on pg 7 becomes (Multiplying both sides by $\frac{-1}{M_\infty}$)

$$\sqrt{M_\infty^2-1}\left(\frac{\sqrt{M_\infty^2-1}}{M_\infty}\frac{\partial}{\partial x} - \frac{1}{M_\infty}\frac{\partial}{\partial y}\right)u$$

$$\mp\left(\mp\frac{\sqrt{M_\infty^2-1}}{M_\infty} \pm \frac{1}{M_\infty}\frac{\partial}{\partial y}\right)v = 0$$

$$\Rightarrow \sqrt{M_\infty^2-1}\left(\frac{\sqrt{M_\infty^2-1}}{M_\infty}\frac{\partial}{\partial x} - \frac{1}{M_\infty}\frac{\partial}{\partial y}\right)u$$

$$+\left(\frac{\sqrt{M_\infty^2-1}}{M_\infty}\frac{\partial}{\partial x} + \frac{1}{M_\infty}\frac{\partial}{\partial y}\right)v = 0$$

$$\Rightarrow \cot\mu\left(\cos\mu\,\partial_x - \sin\mu\,\partial_y\right)u + \left(\cos\mu\,\partial_x + \sin\mu\,\partial_y\right)v = 0$$

Pg 148 Hirsh Vol I

$(L^{-1})^{i\alpha} = l^{i(\alpha)}$

$l^{i(\alpha)} k_{ij} = l^i \delta_{ij} \lambda_{(\alpha)} = \lambda_{(\alpha)} l^{j(\alpha)}$

Thus vector $l^i k_{ij} = \lambda_{(\alpha)} \vec{l}^{(\alpha)}$

$l^i k_{ij}$ = Multiply matrix k by

$$(L^{-1}) K = \begin{pmatrix} \lambda_{(1)} \vec{l}^{(1)T} \\ \lambda_{(2)} \vec{l}^{(2)T} \\ \vdots \\ \lambda_{(n)} \vec{l}^{(n)T} \end{pmatrix}$$

Then $L^{-1} k = \Lambda L^{-1}$

3, 2, 21

$[A^k n_k] \hat{U} = 0$ what is $\hat{U}$?

$A^m_{ij} = \delta_{ij}$ 3.4.1

$\det | k - \lambda I | = 0$ 3.4.5

$[I n_t + \underbrace{A^k n_k}_{K}] \hat{U} = 0$

$\Rightarrow \hat{U}$ is a Right eigen vector of k_{ij}

Pg 149 Hirsh Vol I

$$\frac{\partial U}{\partial t} + (\vec{a} \circ \nabla) U = 0$$

$U = \hat{U} e^{i(\vec{k} \circ \vec{x} - \omega t)}$ put in above

$$-i\omega \hat{U} + (\vec{a} \circ (i\vec{k}))\hat{U} = 0$$

$\omega = \vec{a} \circ \vec{k}$ Solve this for $\vec{a}$.? w/ $\vec{k}$ arb $\Rightarrow$ only solution is

$$\vec{a} = \vec{k}\,\frac{\omega}{k^2}$$

let $\vec{k} = (k_1 0, \ldots, 0)$

$\omega = a_1 k_1$ let $\vec{k} = (0, k_2, 0 \ldots)$

$a_1 = \frac{\omega}{k_1}$, $a_2 = \frac{\omega}{k_2}$ ..., a

$\vec{a} = \left(\frac{\omega}{k_1}, \frac{\omega}{k_2}, \ldots, \frac{\omega}{k_n}\right)$ $\vec{a} \circ \vec{k} = n\omega. \Rightarrow ?$ $n = 1$

$= \frac{\omega}{k^2}$ Then

$$\left(\frac{\partial U}{\partial t} + A^k \frac{\partial U}{\partial x^k}\right) = 0 \qquad U = \hat{U} e^{i(\vec{k} \circ \vec{x} - \omega t)}$$

$$I(-i\omega)\hat{U} + A^k i k_k \hat{U} = 0$$

$$-i\omega \delta_{ij} \hat{U}^j + i k_k A^k_{ij} \hat{U}^j = 0$$

$$\Rightarrow -\omega \delta_{ij} \hat{U}^j + k_k A^k_{ij} \hat{U}^j = 0$$

$$\det\left(-\omega \delta_{ij} + k_k A^k_{ij}\right) = 0$$

<u>Pg 150 Hirsh Vol I</u>

From 3.4.13 $\quad U = \hat{U} e^{i(\vec{k} \circ \vec{x} - \omega t)}$

if $\omega_1 = \xi + i\eta$

$$U = \hat{U}_1 e^{i(\vec{k} \circ \vec{x})} e^{-i\xi t} e^{+\eta t}$$

<u>Pg 150-151 Hirsh Vol I</u>

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} u & h \\ g & u \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$

<u>Characteristic</u> velocities,

For this problem $\hat{K} = \vec{A} \circ l_k = \begin{pmatrix} u & h \\ g & u \end{pmatrix} \frac{k_x}{|k_x|} = \begin{pmatrix} u & h \\ g & u \end{pmatrix}$

$$\begin{vmatrix} -a + u & h \\ g & -a + u \end{vmatrix} = 0$$

$$(a-u)^2 - gh = 0$$

$$a = u \pm \sqrt{gh}$$

$$U = \hat{U} e^{i(\vec{x}\circ\nabla S + tS_t)}$$

put into $\dfrac{\partial U}{\partial t} + A^k \dfrac{\partial U}{\partial x^k} = 0$

$$\frac{\partial U}{\partial t} = \hat{U}(iS_t) e^{i(\vec{x}\circ\nabla S + tS_t)}$$

$$\frac{\partial U}{\partial x^k} = \hat{U}\frac{\partial S}{\partial x^k} i e^{i(\vec{x}\cdot\nabla S + tS_t)} \qquad \hat{U} \in \mathbb{R}^n$$

$$\left(iS_t \hat{U} + A^k i \frac{\partial S}{\partial x^k}\hat{U}\right) = 0$$

$$\left(S_t I + A^k \frac{\partial S}{\partial x^k}\right)\hat{U} = 0$$

$$U = \hat{U} e^{i(\vec{x}\circ\nabla S + tS_t)} \qquad \nabla S = \vec{k}.$$

$$S_t = -\omega$$

$$= \hat{U} e^{i(\vec{x}\circ\vec{k} - t\omega)}$$

w/ Non linear relation $\vec{k} = \vec{k}(\vec{x},t)$

$\omega = \omega(\vec{k}(\vec{x},t))$ $\quad \therefore \quad \omega_t = \vec{\nabla}_k\omega \cdot \vec{k}_t$

$\omega_{x^k} = \nabla_k \omega \circ \vec{k}_{x^k}$

$$\frac{\partial U}{\partial t} = i\hat{U}\left[-\omega - t\frac{\partial\omega}{\partial t} + \vec{x}\circ\frac{\partial\vec{k}}{\partial t}\right] = i\hat{U}\left[-\omega - t\nabla_k\omega\circ\frac{\partial\vec{k}}{\partial t} + \vec{x}\circ\frac{\partial\vec{k}}{\partial t}\right]e^{i(\;)}$$

$$\frac{\partial U}{\partial x^k} = i\hat{U}\left[k_k - t\frac{\partial\omega}{\partial x^k} + \vec{x}\circ\frac{\partial\vec{k}}{\partial x^k}\right] = i\hat{U}\left[k_k - t\nabla_k\omega\circ\vec{k}_{x^k} + \vec{x}\circ\frac{\partial\vec{k}}{\partial x^k}\right]e^{i(\;)}$$

$$\Rightarrow \quad \frac{\partial U}{\partial t} = i\hat{U}\left[-\omega + (\vec{x} - t\nabla_k\omega)\circ\frac{\partial\vec{k}}{\partial t}\right]e^{i(\;)}$$

$$\frac{\partial \upsilon}{\partial x^k} = i\hat{\upsilon}\left[(\vec{k})_k + (\vec{x} - t\nabla_{\vec{k}}\omega)\cdot\frac{\partial \vec{k}}{\partial x^k}\right]e^{i(\ \ )}$$

kth component of vector $\vec{k}$

Prob 3.1

Steady potential eq E3.2.1

Ved

$$\left(1-\frac{u^2}{c_0^2}\right)\frac{\partial^2\phi}{\partial x^2}-\frac{2uv}{c_0^2}\frac{\partial^2\phi}{\partial x\,\partial y}+\left(1-\frac{v^2}{c_0^2}\right)\frac{\partial^2\phi}{\partial y^2}=0$$ Is hyperbolic only

For supersonic flows $M>1$ $\quad M^2=\frac{u^2+v^2}{c_0^2}$

1st we have to determine the normals to the characteristic surfaces for this problem. Turn problem into a PDE system.

let $u=\frac{\partial\phi}{\partial x}$ $\quad v=\frac{\partial\phi}{\partial y}$ { Think these are the same u & v from above. Stream fn definition?

Then $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$ equivalence of mixed partials

Ved

$$+\quad\left(1-\frac{u^2}{c_0^2}\right)\frac{\partial u}{\partial x}-\frac{2uv}{c_0^2}\frac{\partial v}{\partial x}+\left(1-\frac{v^2}{c_0^2}\right)\frac{\partial v}{\partial y}=0$$

Ved

$$\begin{pmatrix}1-u^2/c_0^2 & -\frac{2uv}{c_0^2}\\ 0 & -1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_x+\begin{pmatrix}0 & 1-v^2/c_0^2\\ 1 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}_y=\vec{0}$$

consider eq 3.2.21a to determine hyperbolic : Substitute in $U(x,y)=\hat{U}e^{i(\hat{n}\cdot\vec{x})}$

$$\Rightarrow\det\begin{pmatrix}(1-u^2/c_0^2)n_x & -\frac{2uv}{c_0^2}n_x+(1-\frac{v^2}{c_0^2})n_y\\ n_y & -n_x\end{pmatrix}=0$$ Ved

$$-\left(1-\frac{u^2}{c_0^2}\right) n_x^2 + \frac{2uv\, n_x n_y}{c_0^2} - \left(1-\frac{v^2}{c_0^2}\right) n_y^2 = 0$$

÷ by n_y^2

$$-\left(1-\frac{u^2}{c_0^2}\right)\left(\frac{n_x}{n_y}\right)^2 + \frac{2uv}{c_0^2}\left(\frac{n_x}{n_y}\right) - \left(1-\frac{v^2}{c_0^2}\right) = 0$$

$$\frac{n_x}{n_y} = \frac{-\frac{2uv}{c_0^2} \pm \sqrt{\frac{4u^2v^2}{c_0^4} - 4\left(1-\frac{u^2}{c_0^2}\right)\left(1-\frac{v^2}{c_0^2}\right)}}{-2\left(1-\frac{u^2}{c_0^2}\right)}$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{\frac{uv}{c_0^2} \pm \sqrt{\cancel{\frac{u^2v^2}{c_0^4}} - 1 + \frac{u^2}{c_0^2} + \frac{v^2}{c_0^2} - \cancel{\frac{u^2v^2}{c_0^4}}}}{-\left(1-\frac{u^2}{c_0^2}\right)}$$

$$= \frac{\frac{uv}{c_0^2} \pm \sqrt{\frac{u^2+v^2}{c_0^2} - 1}}{-\left(1-u^2/c_0^2\right)} = \boxed{\frac{-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}}{\left(1-u^2/c_0^2\right)} = \frac{n_x}{n_y}} \equiv \lambda$$

normals to characteristic surfaces

$(\lambda, 1)$

(over)

Thus $M^2 > 1$ for the equation to be hyperbolic

Now $\vec{l}^{(\alpha)}$ are left eigenvectors. w/ $\frac{n_x}{n_y}$ = Above.

from eq 3.3.7a w/o a time like direction being singled out one must find $\vec{l}^{(\alpha)} \ni$ (2 such vectors one for the normal to Both characteristic surface)

$$l^i A^k_{ij} n_k = \vec{l}^{(\alpha)T}(A^k n_k) = 0$$

Taking the transpose of Both sides gives

$$(A^k n_k)^T \vec{l}^{(\alpha)} = 0 \Rightarrow (A^1 n_1 + A^2 n_2)^T \vec{l}^{(\alpha)} = 0$$

$$\Rightarrow \begin{pmatrix} (1 - U^2/c_0^2) n_x & n_y \\ -\frac{2UV}{c_0^2} n_x + \left(1 - \frac{V^2}{c_0^2}\right) n_y & -n_x \end{pmatrix} \begin{pmatrix} l_1^{(1)} \\ l_2^{(1)} \end{pmatrix} = \vec{0}$$

✓ed.

But $n_x = \dfrac{-\frac{UV}{c_0^2} + \sqrt{M^2 - 1}}{1 - U^2/c_0^2} n_y$ for $\alpha = 1$ take $+$, $\alpha = 2$ take $-$

Then

$$\begin{pmatrix} \left(-\frac{UV}{c_0^2} \pm \sqrt{M^2-1}\right) n_y & n_y \\ \underbrace{\frac{2UV}{c_0^2} \frac{\left(-\frac{UV}{c_0^2} \pm \sqrt{M^2-1}\right)}{(1 - U^2/c_0^2)} n_y + \left(1 - \frac{V^2}{c_0^2}\right) n_y} & -\frac{\left(\frac{-UV}{c_0^2} \pm \sqrt{M^2-1}\right)}{1 - U^2/c_0^2} n_y \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = \vec{0}$$

Get 1 in position (1,1)

$\div n_y$ assumed $\neq 0$.

✓ed.

$$\Rightarrow \begin{pmatrix} 1 & \frac{1}{-\frac{UV}{c_0^2} \pm \sqrt{M^2-1}} \\ \frac{-\frac{2U^2V^2}{c_0^4} \pm \frac{2UV}{c_0^2}\sqrt{M^2-1}}{(1 - U^2/c_0^2)} n_y + \frac{\left(1 - \frac{U^2}{c_0^2}\right)\left(1 - \frac{V^2}{c_0^2}\right)}{(1 - U^2/c_0^2)} n_y & " \end{pmatrix} \begin{pmatrix} \\ \\ \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \dfrac{1}{-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}} \\[2ex] \dfrac{-\frac{2u^2v^2}{c_0^4} + 1 - \frac{u^2}{c_0^2} - \frac{v^2}{c_0^2} + \frac{u^2v^2}{c_0^4} \pm \frac{2uv}{c_0^2}\sqrt{M^2-1}}{\left(1-u^2/c_0^2\right)} & \dfrac{\frac{uv}{c_0^2} \mp \sqrt{M^2-1}}{1-u^2/c_0^2} \end{pmatrix}$$

simplify position (2,1)

$$\Rightarrow \begin{pmatrix} 1 & \dfrac{1}{-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}} \\[2ex] \dfrac{-\frac{u^2v^2}{c_0^4} + 1 - \frac{u^2}{c_0^2} - \frac{v^2}{c_0^2} \pm \frac{2uv}{c_0^2}\sqrt{M^2-1}}{\left(1-u^2/c_0^2\right)} & \dfrac{\frac{uv}{c_0^2} \mp \sqrt{M^2-1}}{1-u^2/c_0^2} \end{pmatrix}$$ ✓ed

Mult 1st row by negative of element in position (2,1) & Add to 2nd row.

$$\Rightarrow \begin{pmatrix} 1 & \dfrac{1}{-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}} \\[2ex] 0 & \dfrac{\frac{u^2v^2}{c_0^4} - 1 + \frac{u^2}{c_0^2} + \frac{v^2}{c_0^2} \mp \frac{2uv}{c_0^2}\sqrt{M^2-1}}{\left(1-u^2/c_0^2\right)} \dfrac{1}{\left(-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}\right)} + \dfrac{\frac{uv}{c_0^2} \mp \sqrt{M^2-1}}{1-u^2/c_0^2} \end{pmatrix}$$ ✓ed

$$\Rightarrow \begin{pmatrix} 1 & \dfrac{1}{-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}} \\[2ex] 0 & \dfrac{1}{1-u^2/c_0^2}\left(\dfrac{\frac{\cancel{u^2v^2}}{\cancel{c_0^4}} - \cancel{1} + \frac{\cancel{u^2}}{\cancel{c_0^2}} + \frac{\cancel{v^2}}{c_0^2} \mp \frac{\cancel{2uv}}{c_0^2}\sqrt{M^2-1} + -\frac{\cancel{u^2v^2}}{c_0^4} \pm \frac{\cancel{2uv}}{c_0^2}\sqrt{M^2-1} - \cancel{M^2} + \cancel{1}}{\left(-\frac{uv}{c_0^2} \pm \sqrt{M^2-1}\right)}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \dfrac{1}{\frac{-uv}{c_0^2} \pm \sqrt{M^2-1}} \\ 0 & \dfrac{1}{(\ )(\ )} \cdot 0 \end{pmatrix} \begin{pmatrix} l_1^{(1)} \\ l_2^{(1)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow l_1^{(1)} = \frac{+1}{\frac{uv}{c_0^2} \mp \sqrt{M^2-1}}\, l_2^{(1)} = \frac{-1}{\frac{-uv}{c_0^2} \pm \sqrt{M^2-1}}\, l_2 = \frac{-1}{b \pm \sqrt{1-a-c}}$$

$$\boxed{\vec{l}^{(1),(2)} \propto \begin{pmatrix} 1 \\ \frac{uv}{c_0^2} \mp \sqrt{M^2-1} \end{pmatrix}}$$

left eigenvectors.

$$= \begin{pmatrix} 1 \\ -(b \pm \sqrt{1-a-c}) \end{pmatrix}$$

Get 2nd left eigenvector. $n_x = \dfrac{-\frac{uv}{c_0^2} - \sqrt{M^2-1}}{1 - u^2/c_0^2}$, n_y

Follow the above w/ sign change in radical.

Thus in summary

$$\vec{l}^{(1)} = \begin{pmatrix} 1 \\ \frac{uv}{c_0^2} - \sqrt{M^2-1} \end{pmatrix} \qquad \vec{l}^{(2)} = \begin{pmatrix} 1 \\ \frac{uv}{c_0^2} + \sqrt{M^2-1} \end{pmatrix}$$

consider

$$\vec{l}^{(1)} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{uv}{c_0^2} - \sqrt{M^2-1} & 0 \\ u & v & 0 \end{vmatrix} = \hat{k}\left(v - \frac{u^2 v}{c_0^2} + u\sqrt{M^2-1}\right)$$

Now:

$$\cancel{\frac{|\vec{l}^{(1)} \times \vec{v}|^2}{|\vec{l}^{(1)}|^2 |\vec{v}|^2} = \sin^2 \mu}$$

$$\cancel{|\vec{l}^{(1)}|^2 = \frac{u^2 v^2}{c^4} - \frac{2uv}{c^2}\sqrt{M^2-1}}$$

stopping to follow the hint.

I obtained

$$\frac{n_x}{n_y} = \frac{-\frac{uv}{c^2} \pm \sqrt{M^2-1}}{1 - u^2/c^2} = \frac{b \pm \sqrt{1-a-c}}{a}$$

If $n_x = 1$ As suggested $n_y = \frac{a}{b \pm \sqrt{1-a-c}}$

$$\frac{1}{n_y} = \frac{-\frac{uv}{c^2} \pm \sqrt{M^2-1}}{1 - u^2/c^2}$$

Idea: to show

$n_y = -\cot(\beta \pm \mu) = -\frac{1}{\tan(\beta \pm \mu)}$

show $\frac{1}{n_y} = -\tan(\beta \pm \mu)$

$$\frac{1}{\cot(\beta \pm \mu)} = \tan(\beta \pm \mu) = \frac{\tan\beta \pm \tan\mu}{1 \mp \tan\beta \tan\mu}$$

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin(\alpha)\cos\beta \pm \sin\beta\cos\alpha}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta} \div \frac{1}{\cos\beta \cos\alpha}$$

$$= \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\boxed{\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}}$$

$$\tan\beta = \frac{v}{u} \qquad \sin\mu = \frac{1}{M}$$

$$\tan\mu = ? = \frac{1}{\sqrt{M^2-1}}$$

$$\therefore \frac{1}{\operatorname{cotan}(\beta \pm \mu)} = \frac{\frac{v}{u} \pm \frac{1}{\sqrt{M^2-1}}}{1 \mp \frac{v}{u}\frac{1}{\sqrt{M^2-1}}}$$

$$\tan(\beta \pm \mu) = \frac{\frac{v}{u} \pm \frac{1}{\sqrt{M^2-1}}}{1 \mp \frac{v}{u}\frac{1}{\sqrt{M^2-1}}} \cdot \frac{\sqrt{M^2-1}}{\sqrt{M^2-1}}$$ clear fractional sqrt

$$= \frac{\frac{v}{u}\sqrt{M^2-1} \pm 1}{\sqrt{M^2-1} \mp \frac{v}{u}} \cdot \frac{\left(-\sqrt{M^2-1} \mp \frac{v}{u}\right)}{\left(-\sqrt{M^2-1} \mp \frac{v}{u}\right)}$$ remove sqrt from denom

$$= \frac{-\frac{v}{u}(M^2-\cancel{1}) \mp \frac{v^2}{u^2}\sqrt{M^2-1} \mp \sqrt{M^2-1} - \cancel{\frac{v}{u}}}{-(M^2-1) + \frac{v^2}{u^2}}$$

$$= \frac{-\frac{v}{u}M^2 \mp \left(1+\frac{v^2}{u^2}\right)\sqrt{M^2-1}}{1-M^2+\frac{v^2}{u^2}} = -\frac{\left(\frac{v}{u}M^2 \pm \left(1+\frac{v^2}{u^2}\right)\sqrt{M^2-1}\right)}{1-M^2+\frac{v^2}{u^2}}$$

Mult top & bottom by $\frac{1}{(1+V^2/U^2)}$

$$= -\frac{\left(\frac{V}{U}\frac{M^2}{(1+V^2/U^2)} \pm \sqrt{M^2-1}\right)}{\frac{1-M^2+\frac{V^2}{U^2}}{(1+V^2/U^2)}}$$

$$= -\frac{\left(\frac{V}{\cancel{U}}\frac{\cancel{(U^2+V^2)}/c_0^2\, U^2}{\cancel{(U^2+V^2)}} \pm \sqrt{M^2-1}\right)}{\frac{\left(1-\frac{U^2+V^2}{c_0^2}+\frac{V^2}{U^2}\right)U^2}{U^2+V^2}}$$

$$= -\frac{\left(\frac{VU}{c_0^2} \pm \sqrt{M^2-1}\right)}{\frac{U^2+V^2-\left(\frac{U^2+V^2}{c_0^2}\right)U^2}{U^2+V^2}} = -\frac{\left(\frac{VU}{c_0^2} \pm \sqrt{M^2-1}\right)}{1-U^2/c_0^2}$$

$$= -\frac{1}{n_y}$$

~~Now w/ $l_1 = 1$ (I got)~~

~~$l_2 = -$ any Not c?~~

From pg 8

$$-\frac{\left(\frac{uv}{c_0^2} \pm \sqrt{M^2-1}\right)}{1-u^2/c_0^2} = -\frac{1}{n_y}$$

$$\Rightarrow -\frac{(l_2)}{1-u^2/c_0^2} = -\frac{1}{n_y}$$

$$\boxed{l_2 = \frac{-a}{n_y}}$$

Thus I got $\vec{l} = \begin{pmatrix} 1 \\ -\frac{a}{n_y} \end{pmatrix}$

Then

$\bar{Z} = ?$ $\qquad Z_j^k = l^i A_{ij}^k$

$$\vec{l}^T A^1 = \left(1 \quad -\frac{a}{n_y}\right)\begin{pmatrix} 1-u^2/c_0^2 & -2\frac{uv}{c_0^2} \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1-u^2/c_0^2 \\ -2\frac{uv}{c_0^2} + \frac{a}{n_y} \end{pmatrix}$$

$$= \begin{pmatrix} a \\ 2b + \frac{a}{n_y} \end{pmatrix}$$

$$\vec{l}^T A^2 = \left(1 \quad -a/n_y\right)\begin{pmatrix} 0 & c \\ 1 & 0 \end{pmatrix} = \left(-a/n_y \quad c\right)$$

Assuming $\vec{Z}_1 = \begin{pmatrix} a \\ -2b+cn_y \end{pmatrix}$ & $\vec{Z}_2 = \begin{pmatrix} -cn_y \\ c \end{pmatrix}$

The compact relations become $d_j u^j = 0$

$$(\vec{Z}_j \circ \vec{\nabla}) u^j = 0$$

$$\Rightarrow \left(a\frac{\partial}{\partial x} + (-2b+cn_y)\frac{\partial}{\partial y} \right) u + \left(-cn_y \frac{\partial}{\partial x} + c\frac{\partial}{\partial y} \right) v = 0$$

$$\Rightarrow a\frac{\partial u}{\partial x} + (cn_y - 2b)\frac{\partial u}{\partial y} - cn_y\frac{\partial v}{\partial x} + c\frac{\partial v}{\partial y} = 0.$$

Problem 3.2

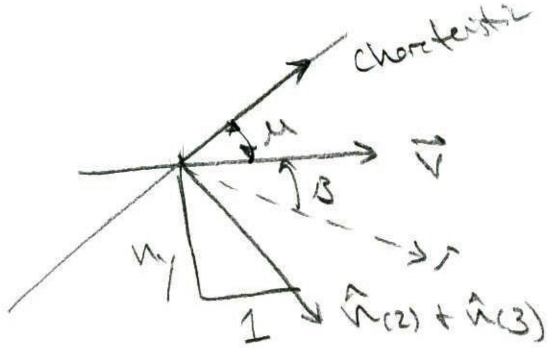
We must find the 2 characteristic normals ~~to~~ for $\lambda_{(2)}$ & $\lambda_{(3)}$

$\lambda = \dfrac{n_x}{n_y}$ from pg 142

$\therefore$ normal directions are $(\lambda_{(2)}, 1)$ & $(\lambda_{(3)}, 1)$

or if $n_x = 1$

normal directions can be taken $(1, n_{y(2)})$ & $(1, n_{y(3)})$



let β again be the $\measuredangle$ of the velocity vector makes w/ positive x-axis.

$\cos\beta = \dfrac{u}{|\vec{V}|}$; $\sin\beta = \dfrac{v}{|\vec{V}|}$

? How finish ?

Prob 3.3

eq 3.3.8 w/ no time like variable singled out

$\Rightarrow \quad Z^k_j = \ell^i A^k_{ij}$

If $\quad A^m_{ij} = \delta_{ij}$

$Z^k_j = \ell^i A^k_{ij} \qquad 1 \le k \le m-1 \qquad Z^k \in \mathbb{R}^n$

$+ \quad Z^m_j = \ell^i \delta_{ij} = \ell^j \quad \in \mathbb{R}^n$

Typo in Book $\underset{\text{last}}{\wedge}$ column of matrix Z^k_j is $\vec{\ell}$

3.3.12

$\vec{Z}_j \circ \vec{n} = Z^k_j n_k = 0 \qquad j = 1, \dots, n$

$\qquad\qquad ||$

$Z^m_j n_m + \underbrace{\sum_{k=1}^{m-1} Z^k_j n_k}_{} = 0$

$\ell^j n_m + \ell^i A^k_{ij} n_k = 0$

$||$

n_t

$$= \ell^i \delta_{ij} n_t + \ell^i A^k_{ij} n_k = 0 \qquad k = 1, \dots, m-1$$

$$= \ell^i (A^k_{ij} n_k) = -\ell^i (\delta_{ij} n_t)$$

$$= \underline{-n_t}\, \ell^i \delta_{ij}$$

$$= \lambda_{(\alpha)}$$

Prob 3.4

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$

$$\begin{pmatrix} h \\ u \end{pmatrix}_t + \begin{pmatrix} u & h \\ g & u \end{pmatrix}\begin{pmatrix} h \\ u \end{pmatrix}_x = 0$$ in quasi linear form.

For these eqs to have plane wave solutions. As a time like variable has already been selected.

$$U = \hat{U} e^{i(k_x x + k_y y)} = \hat{U} e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$
$$= \hat{U} e^{i(kx - \omega t)}$$

$$\Rightarrow \hat{U} e^{i(kx-\omega t)}(-i\omega) + \begin{pmatrix} u & h \\ g & u \end{pmatrix}\hat{U} e^{i(kx-\omega t)}(ik) = 0$$

$$\Rightarrow \left(-i\omega I + ik\begin{pmatrix} u & h \\ g & u \end{pmatrix}\right)\hat{U} = 0$$

$$\Rightarrow \left(-\omega I + k\begin{pmatrix} u & h \\ g & u \end{pmatrix}\right)\hat{U} = 0$$ must have nontrivial solutions

$$\Rightarrow \left|\begin{pmatrix} -\omega & 0 \\ 0 & -\omega \end{pmatrix} + \begin{pmatrix} Uk & hk \\ gk & kU \end{pmatrix}\right| = \begin{vmatrix} Uk-\omega & hk \\ gk & kU-\omega \end{vmatrix} = 0$$

$$\Rightarrow (Uk-\omega)^2 - ghk^2 = 0$$

phase speed $= \frac{\omega}{k}$

$\equiv$ characteristic speed

also

$$\Rightarrow \left(U - \frac{\omega}{k}\right)^2 - gh = 0$$

$$\Rightarrow \frac{\omega}{k} = U \pm \sqrt{gh}$$

$\frac{\omega}{k}$ = phase speed = $\boxed{a = U \pm \sqrt{gh}}$

$$\underbrace{\omega}_{= -n_t \text{ eq } 3.4.29} = (U \pm \sqrt{gh})\underbrace{k}_{= n_x}$$

dispersion relation. Not phase/characteristic speed $= \frac{\partial\omega}{\partial k}$ = group speed

let eigenvectors given by 3.4.7 & 3.4.8

$$\Rightarrow l^i k_{ij} = \lambda_{(\alpha)} l^i \delta_{ij} = -n^{(\alpha)}$$

$$\| $$

$$\vec{l}^T (A n_x) = +n^{(\alpha)} l^T$$

$$A = \begin{pmatrix} U & h \\ g & U \end{pmatrix}$$

$$(l_1 \quad l_2)\begin{pmatrix} U & h \\ g & U \end{pmatrix} = +(U \pm \sqrt{gh})(l_1 \quad l_2)$$

$$\Rightarrow \begin{pmatrix} l_1 & l_2 \end{pmatrix} \left(\begin{pmatrix} u & h \\ g & u \end{pmatrix} - (u \pm \sqrt{gh}) I \right) = 0$$

$$\begin{pmatrix} l_1 & l_2 \end{pmatrix} \begin{pmatrix} \mp\sqrt{gh} & h \\ g & \mp\sqrt{gh} \end{pmatrix} = 0$$ taking transpose ...

$$\begin{pmatrix} \mp\sqrt{gh} & g \\ h & \mp\sqrt{gh} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & \mp g/\sqrt{gh} \\ h & \mp\sqrt{gh} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & \mp\sqrt{\frac{g}{h}} \\ 1 & \mp\sqrt{\frac{g}{h}} \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \vec{0}$$

$$l_1 = \pm\sqrt{\frac{g}{h}}\, l_2 \qquad \therefore \quad \vec{l} = \begin{pmatrix} 1 \\ \pm\sqrt{\frac{h}{g}} \end{pmatrix}$$

$$\Rightarrow l_2 = \pm\sqrt{\frac{h}{g}}\, l_1$$

$$\therefore \boxed{\vec{l}_{\pm} = \begin{pmatrix} 1 \\ \pm\sqrt{\frac{h}{g}} \end{pmatrix}}$$

Pg 146 Hirsch gives $Z^k_j = l^i A^k_{ij}$

$\vec{Z}_j$ are the vectors index on $K = 1, 2$

consider $\vec{l}^T_\pm A^1 = \left(1 \quad \pm\sqrt{\frac{h}{g}}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(1 \quad \pm\sqrt{\frac{h}{g}}\right)$

$+ \quad \vec{l}^T_\pm A^2 = \left(1 \quad \pm\sqrt{\frac{h}{g}}\right)\begin{pmatrix} u & h \\ g & u \end{pmatrix} = \left(u \pm \sqrt{gh},\ h \pm u\sqrt{\frac{h}{g}}\right)$

$$= \left(u \pm \sqrt{gh}, \pm\sqrt{\frac{h}{g}}\left(u \pm \sqrt{g}\sqrt{h}\right)\right)$$

$$= \left(u \pm \sqrt{gh}\right)\left(1, \ \pm\sqrt{\frac{h}{g}}\right)$$

$Z^{k=1}_{j=1} = 1 \quad ; \quad Z^{k=2}_{j=1} = u \pm \sqrt{gh} \Rightarrow \boxed{\vec{Z}_1 = \begin{pmatrix} 1 \\ u \pm \sqrt{gh} \end{pmatrix}}$

$Z^{k=1}_{j=2} = \pm\sqrt{\frac{h}{g}} \quad ; \quad Z^{k=2}_{j=2} = \left(u \pm \sqrt{gh}\right)\left(\pm\sqrt{\frac{h}{g}}\right)$

$$\Rightarrow \boxed{\vec{Z}_2 = \pm\sqrt{\frac{h}{g}}\begin{pmatrix} 1 \\ u \pm \sqrt{gh} \end{pmatrix}}$$

Then Compatibility Relations become:

$$d_j \equiv z_j^k \frac{\partial}{\partial x^k} =$$

$$d_j u^j = 0$$

$$\Rightarrow \quad d_1 u^1 + d_2 u^2 = 0$$

$$\Rightarrow \quad d_1 h + d_2 u = 0$$

$$\Rightarrow \quad \left(\frac{\partial}{\partial t} + (u \pm \sqrt{gh})\frac{\partial}{\partial x}\right)h \pm \sqrt{\frac{h}{g}}\left(\frac{\partial}{\partial t} + (u \pm \sqrt{hg})\frac{\partial}{\partial x}\right)u = 0$$

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Prob 3.5 Show:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad \text{are elliptic}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y = \vec{0}$$

consider sol of the form $U = \hat{U} e^{i(\vec{n}\cdot\vec{x})}$

$$\Rightarrow \begin{pmatrix} n_x & n_y \\ -n_y & n_x \end{pmatrix}\begin{pmatrix} \hat{U}_x \\ \hat{U}_y \end{pmatrix} = 0$$

to have a nontrivial solution to this eq $\begin{vmatrix} n_x & n_y \\ -n_y & n_x \end{vmatrix} = 0$

Both n_x & n_y must be real $\Rightarrow$

$$n_x^2 + n_y^2 = 0 \Rightarrow \left(\frac{n_x}{n_y}\right)^2 + 1 = 0$$

$$\frac{n_x^2}{n_y^2} = -1 \Rightarrow \frac{n_x}{n_y} = \pm\sqrt{-1} = \pm i$$

From pg 137 Hirsh Vol I

n_y real $\forall$ real values of n_x

disc $a = 1,\ b = 0,\ c = 1$

$b^2 - ac = 0 - 1 < 0$ 2 sol are complex conj eq. is elliptic combining to get one eq gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

$\left(\frac{\partial}{\partial x} \text{ of 1st eq}\right)$ $\quad$ $\left(\frac{\partial}{\partial y} \text{ of 2nd eq}\right)$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{Laplaces eq.}$$

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Prob 3.6

H = total Enthalpy.

$$= E + \frac{p}{\rho} = e + \frac{V^2}{2} + \frac{p}{\rho}$$

Assuming ideal gas $e = c_v T$ & $p = \rho r T$

$$\Rightarrow T = \frac{p}{\rho r}$$

$$e = \frac{c_v p}{\rho r} = \frac{c_v}{r} \frac{p}{\rho}$$

$$\therefore H = \left(1 + \frac{c_v}{r}\right)\frac{p}{\rho} + \frac{V^2}{2}$$

H is now in terms of p (as asked) & ρ & V (we have eqs for these)

$$\boxed{R = c_p - c_v \quad \& \quad \frac{c_p}{c_v} = \gamma \quad \Leftarrow \text{How get?}}$$

(Definition of γ) Abot R?

$$\frac{r}{c_p} = 1 - \frac{c_v}{c_p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}$$

$$\frac{r}{c_v} = \frac{\gamma - 1}{\gamma} \frac{c_p}{c_v} = \gamma - 1$$

$$\therefore \frac{c_v}{r} = \frac{1}{\gamma - 1}$$

$$1 + \frac{c_v}{r} = \frac{\gamma - 1 + 1}{\gamma - 1} = \frac{\gamma}{\gamma - 1}$$

$$\therefore H = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \frac{V^2}{2}$$

Now $\quad \frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial t}$

$$\Rightarrow \frac{\gamma}{\gamma-1}\frac{\partial}{\partial t}\left(\frac{p}{\rho}\right) + \frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + u\frac{\gamma}{\gamma-1}\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) + u\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \frac{1}{\rho}\frac{\partial p}{\partial t}$$

$$\Rightarrow \frac{\gamma}{\gamma-1}\frac{p_t}{\rho} - \frac{\gamma}{\gamma-1}\frac{p}{\rho^2}\rho_t + \underline{uu_t} + u\frac{\gamma}{\gamma-1}\frac{p_x}{\rho} - \frac{u\gamma}{\gamma-1}\frac{p}{\rho^2}\rho_x + \underline{uuu_x} = \frac{1}{\rho}p_t$$

Now $u \cdot$ 2nd eq (mmt eq) becomes

$$uu_t + u^2u_x = -\frac{u}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \frac{p_t}{\rho} - \frac{p}{\rho^2}\rho_t + \frac{\gamma-1}{\gamma}\underbrace{(uu_t + uuu_x)}_{-\frac{u}{\rho}p_x} + u\frac{p_x}{\rho} - u\frac{p}{\rho^2}\rho_x = \frac{\gamma-1}{\gamma}\frac{p_t}{\rho}$$

$$\Rightarrow \frac{p_t}{\rho}\underbrace{\left(1 - \frac{\gamma-1}{\gamma}\right)}_{\frac{1}{\gamma}} + \left(\frac{-p}{\rho^2}\right)\rho_t + \underbrace{\left(-\frac{(\gamma-1)}{\gamma} + 1\right)}_{\frac{1}{\gamma}}\frac{up_x}{\rho} + \frac{-up}{\rho^2}\rho_x = 0$$

$$\Rightarrow \frac{1}{\gamma}\frac{p_t}{\rho} - \frac{p}{\rho^2}\rho_t + \frac{u}{\rho\gamma}p_x - \frac{up}{\rho^2}\rho_x = 0$$

$$\frac{p_t}{\gamma\rho} + \frac{u}{\gamma\rho}p_x - \frac{p}{\rho^2}\underbrace{(\rho_t + u\rho_x)}_{-\rho u_x} = 0$$

$$\Rightarrow \quad \frac{p_t}{\cancel{\rho}\gamma} + \frac{u p_x}{\gamma\cancel{\rho}} + \frac{p}{\cancel{\rho}}u_x = 0$$

$$p_t + u p_x + \underbrace{\frac{\gamma p}{\rho}}_{c^2}\rho u_x = 0$$

$$c^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_S = \cancel{rT} = \frac{p}{\rho}$$

get entropy as a fn of $p + \rho$ (p, v)

$$\begin{array}{ccccc} T\,dS & = & de & + & \underbrace{p\,d(1/\rho)}_{p,\rho} \\ \downarrow & & \downarrow & & \\ \frac{p}{\gamma\rho} & & c_v T & & \\ & & \downarrow & & \\ & & \frac{1}{\gamma-1}\frac{p}{\rho} & & \end{array}$$

integrate to get S

$$\Rightarrow \quad p_t + u p_x + c^2\rho u_x = 0$$

Thus in primitive variables (A 1st order system)

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix}\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_x = \vec{0}$$

To see if this eq is hyperbolic. consider wave like solutions

$v = \hat{v}e^{i(\vec{n}\circ\vec{x})}$ $\quad \div$ by $e^{i(\vec{n}\circ\vec{x})}$

$$\Rightarrow \begin{pmatrix} n_t + u n_x & \rho n_x & 0 \\ 0 & n_t + u n_x & \frac{1}{\rho} n_x \\ 0 & \rho c^2 n_x & n_t + u n_x \end{pmatrix}\begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{v}_3 \end{pmatrix} = \vec{0}$$

$\div$ by n_x gives w/ $\lambda = \frac{n_t}{n_x}$

$$\begin{pmatrix} \lambda + u & \rho & 0 \\ 0 & \lambda + u & 1/\rho \\ 0 & \rho c^2 & \lambda + u \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \bar{0}$$

For a non trivial solution, the determinant must vanish

$$\Rightarrow \quad (\lambda + u)\left[(\lambda + u)^2 - c^2\right] = 0$$

$$\Rightarrow \quad \lambda = -u \quad \lambda = -u \pm c.$$

taking λ to be the negative $\lambda = u$ $\lambda = u \pm c$ Ask.

This also follows from the definition of time like variables on

pg 14B.

To obtain the left eigenvectors for a system w/ a time direction specified.

$$l^i(\delta_{ij} n_t + A^t_{ij} n_t) = 0$$

$$l^i k_{ij} = \lambda_{(\alpha)} l^i \delta_{ij}$$

$$\Rightarrow \vec{l}^T k = \lambda_{(\alpha)} \vec{l}^T \quad \text{taking the transpose}$$

$$(K^T - \lambda_{(\alpha)} I)\vec{l}^{(\alpha)} = 0 \qquad \left\{ \begin{pmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \rho c^2 & u-\lambda \end{pmatrix}^T \right.$$

$$\lambda_{(1)} = u$$

$$\begin{pmatrix} 0 & 0 & 0 \\ \rho & 0 & \rho c^2 \\ 0 & 1/\rho & 0 \end{pmatrix} \begin{pmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & c^2 \\ 0 & 1/\rho & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$l_1^{(1)} = -c^2 l_3^{(1)} \qquad l_3^{(1)} \text{ arb}$$

$$l_2^{(1)} = 0$$

$$\vec{l}^{(1)} = \begin{pmatrix} -c^2 \\ 0 \\ 1 \end{pmatrix} \vec{l}_3^{(1)}$$

$$\boxed{\vec{l}^{(1)} = \begin{pmatrix} -c^2 \\ 0 \\ 1 \end{pmatrix}}$$

$\vec{\ell}^{(2)} = ?$ $\lambda_{(2)} = u + c$

$$\begin{pmatrix} -c & 0 & 0 \\ \rho & -c & \rho c^2 \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \rho & -c & \rho c^2 \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -c & \rho c^2 \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\rho c \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\rho c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0} \Rightarrow \begin{array}{l} \ell_1^{(2)} = 0 \\ \ell_2^{(2)} = \rho c \, \ell_3^{(2)} \end{array}$$

$$\Rightarrow \vec{\ell}^{(2)} = \begin{pmatrix} 0 \\ \rho c \\ 1 \end{pmatrix} \ell_3^{(2)} = \begin{pmatrix} 0 \\ \rho c \\ 1 \end{pmatrix} \qquad \boxed{\vec{\ell}^{(2)} = \begin{pmatrix} 0 \\ \rho c \\ 1 \end{pmatrix}}$$

- - - - - - - - - - - - -

$\vec{\ell}^{(3)} = ?$

$$\begin{pmatrix} c & 0 & 0 \\ \rho & c & \rho c^2 \\ 0 & 1/\rho & c \end{pmatrix} \begin{pmatrix} \ell_1^{(3)} \\ \ell_2^{(3)} \\ \ell_3^{(3)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho c \\ 0 & 1/\rho & c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\Rightarrow \quad \ell_1^{(3)} = 0 \qquad \ell_2^{(3)} = -\rho c\, \ell_3^{(3)}$$

$$\vec{\ell}^{(3)} = \begin{pmatrix} 0 \\ -\rho c \\ 1 \end{pmatrix} \ell_3^{(3)} \quad \checkmark \qquad \boxed{\vec{\ell}^{(3)} = \begin{pmatrix} 0 \\ -\rho c \\ 1 \end{pmatrix}}$$

Right eigenvectors. $K_{ij} r^j = \lambda_{(\alpha)} r^j \delta_{ij}$

$$\begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix} \begin{pmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_3^{(1)} \end{pmatrix} = u \begin{pmatrix} r_1^{(1)} \\ r_2^{(1)} \\ r_3^{(1)} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & 1/\rho \\ 0 & \rho c^2 & 0 \end{pmatrix} \begin{pmatrix} \vec{r} \end{pmatrix} = \vec{0} \quad \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$\vec{r}_2^{(1)} = 0 = \vec{r}_3^{(1)}$; $\vec{r}_1^{(1)}$ arb $\boxed{\vec{r}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\lambda_{(2)} = U + C$

$$\begin{pmatrix} -C & \rho & 0 \\ 0 & -C & 1/\rho \\ 0 & \rho C^2 & -C \end{pmatrix} \vec{r}^{(2)} = \vec{0}$$

{ ÷ 3rd row w/ $-\rho C$ }

$$\Rightarrow \begin{pmatrix} 1 & -\rho/C & 0 \\ 0 & -C & 1/\rho \\ 0 & -C & 1/\rho \end{pmatrix} \vec{r}^{(2)} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & -\rho/C & 0 \\ 0 & 1 & -1/C\rho \\ 0 & 0 & 0 \end{pmatrix} \quad " \quad = \quad "$$

$\frac{\rho}{C} \cdot \frac{-1}{C\rho} = \frac{-1}{C^2}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & \frac{-1}{C^2} \\ 0 & 1 & -1/C\rho \\ 0 & 0 & 0 \end{pmatrix} \quad " \quad = \quad "$$

$\vec{r}_1^{(2)} = \frac{1}{C^2} \vec{r}_3^{(2)}$

$\vec{r}_2^{(2)} = \frac{1}{C\rho} \vec{r}_3^{(2)}$

$$\vec{r}^{(2)} = \begin{pmatrix} 1/C^2 \\ 1/C\rho \\ 1 \end{pmatrix} \vec{r}_3^{(2)} = \frac{2}{\rho C} \begin{pmatrix} \frac{\rho}{2C} \\ \frac{1}{2} \\ \frac{\rho C}{2} \end{pmatrix} \vec{r}_3^{(2)}$$

$$\boxed{\vec{r}^{(2)} = \begin{pmatrix} 1/C^2 \\ 1/C\rho \\ 1 \end{pmatrix}}$$

$$\lambda_{(3)} = u - c$$

$$\begin{pmatrix} c & \rho & 0 \\ 0 & c & 1/\rho \\ 0 & \rho c^2 & c \end{pmatrix} \vec{r}^{\,(3)} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & \rho/c & 0 \\ 0 & c & 1/\rho \\ 0 & \rho c^2 & c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & \rho/c & 0 \\ 0 & 1 & 1/\rho c \\ 0 & \rho c^2 & c \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 0 & 0 \end{pmatrix}$$

$$\vec{r}^{\,(3)}_1 = \frac{1}{c^2} \vec{r}^{\,(3)}_3$$

$$\vec{r}^{\,(3)}_2 = -\frac{1}{\rho c} \vec{r}^{\,(3)}_3$$

$$\vec{r}^{\,(3)} = \vec{r}^{\,(3)}_3 \begin{pmatrix} 1/c^2 \\ -1/\rho c \\ 1 \end{pmatrix} = -2\frac{\vec{r}^{\,(3)}_3}{\rho c} \begin{pmatrix} \frac{-\rho}{2c} \\ \frac{1}{2} \\ -\frac{1}{2}\rho c \end{pmatrix}$$

As All eigenvalues are real we have a hyperbolic system

$$\boxed{\vec{r}^{\,(3)} = \begin{pmatrix} 1/c^2 \\ -1/\rho c \\ 1 \end{pmatrix}}$$

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Prob 3.7 $(\rho u)_t + \partial_x(\rho u^2) = \ldots$

let $v = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}$

$= \alpha \frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial v}{\partial x} = 0$

2 eqs are:

$\frac{\partial u}{\partial x} = v$

$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial v}{\partial x} = 0$

$\vec{U} = \begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ u & -\alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} v \\ 0 \end{pmatrix}$$ From pg 135

$\Rightarrow$ Call hyperbolic if homogenous part admits wave like solution.

consider eq 3.2.21a then

$$\begin{pmatrix} n_x & 0 \\ n_t + u n_x & -\alpha n_x \end{pmatrix}$$ Should be not invertible for choices of n_x and n_t both real.

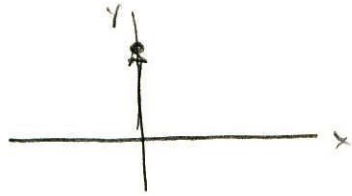
$$\Rightarrow \det\begin{vmatrix} n_x & 0 \\ n_t + u n_x & -\alpha n_x \end{vmatrix} = 0$$

$\Rightarrow -\alpha n_x^2 = 0 \Rightarrow n_x = 0$ or $\alpha = 0$ $\quad \alpha = 0$ eq changes.

$\alpha \neq 0 \quad n_x = 0$

$$\det\begin{vmatrix} 0 & 0 \\ n_t & 0 \end{vmatrix} = 0 \Rightarrow n_t = \text{anything}.$$

$\vec{n} = \beta(1,0)$



From 141 $\exists$ only $\underline{\underline{1}}$ characteristic normal $\vec{n} = \beta(1,0)$

There should be $\underline{2}$ for a hyperbolic system $\Rightarrow$ system is parabolic

$|A^k n_k|$ is not of rank 2, because $n_x \equiv 0$.

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Prob 3,8

(a) $$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \vec{0}$$

(b) To show the system is hyperbolic consider. $U = \hat{U} e^{i(\vec{n}\cdot\vec{x})}$ $\quad \vec{n} = (n_t, n_x)$, $\vec{x} = (t, x)$

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} n_t + \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix} n_x\right)\begin{pmatrix} \hat{U}_1 \\ \hat{U}_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} n_t + n_x/2 & n_x \\ n_x & n_t + \frac{n_x}{2} \end{pmatrix}\begin{pmatrix} \hat{U}_1 \\ \hat{U}_2 \end{pmatrix} = \vec{0}$$

There must be certain directions called characteristic directions $\ni \hat{U} \neq \vec{0}$.

$$\Rightarrow \begin{vmatrix} n_t + n_x/2 & n_x \\ n_x & n_t + n_x/2 \end{vmatrix} = 0$$

$\div\, n_x$ both rows

$+$ define $\lambda \equiv -\frac{n_t}{n_x}$

Or since one direction is singled out as time like (How know?)
we obtain from eq 3,4,5

$$\begin{vmatrix} 1/2 - \lambda & 1 \\ 1 & 1/2 - \lambda \end{vmatrix} = 0$$

$$(1/2 - \lambda)^2 = 1$$

$$\lambda = \tfrac{1}{2} \pm 1 = \begin{matrix} 3/2 \\ -1/2 \end{matrix}$$

Both are real $+$ $\exists$ 2 $\Rightarrow$ system is hyperbolic.

left eigenvectors follow from eq 3.4.7 $\left\{ \vec{\ell}^T \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix} = \lambda \vec{\ell}^T \right\}$

$\Rightarrow \quad \ell^i \left(\delta_{ij} n_t + A^k_{ij} n_k \right) = 0$

taking transpose

$\Rightarrow \quad \left(\delta_{ij} n_t + A^k_{ij} n_k \right) \ell^i = 0$

From eq 3.4.8

$\ell^T k = \lambda_{(\alpha)} \ell^T$

$k^T \ell = \lambda_{(\alpha)} \ell$

$\underline{\lambda = -\frac{1}{2}}$ $\qquad \frac{n_t}{n_x} = -\lambda_{(\alpha)}$

Then

$$\left(\tfrac{1}{2} I + A^1 \right) \begin{pmatrix} \ell^{(1)}_1 \\ \ell^{(1)}_2 \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \ell^{(1)}_1 \\ \ell^{(1)}_2 \end{pmatrix} = \vec{0}$$

$\vec{\ell}^{(1)}_1 = -\vec{\ell}^{(1)}_2$ & normalizing one gets

$$\boxed{\vec{\ell}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

$\lambda = 3/2$ $\qquad$ consider $\left(k^T - \lambda_{(\alpha)} I \right) \ell = 0$

$$\Rightarrow \left(\begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix} - \tfrac{3}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \vec{\ell} = 0$$

$$= \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \ell^{(2)}_1 \\ \ell^{(2)}_2 \end{pmatrix} = \vec{0} \quad \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \; \end{pmatrix} = 0$$

$\vec{l}_1^{(2)} = \vec{l}_2^{(2)}$ normalizing one gets

$$\therefore \boxed{\vec{l}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}$$

Right eigenvectors $A\vec{r} = \lambda(\alpha)\, r^j \delta_{ij}$

$$\Rightarrow \begin{pmatrix}1/2 & 1\\1 & 1/2\end{pmatrix}\begin{pmatrix}r_1\\r_2\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}r_1\\r_2\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}r_1^{(1)}\\r_2^{(1)}\end{pmatrix} = \vec{0}$$

$$\Rightarrow \boxed{\vec{r}^{(1)} = \begin{pmatrix}1\\-1\end{pmatrix}}$$

$$\begin{pmatrix}1/2 & 1\\1 & 1/2\end{pmatrix}\begin{pmatrix}r_1^{(2)}\\r_2^{(2)}\end{pmatrix} = \frac{3}{2}\begin{pmatrix}r_1^{(2)}\\r_2^{(2)}\end{pmatrix}$$

$$\begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}r_1^{(2)}\\r_2^{(2)}\end{pmatrix} = \vec{0}$$

$$L^{-1}AL = \Lambda$$

$$\boxed{\vec{r}^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}}$$

The Matrix which diagonalizes A is obtained by taking the rows of L^{-1} to be the left eigenvectors

$$\Rightarrow L^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix} \Rightarrow L = \sqrt{2}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}^{-1}$$

$L = ?$

$$\Rightarrow \left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & \tfrac{1}{2} & \tfrac{1}{2} \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\tfrac{1}{2} & \tfrac{1}{2} \\ 0 & 1 & \tfrac{1}{2} & \tfrac{1}{2} \end{array}\right)$$

$$\therefore \quad L = \frac{\sqrt{2}}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

Left + Right eivectors are the same because the matrix A is symmetric

(d) Characteristic variables $Z^k_j = \ell^i A^k_{ij}$

For gen hyperbolic system w/ no time like variable singled out.

$$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^2 = \begin{pmatrix} \tfrac{1}{2} & 1 \\ 1 & \tfrac{1}{2} \end{pmatrix}$$

$$\vec{l}^{(1)T} A^1 = \frac{1}{\sqrt{2}}(1 \quad -1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}(1 \quad -1)$$

$$\vec{l}^{(1)T} A^2 = \frac{1}{\sqrt{2}}(1 \quad -1)\begin{pmatrix}\frac{1}{2} & 1\\ 1 & \frac{1}{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left(-\frac{1}{2} \quad \frac{1}{2}\right) = \frac{1}{2\sqrt{2}}(-1 \quad 1)$$

$$\therefore \quad z_1^1 = \frac{1}{\sqrt{2}} \qquad z_1^2 = \frac{-1}{2\sqrt{2}} \qquad \vec{z}_1 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{2\sqrt{2}}\right)$$

$$z_2^1 = -\frac{1}{\sqrt{2}} \qquad z_2^2 = \frac{1}{2\sqrt{2}} \qquad \therefore \quad \vec{z}_2 = \left(\frac{-1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$$

index on z_j of j corresponds to diff $\vec{z}$ vector.

$$\vec{l}^{(2)T} A^1 = \frac{1}{\sqrt{2}}(1 \quad 1)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}}(1 \quad 1)$$

$$\vec{l}^{(2)T} A^2 = \frac{1}{\sqrt{2}}(1 \quad 1)\begin{pmatrix}\frac{1}{2} & 1\\ 1 & \frac{1}{2}\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{3}{2} \quad \frac{3}{2}\right) = \frac{3}{2\sqrt{2}}(1 \quad 1)$$

$$z_1^1 = \frac{1}{\sqrt{2}} \qquad z_1^2 = \frac{3}{2\sqrt{2}} \qquad \vec{z}_1 = \left(\frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$$

$$\therefore$$

$$z_2^1 = \frac{1}{\sqrt{2}} \qquad z_2^2 = \frac{3}{2\sqrt{2}} \qquad \vec{z}_2 = \left(\frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$$

Compatibility relations become: $d_j u^j = 0$

$$\Rightarrow \qquad (\vec{z}_j \cdot \nabla) u^j = 0$$

$$\left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial t} - \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x}\right)u + \left(\frac{-1}{\sqrt{2}}\frac{\partial}{\partial t} + \frac{1}{2\sqrt{2}}\frac{\partial}{\partial x}\right)v = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} - \frac{(1)\partial u}{2\partial x} - \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} - \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) = 0$$

$$= \quad \frac{\partial}{\partial t}(u-v) - \frac{1}{2}\frac{\partial}{\partial x}(u-v) = 0$$

2nd compatibility Relation becomes: $(Z_j \circ \nabla) u^j = 0$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial t} + \frac{3}{2\sqrt{2}}\frac{\partial}{\partial x}\right)u + \left(\frac{1}{\sqrt{2}}\frac{\partial}{\partial t} + \frac{3}{2\sqrt{2}}\frac{\partial}{\partial x}\right)v = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \frac{3}{2}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} + \frac{3}{2}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(u+v) + \frac{3}{2}\frac{\partial(u+v)}{\partial x} = 0$$

From This we see the characteristic variables are $w_1 = \frac{u+v}{2}$

$w_2 = \frac{u-v}{2}$

Note: $L^{-1} \longrightarrow \frac{\partial \bar{u}}{\partial t} + \Lambda \frac{\partial \bar{u}}{\partial x} = 0$ gives them also

$$L^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -u+v \\ u+v \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

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Prob 3.9 eqs given become

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

let $\vec{U} = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 \\ u & 0 & 1/\rho \\ 0 & u & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix}_x + \begin{pmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & 1/\rho \end{pmatrix}\begin{pmatrix} u \\ v \\ p \end{pmatrix}_y = 0$$

To see if these eqs are hyperbolic we look for a traveling wave sol

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \hat{U} e^{i(n_x x + n_y y)}$$

$$\Rightarrow \left(\begin{pmatrix} 1 & 0 & 0 \\ u & 0 & \rho^{-1} \\ 0 & u & 0 \end{pmatrix} n_x + \begin{pmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & \rho^{-1} \end{pmatrix} n_y \right) \hat{U} = 0$$

To have non trivial solutions the det. of the above must vanish (1st $\div$ by n_x & letting $\lambda = \frac{n_y}{n_x}$)

$$\Rightarrow \left| \begin{pmatrix} 1 & 0 & 0 \\ u & 0 & \rho^{-1} \\ 0 & u & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ v & 0 & 0 \\ 0 & v & \rho^{-1} \end{pmatrix} \lambda \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & \lambda & 0 \\ u+\lambda v & 0 & \rho^{-1} \\ 0 & u+\lambda v & \rho^{-1}\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1\left(-\cancel{\rho^{-1}}(u+\lambda v)\right) - \lambda\left(\cancel{\lambda\rho^{-1}}(u+\lambda v)\right) = 0$$

$$-(u+\lambda v) - \lambda^2(u+\lambda v) = 0$$

$$u+\lambda v = 0 \qquad -1-\lambda^2 = 0$$

$$\lambda^2 = -1$$

$$\lambda = -\frac{u}{v} \qquad \lambda = \pm i$$

To be hyperbolic all e.v. would have to be real.
To be elliptic all e.v.s would have to be imaginary.
As we have both $\Rightarrow$ system is hybrid.