A Solution Manual and Notes for: Ordinary Differential Equations by Edward L. Ince.

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Chapter 1 (Introductory)

Notes on the text

An example differential identity

The suggested differential identity can be obtained as follows. We first recall the identity

$$\frac{dy}{dx}\frac{dx}{dy} = 1\,. \tag{1}$$

Then taking the y derivative of this expression and using the product rule gives

$$\frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dx}{dy} + \frac{dy}{dx}\frac{d^2x}{dy^2} = 0\,.$$

Using $\frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx}$ on the first term we get

$$\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dy^2} = 0\,.$$

Now taking the x derivative of this expression gives four terms

$$\frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 + \frac{d^2y}{dx^2}\frac{d}{dx}\left[\left(\frac{dx}{dy}\right)^2\right] + \frac{d^2y}{dx^2}\frac{d^2x}{dy^2} + \frac{dy}{dx}\frac{d}{dx}\left(\frac{d^2x}{dy^2}\right) = 0\,.$$

If we consider the second term on the left-hand-side we get

$$\frac{d^2y}{dx^2}\frac{d}{dx}\left[\left(\frac{dx}{dy}\right)^2\right] = 2\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)\cdot\frac{d^2x}{dy^2}\frac{dy}{dx}\,.$$

Using Equation 1 this becomes

$$2\frac{d^2y}{dx^2}\frac{d^2x}{dy^2}\,,$$

and we finally get after combining terms

$$\frac{d^3x}{dy^3}\left(\frac{dy}{dx}\right)^2 + 3\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} + \frac{d^3y}{dx^3}\left(\frac{dx}{dy}\right)^2 = 0\,,$$

which is the quoted expression.

Notes on classification of differential equations

To classify the differential equation given by

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2} = 3\frac{d^2y}{dx^2}\,,$$

we need to make sure it is polynomial in all of the differential coefficients. We can do this by by squaring both sides to remove the square root and we get

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = 9\left(\frac{d^2y}{dx^2}\right)^2 .$$

Since the highest differential coefficient is the one corresponding to $\frac{d^2y}{dx^2}$, this equation is second order and since it is "squared" in the above expression the *degree* of this equation is two.

Notes on the Genesis of an Ordinary Differential Equation

When the constraint on x and y is given by the following constraint

$$f(x, y, c_1, c_2, \cdots, c_n) = 0 . \tag{2}$$

We can derive an ordinary differential equation that is satisfied for every possible settings of the coefficients c_i as follows. We taking the first x derivative of this expression to get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0 . \tag{3}$$

A x derivatives of this equation using the chain rule gives

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x} y' + \frac{\partial^2 f}{\partial x \partial y} y' + \frac{\partial^2 f}{\partial y^2} y'^2 + \frac{\partial f}{\partial y} y'' = 0 ,$$

or combining terms we have

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} y' + \frac{\partial^2 f}{\partial y^2} y'^2 + \frac{\partial f}{\partial y} y'' = 0 . \tag{4}$$

We would continue taking x derivatives until we had evaluated n derivatives. Then using all $n+1$ equations (the n derivatives) plus the original constraint $f(x, y, c_1, c_2, \cdots, c_n) = 0$ we can eliminate the coefficients c_i and obtain an expression of the form

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0 .$$

In some cases the primitive Equation 2 is reducible and while it looks like there may be n coefficients requiring n derivatives in the ordinary differential equation in the case when the primitive is reducible this is not true. For example the primitive

$$y^2 - (a+b)y + abx^2 = 0 ,$$

can be factored as

$$(y - ax)(y - bx) = 0 .$$

Thus the original primitive is really two primitives $y - ax = 0$ or $y - bx = 0$. For each primitive we have a constraint function f like $f(x, y, c) = y - cx$ and thus Equation 3 is $-c + y' = 0$ or since $c = \frac{y}{x}$ Thus $y' - a = 0$ or $y' = a = \frac{y}{x}$ is the differential equation for y (note is is of first order).

Notes on the Differential Equation for Confocal Conics

As an example of obtaining the differential equation from a primitive consider the equation

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1\,. \tag{5}$$

We assume that a and b are fixed (given say) and that λ is a parameter used to index or specify solutions $y = y(x)$ and we wish to derive the differential equation for $y(x)$. Since there is only one parameter we expect the differential equation to be of first order. Taking the x derivative of the primitive gives

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0\,.$$

Next we solve for $\frac{1}{a^2+\lambda}$ and $\frac{1}{b^2+\lambda}$ in the primitive Equation 5 to get

$$\begin{aligned} \frac{1}{a^2+\lambda} &= \left(1 - \frac{y^2}{b^2+\lambda}\right)\frac{1}{x^2} \\ \frac{1}{b^2+\lambda} &= \left(1 - \frac{x^2}{a^2+\lambda}\right)\frac{1}{y^2}\,. \end{aligned}$$

First put $\frac{1}{a^2+\lambda}$ in the derivative of the primitive as

$$\frac{2x}{x^2}\left[1 - \frac{y^2}{b^2+\lambda}\right] + \frac{2yy'}{b^2+\lambda} = 0\,.$$

or

$$\frac{1}{x} - \frac{y^2}{x}\frac{1}{b^2+\lambda} + \frac{yy'}{b^2+\lambda} = 0\,.$$

or

$$\frac{b^2+\lambda}{x} = \frac{y^2}{x} - yy'\,,$$

or

$$b^2+\lambda = y^2 - xyy' \tag{6}$$

when we solve for $b^2+\lambda$. Next put $\frac{1}{b^2+\lambda}$ into the derived primitive to get

$$\frac{x}{a^2+\lambda} + \frac{yy'}{y^2}\left[1 - \frac{x^2}{a^2+\lambda}\right] = 0\,.$$

or

$$\left(x - \frac{x^2}{y^2}yy'\right)\frac{1}{a^2+\lambda} = -\frac{yy'}{y^2} = -\frac{y'}{y}\,,$$

or

$$a^2+\lambda = -\frac{y}{y'}\left(x - \frac{x^2}{y}y'\right) = \frac{x^2y' - xy}{y'}\,, \tag{7}$$

Subtracting Equation 7 from Equation 6 to eliminate λ gives

$$a^2 - b^2 = \frac{x^2y' - xy}{y'} - y^2 + xyy'\,,$$

multiply this expression by y' to get

$$xyy'^2 + (-y^2 + b^2 - a^2)y' + x^2y' - xy = 0\,.$$

or

$$xyy'^2 + (x^2 - y^2 - a^2 + b^2)y' - xy = 0\,.$$

Which is an equation of the first order (and second degree).

Notes on the Formation of Partial Differential Equations

The first derivative of $f(x_1, x_2, \cdots, x_m; z; c_1, c_2, \cdots, c_n) = 0$ with respect to the independent variable x_r is given by

$$\frac{\partial f}{\partial x_r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x_r} = 0\,.$$

Taking the x_s derivative of the above assuming $x_s \neq x_r$ and using the chain rule gives

$$\frac{\partial^2 f}{\partial x_s \partial x_r} + \frac{\partial^2 f}{\partial z \partial x_r}\frac{\partial z}{\partial x_s} + \frac{\partial^2 f}{\partial x_s \partial z}\frac{\partial z}{\partial x_r} + \frac{\partial^2 f}{\partial z^2}\left(\frac{\partial z}{\partial x_s}\right)\frac{\partial z}{\partial x_r} + \frac{\partial f}{\partial z}\frac{\partial^2 z}{\partial x_s \partial x_r} = 0\,.$$

Notes on the Partial Differential Equations of the most general sphere

Consider the primitive for the most general sphere

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2\,, \tag{8}$$

where we consider z a function of x and y. Then taking the x and y derivatives and recalling the following derivative shorthands

$$\begin{aligned} p &\equiv \frac{\partial z}{\partial x} & (9)\\ q &\equiv \frac{\partial z}{\partial y} & (10)\\ r &\equiv \frac{\partial^2 z}{\partial x^2} & (11)\\ s &\equiv \frac{\partial^2 z}{\partial x \partial y} & (12)\\ t &\equiv \frac{\partial^2 z}{\partial y^2}\,, & (13) \end{aligned}$$

we have

$$\begin{aligned} (x-a) + (z-c)p &= 0 & (14)\\ (y-b) + (z-c)q &= 0\,. & (15) \end{aligned}$$

Taking the x derivative of Equation 14 gives

$$1 + \frac{\partial z}{\partial x} p + (z - c)\frac{\partial p}{\partial x} = 0\,,$$

or

$$1 + p^2 + (z - c)r = 0\,. \tag{16}$$

Taking the y derivative of Equation 14 one gets

$$1 + \frac{\partial z}{\partial y} p + (z - c)\frac{\partial p}{\partial y} = 0\,.$$

or

$$pq + (z - c)s = 0\,. \tag{17}$$

Taking the y derivative of Equation 15 one gets

$$1 + \frac{\partial z}{\partial y} + (z - c)\frac{\partial q}{\partial y} = 0\,.$$

or

$$1 + q^2 + (z - c)t = 0\,. \tag{18}$$

Solving for $z - c$ in Equations 16, 17, and 18 and let the common expression be λ and we find

$$\lambda = \frac{1 + p^2}{r} = \frac{pq}{s} = \frac{1 + q^2}{t}\,,$$

is the differential equation for z. Solving for r, s, and t in terms of λ gives

$$r = \frac{1 + p^2}{\lambda}\,, \quad s = \frac{pq}{\lambda}\,, \quad \text{and} \quad t = \frac{1 + q^2}{\lambda}\,,$$

so the expression $\lambda^2(rt - s^2)$ is given by

$$\lambda^2(rt - s^2) = \lambda^2 \left[\frac{(1 + p^2)(1 + q^2)}{\lambda^2} - \frac{p^2 q^2}{\lambda^2} \right] = 1 + p^2 + q^2 > 0\,.$$

Thus if z is to be real valued then we must have $rt - s^2 > 0$ or

$$rt > s^2\,,$$

true.

Notes on a property of Jacobians

In part of this proof we assume that $\phi(u_1, u_2, \cdots, u_p) = 0$ holds while all determinants of a higher order $p + 1$ or greater vanish. Take the $x_1, x_2, \cdots, x_p$ derivatives of this functional relationship using the chain rule to get

$$\begin{aligned}
\frac{\partial \phi}{\partial x_1} &= \frac{\partial \phi}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \cdots + \frac{\partial \phi}{\partial u_p}\frac{\partial u_p}{\partial x_1} = 0 \\
\frac{\partial \phi}{\partial x_2} &= \frac{\partial \phi}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \cdots + \frac{\partial \phi}{\partial u_p}\frac{\partial u_p}{\partial x_2} = 0 \\
&\vdots \\
\frac{\partial \phi}{\partial x_p} &= \frac{\partial \phi}{\partial u_1}\frac{\partial u_1}{\partial x_p} + \frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial x_p} + \cdots + \frac{\partial \phi}{\partial u_p}\frac{\partial u_p}{\partial x_p} = 0\,,
\end{aligned}$$

or as a matrix expression this is

$$\begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_p}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_p}{\partial x_2} \\ \vdots & & & \vdots \\ \frac{\partial u_1}{\partial x_p} & \frac{\partial u_2}{\partial x_p} & \cdots & \frac{\partial u_p}{\partial x_p} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi}{\partial u_1} \\ \frac{\partial \phi}{\partial u_2} \\ \cdots \\ \frac{\partial \phi}{\partial u_p} \end{bmatrix} = 0\,.$$

Since we assume that $\frac{\partial \phi}{\partial u_i} \neq 0$ for all i (otherwise ϕ is identically the zero function) we must have the determinant of the matrix of leading coefficients zero, which is in contrast to the hypothesis that the determinant is nonzero.

Notes on partial differential equations through elimination of a function

This section presents a method for deriving a partial differential equation for a function z if we assume that each function u_i is implicitly a function of all the independent variables $x_1, x_2, \cdots, x_n$ and the dependent variable z, and a constraint (primitive) on the set of u_i. If we think of each u_i with z replaced by its expression in terms of $z(x_1, x_2, \cdots, x_n)$ we have

$$u_i(x_1, x_2, \cdots, x_n, z) = u_i(x_1, x_2, \cdots, x_n, z(x_1, x_2, \cdots, x_n))\,.$$

Then given the constraint

$$F(u_1, u_2, \cdots, u_n) = 0\,,$$

we can use the same logic as in the previous section to obtain that

$$\begin{vmatrix} D_1 u_1 & D_1 u_2 & \cdots & D_1 u_n \\ D_2 u_1 & D_2 u_2 & \cdots & D_2 u_n \\ \vdots & & & \vdots \\ D_n u_1 & D_n u_2 & \cdots & D_n u_n \end{vmatrix} = 0\,. \tag{19}$$

Viewing each function u_i above as composed of two terms, one that depend on x_r directly and one that depends on x_r thought z we can write the partial derivatives $D_r u_s$ needed in the above determinant as

$$D_r u_s = \frac{\partial u_s}{\partial x_r} + \frac{\partial u_s}{\partial z}\frac{\partial z}{\partial x_r}\,.$$

Taking the transpose of the determinant expression and using the previous derivative identity we find

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial z}\frac{\partial z}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial z}\frac{\partial z}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} + \frac{\partial u_1}{\partial z}\frac{\partial z}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial z}\frac{\partial z}{\partial x_1} & \frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial z}\frac{\partial z}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} + \frac{\partial u_2}{\partial z}\frac{\partial z}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial u_n}{\partial x_1} + \frac{\partial u_n}{\partial z}\frac{\partial z}{\partial x_1} & \frac{\partial u_n}{\partial x_2} + \frac{\partial u_n}{\partial z}\frac{\partial z}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} + \frac{\partial u_n}{\partial z}\frac{\partial z}{\partial x_n} \end{vmatrix} = 0\,.$$

This later expression gives the partial differential equation desired.

The partial differential equation of a surface of revolution

For a surface of revolution we start with the constraint $F(z, x^2 + y^2) = 0$ and we can derive a partial differential equation for z by considering this to be in the form of $F(u_1, u_2)$, where $u_1 = z$ and $u_2 = x^2 + y^2$ and $x_1 = x$ and $x_2 = y$. In that case Equation 19 gives

$$\begin{vmatrix} D_1 u_1 & D_2 u_1 \\ D_1 u_2 & D_2 u_2 \end{vmatrix} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ 2x & 2y \end{vmatrix} = 0\,.$$

or when we simplify

$$y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0\,.$$

Notes on Euler's theorem on homogeneous functions

We can apply the results form the previous section to derive the differential equation satisfied by a function z that his homogeneous of degree n. In that case the function $z = \phi(x, y)$ must satisfy

$$z = \phi(x, y) = x^{-n} \psi\left(\frac{y}{x}\right)\,.$$

or

$$z x^n - \psi\left(\frac{y}{x}\right) = 0\,.$$

Which is of the form $F(u_1, u_2) = 0$ if we take $u_1 = z x^n$, $u_2 = \psi\left(\frac{y}{x}\right)$ and $F = u_1 - u_2$. With $x_1 = x$ and $x_2 = y$ we apply the results from above to get

$$\begin{vmatrix} D_1 u_1 & D_2 u_1 \\ D_1 u_2 & D_2 u_2 \end{vmatrix} = \begin{vmatrix} -n x^{-n-1} z + x^{-n} \frac{\partial z}{\partial x} & x^{-n} \frac{\partial z}{\partial y} \\ -\frac{y}{x^2} \psi' & \frac{1}{x} \psi' \end{vmatrix} = 0\,.$$

or when we simplify this

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n z\,. \qquad (20)$$

As an extension to this, if a function is homogeneous of degree n in the variables $x_1, x_2, \cdots, x_n$ it satisfies the following partial differential equation

$$\sum_{k=1}^{n} x_i \frac{\partial z}{\partial x_k} = n z\,.$$

Notes on the formulation of a total differential equation of three variables

We start with the constraint on x, y, and z given by $\phi(x, y, z) = c$ which represents a surface in three dimensions. We then increment x, y, and z slightly such that we remain on the given surface. Since $\phi = c$ at both the point (x, y, z) and $(x + \delta x, y + \delta y, z + \delta z)$ when we subtract ϕ at both these points we get 0. In addition, Taylor's theorem on this difference $\phi(x+\delta x, y+\delta y, z+\delta z) - \phi(x,y,z)$ for infinitesimal steps gives

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0\,. \tag{21}$$

In the case where each of these derivatives is related to a scale factor μ as

$$\frac{\partial \phi}{\partial x} = \mu P\,, \quad \frac{\partial \phi}{\partial y} = \mu Q\,, \quad \frac{\partial \phi}{\partial z} = \mu R\,,$$

then Equation 21 becomes

$$Pdx + Qdy + Rdz = 0\,.$$

Consider the analogous relationship for two variables as (x, y). In this case $\phi(x, y) \equiv y - f(x) = c$, and the needed partial derivatives for Equation 21 are

$$\frac{\partial \phi}{\partial x} = -f'(x)\,, \quad \frac{\partial \phi}{\partial y} = +1\,.$$

So Equation 21 becomes

$$-f'(x)dx + dy = 0\,,$$

which is a standard way of writing the differential of $y - f(x)$.

As an example, of these manipulations lets find the total derivative of the primitive

$$\frac{(x+z)(y+z)}{(x+y)} = c\,.$$

The needed partial derivatives are

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{y+z}{x+y} - \frac{(x+z)(y+z)}{(x+y)^2} = \frac{(y+z)(y-z)}{(x+y)^2} \\
\frac{\partial \phi}{\partial y} &= \frac{x+z}{x+y} - \frac{(x+z)(y+z)}{(x+y)^2} = \frac{(x+z)(x-z)}{(x+y)^2} \\
\frac{\partial \phi}{\partial z} &= \frac{y+z+x+z}{x+y} = \frac{x+y+2z}{x+y}\,.
\end{aligned}$$

When we put these into Equation 21 and cancel $(x+y)^2$ we get

$$(y+z)(y-z)dx + (x+z)(x-z)dy + (x+y+2z)(x+y)dz = 0\,,$$

the same expression as in the book.

Notes on the solutions of an ordinary differential equation

In this section the book makes the statement *"It must not, however, be concluded that no solution exists which is not a mere particular case of the general solution"*. What I take this to mean is that even after one has the general solution to the differential equation of order n i.e. a solution that depends on n arbitrary constants there is still the chance that a solution exists that is *not* constructable from this general solution. That is, there may still be work to do in looking for solutions to the given equation. These additional solutions are called *singular solutions*. As an example of this consider the constraint (primitive)

$$c^2 + 2cy + a^2 - x^2 = 0\,, \tag{22}$$

then the derived equation holding c as a constant is given by

$$2cdy - 2xdx = 0 \quad \text{or} \quad cdy - xdx = 0\,.$$

Thus $c = x\frac{dx}{dy}$. Before we use this information we add and subtract y^2 to Equation 22 as

$$c^2 + 2cy + y^2 - y^2 - x^2 + a^2 = 0\,,$$

or

$$(c + y)^2 - y^2 - x^2 + a^2 = 0\,.$$

When we put in $c = x\frac{dy}{dx}$ into this form of the primitive we get

$$\left(x\frac{dx}{dy} + y\right)^2 - y^2 - x^2 + a^2 = 0\,.$$

or

$$x\frac{dx}{dy} + y = \pm(y^2 + x^2 - a^2)^{1/2}\,,$$

or

$$xdx + ydy = \pm(y^2 + x^2 - a^2)^{1/2}dy\,,$$

or

$$[\pm(y^2 + x^2 - a^2)^{1/2} - y]dy - xdx = 0 \tag{23}$$

which is the equation quoted in the book if we take the positive sign. Now lets turn the situation around, consider the situation where we are *given* the differential equation specified by Equation 23 and are asked about its solutions. Since in this case we know the differential equation originated from the primitive Equation 22 we might conclude that we have all the solutions and remark that $y(x)$ given via Equation 22 namely

$$y(x) = \frac{1}{2c}(x^2 - a^2 - c^2)\,, \tag{24}$$

represents all of the solutions. To show that we are in fact missing a singular solution consider varying x, y, and c simultaneously in Equation 22. We then get

$$2cdc + 2ydc + 2cdy - 2xdx = 0\,.$$

or

$$(c + y)dc + cdy - xdx = 0\,,$$

the quoted expression. When we eliminate c using $c = x\frac{dx}{dy}$ we get

$$\left(x\frac{dx}{dy} + y\right)dc + x\frac{dx}{dy}dy - xdx = 0\,.$$

Using Equation 23 to solve for $x\frac{dy}{dx}$ we have $x\frac{dx}{dy} = \pm\left(y^2 + x^2 - a^2\right)^{1/2} - y$, which when we put this into the above we get

$$(\pm(y^2 + x^2 - a^2)^{1/2})dc + (\pm(y^2 + x^2 - a^2)^{1/2} - y)dy - xdx = 0\,, \tag{25}$$

the last two terms are zero since $(\pm(y^2 + x^2 - a^2)^{1/2} - y)dy - xdx = 0$, by Equation 23 so we end with

$$(\pm(y^2 + x^2 - a^2)^{1/2})dc = 0\,.$$

This can be made true if there exists a function $y(x)$ that satisfies $y^2 + x^2 = a^2$, or

$$y(x) = \pm\sqrt{a^2 - x^2}\,. \tag{26}$$

Notice that an expression $y(x)$ that satisfies $y^2 + x^2 = a^2$ is *not* a member of the general solution Equation 24. However by considering the function $y(x)$ that satisfies $y^2 + x^2 = a^2$ by putting it into the differential Equation 23 or $(\pm(y^2 + x^2 - a^2)^{1/2} - y)dy - xdx = 0$ we have

$$-ydy - xdx = 0\,,$$

which is the first differential of $x^2 + y^2 = a^2$.

Main Point: It can sometimes be hard to see the forest from the trees which is why it can be good to recap what we are trying to show. The point of this section is that when *given* a differential equation we would like to solve (in this case Equation 23) we may work hard and find a general solution (in this case Equation 24). Even though we have a general solution we *cannot* then conclude that we have all solutions. Certain solutions *not* obtainable from the general solution (in this case Equation 26) may still exist.

Exercises

Exercise 1 (from primitives to differential equations)

Note we will work the first few of these problems in some detail and then automate this procedure for additional problems using Mathematica. Once the general idea has been understood there seems no reason to perform the tedious calculations by hand.

Part (i): Consider the primitive f defined as

$$f(x, y; A, B) \equiv y - Ax^m - Bx^n = 0\,. \tag{27}$$

Then to eliminate the two constants A and B we will need to take two derivatives of this expression and then use all three expressions (the primitive itself and its two derivatives) to eliminate A and B. The first and second derivative are given by

$$y' - Amx^{m-1} - Bnx^{n-1} = 0 \tag{28}$$

$$y'' - Am(m-1)x^{m-2} - Bn(n-1)x^{n-2} = 0\,. \tag{29}$$

Now using Equations 27, 28, and 29, we mean to eliminate A and B. To do that we can take Equations 27 and 28, solve them for A and B and then put the resulting expressions into Equation 29 and derive our differential equation. Putting Equations 27 and 28, in matrix form, which can make solving for A and B easier we have

$$\begin{bmatrix} -x^m & -x^n \\ -mx^{m-1} & -nx^{n-1} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} -y \\ -y' \end{bmatrix},$$

To solve this system for A and B using Crammer's rule we need to compute the determinant of the coefficient matrix

$$D = \begin{vmatrix} -x^m & -x^n \\ -mx^{m-1} & -nx^{n-1} \end{vmatrix} = nx^{m+n-1} - mx^{m+n-1} = (n-m)x^{m+n-1}.$$

Then A and B are given by

$$\begin{aligned} A &= \frac{1}{D}\begin{vmatrix} -y & -x^n \\ -y' & -nx^{n-1} \end{vmatrix} = \frac{nx^{n-1}y - x^n y'}{(n-m)x^{m+n-1}} \\ B &= \frac{1}{D}\begin{vmatrix} -x^m & -y \\ -mx^{m-1} & -y' \end{vmatrix} = \frac{x^m y' - mx^{m-1}y}{(n-m)x^{m+n-1}}. \end{aligned}$$

When we put these two expressions for A and B into Equation 29 we get

$$y'' - m(m-1)x^{m-2}\left[\frac{nx^{n-1}y - x^n y'}{(n-m)x^{m+n-1}}\right] - n(n-1)x^{n-2}\left[\frac{x^m y' - mx^{m-1}y}{(n-m)x^{m+n-1}}\right] = 0.$$

Next multiplying by $(n-m)x^{m+n-1}$ gives

$$(n-m)x^{m+n-1}y'' + (m(m-1) - n(n-1))x^{m+n-2}y' + (-mn(m-1) + mn(n-1))x^{m+n-3}y = 0,$$

or simplifying some

$$(n-m)x^{m+n-1}y'' - (n-m)(n+m+1)x^{m+n-2}y' + nm(n-m)x^{m+n-3}y = 0,$$

On dividing by $(n-m)x^{m+n-3}$ gives

$$x^2y'' - (n+m+1)xy' + nmy = 0. \quad (30)$$

Part (ii): Consider the primitive

$$y = Ae^{mx} + Be^{nx}. \quad (31)$$

Taking the first two x derivatives gives

$$\begin{aligned} y' &= Ame^{mx} + Bne^{nx} & (32) \\ y'' &= Am^2e^{mx} + Bn^2e^{nx}. & (33) \end{aligned}$$

We next solve for A and B in Equations 31 and 32 using Cramer's rule. Thus we need

$$D = \begin{vmatrix} e^{mx} & e^{nx} \\ me^{mx} & ne^{nx} \end{vmatrix} = ne^{(m+n)x} - me^{(n+m)x} = (n-m)e^{(m+n)x}.$$

Then

$$
\begin{aligned}
A &= \frac{1}{D}\begin{vmatrix} y & e^{nx} \\ y' & ne^{nx} \end{vmatrix} = \frac{ny - y'}{n-m} e^{-mx} \\
B &= \frac{1}{D}\begin{vmatrix} e^{mx} & y \\ me^{mx} & y' \end{vmatrix} = \frac{y' - my}{n-m} e^{-nx} \,.
\end{aligned}
$$

When we put these two expressions into Equation 33 we obtain

$$
y'' = \left(\frac{ny - y'}{n-m} e^{-mx} \right) m^2 e^{mx} + \left(\frac{y' - my}{n-m} e^{-nx} \right) n^2 e^{nx} \,.
$$

On simplifying some we get

$$
y'' + (m+n)y' + mny = 0 \,.
$$

Part (iii): Since the given primitive has two constants we will need two take two derivatives of it. We find

$$
\begin{aligned}
y &= A\cos(nx) + B\sin(nx) && (34) \\
y' &= -An\sin(nx) + Bn\cos(nx) && (35) \\
y'' &= -An^2\cos(nx) - Bn^2\sin(nx) \,. && (36)
\end{aligned}
$$

Solving for A and B using Cramer's rule using Equations 34 and 35 we need to compute

$$
D = \begin{vmatrix} \cos(nx) & \sin(nx) \\ -n\sin(nx) & n\cos(nx) \end{vmatrix} = n\cos^2(nx) + n\sin^2(nx) = n \,.
$$

Then A and B are given by

$$
\begin{aligned}
A &= \frac{1}{D}\begin{vmatrix} y & \sin(nx) \\ y' & n\cos(nx) \end{vmatrix} = \frac{n\cos(nx)y - \sin(nx)y'}{n} \\
B &= \frac{1}{D}\begin{vmatrix} \cos(nx) & y \\ -n\sin(nx) & y' \end{vmatrix} = \frac{\cos(nx)y' + n\sin(nx)y}{n} \,.
\end{aligned}
$$

We then take these expressions for A and B and put them in Equation 36 to get

$$
\begin{aligned}
y'' &= -\left(\frac{n\cos(nx)y - \sin(nx)y'}{n} \right) n^2\cos(nx) - \left(\frac{\cos(nx)y' + n\sin(nx)y}{n} \right) n^2\sin(nx) \\
&= -n^2\cos^2(nx)y + n\cos(nx)\sin(nx)y' - n\sin(nx)\cos(nx)y' - n^2\sin^2(nx)y \\
&= -n^2 y \,,
\end{aligned}
$$

Thus we get

$$
y'' + n^2 y = 0 \,.
$$

Part (iv): Since this primitive has two constants we will need it and two derivatives. We find

$$
\begin{aligned}
y &= e^{mx}(A\cos(nx) + B\sin(nx)) && (37) \\
y' &= mAe^{mx}\cos(nx) - nAe^{mx}\sin(nx) + mBe^{mx}\sin(nx) + nBe^{mx}\cos(nx) \\
&= (mA + nB)e^{mx}\cos(nx) + (-nA + mB)e^{mx}\sin(nx) && (38) \\
y'' &= (m^2A + mnB)e^{mx}\cos(nx) - (mnA + n^2B)e^{mx}\sin(nx) \\
&+ (-nmA + m^2B)e^{mx}\sin(nx) + (-n^2A + mnB)e^{mx}\cos(nx) \\
&= [(m^2 - n^2)A + 2mnB]e^{mx}\cos(nx) + [-2mnA + (m^2 - n^2)B]e^{mx}\sin(nx) \,. && (39)
\end{aligned}
$$

We will solve for A and B using Equations 37 and 38. To do this we write y' as

$$y' = (m\cos(nx) - n\sin(nx))e^{mx}A + (m\sin(nx) + n\cos(nx))e^{mx}B\,.$$

Then to use Cramer's rule for this problem we need to compute

$$\begin{aligned}
D &= \begin{vmatrix} \cos(nx)e^{mx} & \sin(nx)e^{mx} \\ (m\cos(nx) - n\sin(nx))e^{mx} & (m\sin(nx) + n\cos(nx))e^{mx} \end{vmatrix} \\
&= e^{2mx}[m\sin(nx)\cos(nx) + n\cos^2(nx) - m\sin(nx)\cos(nx) + n\sin^2(nx)] \\
&= ne^{2mx}\,.
\end{aligned}$$

Then A is given by

$$\begin{aligned}
A &= \frac{1}{D}\begin{vmatrix} y & \sin(nx)e^{mx} \\ y' & (m\sin(nx) + n\cos(nx))e^{mx} \end{vmatrix} \\
&= \frac{1}{ne^{2mx}}[(m\sin(nx) + n\cos(nx))e^{mx}y - \sin(nx)y'e^{mx}] \\
&= \frac{1}{n}[(m\sin(nx) + n\cos(nx))y - \sin(nx)y']e^{-mx}\,,
\end{aligned}$$

and B is given by

$$\begin{aligned}
B &= \frac{1}{D}\begin{vmatrix} \cos(nx)e^{mx} & y \\ (m\cos(nx) - n\sin(nx))e^{mx} & y' \end{vmatrix} \\
&= \frac{1}{ne^{2mx}}[\cos(nx)y'e^{mx} - (m\cos(nx) - n\sin(nx))ye^{mx}] \\
&= \frac{1}{n}[\cos(nx)y' - (m\cos(nx) - n\sin(nx))y]e^{-mx}\,.
\end{aligned}$$

When we put these into Equation 39 written as

$$y'' = [(m^2 - n^2)\cos(nx) - 2mn\sin(nx)]e^{mx}A + [(m^2 - n^2)\sin(nx) + 2mn\cos(nx)]e^{mx}B\,,$$

we get

$$y'' = 2my' - (n^2 + m^2)y\,.$$

Part (v): We find the first two derivatives of y given by

$$y = A\cosh\left(\frac{x}{A}\right) \tag{40}$$

$$y' = A\sinh\left(\frac{x}{A}\right)\frac{1}{A} = \sinh\left(\frac{x}{A}\right) \tag{41}$$

$$y'' = \frac{1}{A}\cosh\left(\frac{x}{A}\right)\,. \tag{42}$$

From these we see that

$$yy'' = \cosh^2\left(\frac{x}{A}\right) = 1 + \sinh^2\left(\frac{x}{A}\right) = 1 + y'^2\,,$$

is a differential equation satisfied by y.

Part (vi): We find the first two derivatives of y given by

$$\begin{aligned}
y &= x^n(A + B\log(x)) = Ax^n + Bx^n\log(x) & (43)\\
y' &= nAx^{n-1} + B[nx^{n-1}\log(x) + x^{n-1}] & (44)\\
y'' &= n(n-1)Ax^{n-2} + B[n(n-1)x^{n-2}\log(x) + nx^{n-2} + (n-1)x^{n-2}]\,. & (45)
\end{aligned}$$

Considering Equations 44 and 45 as a system we have

$$\left[\begin{array}{cc} n & n\log(x)+1 \\ n(n-1) & n(n-1)\log(x)+2n-1 \end{array}\right]\left[\begin{array}{c} A \\ B \end{array}\right] = \left[\begin{array}{c} \frac{y'}{x^{n-1}} \\ \frac{y''}{x^{n-2}} \end{array}\right].$$

When we solve fore A and B and then put these two expression into Equation 43 we get

$$x^2y'' + (1-2n)xy' + n^2y = 0\,.$$

Part (vii): We find the first two derivatives of y are given by

$$\begin{aligned}
y &= Ae^{mx} + Bxe^{mx} & (46)\\
y' &= Ame^{mx} + Be^{mx} + Bmxe^{mx} = Ame^{mx} + B(1+mx)e^{mx} & (47)\\
y'' &= Am^2e^{mx} + B(2m + m^2x)e^{mx}\,. & (48)
\end{aligned}$$

When we consider Equations 47 and 48 as a system we get

$$\left[\begin{array}{cc} m & (1+mx) \\ m^2 & 2m+m^2x \end{array}\right]\left[\begin{array}{c} A \\ B \end{array}\right] = \left[\begin{array}{c} y'e^{-mx} \\ y''e^{-mx} \end{array}\right].$$

When we solve for A and B in this last system and then put the resulting expressions into Equation 46 we get

$$y'' - 2my' + m^2y = 0\,.$$

Exercise 2 (a differential equation for $y = \frac{ax+b}{cx+d}$)

Part (a): First we write the given expression for $y(x)$ as

$$(cx+d)y = ax+b\,,$$

and take several derivatives with respect to x of this expression. The first derivative gives

$$cy + (cx+d)y' = a\,. \quad (49)$$

The next derivative gives

$$cy' + cy' + (cx+d)y'' = 0\,,$$

or combining terms

$$2cy' + (cx+d)y'' = 0\,. \quad (50)$$

The x derivative of this gives

$$2cy'' + cy'' + (cx+d)y''' = 0\,,$$

or combining terms

$$3cy'' + (cx + d)y''' = 0\,. \tag{51}$$

We can solve this last equation for y''' and we get

$$y''' = -\frac{3cy''}{cx + d}\,.$$

If we solve for the expression $cx + d$ in Equation 50 we find $cx + d = -\frac{2cy'}{y''}$. Putting that in the expression for y''' above gives

$$y''' = \frac{3}{2}\frac{y''^2}{y'}\,.$$

or

$$2y'y''' = 3y''^2\,,$$

the desired equation.

Part (b): If $a + d = 0$ then $d = -a$ and y given above becomes $y = \frac{ax+b}{cx-a}$ or

$$(cx - a)y = ax + b\,. \tag{52}$$

The first derivative of this expression gives

$$cy + (cx - a)\frac{dy}{dx} = a\,. \tag{53}$$

Taking another derivative gives

$$c\frac{dy}{dx} + c\frac{dy}{dx} + (cx - a)\frac{d^2y}{dx^2} = 0 \Rightarrow 2c\frac{dy}{dx} + (cx - a)\frac{d^2y}{dx^2} = 0\,. \tag{54}$$

If we write the three Equations 52, 53, and 54 at a linear system in terms of a, b, c we have

$$\left[\begin{array}{ccc} -x - y & -1 & xy \\ -1 - y' & 0 & y + xy' \\ -y'' & 0 & 2y' + xy'' \end{array}\right] \left[\begin{array}{c} a \\ b \\ c \end{array}\right] = 0\,.$$

In order for this system to have a nontrivial solution for a, b, and c the determinant of the leading coefficients matrix must be zero or

$$\left|\begin{array}{ccc} -x - y & -1 & xy \\ -1 - y' & 0 & y + xy' \\ -y'' & 0 & 2y' + xy'' \end{array}\right| = 0\,.$$

Evaluating this determinant we find

$$\begin{aligned} -(-1)\left|\begin{array}{cc} -1 - y' & y + xy' \\ -y'' & 2y' + xy'' \end{array}\right| &= (-1 - y')(2y' + xy'') + y''(y + xy') \\ &= -2y' - xy'' - 2y'^2 - xy'y'' + yy'' + xy'y'' \\ &= -2y'(1 + y') + (y - x)y''\,. \end{aligned}$$

When we set this equal to zero we get

$$(y - x)y'' = 2y'(1 + y')\,.$$

Note that this is different from the result in the book in that the right-hand-side has a $2y'$ factor rather than a $2y$ factor. If anyone sees anything wrong with what I have done please contact me.

Exercise 3 (from primitive to differential equation)

Consider the primitive

$$y^3 - 3ax^2 + x^3 = 0\,. \tag{55}$$

We first take the x derivative of this expression

$$3y^2\frac{dy}{dx} - 6ax + 3x^2 = 0 \quad \text{or} \quad \frac{dy}{dx} = \frac{2ax - x^2}{y^2}\,. \tag{56}$$

Taking a second derivative gives

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{2a-2x}{y^2} - \frac{2(2ax-x^2)}{y^3}\left(\frac{dy}{dx}\right) \\
&= \frac{2a-2x}{y^2} - \frac{2(2ax-x^2)}{y^3}\left(\frac{2ax-x^2}{y^2}\right) \\
&= \frac{2(a-x)(3ax^2-x^3) - 2(4a^2x^2 - 4ax^3 + x^4)}{y^5} = -\frac{2a^2x^2}{y^5}\,,
\end{aligned}$$

when we simplify. From the primitive Equation 55 we see that

$$y^3 = 3ax^2 - x^3 = (3a-x)x^2\,.$$

Thus if $x < 3a$ we have that $y > 0$ and if $x > 3a$ we have $y < 0$. From the above expression for the second derivative $\frac{d^2y}{dx^2}$ we see that if $x < 3a$ then $\frac{d^2y}{dx^2} < 0$ and if $x > 3a$ then $\frac{d^2y}{dx^2} > 0$. Thus across the point $x = 3a$ the *sign* of the second derivative changes. This is the definition of a point on inflection.

Exercise 4 (recursive calculation of derivatives)

Given the equation

$$x(1-x)\frac{d^2y}{dx^2} - (4-12x)\frac{d^1y}{dx^1} - 36\frac{d^ny}{dx^n} = 0\,,$$

we want to show

$$x(1-x)\frac{d^{n+2}y}{dx^{n+2}} - \{4-n-(12-2n)x\}\frac{d^{n+1}y}{dx^{n+1}} - (4-n)(9-n)\frac{d^ny}{dx^n} = 0\,. \tag{57}$$

If we take $n = 0$ in this last equation we get the initial equation. We consider the truth of Equation 57 when $n = 0$ as the initial condition for an induction proof. Assume that Equation 57 holds for all n up to some point. Then taking an x derivative of Equation 57 we get

$$\begin{aligned}
& (1-x)\frac{d^{n+2}y}{dx^{n+2}} - x\frac{d^{n+2}y}{dx^{n+2}} + x(1-x)\frac{d^{n+3}y}{dx^{n+3}} \\
+ \quad & \{12-2n\}\frac{d^{n+1}y}{dx^{n+1}} - \{4-n-(12-2n)x\}\frac{d^{n+2}y}{dx^{n+2}} - (4-n)(9-n)\frac{d^{n+1}y}{dx^{n+1}} = 0\,.
\end{aligned}$$

Or grouping terms

$$
\begin{aligned}
x(1-x)\frac{d^{n+3}y}{dx^{n+3}} &+ [1-x-x-(4-n)+(12-2n)x]\,\frac{d^{n+2}y}{dx^{n+2}} \\
&- [(4-n)(9-n)-(12-2n)]\,\frac{d^{n+1}y}{dx^{n+1}} = 0\,.
\end{aligned}
$$

If we consider the coefficient of $\frac{d^{n+2}y}{dx^{n+2}}$ we find

$$
\begin{aligned}
(-3+n)+(10-2n)x &= -(4-(n+1))+12-2(n+1) \\
&= -\{4-(n+1)-(12-2(n+1))x\}\,.
\end{aligned}
$$

Now consider the coefficient of $\frac{d^{n+1}y}{dx^{n+1}}$ we find

$$
36-13n+n^2-12+2n = 24-11n+n^2\,.
$$

While if we consider the expression $(4-(n+1))(9-(n+1))$ we see that it expands to

$$
\begin{aligned}
(4-(n+1))(9-(n+1)) &= (3-n)(8-n) \\
&= 24-11n+n^2\,.
\end{aligned}
$$

the same expression. Thus we have

$$
\begin{aligned}
x(1-x)\frac{d^{n+3}y}{dx^{n+3}} &- \{4-(n+1)-(12-2(n+1))x\}\,\frac{d^{n+2}y}{dx^{n+2}} \\
&- ((4-(n+1))(9-(n+1)))\frac{d^{n+1}y}{dx^{n+1}} = 0\,.
\end{aligned}
$$

Which is Equation 57 with $n \to n+1$ completing the inductive proof.

Exercise 7 (some derivative identities)

For $z = 3xy - y^2 + (y^2-2x)^{3/2}$ we find the x and y partial derivatives of z given by

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= 3y + \frac{3}{2}(y^2-2x)^{1/2}(-2) = 3y - 3(y^2-2x)^{1/2} \\
\frac{\partial z}{\partial y} &= 3x - 2y + \frac{3}{2}(y^2-2x)^{1/2}(2y) = 3x - 2y + 3y(y^2-2x)^{1/2} \\
\frac{\partial^2 z}{\partial y \partial x} &= 3 - \frac{3}{2}(y^2-2x)^{-1/2}(2y) = 3 - 3y(y^2-2x)^{-1/2} \\
\frac{\partial^2 z}{\partial x \partial y} &= 3 + 3y(y^2-2x)^{-1/2}\left(\frac{1}{2}(-2)\right) = 3 - 3y(y^2-2x)^{-1/2} \\
\frac{\partial^2 z}{\partial x^2} &= -\frac{3}{2}(y^2-2x)^{-1/2}(-2) = 3(y^2-2x)^{-1/2} \\
\frac{\partial^2 z}{\partial y^2} &= -2 + 3(y^2-2x)^{1/2} + \frac{3}{2}y(y^2-2x)^{-1/2}(2y) \\
&= -2 + 3(y^2-2x)^{1/2} + 3y^2(y^2-2x)^{-1/2}\,.
\end{aligned}
$$

Note that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ as we were to show.

Chapter 2 (Elementary Methods of Integration)

Notes on the text

Exact Equations of First Order and Degree

Consider the equation

$$P(x,y)dx + Q(x,y)dy = 0\,. \tag{58}$$

We want to show that the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is sufficient for integrability. Define u as

$$u \equiv \int_{x_0}^{x} P(x,y)dx + \phi(y)\,. \tag{59}$$

Then

$$\frac{\partial u}{\partial x} = P \quad \text{and} \quad \frac{\partial u}{\partial y} = Q\,.$$

$$\frac{\partial u}{\partial x} = P \quad \text{and} \quad \frac{\partial u}{\partial y} = \int_{x_0}^{x} \frac{\partial P(x,y)}{\partial y}dx + \phi'(y) = Q(x,y)\,.$$

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then this becomes

$$\begin{aligned} \frac{\partial u}{\partial y} &= \int_{x_0}^{x} \frac{\partial Q(x,y)}{\partial x}dx + \phi'(y) \\ &= Q(x,y) - Q(x_0,y) + \phi'(y)\,. \end{aligned}$$

Thus to have this equal $Q(x,y)$ exactly we must take $\phi'(y) = Q(x_0,y)$ so

$$\phi(y) = \int_{y_0}^{y} Q(x_0,y)dy\,.$$

Thus with the form of u given by Equation 59 and the above computed value of $\phi(y)$ we have

$$u = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy = c\,. \tag{60}$$

For the general solution to the exact equation $Pdx + Qdy = 0$ we can pick x_0 and y_0 to make the specified initial conditions on $y(x)$ correct.

As an example, consider the equation

$$\frac{2x-y}{x^2+y^2}dx + \frac{2y+x}{x^2+y^2}dy = 0\,.$$

We first check if this problem is exact. That is if

$$\frac{\partial}{\partial y}\left(\frac{2x-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{2y+x}{x^2+y^2}\right)\,,$$

is true. Now evaluate the left-hand-side of this expression where we get

$$-\frac{1}{x^2+y^2} - \frac{(2x-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2-4xy+2y^2}{(x^2+y^2)^2} = \frac{-x^2-4xy+y^2}{(x^2+y^2)^2}\,.$$

Now evaluate the right-hand-side of this expression where we get

$$\frac{1}{x^2+y^2} - \frac{2x(2y+x)}{(x^2+y^2)^2} = \frac{x^2+y^2-4xy-2x^2}{(x^2+y^2)^2} = \frac{-x^2-4xy+y^2}{(x^2+y^2)^2}\,.$$

Since these two are equal the given expression is an exact differential. The solution is then

$$u = \int_{x_0}^{x} \frac{2x-y}{x^2+y^2}dx + \int_{y_0}^{y} \frac{2y+x_0}{x_0^2+y^2}dy = c \tag{61}$$

As discussed in the text there are not three arbitrary constant x_0, y_0 and c but in fact only one. Thus we can still find solutions by specifying some of these. If we take $x_0 = 0$ then the second integral in Equation 61 becomes

$$\int_{y_0}^{y} \frac{2y}{y^2}dy = 2\log(y)|_{y_0}^{y} = 2\log\left(\frac{y}{y_0}\right)\,.$$

Lets take $y_0 = 1$ and this becomes $2\log(y)$. The first integral in Equation 61 becomes

$$\begin{aligned}
\int_0^x \frac{2x-y}{x^2+y^2}dx &= \int_0^x \frac{2x}{x^2+y^2}dx - y\int_0^x \frac{dx}{x^2+y^2} \\
&= \log(x^2+y^2)|_0^x - y\int_0^x \frac{dx}{x^2+y^2}\,.
\end{aligned}$$

The integral of the expression $\int_0^x \frac{dx}{x^2+y^2}$ is given by $\frac{1}{y}\arctan\left(\frac{x}{y}\right)$ and thus we are left with

$$\begin{aligned}
u(x,y) &= \log(x^2+y^2) - \log(y^2) - y\left(\frac{1}{y}\arctan\left(\frac{x}{y}\right)\right) + 2\log(y) \\
&= \log(x^2+y^2) - \arctan\left(\frac{x}{y}\right) = c\,.
\end{aligned}$$

Notes on Separation of Variables

As an example of an exact equation where P is a function of only x and Q is a function of only y consider the differential equation

$$(x^2+1)(y^2-1)dx + xydy = 0\,. \tag{62}$$

Dividing by $x(y^2-1)$ we get

$$\frac{x^2+1}{x}dx + \frac{y}{y^2-1}dy = 0\,.$$

If we integrate this expression we find

$$\log(x) + \frac{x^2}{2} + \frac{1}{2}\log(y^2 - 1) = c\,.$$

On multiplying by 2 and combining logarithms gives

$$\log(c(x^2(y^2 - 1))) = -x^2\,,$$

or

$$cx^2(y^2 - 1) = e^{-x^2}\,.$$

Solving for y^2 we get

$$y^2 = 1 + \frac{c}{x}e^{-x^2}\,.$$

From Equation 62 we see when we divided by the expression $x(y^2 - 1)$, we lost the constant solutions $x = 0$ and $y = \pm 1$. When ever one divides by a function $F(x, y)$ say, one must go back and verify that it is never the case that $F(x, y) = 0$.

Notes on Homogeneous Equations

In case the functions $P(x, y)$ and $Q(x, y)$ found in the first order and first degree equation

$$P(x, y)dx + Q(x, y)dy = 0\,,$$

are *homogeneous* meaning that if we let $y = vx$ then $P(x, y)$ and $Q(x, y)$ transform as

$$\begin{aligned} P(x, y) &= P(x, vx) = x^n P(1, v) \\ Q(x, y) &= Q(x, vx) = x^n Q(1, v)\,. \end{aligned}$$

and we can obtain a reducible equation (one that can be solved by quadrature or integration) in terms of the variable v. Considering v as a function of x we have $dy = xdv + vdx$ and the original equation $P(x, y)dx + Q(x, y)dy = 0$ in terms of x and v becomes

$$x^n P(1, v)dx + x^n Q(1, v)(xdv + vdx) = 0\,.$$

Assuming that $x \neq 0$ and grouping by the differentials dx and dv we have

$$(P(1, v) + vQ(1, v))dx + xQ(1, v)dv = 0\,.$$

Assuming $P(1, v) + vQ(1, v) \neq 0$ so that we can divide by it we get

$$\frac{dx}{x} + \left(\frac{Q(1, v)}{P(1, v) + vQ(1, v)}\right) dv = 0\,. \tag{63}$$

This can be written as

$$\frac{dx}{x} + \frac{dv}{v + \frac{P(1,v)}{Q(1,v)}} = 0\,.$$

When we define the function $\phi(v)$ as $\phi(v) \equiv v + \frac{P(1,v)}{Q(1,v)}$ this last expression is

$$\frac{dx}{x} + \frac{dv}{\phi(v)} = 0\,.$$

This is a separable equation and has a solution given by

$$\log(x) + \int \frac{dv}{\phi(v)} = c$$

or

$$\log\left(\frac{c}{x}\right) = \int \frac{dv}{\phi(v)}\,. \tag{64}$$

As an example of a homogeneous equation consider

$$(y^4 - 2x^3 y)dx + (x^4 - 2xy^3)dy = 0\,.$$

From the standard form for first order first degree equations given by Equation 58 and the definitions of $P(x,y)$ and $Q(x,y)$ we see that both functions $P(x,y)$ and $Q(x,y)$ are homogeneous of degree 4. Let $y = vx$ then

$$P(1,v) = v^4 - 2v \quad \text{with} \quad Q(1,v) = 1 - 2v^3\,,$$

and ϕ introduced above becomes

$$\phi(v) = v + \frac{v^4 - 2v}{1 - 2v^3} = \frac{v - 2v^4 + v^4 - 2v}{1 - 2v^3} = \frac{-v - v^4}{1 - 2v^3}\,.$$

Then from Equation 64 we have

$$\log\left(\frac{x}{c}\right) = \int \frac{1 - 2v^3}{v + v^4} dv\,.$$

Note that the fraction in the integral can be written as

$$\begin{aligned} \frac{1 - 2v^3}{v + v^4} &= \frac{1}{v}\frac{1 - 2v^3}{1 + v^3} = \frac{1}{v}\frac{1 + v^3 - v^3 - 2v^3}{1 + v^3} = \frac{1}{v}\frac{1 + v^3 - 3v^3}{1 + v^3} \\ &= \frac{1}{v}\left(1 - \frac{3v^3}{1 + v^3}\right) = \frac{1}{v} - \frac{3v^2}{1 + v^3}\,. \end{aligned}$$

Thus by integrating these two expressions we have

$$\log\left(\frac{x}{c}\right) = \log(v) - \log(1 + v^3)\,,$$

or

$$\log\left(\frac{x(1 + v^3)}{c}\right) = \log(v)\,,$$

or

$$\frac{x(1 + v^3)}{c} = v\,.$$

Using $v = \frac{y}{x}$ this gives

$$x\left(1 + \frac{y^3}{x^3}\right) = \frac{cy}{x}.$$

Multiplying by x^2 we get

$$x^3 + y^3 = cxy,$$

for the solution.

A particular special case and one that is very easy to solve happens when the equation we are considering $Pdx + Qdy = 0$ is both homogeneous and exact. In that case this equation is immediately integrable without quadratures (assuming that P and Q are not homogeneous of degree -1) and the solution is given by

$$Px + Qy = c. \tag{65}$$

To show this consider the proposed solution u given by $u = Px + Qy$ then the x derivative is given by

$$\frac{\partial u}{\partial x} = P + \frac{\partial P}{\partial x}x + y\frac{\partial Q}{\partial x}.$$

But our original Equation 58 is exact which means that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ so that replacing the second term in the above we get

$$\frac{\partial u}{\partial x} = P + \frac{\partial P}{\partial x}x + y\frac{\partial P}{\partial y}.$$

Now P is homogeneous of degree $n \neq -1$ thus by Euler's theorem 20 we have $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = nP$ and the x derivative of u becomes

$$\frac{\partial u}{\partial x} = (n+1)P.$$

In the same way we have for the y derivative of u

$$\frac{\partial u}{\partial y} = \frac{\partial P}{\partial y}x + \frac{\partial Q}{\partial y}y + Q = x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + Q = (n+1)Q.$$

Thus the total differential of u is given by

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (n+1)Pdx + (n+1)Qdy = (n+1)(Pdx + Qdy).$$

Thus recalling what u is and solving for $Pdx + Qdy$ this last expression is

$$Pdx + Qdy = \frac{1}{n+1}d(Px + Qy).$$

This assumes $n \neq -1$. If we take $Px + Qy = c$ a constant, the right-hand-side of this expression is zero and we have a solution.

As an example of an exact homogeneous equation consider

$$x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0.$$

We first check that this equation is exact. To do that consider

$$\frac{\partial}{\partial y}(x^3 + 3xy^2) = 6xy$$
$$\frac{\partial}{\partial x}(y^3 + 3yx^2) = 6xy\,.$$

since these two derivatives are equal this equation is exact. Next we check that P and Q are homogeneous. This is true since they are polynomials of degree 3. The solution to this problem is then $xP + Qy = c$ or

$$x^4 + 3x^2y^2 + y^4 + 3y^2x^2 = c\,,$$

or by grouping terms we get

$$x^4 + y^4 + 6x^2y^2 = 0\,.$$

We now consider the special case excluded in the above derivation. That is the case where P and Q are homogeneous of degree -1. We widen our discussion at this point to not necessarily require $Pdx + Qdy = 0$ to be exact as we did in the previous section. To solve our differential equation in this case consider the original differential Equation 58 divided by $Px + Qy$ or

$$\frac{Pdx + Qdy}{Px + Qy} = \frac{P}{Px + Qy}dx + \frac{Q}{Px + Qy}dy = 0\,. \tag{66}$$

We now show that this new equation is exact and can therefore be integrated by the methods on Page 19. For this equation to be exact we must show that

$$\frac{\partial}{\partial y}\left(\frac{P}{Px + Qy}\right) = \frac{\partial}{\partial x}\left(\frac{Q}{Px + Qy}\right)\,.$$

The left-hand-side of this expression is given by

$$\begin{aligned}
\frac{\partial}{\partial y}\left(\frac{P}{Px + Qy}\right) &= -\frac{P}{(Px + Qy)^2}\left[\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial y}y + Q\right] + \frac{1}{Px + Qy}\frac{\partial P}{\partial y} \\
&= \frac{1}{(Px + Qy)^2}\left[-P\frac{\partial P}{\partial y}x - P\frac{\partial Q}{\partial y}y - PQ + P\frac{\partial P}{\partial y} + yQ\frac{\partial P}{\partial y}\right] \\
&= \frac{1}{(Px + Qy)^2}\left[-Py\frac{\partial P}{\partial y} + Qy\frac{\partial P}{\partial y} - PQ\right]\,.
\end{aligned}$$

While the right-hand-side of this expression is given by

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{Q}{Px + Qy}\right) &= -\frac{Q}{(Px + Qy)^2}\left[\frac{\partial P}{\partial x}x + P + \frac{\partial Q}{\partial x}y\right] + \frac{1}{Px + Qy}\frac{\partial Q}{\partial x} \\
&= \frac{1}{(Px + Qy)^2}\left[-Q\frac{\partial P}{\partial x}x - QP - Q\frac{\partial Q}{\partial x}y + Px\frac{\partial Q}{\partial x} + yQ\frac{\partial Q}{\partial x}\right] \\
&= \frac{1}{(Px + Qy)^2}\left[-Qx\frac{\partial P}{\partial x} + xP\frac{\partial Q}{\partial x} - PQ\right]\,.
\end{aligned}$$

Setting these two expressions equal would require

$$-Qx\frac{\partial P}{\partial x} + xP\frac{\partial Q}{\partial x} = -Py\frac{\partial Q}{\partial y}Qy\frac{\partial P}{\partial y}\,,$$

or moving things around

$$Q\left[y\frac{\partial P}{\partial y}+x\frac{\partial P}{\partial x}\right]=P\left[x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}\right] .$$

Since P and Q are homogeneous of degree -1 using Euler's theorem 20 we get $Q(-1)P = P(-1)Q$ which is an exact equality verifying that our equation is indeed exact which we know how to solve using methods discussed on Page 19.

We should note that this transformation (dividing our equation by the expression $xP + yQ$) results in a new differential equation that is also homogeneous of degree -1. To show this recall that if $P(x, y)$ and $Q(x, y)$ are homogeneous of the same degree n, this means that when we take $y = vx$ we get

$$P(x,y)=x^nP(1,v) \quad \text{and} \quad Q(x,y)=x^nQ(1,v) .$$

Note when we let $y = vx$ in the coefficients of dx and dy in our new Equation 66 we get

$$\begin{aligned}
\frac{P}{Px+Qy} &= \frac{x^nP(1,v)}{x^{n+1}P(1,v)+x^{n+1}vQ(1,v)} = \frac{1}{x}\frac{P(1,v)}{P(1,v)+vQ(1,v)} \\
\frac{Q}{Px+Qy} &= \frac{x^nQ(1,v)}{x^{n+1}P(1,v)+x^{n+1}vQ(1,v)} = \frac{1}{x}\frac{Q(1,v)}{P(1,v)+vQ(1,v)} ,
\end{aligned}$$

showing both are homogeneous of degree -1. Thus this transform did not change the homogeneity of the equation but only made it exact.

As another equation that we can solve consider $\frac{dy}{dx}$ equal to a function of a rational expression of x and y or

$$\frac{dy}{dx}=F\left(\frac{Ax+By+C}{ax+by+c}\right) . \tag{67}$$

Let $x = h + \xi$ and $y = k + \eta$ with ξ and η the new independent and dependent variables and h and k constants chosen to satisfy

$$\begin{aligned}
Ah+Bk+C &= 0 \\
ah+bk+c &= 0 .
\end{aligned} \tag{68}$$

That this last equation is solvable requires that $Ab - Ba \neq 0$ (the above equations must be linearly independent). When we make this substitution Equation 67 becomes

$$\frac{d\xi}{d\eta}=F\left(\frac{Ah+A\xi+Bk+B\eta+C}{ah+a\xi+bk+b\eta+c}\right)=F\left(\frac{A\xi+B\eta}{a\xi+b\eta}\right) .$$

Note that the right-hand-side of this equation is homogeneous of degree zero. That is, if we let $\eta = v\xi$ we get

$$F\left(\frac{A\xi+Bv\xi}{a\xi+bv\xi}\right)=F\left(\frac{A+Bv}{a+bv}\right) ,$$

a constant. We know how to integrate homogeneous equation, see Page 21. In order to solve Equation 67 in the case where $Ab - aB = 0$ not that in that case

$$\frac{B}{A}=\frac{b}{a} ,$$

and we introduce a new *dependent* variable η as

$$\eta = x + \frac{B}{A}y = x + \frac{b}{a}y\,.$$

This transformation gives for y in terms of η and x

$$y = \frac{A}{B}(\eta - x) = \frac{a}{b}(\eta - x)\,.$$

With this transformation we have $d\eta = dx + \frac{B}{A}\frac{dy}{dx}dx$ so $\frac{d\eta}{dx}$ becomes

$$\begin{aligned}\frac{d\eta}{dx} &= 1 + \frac{b}{a}\frac{dy}{dx} = 1 + \frac{b}{a}F\left(\frac{Ax + A(\eta - x) + C}{ax + b(\eta - x) + c}\right)\\ &= 1 + \frac{b}{a}\frac{dy}{dx} = 1 + \frac{b}{a}F\left(\frac{A\eta + C}{b\eta + c}\right)\,,\end{aligned}$$

which is a separable equation. That is one can write it as

$$\frac{d\eta}{1 + \frac{b}{a}F\left(\frac{A\eta + C}{b\eta + c}\right)} = dx\,,$$

and integrate both sides to get the solution.

As an example of this technique consider the differential equation

$$(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0\,, \tag{69}$$

or

$$\frac{dy}{dx} = \frac{-7x + 3y + 7}{-3x + 7y + 3}\,.$$

When we compare to Equation 67 we have $A = -7$, $B = 3$, $C = 7$, $a = -3$, $b = 7$, and $c = 3$. Then

$$Ab - aB = (-7)(7) - (-3)(3) \neq 0\,.$$

We need to solve for h and k in Equation 68 or

$$\begin{aligned}-7h + 3k + 7 &= 0\\ -3h + 7k + 3 &= 0\,,\end{aligned}$$

where we find $h = 1$ and $k = 0$. We then let $x = \xi + 1$ and $y = \eta$ in Equation 69 to get

$$(3\eta - 7\xi)d\xi + (7\eta - 3\xi)d\eta = 0\,,$$

which is a homogeneous equation of degree 1. Let $\eta = v\xi$ where $d\eta = \xi dv + vd\xi$ to get

$$(3v\xi - 7\xi)d\xi + (7v\xi - 3\xi)(\xi dv + vd\xi) = 0\,,$$

or when we simplify

$$7(v^2 - 1)d\xi + (7v - 3)dv = 0\,,$$

which is a separable equation. To integrate this we write it as

$$\frac{7}{\xi}d\xi + \left(\frac{7v-3}{v^2-1}\right) dv = 0\,,$$

and use partial fractions on the expression $\frac{7v-3}{v^2-1}$ we find

$$\frac{7v-3}{v^2-1} = \frac{A}{v-1} + \frac{B}{v+1}\,,$$

where

$$A = \left.\frac{7v-3}{v+1}\right|_{v=1} = 2 \quad \text{and} \quad B = \left.\frac{7v-3}{v-1}\right|_{v=-1} = 5\,.$$

In this case we have

$$\frac{7}{\xi} + \left[\frac{2}{v-1} + \frac{5}{v+1}\right] dv = 0\,.$$

Integrating gives

$$7\log(\xi) + 2\log(v-1) + 5\log(v+1) = c\,,$$

or combining terms we get

$$\xi^7(v-1)^2(v+1)^5 = c\,.$$

When we replace v with $v = \frac{\eta}{\xi} = \frac{y}{x-1}$ and ξ with $x-1$ we get

$$(x-1)^7\left[\frac{y}{x-1} - 1\right]^2\left[\frac{y}{x-1} + 1\right]^5 = c\,.$$

Finally combining everything gives $(y-x+1)^2(y+x-1)^5 = c$ as our primitive.

Linear Equations of First Order

The most common differential equation is the linear equation of first order given by

$$\frac{dy}{dx} + \phi y = \psi\,, \tag{70}$$

where ϕ and ψ are arbitrary functions of x. The homogeneous problem given by taking the right-hand-side of the above equation equal to zero and we get

$$\frac{dy}{y} = -\phi dx\,,$$

which can be integrated to give

$$\ln(|y|) = -\int \phi dx\,.$$

This later expression can be solved for y

$$y = ce^{-\int \phi dx}\,. \tag{71}$$

Motivated by the homogeneous solution we consider $y = ve^{-\int \phi dx}$ where v is a function of x yet to be determined. This procedure is generally called *variation of parameters.* The derivative of this expression we have

$$y' = v'e^{-\int \phi dx} - v\phi e^{-\int \phi dx}\,.$$

Replacing $\frac{dy}{dx}$ in the non-homogeneous Equation 70 we get

$$v'e^{-\int \phi dx} - v\phi e^{-\int \phi dx} + \phi v e^{-\int \phi dx} = \psi\,,$$

or $v'e^{-\int \phi dx} = \psi$ or

$$\frac{dv}{dx} = \psi e^{\int \phi dx}\,. \tag{72}$$

Integrating this to compute the function v we get

$$v = C + \int \psi e^{\int \phi dx} dx\,.$$

When we use this in $y = ve^{-\int \phi dx}$ we find the general solution for y of

$$y = Ce^{-\int \phi dx} + e^{-\int \phi dx} \int \psi e^{\int \phi dx} dx\,. \tag{73}$$

A general function, y, that is linearly dependent on a constant C as $y = Cf(x) + g(x)$ has the derivative $y' = Cf'(x) + g'(x)$. If we next eliminate the constant C in the derivative expression by replacing C with $C = \frac{1}{f(x)}(y - g(x))$ we get

$$y' = \frac{1}{f(x)}(y - g(x))f'(x) + g'(x)\,,$$

or

$$f(x)y' - f'(x)y = -g(x)f'(x) + f'(x)g(x)\,.$$

which is a linear differential equation for y.

If we assume we know *one* particular solution say y_1 to Equation 70 that is y_1 satisfies $\frac{dy_1}{dx} + \phi y_1 = \psi$, then subtracting the differential equation for the general solution y of $\frac{dy}{dx} + \phi y = \psi$ we get a homogeneous equation for the difference $y - y_1$ of

$$\frac{d}{dx}(y - y_1) + \phi(y - y_1) = 0\,,$$

so

$$y = Ce^{-\int \phi dx} + y_1\,.$$

which gives the general solution y in terms of the one solution y_1 we know.

Since the general solution to Equation 70 is of the form $y = Cf(x) + g(x)$ for two functions f and g if we know *two* particular solutions y_1 and y_2 then we can write them in terms of f and g as

$$\begin{aligned} y_1 &= C_1 f(x) + g(x) \\ y_2 &= C_2 f(x) + g(x)\,, \end{aligned}$$

from which we see that $y - y_1 = (C - C_1)f(x)$ and $y_2 - y_1 = (C_2 - C_1)f(x)$ thus

$$\frac{y - y_1}{y_2 - y_1} = \frac{C - C_1}{C_2 - C_1} \, .$$

Since the right-hand-side of this is another constant say D, our general solution in the case we know two particular solutions y_1 and y_2 is given by solving the above for y

$$y = D(y_2 - y_1) + y_1 \, .$$

Linear Equations of First Order: Examples

For the differential equation $y' - ay = e^{mx}$ the homogeneous solution is $y = ce^{ax}$ and to find the non-homogeneous solution we take $y = ve^{ax}$ to find from Equation 72 that v must satisfy

$$\frac{dv}{dx} = e^{mx}e^{-ax} = e^{(m-a)x} \, .$$

Thus when $m \neq a$ we have

$$v(x) = C + \frac{e^{(m-a)x}}{m - a} \, ,$$

so the general solution $y(x)$ is given by

$$y(x) = Ce^{ax} + \frac{1}{m - a}e^{mx} \, .$$

If $m = a$ then the homogeneous solution is the same but the equation for $v(x)$ becomes $\frac{dv}{dx} = 1$, so $v(x) = x + C$ which gives the total solution of $y(x) = Ce^{ax} + xe^{ax}$.

For the differential equation

$$y' - \frac{2x}{x^2 + 1}y = 2x(x^2 + 1) \, .$$

Then the homogeneous equation has the solution

$$y(x) = ce^{-\int -\frac{2x}{x^2+1}dx} = ce^{\ln(x^2+1)} = c(x^2 + 1) \, .$$

The total solution is given by $y(x) = v(x)(x^2 + 1)$ where v satisfies Equation 72 or

$$\frac{dv}{dx} = \frac{2x(x^2 + 1)}{x^2 + 1} = 2x \, ,$$

or $v = x^2 + C$. Thus the total solution is

$$\begin{aligned} y(x) &= C(x^2 + 1) + x^2(x^2 + 1) = C(x^2 + 1) + (x^2 + 1 - 1)(x^2 + 1) \\ &= (C - 1)(x^2 + 1) + (x^2 + 1)^2 = C'(x^2 + 1) + (x^2 + 1)^2 \, . \end{aligned}$$

For the differential equation

$$y' \cos(x) + y \sin(x) = 1 \, ,$$

to use the results from this section we first have to write it in the same form as Equation 70 meaning it has a leading coefficient of y' of 1. That is as

$$y' + \frac{\sin(x)}{\cos(x)} y = \frac{1}{\cos(x)} \, .$$

The homogeneous solution then is given by

$$y(x) = c e^{-\int \tan(x) dx} = c e^{\ln(|\cos(x)|)} = c |\cos(x)| \, .$$

Note that $y = \cos(x)$ satisfies the homogeneous equation and thus we can drop the absolute values. Then let $y = v(x)\cos(x)$ be the total solution and we see that $v(x)$ satisfies Equation 72 or

$$\frac{dv}{dx} = \frac{\frac{1}{\cos(x)}}{\cos(x)} = \sec^2(x) \, .$$

Thus integrating this we find $v(x) = \tan(x) + C$ and the total solution for y is given by

$$y(x) = v(x)\cos(x) = C\cos(x) + \sin(x) \, .$$

The Equations of Bernoulli and Jacobi

The differential equation

$$\frac{dy}{dx} + \phi y = \psi y^n \, , \tag{74}$$

where ϕ and ψ are arbitrary functions of x is known as a **Bernoulli** equation. It is nonlinear due to the y^n term but can be transformed into a linear equation with the transformation

$$z = y^{1-n} \, .$$

In this case we find

$$\frac{dz}{dx} = (1-n) y^{-n} \frac{dy}{dx} \, ,$$

which when we replace $\frac{dy}{dx}$ with the expression from Equation 74 we get

$$\frac{dz}{dx} = (1-n) y^{-n} (\psi y^n - \phi y) = (1-n)\psi - (1-n)\phi y^{1-n} = (1-n)\psi - (1-n)\phi z \, ,$$

or

$$\frac{dz}{dx} + (1-n)\phi z = (1-n)\psi \, , \tag{75}$$

which is a linear equation and is solvable via the methods discussed earlier.

The **Jacobi** equation is defined as

$$\mathcal{J} \equiv (a_1 + b_1 x + c_1 y)(x dy - y dx) - (a_2 + b_2 x + c_2 y) dy + (a_3 + b_3 x + c_3 y) dx = 0 \, . \tag{76}$$

Here a, b, and c are constants. We next transform this equation into a Bernoulli equation. The algebra in this section is a bit long and can be quickly skimmed if the reader is uninterested. The point of such a detailed treatment is to be able to "read" the resulting text and quickly verify the truthfulness of the final statements and also to shed light on any transformations that need to be performed and may not be clear from the text. In the Jacobi Equation 76 we begin by letting $x = X + \alpha$ and $y = Y + \beta$ for constant values of α and β and then grouping the resulting expression by dX, dY and $XdY - YdX$. To begin when $x = X + \alpha$ and $y = Y + \beta$ we get

$$\begin{aligned} \mathcal{J} &= (a_1 + b_1 X + \alpha b_1 + c_1 Y + \beta c_1)((X+\alpha)dY - (Y+\beta)dX) \\ &- (a_2 + b_2 X + \alpha b_2 + c_2 Y + \beta c_2)dY + (a_3 + b_3 X + \alpha b_3 + c_3 Y + \beta c_3)dX \,. \end{aligned}$$

Extracting the αdY and the βdX from the first term gives

$$\begin{aligned} \mathcal{J} &= (a_1 + b_1 X + \alpha b_1 + c_1 Y + \beta c_1)(XdY - YdX) \\ &+ (a_1 + b_1 X + \alpha b_1 + c_1 Y + \beta c_1)\alpha dY - (a_1 + b_1 X + \alpha b_1 + c_1 Y + \beta c_1)\beta dX \\ &- (a_2 + b_2 X + \alpha b_2 + c_2 Y + \beta c_2)dY + (a_3 + b_3 X + \alpha b_3 + c_3 Y + \beta c_3)dX \,. \end{aligned}$$

Grouping by dX, dY and $XdY - YdX$ and changing the order of some terms gives

$$\begin{aligned} \mathcal{J} &= (b_1 X + c_1 Y + a_1 + \alpha b_1 + \beta c_1)(XdY - YdX) \\ &+ (\alpha a_1 + \alpha^2 b_1 + \alpha\beta c_1 - a_2 - \alpha b_2 - \beta c_2 + (\alpha b_1 - b_2)X + (\alpha c_1 - c_2)Y)dY \\ &+ (-\beta a_1 - \beta\alpha b_1 - c_1\beta^2 + a_3 + \alpha b_3 + \beta c_3 - (\beta b_1 - b_3)X - (\beta c_1 - c_3)Y)dX = 0 \,. \end{aligned}$$

Factoring these terms in a slightly different manner we have

$$\begin{aligned} \mathcal{J} &= (b_1 X + c_1 Y + a_1 + \alpha b_1 + \beta c_1)(XdY - YdX) \\ &+ (\alpha(a_1 + \alpha b_1 + \beta c_1) - (a_2 + \alpha b_2 + \beta c_2) + (\alpha b_1 - b_2)X + (\alpha c_1 - c_2)Y)dY \\ &+ (-\beta(a_1 + \alpha b_1 + \beta c_1) + (a_3 + \alpha b_3 + \beta c_3) - (\beta b_1 - b_3)X - (\beta c_1 - c_3)Y)dX = 0 \,. \end{aligned}$$

Based on these expressions define some new variables A_i as

$$A_i = a_i + \alpha b_i + \beta c_i \,, \tag{77}$$

for $i = 1, 2, 3$. Then in terms of the A_i variables we get

$$\begin{aligned} \mathcal{J} &= (b_1 X + c_1 Y + A_1)(XdY - YdX) \\ &+ (\alpha A_1 - A_2 + (\alpha b_1 - b_2)X + (\alpha c_1 - c_2)Y)dY \\ &+ (-\beta A_1 + A_3 - (\beta b_1 - b_3)X - (\beta c_1 - c_3)Y)dX = 0 \,, \end{aligned}$$

or moving the term $A_1(XdY - YdX)$ into the dX and dY terms gives

$$\begin{aligned} \mathcal{J} &= (b_1 X + c_1 Y)(XdY - YdX) \\ &- (A_2 + b_2 X + c_2 Y - \alpha(A_1 + b_1 X + c_1 Y) - A_1 X)dY \\ &+ (A_3 + b_3 X + c_3 Y - \beta(A_1 + b_1 X + c_1 Y) - A_1 Y)dX = 0 \,. \end{aligned}$$

Note that the coefficient of $XdY - YdX$ is homogeneous. We can make the coefficients of dY and dX become homogeneous if we take $A_2 - \alpha A_1 = 0$ and $A_3 - \beta A_1 = 0$. Another way to say this is to take $A_1 = \lambda$ (a new unknown) and then from the previous two equations A_2 and A_3 are determined. In this view these expressions become

$$\begin{aligned} A_1 &= \lambda \\ A_2 &= \alpha\lambda \\ A_3 &= \beta\lambda \,. \end{aligned}$$

In terms of the original variables a_r, b_r, c_r, α, and β these equations are

$$\begin{aligned} a_1 + \alpha b_1 + \beta c_1 - \lambda &= 0 \\ a_2 + \alpha b_2 + \beta c_2 - \alpha\lambda &= 0 \\ a_3 + \alpha b_3 + \beta c_3 - \beta\lambda &= 0 \,, \end{aligned}$$

Warning: I'm not sure how to show that this requires that

$$\begin{vmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_2 - \lambda \end{vmatrix} = 0 \,.$$

That is that the solution for λ is given by the above cubic equation. If anyone know the logic behind this statement please contact me. Once we pick λ to satisfy the above determinant and α and β accordingly then the differential equation becomes

$$\begin{aligned} \mathcal{J} &= (b_1 X + c_1 Y)(XdY - YdX) \\ &- (b_2 X + c_2 Y - \alpha(b_1 X + c_1 Y) - A_1 X)dY \\ &+ (b_3 X + c_3 Y - \beta(b_1 X + c_1 Y) - A_1 Y)dX = 0 \,. \end{aligned}$$

If we divide by $b_1 X + c_1 Y$ we get

$$XdY - YdX - \left[\frac{b_2 X + c_2 Y}{b_1 X + c_1 Y} - \alpha - \frac{A_1 X}{b_1 X + c_1 Y}\right] dY + \left[\frac{b_3 X + c_3 Y}{b_1 X + c_1 Y} - \beta - \frac{A_1 Y}{b_1 X + c_1 Y}\right] dX = 0 \,.$$

Lets write this equation as

$$XdY - YdX - \Phi dY + \Psi dX = 0 \,,$$

where we have defined Φ and Ψ in the expression above. Namely we have that Φ and Ψ are functions of the fraction $\frac{Y}{X}$ as

$$\begin{aligned} \Phi\left(\frac{Y}{X}\right) &= \frac{b_2 + c_2\left(\frac{Y}{X}\right)}{b_1 + c_2\left(\frac{Y}{X}\right)} - \alpha - \frac{A_1}{b_1 + c_2\left(\frac{Y}{X}\right)} \\ \Phi\left(\frac{Y}{X}\right) &= \frac{b_3 + c_3\left(\frac{Y}{X}\right)}{b_1 + c_2\left(\frac{Y}{X}\right)} - \beta - \frac{A_1}{b_1\left(\frac{Y}{X}\right)^{-1} + c_1} \,. \end{aligned}$$

or

$$XdY - YdX - \Phi\left(\frac{Y}{X}\right) dY + \Psi\left(\frac{Y}{X}\right) dX = 0$$

Based on the ratios of $\frac{Y}{X}$ as the arguments of the functions Φ and Ψ let $Y = Xu$ then $dY = Xdu + udX$ and we get

$$X(Xdu + udX) - XudX - \Phi(u)(Xdu + udX) + \Psi(u)dX = 0 \,.$$

or canceling $uXdX$ and dividing by du to get

$$X^2 - \Phi(u)\left(X + u\frac{dX}{dU}\right) + \Psi(u)\frac{dX}{du} = 0 \,.$$

or changing the order of some terms we have

$$\frac{dX}{du} + \frac{\Phi(u)}{\Psi(u) - u\Phi(u)} X + \frac{1}{(\Psi(u) - u\Phi(u))} X^2 = 0 \,,$$

which is a Bernoulli equation in the dependent variable X.

The Riccati Equation

The generalized Riccati equation is defined as

$$\frac{dy}{dx} + \psi y^2 + \phi y + \chi = 0\,, \tag{78}$$

where ψ, ϕ, and χ are arbitrary functions of x. To begin lets assume that we *have* a solution (denoted as y_1) to Equation 78. This can be gotten in several ways. One way is simply trial and error. Another method that might work in some problems is too look for a *constant* solution to the above equation, that is a solution y such that $\frac{dy}{dx} = 0$ and thus must satisfy

$$\psi y^2 + \phi y + \chi = 0\,.$$

This last technique will work if ψ, ϕ, and χ are not really functions of x but are in fact constants. Now assuming we have y_1 a solution to Equation 78 then lets look for another solution y to Equation 78 this time defined as

$$y = y_1 + z\,. \tag{79}$$

Then since $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dz}{dx}$ and when we put this into Equation 78 we get

$$\frac{dy_1}{dx} + \frac{dz}{dx} + \psi(y_1^2 + 2y_1 z + z^2) + \phi(y_1 + z) + \chi = 0\,,$$

or since y_1 is a solution to Equation 78 we find that the function z satisfies

$$\frac{dz}{dx} + (2\psi y_1 + \phi)z + \psi z^2 = 0\,.$$

This later equation can be transformed into a *Bernoulli* equation for another function $u(x)$ when we define $u(x)$ in terms of $z(x)$ as

$$z(x) = \frac{1}{u(x)}\,.$$

With this definition of $z(x)$ we have that $z' = -\frac{1}{u^2}u'$, and so the Bernoulli equation above becomes

$$-\frac{1}{u^2}u' + (2\psi y_1 + \phi)\frac{1}{u} + \frac{\psi}{u^2} = 0\,,$$

or when we multiply by $-u^2$ we find

$$u' - (2\psi y_1 + \phi)u + \psi = 0\,, \tag{80}$$

as the linear equation to be solved for $u(x)$. After we have solved for $u(x)$ we can then use Equation 79 since $z = 1/u$ to get another solution to the generalized Riccati Equation 78 namely $y_1 + \frac{1}{u}$.

If we now solve for u as a function of y in that relationship we note that

$$y = y_1 + \frac{1}{u} \quad \text{or} \quad u = \frac{1}{y - y_1}\,.$$

We can now replace the general solution y in the above expression with two *other* particular solutions y_2 and y_3 and introduce two new functions u_1 and u_2 as

$$u_1 = \frac{1}{y_2 - y_1} \quad \text{and} \quad u_2 = \frac{1}{y_3 - y_1} \,.$$

Each of these three functions u, u_1, and u_2 must satisfy the first order differential Equation 80. Thus they must be linearly dependent or satisfy

$$u - u_1 = C(u_2 - u_1) \,.$$

for some constant C. If we isolate the constant C, we see that this last equation is equivalent to

$$\frac{u - u_1}{u_2 - u_1} = C \,,$$

which when we use the above expressions for u in terms of y (that is $u = \frac{1}{y-y_1}$) we find the left-hand-side of the above given by

$$\frac{u - u_1}{u_2 - u_1} = \frac{\frac{1}{y-y_1} - \frac{1}{y_2-y_1}}{\frac{1}{y_3-y_1} - \frac{1}{y_2-y_1}} = \frac{\frac{y_2-y_1-(y-y_1)}{(y-y_1)(y_2-y_1)}}{\frac{y_2-y_1-(y_3-y_1)}{(y-y_1)(y_2-y_1)}} = \left(\frac{y_2 - y}{y - y_1}\right)\left(\frac{y_3 - y_1}{y_2 - y_3}\right) \,.$$

or

$$\frac{y - y_2}{y - y_1} = C\frac{y_3 - y_2}{y_3 - y_1} \,.$$

Solving this last equation for y we find that the *general* solution to Riccati's equation can be expressed as

$$y = \frac{C\left(\frac{y_2-y_3}{y_3-y_1}\right) y_1 + y_2}{C\left(\frac{y_2-y_3}{y_3-y_1}\right) - 1} = \frac{C(y_2 - y_3)y_1 + y_2(y_3 - y_1)}{C(y_2 - y_3) - (y_3 - y_1)} \,.$$

This expression is important in several ways. First it shows that the *general* solution to the Riccati equation is expressible rationally in terms of *three* particular solutions. Thus if when confronted with a Riccati equation if we can find three solutions, we are done as far as solving our differential equation is concerned, we can construct the general solution as expressed above. Motivated by this expression we hypothesize that an expression for y of the form

$$y = \frac{Cf_1 + f_2}{Cf_3 + f_4} \,, \tag{81}$$

with C a constant and f_i are arbitrary functions of x might be a solution to a Riccati type equation. The proof of this will now be shown. When we solve for C in terms of y in the above expression for y in terms of f_i we get

$$C = \frac{-f_4 y + f_2}{+f_3 y - f_1} \,. \tag{82}$$

Next, taking the first derivative of $y(x)$ we find

$$y' = \frac{Cf_1' + f_2'}{Cf_3 + f_4} - \frac{Cf_1 + f_2}{(Cf_3 + f_4)^2}(Cf_3' + f_4')$$

We next check what the value of the right-hand-side of this expression simplifies to a Riccati form when we use the expression for C in terms of y given by Equation 82. Since there is a lot of algebra in this calculation, we verify that y' can be written in the form given by Equation 78 in the Mathematica file `fractional_f_is_a_Riccati_solution.nb`. The summary of these calculations is that an expression for $y(x)$ given by Equation 81 satisfies the Riccati Equation 78 with

$$\begin{aligned}
\psi &= \frac{f_4(x)f_3'(x) - f_3(x)f_4'(x)}{f_1(x)f_4(x) - f_2(x)f_3(x)} \\
\phi &= \frac{f_4(x)f_1'(x) - f_3(x)f_2'(x) + f_2(x)f_3'(x) - f_1(x)f_4'(x)}{f_1(x)f_4(x) - f_2(x)f_3(x)} \\
\chi &= \frac{f_2(x)f_1'(x) - f_1(x)f_2'(x)}{f_1(x)f_4(x) - f_2(x)f_3(x)} .
\end{aligned}$$

These equations can now be used to solve a given Riccati equation. For example, given a differential equation of the form given by Equation 78 the three equations above provide three constraints on the functions $f_i(x)$. One could use the above equations to guide the selection of $f_i(x)$. Once suitable values for $f_i(x)$ are found that satisfy all the above equations by using Equation 81 gives the general solution to 78.

We next show that the solution to the generalized Riccati Equation 78 can be expressed as the solution of a second order linear equation. We assume that $\psi \neq 0$ (otherwise we are dealing with a linear equation) and let

$$y = \frac{v}{\psi} , \tag{83}$$

for some yet to be determined function v. We find the first derivative of y is given by

$$\frac{dy}{dx} = \frac{1}{\psi}\frac{dv}{dx} - \frac{v}{\psi^2}\frac{d\psi}{dx} ,$$

so that Equation 78 in term of the unknown function v becomes

$$\frac{1}{\psi}\frac{dv}{dx} - \frac{v}{\psi^2}\frac{d\psi}{dx} + \psi\left(\frac{v^2}{\psi^2}\right) + \phi\left(\frac{v}{\psi}\right) + \chi = 0 .$$

or multiplying by ψ we have

$$\frac{dv}{dx} + v^2 - \frac{v}{\psi}\frac{d\psi}{dx} + \phi v + \chi\psi = 0 .$$

On grouping terms we have

$$\frac{dv}{dx} + v^2 + \left(\phi - \frac{1}{\psi}\frac{d\psi}{dx}\right) v + \chi\psi = 0 \tag{84}$$

To simplify the following notation define the functions $P(x)$ and $Q(x)$ as

$$\begin{aligned}
P(x) &\equiv \phi - \frac{1}{\psi}\frac{d\psi}{dx} \\
Q(x) &\equiv \chi\psi .
\end{aligned}$$

Now introduce a function $u(x)$ derived from $v(x)$ as

$$v = \frac{u'}{u}\,.$$

From this expression we find

$$\frac{dv}{dx} = \frac{u''}{u} - \frac{u'^2}{u^2}\,,$$

so that when we put this into equation 84 we find

$$\frac{u''}{u} - \frac{u'^2}{u^2} + \frac{u'^2}{u^2} + P(x)\frac{u'}{u} + Q(x) = 0\,.$$

or on canceling terms

$$u'' + P(x)u' + Q(x)u = 0\,, \tag{85}$$

which is the second order linear equation quoted in the book.

The equation *originally* studied by Riccati or

$$\frac{dy}{dx} + ay^2 = bx^m\,,$$

has $\psi = a$, $\phi = 0$, and $\chi = -bx^m$, so that $P(x) = 0 - \frac{1}{a}0 = 0$, and $Q(x) = -abx^m$. Our second order Equation 85 for u in this case becomes

$$u'' - abx^m u = 0\,,$$

as claimed in the book.

Exercises

Exercise 1 (differential equations to integrate)

Part (i): For the equation

$$(1-x^2)^{1/2}dx + (1-y^2)^{1/2}dy = 0\,,$$

considered of the form in Equation 58 we have $P(x,y) = (1-x^2)^{1/2}$ which is a function of x only and $Q(x,y) = (1-y^2)^{1/2}$ which is a function of y only. This equation is exact trivially. The solution is then

$$\int \sqrt{1-x^2}dx + \int \sqrt{1-y^2}dy = c\,,$$

or

$$\frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin(x)\right] + \frac{1}{2}\left[y\sqrt{1-y^2} + \arcsin(y)\right] = c\,.$$

Part (ii): For the equation

$$x(1+y^2)^{1/2}dx + y(1+x^2)^{1/2}dy = 0\,,$$

this is separable since we can write it as

$$\frac{x}{(1+x^2)^(1/2)}dx + \frac{y}{(1+y^2)^(1/2)}dy = 0\,.$$

This later equation can be integrated to give

$$(1+x^2)^{1/2} + (1+y^2)^{1/2} = c\,,$$

as its primitive.

Chapter 7 (The Solution of Linear Differential Equations In An Infinte Form)

Notes on the text

$$y'' + (a - 2\theta\cos(2x))y = 0\,,$$

with $p(x) = a - 2\theta\cos(2x)$ of period π. The $y(x)$ is even then $y'(x=0) = 0$

$$c_0 = \sum_{r=0}^{\infty} c_r \cos((2r+1)x)\,,$$

is periodic of period 2π.

$$c_0'' = \sum_{r=0}^{\infty} -(2r+1)^2 \cos((2r+1)x)\,.$$

when we put these in we get

$$\sum_{r=0}^{\infty} -(2r+1)^2 \cos((2r+1)x) + a\sum_{r=0}^{\infty} c_r \cos((2r+1)x) - 2\theta\sum_{r=0}^{\infty} c_r \cos(2x)\cos((2r+1)x) = 0\,.$$

but

$$\cos(2x)\cos((2r+1)x) = \frac{1}{2}[\cos((2r+3)x) + \cos((2r-1)x)]\,,$$

using

$$\cos(\alpha)\cos(\beta) = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]\,.$$

Then

$$\sum_{r=0}^{\infty} c_r\left(-(2r+1)^2 + a\right)\cos((2r+1)x)$$

Chapter 10 (The Sturmian Theory and its Later Developments)

Notes on the text

Regular Sturmian theor 10.2 $a < x_n < b$ for all n by the Bolzano-Weirstrass theorm. Thus there exists a limit poitn $y(x_n) = 0$ for all n. Why $y(c) = 0$ and not $y(c) = \epsilon \ll 1$

$$y(c+h) = y(c) + hy'(c+\theta h) = hy'(c+\theta h)\,,$$

for $0 \le \theta < 1$. Continuity of y gives us that $y(c) = 0$ and existance gives one continious solution with one continious deriative $y'(c+\theta h) = 0$ h small $h \ll 1$.

If y_2 vanished at $y_1(x_1)$ with $L(y) = 0$ and for y_2 $y(x_1) = 0$, $y'(x_1) = \gamma_1$ and for y_2 $L(y) = 0$ with $y(x_2) = 0$ and $y'(x_1) = \gamma_2$ implys that y_1 and y_2 are multplies of each other. Thus

$$\frac{y_1}{y_2}\,,$$

vanishes x_1 and x_2 for all points in (x_1, x_2). Consider

$$\frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{y_1'}{y_2} - \frac{y_1 y_2'}{y_2^2} = \frac{y_1' y_2 - y_1 y_2'}{y_2^2}\,.$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0\,.$$

Now

$$\begin{aligned}
\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} A\sin(x) + B\cos(x) & C\sin(x) + D\cos(x) \\ A\cos(x) - B\sin(x) & C\cos(x) - D\sin(x) \end{vmatrix} \\
&= AC\sin(x)\cos(x) - DA\sin^2(x) + BC\cos^2(x) - DB\cos(x)\sin(x) \\
&- (AC\sin(x)\cos(x) + AD\cos^2(x) - BC\sin^2(x) - BD\sin(x)\cos(x)) \\
&= -DA + BC \neq 0\,.
\end{aligned}$$

$k(x) > 0$ and $a \le x \le b$ with x_1 and x_2 consecutive zeros of u and assume that v has no zero between u. We hope to reach a contradiction. Without loss of generality we assume that u and v are greater than zero on (x_1, x_2) and $G_1 - G_2 \ge 0$. Then the left-hand-side LHS gives

$$\begin{aligned}
\text{LHS} &= k(x_2)[u'(x_2)v(x_2) - 0] - k(x_1)[u'(x_1)v(x_1)] \\
&= k(x_1)[u'(x_1)v(x_1)] \\
&= k(x_2)u'(x_2)v(x_2) - k(x_1)u'(x_1)v(x_1)\,.
\end{aligned}$$

Now $u'(x_2) < 0$ and $u'(x_1) > 0$ therefore the left-hand-side is a negative minus a postive number and thus is negative quantitie.

Where as the right-hand-side is postive. This is a contradiction. If u and $v(x_1) = 0$ and x_2 is the next zero of u then the left-hand-side is given by

$$k(x_2)[u'(x_2)v(x_2) - u(x_2)] = -k(x_1)u'(x_1) = 0\,,$$

therefore $k(x_2)u'(x_2)v(x_2) < 0$ which is a contradiction. Therefore there exists a zero of v before the next zero of $u(x)$.

Solution to the example problem is

$$\begin{aligned} u &= A\cos(mx) + B\sin(mx) \\ v &= A'\cos(nx) + B'\sin(nx)\,. \end{aligned}$$

If $m > n$ then u osscilates more rappidly than v. I wanted something like if $G_1(x) > G_2(x)$ then $u(x)$ will osscilate less rapidly than $v(x)$ or $v(x)$ will osscilate more rapidly than $u(x)$. Multiply by v in the first and $u(x)$ in the second an integrate from x_1 to x_2 to get the given expression. Now x_1 and x_2 are consectutive zero of $u(x)$. Now $u(x)$ and $v(x)$ can both be positive because if they are negative then $-u(x)$ is postive and also a solution to the ODE. Integral is positive. If $u9x)$ looks like the picture to the right then $u'(x_1) > 0$ and $u'(x_2) < 0$. Also if $v(x)$ does not have a zero between x_1 and x_2 then it looks like the following and both $v(x_1)$ and $v(x_2)$ are postive. The left-hand-side becomes

$$k(x_2)(u'(x_2)v(x_2)) - k(x_1)u'(x_1)v(x_1)\,,$$

which is a negative number from which we subtract an postive number, the result must be a negative number and we have a contradiction if $v(x)$ does not cross the x-axis between x_1 and x_2. Thus $v(x)$ must vanish between x_1 and x_2 if both $u(x)$ and $v(x)$ vanish before x_2 therefore $v(x)$ osscilates more than $u(x)$. As P_n satisfies

$$\frac{d}{dx}((1-x^2)y') + n(n+1)y = 0\,,$$

and P_{n+1} satisfies

$$\frac{d}{dx}((1-x^2)y') + (n+1)(n+2)y = 0\,,$$

we see that P_{n+1} must osscilate more rapidly than P_n. This can be verified by looking at plots of $P_n(x)$ and $P_{n+1}(x)$ for $n = 10$ say We see that $P_{10}(x)$ crosses the x-axis 10 times while $P_{11}(x)$ crosses the x-axis 11 times.

Chapter 15 (Linear Equations in the Complex Domain)

Notes on the text

Why want solution to $u(z+2w) = su(z)$. No loss in general assuming that $w \in R$. If s_1 repeated $u_1(z), u_2(z), \cdots, u_\mu(z)$ such that

$$\begin{aligned}
u_1(z+2w) &= s_1 u_1(z) \\
u_2(z+2w) &= s_1(u_2(z) + u_1(z)) \\
&\vdots \\
u_\mu(z+2w) &= s_1(u_\mu(z) + u_{\mu-1}(z)) .
\end{aligned}$$

Then

$$e^{-a(z+2w)} = s_1 u_1(z) e^{-az} e^{-2aw} = [s_1 e^{-2aw}] e^{-az} u_1(z) .$$

therefore $e^{-az}u_1(z)$ if $s_1 = e^{2aw}$ if the n roots of characteristic are equal.

With $u_\nu(z) = e^{a_1 z} v_\nu(z)$ we have

$$\begin{aligned}
e^{a_1(z+2w)} v_1(z+2w) &= s_1 e^{a_1 z} v_1(z) \\
e^{a_1(z+2w)} v_2(z+2w) &= s_1 e^{a_1 z}(v_1(z) + v_2(z)) \\
&\vdots \\
e^{a_1(z+2w)} v_\mu(z+2w) &= s_1 e^{a_1 z}(v_\mu(z) + v_{\mu-1}(z)) .
\end{aligned}$$

with $e^{2wa_1} = s_1$ then we have

$$\begin{aligned}
v_1(z+2w) &= v_1(z) \\
v_2(z+2w) &= v_1(z) + v_2(z) \\
&\vdots \\
v_\mu(z+2w) &= v_\mu(z) + v_{\mu-1}(z) .
\end{aligned}$$

so

$$\frac{v_2(z+2w)}{v_1(z+2w)} = \frac{v_2(z)}{v_1(z)} + 1 .$$

Consider $\frac{v_2(z)}{v_1(z)} - \frac{z}{2w}$, then

$$\begin{aligned}
\frac{v_2(z+2w)}{v_1(z+2w)} - \frac{z+2w}{2w} &= \frac{v_2(z)}{v_1(z)} + 1 - \frac{z}{2w} - 1 \\
&= \frac{v_2(z)}{v_1(z)} - \frac{z}{2w} .
\end{aligned}$$

distinct n solution $u_r(x) = e^{\alpha}$

$$\frac{d^2w}{dz^2} = p(z)w\,.$$

$$\begin{vmatrix} a_{11}-s & a_{12} \\ a_{21} & a_{22}-s \end{vmatrix} = 0\,.$$

or

$$(a_{11}-s)(a_{22}-s) - a_{12}a_{21} = 0\,.$$

or

$$a_{11}a_{22} - (a_{11}+a_{22})s + s^2 - a_{12}a_{21} = 0\,.$$

or

$$s^2 - (a_{11}+a_{22})s + s^2 + (a_{11}a_{22} - a_{12}a_{21}) = 0\,.$$

as $|A| \neq 0$ we can scale $|A| = 1$ to get

$$s^2 - As + 1 = 0\,.$$

Idea. Given

$$\frac{d^nw}{dz^n} + p_1(z)\frac{d^{n-1}w}{dz^{n-1}} + p_2(w)\frac{d^{n-2}w}{dz^{n-2}} + \cdots + p_n(z)w = 0\,.$$

with p_i periodic with period $2w$. What type of solutions do we get. Is the solution periodic? The anwser depends on the n characteristic number of the equations

$$u_i(z) = e^{a_i z}\phi_i(z)\,.$$

with ϕ_i periodic with period $2w$ if all charteristic number are distinct. If not. Write a code to calculate these Floquet multipliers. Compute the Floquet exponentials for

$$\frac{d^2w}{dz^2} = p(z)w\,.$$

There exist two numbers which are the solution to

$$s^2 - As + 1 = 0\,.$$

If $p(z) > 0$ for all $z \in R$ the characteristic exponents are real then the system is unstable. Therefor $p(z)$ must be negative. Try $p(z) = \sin(z)$ not negative for all values, i.e. solve

$$\frac{d^2w}{dz^2} = \sin(z)w\,.$$

then $w = \pi$ in this case. Lets solve for A

$$\begin{aligned} A &= f(2w) + g'(2w) \\ &= 2 + \sum_{n=1}^{\infty}(f_n(2\pi) + g_n'(2\pi)) \end{aligned}$$

Approximate mode in MMA. Sum to just 10.

If

$$\begin{aligned} f(z,\lambda) &= 1+\lambda f_1(z)+\cdots+\lambda^n f_n(z)+\cdots = \sum_{n=0}^{\infty}\lambda^n f_n(z) \\ g(z,\lambda) &= z+\lambda g_1(z)+\cdots+\lambda^n g_n(z)+\cdots = \sum_{n=0}^{\infty}\lambda^n g_n(z)\,. \end{aligned}$$

with $f(z,\lambda)$ the solution to

$$\frac{d^2w}{dz^2}=\lambda p(z)w\,,$$

$g(0)=0$ and $g'(0)=1$. Then

$$\frac{d^2f}{dz^2}=\lambda p(z)$$

gives since $n=0$ is a constant

$$\begin{aligned} \sum_{n=1}^{\infty}\lambda^n f_n''(z) &= \lambda p(z)\sum_{n=0}^{\infty}\lambda^n f_n(z) \\ &= \sum_{n=0}^{\infty}\lambda^{n+1}p(z)f_n(z) \\ &= \sum_{n=1}^{\infty}\lambda^n p(z)f_{n-1}(z)\,. \end{aligned}$$

or

$$f_n''(z)=p(z)f_{n-1}(z)\,.$$

$g(z)$ is similar.

$$w''+pw'+qw=0\,.$$

Let

$$w=We^{-\frac{1}{2}\int p\,dz}\,. \tag{86}$$

Then

$$\begin{aligned} w' &= W'e^{-\frac{1}{2}\int p\,dz}+We^{-\frac{1}{2}\int p\,dz}\left(-\frac{1}{2}p\right) \\ w'' &= W''e^{-\frac{1}{2}\int p\,dz}+W'\left(-\frac{1}{2}p\right)e^{-\frac{1}{2}\int p\,dz}+W'\left(-\frac{1}{2}p\right)e^{-\frac{1}{2}\int p\,dz} \\ &+ W\left(-\frac{1}{2}p'\right)e^{-\frac{1}{2}\int p\,dz}+We^{-\frac{1}{2}\int p\,dz}\left(\frac{1}{4}p^2\right) \\ &= \left(W''-pW'+\left(-\frac{1}{2}p'+\frac{1}{4}p^2\right)W\right)e^{-\frac{1}{2}\int p\,dz}\,. \end{aligned}$$

Thus adding together and dividing by $e^{-\frac{1}{2}\int p\,dz}$ gives

$$W''-pW'+\left(-\frac{p'}{2}+\frac{p^2}{4}\right)W+pW'-\frac{p^2}{2}W+qW=0\,.$$

or

$$W''+\left(q-\frac{p^2}{4}-\frac{p'}{4}\right)W=0\,.$$