Problem Solutions For Numerical Methods for Conservation Laws by Randall J. LeVeque

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Additional Notes And Derivations

An Example Calculation of the Rankine-Hugoniot Curve for the Isothermal Equations (Page 71)

The isothermal equation are given by

$$\begin{aligned} \rho_t + m_x &= 0 \quad (1) \\ m_t + \left(\frac{m^2}{\rho} + a^2 \rho \right)_x &= 0 \quad (2) \end{aligned}$$

Here $m = \rho u$, $p = a^2 \rho$, and $a^2 = R\bar{T}$. Defining the vector u as

$$u = \begin{bmatrix} \rho \\ m \end{bmatrix} \quad (3)$$

we see the Jacobian $f'(u)$ of the above flux is given by

$$f'(u) = \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & \frac{2m}{\rho} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - u^2 & 2u \end{bmatrix} \quad (4)$$

To determine the eigenvalues we must solve for λ in

$$|\lambda I - f'(u)| = 0 \quad (5)$$

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or

$$\begin{vmatrix} \lambda & -1 \\ -a^2+u^2 & -2u+\lambda \end{vmatrix} = 0 \tag{6}$$

or evaluating the determinate we have

$$-2u\lambda + \lambda^2 - a^2 + u^2 = 0 \tag{7}$$

or

$$\lambda^2 - 2u\lambda + u^2 - a^2 = 0 \tag{8}$$

or on completing the square we have

$$(\lambda - u)^2 - a^2 = 0 \tag{9}$$

or

$$\lambda = u \pm a \tag{10}$$

Now the Rankine-Hugoniot equations for the isothermal equations are (we are assuming that $\hat{\rho}$ and $\hat{m}$ are the state *ahead* of the shock wave)

$$\tilde{m} - \hat{m} = s(\tilde{\rho} - \hat{\rho}) \tag{11}$$

$$\frac{\tilde{m}^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s(\tilde{m} - \hat{m}) \tag{12}$$

This is a system of two equations with three unknowns (the state behind the shock wave of $(\tilde{\rho}, \tilde{m})$ and the shock speed s). We will choose to solve for $\tilde{m}$ and s in terms of $\tilde{\rho}$. One might ask why we choose these two variables over any other. One reasons to try this choice might be that $\tilde{\rho}$ appears *linearly* in both equations. We first solve the first equation for $\tilde{m}$ and put this into the second equation. Solving the first equation for $\tilde{m}$ gives

$$\tilde{m} = \hat{m} + s(\tilde{\rho} - \hat{\rho}) \tag{13}$$

when put into the second equation we obtain

$$\frac{(\hat{m} + s(\tilde{\rho} - \hat{\rho}))^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s(s(\tilde{\rho} - \hat{\rho})) \tag{14}$$

or expanding the square

$$\frac{\hat{m}^2 + 2\hat{m}(\tilde{\rho} - \hat{\rho})s + (\tilde{\rho} - \hat{\rho})^2 s^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s^2(\tilde{\rho} - \hat{\rho}) \tag{15}$$

or multiplying by $\tilde{\rho}$

$$(\tilde{\rho}-\hat{\rho})^2 s^2 + 2\hat{m}(\tilde{\rho}-\hat{\rho}) + \hat{m}^2 + a^2\tilde{\rho}^2 - \frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} - a^2\tilde{\rho}\hat{\rho} = s^2(\tilde{\rho}-\hat{\rho})\tilde{\rho} \qquad (16)$$

or grouping together powers of the variable s

$$\left[(\tilde{\rho}-\hat{\rho})^2 - \tilde{\rho}(\tilde{\rho}-\hat{\rho})\right] s^2 + 2\hat{m}(\tilde{\rho}-\hat{\rho})s + \hat{m}^2 + a^2\tilde{\rho}^2 - \frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} - a^2\tilde{\rho}\hat{\rho} = 0\,. \quad (17)$$

The coefficient of s^2 expands to

$$-\hat{\rho}(\tilde{\rho}-\hat{\rho})\,.$$

The above can be recognized as a quadratic equation for s. Dividing by the leading coefficient of the s^2 term we obtain

$$s^2 - 2\frac{\hat{m}}{\hat{\rho}}s - \frac{1}{\hat{\rho}(\tilde{\rho}-\hat{\rho})}\left[\hat{m}^2 + a^2\hat{\rho}^2 - \left(\frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} + a^2\tilde{\rho}\hat{\rho}\right)\right] = 0 \qquad (18)$$

To complete the square we take the coefficient of s divide it by two, square it and add this to both sides, this gives the value of $\frac{\hat{m}^2}{\hat{\rho}^2}$ giving

$$s^2 - 2\frac{\hat{m}}{\hat{\rho}}s + \frac{\hat{m}^2}{\hat{\rho}^2} - \frac{\hat{m}^2}{\hat{\rho}^2} - \frac{1}{\hat{\rho}(\tilde{\rho}-\hat{\rho})}\left[\hat{m}^2 + a^2\tilde{\rho}^2 - \left(\frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} + a^2\tilde{\rho}\hat{\rho}\right)\right] = 0 \quad (19)$$

or

$$(s - \frac{\hat{m}}{\hat{\rho}})^2 - \frac{\hat{m}^2}{\hat{\rho}^2} - \frac{1}{\hat{\rho}(\tilde{\rho}-\hat{\rho})}\left[\frac{\hat{m}^2}{\hat{\rho}}(\hat{\rho}-\tilde{\rho}) + a^2\tilde{\rho}(\tilde{\rho}-\hat{\rho})\right] = 0 \qquad (20)$$

or

$$(s - \frac{\hat{m}}{\hat{\rho}})^2 - \frac{a^2\tilde{\rho}}{\hat{\rho}} = 0 \qquad (21)$$

or giving

$$s = \frac{\hat{m}}{\hat{\rho}} \pm a\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}}\,. \qquad (22)$$

Which is equation 7.10 in the book. Then we have for $\tilde{m}$ the following

$$\tilde{m} = \hat{m} + (\tilde{\rho}-\hat{\rho})\left(\frac{\hat{m}}{\hat{\rho}} \pm a\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}}\right) \qquad (23)$$

or

$$\tilde{m} = \frac{\hat{m}}{\hat{\rho}}\left(\hat{\rho} + \tilde{\rho} - \hat{\rho}\right) \pm a(\tilde{\rho}-\hat{\rho})\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}} = \frac{\tilde{\rho}}{\hat{\rho}}\hat{m} \pm a(\tilde{\rho}-\hat{\rho})\sqrt{\frac{\tilde{\rho}}{\hat{\rho}}} \qquad (24)$$

which is LeVeque Eq. 7.9 in the book.

An Example Calculation of the Rarefaction Waves and Integral Curves for the Isothermal Equations (Page 71)

Problem Solutions

Chapter 8:

Exercise 8.4

We first consider the Riemann problem between the states u_1 and u_2 and then the corresponding Riemann problem between the states u_2 and u_3 as defined in the text. To aid in this process we will use the FORTRAN code developed to solve the Riemann problem for the isothermal equations which can be found at:

`http://web.mit.edu/wax/www/Software`

The first Riemann problem is represented in the $(\rho, m) = (\rho, \rho u)$ plane in Figure XXX