Additional Notes And Derivations

An Example Calculation of the Rankine-Hugoniot Curve for the Isothermal Equations (Page 71)

The isothermal equation are given by

\[ \rho_t + m_x = 0 \]  \hspace{1cm} (1)

\[ m_t + \left( \frac{m^2}{\rho} + a^2 \rho \right)_x = 0 \]  \hspace{1cm} (2)

Here \( m = \rho u, \ p = a^2 \rho, \) and \( a^2 = RT. \) Defining the vector \( u \) as

\[ u = \begin{bmatrix} \rho \\ m \end{bmatrix} \]  \hspace{1cm} (3)

we see the Jacobian \( f'(u) \) of the above flux is given by

\[ f'(u) = \begin{bmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & \frac{2m}{\rho} \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ a^2 - u^2 & 2u \end{bmatrix} \]  \hspace{1cm} (4)

To determine the eigenvalues we must solve for \( \lambda \) in

\[ |\lambda I - f'(u)| = 0 \]  \hspace{1cm} (5)
or
\[
\begin{vmatrix}
\lambda & -1 \\
-a^2 + u^2 & -2u + \lambda
\end{vmatrix} = 0
\]  
(6)

or evaluating the determinate we have

\[-2u\lambda + \lambda^2 - a^2 + u^2 = 0\]  
(7)

or
\[\lambda^2 - 2u\lambda + u^2 - a^2 = 0\]  
(8)

or on completing the square we have

\[(\lambda - u)^2 - a^2 = 0\]  
(9)

or
\[\lambda = u \pm a\]  
(10)

Now the Rankine-Hugoniot equations for the isothermal equations are (we are assuming that \(\hat{\rho}\) and \(\hat{m}\) are the state *ahead* of the shock wave)

\[\frac{\dot{m}^2}{\hat{\rho}} + a^2 \hat{\rho} - \frac{\dot{m}^2}{\hat{\rho}} - a^2 \hat{\rho} = s(\dot{m} - \hat{m})\]  
(11)

This is a system of two equations with three unknowns (the state behind the shock wave of \((\hat{\rho}, \hat{m})\) and the shock speed \(s\)). We will choose to solve for \(\hat{m}\) and \(s\) in terms of \(\hat{\rho}\). One might ask why we choose these two variables over any other. One reasons to try this choice might be that \(\hat{\rho}\) appears linearly in both equations. We first solve the first equation for \(\hat{m}\) and put this into the second equation. Solving the first equation for \(\hat{m}\) gives

\[\hat{m} = \dot{m} + s(\hat{\rho} - \hat{\rho})\]  
(13)

when put into the second equation we obtain

\[\frac{(\dot{m} + s(\hat{\rho} - \hat{\rho}))^2}{\hat{\rho}} + a^2 \hat{\rho} - \frac{\dot{m}^2}{\hat{\rho}} - a^2 \hat{\rho} = s(s(\hat{\rho} - \hat{\rho}))\]  
(14)

or expanding the square

\[\frac{\dot{m}^2 + 2\hat{m}(\hat{\rho} - \hat{\rho})s + (\hat{\rho} - \hat{\rho})^2s^2}{\hat{\rho}} + a^2 \hat{\rho} - \frac{\dot{m}^2}{\hat{\rho}} - a^2 \hat{\rho} = s(\hat{\rho} - \hat{\rho})\]  
(15)
or multiplying by \( \tilde{\rho} \)

\[
(\tilde{\rho} - \tilde{\rho})^2 s^2 + 2\tilde{m}(\tilde{\rho} - \tilde{\rho}) + \tilde{m}^2 + a^2\tilde{\rho}^2 - \frac{\tilde{m}^2\tilde{\rho}}{\tilde{\rho}} - a^2\tilde{\rho}\tilde{\rho} = s^2(\tilde{\rho} - \tilde{\rho})\tilde{\rho} 
\]  
(16)

or grouping together powers of the variable \( s \)

\[
[(\tilde{\rho} - \tilde{\rho})^2 - \tilde{\rho}(\tilde{\rho} - \tilde{\rho})] s^2 + 2\tilde{m}(\tilde{\rho} - \tilde{\rho})s + \tilde{m}^2 + a^2\tilde{\rho}^2 - \frac{\tilde{m}^2\tilde{\rho}}{\tilde{\rho}} - a^2\tilde{\rho}\tilde{\rho} = 0. 
\]  
(17)

The coefficient of \( s^2 \) expands to

\[-\tilde{\rho}(\tilde{\rho} - \tilde{\rho}).\]

The above can be recognized as a quadratic equation for \( s \). Dividing by the leading coefficient of the \( s^2 \) term we obtain

\[
s^2 - 2\frac{\tilde{m}}{\tilde{\rho}}s - \frac{1}{\tilde{\rho}(\tilde{\rho} - \tilde{\rho})} \left[ \tilde{m}^2 + a^2\tilde{\rho}^2 - \left( \frac{\tilde{m}^2\tilde{\rho}}{\tilde{\rho}} + a^2\tilde{\rho}\tilde{\rho} \right) \right] = 0 \]  
(18)

To complete the square we take the coefficient of \( s \) divide it by two, square it and add this to both sides, this gives the value of \( \tilde{m}^2/\tilde{\rho} \) giving

\[
s^2 - 2\frac{\tilde{m}}{\tilde{\rho}}s + \frac{\tilde{m}^2}{\tilde{\rho}^2} - \frac{1}{\tilde{\rho}(\tilde{\rho} - \tilde{\rho})} \left[ \tilde{m}^2 + a^2\tilde{\rho}^2 - \left( \frac{\tilde{m}^2\tilde{\rho}}{\tilde{\rho}} + a^2\tilde{\rho}\tilde{\rho} \right) \right] = 0 \]  
(19)

or

\[
(s - \frac{\tilde{m}}{\tilde{\rho}})^2 - \frac{\tilde{m}^2}{\tilde{\rho}^2} - \frac{1}{\tilde{\rho}(\tilde{\rho} - \tilde{\rho})} \left[ \frac{\tilde{m}^2}{\tilde{\rho}} (\tilde{\rho} - \tilde{\rho}) + a^2\tilde{\rho}(\tilde{\rho} - \tilde{\rho}) \right] = 0 \]  
(20)

or

\[
(s - \frac{\tilde{m}}{\tilde{\rho}})^2 - \frac{a^2\tilde{\rho}}{\tilde{\rho}} = 0 \]  
(21)

or giving

\[
s = \frac{\tilde{m}}{\tilde{\rho}} \pm a\sqrt{\frac{\rho}{\tilde{\rho}}}. \]  
(22)

Which is equation 7.10 in the book. Then we have for \( \tilde{m} \) the following

\[
\tilde{m} = \tilde{m} + (\tilde{\rho} - \tilde{\rho}) \left( \frac{\tilde{m}}{\tilde{\rho}} \pm a\sqrt{\frac{\rho}{\tilde{\rho}}} \right) \]  
(23)

or

\[
\tilde{m} = \frac{\tilde{m}}{\tilde{\rho}} (\tilde{\rho} + \tilde{\rho} - \tilde{\rho}) \pm a(\tilde{\rho} - \tilde{\rho})\sqrt{\frac{\rho}{\tilde{\rho}}} = \frac{\rho}{\tilde{\rho}} \tilde{m} \pm a(\tilde{\rho} - \tilde{\rho})\sqrt{\frac{\rho}{\tilde{\rho}}} \]  
(24)

which is LeVeque Eq. 7.9 in the book.
An Example Calculation of the Rarefaction Waves and Integral Curves for the Isothermal Equations (Page 71)

Problem Solutions

Chapter 8:
Exercise 8.4

We first consider the Riemann problem between the states $u_1$ and $u_2$ and then the corresponding Riemann problem between the states $u_2$ and $u_3$ as defined in the text. To aid in this process we will use the FORTRAN code developed to solve the Riemann problem for the isothermal equations which can be found at:

http://web.mit.edu/wax/www/Software

The first Riemann problem is represented in the $(\rho, m) = (\rho, \rho u)$ plane in Figure XXX