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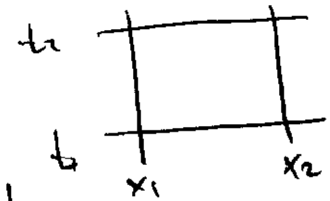

Pg 16 ⇒

$$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt$$

w/ $x_1 = -M$; $x_2 = M$; $t_1 = 0$; $t_2 = 1$

$$= \int_{-M}^{M} u(x,1)\,dx = \int_{-M}^{M} u(x,0)\,dx + \int_0^1 f(u(-M,t))\,dt - \int_0^1 f(u(M,t))\,dt$$

M large enough $u(-M,t) \to u_l$

$u(M,t) \to u_R$

$$\Rightarrow \int_{-M}^{M} u(x,1)\,dx = \int_{-M}^{M} u(x,0)\,dx + f(u_l) - f(u_r)$$

$$= \int_{-M}^{0} u_l + \int_0^M u_r + \cdots$$

$$= u_l(M) + u_r M + \cdots$$

$u(x,1) = w(x/1)$

$$= M(u_l + u_r)$$

[illegible]

let

$x_1 = 0$; $x_2 = M$; $t_1 = 0$; $t_2 = 1$

$$\int_0^M u(x,1)\,dx = \underbrace{\int_0^M u(x,0)\,dx}_{u_r M} + \int_0^1 f(u(0,t))\,dt - \underbrace{\int_0^1 f(u(M,t))\,dt}_{f(u_r)} \quad *$$

$$+ \underbrace{\int_0^1 f(u(0,t))\,dt}_{F(u_l,u_r)}$$

$$\therefore \int_0^M \hat{w}(\xi)\,d\xi = M u_r + F(u_l,u_r) - f(u_r)$$

let $x_1 = -M$; $x_2 = 0$; $t_1 = 0$; $t_2 = 1$

$$\Rightarrow \int_{-M}^0 u(x,1)\,dx = \underbrace{\int_{-M}^0 u(x,0)\,dx}_{u_l M} + \underbrace{\int_0^1 f(u(-M,t))\,dt}_{f(u_l)} - \underbrace{\int_0^1 f(u(0,t))\,dt}_{F(u_l,u_r)} \quad **$$

From 14.6

$$\int_{-M}^0 \hat{w}\,d\xi + \int_0^M \hat{w}\,d\xi = M(u_l + u_r) + f(u_l)\cancel{\xi_l} - f(u_r)$$

put in ↑ from last eq (eq **)

$$\Rightarrow u_l \cancel{M} + \cancel{f(u_l)} - F(u_l,u_r) = M(\cancel{u_l} + u_r) + \cancel{f(u_l)} - f(u_r) - \int_0^M \hat{w}\,d\xi$$

$$\Rightarrow \quad F(u_\ell, u_r) = \cancel{f(u_r)+\ldots}$$

$$= f(u_r) - Mu_r + \int_0^M \hat{w}\, d\xi$$

put in $\int_0^M \hat{w}\, d\xi$ from eq \# to get

$$\int_{-M}^{0} \hat{w}\, d\xi + \cancel{u_r M} + F(u_\ell, u_r) - \cancel{f(u_r)} = M(u_\ell + \cancel{u_r}) + \underline{f(u_\ell)} - \cancel{f(u_r)}$$

$$\Rightarrow \quad F(u_\ell, u_r) = f(u_\ell) + Mu_\ell - \int_{-M}^{0} \hat{w}\, d\xi$$

12.55

$$\eta(U_j^{n+1}) \le \eta(U_j^n) - \frac{k}{h}\left[F(U^n;j) - F(U^n;j-1)\right]$$ discrete entropy eq:

Thus inforce on sol. $\hat{w}(\xi)$ ∋ it

$$\int_{-M}^{M} \eta(\hat{w}(\xi))\,d\xi \le M(\eta(u_l) + \eta(u_r)) + (f(u_l) - f(u_r))$$

integral form of entropy condition

$$\int_{x_1}^{x_2} \eta(u(x,t_2))\,dx \le \int_{x_1}^{x_2} \eta(u(x,t_1))\,dx - \int_{t_1}^{t_2} f(u(x_2,t))\,dt + \int_{t_1}^{t_2} f(u(x_1,t))\,dt$$

Then let $x_1 = -M$; $x_2 = M$; $t_1 = 0$; $t_2 = 1$

$$\Rightarrow \int_{-M}^{M} \eta(u(x,1))\,dx \le \int_{-M}^{M} \eta(u(x,0))\,dx - f(u_r) + f(u_l)$$

$$\int_{-M}^{0} \eta(u_l) + \int_{0}^{M} \eta(u_r)$$

$$M\eta(u_l) + M\eta(u_r)$$

$$\int_{-M}^{M} \eta(u(x,1))\,dx \le M(\eta(u_l) + \eta(u_r)) + f(u_l) - f(u_r)$$

As before let $x_1 = 0;\ x_2 = M;\ t_1 = 0;\ t_2 = 1$
to get

$$\int_0^M \eta(\hat{w}(\xi))\,d\xi \le M\eta(u_r) + \int_0^1 f(u(0,t))\,dt - \underbrace{\int_0^1 f(u(M,t))\,dt}_{f(u_r)}$$

$$\le M\eta(u_r) + F(u_\ell, u_r) - f(u_r)$$

& let $x_1 = -M;\ x_2 = 0;\ t_1 = 0;\ t_2 = 1$ to get

$$\int_{-M}^{0} \eta(\hat{\omega}(\xi))\,d\xi \le \eta(u_\ell)M$$

All steps follow as before to get 2 expression for the numerical entropy fn.

$$F(u_\ell, u_r) = f(u_\ell) + M\eta(u_\ell) - \int_{-M}^{0} \eta(\hat{w}(\xi))\,d\xi$$

or

$$F(u_\ell, u_r) = f(u_r) - M\eta(u_r) + \int_0^M \eta(\hat{\omega}(\xi))\,d\xi.$$

pg 149 h.V.

~~[illegible]~~ The Easy Hyperbolic eq satisfies

$$\int_{-M}^{M} \hat{w}(\xi)\,d\xi = M(u_l + u_r) + \hat{f}(u_l) - \hat{f}(u_r)$$

For M large enough. But ~~so~~ we require ~~[illegible]~~

Sol $$\int_{-M}^{M} w(\xi)\,d\xi = M(u_l + u_r) + f(u_l) - f(u_r)$$

The Approx Sol to h $\Rightarrow$ $\int_{-M}^{M} \hat{w}(\xi)\,d\xi = M(\quad) + f(u_l) - f(u_r)$

By 14.6

"$w = u(x,1)$"

$\therefore$ $f(u_l) - f(u_r) = \hat{f}(u_l) - \hat{f}(u_r)$

Then by 14.9

$$F(u_l, u_r) = f(u_r) - Mu_r + \int_{0}^{M} \hat{w}(\xi)\,d\xi$$

$f(u_l) = f(u_r) + \hat{f}(u_l) - \hat{f}(u_r)$ from 14.13.

then $F(u_l, u_r) = \underline{f(u_r)} + \hat{f}(u_l) - \underline{\hat{f}(u_r)} + Mu_l - \int_{-M}^{0} \hat{w}(\xi)\,d\xi$

$$F(u_\ell; u_r) = f(u_r) - \hat{f}(u_r) + \underbrace{\hat{f}(u_\ell) + M u_\ell - \int_{-M}^{0} \hat{w}(\xi)\, d\xi}_{\hat{F}(u_\ell, u_r)}$$

By 14.8

But $\hat{F}(u_\ell, u_r) = \hat{f}(\hat{w}(0))$

$$\therefore \quad F(u_\ell, u_r) = f(u_r) - \hat{f}(u_r) + \hat{f}(\hat{w}(0)) \qquad (14.14)$$

Also. $f(u_r) = \hat{f}(u_r) - \hat{f}(u_\ell) + f(u_\ell)$ put in eq

14.9

$$F(u_\ell, u_r) = \hat{f}(u_r) - \hat{f}(u_\ell) + f(u_\ell) - M u_r + \int_0^M \hat{w}(\xi)\, d\xi$$

$$= f(u_\ell) - \hat{f}(u_\ell) + \underbrace{\hat{f}(u_r) - M u_r + \int_0^M \hat{w}\, d\xi}_{\hat{F}(u_\ell, u_r) \,=\, \hat{f}(\hat{w}(0))}$$

$$= f(u_\ell) - \hat{f}(u_\ell) + \hat{f}(\hat{w}(0))$$

4.6 Approx $\hat{w}$) $\displaystyle\int_{-M}^{M} \hat{w}(\xi)\, d\xi = M(U_\ell + U_r) + \hat{f}(U_\ell) - \hat{f}(U_r)$ ~~Rec-hatted~~ eq. **

if $\hat{A}(U_\ell, U_r)(U_r - U_\ell) = f(U_r) - f(U_\ell)$

Claim $\Rightarrow$ * is true
if $\hat{f}(U) = \hat{A} U$. know
$$\hat{W}(\xi) = U_\ell + \sum_{\hat{\lambda}_p < \xi} \alpha_p \hat{r}_p$$

$$\therefore \int_{-M}^{M} \hat{W}(\xi)\, d\xi = U_\ell 2M + \int_{\xi=-M}^{M} \sum_{\hat{\lambda}_p < \xi} \alpha_p \hat{r}_p \, d\xi$$

$\alpha_1 r_1$

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n.$$

$$+ \int_{\xi=\lambda_1}^{\lambda_2} + \int_{\lambda_2}^{\lambda_3} + \int_{\lambda_3}^{\lambda_4} + \cdots + \int_{\lambda_{n-1}}^{\lambda_n} + \int_{\lambda_n}^{M}$$

$$+ \alpha_1 \hat{r}_1 (\cancel{\lambda_2} - \lambda_1) + (\alpha_1 \hat{r}_1 + \alpha_2 \hat{r}_2)(\lambda_3 - \lambda_2)$$

$$+ (\alpha_1 \hat{r}_1 + \alpha_2 \hat{r}_2 + \alpha_3 \hat{r}_3)(\lambda_4 - \lambda_3) + \cdots$$

$$+ (\alpha_1 \hat{r}_1 + \alpha_2 \hat{r}_2 + \cdots + \alpha_{n-1} \hat{r}_{n-1})(\lambda_n - \lambda_{n-1})$$

$$+ (\alpha_1 \hat{r}_1 + \alpha_2 \hat{r}_2 + \cdots + \alpha_{n-1} \hat{r}_{n-1} + \alpha_n \hat{r}_n)(M - \lambda_n)$$

$$= \alpha_1 \hat{r}_1 \left[(\cancel{\lambda_2} - \lambda_1) + (\cancel{\lambda_3} - \cancel{\lambda_2}) + (\cancel{\lambda_4} - \cancel{\lambda_3}) + \cdots + \underset{(\cancel{\lambda_n} - \cancel{\lambda_{n-1}})}{\wedge} (M - \cancel{\lambda_n}) \right]$$

$$+ \alpha_2 \hat{r}_2 \left[\cancel{\lambda_3} - \lambda_2 + \cancel{\lambda_4} - \cancel{\lambda_3} + \cdots + (\cancel{\lambda_n} - \cancel{\lambda_{n-1}}) + (M - \cancel{\lambda_n}) \right]$$

$$+ \cdots +$$

$$+ \alpha_{n-1} \hat{r}_{n-1} \left[(\cancel{\lambda_n} - \lambda_{n-1}) + (M - \cancel{\lambda_n}) \right]$$

$$+ \alpha_n \hat{r}_{n-1} \left[M - \lambda_n \right]$$

$$= \alpha_1 \hat{r}_1 [M - \lambda_1] + \alpha_2 \hat{r}_2 [M - \lambda_2] + \cdots$$

$$\alpha_{n-1} \hat{r}_{n-1} [M - \lambda_{n-1}] + \alpha_n \hat{r}_n [M - \lambda_n]$$

$$= M \sum_p \alpha_p \hat{r}_p - \sum_p \lambda_p \alpha_p \hat{r}_p$$

$$= M(u_r - u_l) - \hat{A}(u_r - u_l).$$

$$\int_{-M}^{M} \hat{w}(\xi)\, d\xi = 2Mu_l + Mu_r - Mu_l - \hat{A}(u_r - u_l)$$

$$= M(u_l + u_r) + \hat{A}(u_l - u_r)$$

if $\hat{A}(u_l - u_r) = f(u_l) - f(u_r)$

14.6 stu

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R.H. cond ⇒ $f(u_r) - f(u_l) = s(u_r - u_l)$

14.19: ⇒ $A(u_r - u_l) = s(u_r - u_l)$

⇒ $u_r - u_l$ e.vector of A s evalue A ∴ $\hat{u}(x,t)$

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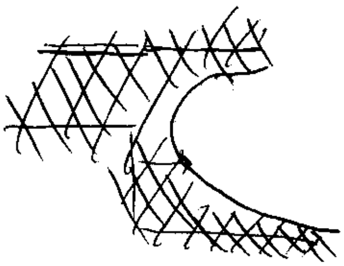

$$F(u_l, u_r) = \hat{A}\hat{w}(0) + f(u_r) - \hat{A}u_r$$

$$= f(u_r) + \hat{A}\Big[\cancel{u_r} - \sum_{\lambda_p>0} \alpha_p \hat{r}_p\Big] - \cancel{\hat{A}u_r}$$

$$= f(u_r) - \hat{A}\sum_{\lambda_p>0} \alpha_p \hat{r}_p$$

$$= f(u_r) - \sum_{p=1}^{m} \hat{\lambda}_p^{+} \alpha_p \hat{r}_p .$$

or $$F(u_l, u_r) = f(u_l) - \cancel{\hat{A}u_l} + \hat{A}\Big[\cancel{u_l} + \sum_{\lambda_p<0} \alpha_p \hat{r}_p\Big]$$

$$= f(u_l) + \hat{A}\sum_{\lambda_p<0} \alpha_p \hat{r}_p$$

$$= f(u_l) + \sum_{p=1}^{m} \hat{\lambda}_p^{-} \alpha_p \hat{r}_p .$$

$$F(u_r, u_l) = \frac{1}{2}\left(f(u_l) + f(u_r)\right) + \frac{1}{2}\left(\sum \hat{\lambda}_p^- \alpha_p \hat{r}_p - \sum \hat{\lambda}_p^+ \alpha_p \hat{r}_p\right)$$

$$= \frac{1}{2}(f_l + f_r) - \frac{1}{2}\left(\sum \hat{\lambda}_p^+ \alpha_p \hat{r}_p - \sum \hat{\lambda}_p^- \cdots\right)$$

$$-\frac{1}{2} \sum_{p=1}^{m} |\hat{\lambda}_p| \alpha_p \hat{r}_p$$

14.22 $\xRightarrow{\text{Scalar case}}$ $f(u_l) + \hat{a}^-($

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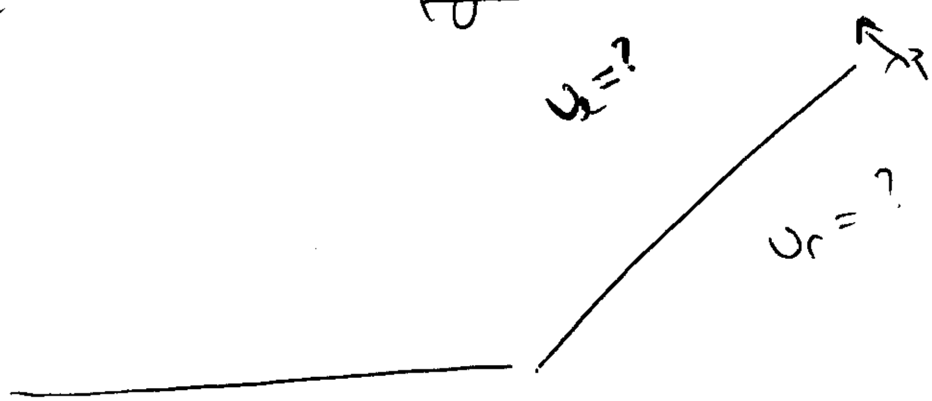

$$U_{pl} = U_l + \sum_{i=1}^{p-1} \alpha_i \hat{r}_i \qquad \text{and} \qquad U_{pr} = U_{pl} + \alpha_i \hat{r}_i .$$

predicted + By Roe.

$$(\lambda_{qr} - \lambda_{ql}) U_{qm} = (\hat{\lambda}_q - \lambda_{ql}) u_{ql} + (\lambda_{qr} - \hat{\lambda}_q) u_{qr}$$

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$$U_{qm} - U_{ql} = \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}} - \frac{(\lambda_{qr} - \lambda_{ql})}{(\lambda_{qr} - \lambda_{ql})} \right) U_{ql} + \left(\frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}} \right) U_{qr}$$

$$= \frac{\hat{\lambda}_q - \lambda_{qr}}{\lambda_{qr} - \lambda_{ql}} U_{ql} + \frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}} U_{qr}$$

$$= \left(\frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}} \right) (U_{qr} - U_{ql})$$

$$+ \quad U_{qr} - U_{qm} = \frac{\lambda_{qr} - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\left(-\alpha_q \hat{r}_q\right) \quad \text{from } 14.26.$$

$$U_{qr} - U_{qm} = -\left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)U_{ql} - \left(\frac{\lambda_{qr} - \hat{\lambda}_q}{\lambda_{qr} - \lambda_{ql}}\right)U_{qr} + \left(\frac{\lambda_{qr} - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)U_{qr}$$

$$= -\left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)U_{ql} + \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)U_{qr}$$

$$= \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)\left(U_{qr} - U_{ql}\right) = \left(\frac{\hat{\lambda}_q - \lambda_{ql}}{\lambda_{qr} - \lambda_{ql}}\right)\left(\alpha_q \hat{r}_q\right) \quad \text{from } 14.26$$

pg 153 Leblique

eq 14.34

$$\hat{\lambda}_{pe} = \lambda_{pe}^{-} \left(\frac{\lambda_{pr}^{+} - \hat{\lambda}_{p}}{\lambda_{pr}^{+} - \lambda_{pe}^{-}} \right)$$

$$\hat{\lambda}_{pr} = \lambda_{pr}^{+} \left(\frac{\hat{\lambda}_{p} - \lambda_{pe}^{-}}{\lambda_{pr}^{+} - \lambda_{pe}^{-}} \right)$$

$p = q$

$$\hat{\lambda}_{qe} = \lambda_{qe} \left(\frac{\lambda_{qr} - \hat{\lambda}_{q}}{\lambda_{qr} - \lambda_{qe}} \right)$$

eq 14.30 ✓

$$\hat{\lambda}_{qr} = \lambda_{qr} \left(\frac{\hat{\lambda}_{q} - \lambda_{qe}}{\lambda_{qr} - \lambda_{qe}} \right)$$

eq 14.31 ✓

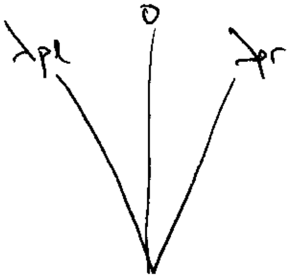
$p \neq q$

)))

4

$$\hat{\lambda}_{pl} \longrightarrow \hat{\lambda}_p^- \quad \Big] \text{ when not in a sonic rare wave.}$$

$$\& \ \hat{\lambda}_{pr} \longrightarrow \hat{\lambda}_p^+$$



$\Rightarrow$ Sonic rare Case

$0 < \lambda_{pl} < \lambda_{pr}$ $\qquad \lambda_{pl}^- = 0$

$$\lambda_{pr}^+ = \lambda_{pr}$$

$$\therefore \quad \hat{\lambda}_{pl} = 0$$

$$\& \quad \hat{\lambda}_{pr} = \lambda_{pr}\left(\frac{\hat{\lambda}_p - 0}{\lambda_{pr} - 0}\right) = \hat{\lambda}_p \quad \checkmark$$

$\lambda_{pl} < \lambda_{pr} < 0$ $\qquad \lambda_{pl}^- = -\lambda_{pl}$

$$\lambda_{pr}^+ = 0$$

$$\therefore \quad \hat{\lambda}_{pl} = -\lambda_{pl}\left(\frac{0 - \hat{\lambda}_p}{0 + \lambda_{pl}}\right) = \hat{\lambda}_p \quad \checkmark$$

$$\& \quad \hat{\lambda}_{pr} = 0$$

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$$\hat{\lambda}_q = \hat{a} = (f(u_r) - f(u_l))/(u_r - u_l)$$

$$14.27 \Rightarrow \quad u_m = \frac{(\hat{a} - f'(u_l))u_l + (f'(u_r) - \hat{a})u_r}{f'(u_r) - f'(u_l)}$$

$$= \quad \cancel{}$$

$$= \frac{(\hat{a} - f'(u_l) + f'(u_r))u_l}{f'(u_r) - f'(u_l)} + \frac{-f'(u_r)u_l + (f'(u_r) - \hat{a})u_r}{f'(u_r) - f'(u_l)}$$

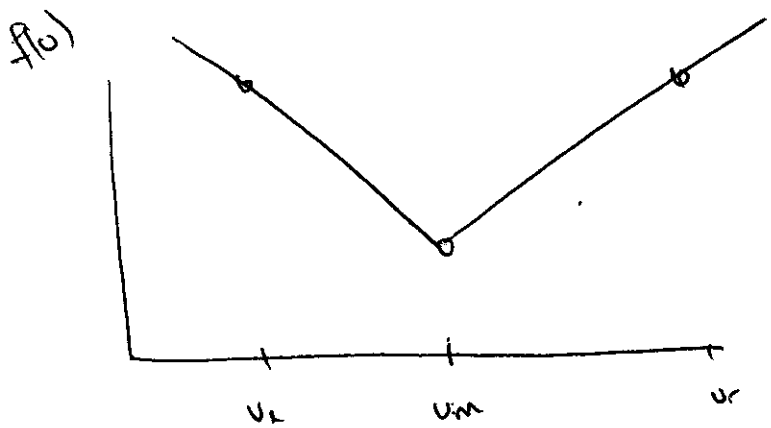
$$= u_l + \frac{\hat{a}u_l - f'(u_r)u_l + (f'(u_r) - a)u_r}{\Delta f'}$$

$$= u_l + \frac{(f'(u_r) - \hat{a})(u_r - u_l)}{\Delta f'}$$

Pg 154 LV6

$$\hat{w}(x/t) = \begin{cases} u_l & x/t < \hat{a} \\ u_r & x/t > \hat{a} \end{cases}$$

$$\hat{w}(x/t) = \begin{cases} u_l & x/t < f'(u_l) \\ u_m & f'(u_l) < x/t < f'(u_r) \\ u_r & x/t > f'(u_r) \end{cases}$$



How show 14.41?

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to get pt of intersection in 14.41

$$f(u_l) + (u_m - u_l) f'(u_l) = f(u_r) + (u_m - u_r) f'(u_r)$$

$$\Rightarrow \quad u_m = \frac{f(u_l) - f(u_r) + u_r f'(u_r) - u_l f'(u_l)}{-f'(u_l) + f'(u_r)}$$

$$= \frac{-u_l f'(u_l) + u_l f'(u_r) \quad (u_r - u_l) f'(u_r)}{-f'(u_l) + f'(u_r)}$$

$$= u_l + \frac{(f'(u_r) - \hat{a})}{f'(u_r) - f'(u_l)} (u_r - u_l) + \frac{f(u_l) - f(u_r)}{-f'(u_l) + f'(u_r)}$$

)

Think should be able to write as $u_r - \ldots$

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14.32

$$J(u_l,u_r) = f(u_l) + \sum_{p\neq q} \hat{\mathcal{I}}_p \alpha_p \hat{r}_p + \hat{\mathcal{I}}_q \alpha_q \hat{r}_q$$

$$\Rightarrow \quad f(u_l) + \quad (1) \quad \text{using def } 14.34$$

$$+ \quad f'(u_l)\left(\frac{f'(u_r) - \hat{a}}{f'(u_r) - f'(u_l)}\right)(u_r - u_l)(1)$$

14.14

$$F(u_l,u_r) =$$

$$= \hat{f}(\hat{\omega}(0)) + f(u_r) - \hat{f}(u_r)$$

$$= \hat{f}(u_m) + f(u_r) - \underbrace{\hat{f}(u_r)}_{f(u_r)}$$

$$= \hat{f}(u_m)$$

$$= f(u_l) + (u_m - u_l) f'(u_l)$$

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$\vec{v} = \begin{pmatrix} \rho \\ m \end{pmatrix}$ $f(u) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + a^2\rho \end{pmatrix}$ eqs 5.32 ✓

$f'(u) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & \frac{2m}{\rho} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a^2 - v^2 & 2v \end{pmatrix}$ eq 14.48

$\vec{z} = \rho^{-1/2}\vec{v} \Rightarrow \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \rho^{1/2} \\ m/\rho^{1/2} \end{pmatrix} = \begin{pmatrix} \rho^{1/2} \\ v\rho^{1/2} \end{pmatrix}$ eq 14.49

Then $\vec{v} = z_1 z = \begin{pmatrix} z_1^2 \\ z_1 z_2 \end{pmatrix}$ $\vec{f}(\vec{v}) = \begin{pmatrix} z_1 z_2 \\ z_2^2 + a^2 z_1^2 \end{pmatrix}$ eq 14.50

$\bar{z} = \frac{1}{2}(z_l + z_r) = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \bar{\rho}_l^{1/2} + \bar{\rho}_r^{1/2} \\ \frac{m_l}{\rho_l^{1/2}} + \frac{m_r}{\rho_r^{1/2}} \end{pmatrix}$ eq 14.51

Then $v_l - v_r = \begin{pmatrix} z_{1l}^2 \\ z_{1l} z_{2l} \end{pmatrix} - \begin{pmatrix} z_{1r}^2 \\ z_{1r} z_{2r} \end{pmatrix} = \begin{pmatrix} z_{1l}^2 - z_{1r}^2 \\ z_{1l} z_{2l} - z_{1r} z_{2r} \end{pmatrix}$

Cheating a bit (How do directly?) we notice that
see notes that go w/ 8-14-01 4

$\bar{z}_2(z_{1l} - z_{1r}) + \bar{z}_1(z_{2l} - z_{2r})$

$= \frac{1}{2}(z_{2l} + z_{2r})(z_{1l} - z_{1r}) + \frac{1}{2}(z_{1l} + z_{1r})(z_{2l} - z_{2r})$

$= \frac{1}{2}\left[z_{2l} z_{1l} - \cancel{z_{2l} z_{1r}} + \cancel{z_{2r} z_{1l}} - z_{2r} z_{1r} \right]$

$$+\frac{1}{2}\left[z_{1l} z_{2l} - \cancel{z_{1l} z_{2r}} + \cancel{z_{1r} z_{2l}} - z_{1r} z_{2r} \right]$$

$$= z_{2l} z_{1l} - z_{1r} z_{2r}$$

Then

$$u_l - u_r = \begin{pmatrix} (z_{1l} - z_{1r})(z_{1l} + z_{1r}) \\ \bar{z}_2 (z_{1l} - z_{1r}) + \bar{z}_1 (z_{2l} - z_{2r}) \end{pmatrix} = \begin{pmatrix} 2\bar{z}_1 (z_{1l} - z_{1r}) \\ \bar{z}_2 (z_{1l} - z_{1r}) + \bar{z}_1 (z_{2l} - z_{2r}) \end{pmatrix}$$

$$= \begin{pmatrix} 2\bar{z}_1 & 0 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} z_{1l} - z_{1r} \\ z_{2l} - z_{2r} \end{pmatrix} = \begin{pmatrix} 2\bar{z}_1 & 0 \\ \bar{z}_2 & \bar{z}_1 \end{pmatrix} (z_l - z_r)$$

Also

$$f(u_l) - f(u_r) = \begin{pmatrix} z_{1l} z_{2l} - z_{1r} z_{2r} \\ a^2 (z_{1l}^2 - z_{1r}^2) + z_{2l}^2 - z_{2r}^2 \end{pmatrix} \longleftarrow$$ same as last component Above

$$= \begin{pmatrix} \bar{z}_2 (z_{1l} - z_{1r}) + \bar{z}_1 (z_{2l} - z_{2r}) \\ a^2 2\bar{z}_1 (z_{1l} - z_{1r}) + 2\bar{z}_2 (z_{2l} - z_{2r}) \end{pmatrix}$$

$$= \begin{pmatrix} \bar{z}_2 & \bar{z}_1 \\ 2a^2 \bar{z}_1 & 2\bar{z}_2 \end{pmatrix} \begin{pmatrix} z_{1l} - z_{1r} \\ z_{2l} - z_{2r} \end{pmatrix}$$

$$[z] = \hat{B}^{-1}[u]$$

$$\therefore [f] = \hat{C}\hat{B}^{-1}[u]$$

$$\hat{A}(u_e, u_r) = \begin{bmatrix} \bar{z}_2 & \bar{z}_1 \\ 2a^2\bar{z}_1 & 2\bar{z}_2 \end{bmatrix} \frac{1}{2\bar{z}_1^2} \begin{bmatrix} \bar{z}_1 & 0 \\ -\bar{z}_2 & 2\bar{z}_1 \end{bmatrix}$$

$$= \frac{1}{2\bar{z}_1^2} \begin{bmatrix} 0 & 2\bar{z}_1^2 \\ 2a^2\bar{z}_1^2 - 2\bar{z}_2^2 & 2\bar{z}_1\bar{z}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ a^2 - \left(\frac{\bar{z}_2}{\bar{z}_1}\right)^2 & \left(\frac{\bar{z}_2}{\bar{z}_1}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a^2 - \bar{v}^2 & \bar{v} \end{bmatrix} \quad \text{eq } 14.53$$

w/ $\bar{v} = \frac{\bar{z}_2}{\bar{z}_1} = \frac{\rho_e^{1/2} v_e + \rho_r^{1/2} v_r}{\rho_e^{1/2} + \rho_r^{1/2}}$ eq 14.54

w/ $\hat{A}$ defined by $\hat{A}(u_e, u_r) = \begin{pmatrix} 0 & 1 \\ a^2 - \bar{v}^2 & 2\bar{v} \end{pmatrix}$

General Method to obtain Roe matrix terms : 8-14-01

Given $z_{1l} z_{2l} - z_{1r} z_{2r}$ we want to write this in terms of $\bar{z}_1, \bar{z}_2, (z_{1l} - z_{1r}), (z_{2l} - z_{2r})$ This is equivalent to the following problem

$x_l y_l - x_r y_r$ written in terms of

$\frac{1}{2}(x_l + x_r), \frac{1}{2}(y_l + y_r), x_l - x_r, y_l - y_r$.

Thus Notice that

$$x_l = \frac{1}{2}(x_l + x_r) + \frac{1}{2}(x_l - x_r)$$
$$y_l = \frac{1}{2}(y_l + y_r) + \frac{1}{2}(y_l - y_r)$$
$$x_r = \frac{1}{2}(x_l + x_r) - \frac{1}{2}(x_l - x_r)$$
$$y_r = \frac{1}{2}(y_l + y_r) - \frac{1}{2}(y_l - y_r)$$

putting these in the Above expression we get

$$x_l y_l - x_r y_r = \left(\bar{x} + \frac{1}{2}(\Delta x)\right)\left(\bar{y} + \frac{1}{2}\Delta y\right) - \left(\bar{x} - \frac{1}{2}\Delta x\right)\left(\bar{y} - \frac{1}{2}\Delta y\right)$$

$$= \cancel{\bar{x}\bar{y}} + \frac{\bar{x}}{2}\Delta y + \frac{\bar{y}}{2}\Delta x + \cancel{\frac{1}{4}\Delta x \Delta y}$$

$$- \cancel{\bar{x}\bar{y}} + \frac{\bar{x}}{2}\Delta y + \frac{\bar{y}}{2}\Delta x - \cancel{\frac{1}{4}\Delta x \Delta y}$$

$$= \bar{x}\Delta y + \bar{y}\Delta x$$

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Ex 14.4 eq 5.38 are

$$\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v^2/2 + \phi \\ \phi v \end{pmatrix}_x = 0$$

$\{\phi = gh\}$

Then following the development of a Roe matrix for the isotherme eqs we note that a characteristic of the transformation done there is that the flux is rewritten in terms of products thus let

$$\vec{Z} = \phi^{-1/2}\vec{U} = \begin{pmatrix} \frac{v}{\phi^{1/2}} \\ \phi^{1/2} \end{pmatrix} \equiv \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

Then $\vec{U} = \phi^{1/2}\vec{Z} = z_2\vec{Z} = \begin{pmatrix} z_1 z_2 \\ z_2^2 \end{pmatrix}$ good two double products

Also $\bar{f}(\bar{U}) = \begin{pmatrix} v^2/2 + \phi \\ \phi v \end{pmatrix} = \begin{pmatrix} \frac{z_1^2 z_2^2}{2} + z_2^2 \\ z_2^2 z_1 z_2 \end{pmatrix}$ Not so good. Many products of high order maybe difficult to factor $z_l - z_r$ out of this expression.

Based on simple form of 14.55 - 14.56 we look at eigenvalues of $f(u) = \begin{pmatrix} v^2/2 + \phi \\ \phi v \end{pmatrix}$

$$f'(u) = \begin{pmatrix} v & 1 \\ \phi & v \end{pmatrix} \qquad \lambda_{1,2} = v \pm \sqrt{\phi}$$

Now $v_\ell - v_r = \begin{pmatrix} z_{1\ell} z_{2\ell} - z_{1r} z_{2r} \\ z_{2\ell}^2 - z_{2r}^2 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 (z_{1\ell} - z_{1r}) + \bar{z}_1 (z_{2\ell} - z_{2r}) \\ 2\bar{z}_2 (z_{2\ell} - z_{2r}) \end{pmatrix}$

$$= \begin{pmatrix} \bar{z}_2 & \bar{z}_1 \\ 2\bar{z}_2 & 0 \end{pmatrix} \begin{pmatrix} z_{1\ell} - z_{1r} \\ z_{2\ell} - z_{2r} \end{pmatrix} = \begin{pmatrix} \bar{z}_2 & \bar{z}_1 \\ 2\bar{z}_2 & 0 \end{pmatrix} (\bar{z}_\ell - \bar{z}_r)$$

Now $H(v_\ell) - H(v_r) = \begin{pmatrix} \frac{z_{1\ell}^2 z_{2\ell}^2}{2} - \frac{z_{1r}^2 z_{2r}^2}{2} + z_{2\ell}^2 - z_{2r}^2 \\ z_{2\ell}^2 z_{1\ell} z_{2\ell} - z_{2r}^2 z_{1r} z_{2r} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2}(z_{1\ell} z_{2\ell} - z_{1r} z_{2r})(z_{1\ell} z_{2\ell} + z_{1r} z_{2r}) + 2\bar{z}_2 (z_{2\ell} - z_{2r}) \\ z_{2\ell}^3 z_{1\ell} - z_{2r}^3 z_{1r} \end{pmatrix}$$

\+ $z_{1\ell} z_{2\ell} - z_{1r} z_{2r} = \bar{z}_2 (z_{1\ell} - z_{1r}) + \bar{z}_1 (z_{2\ell} - z_{2r})$ from LeVeque book

$\frac{1}{2}\left(z_{1l}^2 z_{2l}^2 - z_{1r}^2 z_{2r}^2\right)$ Following notes on pg 8-14-01 4

$= \frac{1}{2}\left(x_l^2 y_l^2 - x_r^2 y_r^2\right) = \frac{1}{2}\left(\left(\bar{x}+\frac{1}{2}\Delta x\right)^2\left(\bar{y}+\frac{1}{2}\Delta y\right)^2 - \left(\bar{x}-\frac{\Delta x}{2}\right)^2\left(\bar{y}-\frac{\Delta y}{2}\right)^2\right)$

This does not simplify as required $\Rightarrow$ perhaps our definition of $\vec{z}$ is incorrect. What is a better choice for z?

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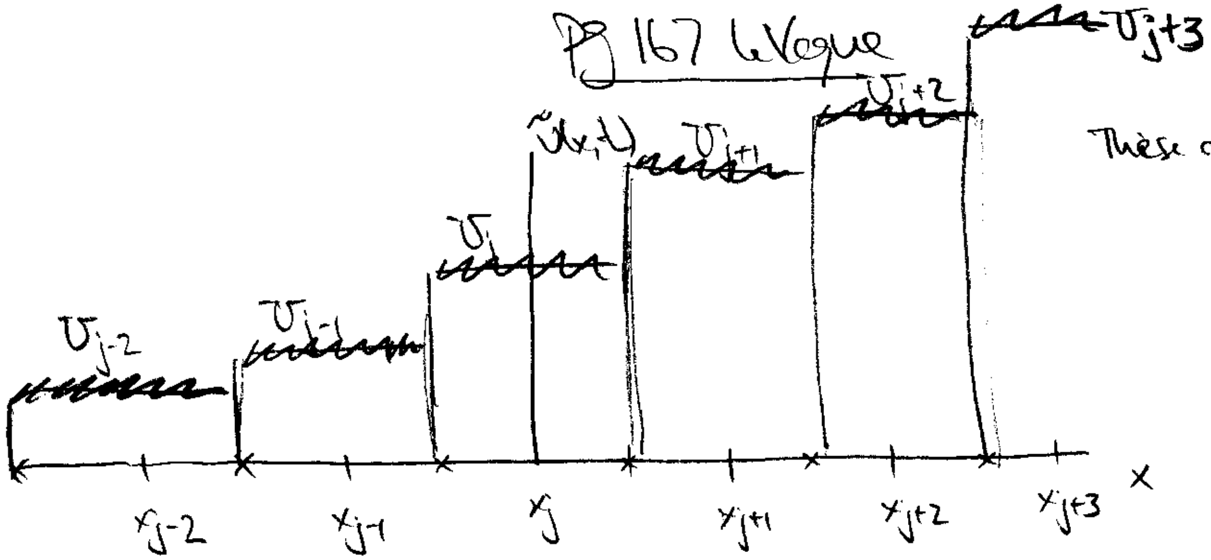
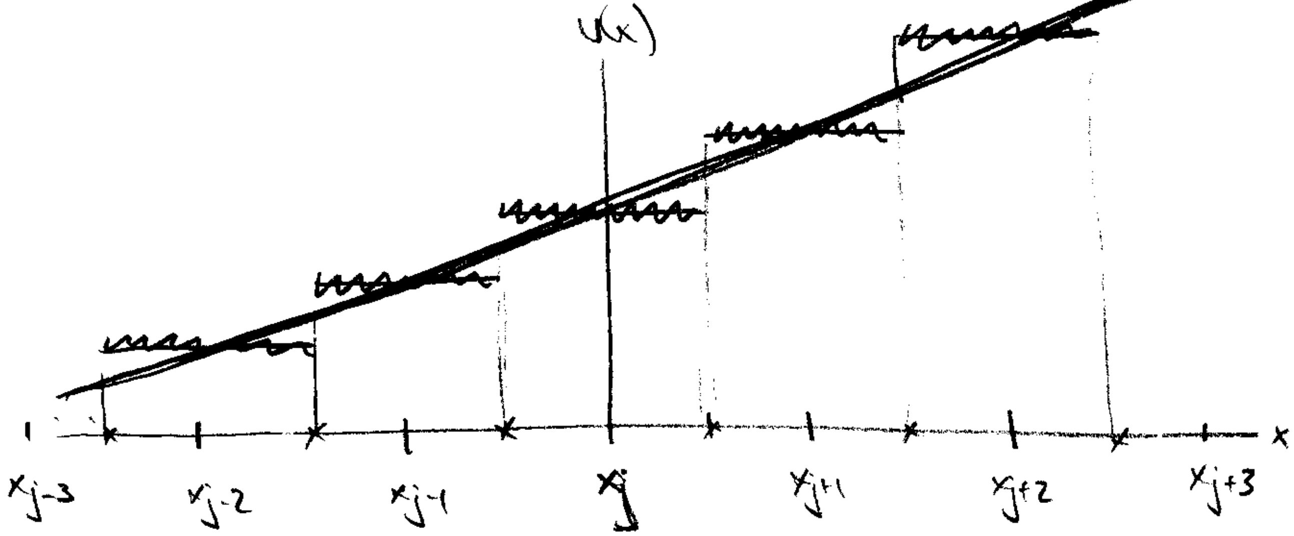


These are the fns $\tilde{v}(x)$

$$\|\tilde{v}(x)\|_1 = \int_{-\infty}^{\infty} |\tilde{v}(x)|\,dx = \sum_{j=-\infty}^{+\infty} \int_{x_j-h/2}^{x_j+h/2} |v(x)|\,dx = \sum_{j=-\infty}^{\infty} |v_j| h = h \sum_{j=-\infty}^{+\infty} |v_j|$$

$= \|v\|_1$ eq 15.25



Step fn is Restriction to constant cells given by Avg of

$$\|v\|_1 \equiv h \sum_{j=-\infty}^{+\infty} |v_j| = \sum_{j=-\infty}^{+\infty} h|\bar{v}_j|$$

Now consider

$$\int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx \geq \int_{x_j-h/2}^{x_j+h/2} u(x)\,dx = h\,\frac{1}{h}\int_{x_j-h/2}^{x_j+h/2} u(x)\,dx$$

$$= h\bar{u}_j$$

Then since $\int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx \geq 0$ $\quad \left|\int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx\right| = \int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx$

So that

$$h|\bar{u}_j| \leq \int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx$$

Then

$$\|\bar{u}\|_1 \leq \sum_{j=-\infty}^{+\infty} \int_{x_j-h/2}^{x_j+h/2} |u(x)|\,dx = \|u\|_1 \qquad \text{eq } 15.26$$

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$$TV(u) = \sum_{j=-\infty}^{\infty} |u_{j+1}^n - u_j^n| = \sum_{j=-\infty}^{\infty} |u_j^n - \underbrace{u_{j-1}^n}_{= v_j^n}| = \sum_{j=-\infty}^{\infty} |u_j^n - v_j^n|$$

$$= \frac{1}{h} \|u - v\|_1$$

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Ex 15.2 Lax-Friedrich 12.15 is

$$U_j^{n+1} = \frac{1}{2}(U_{j-1}^n + U_{j+1}^n) - \frac{k}{2h}(f(U_{j+1}^n) - f(U_{j-1}^n))$$

& for V_j^n is

$$V_j^{n+1} = \frac{1}{2}(V_{j-1}^n + V_{j+1}^n) - \frac{k}{2h}(f(V_{j+1}^n) - f(V_{j-1}^n))$$

Defining $W_j^n = U_j^n - V_j^n$ we get

$$W_j^{n+1} = \frac{1}{2}(W_{j-1}^n + W_{j+1}^n) - \frac{k}{2h}\Big[(f(U_{j+1}^n) - f(V_{j+1}^n))$$
$$- (f(U_{j-1}^n) - f(V_{j-1}^n))\Big]$$

Again by smoothness of f (mean value thm)

$$f(U_j^n) - f(V_j^n) = f'(\theta_j^n)(U_j^n - V_j^n) \qquad \theta_j^n \text{ is } \underline{\text{between}}\ U_j^n \ \&\ V_j^n$$

$\Rightarrow$

$$W_j^{n+1} = \frac{1}{2}(W_{j-1}^n + W_{j+1}^n) - \frac{k}{2h}f'(\theta_{j+1}^n)W_{j+1}^n + \frac{k}{2h}f'(\theta_{j-1}^n)W_{j-1}^n$$

$$= \frac{1}{2}\Big[1 + \frac{k}{h}f'(\theta_{j-1}^n)\Big]W_{j-1}^n + \frac{1}{2}\Big[1 - \frac{k}{h}f'(\theta_{j+1}^n)\Big]W_{j+1}^n$$

$$-1 \le \frac{k}{h}f'(u) \le +1 \qquad \forall \quad \min_j(U_j^n, V_j^n) \le u \le \max_j(U_j^n, V_j^n)$$

$$\therefore \quad 0 \leq 1 + \frac{k}{h} f'(\theta_{j-1}^n) \leq 2 \quad \Rightarrow \quad 0 \leq \frac{1}{2}\left(1 + \frac{k}{h} f'(\theta_{j-1}^n)\right) \leq 1$$

$$+ \quad 0 \leq 1 - \frac{k}{h} f'(\theta_{j+1}^n) \leq 2 \qquad 0 \leq \frac{1}{2}\left(1 - \frac{k}{h} f'(\theta_{j+1}^n)\right) \leq 1$$

So that

$$|\bar{w}_j^{n+1}| \leq \frac{1}{2}(1+\alpha_{j-1})|\bar{w}_{j-1}^n| + \frac{1}{2}(1-\alpha_{j+1})|\bar{w}_{j+1}^n|$$

$$= \frac{1}{2}|\bar{w}_{j-1}^n| + \frac{1}{2}|\bar{w}_{j+1}^n| + \frac{\alpha_{j-1}}{2}|\bar{w}_{j-1}^n| - \frac{\alpha_{j+1}}{2}|\bar{w}_{j+1}^n|$$

Multiplying by h & summing from $j = -\infty$ to $+\infty$ gives

$$h\sum_{j=-\infty}^{+\infty} |\bar{w}_j^{n+1}| \leq h\sum_{j=-\infty}^{+\infty} |\bar{w}_j^n| + 0$$

$\rightsquigarrow$

$$\|\bar{w}^{n+1}\|_1 \leq \|\bar{w}^n\|_1$$

LW

$$U_j^{n+1} = U_j^n - \frac{\nu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}\nu^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$= U_j^n - \nu(U_j^n - U_{j-1}^n) + \nu U_j^n - \frac{\nu}{2}U_{j-1}^n - \frac{\nu}{2}U_{j+1}^n$$

$$+ \frac{1}{2}\nu^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$= U_j^n - \nu(U_j^n - U_{j-1}^n) - \frac{\nu}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$+ \frac{1}{2}\nu^2(\Delta_j^2 U_j^n)$$

$$= U_j^n - \nu(U_j^n - U_{j-1}^n) - \frac{1}{2}\nu(1-\nu)(U_{j+1}^n - 2U_j^n + U_{j-1}^n)$$

$$\nu = \frac{ak}{h}$$

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[\left(aU_j^n + \frac{1}{2}a(1-\nu)(U_{j+1}^n - U_j^n)\right)\right.$$

$$\left. - \left(aU_{j-1}^n + \frac{1}{2}a(1-\nu)(U_j^n - U_{j-1}^n)\right)\right]$$

$$\therefore \quad F(U;j) = aU_j + \frac{1}{2}a(1-\nu)(U_{j+1} - U_j)$$

)))

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w/o U_j^n

$$-\nu(U_j^n) + \nu U_{j-1}^n - \frac{1}{2}\Big[\nu U_{j+1}^n - 2\nu U_j^n + \nu U_{j-1}^n$$
$$- \nu^2 U_{j+1}^n + 2\nu^2 U_j^n - \nu^2 U_{j-1}^n\Big]$$

✓

$$= \nu(\underbrace{+1 - \tfrac{1}{2}}_{1/2} + \tfrac{1}{2}\nu)U_{j-1}^n + \nu(\cancel{-1} + \cancel{1} - \nu)U_j^n + \nu(-\tfrac{1}{2} + \tfrac{\nu}{2})U_{j+1}^n$$

$$= -\frac{1}{2}\nu(U_{j+1}^n - U_{j-1}^n) + \frac{1}{2}\nu^2(U_{j+1}^n - 2U_j^n + U_{j-1}^n) \quad ✓$$

Ex 16.1

$$F(U;j) = aU_j + \frac{1}{2}a(1-\nu)(U_{j+1} - U_j)\phi(\theta_j)$$

Then method is

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U^n_{j-p}, U^n_{j-p+1}, \ldots, U^n_{j+q}) - F(U^n_{j-p-1}, U^n_{j-p}, \ldots, U^n_{j+q+1})\right]$$

thus

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Big\{ aU_j^n + \frac{1}{2}a(1-\nu)(U^n_{j+1} - U^n_j)\phi(\theta_j)$$

$$- aU^n_{j-1} - \frac{1}{2}a(1-\nu)(U^n_j - U^n_{j-1})\phi(\theta_{j-1})\Big\}$$ ✓

$$\Rightarrow \cancel{U^n_j} + k(U_t)^n_j + \frac{k^2}{2}(U_{tt})^n_j + \frac{k^3}{6}(U_{ttt})^n_j + O(k^4)$$

$$= \cancel{U^n_j} - \frac{k}{h}\Big\{ a\cancel{U^n_j} + \frac{1}{2}a(1-\nu)\left(h(U_x)^n_j + \frac{h^2}{2}(U_{xx})^n_j + \frac{h^3}{6}(U_{xxx})^n_j + O(h^4)\right)\phi(\theta_j)$$

$$-a\left[\cancel{U^n_j} - h(U_x)^n_j + \frac{h^2}{2}(U_{xx})^n_j - \frac{h^3}{6}(U_{xxx})^n_j + O(h^4)\right]$$

$$-\frac{1}{2}a(1-\nu)\left(\cancel{U^n_j} - \left(\cancel{U^n_j} - h(U_x)^n_j + \frac{h^2}{2}(U_{xx})^n_j - \frac{h^3}{6}(U_{xxx})^n_j + O(h^4)\right)\right)\phi(\theta_{j-1})\Big\}$$ ✓

$$\Rightarrow (v_t)_j^n + \frac{k}{2}(v_{tt})_j^n + \frac{k^2}{6}(v_{ttt})_j^n + O(k^3)$$

$$= -\frac{1}{h}\Big\{ \frac{1}{2}a(1-\nu)\Big(h(v_x)_j^n + \frac{h^2}{2}(v_{xx})_j^n + \frac{h^3}{6}(v_{xxx})_j^n + O(h^4)\Big)\phi(\theta_j)$$

$$+ ah(v_x)_j^n - \frac{ah^2}{2}(v_{xx})_j^n + \frac{ah^3}{6}(v_{xxx})_j^n + O(h^4)$$

$$+ \frac{1}{2}a(1-\nu)\Big(-h(v_x)_j^n + \frac{h^2}{2}(v_{xx})_j^n - \frac{h^3}{6}(v_{xxx})_j^n + O(h^4)\Big)\phi(\theta_{j-1})\Big\} \checkmark$$

$\Rightarrow$

$$(v_t)_j^n + \frac{k}{2}(v_{tt})_j^n + \frac{k^2}{6}(v_{ttt})_j^n + O(k^3)$$

$$= -\Big\{ \frac{1}{2}a(1-\nu)\Big((v_x)_j^n + \frac{h}{2}(v_{xx})_j^n + \frac{h^2}{6}(v_{xxx})_j^n + O(h^3)\Big)\phi(\theta_j)$$

$$+ a(v_x)_j^n - \frac{ah}{2}(v_{xx})_j^n + \frac{ah^2}{6}(v_{xxx})_j^n + O(h^3)$$

$$- \frac{1}{2}a(1-\nu)\Big((v_x)_j^n - \frac{h}{2}(v_{xx})_j^n + \frac{h^2}{6}(v_{xxx})_j^n + O(h^3)\Big)\phi(\theta_{j-1})\Big\} \checkmark$$

$$\Rightarrow (v_t)_j^n + a(v_x)_j^n = -\frac{k}{2}(v_{tt})_j^n - \frac{k^2}{6}(v_{ttt})_j^n + O(k^3)$$

$$+ \frac{ah}{2}(v_{xx})_j^n - \frac{ah^2}{6}(v_{xxx})_j^n + O(h^3)$$

$$-\frac{a}{2}(1-\nu)\Big[(v_x)_j^n(\phi(\theta_j)-\phi(\theta_{j-1}))+\frac{h}{2}(v_{xx})_j^n(\phi(\theta_j)+\phi(\theta_{j-1}))$$

$$+\frac{h^2}{6}(v_{xxx})_j^n(\phi(\theta_j)-\phi(\theta_{j-1}))+O(h^3)\Big]$$

Now $\phi(\theta_j)-\phi(\theta_{j-1})$

$$=\phi\left(\frac{v_j-v_{j-1}}{v_{j+1}-v_j}\right)-\phi\left(\frac{v_{j-1}-v_{j-2}}{v_j-v_{j-1}}\right)$$

So expanding θ_j 1st

$$\theta_j=\frac{v_j-v_{j-1}}{v_{j+1}-v_j}=\frac{v_j-\left(v_j-h(v_x)_j+\frac{h^2}{2}(v_{xx})_j+O(h^3)\right)}{h(v_x)_j+\frac{h^2}{2}(v_{xx})_j+O(h^3)}$$

$$=\frac{h(v_x)_j-\frac{h^2}{2}(v_{xx})_j+O(h^3)}{h(v_x)_j+\frac{h^2}{2}(v_{xx})_j+O(h^3)}$$

$$=\frac{(v_x)_j-\frac{h}{2}(v_{xx})_j+O(h^2)}{(v_x)_j+\frac{h}{2}(v_{xx})_j+O(h^2)}$$

Assuming $(v_x)_j \neq 0$

$$=\frac{1-\frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j}+O(h^2)}{1+\frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j}+O(h^2)}$$

$$\theta_j \approx \left(1 - \frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j} + O(h^2)\right)\left(1 - \frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j} + O(h^2)\right)$$

$$= 1 - \frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j} - \frac{h}{2}\frac{(v_{xx})_j}{(v_x)_j} + O(h^2)$$

Now: $v_t + a v_x = 0$

$v_{tt} + a(v_t)_x = 0$

$v_{tt} - a^2 v_{xx} = 0$

Then $-\frac{ka^2}{2} v_{xx} + \frac{ah}{2} v_{xx}$

$(ka - h) v_{xx}$

$\nu = \frac{k}{h} a$

Method: $U_j^{n+1} = U_j^n - \frac{k}{h}\left(F(U^n, j) - F(U^n, j-1) \right)$

w/ Flux $F(U^n, j) = aU_j + \frac{1}{2} a(1-\nu)(U_{j+1} - U_j)\phi_j$

Then

$$U_j^{n+1} = U_j - \frac{k}{h}\left[aU_j + \frac{1}{2}a(1-\nu)(U_{j+1} - U_j)\phi_j - aU_{j-1} - \frac{1}{2}a(1-\nu)(U_j - U_{j-1})\phi_{j-1} \right]$$

$$= U_j - \frac{k}{h}\left(a(U_j - U_{j-1}) + \frac{1}{2}a(1-\nu)\left[(U_{j+1} - U_j)\phi_j - (U_j - U_{j-1})\phi_{j-1} \right] \right)$$

$$= U_j - \nu(U_j - U_{j-1}) - \frac{1}{2}\nu(1-\nu)\left[(U_{j+1} - U_j)\phi_j - (U_j - U_{j-1})\phi_{j-1} \right]$$

$$= U_j + \left[-\nu + \frac{\nu}{2}(1-\nu)\phi_{j-1} \right](U_j - U_{j-1})$$

$$+ \left[-\frac{1}{2}\nu(1-\nu)\phi_j \right](U_{j+1} - U_j)$$

$$\Rightarrow U_j^{n+1} = U_j - \left(\nu - \frac{\nu}{2}(1-\nu)\phi_{j-1}\right)(U_j - U_{j-1})$$

$$- \frac{1}{2}\nu(1-\nu)\phi_j (U_{j+1} - U_j) \qquad \text{eq } 16.17$$

Consider

$$U_{j+1}^{n+1} - U_j^{n+1} = U_{j+1} - C_j(U_{j+1} - U_j) + D_{j+1}(U_{j+2} - U_{j+1})$$

$$- U_j + C_{j-1}(U_j - U_{j-1}) - D_j(U_{j+1} - U_j)$$

$$\therefore \quad v_{j+1}^{n+1} - v_j^{n+1} = C_{j-1}(v_j - v_{j-1})$$

$$+ \; v_{j+1} - v_j - (C_j + D_j)(v_{j+1} - v_j)$$

$$+ \; D_{j+1}(v_{j+2} - v_{j+1})$$

$$= (1 - C_j - D_j)(v_{j+1} - v_j) + D_{j+1}(v_{j+2} - v_{j+1})$$

$$+ \; C_{j-1}(v_j - v_{j-1})$$

$$|v_{j+1}^{n+1} - v_j^{n+1}| \leq |1 - C_j - D_j||v_{j+1} - v_j| + |D_{j+1}||v_{j+2} - v_{j+1}|$$

$$+ |C_{j-1}||v_j - v_{j-1}|$$

Summing over j

$$\sum_{j=-\infty}^{+\infty} |v_{j+1}^{n+1} - v_j^{n+1}| \leq \sum_{j=-\infty}^{+\infty} |1 - C_j - D_j||v_{j+1} - v_j| + \sum_{j=-\infty}^{+\infty} |D_{j+1}||v_{j+2} - v_{j+1}|$$

$$+ \sum_{j=-\infty}^{+\infty} |C_{j-1}||v_j - v_{j-1}|$$

& If $\quad C_j + D_j \leq 1 \quad \forall j$

$$D_j \geq 0 \quad \forall j$$

$$\& \; C_j \geq 0 \quad \forall j$$

Then $$\sum_{j=-\infty}^{+\infty} \left| v^{n+1}_{j+1} - v^{n+1}_{j} \right| \le \sum_{j=-\infty}^{+\infty} (1 - C_j - D_j) \left| v_{j+1} - v_j \right|$$

$$+ \sum_{j=-\infty}^{+\infty} D_{j+1} \left| v_{j+2} - v_{j+1} \right| + \sum_{j=-\infty}^{+\infty} C_{j-1} \left| v_j - v_{j-1} \right|$$

$$\sum_{j=-\infty}^{+\infty} \left| v^{n+1}_{j+1} - v^{n+1}_{j} \right| \le \sum_{j=-\infty}^{+\infty} (1 - C_j - D_j) \left| v_{j+1} - v_j \right| + \sum_{j=-\infty}^{+\infty} D_j \left| v_{j+1} - v_j \right|$$

$$+ \sum_{j=-\infty}^{+\infty} C_j \left| v_{j+1} - v_j \right|$$

$$= \sum_{j=-\infty}^{+\infty} \left| v_{j+1} - v_j \right|$$

Ex 16.2 see notes above

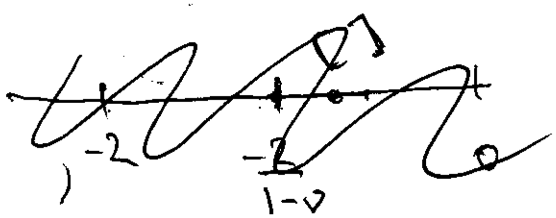
$$0 \le \nu \left\{ 1 + \tfrac{1}{2}(1-\nu)\left[\frac{\phi(\theta_j)}{\theta_j} - \phi(\theta_{j-1}) \right] \right\} \le 1$$

$$|\nu| \le 1$$

$$-1 < \nu < 0$$

$$\Rightarrow \quad 0 \ge \left\{ 1 + \tfrac{1}{2}(1-\nu)\left[\quad \right] \right\} \ge \frac{1}{\nu} = \frac{-1}{|\nu|}$$

$$\frac{-2}{1-\nu} \ge \left[\quad \right] \ge \frac{2\left(1 - \frac{1}{|\nu|}\right)}{1-\nu} = \frac{2(-\nu-1)}{-\nu(1-\nu)}$$



$$\frac{2(|\nu|-1)}{|\nu|(1+|\nu|)} \le [\quad\quad] \le \frac{-2}{1+|\nu|} \qquad \text{w/ } |\nu|<1$$

$$\Rightarrow \quad \frac{-2(1-|\nu|)}{|\nu|(1+|\nu|)} \le [\quad\quad] \le \frac{-2}{1+|\nu|}$$

$$-2 \le \frac{-2\left(\frac{1}{|\nu|}-1\right)}{1+|\nu|} \le [\quad] \le \frac{-2}{1+|\nu|} \le 0$$

↑

to show need that $0 < \dfrac{\frac{1}{|\nu|}-1}{1+|\nu|} \overset{?}{<} 1 \Leftarrow 1-|\nu| < |\nu|+|\nu|^2$

$$\Leftarrow \quad |\nu|^2+2|\nu|-1 \overset{?}{>} 0$$

$(|\nu|-1)^2 > 0$ ✓. true.

If $0 < \nu < 1$ $\quad 0 \le g_{-1} \le 1$

$\Rightarrow$

$$\frac{2(0-1)}{1-\nu} \le [\quad] \le \frac{2(1/\nu-1)}{1-\nu}$$

$$\frac{-2}{1-\nu} \le [\quad] \le 2 \quad ?$$

or $\frac{2}{1-\nu} \ge -[\quad] \ge \frac{-2(1/\nu-1)}{1-\nu}$

$\cancel{\ldots}^2$

$\nu^2 \cancel{-2\nu+1<0}$

$\cancel{(\nu-1)^2<0}$ not!

know $\frac{\frac{1}{\nu}-1}{1-\nu} \not< 1$

$\Rightarrow \frac{\frac{1}{\nu}-1}{1-\nu} > 1$

Sec 7? Still Don't

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$1 > \nu > 0$

$$\frac{-2}{1-\nu} = \frac{2(0-1)}{(1-\nu)} \leq \frac{\phi(\theta_j)}{\theta_j} - \phi(\theta_{j-1}) \leq \frac{2(1/\nu - 1)}{(1-\nu)} = \frac{2}{\nu}$$

$1 > 1-\nu > 0$

Then

$$\frac{-2}{\nu} \leq -\left(\frac{\phi(\theta_j)}{\theta} - \phi(\theta_{j-1})\right) \leq \frac{2}{1-\nu}$$

)))

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$$\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{U}_j^n(x,t_n)\,dx = U_j^n + \sigma_j^n \left(\frac{x^2}{2} - x_j x \right|_{x_{j-1/2} = x_j - \frac{h}{2}}^{x_j + \frac{h}{2}}$$

$$= U_j^n + \sigma_j^n \left[\frac{(x_j + \frac{h}{2})^2 - (x_j - \frac{h}{2})^2}{2} - x_j(x_j + \tfrac{h}{2}) + x_j(x_j - \tfrac{h}{2}) \right]$$

$$\| $$

$$0 \checkmark .$$

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$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\right]$$

w/ $F(U_j^n, U_{j+1}^n) = \frac{1}{k}\int_{t_n}^{t_{n+1}} f(\tilde{u}^n(x_{j+1/2}, t))\,dt$

let $\tilde{u}^n(x, t_n) = U_j^n + \sigma_j^n (x - x_j)$

If $\sigma_j^n = \dfrac{U_{j+1}^n - U_j^n}{h}$

Then if $f(u) = au$ $(a>0)$

$$F(U_j^n, U_{j+1}^n) = \frac{a}{k}\Big[hU_j^n + \sigma_j^n\left(x_j + \tfrac{h}{2} - x_j\right)$$

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$$\nu = \frac{ka}{h}$$

$$F(U;j) = aU_j + \frac{1}{2}a(1-\nu)h\sigma_j$$

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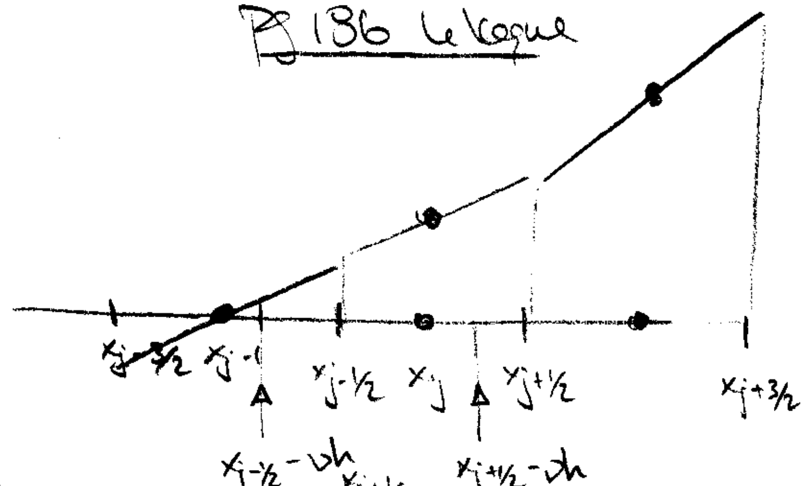
$$F(U;j) = aU_{j_1} + \frac{1}{2}a(\mathrm{sgn}(\nu) - \nu)(h\sigma_{j_1})$$

Ex 16.3



By step 3

$$U_j^{n+1} = \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}(x, t_{n+1})\,dx = \int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{u}^n(x - ak, t_n)\,dx$$

$$= \int_{x_{j-1/2}-ak}^{x_{j+1/2}-ak} \tilde{u}^n(x, t_n)\,dx = \int_{x_{j-1/2}-\nu h}^{x_{j+1/2}-\nu h} \tilde{u}^n(x, t_n)\,dx$$

If $a > 0$.

since $\left.\begin{matrix} \\ ak = \nu h \end{matrix}\right\}$

& $\nu < 1$

so $\nu h < h$

$$= \int_{x_{j-1/2}-\nu h}^{x_{j-1/2}} \left(U_{j-1}^n + \sigma_{j-1}^n (x - x_{j-1})\right) dx$$

$$+ \int_{x_{j-1/2}}^{x_{j+1/2}-\nu h} \left(U_j^n + \sigma_j^n (x - x_j)\right) dx$$

$$= U_{j-1}^n \left(x_{j-1/2} - x_{j-1/2} + \nu h\right) + \sigma_{j-1}^n \int_{x_{j-1/2}-\nu h}^{x_{j-1/2}} (x - x_{j-1})\,dx$$

$$+ \sigma_j^n (x_{j+1/2} - \nu h - x_{j-1/2}) + \delta_j^n \int_{x_{j-1/2}}^{x_{j+1/2}-\nu h} (x - x_j)\, dx$$

$$= \sigma_{j-1}^n \nu h + \delta_{j-1}^n \int_{x_{j-1/2}-\nu h - x_{j-1}}^{x_{j-1/2} - x_{j-1}} r\, dr \quad + \sigma_j^n (h - \nu h) + \delta_j^n \int_{x_{j-1/2}-x_j}^{x_{j+1/2}-\nu h - x_j} r\, dr$$

$$= \sigma_{j-1}^n \nu h + \delta_{j-1}^n \int_{h/2-\nu h}^{h/2} r\, dr \quad + \sigma_j^n h(1-\nu) + \delta_j^n \int_{-h/2}^{h/2-\nu h} r\, dr$$

$$= \sigma_{j-1}^n \nu h + \delta_{j-1}^n \left(\frac{h^2}{2\cdot 4} - \frac{1}{2}\left(\frac{h}{2} - \nu h\right)^2 \right)$$

$$+ \sigma_j^n h(1-\nu) + \delta_j^n \left(\frac{1}{2}\left(\frac{h}{2} - \nu h\right)^2 - \frac{h^2}{2\cdot 4} \right)$$

$$= \nu h \sigma_{j-1}^n + \delta_{j-1}^n h^2 \left[\frac{1}{8} - \frac{1}{8}(1-2\nu)^2 \right]$$

$$+ \sigma_j^n h(1-\nu) + \frac{\delta_j^n}{8}\left[(1-2\nu)^2 - 1 \right] h^2$$

$$= \nu h U_{j-1}^n + \frac{h^2 \sigma_{j-1}^n}{8}\left[\cancel{1} - 1 + 4\nu - 4\nu^2\right]$$

$$+ h(1-\nu)U_j^n + \frac{h^2 \sigma_j^n}{8}\left[\cancel{1} - 4\nu + 4\nu^2 - \cancel{1}\right]$$

$$= \nu h U_{j-1}^n + \frac{h^2}{2}\nu \sigma_{j-1}^n (1-\nu) + hU_j^n - h\nu U_j^n$$

$$- \frac{h^2}{2}\nu(1-\nu)\sigma_j^n$$

Since Average is defined as $\frac{1}{h}\int_{x_{j-1/2}}^{x_{j+1/2}} \tilde{U}(x, t_{n+1})\,dx$ & I forgot the $\frac{1}{h}$

we get

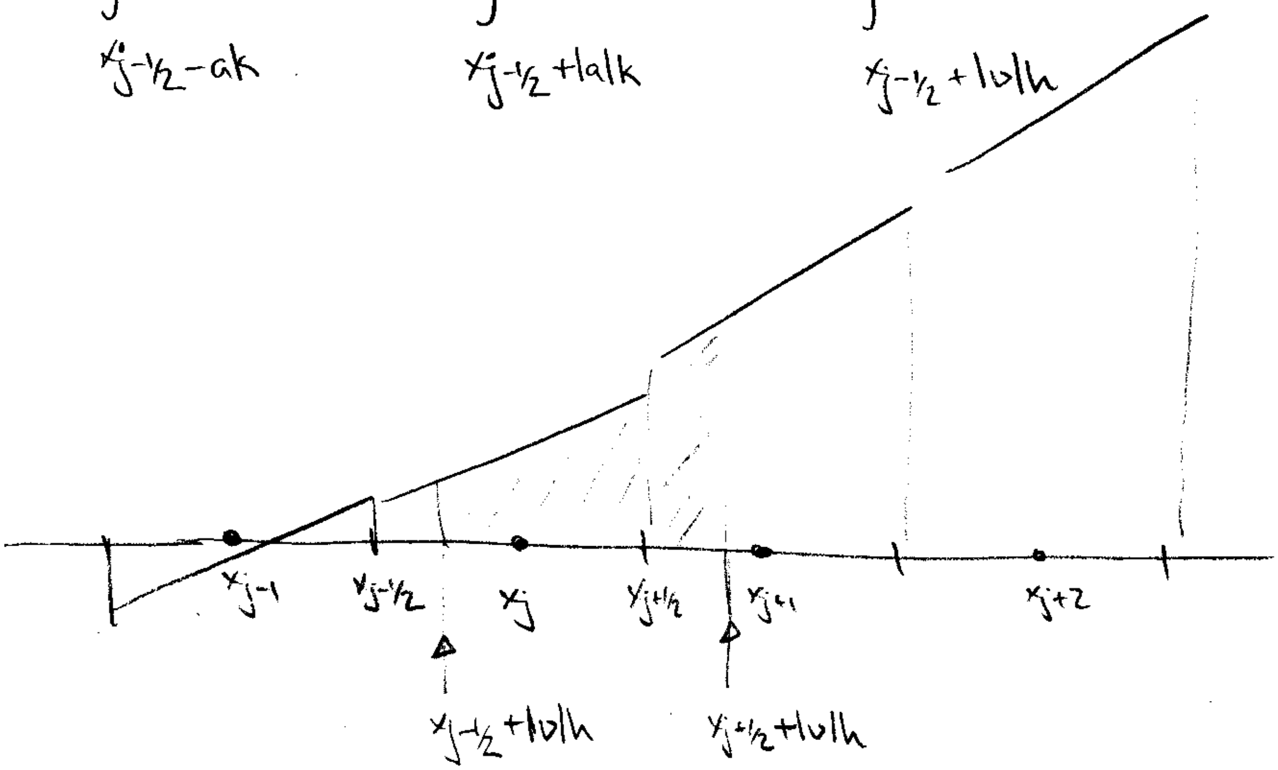
$$U_j^{n+1} = U_j^n + \nu(U_{j-1}^n - U_j^n) + \frac{h}{2}\nu(1-\nu)(\sigma_{j-1}^n - \sigma_j^n)$$

or

$$U_j^{n+1} = U_j^n - \nu(U_j^n - U_{j-1}^n) - \frac{1}{2}\nu(1-\nu)(h\sigma_j^n - h\sigma_{j-1}^n) \quad \text{eq 16.45}$$

If $a < 0$ Then

$$v_j^{n+1} = \int_{x_{j-1/2}-ak}^{x_{j+1/2}-ak} \tilde{v}^n(x,t_n)\,dx = \int_{x_{j-1/2}+|a|k}^{x_{j+1/2}+|a|k} \tilde{v}^n(x,t_n)\,dx = \int_{x_{j-1/2}+|\nu|h}^{x_{j+1/2}+|\nu|h} \hat{v}^n(x,t_n)\,dx$$



$$= \int_{x_{j-1/2}+|\nu|h}^{x_{j+1/2}} \left(v_j^n + \sigma_j^n(x-x_j)\right)dx + \int_{x_{j+1/2}}^{x_{j+1/2}+|\nu|h} \left(v_{j+1}^n + \sigma_{j+1}^n(x-x_{j+1})\right)dx$$

$$= v_j^n\left(x_{j+1/2} - x_{j-1/2} - |\nu|h\right) + \sigma_j^n \int_{x_{j-1/2}+|\nu|h}^{x_{j+1/2}} (x-x_j)\,dx$$

$$+ v_{j+1}^n\left(x_{j+1/2} + h|\nu| - x_{j+1/2}\right) + \sigma_{j+1}^n \int_{x_{j+1/2}}^{x_{j+1/2}+|\nu|h} (x-x_{j+1})\,dx$$

$$= v_j^n(h-|\nu|h) + \sigma_j^n \int_{-\frac{h}{2}+|\nu|h}^{h/2} v\,dv + v_{j+1}^n h|\nu| + \sigma_{j+1}^n \int_{h/2}^{-h/2+|\nu|h} v\,dv$$

$$= h v_j^n(1-|\nu|) + \frac{\sigma_j^n}{2}\left(\frac{h^2}{4} - \left(-\frac{h}{2}+|\nu|h\right)^2\right) + v_{j+1}^n h|\nu| + \frac{\sigma_{j+1}^n}{2}\left[\left(-\frac{h}{2}+|\nu|h\right)^2 - \frac{h^2}{4}\right]$$

$$= h v_j^n(1-|\nu|) + \frac{\sigma_j^n}{2}h^2\left[\cancel{\tfrac{1}{4}} - \left(\cancel{\tfrac{1}{4}} - |\nu| + |\nu|^2\right)\right] + v_{j+1}^n h|\nu|$$

$$+ \frac{\sigma_{j+1}^n}{2}h^2\left[\left(\cancel{\tfrac{1}{4}} - |\nu| + |\nu|^2\right) - \cancel{\tfrac{1}{4}}\right]$$

$$= h v_j^n(1-|\nu|) + \frac{\sigma_j^n h^2}{2}|\nu|(1-|\nu|) + v_{j+1}^n h|\nu| - \frac{\sigma_{j+1}^n h^2}{2}|\nu|(1-|\nu|)$$

$\div$ by h give

$$v_j^{n+1} = v_j^n(1-|\nu|) + v_{j+1}^n|\nu| + \frac{h}{2}\sigma_j^n|\nu|(1-|\nu|) - \frac{h}{2}\sigma_{j+1}^n|\nu|(1-|\nu|)$$

$$= v_j^n + |\nu|(v_{j+1}^n - v_j^n) + \frac{1}{2}|\nu|(1-|\nu|)(h\sigma_j^n - h\sigma_{j+1}^n)$$

$$\Rightarrow v_j^{n+1} = v_j^n + |\nu|(v_{j+1}^n - v_j^n) - \frac{|\nu|}{2}(1-|\nu|)(h\sigma_{j+1}^n - h\sigma_j^n)$$

Now: when $a>0$, $\nu>0$

$$v_j^{n+1} = v_j^n - \nu(v_j^n - v_{j-1}^n) - \frac{1}{2}\nu(1-\nu)(h\sigma_j^n - h\sigma_{j-1}^n)$$

& when $a<0$ $\nu<0$

$$v_j^{n+1} = v_j^n + |\nu|(v_{j+1}^n - v_j^n) - \frac{|\nu|}{2}(1-|\nu|)(h\sigma_{j+1}^n - h\sigma_j^n)$$

These can be combined into 1 formula as

$$v_j^{n+1} = v_j^n - \nu(v_{j_1}^n - v_{j_1-1}^n) - \frac{|\nu|}{2}(\operatorname{sign}(\nu) - \nu)(h\sigma_{j_1}^n - h\sigma_{j_1-1}^n)$$

$$j_1 = \begin{cases} j & a>0 \\ j+1 & a<0 \end{cases}$$

I got $|\nu|$ rather than ν in the choice

Method 16.4B is $U_j^{n+1} = U_j^n - \nu(U_{j1}^n - U_{j1-1}^n) - \frac{1}{2}\nu(\mathrm{sgn}(\nu) - \nu)(h\sigma_{j1} - h\sigma_{j1-1})$

Used to update the pth family.

$$V_{pj}^{n+1} = V_{pj} - \nu_p(V_{p,jp} - V_{p,jp-1}) - \frac{1}{2}\nu_p(\mathrm{sgn}(\nu_p) - \nu_p)(h\beta_{p,jp} - h\beta_{p,jp-1})$$

?

$$\beta_{pj} = \frac{1}{h}\,\mathrm{minmod}(V_{p,j+1} - V_{p,j}, V_{pj} - V_{p,j-1}) = \frac{1}{h}\,\mathrm{minmod}(\alpha_{pj}, \alpha_{p,j-1})$$

Multiply by r_p + sum over p.

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Bigg[\sum_{p=1}^{m}(V_{p,jp}\lambda_p r_p - V_{p,jp-1}\lambda_p r_p)$$

$$+\frac{1}{2}\sum_{p=1}^{m}\lambda_p(\mathrm{sgn}(\nu_p) - \nu_p)(h\beta_{p,jp}r_p - h\beta_{p,jp-1}r_p)\Bigg]$$

Defining $\beta_{p,jp}r_p = \sigma_{pj}$ + renum

$$F_L(U;j) = \sum_{p=1}^{m} V_{p,jp}\lambda_p r_p$$

$$U_j^{n+1} = U_j^n - \frac{k}{h}\Bigg[\sum_{p=1}^{m}V_{p,jp}\lambda_p r_p + \frac{1}{2}\sum_{p=1}^{m}\lambda_p(\mathrm{sgn}(\nu_p) - \nu_p)h\sigma_{pjp}$$

$$-\sum_{p=1}^{m}V_{p,jp-1}\lambda_p r_p - \frac{1}{2}\sum_{p=1}^{m}\lambda_p(\mathrm{sgn}(\nu_p) - \nu_p)h\sigma_{p,jp-1}\Bigg]$$

Thus flux $F(U;j) = F_L(U;j)$

$$+ \frac{1}{2}\sum_{p=1}^{m} \lambda_p(\mathrm{sgn}(\nu_p) - \nu_p) h \sigma_{pjp}$$

Flux 16.40 is

$$F(U;j) = F(U;j) + \frac{1}{2}\sum_{p=1}^{m} \phi(\theta_{pj})(\mathrm{sgn}(\nu_p) - \nu_p)\lambda_p \alpha_{pj} r_p$$

& the same flux is obtained if we choose $\sigma_{pjp} = \frac{\phi(\theta_{pj})}{h}\alpha_{pj} r_p$

eq 16.61

Ex 16.4

By 16.58 & 16.56

$$\sigma_{pj} = \beta_{pj} r_p = \frac{1}{h}\mathrm{minmod}(\alpha_{pj}, \alpha_{pj-1}) r_p = \text{LHS.}$$

By 16.37 & ϕ minmod limiter 16.53

$$\theta_{pj} = \frac{\alpha_{pj'}}{\alpha_{pj}}$$

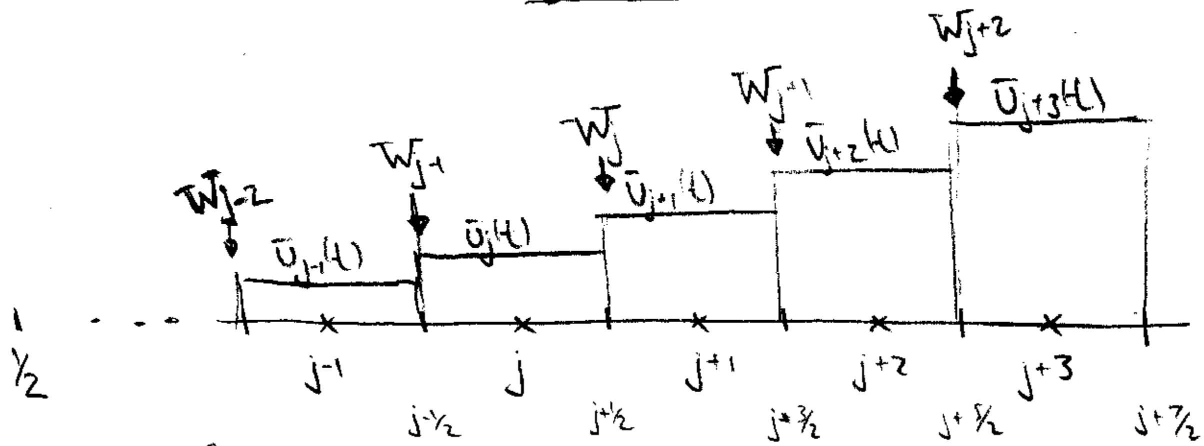
w/ $j' = j - \mathrm{sgn}(\nu_p)$

Then RHS is

$$\frac{1}{h}\phi_{minmod}\left(\frac{\alpha_{pj'}}{\alpha_{pj}}\right)\alpha_{pj}\ r_p$$

$$\phi\left(\frac{\alpha_{pj}'}{\alpha_{pj}}\right) = \begin{cases} 0 & \text{if } \frac{\alpha_{pj}'}{\alpha_{pj}} < 0 \\ \frac{\alpha_{pj}'}{\alpha_{pj}} & \text{if } 0 \le \frac{\alpha_{pj}'}{\alpha_{pj}} \le 1 \qquad \alpha_{pj}' \le \alpha_{pj} \\ 1 & \text{if } \frac{\alpha_{pj}'}{\alpha_{pj}} \geq 1 \qquad \alpha_{pj}' \geq \alpha_{pj} \end{cases}$$

This doesn't seem to become $\text{minmod}(\alpha_{pj}', \alpha_{pj})$ as I think it should.



Given $\{\bar{U}_j(t)\,;\, j \in (-\infty, +\infty)\}$ how can I reconstruct accurately values of U to high order?

Define an auxilliary fn $w(x)$ as $w(x) = \int_{x_{1/2}}^{x} U(\xi, t)\, d\xi$

Note then $w'(x) = U(x,t)$ Thus if one can approximate $w(x)$ to high order one can differentiate this fn & obtain an accurate approximation to $U(x,t)$.

define $W_j = w(x_{j+1/2}) = \int_{x_{1/2}}^{x_{j+1/2}} U(\xi,t)\, d\xi = h \sum_{i=1}^{j} \bar{U}_i(t)$ & W_j is exact at the ½ nodes if $\bar{U}_i(t)$ is known exactly.

Now it is obvious that we have the exact values for $w(x)$ at the ½ nodes.

8-20-011

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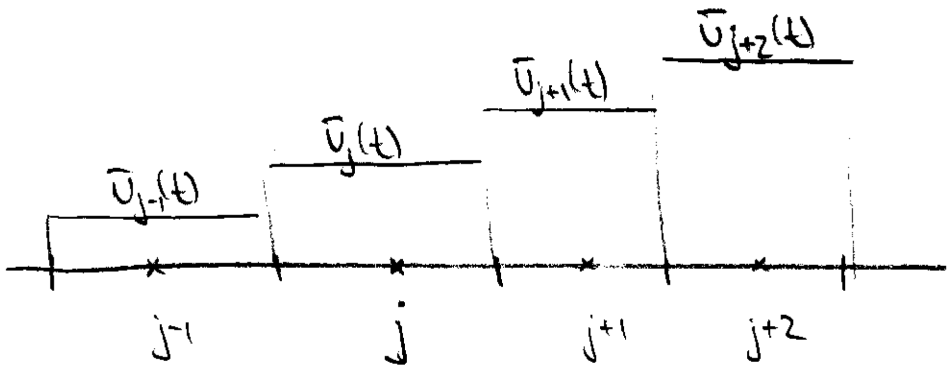
Ex 17.1



$q=2 \Rightarrow$ We wish to approximate u on $[x_{j-1/2}, x_{j+1/2}]$ based on a derivative of an interpolating polynomial for $w(x) = \int_{x_{1/2}}^{x} u(\xi,t)\,d\xi$

The ENO scheme requires that we choose the minimum between the divided difference of

$$\{w_{j-2}, w_{j-1}, w_j\} \quad \& \quad \{w_{j-1}, w_j, w_{j+1}\}$$

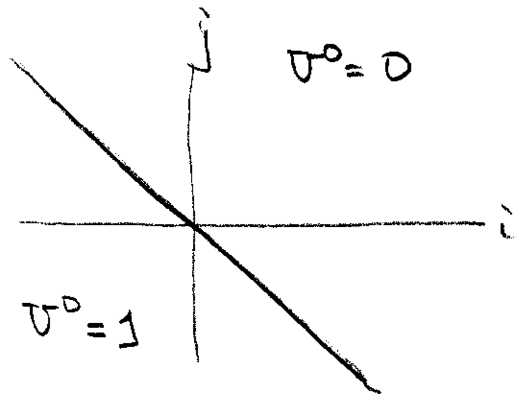
$$\begin{matrix} w_{j-2} & & & & \\ & \dfrac{w_{j-2}-w_{j-1}}{h} & & & \\ w_{j-1} & & \dfrac{w_{j-2}-w_{j-1}-w_{j-1}+w_j}{h^2} & = & \dfrac{w_{j-2}-2w_{j-1}+w_j}{h^2} \\ & \dfrac{w_{j-1}-w_j}{h} & & & \\ w_j & & & & \end{matrix}$$

performing the same ÷ difference we obtain

Ex 18.1

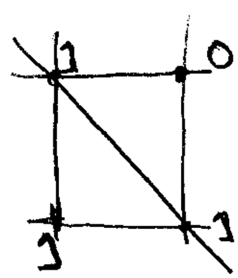
$$u_t + u_x + u_y = 0$$

$$U^0_{ij} = \begin{cases} 1 & i+j < 0 \\ 0 & i+j \geq 0 \end{cases}$$



Solving 1st $u_t + f(u)_x = 0$ + then

$$u_t + g(u)_y = 0$$



Not sure how to do?

$$e^{-tB\partial_y} e^{-tA\partial_x} = \left(1 - tB\partial_y + \frac{t^2 B^2 \partial_y^2}{2} \cdots\right)\left(1 - tA\partial_x + \frac{t^2 A^2 \partial_x^2}{2} - \cdots\right)$$

$$= 1 - t(A\partial_x + B\partial_y) + \frac{t^2}{2}\left(A^2\partial_x^2 + B^2\partial_y^2 + 2BA\partial_y\partial_x\right) + O(t^3)$$

Now

$$(A\partial_x + B\partial_y)^2 = (A\partial_x + B\partial_y)(A\partial_x + B\partial_y)$$

$$= A^2\partial_x^2 + AB\partial_x\partial_y + BA\partial_y\partial_x + B^2\partial_y^2$$

Thus we shall add $AB\partial_x\partial_y$ to attempt to get this second term

$$\Rightarrow \quad e^{-tB\partial_y} e^{-tA\partial_x} = 1 - t(A\partial_x + B\partial_y) + \frac{t^2}{2}\left(A^2\partial_x^2 + AB\partial_x\partial_y + BA\partial_y\partial_x + B^2\partial_y^2 - AB\partial_x\partial_y + BA\partial_y\partial_x\right) + O(t^3)$$

$$= 1 - t(A\partial_x + B\partial_y) + \frac{t^2}{2}\left[(A\partial_x + B\partial_y)^2 - (AB - BA)\partial_x\partial_y\right] + O(t^3)$$

$$= 1 - t(A\partial_x + B\partial_y) + \frac{t^2}{2}(A\partial_x + B\partial_y)^2 - \frac{t^2}{2}(AB - BA)\partial_x\partial_y + O(t^3)$$

$$= e^{-t(A\partial_x + B\partial_y)} - \frac{t^2}{2}(AB - BA)\partial_x\partial_y + O(t^3)$$

Because I can add all the higher order terms I need w/ the $O(t^3)$

$$e^{-\frac{1}{2}kA\partial_x} e^{-kB\partial_y} e^{-\frac{k}{2}A\partial_x} =$$

$$\left(1 - \frac{k}{2}A\partial_x + \frac{(\frac{1}{2}kA\partial_x)^2}{2} - \frac{1}{3!}(\tfrac{1}{2}kA\partial_x)^3 + \dots\right) \cdot$$

$$\left(1 - kB\partial_y + \frac{1}{2}(kB\partial_y)^2 - \frac{1}{3!}(kB\partial_y)^3 + \dots\right) \cdot$$

$$\left(1 - \frac{1}{2}kA\partial_x + \frac{1}{2}(\qquad)\right)$$

$$= 1 - \frac{k}{2}A\partial_x - kB\partial_y - \frac{1}{2}kA\partial_x + 2\frac{1}{2}k$$

$$(A\partial_x + B\partial_y)^2 = A^2\partial_x^2 + AB\partial_x\partial_y + BA\partial_y\partial_x + B^2\partial_y^2$$

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$$U^n = \mathcal{H}^x_{k/2}\mathcal{H}^y_k\mathcal{H}^x_{k/2}U^{n-1} = \mathcal{H}^x_{k/2}\mathcal{H}^y_k\mathcal{H}^x_{k/2}\left(H^x_{k/2}H^y_k H^x_{k/2}\right)U^{n-2}$$

$$= H^x_{k/2}\left(H^y_k H^x_k\right)H^y_k H^x_{k/2}U^{n-2}$$

$$= H^x_{k/2}\left(H^y_k H^x_k\right)\left(H^y_k H^x_k H^y_k H^x_{k/2}\right)U^{n-3}$$

$$= \left(H^y_k H^x_k\right)^2 H^y_k H^x_{k/2}U^{n-3}$$

. . .

$$U^n = H^x_{k/2}\left(H^y_k H^x_k\right)^{n-1}H^y_k H^x_{k/2}U^0.$$

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Ex 18.2

$$U^{n+1} = \mathcal{H}^x_{k/2}\mathcal{H}^y_k\mathcal{H}^x_{k/2}U^n$$

we are told that $\mathcal{H}^x_k = H^x_k + O(k^2)$ w/ H^x_k the exact solution opperator.

Thus

$$U^{n+1} = \left(H^x_{k/2} + O(k^2/4)\right)\left(H^y_k + O(k^2)\right)\left(H^x_{k/2} + O(k^2/4)\right)$$

$$= H^x_k H^y_k H^x_{k/2} + H^x_{k/2}$$