General Derivation of implicit finite difference Formulas for 1st + 2nd derivatives.

To Derive eqs 4.3.14. pg 134 Hirsch Vol I

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$$a_+ U_{i+1} + a_0 U_i + a_- U_{i-1} + b_+ (U_x)_{i+1} + b_0 (U_x)_i + b_- (U_x)_{i-1}$$

$$+ C_+ (U_{xx})_{i+1} + C_0 (U_{xx})_i + C_- (U_{xx})_{i-1} = 0 \qquad \text{eq 4.3.10} \qquad (1)$$

Expanding all variables in a taylor series gives

$$\Rightarrow \underline{a_+ U_i} + a_+ \underline{\underline{\Delta x (U_x)_i}} + a_+ \frac{\Delta x^2}{2} \underline{\underline{(U_{xx})_i}} + a_+ \frac{\Delta x^3}{6} \underset{IV}{(U_{xxx})_i} + O(\Delta x^4)$$

$$+ \underline{a_0 U_i}$$

$$+ \underline{a_- U_i} - a_- \underline{\underline{\Delta x (U_x)_i}} + a_- \frac{\Delta x^2}{2} \underline{\underline{(U_{xx})_i}} - a_- \frac{\Delta x^3}{6} \underset{IV}{(U_{xxx})_i} + a_- \frac{\Delta x^4}{24} (U_{xxxx})_i + O(\Delta x^5)$$

$$+ \underline{\underline{b_+ (U_x)_i}} + b_+ \Delta x \underline{\underline{(U_{xx})_i}} + b_+ \frac{\Delta x^2}{2} \underset{IV}{(U_{xxx})_i} + b_+ \frac{\Delta x^3}{6} (U_{xxxx})_i + O(\Delta x^4)$$

$$+ \underline{\underline{b_0 (U_x)_i}}$$

$$+ \underline{\underline{b_- (U_x)}} - b_- \Delta x \underline{\underline{(U_{xx})_i}} + b_- \frac{\Delta x^2}{2} (U_{xxx})_i - b_- \frac{\Delta x^3}{6} \underset{IV}{(U_{xxxx})_i} + b_- \frac{\Delta x^4}{24} (U_{xxxxx})_i + O(\Delta^5)$$

$$+ \underline{\underline{C_+ (U_{xx})_i}} + C_+ \underset{IV}{(U_{xxx})_i} \Delta x + C_+ (U_{xxxx})_i \frac{\Delta x^2}{2} + C_+ (U_{xxxxx})_i \frac{\Delta x^3}{6} +$$

$+ O(\Delta x^4)$

$+ \quad c_0 (u_{xx})_i$

$+ \; c_-(u_{xx})_i - c_-(u_{xxx})_i \Delta x + c_-(u_{xxxx})_i \frac{\Delta x^2}{2} - c_-(u_{xxxxx})_i \frac{\Delta x^3}{6} + O(\Delta x^4)$

$= 0$

$+ \; O(u_i)$ Goal is to solve for $(u_{xx})_i = ?$

$a_+ + a_0 + a_- = 0$

$+ \; O((u_x)_i)$

$a_+ \Delta x - a_- \Delta x + b_+ + b_0 + b_- = 0$

$O((u_{xx})_i)$

$\frac{\Delta x^2}{2} a_+ + \frac{\Delta x^2}{2} a_- + \Delta x \, b_+ - b_- \Delta x + c_+ + c_0 + c_- = 0$

$O((u_{xxx})_i)$

$a_+ \frac{\Delta x^3}{6} - a_- \frac{\Delta x^3}{6} + b_+ \frac{\Delta x^2}{2} + b_- \frac{\Delta x^2}{2} + \Delta x c_+ - \Delta x c_- = 0$

$$\Rightarrow \quad a_+ + a_0 + a_- = 0 \qquad O(u_i)$$

$$\Delta x(a_+ - a_-) + b_+ + b_0 + b_- = 0 \qquad O((u_x)_i)$$

$$\frac{\Delta x^2}{2}(a_+ + a_-) + \Delta x(b_+ - b_-) + c_+ + c_0 + c_- = 0 \qquad * \quad O(u_{xx})$$

$$\frac{\Delta x^3}{6}(a_+ - a_-) + \frac{\Delta x^2}{2}(b_+ + b_-) + \Delta x(c_+ - c_-) = 0 \qquad * \quad O(u_{xxx})$$

4 eqs 3 a's, 3 b's, 3 c's $\Rightarrow$ 9 unknowns.

Solve for a_+, a_0, a_- + b_0

Both *ed eqs only depend on 2 of the quantities of interest $\therefore$ we will solve one of them for the other. Multiplying the 1st eq by $\frac{\Delta x}{3}$ + Adding to the second gives

$$+ \frac{2\Delta x^3}{6} a_+ + \frac{\Delta x^2}{3}(b_+ - b_-) + \frac{\Delta x^2}{2}(b_+ + b_-) + \frac{\Delta x}{3}(c_+ + c_0 + c_-)$$
$$+ \Delta x(c_+ - c_-) = 0.$$

4 eqs in 4.3.14 multiply 3rd eq by $\frac{\Delta x}{3}$ & Add to 4nd eq

$$\frac{\Delta x}{3}\cdot(3rd) \Rightarrow \frac{\Delta x^3}{6}(a_+ + \cancel{a_-}) + \frac{\Delta x^2}{3}(b_+ - b_-) + \frac{\Delta x}{3}(c_+ + c_0 + c_-) = 0$$

$$\frac{\Delta x^3}{6}(a_+ - \cancel{a_-}) + \frac{\Delta x^2}{2}(b_+ + b_-) + \Delta(c_+ - c_-) = 0$$

$+$

$$\frac{2\Delta x^3}{6}a_+ + \Delta x^2\left(\left(\frac{1}{3}+\frac{1}{2}\right)b_+ + \left(\frac{1}{2}-\frac{1}{3}\right)b_-\right) + \Delta x\left(\frac{4}{3}c_+ + \frac{1}{3}c_0 + \left(\frac{1}{3}-1\right)c_-\right) = 0$$

$$\Rightarrow \frac{\Delta x^3}{3}a_+ \quad \Delta x^2\left(\frac{5}{6}b_+ + \frac{1}{6}b_-\right) + \frac{\Delta x}{3}(4c_+ + c_0 - 2c_-) = 0$$

$$a_+ = -\frac{3}{\Delta x^2}\left[\frac{\Delta x}{6}(5b_+ + b_-) + \frac{1}{3}(4c_+ + c_0 - 2c_-)\right]$$

$$= + \frac{1}{2\Delta x}(-5b_+ - b_-) - \frac{1}{\Delta x^2}(4c_+ + c_0 - 2c_-)$$

$$= \frac{2}{\Delta x}\left[-5b_+ - b_- + \frac{2}{\Delta x}(2c_- - 4c_+ - c_0)\right]$$

put this eq into 3rd eq to solve for a_-

$$\frac{\Delta x^2}{\cancel{2}} \frac{\cancel{2}}{\cancel{\Delta x}}\left[-5b_+ - b_- + \frac{2}{\Delta x}(2c_- - 4c_+ - c_0)\right] + \frac{\Delta x^2}{2} a_-$$

$$= -\Delta x(b_+ - b_-) - c_+ - c_0 - c_-$$

$$\Rightarrow \frac{\Delta x^2}{2} a_- = \Delta x\left[5b_+ + b_- - \frac{2}{\Delta x}(2c_- - 4c_+ - c_0)\right]$$

$$- \Delta x b_+ + \Delta x b_- - c_+ - c_0 - c_-$$

$$\Rightarrow \frac{\Delta x^2}{2} a_- = \Delta x(4b_+) + \Delta x 2b_- - 4c_- + 8c_+ + 2c_0$$

$$- c_+ - c_0 - c$$

$$= \Delta x(4b_+ + 2b_-) - 5c_- + c_0 + 7c_+$$

Didn't quite get eq 4.3.16

$b_+ = b_- = b_0 = 0$

$c_+ = \frac{1}{12} \qquad c_0 = \frac{10}{12} \qquad c_- = \frac{1}{12}$

$\Rightarrow$

$$a_+ = \frac{1}{\cancel{2}\Delta x}\left[\frac{\cancel{2}}{\Delta x}\left(\frac{2}{12} - \frac{4}{12} - \frac{10}{12}\right)\right] = \frac{1}{\Delta x^2}\left[\frac{1}{6} - \frac{1}{3} - \frac{5}{6}\right]$$

$$= \frac{-1}{\Delta x^2}\left[1\right] \qquad -\frac{2}{3} \Rightarrow -1$$

7 unknowns in a homogen eq $\Rightarrow$ 8 true unknown

4 eqs from 4.3.14

$$2(b_+ - b_-) + \frac{10}{\Delta x}(c_+ + c_-) - \frac{2}{\Delta x} c_0 = 0$$

$2(b_+ + b_-)$

unknowns b_+, b_-, c_+, c_-, c_0 eqs $= 3$

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$$a_+ U_{i+1} + a_0 U_i + a_- U_{i-1} + b_+ U'_{i+1} + b_0 U'_i + b_- U'_{i-1}$$

$$+ c_+ U''_{i+1} + c_0 U''_i + c_- U''_{i-1} = 0$$

Involving 1st order derivs only $\Rightarrow$ $c_+, c_-, c_0 = 0$.

$$a_+ U_{i+1} + a_0 U_i + a_- U_{i-1} + b_+ U'_{i+1} + b_0 U'_i + b_- U'_{i-1} = 0$$

Thus # of unique coeff is 5

$\therefore$ can set Δx^0, Δx^4
$= 0$

$\Rightarrow$ Maximum order of scheme is ~~4th~~ 5th. Why is this incorrect? Also

Why don't we just expand out U_i's & equate powers of Δx & instead we expand out & equate powers of ~~Δx~~ $\frac{\partial^i U}{\partial x^i}$?

If $\tau = \overline{O}$

$$a_+ = \frac{-1}{2\Delta x}(b_- + 5b_+)$$

$$a_0 = \frac{2}{\Delta x}(b_+ - b_-)$$

$$a_- = \frac{1}{2\Delta x}(b_+ + 5b_-)$$

Note the symmetry!

$$\# \quad R = \text{LTE} = \frac{\Delta x^3}{24}\left[2(b_+ - b_-)\right] u_{xxxx}$$

$$+ \frac{\Delta x^4}{120}\left[2(b_+ + b_-)\right]\frac{\partial^5 u}{\partial x^5}$$

$$+ O(\Delta x^5)$$

Req'g 1 more eq $\beta = \frac{b_-}{b_+}$

$$\frac{R}{\cancel{b_+}} = \frac{\Delta x^3}{12}(1-\beta)\, u_{xxxx} + \frac{\Delta x^4}{60}(1+\beta)\, u_{xxxxx}$$

$\beta = 1 \Rightarrow$

$$(u_x)_{i+1} + 4(u_x)_i + (u_x)_{i-1} = 6\frac{u_{i+1}}{2\Delta x} - 6\frac{u_{i-1}}{2\Delta x} \qquad 4.33 -$$

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2 pt difference formulas

$$a_+ = \frac{1}{2\Delta x}\left[-5b_+ + \frac{2}{\Delta x}(-4c_+ - c_0)\right]$$

$$a_0 = \frac{2}{\Delta x}\left[b_+ - b_-\right]$$

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$$a_- = \frac{1}{2\Delta x}\left[b_+ + \frac{2}{\Delta x}(2c_+ - c_0)\right] = 0$$

$$b_+ = \frac{2}{\Delta x}(c_0 - 2c_+)$$

$$b_0 = 2b_+ + \frac{6}{\Delta x}c_+$$

put in a_+, a_0, b_0 eqs

get sol

$$a_+ = \frac{1}{2\Delta x}\left[-\frac{10}{\Delta x}(c_0 - 2c_+)\right.$$

For truncation error I get

$$R = \frac{\partial^4 u}{\partial x^4}\frac{\Delta x^3}{24}\left[2(b_+ - b_-) + \frac{10}{\Delta x}(c_+ + c_-) - \frac{2}{\Delta x}b\right] \quad ✓$$

$$+$$

$$+ \quad \frac{\partial^7 u}{\partial x^7}\frac{\Delta x^6}{7!}\left[4(b_- + b_+) + \frac{36}{\Delta x}(c_+ - c_-)\right] \quad ✓$$

b(u^8_x) not same?

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$$\Delta^{(2)} = \frac{1}{\Delta x^2}(\mu_y \delta_x)^2 + \frac{1}{\Delta y^2}(\mu_x \delta_y)^2$$

$$\mu_y = \frac{1}{2}\left(E_y^{1/2} + E_y^{-1/2}\right)$$

$$\delta_x = E_x \cancel{{}^{1} -} I \qquad \delta_y = \cancel{E_y}^{1} - I$$

$\delta u_i = u_{i+1/2} - u_{i-1/2}$
$= (E^{1/2} - E^{-1/2})u_i$

$$\Rightarrow \Delta^{(2)} = \frac{1}{\Delta x^2}\left[\frac{1}{2}\left(E_y^{1/2} + E_y^{-1/2}\right)\left(E_x - I\right)\right]^2 + \frac{1}{\Delta y^2}\left[\frac{1}{2}\left(E_x^{1/2} + E_x^{-1/2}\right)\left(E_y - I\right)\right]^2$$

Text is wrong

$$= \frac{1}{4\Delta x^2}\left[E_y^{1/2}E_x - E_y^{1/2} + E_y^{-1/2}E_x - E_y^{-1/2}\right]^2 + \frac{1}{4\Delta y^2}\left[E_x^{1/2}E_y - E_x^{1/2} + E_x^{-1/2}E_y - E_x^{-1/2}\right]^2$$

$$= \frac{1}{4\Delta x^2}\left[E_y E_x^2 \quad \ldots\right.$$

or see following pg

or simpler is to 1st square each term

$$\Delta^{(2)} = \frac{1}{4\Delta x^2}\left(E_y + 2 + E_y^{-1}\right)\left(E_x^2 - 2E_x + I\right)$$

$$+ \frac{1}{4\Delta y^2}\left(E_x + 2 + E_x^{-1}\right)\left(E_y^2 - 2E_y + I\right)$$

3+3+3
= 3.3 = 9

$$= \frac{1}{4\Delta x^2}\Big[E_y E_x^2 - 2E_y E_x + E_y + 2E_x^2 - 4E_x + 2$$
$$+ E_y^{-1}E_x^2 - 2E_y^{-1}E_x + E_y^{-1}\Big]$$

$$+ \frac{1}{4\Delta y^2}\Big[E_x E_y^2 - 2E_x E_y + E_x + 2E_y^2 - 4E_y + 2$$
$$+ E_x^{-1}E_y^2 - 2E_x^{-1}E_y + E_x^{-1}\Big]$$

$$= \frac{1}{4\Delta x^2}\Big\{E_y E_x^2 - 2E_y E_x + 2E_x \quad \ldots$$

if $\Delta x = \Delta y$

$$\Rightarrow \Delta^{(2)} = \frac{1}{4\Delta x^2}$$

$$\Delta^{(2)} = \frac{1}{\Delta x^2}(\mu_y \delta_x)^2 + \frac{1}{\Delta y^2}(\mu_x \delta_y)$$

$$\mu = (E_y^{1/2} + E_y^{-1/2})\frac{1}{2} \qquad \delta = (E^{1/2} - E^{-1/2})$$

$$\Delta^{(2)} = \frac{1}{4\Delta x^2}(E_y + 2 + E_y^{-1})(E_x - 2 + E_x^{-1})$$
$$+ \frac{1}{4\Delta y^2}(E_x + 2 + E_x^{-1})(E_y - 2 + E_y^{-1})$$

$$= \frac{1}{4\Delta x^2}(E_y E_x - \cancel{2E_y} + E_y E_x^{-1} + \cancel{2E_x} - \underline{4} + \cancel{2E_x^{-1}}$$
$$E_y^{-1} E_x - \cancel{2E_y^{-1}} + E_y^{-1} E_x^{-1})$$

$$+ \frac{1}{4\Delta y^2}(E_x E_y - \cancel{2E_x} + E_x E_y^{-1} + \cancel{2E_y} - \underline{4} + \cancel{2E_y^{-1}}$$
$$+ \underline{E_x^{-1}} E_y - \cancel{2E_x^{-1}} + E_x^{-1} E_y^{-1})$$

if $\Delta x = \Delta y$ $\qquad$ & $E_x E_y = E_y E_x$

$$= \frac{1}{4\Delta x^2}\{2E_x E_y + 2E_y E_x^{-1} - 8 + 2E_x E_y^{-1}$$
$$+ 2E_x^{-1} E_y^{-1}\}$$

$$= \frac{1}{2\Delta x^2}\{E_x E_y + E_y E_x^{-1} + E_x E_y^{-1} + E_x^{-1} E_y^{-1} - 4\}$$

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$$\Delta^{(2)} u_{ij} = \left[\left(\frac{1}{\Delta x} \mu_y \delta_x \right)^2 + \left(\frac{1}{\Delta y} \mu_x \delta_y^2 \right) \right] u_{ij}$$

Pg 175 gives $\mu^2 = 1 + \delta^2/4$

Pg 174 gives $\Delta x D = \delta - \frac{1}{3}\frac{\delta^3}{8} + O(\delta^5)$

$$\Downarrow$$

~~$\Delta x D \sim \delta \quad \delta \to 0$~~ $\quad \delta = \Delta x D + \frac{1}{24}\delta^3 + O(\delta^5)$

$\Rightarrow$ ~~$\Delta x D = \delta - \frac{1}{24}\Delta x$~~ $\Rightarrow \delta \sim \Delta x D \quad \delta \to 0$

$$\Rightarrow \delta = \Delta x D + \frac{1}{24}\Delta x^3 D^3 + O(\Delta x^5)$$

Thus $\mu^2 = 1 + \frac{1}{4}\delta^2 = 1 + \frac{1}{4}\left(\Delta x^2 D^2 + O(\Delta x^4) \right)$

$$= 1 + \frac{\Delta x^2}{4} D^2 + O(\Delta x^4)$$

$\therefore$

$$\Delta^{(2)} u_{ij} = \left(\left(1 + \frac{\Delta y^2 D_y^2}{4} + O(\Delta y^4)\right)\left(\frac{\partial}{\partial x} + \frac{1}{24}\Delta x^2 \frac{\partial^3}{\partial x^3} + O(\Delta x^4) \right) \right)^2$$

$$+ \left(\left(1 + \frac{\Delta x^2 D_x^2}{4} + O(\Delta x^4)\right)\left(D_y + \frac{1}{24}\Delta y^2 D_y^3 + O(\Delta y^4) \right) \right)^2$$

$$\Delta^{(2)} u_{ij} = \left(1 + \frac{\Delta y^2 D_y^2}{4}\right)\left(D_x^2 + \frac{1}{12}\Delta x^2 D_x^4\right) u_{ij}$$

$$+ \left(1 + \frac{\Delta x^2 D_x^2}{4}\right)\left(D_y^2 + \frac{1}{12}\Delta y^2 D_y^4\right) u_{ij} + O(\Delta x^4, \Delta y^4)$$

$$\Rightarrow \Delta^{(2)} u_{ij} = \Big\{ D_x^2 + \frac{1}{12}\Delta x^2 D_x^4 + \frac{\Delta y^2}{4} D_y^2 D_x^2$$

$$+ D_y^2 + \frac{\Delta y^2}{12} D_y^4 + \frac{\Delta x^2 D_x^2 D_y^2}{4} \Big\} u_{ij} + O(\Delta x^4 \Delta y^2, \Delta x \Delta y^3, \Delta x^2 \Delta y^2, \Delta x^3, \Delta y)$$

$$= \Delta u_{ij} + \frac{1}{12}\Delta x^2 D_x^4 u_{ij} + \frac{1}{12}\Delta y^2 D_y^4 u_{ij}$$

$$+ \frac{(\Delta x^2 + \Delta y^2)}{4} D_x^2 D_y^2 u_{ij} + O(\text{order } 4)$$

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w/ $\Delta^{(2)}$ if values of u can become → in Figure 4.4.4 Then the sum of the values w/ a's becomes 0 & the sum of the values w/ b's becomes 0.

If we put this initial condition into eq 4.4.6 then a & b must satisfy

$$(-4b)2 + b(2) + (-4a)4 = 0$$

$$-8b + 2b - 16a = 0$$

$$-6b = 16a \quad \Rightarrow \quad a = \frac{3}{8}b.$$

$$\Delta^{(3)} = a\Delta^{(1)} + b\Delta^{(2)} = \frac{a\delta_x^2}{\Delta x^2} + \frac{a\delta_y^2}{\Delta y^2} + \frac{b}{\Delta x^2}(\mu_y\delta_x)^2$$

$$+ \frac{b}{\Delta y^2}(\mu_x\delta_y)^2$$

w/ $\mu_x^2 = 1 + \frac{1}{4}\delta_x^2 \qquad \mu_y^2 = 1 + \frac{1}{4}\delta_y^2$

$$\Delta^{(3)} = a\frac{\delta_x^2}{\Delta x^2} + a\frac{\delta_y^2}{\Delta y^2} + \frac{b}{\Delta x^2}\left(1 + \frac{\delta_y^2}{4}\right)\delta_x^2 + \frac{b}{\Delta y^2}\left(1 + \frac{1}{4}\delta_x^2\right)\delta_y^2$$

$$= \frac{1}{\Delta x^2}\delta_x^2 + \frac{1}{\Delta y^2}\delta_y^2 + \frac{b}{4\Delta x^2}\delta_y^2\delta_x^2 + \frac{b}{4\Delta y^2}\delta_x^2\delta_y^2$$

if $\Delta x = \Delta y$ $\qquad\qquad \delta_y\delta_x = \delta_x\delta_y$

$$= \frac{1}{\Delta x^2}\left[(\delta_x^2 + \delta_y^2) + \frac{b}{2}\delta_x^2\delta_y^2\right]$$

$$= \Delta^{(1)} + \frac{b}{2\Delta x^2}\delta_x^2\delta_y^2$$

$$= \Delta^{(1)} + \frac{b}{2\Delta x^2}\left[\Delta x D_x + \frac{1}{24}\Delta x^3 D_x^3 + O(\Delta x^5)\right]^2 \cdot \left[\Delta x D_y + \frac{1}{24}\Delta x^3 D_y^3 + O(\Delta^5)\right]^2$$

$$\Rightarrow \Delta^{(3)} = \Delta^{(1)} + \frac{b}{2\Delta x^2}\left[\Delta x^2 D_x^2 + \frac{1}{12}\Delta x^4 D_x^4 + O(\Delta x^5)\right]$$

$$\left[\Delta x^2 D_y^2 + \frac{1}{12}\Delta x^4 D_y^4 + O(\Delta x^5)\right]$$

$$= \Delta^{(1)} + \frac{b}{2\Delta x^2}\left[\Delta x^4 D_x^2 D_y^2 + \frac{\Delta x^6 D_x^2 D_y^4}{12} + \frac{1}{12}\Delta x^6 D_x^4 D_y^2 + O(\Delta x^7)\right]$$

$$= \Delta^{(1)} + \frac{b}{2}\Delta x^2 D_x^2 D_y^2 + O(\Delta x^4)$$

$$\cancel{\frac{b}{2}\Delta x^2 D_x^2 D_y^2 + \frac{b}{2}\frac{\Delta x^4}{12}}$$

eq trying to solve

But $D_x^2 u_{ij} + D_y^2 u_{ij} = 0$ take D_x^2 on eq

$\Rightarrow \cancel{\Delta^{(3)} = \Delta^{(1)} + \frac{b}{2}\Delta x^2}$ & take D_y^2 on eq

$$\Rightarrow \quad D_x^4 u_{ij} + D_x^2 D_y^2 u_{ij} = 0$$

$$\& \quad D_y^2 D_x^2 u_{ij} + D_y^4 u_{ij} = 0 \qquad \text{Then Add}$$

$$\Rightarrow \quad 2D_y^2 D_x^2 u_{ij} + D_x^4 u_{ij} + D_y^4 u_{ij} = 0$$

$$\Rightarrow \Delta^{(3)} = \Delta^{(1)} + \frac{b}{}$$

$$\therefore \quad \Delta^{(3)} = \Delta^{(1)} - \frac{b}{4}\Delta x^2 (D_x^4 + D_y^4) + O(\Delta x^4)$$

$(D_x^2 + D_y^2)U = 0$ Apply $D_x^2 + D_y^2$ onto this eq

$D_x^4 + 2D_x^2 D_y^2$ ... same eq as Above.

Now $\Delta^{(3)} = \Delta^{(1)} + \frac{b}{2}\Delta x^2 D_x^2 D_y^2 + O(\Delta x^4)$

$$\Delta^{(1)} = \frac{1}{\Delta x^2}(\delta_x^2 + \delta_y^2) = \left(D_x + \frac{1}{24}\Delta x^2 D_x^3 + O(\Delta x^4)\right)^2 + \left(D_y + \frac{1}{24}\Delta y^2 D_y^3 + O(\Delta y^4)\right)^2$$

$$= D_x^2 + \frac{1}{12}\Delta x^2 D_x^4 + O(\Delta x^4)$$

$$+ D_y^2 + \frac{1}{12}\Delta y^2 D_y^4 + O(\Delta y^4)$$

$\therefore$ w/ $\Delta x = \Delta y$

$$\Delta^{(3)} = D_x^2 + D_y^2 + \frac{1}{12}\Delta x^2 \left\{ D_x^4 + D_y^4 + 6b D_x^2 D_y^2 \right\} + O(\Delta x^4)$$

eq 4.4.13

If $b = 2/3$ $a = 1/3$ & $\Delta^{(3)}$

$$\Delta^{(3)} = \frac{1}{3}\Delta^{(1)} + \frac{2}{3}\Delta^{(2)} \Rightarrow \frac{1}{3\Delta x^2}\left\{ U_{i+1,j} + U_{i-1,j} + U_{i,j-1} + U_{i,j+1} - 4U_{i,j} \right\} +$$

$$\frac{1}{2\cdot 3\Delta x^2}\left(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j-1} + U_{i-1,j+1} - 4U_{i,j}\right)$$

$$\Rightarrow \delta^{(3)}U_{i,j} = \frac{1}{3\Delta x^2}\Big\{ U_{i+1,j} + U_{i-1,j} + U_{i,j-1} + U_{i,j+1}$$

$$-4U_{i,j} - 2U_{i,j} + \tfrac{1}{2}U_{i+1,j+1} + \tfrac{1}{2}U_{i+1,j-1}$$

$$+ \tfrac{1}{2}U_{i-1,j-1} + \tfrac{1}{2}U_{i-1,j+1} \Big\}$$

$$= \frac{1}{3\Delta x^2}\Big\{$$

...

w/ General b value ...

$$\Delta^{(3)}U_{ij} = (a\Delta^{(1)} + b\Delta^{(2)})U_{ij} \qquad a+b=1$$

$$= \frac{1}{\Delta x^2}\left(aU_{i+1,j} + aU_{i-1,j} + aU_{i,j-1} + aU_{i,j+1} - 4aU_{i,j}\right)$$

$$\frac{1}{4\Delta x^2}\left(bU_{i+1,j+1} + bU_{i+1,j-1} + bU_{i-1,j-1} + bU_{i-1,j+1} - 4bU_{i,j}\right)$$

$$= \frac{1}{4\Delta x^2}\Big(4aU_{i+1,j} + 4aU_{i-1,j} + 4aU_{i,j-1} + 4aU_{i,j+1}$$

$$-16aU_{i,j} + bU_{i+1,j+1} + bU_{i+1,j-1} + bU_{i-1,j-1} + bU_{i-1,j+1}$$

$$-4bU_{i,j}\Big)$$

$$\Delta^{(3)} u_{ij} = \frac{1}{4\Delta x^2}\Big(4a u_{i+1,j} + 4a u_{i-1,j} + 4a u_{i,j-1} + 4a u_{i,j+1}$$
$$- (16a + 4b) u_{ij} + b u_{i+1,j+1} + b u_{i+1,j-1}$$
$$+ b u_{i-1,j-1} + b u_{i-1,j+1} \Big)$$

... How get Figures 4.4.6 ...?

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$b = 1/3$ truncation error =

$$\Delta^{(3)} \cancel{u_{ij}} \; \Delta u_{ij} = \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)$$

$$\Delta u - \Delta^{(3)} u = -\frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)$$

$$= -\frac{\Delta x^2}{12}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u$$

$$= -\frac{\Delta x^2}{12}\Delta^2 u$$

Thus solving $\Delta^2 u = \Delta u$ using $\Delta^{(3)} = \frac{2}{3}\Delta^{(1)} + \frac{1}{3}\Delta^{(2)}$

gives truncation error of $\Delta u - \Delta^{(3)} u = -\frac{\Delta x^2}{12}\Delta^2 u$

$\therefore$ The scheme $\Delta^{(3)} u$ solves

$$\Delta^{(3)} u_{ij} = \Delta u + \frac{\Delta x^2}{12}\Delta^2 u$$

$$\parallel$$

$$\Delta u$$

$\Rightarrow \Delta^{(3)} u_{ij} \overset{?}{=} \left(1 + \frac{\Delta x^2 \Delta^2}{12}\right) u$ to 4th order?

How get 4th & not 3rd

PS 19.2 Hub VolI

$$U_{xy} = D_x D_y U = \frac{1}{\Delta x} \mu_x \left(\delta_x - \frac{1}{6} \delta_x^3 + O(\delta_x^5) \right)$$

$$\cdot \frac{1}{\Delta y} \mu_y \left(\delta_y - \frac{1}{6} \delta_y^3 + O(\delta_y^5) \right) U$$

By 4.2.30

$$\Delta x D = \mu \delta - \mu \frac{\delta^3}{3!} + O(\delta^5) \Rightarrow \Delta x D \sim \mu \delta \quad \delta \to 0$$

$$\Rightarrow \delta = O(\Delta x)$$

$$\Rightarrow U_{xy} = \frac{1}{\Delta x \Delta y} \mu_x \delta_x \left(1 - \frac{1}{6} \delta_x^2 + O(\Delta x^4) \right) \cdot$$

As $\mu = O(1)$, $D = O(1)$ & value of $\int$ the derivative is order 1.

$$\mu_y \delta_y \left(1 - \frac{1}{6} \delta_y^2 + O(\Delta x^4) \right) U$$

2nd order accuracy for each of D_x & D_y would involve

$\frac{1}{\Delta x} \mu_x \delta x$ & $\frac{1}{\Delta y} \mu_y \delta_y$ respectively $\therefore$ 2nd order

for $U_{xy} = \frac{1}{\Delta x \Delta y} \mu_x \delta x \mu_y \delta_y U + O(\Delta x^2, \Delta y^2, \Delta x \Delta y)$

$$= \mu_x \frac{\delta x}{\Delta x} \cdot \mu_y \frac{\delta y}{\Delta y} + O(\Delta x^2) \qquad \Delta x = \Delta y$$

$$= \frac{1}{2\Delta x} \left(E_x^{1/2} + E_x^{-1/2} \right) \left(E_x^{1/2} - E_x^{-1/2} \right) \cdot$$

$$\frac{1}{2\Delta y} \left(E_y^{1/2} + E_y^{-1/2} \right) \left(E_y^{1/2} - E_y^{-1/2} \right) U + O(\Delta x^2)$$

$$U_{xy} = \frac{1}{4\Delta x \Delta y}\left(E_x^{1} - E_x^{-1}\right)\left(E_y^{1} - E_y^{-1}\right)U_{ij} + O(\Delta x^2)$$

$$= \frac{1}{4\Delta x \Delta y}\left(E_x E_y - E_x E_y^{-1} - E_x^{-1} E_y + E_x^{-1} E_y^{-1}\right)U_{ij} + O(\Delta x^2)$$

$$= \frac{1}{4\Delta x \Delta y}\left(U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}\right) + O(\Delta x^2)$$

$$U_{xy} = D_x D_y U = \frac{\mu_x \delta_x}{\Delta x}\frac{\delta_y^+}{\Delta y}U_{ij} + O(\Delta x^2, \Delta y)$$

2nd order in x

1st order in y

$$= \frac{1}{2\Delta x \Delta y}\left(E_x^{1/2} + E_x^{-1/2}\right)\left(E_x^{1/2} - E_x^{-1/2}\right)\left(E_y - 1\right)U_{ij} + \ldots$$

$$= \frac{1}{2\Delta x \Delta y}\left(E_x - E_x^{-1}\right)\left(E_y - 1\right)U_{ij} + O(\;)$$

$$= \frac{1}{2\Delta x \Delta y}\left(E_x E_y - E_x - E_x^{-1} E_y + E_x^{-1}\right)U_{ij} + O(\;)$$

$$= \frac{1}{2\Delta x \Delta y}\left(U_{i+1,j+1} - U_{i+1,j} - U_{i-1,j+1} + U_{i-1,j}\right) + O(\;)$$

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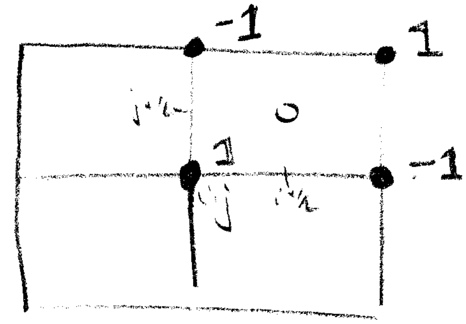
1st order formula is given by using $\Delta_x \approx \frac{\delta_x^+}{\Delta x}$ and $\Delta_y \approx \frac{\delta_y^+}{\Delta y}$

$$\Rightarrow U_{xy} = \frac{1}{\Delta x \Delta y} \delta_x^+ \delta_y^+ U_{ij} + O(\Delta x, \Delta y)$$

$$= \frac{1}{\Delta x \Delta y} (E_x - 1)(U_{i,j+1} - U_{ij}) + O(\quad)$$

$$= \frac{1}{\Delta x \Delta y} (U_{i+1,j+1} - U_{i+1,j} - U_{i,j+1} + U_{ij}) + O(\quad)$$

computational molecule then would be



$$(U_{xy})_{i+1/2,j+1/2} = \frac{1}{\Delta x \Delta y} (U_{i+1,j+1} - U_{i+1,j} - U_{i,j+1} + U_{ij}) + O(\Delta x^2, \Delta y^2)$$

$$= \frac{1}{\Delta x \Delta y} \delta_x \delta_y U_{i+1/2,j+1/2} + O(\quad)$$

Backwards differences in both directions gives

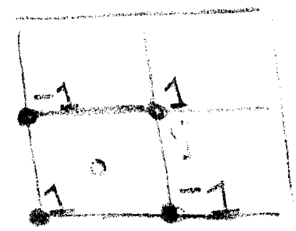
$$(U_{xy})_{ij} = \frac{1}{\Delta x \Delta y} (\delta_x^- \delta_y^-) U_{ij} + O(\Delta x, \Delta y)$$

$$= \frac{1}{\Delta x \Delta y} (1 - E_x^-)(1 - E_y^-) U_{ij} + O(\Delta x, \Delta y)$$

$$(u_{xy}) = \frac{1}{\Delta x \Delta y}(1 - E_x^{-1})(u_{ij} - u_{ij-1}) + O(\Delta x \Delta y)$$

$$= \frac{1}{\Delta x \Delta y}(u_{ij} - u_{ij-1} - u_{i-1j} + u_{i-1j-1}) + O(\quad)$$

$$= \frac{1}{\Delta x \Delta y}\delta_x \delta_y u_{i-\frac{1}{2}, j-\frac{1}{2}} + O(\Delta x^2 \Delta y^2)$$



[illegible] mor 4.4.19 & 4.4.17

$$(u_{xy})_{ij} + (u_{xy})_{ij} = \frac{1}{\Delta x \Delta y}[\delta_x^+ \delta_y^+ + \delta_x^- \delta_y^-]u_{ij} + \underbrace{O(\Delta x^2 \Delta y^2)}_{\text{see problem}}$$

$$= \frac{1}{\Delta x \Delta y}\{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} - u_{ij} +$$

$$u_{i-1,j-1} - u_{i-1,j} - u_{i,j-1} + \underline{u_{ij}}\} + O(\quad)$$

$\Rightarrow$

$$u_{xy} = \frac{1}{2\Delta x \Delta y}\{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i-1,j-1} - u_{i-1,j}$$

$$- u_{i,j-1} + 2u_{ij}\} + O(\quad)$$

$$= \frac{1}{2\Delta x \Delta y}\{$$

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Adding eqs 4.4.19 + 4.4.17 gives

$$(u_{xy})_i = \frac{1}{\Delta x \Delta y}\delta_x^- \delta_y^- u_{ij} + O(\Delta x, \Delta y) =$$

$$= \frac{1}{\Delta x \Delta y}\left(u_{i-1,j-1} - u_{i-1,j} - u_{i,j-1} + u_{ij}\right) + O(\Delta x, \Delta y)$$

$$+ \quad (u_{xy}) = \frac{1}{\Delta x \Delta y}\left(\delta_x^+ \delta_y^+\right) u_{ij} + O(\Delta x, \Delta y) =$$

$$= \frac{1}{\Delta x \Delta y}\left(u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i,j}\right) + O(\Delta x, \Delta y)$$

$$(u_{xy})_i = \frac{1}{2\Delta x \Delta y}\left[\delta_x^+ \delta_y^+ + \delta_x^- \delta_y^-\right] u_{ij} + O(\Delta x^2, \Delta y^2)$$

$$= \frac{1}{2\Delta x \Delta y}\left[u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i-1,j-1} - u_{i-1,j} - u_{i,j-1} + 2u_{ij}\right] + O(\Delta x^2, \Delta y^2)$$

eq 4.4.20.

to Derive eq 4.4.21 ... pg 174 Hirsch Vol I

$$(u_{xy})_i = \frac{1}{\Delta x \Delta y} \delta_x^+ \delta_y^- u_{ij} + O(\Delta x, \Delta y)$$

$$= \frac{1}{\Delta x \Delta y}\left[(E_x - 1)(E_y^{-1} - 1)\right] u_{ij} + O(\Delta x, \Delta y)$$

obtain leading order truncation error for $\delta_x^+ \delta_y^{-1}$ scheme

$$(u_{xy})_i \cong \frac{1}{\Delta x \Delta y} \delta_x^+ \delta_y^- u_{ij} \quad w/ \ \delta^+$$

For forward difference $\Delta x D_x = \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} - \cdots$ pg 173 Hirsch Vol I

$\Rightarrow$ As $\Delta x \to 0$ $\quad \Delta x D_x \cong \delta^+ \quad \Delta x \to 0$

$$\Rightarrow \Delta x D_x = \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} + O(\delta^{+4}) = \delta^+ - \frac{\Delta x^2 D^2}{2} + \frac{\Delta x^3 D^3}{3} + O(\Delta^4)$$

For backwards difference $\Delta x D = \delta^- + \frac{\delta^{-2}}{2} +$

rearranging gives $\quad D = \frac{\delta^-}{\Delta x} + \frac{\Delta x D^2}{2} + \frac{\Delta x^2 D^3}{3} + O(\Delta x^3)$

$$\therefore \frac{\delta^+}{\Delta x} = D_x + \frac{\Delta x D_x^2}{2} - \frac{\Delta x^2 D_x^3}{3} + O(\Delta x^4)$$

$$\frac{\delta^-}{\Delta x} = D_x - \frac{\Delta x D_x^2}{2} - \frac{\Delta x^2 D_x^3}{3} + O(\Delta x^4)$$

$$\Rightarrow \frac{1}{\Delta x}\frac{1}{\Delta y}\delta_x^+\delta_y^- = \frac{1}{\Delta y}\delta_y^-\left[U_x + \frac{\Delta x\, U_x^2}{2} - \frac{\Delta x^2}{3} U_x^3 + O(\Delta x^4) \right]$$ 2

$$= \cancel{U_{xy}}$$

$$= \left[D_y - \frac{\Delta y\, D_y^2}{2} - \frac{\Delta y^2}{3} D_y^3 + O(\Delta y^4) \right]\left[U_x + \frac{\Delta x\, U_x^2}{2} \ldots \right.$$

$$= U_{xy} + \Delta x\, U_x U_{xy} - \Delta x^2 U_x^2 U_{xy} - \frac{\Delta y}{2} U_{xyy}$$

$$\frac{1}{\Delta x \Delta y}\delta_x^+\delta_y^- U = \left\{ D_x + \frac{\Delta x\, D_x^2}{2} - \frac{\Delta x^2}{3} D_x^3 + O(\Delta x^4) \right\}\left\{ D_y + \frac{\Delta y\, D_y^2}{2} - \frac{\Delta y^2}{3} D_y^3 + O(\Delta y^4) \right\} U$$

$$= \left\{ D_{xy} + \Delta y \frac{D_x D_y^2}{2} - \frac{\Delta y^2}{3} D_x D_y^3 + \Delta x \frac{D_x^2 D_y}{2} - \frac{\Delta x \Delta y\, D_x^2 D_y^2}{4} - \frac{\Delta x^2}{3} D_x^3 D_y + O(\Delta x^3, \Delta y^3) \right\} U$$

$$\Rightarrow D_{xy} U = \frac{1}{\Delta x \Delta y}\delta_x^+\delta_y^- - \frac{\Delta y}{2} D_x D_y^2 U - \frac{\Delta x}{2} D_x^2 D_y U + \frac{\Delta x \Delta y\, D_x^2 D_y^2 U}{4}$$

TO Derive 4.4.21 Pg 194 Hoff. Vol I

From eq. 4.2.13 $\Delta x D = \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} - \frac{\delta^{+4}}{4} + O(\delta^{+5})$

$$\& \ \Delta x D = \delta^- + \frac{\delta^{-2}}{2} + \frac{\delta^{-3}}{3} + \frac{\delta^{-4}}{4} + O(\delta^{-5})$$

Thus As $\Delta x \to 0$ $\Delta x D \approx \delta^+$ & $\Delta x D \approx \delta^-$

Thus $\Delta x D = \delta^+ - \frac{\Delta x^2 D^2}{2} + \frac{\Delta x^3 D^3}{3} + O(\Delta x^4)$

$$\& \ \Delta x D = \delta^- + \frac{\Delta x^2 D^2}{2} + \frac{\Delta x^3 D^3}{3} + O(\Delta x^4)$$

Thus

$$\frac{\delta^+}{\Delta x} = D + \frac{\Delta x D^2}{2} - \frac{\Delta x^2 D^3}{3} + O(\Delta x^4)$$

$$\frac{\delta^-}{\Delta x} = D - \frac{\Delta x D^2}{2} - \frac{\Delta x^2 D^3}{3} + O(\Delta x^4)$$

so

$$\frac{\delta_x^+}{\Delta x}\frac{\delta_y^-}{\Delta y} u = \left(D_x + \frac{\Delta x D_x^2}{2} - \frac{\Delta x^2 D_x^3}{3} + O(\Delta x^4)\right)\left(D_y - \frac{\Delta y D_y^2}{2} - \frac{\Delta y^2 D_y^3}{3} + O(\Delta y^4)\right) u$$

$$= \Big(D_x D_y - \frac{\Delta y}{2} D_x D_y^2 - \frac{\Delta y^2}{3} D_x D_y^3 + \frac{\Delta x}{2} D_x^2 D_y - \frac{\Delta x \Delta y}{4} D_x^2 D_y^2$$

$$- \frac{\Delta x \Delta y^2}{6} D_x^2 D_y^3 - \frac{\Delta x^2 D_x^3 D_y}{3} + \frac{\Delta x^2 \Delta y D_x^3 D_y^2}{6} + \frac{\Delta x^2 \Delta y^2 D_x^3 D_y^3}{9}$$

$$+ O(\Delta x^4 \Delta y^4) \Big) u$$

$$\frac{\delta_x^+ \delta_y^-}{\Delta x \Delta y} U = \Big\{ D_x D_y - \frac{\Delta y}{2} D_x D_y^2 - \frac{\Delta x \Delta y}{4} D_x^2 D_y^2 + \frac{\Delta x}{2} D_x^2 D_y$$

$$- \frac{\Delta y^2}{3} D_x D_y^3 \qquad - \frac{\Delta x^2 D_x^3 D_y}{3}$$

$$+ O(\Delta x^3, \Delta y^3) \Big\} U$$

Then $\frac{\delta_x^-}{\Delta x} \frac{\delta_y^+}{\Delta y} U = \Big(D_x - \frac{\Delta x}{2} D_x^2 - \frac{\Delta x^2 D_x^3}{3} + O(\Delta x^4) \Big) \Big(D_y + \frac{\Delta y}{2} D_y^2 - \frac{\Delta y^2}{3} D_y^3$

$+ O(\Delta y^4) \Big) U$

$= D_x D_y + \frac{\Delta y}{2} D_x D_y^2 \quad \ldots\ldots$

replace $y \leftrightarrow x$ in above expression

$$\therefore \frac{\delta_x^-}{\Delta x} \frac{\delta_y^+}{\Delta y} U = \Big\{ D_y D_x - \frac{\Delta x}{2} D_y D_x^2 - \frac{\Delta x \Delta y}{4} D_x^2 D_y^2 + \frac{\Delta y}{2} D_y^2 D_x$$

$$- \frac{\Delta x^2}{3} D_y D_x^3 \qquad - \frac{\Delta y^2}{3} D_y^3 D_x$$

$$+ O(\Delta x^3, \Delta y^3) \Big\} U$$

Then Adding these 2 expressions gives

$$\Big\{ \frac{\delta_x^+}{\Delta x} \frac{\delta_y^-}{\Delta y} + \frac{\delta_x^-}{\Delta x} \frac{\delta_y^+}{\Delta y} \Big\} U = \Big\{ 2 D_x D_y + 0 \Delta y - \frac{\Delta x \Delta y}{2} D_x^2 D_y^2$$

$$+ 0 \Delta x \quad - \frac{2\Delta y^2}{3} D_x D_y^3 - \frac{2}{3} \Delta x^2 D_x^3 D_y + O(\Delta x^3, \Delta y^3) \Big\} U$$

$$\Rightarrow \quad U_{xy} = \frac{1}{2\Delta x \Delta y}\left(\delta_x^+ \delta_y^- + \delta_x^- \delta_y^+ \right) U_{ij}$$

$$+ \frac{\Delta x \Delta y}{2} U_{xxyy} + \frac{2}{3}\Delta y^2 U_{xyyy} + \frac{2}{3}\Delta x^2 U_{xxxy}$$

$$+ O(\Delta x^3, \Delta y^3) \qquad \text{eq 4.4.21}$$

$$= \frac{1}{2\Delta x \Delta y}\Big((E_x - 1)(1 - E_y^{-1}) + (1 - E_x^{-1})(E_y - 1) \Big) U$$

$$+ O(\Delta x^2, \Delta y^2)$$

$$= \frac{1}{2\Delta x \Delta y}\Big(E_x - E_x E_y^{-1} - 1 + E_y^{-1} + E_y - 1 - E_x^{-1} E_y + E_x^{-1} \Big) U$$

$$+ O(\quad)$$

$$= \frac{1}{2\Delta x \Delta y}\Big(\text{[illegible]} - E_x E_y^{-1} \text{[illegible]} E_y - 2 + E_y + E_x^{-1} \Big) U + O(\quad)$$

$$= \frac{1}{2\Delta x \Delta y}\Big(U_{i+1,j} - U_{i+1,j-1} + U_{i,j-1} + U_{i-1,j+1} - 2U_{i,j} + U_{i,j+1}$$

$$+ U_{i-1,j} \Big) + O(\quad) \qquad \text{eqs 4.4.21}$$

$$= \frac{1}{2\Delta x \Delta y}\Big(\delta_x \delta_y U_{i+1/2,j-1/2} + \delta_x \delta_y U_{i-1/2,j+1/2} \Big) + O(\Delta x^2, \Delta y^2)$$

This last expression is just an equivalent expression for the one above

Adding 4.4.20 & 4.4.21 we get.

$$2(u_{xy})_{ij} = \frac{1}{2\Delta x\Delta y}\left[U_{i+1,j+1} + U_{i-1,j-1} - U_{i+1,j-1} - U_{i-1,j+1}\right] + O(\Delta x^2)$$

$$\Rightarrow (U_{xy})_{ij} = \frac{1}{4\Delta x\Delta y}\left[U_{i+1,j+1} + U_{i-1,j-1} - U_{i+1,j-1} - U_{i-1,j+1}\right] + O(\Delta x^2)$$

or 4.4.15

From 4.4.20 one can write

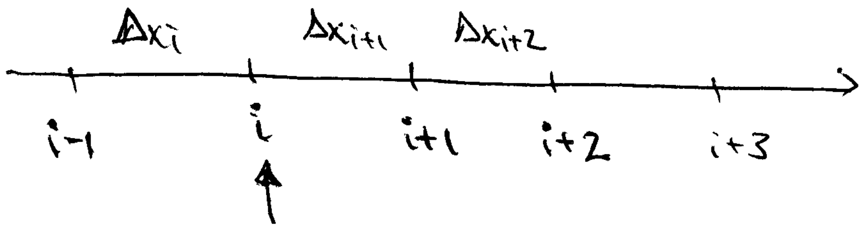
$$(U_{xy})_{ij} = \frac{1}{2\Delta x\Delta y}\left[\delta_x\delta_y U_{i+1/2,j+1/2} + \delta_x\delta_y U_{i-1/2,j-1/2}\right]U_{ij} + O(\Delta x^2\Delta y^2)$$

Thus the most general (How know?) is given by linear combo of 4.4.20 + 4.4.21 giving

$$(U_{xy})_{ij} = a\underbrace{(U_{xy})_{ij}}_{4.4.20} + b\underbrace{(U_{xy})_{ij}}_{4.4.21} \quad w/\ a+b=1$$

$$= \frac{1}{2\Delta x\Delta y}\delta_x\delta_y\left\{aU_{i+1/2,j+1/2} + aU_{i-1/2,j-1/2} + bU_{i+1/2,j-1/2} + bU_{i-1/2,j+1/2}\right\}$$

pg 196 ~~[illegible]~~ Hirsch



$$(U_x)_i = \frac{U_{i+1} - U_i}{\Delta x_{i+1}}$$

Forward difference:

consider

$$\frac{U_{i+1} - U_i}{\Delta x_{i+1}} = \frac{\cancel{U_i} + U_i' \Delta x_{i+1} + U_i'' \frac{\Delta x_{i+1}^2}{2} + O(\Delta x_{i+1}^3) - \cancel{U_i}}{\Delta x_{i+1}}$$

$$= U_i' + U_i'' \frac{\Delta x_{i+1}}{2} + O(\Delta x_{i+1}^2)$$

$$\Rightarrow (U_x)_i = \frac{U_{i+1} - U_i}{\Delta x_{i+1}} - \frac{\Delta x_{i+1}}{2}(U_{xx})_i + O(\Delta x_{i+1}^2) \qquad 4.5.1$$

Backwards differences:

$$(U_x)_i \cong \frac{U_i - U_{i-1}}{\Delta x_i} + ?$$ So consider

$$\frac{\cancel{U_i} - \left[\cancel{U_i} - \Delta x_i (U_x)_i + \frac{\Delta x_i^2}{2}(U_{xx})_i + O(\Delta x_i^3)\right]}{\Delta x_i}$$

$$= (U_x)_i - \frac{\Delta x_i}{2}(U_{xx})_i + O(\Delta x_i^2)$$

$$\Rightarrow (U_x)_i = \frac{U_i - U_{i-1}}{\Delta x_i} + \frac{\Delta x_i}{2}(U_{xx})_i + O(\Delta x_i^2) \qquad 4.5.2$$

to elliminate 1st order terms from 4.5.1 & 4.5.2

mult 4.5.1 by Δx_i & 4.5.2 by Δx_{i+1} & Add

(see problem 4.11 for further taylor expansion of terms)

$$\Rightarrow (\Delta x_i + \Delta x_{i+1})(U_x)_i = \left[\left(\frac{\Delta x_i}{\Delta x_{i+1}}\right)(U_{i+1} - U_i)\right.$$

$$\left. + \left(\frac{\Delta x_{i+1}}{\Delta x_i}\right)(U_i - U_{i-1})\right] - \frac{1}{6}\left[\Delta x_i \Delta x_{i+1}^2 + \Delta x_{i+1}\Delta x_i^2\right](U_{xxx})_i$$

$$+ O(\Delta x^4)$$

$\Rightarrow$

$$\Rightarrow (U_x)_i = \cancel{\frac{\Delta x_i}{\Delta x_{i+1}}(U_{i+1}-U_i) + \frac{\Delta x_{i+1}}{\Delta x_i}(U_i - U_{i-1})}$$

$$= \frac{1}{(\Delta x_i + \Delta x_{i+1})}\left[\frac{\Delta x_i}{\Delta x_{i+1}}(U_{i+1}-U_i) + \frac{\Delta x_{i+1}}{\Delta x_i}(U_i - U_{i-1})\right]$$

$$- \frac{1}{6}\Delta x_i \Delta x_{i+1} (U_{xxx})_i + O(\Delta x^3) \qquad 4.5.4.$$

Forward formula:

$$(U_x)_i \cong aU_i + bU_{i+1} + cU_{i+2}$$

$$\| $$

$$aU_i + b\left[U_i + \Delta x_{i+1}(U_x)_i + \frac{\Delta x_{i+1}^2}{2}(U_{xx})_i + \frac{\Delta x_{i+1}^3}{6}(U_{xxx})_i + \cdots\right]$$

$$+ c\left[U_{i+1} + \Delta x_{i+2}(U_x)_{i+1} + \frac{\Delta x_{i+2}^2}{2}(U_{xx})_{i+1} + \frac{\Delta x_{i+2}^3}{6}(U_{xxx})_{i+1} + \cdots\right]$$

$$= \quad '' \quad + \quad '' \quad +$$

$$c\Big[U_i + \Delta x_{i+1}(U_x)_i + \frac{\Delta x_{i+1}^2}{2}(U_{xx})_i + \cdots$$

$$+ \Delta x_{i+2}\Big[(U_x)_i + \Delta x_{i+1}(U_{xx})_i + \frac{\Delta x_{i+1}^2}{2}(U_{xxx})_i + \cdots \Big]$$

$$+ \frac{\Delta x_{i+2}^2}{2}\Big[(U_{xx})_i + \Delta x_{i+1}(U_{xxx})_i + \frac{\Delta x_{i+1}^2}{2}(U_{xxxx})_i + \cdots \Big] \Big]$$

$$\Rightarrow (U_x)_i = aU_i + bU_i + b\Delta x_{i+1}(U_x)_i + b\frac{\Delta x_{i+1}^2}{2}(U_{xx})_i$$

$$+ b\frac{\Delta x_{i+1}^3}{6}(U_{xxx})_i + \cdots + cU_i + c\Delta x_{i+1}(U_x)_i$$

$$+ c\frac{\Delta x_{i+1}^2}{2}(U_{xx})_i + c\Delta x_{i+2}(U_x)_i + c\Delta x_{i+2}\Delta x_{i+1}(U_{xx})_i$$

$$+ c\Delta x_{i+2}\frac{\Delta x_{i+1}^2}{2}(U_{xxx})_i + c\frac{\Delta x_{i+2}^2}{2}(U_{xx})_i$$

$$+ c\frac{\Delta x_{i+2}^2}{2}\Delta x_{i+1}(U_{xxx})_i + \frac{\Delta x_{i+2}^2}{2}\frac{\Delta x_{i+1}^2}{2}(U_{xxxx})_i + \cdots$$

$$(U_x)_i = (a+b+c)U_i + (b\Delta x_{i+1} + c\Delta x_{i+1} + c\Delta x_{i+2})(U_x)_i$$

$$+ \left(\frac{b\Delta x_{i+1}^2}{2} + \frac{c\Delta x_{i+1}^2}{2} + c\Delta x_{i+2}\Delta x_{i+1} + c\frac{\Delta x_{i+2}^2}{2}\right)(U_{xx})_i$$

$$+\left(\frac{b\Delta x_{i+1}^3}{6} + \frac{c\Delta x_{i+2}\Delta x_{i+1}^2}{2} + \frac{c\Delta x_{i+2}^2\Delta x_{i+1}}{2}\right)(U_{xxx})_i$$

$$\Rightarrow \quad a+b+c = 0$$

$$\Delta x_{i+1} b + (\Delta x_{i+1} + \Delta x_{i+2})c = 1$$

$$\frac{b\Delta x_{i+1}^2}{2} + \frac{1}{2}(\Delta x_{i+1} + \Delta x_{i+2})^2 c = 0$$

$b = -\dfrac{(\Delta x_{i+1} + \Delta x_{i+2})^2}{\Delta x_{i+1}^2}c$ into eq (2)

$$\Rightarrow \left[-\frac{(\Delta x_{i+1} + \Delta x_{i+2})^2}{\Delta x_{i+1}} + (\Delta x_{i+1} + \Delta x_{i+2})\right]c = 1$$

$$\Rightarrow \quad c = \frac{+1}{(\Delta x_{i+1} + \Delta x_{i+2})\left[-\frac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+1}} + \frac{\Delta x_{i+1}}{\Delta x_{i+1}}\right]}$$

$$= \frac{\Delta x_{i+1}}{(\Delta x_{i+1} + \Delta x_{i+2})(-\Delta x_{i+2})} = \frac{-\Delta x_{i+1}}{(\Delta x_{i+2})(\Delta x_{i+1} + \Delta x_{i+2})}$$

Then $b = \dfrac{\cancel{\Delta x_{i+1}}(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2} \quad \Delta x_{i+1}^{\cancel{2}}} = \dfrac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+2}\,\Delta x_{i+1}}$

& $a = -b - c = \dfrac{\Delta x_{i+1}}{\Delta x_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})} - \dfrac{(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2}\,\Delta x_{i+1}}$

$$= \frac{1}{\Delta x_{i+2}\,\Delta x_{i+1}(\Delta x_{i+1} + \Delta x_{i+2})}\left[\underbrace{\Delta x_{i+1}^2 - (\Delta x_{i+1} + \Delta x_{i+2})^2}_{\Delta x_{i+1}}\right]$$

Note: $\Delta x_{i+1} + \Delta x_{i+2} = x_{\cancel{i+1}} - x_i + x_{i+2} - \cancel{x_{i+1}}$

$$= x_{i+2} - x_i$$

Now:

$$(u_x)_i = \frac{\Delta x_{i+1} u_i}{\Delta x_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})} - \frac{(\Delta x_{i+1} + \Delta x_{i+2}) u_i}{\Delta x_{i+2} \Delta x_{i+1}}$$

$$+ \frac{(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2} \Delta x_{i+1}} u_{i+1} - \frac{\Delta x_{i+1}}{\Delta x_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})} u_{i+2}$$

$$+ \left(\frac{\Delta x_{i+1} + \Delta x_{i+2}}{\Delta x_{i+2} \Delta x_{i+1}} \frac{\Delta x_{i+1}^3}{6} + \frac{-\Delta x_{i+1}}{\cancel{\Delta x_{i+2}}(\Delta x_{i+1} + \Delta x_{i+2})} \frac{\cancel{\Delta x_{i+2}} \Delta x_{i+1}^2}{2} \right.$$

$$\left. - \frac{\Delta x_{i+1}}{\cancel{\Delta x_{i+2}}(\Delta x_{i+1} + \Delta x_{i+2})} \frac{\Delta x_{i+2}^2 \Delta x_{i+1}}{2} \right) (u_{xxx})_i$$

$$\Rightarrow \cancel{(u_x)_i = \frac{1}{(\Delta x_{i+1} + \Delta x_{i+2})} \left[\frac{\Delta x_{i+1}}{\Delta x_{i+2}} (u_i + u_{i+2}) \right.}$$

$+$

$$\Rightarrow (U_x)_i = \frac{1}{(\Delta x_{i+1} + \Delta x_{i+2})} \left[\frac{\Delta x_{i+1}}{\Delta x_{i+2}} U_i - \frac{\Delta x_{i+1}}{\Delta x_{i+2}} U_{i+2} \right]$$

$$+ \frac{(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2} \Delta x_{i+1}} (U_{i+1} - U_i)$$

$$+ \left(\frac{(\Delta x_{i+1} + \Delta x_{i+2}) \Delta x_{i+1}^2}{6 \Delta x_{i+2}} - \frac{\Delta x_{i+1}^3}{2(\Delta x_{i+1} + \Delta x_{i+2})} - \frac{\Delta x_{i+1}^2 \Delta x_{i+2}}{2(\Delta x_{i+1} + \Delta x_{i+2})} \right) (U_{xxx})_i$$

$$\Rightarrow (U_x)_i = \frac{\Delta x_{i+1}}{\Delta x_{i+2}(\Delta x_{i+1} + \Delta x_{i+2})} (U_i - U_{i+2})$$

$$+ \frac{(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2} \Delta x_{i+1}} (U_{i+1} - U_i)$$

$$+ \frac{\Delta x_{i+1}^2}{6} \left[\frac{(\Delta x_{i+1} + \Delta x_{i+2})}{\Delta x_{i+2}} - \frac{3 \Delta x_{i+1}}{(\Delta x_{i+1} + \Delta x_{i+2})} - \frac{3 \Delta x_{i+2}}{(\Delta x_{i+1} + \Delta x_{i+2})} \right] (U_{xxx})_i$$

last 2 terms Add to -3

$$\Rightarrow (U_x)_i = \frac{-\Delta x_{i+1}}{\Delta x_{i+2}(\Delta x_{i+1}+\Delta x_{i+2})}(U_{i+2}-U_i) + \frac{(\Delta x_{i+1}+\Delta x_{i+2})}{\Delta x_{i+1}\,\Delta x_{i+2}}(U_{i+1}-U_i)$$

$$+ \frac{\Delta x_{i+1}^2}{6}\left(\underbrace{\frac{\Delta x_{i+1}+\Delta x_{i+2}}{\Delta x_{i+2}} - 3\Delta x_{i+2}}\right)$$

$$\frac{\Delta x_{i+1} - 2\Delta x_{i+2}}{\Delta x_{i+2}}$$

$$1 = \frac{a\Delta x_i^2}{2} + \frac{\Delta x_{i+1}^{\cancel{2}}}{2}\frac{\Delta x_i}{\cancel{\Delta x_{i+1}}}\, a = \frac{a}{2}\left(\Delta x_i^2 + \Delta x_i \Delta x_{i+1}\right)$$

$$a = \frac{2}{\left(\Delta x_i^2 + \Delta x_i \Delta x_{i+1}\right)} = \frac{2}{\Delta x_i\left(\Delta x_i + \Delta x_{i+1}\right)}$$

$$\Rightarrow \quad c = \frac{2}{\Delta x_{i+1}\left(\Delta x_i + \Delta x_{i+1}\right)}$$

$$\& \quad b = \frac{-2}{\left(\Delta x_{i+1} + \Delta x_i\right)}\left[\frac{1}{\Delta x_i} + \frac{1}{\Delta x_{i+1}}\right] = \frac{-2}{\Delta x_i \Delta x_{i+1}}$$

$$\Rightarrow (u_{xx})_i = \frac{2}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} u_{i-1} - \frac{2}{\Delta x_i \Delta x_{i+1}} u_i + \frac{2}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})} u_{i+1}$$

$$- \frac{1}{6}\frac{2}{(\Delta x_i + \Delta x_{i+1})}\left[\frac{\Delta x_{i+1}^3}{\Delta x_{i+1}} - \frac{\Delta x_i^3}{\Delta x_i}\right](u_{xxx})_i$$

$$- \frac{1}{24}\left(\frac{2}{(\Delta x_i + \Delta x_{i+1})}\right)\left(\frac{\Delta x_i^4}{\Delta x_i} + \frac{\Delta x_{i+1}^4}{\Delta x_{i+1}}\right)(u_{xxxx})_i$$

$$\Rightarrow (u_{xx})_i = \frac{2}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} u_{i-1} - \frac{2}{\Delta x_i(\Delta x_i + \Delta x_{i+1})} u_i - \frac{2}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})} u_i$$

$$+ \frac{2}{\Delta x_{i+1}(\Delta x_i + \Delta x_{i+1})} u_{i+1}$$

$$- \frac{1}{3(\Delta x_i + \Delta x_{i+1})}(\Delta x_{i+1} - \Delta x_i)(\Delta x_{i+1} + \Delta x_i)(u_{xxx})_i$$

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$$(U_{xx})_i = \cancel{\quad} \; aU_{i-1} + bU_i + cU_{i+1}$$

$$= a\Big[U_i - \Delta x_i (U_x)_i + \frac{\Delta x_i^2}{2}(U_{xx})_i - \frac{\Delta x_i^3}{6}(U_{xxx})_i + \frac{\Delta x_i^4}{24}(U_{xxxx})_i$$

$$+ O(\Delta x_i^5)\Big] + bU_i$$

$$+ c\Big[U_i + \Delta x_{i+1}(U_x)_i + \frac{\Delta x_{i+1}^2}{2}(U_{xx})_i + \frac{\Delta x_{i+1}^3}{6}(U_{xxx})_i$$

$$+ \frac{\Delta x_{i+1}^4}{24}(U_{xxxx})_i + \cdots\Big]$$

$$= (a+b+c)U_i + \left(-a\Delta x_i + c\Delta x_{i+1}\right)(U_x)_i$$

$$+ \left(a\frac{\Delta x_i^2}{2} + c\frac{\Delta x_{i+1}^2}{2}\right)(U_{xx})_i +$$

$$+ \left(-a\frac{\Delta x_i^3}{6} + c\frac{\Delta x_{i+1}^3}{6}\right)(U_{xxx})_i + \left(\frac{a\Delta x_i^4}{24} + c\frac{\Delta x_{i+1}^4}{24}\right)(U_{xxxx})_i$$

$$+ O(\Delta x_i^5) + O(\Delta x_{i+1}^5)$$

$$\Rightarrow \quad 0 = a+b+c$$

$$0 = -a\Delta x_i + c\Delta x_{i+1} \Rightarrow \quad c = a\frac{\Delta x_i}{\Delta x_{i+1}}$$

$$1 = \frac{a}{2}\Delta x_i^2 + c\frac{\Delta x_{i+1}^2}{2}$$

$$- \frac{1}{12}\frac{(\Delta x_{i+1}^3 + \Delta x_i^3)}{(\Delta x_{i+1} + \Delta x_i)}(u_{xxxx})_i$$

$$\Rightarrow (u_{xx})_i = \left(\frac{u_{i+1} - u_i}{\Delta x_{i+1}} - \frac{u_i - u_{i-1}}{\Delta x_i}\right)\frac{2}{(\Delta x_{i+1} + \Delta x_i)}$$

$$\overset{?}{+} \frac{1}{3}(\Delta x_i - \Delta x_{i+1})(u_{xxx})_i$$

$$- \frac{(\Delta x_{i+1}^3 + \Delta x_i^3)}{12(\Delta x_{i+1} + \Delta x_i)}(u_{xxxx})_i$$

Prob 4.1 3rd order accurate formulas for $(u_x)_i$

$$\Delta x D = \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} - \frac{\delta^{+4}}{4} + O(\delta^{+5}) \qquad \delta^+ \to 0$$

Then $\Delta x \to 0$ $\quad \Delta x D \sim \delta^+$

$$\Rightarrow \quad \Delta x D \simeq \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} + O(\Delta x^4) \qquad \Delta x \to 0$$

$$\Rightarrow \quad D = \frac{\delta^+}{\Delta x} - \frac{\delta^{+2}}{2\Delta x} + \frac{\delta^{+3}}{3\Delta x} + O(\Delta x^3) \qquad \Delta x \to 0$$

$$\therefore \quad u_x = Du = \frac{1}{\Delta x}\left(\delta^+ - \delta^{+2}/2 + \delta^{+3}/3\right)u + O(\Delta x^3) \qquad \Delta x \to 0$$

$$= \frac{1}{\Delta x}\left(E - 1 - \frac{1}{2}(E-1)^2 + \frac{1}{3}(E-1)^3\right)u + O(\Delta x^3)$$

$$= \frac{1}{\Delta x}\left(E - 1 - \frac{1}{2}(E^2 - 2E + 1) + \frac{1}{3}(E^3 - 3E^2 + 3E - 1)\right)u + \cdots$$

$$= \frac{1}{\Delta x}\left(E - 1 - \frac{E^2}{2} + E - \frac{1}{2} + \frac{E^3}{3} - E^2 + E - \frac{1}{3}\right)u + \cdots$$

$-\frac{3}{2} - \frac{1}{3}$ $\quad -\frac{1}{2} - 1$

$-\frac{3}{2}\frac{3}{3} - \frac{2}{6}$

$= -\frac{11}{6}$

$$= \frac{1}{\Delta x}\left(-\frac{11}{6} + 3E - \frac{3}{2}E^2 + \frac{E^3}{3}\right)u + \cdots$$

$$= \frac{1}{6\Delta x}\left(E^3 - 9E^2 + 6E - 11\right)u_i + \cdots$$

$$\therefore\ U_x = \frac{1}{6\Delta x}\left(U_{i+3} - 9U_{i+2} + 6U_{i+1} - 11U_i\right) + O(\Delta_x^3) \quad \Delta x \to 0$$

For Backwards difference use 4.2.20

$$\Delta x D = \delta^- + \frac{\delta^{-2}}{2} + \frac{\delta^{-3}}{3} + \frac{\delta^{-4}}{4} + O(\delta^{-5}) \quad \delta^- \to 0$$

$$\Rightarrow \quad \Delta_x D \sim \delta^- \quad \delta^- \to 0$$

$$\Rightarrow \quad \Delta_x D = \delta^- + \frac{\delta^{-2}}{2} + \frac{\delta^{-3}}{3} + O(\Delta_x^4) \quad \Delta_x \to 0$$

$$\Delta_x D = (1-E^{-1}) + \frac{(1-2E^{-1}+E^{-2})}{2} + \frac{1}{3}(1-3E^{-1}+3E^{-2}-E^{-3}) + O(\ )$$

$$\Rightarrow \Delta x D = 1 - E^{-1} + \frac{1}{2} - E^{-1} + \frac{1}{2}E^{-2} + \frac{1}{3} - E^{-1} + E^{-2} - \frac{1}{3}E^{-3}$$

$$= \frac{11}{6} - 3E^{-1} + \frac{3}{2}E^{-2} - \frac{1}{3}E^{-3}$$

$1+\frac{1}{2}+\frac{1}{3}$	$\frac{1}{2}+1$
$\frac{3}{2}+\frac{1}{3}$	$\frac{3}{2}$
$\frac{9}{6}+\frac{2}{6}$	
$\frac{11}{6}$	

$$\Rightarrow \quad D = \frac{1}{6\Delta x}\left(11 - 18E^{-1} + 9E^{-2} - 2E^{-3}\right) + O(\Delta x^3) \quad \Delta_x \to 0$$

$$\Rightarrow U_x = \frac{1}{6\Delta x}\left(11U_i - 18U_{i-1} + 9U_{i-2} - 2U_{i-3}\right) + O(\Delta_x^3)$$

Prob 4.2

$$(u_x)_i = a u_{i-2} + b u_{i-1} + c u_i + d u_{i+1}$$

$$= a\left(u_i - 2\Delta x u_i' + \frac{4\Delta x^2}{2} u_i'' - \frac{8\Delta x^3}{3\cdot 2} u_i''' + \frac{2^4 \Delta x^4 u_i^{IV}}{4\cdot 3\cdot 2} + O(\Delta x^5)\right)$$

$$+ b\left(u_i - \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' - \frac{\Delta x^3}{6} u_i''' + \frac{\Delta x^4}{4\cdot 3\cdot 2} u_i^{IV} + O(\Delta x^5)\right)$$

$$+ c u_i$$

$$+ d\left(u_i + \Delta x u_i' + \frac{\Delta x^2}{2} u_i'' + \frac{\Delta x^3}{6} u_i''' + \frac{\Delta x^4}{24} u_i^{IV} + O\Delta x^5)\right)$$

$$= (a + b + c + d) u_i$$

$$+ \Delta x u_i' (-2a - b + d)$$

$$+ \frac{\Delta x^2 u_i''}{2} (4a + b + d) + \frac{\Delta x^3 u_i'''}{6} (-8a - b + d)$$

$$+ \frac{\Delta x^4 u_i^{IV}}{24} (16a + b + d) + O(\Delta x^5)$$

$a + b + c + d = 0$

$-2a - b + d = \frac{1}{\Delta x}$ } determine $a, b + d$

$4a + b + d = 0$ } then determine c from Above

$-8a - b + d = 0$

Adding last ~~two~~ equations gives

$\cancel{-4a + 2d = 0 \Rightarrow d = 2a}$ put into eqs 1-3

gives

$\cancel{3a + b + c = 0 \Rightarrow a = \frac{1}{3}\left(\frac{1}{\Delta x}\right.}$ know a already... don't solve for a...

$\cancel{-b = \frac{1}{\Delta x} \Rightarrow b = -\frac{1}{\Delta x}}$

$6a + b = 0 \Rightarrow a = -\frac{b}{6} = \frac{1}{6\Delta x}$

— — — — —

$-2a - b + d = 1/\Delta x$ $\xrightarrow{\text{Add}}$ $2a + 2d = 1/\Delta x$ mult by 2 & Add

$4a + b + d = 0$

$-8a - b + d = 0$ $\xrightarrow{\text{Add}}$ $-4a + 2d = 0$

$= 4d + 2d = \frac{2}{\Delta x} \Rightarrow d = \frac{1}{3\Delta x}$

$\Rightarrow a = d/2 = \frac{1}{6\Delta x}$

$b = -8a + d = \frac{-8}{6\Delta x} + \frac{1}{3\Delta x} = \frac{-4}{3\Delta x} + \frac{1}{3\Delta x} = \frac{-1}{\Delta x}$

$$c = -a - b - d$$

$$= -\frac{1}{6\Delta x} + \frac{1}{\Delta x} - \frac{1}{3\Delta x} = \frac{1}{6\Delta x}(-1+6-2) = \frac{1}{2\Delta x}$$

$\therefore$

$$(u_x)_i = \frac{1}{6\Delta x}\left(u_{i-2} - 6u_{i-1} + 3u_i + 2u_{i-1}\right) + \frac{\Delta x^4}{24} u_i^{IV}(16a+b+d)$$

$$16a+b+d = \frac{8}{3\Delta x} + \frac{-1}{\Delta x} + \frac{1}{3\Delta x} = \frac{1}{3\Delta x}(8-3+1) = \frac{2}{\Delta x}$$

$\therefore$

$$(u_x)_i = \frac{1}{6\Delta x}\left(u_{i-2} - 6u_{i-1} + 3u_i + 2u_{i-1}\right) + \frac{\Delta x^3}{12} u_i^{IV} + O(\Delta x^4)$$

Prob 4.3

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$$(u_x)_i = a u_{i+2} + b u_{i+1} + c u_i + d u_{i-1} + e u_{i-2}$$

$$= a\Big(u_i + 2\Delta x\, u_i' + \frac{4\Delta x^2}{2!}u_i'' + \frac{8\Delta x^3}{3!}u_i''' + \frac{16\Delta x^4}{4!}u_i^{IV} + \frac{32\Delta x^5}{5!}u_i^{V}$$

$$+ \frac{64\Delta x^6}{6!}u_i^{VI} + O(\Delta x^7)\Big)$$

$$+ b\Big(u_i + \Delta x\, u_i' + \frac{\Delta x^2}{2!}u_i'' + \frac{\Delta x^3}{3!}u_i''' + \frac{\Delta x^4}{4!}u_i^{IV} + \frac{\Delta x^5}{5!}u_i^{V} + \frac{\Delta x^6}{6!}u_i^{VI} + O(\Delta$$

$$+ c u_i$$

$$+ d\Big(u_i - \Delta x\, u_i' + \frac{\Delta x^2}{2!}u_i'' - \frac{\Delta x^3}{3!}u_i''' + \frac{\Delta x^4}{4!}u_i^{IV} - \frac{\Delta x^5}{5!}u_i^{V} + \frac{\Delta x^6}{6!}u_i^{VI} + O(\Delta x^7)\Big)$$

$$+ e\Big(u_i - 2\Delta x\, u_i' + \frac{4\Delta^2}{2!}u_i'' - \frac{8\Delta^3}{3!}u_i''' + \frac{16\Delta x^4}{4!}u_i^{IV} - \frac{32\Delta x^5}{5!}u_i^{V}$$

$$+ \frac{64\Delta x^6}{6!}u_i^{VI} + O(\Delta x^7)\Big)$$

$$=$$

$$u_i(a+b+c+d+e) +$$

$$\Delta x\, u_i'(2a+b-d-2e) +$$

$$\frac{\Delta x^2}{2!} u_i''(4a+b+d+4e) +$$

$$\frac{\Delta x^3}{3!} u_i'''(+8a+b-d-8e) +$$

$$\frac{\Delta x^4}{4!} u_i^{IV}(16a+b+d+16e) +$$

$$\frac{\Delta x^5}{5!} u_i^{V}(32a+b-d-32e) +$$ $\Leftarrow$ Residual term for u_x

$$\frac{\Delta x^6}{6!} u_i^{VI}(64a+b+d+64e) + O(\Delta x^7)$$ $\Leftarrow$ Residual term for u_{xx}

$\Rightarrow \quad a+b+c+d+e = 0$

$2a+b-d-2e = 1/\Delta x \sim 1$ + scale every answer by $1/\Delta x$

$4a+b+d+4e = 0$

$8a+b-d-8e = 0$

$16a+b+d+16e = 0$

$$\left(\begin{array}{cccc|c} 2 & 1 & -1 & -2 & 1 \\ 4 & 1 & 1 & 4 & 0 \\ 8 & 1 & -1 & -8 & 0 \\ 16 & 1 & 1 & +16 & 0 \end{array}\right)$$

Note equivalent to finding 1st column of inverse of rather predictable matrix.

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & -1 & \frac{1}{2} \\ 4 & 1 & 1 & 4 & 0 \\ 8 & 1 & -1 & -8 & 0 \\ 16 & 1 & 1 & 16 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -1 & 3 & 8 & -2 \\ 0 & -3 & +3 & 0 & -4 \\ 0 & -7 & 9 & 32 & -8 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -3 & -8 & 2 \\ 0 & -3 & +3 & 0 & -4 \\ 0 & -7 & 9 & 32 & -8 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -3 & -8 & 2 \\ 0 & 0 & -6 & -24 & 2 \\ 0 & 0 & -12 & -24 & 6 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -3 & -8 & 2 \\ 0 & 0 & 1 & 4 & -\frac{1}{3} \\ 0 & 0 & -2 & -4 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -3 & -8 & 2 \\ 0 & 0 & 1 & 4 & -\frac{1}{3} \\ 0 & 0 & 0 & 4 & \frac{1}{3} \end{array}\right)$$

$-21 - 9 = -12$

$-56 + 32 = -24$

21

$14 + -8 = 6$

$8 - 4$

$-\frac{2}{3} + 1 = \frac{1}{3}$

$$\Rightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & -3 & -8 & 2 \\ 0 & 0 & 1 & 4 & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{12} \end{array}\right)$$

Solving on MAPLE gives:

$a = -\frac{1}{12}, \quad b = \frac{2}{3}, \quad c = 0, \quad d = -\frac{2}{3}, \quad e = \frac{1}{12}$

& method becomes

$$(u_x)_i = \frac{-1}{12\Delta x} u_{i+2} + \frac{2}{3\Delta x} u_{i+1} + \frac{-2}{3\Delta x} u_{i-1} + \frac{1}{12\Delta x} u_{i-2}$$

$$+ \frac{-4}{5!} u_i^{V} \frac{\Delta x^5}{\Delta x} + O(\Delta x^5)$$

$$= \frac{(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2})}{12\Delta x} - \frac{1}{30} \Delta x^4 \frac{\partial^5 u}{\partial x^5} + O(\Delta x^5)$$

$$(U_{xx})_i = aU_{i+2} + bU_{i+1} + cU_i + dU_{i-1} + eU_{i-2}$$

$\Rightarrow$ From 1st part eqs become

$$a+b+c+d+e = 0$$

$$2a+b-d-2e = 0$$

$$4a+b+d+4e = \frac{2}{\Delta x^2}$$

$$8a+b-d-8e = 0$$

$$16a+b+d+16e = 0$$

5 eqs + 5 unknowns
Solving on MMA gives

$a = \frac{-1}{12}$, $b = \frac{4}{3}$, $c = -\frac{5}{2}$, $d = \frac{4}{3}$

$e = \frac{-1}{12}$ all scaled by $\frac{1}{\Delta x^2}$

Then truncation error $R = \pm \frac{\Delta x^6}{6!}(64a+b+d+64e)$ Finish ...

$\Rightarrow$ Method becomes

$$(U_{xx})_i = \frac{1}{12\Delta x^2}\left[-U_{i+2} + 16U_{i+1} - 30U_i + 16U_{i-1} - U_{i-2}\right]$$

$$\underbrace{-\frac{8\Delta x^6}{6!}\frac{U^{VI}}{\Delta x^2}}_{} + O(\Delta x^5)$$

$$-\frac{1}{90}\Delta x^4 U^{VI} + O($$

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Prob 4.4

(a) $(U_{xxx})_i = aU_{i+2} + bU_{i+1} + cU_i + dU_{i-1} + eU_{i-2}$

From problem 4.3 taylor expansion of R.H.S. gives, & matching terms

$$0 = a+b+c+d+e$$
$$0 = 2a+b-d-2e$$
$$0 = 4a+b+d+4e$$
$$\frac{3!}{\Delta x^3} = 8a+b-d-8e$$
$$0 = 16a+b+d+16e$$

Notice these are all inverses of the same matrix, choosing different columns & multiplying by a different coefficient.

5 eqs & 5 unknowns. Solving w/ MMA gives

$a = \frac{1}{2}$, $b = -1$, $c = 0$

$d = 1$, $e = -\frac{1}{2}$ all scaled by $\frac{1}{\Delta x^3}$

$$R = \pm \frac{\Delta x^5}{5!}(32a+b-d-32e)\,u^{V}$$

$$\Rightarrow (U_{xxx})_i = \frac{1}{2\Delta x^3}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2})$$

$$+ \frac{\Delta x^5}{\Delta x^3}\frac{30}{5!}\frac{\partial^5 u}{\partial x^5} + O(\Delta x^3)$$

$$= \frac{1}{2\Delta x^3}(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}) + \frac{1}{4}\Delta x^2 u^{V}(x) + O(\Delta x^3)$$

$0 = a + b + c + d + e$ 5 eqs & 5 unknowns

$0 = 2a + b - d - 2e$ $\Rightarrow a = 1,\ b = -4,\ c = 6$

$0 = 4a + b + d + 4e$ $d = -4,\ e = 1$

$0 = 8a + b - d - 8e$ scaled by $\frac{1}{\Delta x^4}$

$\frac{4!}{\Delta x^4} = 16a + b + d + 16e$

$$R = \pm \frac{\Delta x^5}{5!} u^{V} (32a + b - d - 32e) = 0$$

$$u^{IV}(x) = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i+2}}{\Delta x^4}$$

$$+ \frac{1}{6} \Delta x^2 u^{VI} + O(\Delta^3)$$

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Prob 4 I

$U = \cos x$ $U = \sin x$ $U = e^x$

$U' = -\sin x$ $U' = \cos x$ $U' = e^x$

$U'' = -\cos x$ $U'' = -\sin x$ $U'' = e^x$

$U''' = \sin x$ $U''' = -\cos x$ $U''' = e^x$

$U^{IV} = \cos x$ $U^{V} = \sin x$ $U^{IV} = e^x$

Backwards 1st order

$$(U_x)_i = \frac{U_i - U_{i-1}}{\Delta x} + \Delta x (U_{xx})_i$$

2nd order:

$$(U_x)_i = \frac{3U_i - 4U_{i-1} + U_{i-2}}{2\Delta x} + \frac{\Delta x^2}{3} U_{xxx}$$ truncation error may be different.

Forward difference 1st order:

$$\Delta x D = \delta^+ - \delta^{+2}/2$$

$$(U_x)_i = \frac{U_{i+1} - U_i}{\Delta x} - \frac{\Delta x}{2} U_{xx}$$

$$D = \frac{\delta^+}{\Delta x} - \delta^{+2}/2\Delta x = \frac{\delta^+}{\Delta x} - \frac{\Delta x}{2} D^2$$

2nd order:

$$(U_x)_i = \frac{-3U_i + 4U_{i+1} - U_{i+2}}{2\Delta x} + \frac{\Delta x^2}{3} U_{xxx}$$

Centered differencing: 2nd order

$$(U_x)_i = \frac{U_{i+1} - U_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6} U_{xxx}$$

4th order:

$$(U_x)_i = \frac{-U_{i+2} + 8U_{i+1} - 8U_{i-1} + U_{i-2}}{12\,\Delta x} + \frac{\Delta x^4}{30}\frac{\partial^5 U}{\partial x^5}$$

For 2nd derivatives

Backwards differencing: 1st order

$$(U_{xx})_i = \frac{1}{\Delta x^2}(U_i - 2U_{i-1} + U_{i-2}) + \Delta x\, U_{xxx}$$

2nd order

$$= \frac{1}{\Delta x^2}(2U_i - 5U_{i-1} + 4U_{i-2} - U_{i-3}) - \frac{11}{12}\Delta x^2 U_{xxxx}$$

Forward differencing: 1st order

$$(U_{xx})_i = \frac{1}{\Delta x^2}(U_{i+2} - 2U_{i+1} + U_i) - \Delta x\, U_{xxx}$$

2nd order

$$\frac{1}{\Delta x^2}(2U_i - 5U_{i+1} + 4U_{i+2} - U_{i+3}) + \frac{11}{12}\Delta x^2 U_{xxxx}$$

Central differencing: 2nd order

$$(U_{xx})_i = \frac{1}{\Delta x^2}(U_{i+1} - 2U_i + U_{i-1}) - \frac{\Delta x^2}{12} U_{xxxx}$$

4th order

$$(U_{xx})_i = \frac{1}{12\Delta x^2}(-U_{i+2} + 16U_{i+1} - 30U_i + 16U_{i-1} - U_{i-2}) + \frac{\Delta x^4}{90} U^{VI}$$

At $x=0$ for all schemes considered we require the evaluations at nodal pts $i-3, i-2, i-1, i, i+1, i+2, i+3$

$\Rightarrow$ need the evaluations at

$$-3\Delta x, -2\Delta x, -\Delta x, 0, \Delta x, 2\Delta x, 3\Delta x$$

enter coefficients of scheme a, b, c, d, e, f, power of Δx, & coeff of Δx in Remainder, derivative in Remainder.

output $\{$exact derivative, approximation to derivative, exact error, approximation to error$\}$

Exact answers:

	e^x	$\cos x$	$\sin x$
$f(0)$	1	1	0
$f'(0)$	1	0	1
$f''(0)$	1	-1	0
$f'''(0)$	1	0	-1
$f^{IV}(0)$	1	1	0
$f^{V}(0)$		0	1
$f^{VI}(0)$		-1	0

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(Prob 4.6) elliminate a_+, b_+, c_+ & a_0 from 4.3.14

4.3.14 ⇒ $\underline{a_+} + \underline{a_0} - a_- = 0$

$$\Delta x(\underline{a_+} - a_-) + \underline{b_+} + b_0 + b_- = 0$$

$$\frac{\Delta x^2}{2}(\underline{a_+} + a_-) + \Delta x(\underline{b_+} - b_-) + \underline{c_+} + c_0 + c_- = 0$$

$$\frac{\Delta x^3}{6}(\underline{a_+} + a_-) + \frac{\Delta x^2}{2}(\underline{b_+} + b_-) + \Delta x(\underline{c_+} - c_-) = 0$$

w/ MMA gives

$$a_0 = -\frac{1}{\Delta x^2}\left[8\Delta x^2 a_- - (12b_- + 3b_0)\Delta x + (12c_- + 6c_0)\right]$$

$$a_+ = -\frac{1}{\Delta x^2}\left[-7\Delta x^2 a_- + (12b_- + 3b_0)\Delta x - (12c_- + 6c_0)\right]$$

$$c_+ = \frac{1}{2}\left[4\Delta x^2 a_- - (8b_- + b_0)\Delta x + 10c_- + 4c_0\right]$$

$$b_+ = -\frac{1}{\Delta x}\left[6\Delta x^2 a_- - (11b_- + 2b_0)\Delta x + 12c_- + 6c_0\right]$$

Truncation error given in MMA notebook.

$$R = \frac{\Delta x^2}{24}(8\Delta x^2 a_- - 20 \qquad) \cdot u_i^{IV}$$

$+ \ldots$

Solving to obtain the highest order accuracy note we

have c_-, c_0, b_-, b_0 + a_- as unknowns = 5 unknowns

but looking at eq 4.3.10 we see that one unknown is really degenerate

(one could ÷ by a coefficient + loose the information that

it existed)

→ 4 unknowns make terms w/ U^{IV}, U^{V}, U^{VI}, U^{VII}

vanish w/

get

$$= -\frac{\Delta x^6}{5040} b_0 \frac{\partial^{VIII} U}{\partial x^{VIII}} + O(\Delta x^7)$$

⇒ $b_0 = 0$ is highest order why does it not depend on the ratio?

Prob 4.7 Following the hint eqs 4.3.15 become

$\{ c_+ = c_- = 0 \,,\; \underbrace{c_0 = 1}$

$$a_+ = \frac{1}{2\Delta x}\left[-5b_+ - b_- - \frac{2}{\Delta x}\right]$$

$$a_0 = \frac{2}{\Delta x}\left[b_+ - b_- + \frac{1}{\Delta x}\right]$$

$$a_- = \frac{1}{2\Delta x}\left[b_+ + 5b_- - \frac{2}{\Delta x}\right]$$

$$b_0 = 2(b_+ + b_-)$$

Then truncation error R becomes

$$R = \frac{\Delta x^3}{24}\left[2(b_+ - b_-) - \frac{2}{\Delta x}\right]\frac{\partial^4 u}{\partial x^4}$$

$$+ \frac{\Delta x^4}{120}\left[2(b_+ + b_-)\right]\frac{\partial^5 u}{\partial x^5} + \frac{\Delta x^5}{6!}\left[4(b_+ - b_-) - \frac{2}{\Delta x}\right]\frac{\partial^6 u}{\partial x^6}$$

$$+ \frac{\Delta x^6}{7!}\left[4(b_+ - b_-)\right]\frac{\partial^7 u}{\partial x^7} + \frac{\Delta x^7}{8!}\left[6(b_+ - b_-) - \frac{2}{\Delta x}\right]\frac{\partial^8 u}{\partial x^8} + \cdots$$

solve $\left.\begin{aligned} 2(b_+ - b_-) - \frac{2}{\Delta x} &= 0 \\ 2(b_+ + b_-) &= 0 \end{aligned}\right\} \Rightarrow b_+ = \frac{1}{2\Delta x},\; b_- = \frac{-1}{2\Delta x}$

$$\Rightarrow \quad a_+ = \frac{1}{2\Delta x}\left[-\frac{5}{2\Delta x} + \frac{1}{2\Delta x} - \frac{2}{\Delta x}\right] = \frac{1}{4\Delta x^2}\left[-5 + 1 - 4\right] = \frac{-2}{\Delta x^2}$$

$$a_0 = \frac{2}{\Delta x}\left[\frac{1}{2\Delta x} + \frac{1}{2\Delta x} + \frac{2}{2\Delta x}\right] = \frac{4}{\Delta x^2}$$

$$a_- = \frac{1}{2\Delta x}\left[\frac{1}{2\Delta x} - \frac{5}{2\Delta x} - \frac{2}{\Delta x}\right] = \frac{1}{4\Delta x^2}\left[1-5-4\right] = \frac{-2}{\Delta x^2}$$

$$b_0 = \frac{\cancel{2}}{\cancel{2\Delta x}} = 0.$$

Thus eq becomes

$$(u_{xx})_i + \frac{-2}{\Delta x^2}u_{i+1} + \frac{4}{\Delta x^2}u_i - \frac{2}{\Delta x^2}u_{i-1} + \frac{1}{2\Delta x}(u_x)_{i+1} - \frac{1}{2\Delta x}(u_x)_{i-1}$$

$$- \frac{\Delta x^5}{6!}\left[4\left(\frac{1}{2\Delta x}+\frac{1}{2\Delta x}\right) - \frac{2}{\Delta x}\right]\frac{\partial^6 u}{\partial x^6} + O(\Delta x^5) = 0$$

$$\Rightarrow (u_{xx})_i = -\frac{1}{2\Delta x}\left[(u_x)_{i+1} - (u_x)_{i-1}\right] + \frac{2}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1})$$

$$+ \frac{\Delta x^4}{6!}(4-2)\frac{\partial^6 u}{\partial x^6} + O(\Delta x^5)$$

$$= -\frac{1}{2\Delta x}\left[(u_x)_{i+1} - (u_x)_{i-1}\right] + \frac{2}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1})$$

$$+ \frac{1}{360}\Delta x^4\frac{\partial^6 u}{\partial x^6} + O(\Delta x^5)$$

Prob 4.3 No 2nd derivatives : $c_+ = c_0 = c_- = 0$

$\alpha = b_+/b_-$

This gives

$$a_+ = \frac{1}{2\Delta x}\left[-5b_+ - b_-\right]$$

$$a_0 = \frac{2}{\Delta x}\left[b_+ - b_-\right]$$

$$a_- = \frac{1}{2\Delta x}\left[b_+ + 5b_-\right]$$

$$b_0 = 2(b_+ + b_-)$$

+ Truncation error is $R = \frac{\Delta x^3}{24} 2(b_+ - b_-)\frac{\partial^4 u}{\partial x^4}$

$$+ \frac{\Delta x^4}{120} 2(b_+ + b_-)\frac{\partial^5 u}{\partial x^5} + \frac{\Delta x^5}{6!} 4(b_+ - b_-)\frac{\partial^6 u}{\partial x^6} + \cdots$$

Then eq 4.3.10 becomes

$$\frac{1}{2\Delta x}\left[-5b_+ - b_-\right]u_{i+1} + \frac{2}{\Delta x}\left[b_+ - b_-\right]u_i + \frac{1}{2\Delta x}\left[b_+ + 5b_-\right]u_{i-1}$$

$$+ b_+(u_x)_{i+1} + 2(b_+ + b_-)(u_x)_i + b_-(u_x)_{i-1} = 0$$

w/ $R = \frac{\Delta x^3}{12}(b_+ - b_-)\frac{\partial^4 u}{\partial x^4} + \frac{\Delta x^4}{60}(b_+ + b_-)\frac{\partial^5 u}{\partial x^5}$

$+ \cdots$

$\div$ by b_- gives:

$$\frac{1}{2\Delta x}\left[-5\alpha - 1\right]u_{i+1} + \frac{2}{\Delta x}(\alpha - 1)u_i + \frac{1}{2\Delta x}(\alpha + 5)u_{i-1}$$

$$+ \alpha(u_x)_{i+1} + 2(\alpha+1)(u_x)_i + (u_x)_{i-1} = 0$$

w/ $R = \frac{\Delta x^3}{12}(\alpha - 1)u^{IV} + \frac{\Delta x^4}{60}(\alpha+1)u^{V} + \cdots$

Prob 4.9 eq 4.4.13 is

$$\Delta^{(3)} U_{ij} = \frac{1}{\Delta x^2}\left[(\delta_x^2+\delta_y^2) + \frac{b}{2}\delta_x^2\delta_y^2\right]U_{ij} = \Delta^{(1)}U_{ij} + \frac{b}{2}\delta_x^2\delta_y^2 U_{ij}$$

$\Delta^{(1)} U_{ij}$ is given on pg 138 Fig 4.4.2

To describe the Additional term $\delta_x^2\delta_y^2$

$$= (E_x^{1/2} - E_x^{-1/2})^2 (E_y^{1/2}-E_y^{-1/2})^2 = (E_x - 2 + E_x^{-1})(E_y - 2 + E_y^{-1})$$

$$= E_xE_y - 2E_x + E_xE_y^{-1} - 2E_y + 4 - 2E_y^{-1} + E_x^{-1}E_y - 2E_x^{-1} + E_x^{-1}E_y^{-1}$$

$$\Rightarrow \quad \delta_x^2\delta_y^2 U_{ij} = U_{i+1,j+1} - 2U_{i+1,j} + U_{i+1,j-1} - 2U_{i,j+1} + 4U_{ij}$$
$$- 2U_{i,j-1} + U_{i-1,j+1} - 2U_{i-1,j} + U_{i-1,j-1}$$

Thus computational molecule looks like:

1	-2	1
-2	4	-2
1	-2	1

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Prob 4.10 Note: truncation error for 4.4.21 is worked at in detail w/ Notes that accompany pg 194

$~~1 = \mu\left(1+\frac{\delta^2}{4}\right)^{-1/2}~~$

$~~= \mu\left(1-\frac{\delta^2}{8}+\frac{3\delta^4}{128}\dots\right)~~$

4.4.14: $\frac{\mu_x \delta_x}{\Delta x} =$

4.2.30 $~~\Delta x D = \mu\delta - \frac{1}{3!}\mu\delta^3 + \frac{1^2 2^2}{5!}\delta^5 + \dots~~$

$~~\Delta x D \sim \mu\delta~~$ $~~\delta \to 0~~$

$\Rightarrow$ $~~\Delta x D = \mu\delta + \frac{1}{3!}~~$

From eq 4.2.28 $1 = \mu\left(1+\frac{\delta^2}{4}\right)^{-1/2} = \mu\left(1-\frac{\delta^2}{8}+\frac{3\delta^4}{128}+O(\delta^6)\right)$

$\Rightarrow \quad \mu \sim 1 \quad \delta \to 0$

$\therefore$ From 4.2.30 $\Delta x D = \mu\left(\delta - \frac{1}{3!}\delta^3 + O(\delta^5)\right)$

$\Rightarrow \quad \Delta x D \sim \delta \quad \delta \to 0$

$\therefore \quad \Delta x D = \mu\delta - \frac{\mu D^3 \Delta x^3}{3!} + O(\Delta x^5)$

$\Rightarrow \quad D = \frac{\mu\delta}{\Delta x} - \frac{\mu \Delta x^2 D^3}{3!} + O(\Delta x^4) = \frac{\mu\delta}{\Delta x}\left(1 - \frac{D^3 \Delta x^3}{6} + O(\Delta x^4)\right)$

$D_x D_y U = \frac{\mu_x\delta_x}{\Delta x}\left(1 - \frac{D^3 \Delta x^3}{6} + O(\Delta x^4)\right)\frac{\mu_y\delta_y}{\Delta y}\left(1 - \frac{D_y^3 \Delta y^3}{6} + O(\Delta y^4)\right)U_{i,j}$

$$\Rightarrow D_x D_y U = \left(\frac{\mu_x \delta_x}{\Delta x} - \frac{\mu_x \Delta x^2 D_x^3}{6} + O(\Delta x^4) \right) \cdot \left(\frac{\mu_y \delta_y}{\Delta y} - \frac{\mu_y \Delta y^2 D_y^3}{6} + O(\Delta y^4) \right) U_{i,j}$$

$$= \left\{ \frac{\mu_x \delta_x \, \mu_y \delta_y}{\Delta x \, \Delta y} - \frac{\mu_x \mu_y D_x D_y^3 \Delta y^2}{6} + O(\Delta y^4) - \frac{\mu_x \mu_y \Delta x^2 D_y D_x^3}{6} + O(\Delta x^4, \Delta y^4) \right\} U$$

$$= \frac{\mu_x \delta_x \mu_y \delta_y U_{i,j}}{\Delta x \Delta y} - \frac{\Delta x^2 \mu_x \mu_y U_{xxxy}}{6} - \frac{\Delta y^2 \mu_x \mu_y U_{xyyy}}{6} + O(\Delta x^4, \Delta y^4) \qquad \Delta x \to 0$$

$$\Rightarrow U_{xy} = \frac{\mu_x \delta_x \mu_y \delta_y U_{i,j}}{\Delta x \Delta y} \underbrace{- \frac{\Delta x^2 U_{xxxy}}{6} - \frac{\Delta y^2 U_{xyyy}}{6}}_{} \qquad \Delta x \to 0$$

$$+ O(\Delta x^4, \Delta y^4)$$

truncation error for
4.4.14 + 4.4.15

For 4.4.16

From before (This problem) $D = \frac{\mu\delta}{\Delta x} - \frac{\Delta x^2 D^3}{6} + O(\Delta x^4)$ $\Delta x \to 0$

As $\mu \sim 1$ $\Delta x \to 0$

From pg 173, $\Delta x D = \delta^+ - \frac{\Delta x^2 D^2}{2} + O(\Delta x^3)$

$$\Rightarrow D = \frac{\delta^+}{\Delta x} - \frac{\Delta x D^2}{2} + O(\Delta x^2)$$

Then

$$U_{x,y} = D_x D_y U = \left(\frac{\mu_x \delta_x}{\Delta x} - \frac{\Delta x^2 D_x^3}{6} + O(\Delta x^4)\right)\left(\frac{\delta_y^+}{\Delta y} - \frac{\Delta y D_y^2}{2} + O(\Delta y^2)\right) U$$

$$(U_{xy})_{ij} = \frac{\mu_x \delta_x \delta_y^+}{\Delta x \Delta y} U_{ij} - \frac{\Delta y D_x D_y^2}{2} U - \frac{\Delta x^2 D_x^3 D_y U}{6} + O(\Delta x^2 \Delta y) + \text{H.O.T.}$$

$$= \frac{\mu_x \delta_x \delta_y^+}{\Delta x \Delta y} U_{ij} \underbrace{- \frac{\Delta y U_{xyy}}{2} - \frac{\Delta x^2 U_{xxxy}}{6}}_{\text{Truncation error}} + \text{H.O.T.} \qquad \begin{array}{l}\Delta x \to 0 \\ \Delta y \to 0\end{array}$$

For 4.4.17 From before

$$D = \frac{\delta^+}{\Delta x} - \frac{\Delta x D^2}{2} + O(\Delta x^2) \quad \text{then}$$

$$(U_{xy})_{ij} = \frac{\delta_x^+}{\Delta x}\frac{\delta_y^+ U}{\Delta y} - \left(\frac{\delta_x^+}{\Delta x}\right)\frac{\Delta y D_y^2 U}{2} - \left(\frac{\delta_y^+}{\Delta y}\right)\frac{\Delta x D_x^2 U}{2} + O(\Delta x^2, \Delta y^2)$$

$$\Rightarrow (U_{xy})_{ij} = \frac{\delta_x^+ \delta_y^+}{\Delta x\, \Delta y} \underbrace{- \frac{\Delta y}{2} U_{xyy} - \frac{\Delta x}{2} U_{xxy}} + O(\Delta x^2, \Delta y^2) \quad \Delta x \to 0$$

truncation error. for exp 4.4.17

How do I generalize this to get 2nd order truncation in 4.4.18?
w/o having to just do all calculations again

Then when evaluating at midpt one uses centered formulas.

$$(U_x)_i = \frac{U_{i+1/2} - U_{i-1/2}}{\Delta x} - \frac{\Delta x^2}{24} U_{xxx}$$

So

$$(U_x)_{i+1/2} = \frac{U_{i+1} - U_i}{\Delta x} - \frac{\Delta x^2}{24} U_{xxx} + O(\Delta^3)$$

Then $$= \frac{\delta_x^+}{\Delta x} - \frac{\Delta x^2}{24} D^3 + O(\Delta x^3)$$

$$(U_{xy})_{i+1/2, j+1/2} = \left(\frac{\delta_x^+}{\Delta x} - \frac{\Delta x^2}{24} D_x^3 + O(\Delta x^3) \right)\left(\frac{\delta_y^+}{\Delta y} - \frac{\Delta y^2}{24} D_y^3 + O(\Delta y^3) \right) U_{ij}$$

$$= \left(\frac{\delta_x^+}{\Delta x} \frac{\delta_y^+}{\Delta y} - \frac{\delta_x^+}{\Delta x} \frac{\Delta y^2}{24} D_y^3 - \frac{\Delta x^2}{24} D_x^3 \frac{\delta_y^+}{\Delta y} + O(\Delta x^3, \Delta y^3) \right) U_{ij}$$

$$= \frac{\delta_x^+ \delta_y^+}{\Delta x\, \Delta y} \underbrace{- \frac{\Delta y^2}{24} (U_{xyyy})_{ij} - \frac{\Delta x^2}{24} (U_{xxxy})_{ij}} + O(\Delta x^3, \Delta y^3)$$

truncation error for 4.4.18

For 4.4.19 backwards differences would give

$$\Delta x D = \delta^- + \frac{\Delta x^2 D^2}{2} + O(\Delta x^3)$$

$$D = \frac{\delta^-}{\Delta x} + \frac{\Delta x D^2}{2} + O(\Delta x^2)$$

$$\therefore \quad U_{xy} = \left(\frac{\delta_x^-}{\Delta x} + \frac{\Delta_x D_x^2}{2} + O(\Delta_x^2)\right)\left(\frac{\delta_y^-}{\Delta_y} + \frac{\Delta_y D_y^2}{2} + O(\Delta_y^2)\right) U_{ij}$$

$$= \left(\frac{\delta_x^- \delta_y^-}{\Delta x \Delta y} + \frac{\Delta_y D_x D_y^2}{2} + \frac{\Delta_x D_x^2 D_y}{2} + O(\Delta_x^2 \Delta_y^2)\right) U_{ij}$$

$$= \frac{\delta_x^- \delta_y^-}{\Delta x \Delta y} U_{ij} + \underbrace{\frac{\Delta y}{2} U_{xyy} + \frac{\Delta x}{2} U_{xxy}}_{\text{truncation error for 4.4.19}} + O(\Delta_x^2, \Delta_y^2).$$

For All other truncation errors see corresponding notes on the pages when methods were introduced.

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From notes pg 196 we have

$$\frac{U_{i+1}-U_i}{\Delta x_{i+1}} = (U_x)_i + \frac{\Delta x_{i+1}}{2}(U_{xx})_i + \frac{\Delta x_{i+1}^2}{6}(U_{xxx})_i + O(\Delta x_{i+1}^3)$$

$$\& \ \frac{U_i-U_{i-1}}{\Delta x_i} = (U_x)_i - \frac{\Delta x_i}{2}(U_{xx})_i + \frac{\Delta x_i^2}{6}(U_{xxx})_i + O(\Delta x_i^3)$$

$$\Rightarrow (1) \ (U_x)_i = \frac{U_{i+1}-U_i}{\Delta x_{i+1}} - \frac{\Delta x_{i+1}}{2}(U_{xx})_i - \frac{\Delta x_{i+1}^2}{6}(U_{xxx})_i + O(\Delta x_{i+1}^3)$$

$$(2) \ (U_x)_i = \frac{U_i-U_{i-1}}{\Delta x_i} + \frac{\Delta x_i}{2}(U_{xx})_i - \frac{\Delta x_i^2}{6}(U_{xxx})_i + O(\Delta x_i^3)$$

taking Average gives

$$(U_x)_i = \frac{1}{2}\left[\frac{U_{i+1}-U_i}{\Delta x_{i+1}} + \frac{U_i-U_{i-1}}{\Delta x_i}\right]$$

$$- \frac{1}{4}\underbrace{(\Delta x_{i+1} - \Delta x_i)}_{\neq 0 \text{ in general.}}(U_{xx})_i - \frac{1}{12}(\Delta x_{i+1}^2 + \Delta x_i^2)(U_{xxx})_i$$

$$+ O(\Delta x_{i+1}^3 + \Delta x_i^3)$$

$\therefore$ not 2nd order.