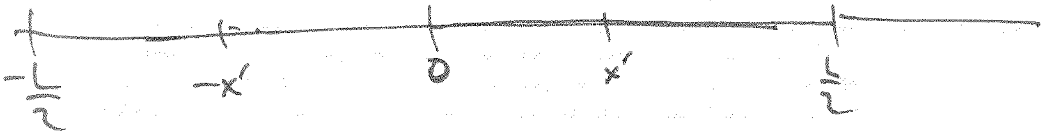
J

pg 121 Hockney / Eastwood



$$\frac{d^2\phi}{dx^2} = \frac{-q}{\epsilon_0}\left(\delta(x-x') - \delta(x+x')\right)$$

charge $+q$ at x' — charge $-q$ at $-x'$

$$\left\{ \frac{d^2\phi}{dx^2} = \frac{-q(x)}{\epsilon_0} \right\}$$

~~Also~~ + $\phi(x+L) = \phi(x)$

Then $\phi(x) = A + Bx$ in each region:

$-\frac{L}{2} \le x \le -x'$

$-x' \le x \le x'$

$x' \le x \le \frac{L}{2}$

+ ~~$\phi(x$~~

+ $\phi(x)$ is continuous at x' & $-x'$ i.e.

+ ~~$\phi(x)$~~ $\lim_{x \to x'^-} \phi(x) = \lim_{x \to x'^+} \phi(x)$ ⇒ ~~same for $\phi(x)$~~ eqs #1 & ~~#2~~

$\lim_{x \to -x'^-} \phi(x) = \lim_{x \to -x'^+} \phi(x)$ ⇒ ~~same for $\phi(x)$~~ eqs #2 & ~~#3~~

+ $\phi\left(\frac{L}{2}\right) = \phi\left(-\frac{L}{2}\right)$ ϕ is periodic of period L.

Then

$\phi_I = A_I + B_I x \qquad -\frac{L}{2} \le x \le -x'$

$\phi_{II} = A_{II} + B_{II} x \qquad -x' \le x \le +x'$

$\phi_{III} = A_{III} + B_{III} x \qquad x' \le x \le \frac{L}{2}$

+ by integrating * over $-x' - \delta$ to $\mp x' + \delta$

& $x' - \delta$ to $+x' + \delta$ $\qquad \delta > 0$

we get

$$\frac{d\phi}{dx}(-x'+\delta) - \frac{d\phi}{dx}(-x'-\delta) = -\frac{q}{\epsilon_0}(-1) \qquad \text{eq \# 3}$$ ✓

& $$\frac{d\phi}{dx}(+x'+\delta) - \frac{d\phi}{dx}(x'-\delta) = -\frac{q}{\epsilon_0}(+1) \qquad \text{eq \#4}$$ ✓

so eqs #1 & #2 require

$$A_I + B_I(-x') = A_{II} + B_{II}(-x')$$ ✓✓

& $$A_{II} + B_{II}(x') = A_{III} + B_{III}(x')$$ ✓✓

eqs #3 & #4 require

$$B_{II} - B_I = +\frac{q}{\epsilon_0}$$ ✓✓

& $$B_{III} - B_{II} = -\frac{q}{\epsilon_0}$$ ✓✓

the periodicity eq $\phi(\frac{L}{2}) = \phi(-\frac{L}{2})$ requires

$$A_I + B_I\left(-\frac{L}{2}\right) = A_{III} + B_{III}\left(+\frac{L}{2}\right)$$

Now have 5 eqs + 6 unknowns

$$A_I - x' B_I - A_{II} + x' B_{II} = 0$$

$$A_{II} + x' B_{II} - A_{III} - x' B_{III} = 0$$

$$-B_I + B_{II} = \frac{q}{\epsilon_0}$$

$$-B_{II} + B_{III} = -\frac{q}{\epsilon_0}$$

$$A_I - \frac{L}{2} B_I - A_{III} - \frac{L}{2} B_{III} = 0$$

performing gaussian ellimination on $A_I, B_I, \ldots, A_{III}, B_{III}$ gives

$$\begin{bmatrix} 1 & -x' & -1 & x' & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & -\frac{L}{2} & 0 & 0 & -1 & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} A_I \\ B_I \\ A_{II} \\ B_{II} \\ A_{III} \\ B_{III} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q/\epsilon_0 \\ -q/\epsilon_0 \\ 0 \end{bmatrix}$$

~~switch rows # ÷ row #1 by~~ x
mult row #1 by −1 & add to row #5

$$\Longleftrightarrow \begin{bmatrix} 1 & -x' & -1 & x' & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -\frac{L}{2}+x' & 1 & -x' & -1 & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} A_I \\ B_I \\ A_{II} \\ B_{II} \\ A_{III} \\ B_{III} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q/G_0 \\ -q/G_0 \\ 0 \end{bmatrix}$$

switch rows #2 & #3 & mult new row #2 by −1

$$\Longleftrightarrow \begin{bmatrix} 1 & -x' & -1 & x' & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -\cancel{\frac{L}{2}}+x' & 1 & -x' & -1 & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} A_I \\ B_I \\ A_{II} \\ B_{II} \\ A_{III} \\ B_{III} \end{bmatrix} = \begin{bmatrix} 0 \\ -q/G_0 \\ 0 \\ -q/G_0 \\ 0 \end{bmatrix}$$

1 x
1 x
1 x
1 x
1 x

mult row #2 by x' & add to row #1
mult row #2 by $\frac{L}{2}-x'$ & add to row #5

2nd eq
mult by $+x'$ Add to 1st

mult 2nd by $\frac{L}{2}-x'$ add to last

$$\Longleftrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & \cancel{} & -1 & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} \\ \\ {''} \\ \\ \end{bmatrix} = \begin{bmatrix} -\frac{q}{G_0}x' \\ -\frac{q}{G_0} \\ 0 \\ -\frac{q}{G_0} \\ -\frac{q}{G_0}\left(\frac{L}{2}-x'\right) \end{bmatrix}$$

$$-x' - \frac{L}{2} + x'$$
$$= -\frac{L}{2}$$

Add row # 3 to row # 1.

mult row # 3 by -1 & add to row # 5

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & x' & -1 & -x' \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -\frac{L}{2}-x' & 0 & -\frac{L}{2}+x' \end{bmatrix} u = \begin{bmatrix} -\frac{q}{6}x' \\ -\frac{q}{6} \\ 0 \\ -\frac{q}{6} \\ -\frac{q}{6}\left(\frac{L}{2}-x'\right) \end{bmatrix}$$

mult row # 4 by -1

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & x' & -1 & -x' \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & x' & -1 & -x' \\ 0 & 0 & 0 & +1 & 0 & -1 \\ 0 & 0 & 0 & -\frac{L}{2}-x' & 0 & -\frac{L}{2}+x' \end{bmatrix} u = \begin{bmatrix} -\frac{q}{6}x' \\ -\frac{q}{6} \\ 0 \\ \frac{q}{6} \\ -\frac{q}{6}\left(\frac{L}{2}-x'\right) \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{L}{2}+x' + (-1)\left(\frac{L}{2}+x'\right) \end{bmatrix} u = \begin{bmatrix} \cancel{-\frac{q}{6}x'} \\ \cancel{-\frac{q}{6}} \\ \cancel{0} \\ \cancel{q/6} \\ \cancel{-\frac{q}{6}\left(\frac{L}{2}-x'\right)} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\frac{q}{6}x' + \frac{q}{6}(-x') \\ 0 \\ -\frac{q}{6}x' \\ \frac{q}{6} \\ \frac{q}{6}\left(\frac{L}{2}+x'\right) - \frac{q}{6}\left(\frac{L}{2}-x'\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2q x'}{6} \\ 0 \\ -\frac{q}{6}x' \\ \frac{q}{6} \\ \frac{2q}{6}x' \end{bmatrix} \quad \checkmark\checkmark$$

+ elt at position 5 × 6 is

$$-\frac{L}{2} + \cancel{x'} - \frac{L}{2} - \cancel{x'} = -L \qquad \text{negative !!} \quad \checkmark$$

$\therefore$

$$\cancel{D_{\boxed{}} = \frac{2q}{6L}x'}$$

∴ get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_I \\ B_I \\ A_{II} \\ B_{II} \\ A_{III} \\ B_{III} \end{bmatrix} = \begin{bmatrix} -\frac{2q}{\epsilon_0}x' \\ 0 \\ -\frac{q}{\epsilon_0}x' \\ \frac{q}{\epsilon_0} \\ -\frac{2q}{\epsilon_0 L}x' \end{bmatrix}$$

$\cancel{[A] = [B]\cdot[X] = \left[\frac{q}{\epsilon_0 L}\right]}$ $\cancel{[B] = \left[\frac{q}{2\epsilon_0}\right]}$

Add row #5 to rows #2 & #4

$$\Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_I \\ B_I \\ A_{II} \\ B_{II} \\ A_{III} \\ B_{III} \end{bmatrix} = \begin{bmatrix} -\frac{2q}{\epsilon_0}x' \\ -\frac{2q}{\epsilon_0 L}x' \\ -\frac{q}{\epsilon_0}x' \\ \frac{q}{\epsilon_0} - \frac{2q}{\epsilon_0}\frac{x'}{L} \\ -\frac{2q}{\epsilon_0 L}x' \end{bmatrix}$$

$$\frac{q}{\epsilon_0} - \frac{2q}{\epsilon_0}\frac{x'}{L} = \frac{q}{\epsilon_0 L}(L - 2x')$$

$$\therefore \quad B_{III} = -\frac{2q}{\epsilon_0 L}x' \; ; \quad B_{II} = \frac{q}{\epsilon_0 L}(L-2x') \; ; \quad B_I = -\frac{2q}{\epsilon_0 L}x'$$

$$+ \quad A_I - A_{III} = -\frac{2q}{\epsilon_0} x'$$

$$A_{II} - A_{III} = -\frac{q}{\epsilon_0} x'$$

$$\Rightarrow \quad A_I = -\frac{2q}{\epsilon_0} x' + A_{III} \quad + \quad A_{II} = -\frac{q}{\epsilon_0} x' + A_{III}.$$

w/ A_{III} undetermd. at this step. thus

$$\phi_I = \left(\frac{-2q}{\epsilon_0} x' + A_{III}\right) + -\frac{2q}{\epsilon_0 L} x' x \qquad -\frac{L}{2} \le x \le -x'$$

$$\phi_{II} = \left(-\frac{q}{\epsilon_0} x' + A_{III}\right) + \frac{q}{\epsilon_0 L} (L - 2x') x \qquad -x' \le x \le x'$$

$$\phi_{III} = A_{III} + -\frac{2q}{\epsilon_0 L} x' x \qquad +x' \le x \le \frac{L}{2}$$

As specified in H-E to make the avg. potential zero

$$\frac{L}{L} \int_{-\frac{L}{2}}^{+\frac{L}{2}} \phi \, dx = 0$$

$$\Rightarrow \int_{-\frac{L}{2}}^{-x'} \phi_I \, dx + \int_{-x'}^{+x'} \phi_{II} \, dx + \int_{x'}^{L/2} \phi_{III} \, dx = 0$$

$$\Rightarrow \left(\frac{-2q}{G_0}x' + A_{III}\right)\left(-x' + \frac{L}{2}\right) - \frac{2q}{G_0 L}\frac{x'}{2}\left[(-x')^2 - \left(-\frac{L}{2}\right)^2\right]$$

$$+ \left(-\frac{q}{G_0}x' + A_{III}\right)(2x') + \frac{q}{G_0 L}\cancel{\frac{(L-2x')}{2}}^{\,0} \cdot 0$$

$$+ A_{III}\left(\frac{L}{2} - x'\right) - \frac{2q}{G_0 L}\frac{x'}{2}\left[\frac{L^2}{4} - x'^2\right] = 0$$

$$\Rightarrow A_{III}\left[\cancel{x'} + \frac{L}{2} + \cancel{2x'} + \frac{L}{2} - \cancel{x'}\right] =$$

$$-\left(-\frac{2q}{G_0}x'\right)\left(-x' + \frac{L}{2}\right) + \cancel{2}\,\frac{qx'}{G_0 L}\left[x'^2 - \frac{L^2}{4}\right] + \frac{q}{G_0}x' \, 2x' +$$

$$+ \frac{qx'}{G_0 L}\left[\frac{L^2}{4} - x'^2\right]$$

$$\Rightarrow L A_{III} = \cancel{\frac{-2qx'^2}{G_0}} + \frac{2qx'L}{2G_0} + \cancel{\frac{qx'^3}{G_0 L}} - \cancel{\frac{qx'L^2}{4G_0 L}} + \cancel{\frac{2qx'^2}{G_0}}$$

$$+ \cancel{\frac{qx'L^2}{4G_0 L}} - \cancel{\frac{qx'^3}{G_0 L}}$$

$$\Rightarrow A_{III} = \frac{2qx'}{2G_0} = \frac{qx'}{G_0} \quad \checkmark$$

Thus

$$\phi_I = \frac{-qx'}{\epsilon_0} + \frac{-2qx'}{\epsilon_0 L}x \qquad -\frac{L}{2} \le x \le -x'$$

$$\phi_{II} = 0 + \frac{q}{\epsilon_0 L}(L-2x')x \qquad -x' \le x \le +x'$$

$$\phi_{III} = \frac{qx'}{\epsilon_0} - \frac{2qx'}{\epsilon_0 L}x \qquad +x' \le x \le \frac{L}{2}$$

or

$$\phi_I = \frac{q}{\epsilon_0 L}(-x'L - 2x'x) \qquad -\frac{L}{2} \le x \le -x'$$

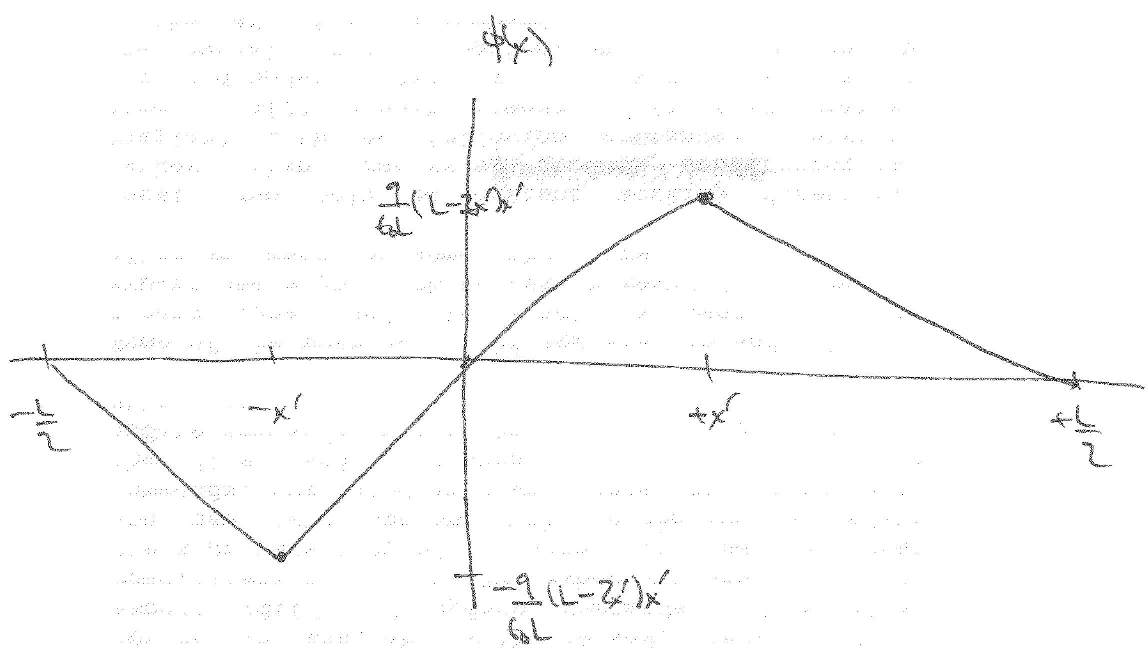
$$\phi_{II} = \frac{q}{\epsilon_0 L}(L-2x')x \qquad ''$$

$$\phi_{III} = \frac{q}{\epsilon_0 L}(x'L - 2x'x) \qquad ''$$

$$\Rightarrow \quad \phi_I = \frac{-q}{\epsilon_0 L}(L+2x)x' \qquad -\frac{L}{2} \le x \le -x'$$

$$\phi_{II} = \frac{q}{\epsilon_0 L}(L-2x')x \qquad -x' \le x \le \cancel{\tfrac{L}{2}} +x' \qquad \text{eq } 5\text{-}3 \checkmark$$

$$\phi_{III} = \frac{q}{\epsilon_0 L}(L-2x)x' \qquad +x' \le x \le \frac{L}{2}$$



Then $E = -\frac{d\phi}{dx}$ so

$$E_{I} = +\frac{q}{\epsilon_0 L}2x' \qquad -\frac{L}{2} \leq x \leq -x'$$

$$E_{II} = -\frac{q}{\epsilon_0 L}(L-2x') \qquad -x' \leq x \leq +x'$$

$$E_{III} = +\frac{q}{\epsilon_0 L}2x' \qquad +x' \leq x \leq +\frac{L}{2}$$

The Average electric field at $-x'$ is

$$\bar{E}(-x') = \frac{1}{2}E_{I}(-x') + \frac{1}{2}E_{II}(-x') = \frac{1}{2}\left(\frac{2x'q}{\epsilon_0 L} - \frac{q}{\epsilon_0 L}(L-2x')\right)$$

$$= \frac{1}{2}\left[\frac{-qL}{\epsilon_0 L} + \frac{4x'q}{\epsilon_0 L}\right]$$

$$= \frac{-q}{2\epsilon_0 L}(L-4x') \qquad \text{eq 5-5}$$

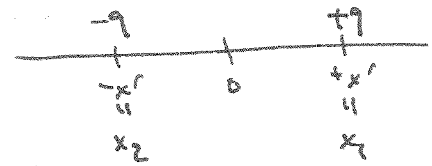
sim.

$$\bar{E}(+x') = \frac{1}{2} E_{II}(+x') + \frac{1}{2} E_{III}(x')$$

$$= \frac{1}{2}\left[\frac{-q}{\epsilon_0 L}(L-2x') + \frac{q}{\epsilon_0 L} 2x'\right]$$

$$= \frac{-q}{2\epsilon_0 L}(L-4x') \qquad \text{eq } 5\text{-}5 \checkmark$$

Then $F_e(x_1) = +q E_e(x_1) = \dfrac{-q^2}{2\epsilon_0 L}(L-4x')$



& $F_e(x_2) = -q E_e(x_2) = \dfrac{+q^2}{2\epsilon_0 L}(L-4x')$ eq 5-6

If $x'=0$ then since $q-q=0$ $F=0$

& sim by periodicity at $x' = +\frac{L}{2}$ $F=0$

$$G = F_e - F$$

$$= \frac{-q^2}{2\epsilon_0 L}(L-2x) - \left[\alpha\left(\frac{-q^2}{2\epsilon_0 L}\right)(L-2x_p) + (1-\alpha)\left(\frac{-q^2}{2\epsilon_0 L}\right)(L-2x_{p+1})\right]$$

$$= \frac{-q^2}{2\epsilon_0 L}\left[\cancel{L}-2x - \alpha(\cancel{L}-2x_p) - (1-\alpha)(\cancel{L}-2x_{p+1})\right]$$

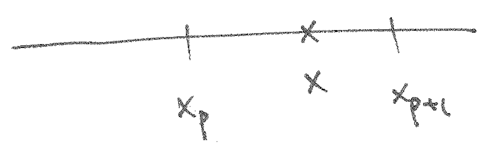
$$= \frac{-q^2}{2\epsilon_0 L}\left[-2x + 2\alpha x_p + (1-\alpha)2x_{p+1}\right]$$

$$= \frac{+q^2}{\epsilon_0 L}\left[x - \alpha x_p - (1-\alpha)x_{p+1}\right]$$

$$= \frac{q^2}{\epsilon_0 L}\left[x - x_{p+1} - \alpha x_p + \alpha x_{p+1}\right]$$

$$= \frac{q^2}{\epsilon_0 L}\left[x - x_{p+1} - \alpha(x_p - x_{p+1})\right] \qquad \text{eq } 5\text{-}18 \ \checkmark$$

pick $\alpha = \dfrac{x - x_{p+1}}{x_p - x_{p+1}}$



$$= \frac{x_{p+1} - x}{x_{p+1} - x_p} \qquad \underbrace{\frac{x - x_p}{x_{p+1} - x_p}}$$

$$= \frac{x_{p+1} - x_p + x_p - x}{x_{p+1} - x_p} = 1 + \frac{x_p - x}{x_{p+1} - x_p} = 1 + \frac{x_p - x}{H} = 1 - \frac{(x - x_p)}{H}$$

then $\varepsilon_1 = 0$ eq 5-19 ✓

2-3

Then take for $F(x) = \alpha F(x_p) + (1-\alpha) F(x_{p+1})$

$$F(x) = \left[1 - \frac{x - x_p}{H}\right] F(x_p) + \left(\frac{x - x_p}{H}\right) F(x_{p+1})$$

$$\parallel$$

$$\cancel{1 - \left[\frac{x_{p+1} - x}{H}\right]}$$

$$= \cancel{\frac{H - x_{p+1} + x}{H}} \qquad x_p = x_{p+1} - H$$

$$= \cancel{\frac{x_{p+1} - x_p - x_{p+1} + x}{H}}$$

$$\parallel$$

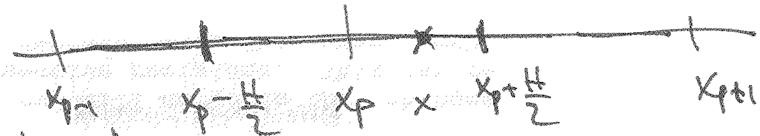
$$\frac{x - (x_{p+1} - H)}{H} = \frac{x - x_{p+1}}{H} + 1$$

$$= +1 + \frac{x - x_{p+1}}{H}$$

$\therefore$

$$F(x) = \left[1 - \frac{x - x_p}{H}\right] F(x_p) + \left[1 + \frac{(x - x_{p+1})}{H}\right] F(x_{p+1})$$

9 5-20 ✓

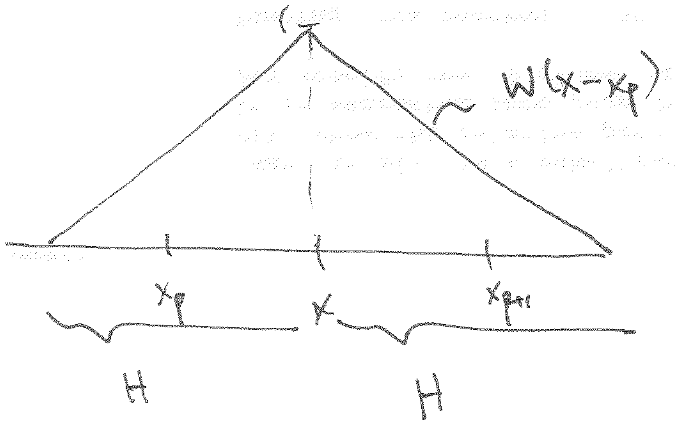


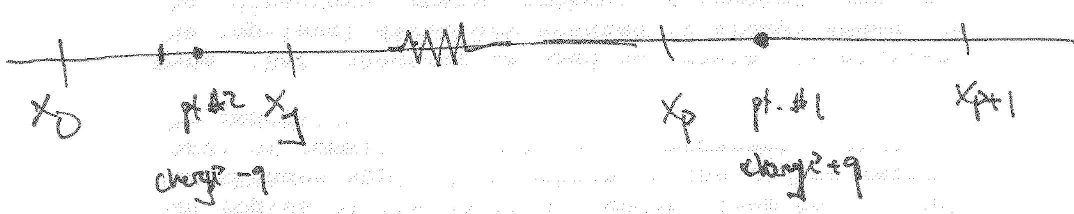
define $W(x-x_p) = \begin{cases} 1 - \dfrac{|x-x_p|}{H} & |x-x_p| \le H \\ 0 & \text{else} \end{cases}$

Then

$$\sum_{p} W(x-x_p)\,F(x_p) = W(x-x_p)F(x_p) + W(x-x_{p+1})F(x_{p+1})$$

$$= \left(1 - \frac{(x-x_p)}{H}\right)F(x_p) + \left(1 + \frac{(x-x_{p+1})}{H}\right)F(x_{p+1})$$





$\rho(x_0) = \dfrac{-q}{H}\beta$ pick $\beta = 1 - \dfrac{x}{H}$

$\rho(x_1) = \dfrac{-q}{H}(1-\beta)$

HJ 157 Hockney / Eastwood

$$\rho^+(x) = Ш\left(\frac{x}{H}\right)\rho'(x)$$

$$\Rightarrow \quad \hat{\rho}^+(k) = H\, Ш\left(\frac{kH}{2\pi}\right) * \hat{\rho}'(k)$$

explicitly evaluating the convolution eq 5-177 gives, from table A-3

if $f(x)g(x) \overset{F.T.}{\longleftrightarrow} \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \hat{f}(k')\hat{g}(k-k')$ then the above becomes:

$$\hat{\rho}^+(k) = \frac{H}{2\pi}\int_{-\infty}^{+\infty} dk'\, \hat{\rho}'(k-k')\, Ш\left(\frac{k'H}{2\pi}\right)$$

$$= \frac{H}{2\pi}\int_{-\infty}^{+\infty} dk'\, \hat{\rho}'(k-k') \sum_n \delta\left(\frac{k'H}{2\pi} - n\right)$$

define $k_g = \frac{2\pi}{H}$

$$\sum_n \delta\left(\frac{k'H}{2\pi} - n\right) = \sum_n \delta\left(\frac{k'}{k_g} - n\right) = \sum_n \delta\left(\frac{k' - nk_g}{k_g}\right)$$

From a property of delta fns $\delta\left(\frac{x}{a}\right) = |a|\,\delta(x)$

So the above becomes:

$$\hat{\rho}^{+}(k) = \frac{H}{2\pi} \cdot |k_g| \sum_n \hat{\rho}'(k - nk_g) = \sum_n \hat{\rho}'(k - nk_g) = \hat{\rho}(k)$$

via. eq 5-172.

eq 5-129 ✓

Equality of $\hat{\rho}$ + $\hat{\rho}^{+}$

$$\hat{\rho}^{+}(k) = \int_{-\infty}^{+\infty} dx\, \rho^{+}(x)\, e^{-ikx} = \int_{-\infty}^{+\infty} dx\, Ш\left(\frac{x}{H}\right)\rho'(x)\, e^{-ikx} = \int_{-\infty}^{+\infty} dx \sum_n \delta\left(\frac{x}{H} - n\right)\rho'(x)\, e^{-ikx}$$

$$= \int_{-\infty}^{+\infty} dx \sum_n \delta\left(\frac{x - nH}{H}\right)\rho'(x)\, e^{-ikx} \qquad w/ \quad \delta\left(\frac{x}{a}\right) = |a|\,\delta(x)$$

$$= \cancel{\sum_n} = H \sum_n \rho'(nH)\, e^{-iknH}$$

let $x_p = p \cdot H$

$$\hat{\rho}^{+}(k) = H \sum_p \rho'(x_p)\, e^{-ikx_p} = \hat{\rho}(k) \qquad \text{eq } 5\text{-}131$$

via. eq 5-173

$$\lambda_{GS} = \rho^2 \qquad \rho = \cos\left(\frac{\pi}{n}\right) \cong 1 - \frac{1}{2}\frac{\pi^2}{n^2}$$

$$\therefore \lambda_{GS} \cong 1 - \frac{\pi^2}{n^2}$$

From 6-24b $\quad t^*_{GS} = \dfrac{-p\ln(10)}{\ln(\lambda_{GS})}$

w/ $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \dfrac{x^k}{k} = x - \dfrac{x^2}{2} \cdots$

So $\ln(\lambda_{GS}) \cong -\dfrac{\pi^2}{n^2} + O\left(\dfrac{1}{n^4}\right)$

$$t^*_{GS} = \frac{-p\ln(10)}{-\frac{\pi^2}{n^2}} = \frac{\ln(10)}{\pi^2}(pn^2) \cong .233\,(pn^2) \cong \frac{pn^2}{4}$$

w/ $n = N_g^{1/d}$

$$t^*_{GS} = \frac{p(N_g)^{2/d}}{4} \quad \text{eq } 6\text{-}30\text{b} \quad \checkmark$$

$\#$ ops in eq 6-26 $= 9$ $\therefore$ on a two dimensional grid $(d=2)$

$$T_{GS} = 9 \cdot n^2\left(\frac{pn^2}{4}\right) \cong 2pn^4 = 2p\left(N_g^{1/d}\right)^4\Big|_{d=2} = 2pN_g^2$$

eq 6-31 $\checkmark$

since $\rho = 1 - \frac{\pi^2}{2n^2}$

+ $\sqrt{1+x} = 1 + \frac{x}{2} + \frac{x^2}{8} + \cdots$

we have

$$\sqrt{1-\rho^2} = \sqrt{1-\left(1-\frac{\pi^2}{2n^2}\right)^2} \cancel{\approx \sqrt{1-\left(1-\frac{\pi^2}{n^2}\right)}}$$

$$= \sqrt{1-\left(1-\frac{\pi^2}{n^2}+\frac{\pi^4}{4n^4}\right)} = \sqrt{\frac{\pi^2}{n^2}-\frac{\pi^4}{4n^4}} = \frac{\pi}{n}\sqrt{1-\frac{\pi^2}{4n^2}}$$

$$= \frac{\pi}{n}\left(1-\frac{\pi^2}{8n^2}\right)$$

so $1+\sqrt{1-\rho^2} = 1 + \frac{\pi}{n} - \frac{\pi^3}{8n^3}$

so $\omega_b = \frac{2}{1+\sqrt{1-\rho^2}} = \frac{2}{1+\frac{\pi}{n}-\frac{\pi^3}{8n^3}} \simeq \frac{2}{1+\frac{\pi}{n}} \doteq 2\left(1-\frac{\pi}{n}\right)$ eq 6-34b ✓

$\therefore \; \lambda_{SOR} = \omega_b - 1 = 1 - \frac{2\pi}{n}$ eq 6-34c ✓

since $\lambda_{SOR}^{t} = (1 - \frac{2\pi}{n})^{t} = 10^{-p}$

$\Rightarrow \quad t^{*}_{SOR} = \dfrac{-p \ln(10)}{\ln(1 - \frac{2\pi}{n})}$

+ w/ $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k}}{k}$

$\ln(1 - \frac{2\pi}{n}) \cong -\frac{2\pi}{n}$ so

$t^{*}_{SOR} = \dfrac{-p\ln(10)}{(-\frac{2\pi}{n})} = (pn)\,\dfrac{\ln(10)}{2\pi}$

Now $\dfrac{\ln(10)}{2\pi} = .36\cdots \cong \frac{1}{3}$

$t^{*}_{SOR} \cong \dfrac{pn}{3}$

$n \cong 100$ eq 6-26a is 9 opperation

eq 6-26b is 4 oppe as is or 3 opperations

w/ eq 6-27b

for a total of 13 or 12 opperations

so $T_{SOR} = 12 N_g \frac{p}{3} (N_g)^{1/2} \cong 24 N_g^{1+1/2}$ eq 6-36

Assuming

$$\rho = \cos(\pi/n)(2-\cos(\pi/n)) \cong \left(1-\frac{\pi}{n}\right)\left(2-\left(1-\frac{\pi}{n}\right)\right)$$

$$= \left(1-\frac{\pi}{n}\right)\left(1+\frac{\pi}{n}\right) = 1-\frac{\pi^2}{n^2} \quad \text{eq 6-39a} ✓$$

$$\omega_b = \frac{2}{1+\sqrt{1-\rho^2}} = ?$$

since $\rho^2 \cong 1 - \frac{2\pi^2}{n^2}$

$$1-\rho^2 \cong \frac{2\pi^2}{n^2}$$

$$\sqrt{1-\rho^2} \cong \frac{\sqrt{2}\,\pi}{n}$$

so $$\omega_b = \frac{2}{1+\frac{\sqrt{2}\pi}{n}} \cong 2\left(1-\frac{\sqrt{2}\pi}{n}\right) = 2-\frac{2\sqrt{2}\pi}{n} \quad \text{eq 6-39b} ✓$$

$$\lambda_{SLOR} = \omega_b - 1 = 1-\frac{2\sqrt{2}\pi}{n} \quad \text{eq 6-39c}$$

$$t^*_{SLOR} = \frac{-p\ln(10)}{\ln(\lambda)} = \frac{-p\ln(10)}{-\frac{2\sqrt{2}\pi}{n}} = \frac{\ln(10)}{2\sqrt{2}\pi}(np)$$

Now $\frac{\ln(10)}{2\sqrt{2}\pi} = .2591 \cong \frac{1}{4}$ $\therefore$ $t^*_{SLOR} = \frac{np}{4}$ eq 6.39d ✓

$T_{SLOR} \cong 11\cdot N_g \frac{pn}{4} \cong 3pn^3$ w/ $N_g = n^2$ in a 2D problem.

$$\frac{\phi^{n+1/2}-\phi^{n}}{\Delta t/2} = -L_x\phi^{n+1/2} - L_y\phi^{n} + q$$

$$\frac{\phi^{n+1}-\phi^{n+1/2}}{\Delta t/2} = -L_x\phi^{n+1/2} - L_y\phi^{n+1} + q$$

$$\Rightarrow \quad \phi^{n+1/2} - \phi^{n} = -rL_x\phi^{n+1/2} - rL_y\phi^{n} + rq$$

$$+ \quad \phi^{n+1} - \phi^{n+1/2} = -rL_x\phi^{n+1/2} - rL_y\phi^{n+1} + rq$$

$$\Rightarrow \quad (I+rL_x)\,\phi^{n+1/2} = [I-rL_y]\,\phi^{n} + rq \qquad \text{eq 6-48a} \checkmark$$

$$+ \quad [I+rL_y]\,\phi^{n+1} = [I - rL_x]\,\phi^{n+1/2} + rq \qquad \text{eq 6-48b} \checkmark$$

$$\Rightarrow \quad \phi^{n+1/2} = [I+rL_x]^{-1}[I-rL_y]\,\phi^{n} + r[I+rL_x]^{-1}q$$

then

$$\phi^{n+1} = [I+rL_y]^{-1}[I-rL_x][I+rL_x]^{-1}[I-rL_y]\,\phi^{n} + r[I+rL_y]^{-1}[I-rL_x][I+rL_x]^{-1}q$$

$$\cancel{+rq} \; [I+rL_y]^{-1} rq$$

$$\equiv M\phi^{n} + N$$

w/ $M = [I + rL_y]^{-1}[I - rL_x][I + rL_x]^{-1}(I - rL_y)$

$N = r[I + rL_y]^{-1}[I - rL_x][I + rL_x]^{-1}q + [I + rL_y]^{-1}rq$

Now consider

$[I - rL_x][I + rL_x]^{-1} \stackrel{?}{=} [I + rL_x]^{-1}[I - rL_x]$ i.e. do these two

matrices commute?

$\Leftrightarrow \quad [I + rL_x][I - rL_x] \stackrel{?}{=} [I - rL_x][I + rL_x]$

Yes since each side equals $I - rL_x + rL_x - r^2L_x^2$ thus M can be

written as

$M = [I + rL_y]^{-1}[I + rL_x]^{-1}[I - rL_x][I - rL_y]$ eq 6-49 b ✓

Sim w/ N, N can be written as

$N = \cancel{[I + rL_y]^{-1}[I + rL_x]^{-1}}[$

$= [I + rL_y]^{-1}\left\{[I + rL_x]^{-1}[I - rL_x] + \underset{\shortparallel}{I}\right\}rq$

$[I + rL_x]^{-1}[I + rL_x]$

$$\Rightarrow \quad N = [I + rL_y]^{-1}[I + rL_x]^{-1}(2I)\, rq$$

$$= 2[I + rL_y]^{-1}[I + rL_x]^{-1} rq \qquad \text{eq } 6\text{-}49c \ \checkmark$$

pg 194 Hockney / Eastwood

Given

$$(A+N)\phi^{t+1} = N\phi^t + \rho$$

$$\Leftrightarrow \quad LU\phi^{t+1} = N\phi^t + \rho$$

$$LU\phi^{t+1} - LU\phi^t = \underbrace{(N-LU)}_{-A}\phi^t + \rho$$

$$= -(A\phi^t - \rho)$$

$$= -r$$

$$\rightarrow \quad LU\Delta\phi^{(t)} = -r$$

let $y \ni Ly = -r$

$\&$ $\quad U\Delta\phi^{(t)} = y$

$\&$ $\quad \phi^{(t+1)} = \phi^{(t)} + \Delta\phi^{(t)}$ $\qquad$ q 6-74e.