Worked Examples and Solutions for the Book: Computational Finance Using C and C# by George Levy

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May 8, 2009

Introduction

As a former applied mathematician, I found this book a very nice addition to the computational finance literature. The mathematical formulation of the problems discussed is clear without sacrificing rigor. Helpful but not overly theoretical proofs are provided, and a nice summary appendix of useful background mathematical results is given. One nice aspect of the book is that it provides somewhat more advanced algorithms (and source code) than other introductory books. This is beneficial in that this provides a more readable introduction to the various papers one would need to read to understand the same material. In these notes you'll find the solutions to the problems for Chapter 2 and any additional mathematical derivations I performed as I worked thought this book.

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Chapter 2 (Introduction to Stochastic Processes)

Problem Solutions

Problem 1 (evaluating $\beta_t^k = E[W_t^k]$)

Part (a): As an explanation of the solution process we will take for this problem we will derive a differential equation for W_t^k which we will then solve. Having solved this differential equation we will take the expectation of the solution to derive a recursion relationship for the requested $\beta_t^k = E[W_t^k]$. We begin by defining the function ϕ as $\phi(W_t) = W_t^k$. Since ϕ is only a function of only a stochastic component W_t (and not time) from Ito's formula we have for its differential the following

$$d\phi = \frac{\partial \phi}{\partial W} dW_t + \frac{1}{2} \frac{\partial^2 \phi}{\partial W^2} dt \,.$$

Thus for this specific $\phi(W_t)$ we find

$$d(W_t^k) = k(W_t)^{k-1} dW_t + \frac{1}{2} k(k-1)(W_t)^{k-2} dt \,.$$

Integrating both sides of this last expression from 0 to t gives

$$\int_{s=0}^{t} d(W_s^k) = k \int_{s=0}^{t} (W_s)^{k-1} dW_s + \frac{1}{2} k(k-1) \int_{s=0}^{t} (W_s)^{k-2} ds \,,$$

or

$$W_t^k - W_0^k = k \int_{s=0}^{t} (W_s)^{k-1} dW_s + \frac{1}{2} k(k-1) \int_{s=0}^{t} (W_s)^{k-2} ds \,.$$

Since $W_0^k = 0$ we can take the expectation of the above to get

$$\beta_t^k = E[W_t^k] = kE\left[\int_{s=0}^{t} (W_s)^{k-1} dW_s\right] + \frac{1}{2} k(k-1) E\left[\int_{s=0}^{t} (W_s)^{k-2} ds\right]$$

Since $E\left[\int_{s=0}^{t} (W_s)^{k-1} dW_s\right] = 0$ then passing the second expectation into the integral above gives the desired expression for β_t^k of

$$\beta_t^k = \frac{1}{2} k(k-1) \int_{s=0}^{t} \beta_s^{k-2} ds \,, \tag{1}$$

the desired expression.

Part (b): From Equation 1 above we have that when $k = 4$ that

$$E[W_t^4] = \beta_t^4 = \frac{4}{2}(3)\int_{s=0}^t \beta_s^2 ds\,,$$

which depends on β_s^2. When $k = 2$ Equation 1 gives

$$\beta_t^2 = \frac{1}{2}(2)\int_{s=0}^t ds = t\,.$$

So with these two results we see that

$$E[W_t^4] = 6\int_{s=0}^t s^2 ds = 2s^3\Big|_0^t = 2t^3\,.$$

Part (c): In the same way as above we have when $k = 6$ that

$$E[W_t^6] = \frac{1}{2}6(5)\int_{s=0}^t \beta_s^4 ds = 15\int_{s=0}^t 3s^2 ds = 15t^3\,.$$

Problem 2 (solving $dX_t = X_t dt + dW_t$)

From the given differential equation $dX_t = X_t dt + dW_t$ we can write it trivially but emphasizing that the random component W_t can be considered like a forcing term to the linear system represented by the left hand side as

$$dX_t - X_t dt = dW_t\,.$$

This also suggest multiplying both sides by the integration factor of e^{-t} to get

$$dX_t e^{-t} - X_t e^{-t} dt = e^{-t} dW_t\,.$$

In this expression the left hand side is equivalent to an exact differential and our equation becomes

$$d(X_t e^{-t}) = e^{-t} dW_t\,.$$

Upon integrating both sides of this from t_0 to t we obtain

$$\int_{t_0}^t d(X_t e^{-t}) = \int_{t_0}^t e^{-t} dW_t\,,$$

or performing the integration on the left hand side gives

$$X_t e^{-t} - X_{t_0} e^{-t_0} = \int_{t_0}^{t} e^{-s} dW_s \,.$$

Solving for X_t we finally find

$$X_t = X_{t_0} e^{-t_0+t} + \int_{t_0}^{t} e^{t-s} dW_s \,.$$

Problem 3 (solving $dX_t = -X_t dt + e^{-t} dW_t$)

Writing the given differential equation as

$$dX_t + X_t dt = e^{-t} dW_t \,,$$

we can simplify it by multiplying by an integrating factor of e^t to get

$$e^t dX_t + e^t X_t dt = dW_t \,,$$

or

$$d(e^t X_t) = dW_t \,.$$

When we integrate both sides from t_0 to t we find

$$e^t X_t - e^{t_0} X_{t_0} = \int_{t_0}^{t} dW_s = W_t - W_{t_0} \,.$$

Since $W_{t_0} = 0$ when we solve for X_t we find

$$X_t = e^{t_0 - t} X_{t_0} + e^{-t} W_t \,.$$

Problem 4 (integrating $\int W_s^2 dW_s$)

Consider a function ϕ defined as $\phi = W_t^3$. Then Ito's formula in this case since our stochastic variable, X, written in terms of W is so simple (X is *equal* to W) the general expression has $a = 0$, $b = 1$, and we find

$$\begin{aligned} d\phi &= \left(\frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \phi}{\partial W^2} \right) dt + \frac{\partial \phi}{\partial W} dW \\ &= \left(0 + \frac{6}{2} W_t \right) dt + 3 W_t^2 dW \\ &= 3 W_t dt + 3 W_t^2 dW \,, \end{aligned}$$

or expressing this in terms of $\frac{1}{3}W_t^3$ we find

$$\frac{1}{3}d(W_t^3) = W_t dt + W_t^2 dW \,.$$

On integrating both sides from t_0 to t since $W_{t_0} = 0$ we see that

$$\frac{1}{3}W_t^3 = \int_{s=0}^{t} W_s ds + \int_{s=0}^{t} W_s^2 dW_s \,,$$

which is the required identity.

Problem 5 (solving $dY_t = rdt + \alpha Y_t dW_t$)

To solve

$$dY_t = rdt + \alpha Y_t dW_t \,, \tag{2}$$

as a hint we are told to consider an integrating factor $F_t = e^{-\alpha W_t - \left(\frac{\alpha^2}{2}\right)t}$. Note that there is a typo in the sign of the t term in the expression the books provides for F_t. We begin by writing Equation 2 as

$$dY_t - \alpha Y_t dW_t = rdt \,.$$

We next multiply both sides by the suggested integrating factor F_t to obtain

$$F_t dY_t - \alpha Y_t F_t dW_t = rF_t dt \,. \tag{3}$$

Now with Ito's formula lets evaluate the differential of the integrating factor F_t. We find

$$\begin{aligned}
dF_t &= \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\right) dt + \frac{\partial F}{\partial W} dW \quad \text{where} \\
\frac{\partial F}{\partial t} &= -\left(\frac{\alpha^2}{2}\right) F \\
\frac{\partial F}{\partial W} &= -\alpha F \\
\frac{\partial^2 F}{\partial W^2} &= \alpha^2 F \,.
\end{aligned}$$

So we see that

$$dF = \left(-\frac{\alpha^2}{2} + \frac{1}{2}\alpha^2\right) F dt - \alpha F dW = -\alpha F dW \,. \tag{4}$$

Using Equation 4 we see that the differential of the product $Y_t F_t$ is

$$\begin{aligned} d(Y_t F_t) &= (dY_t)F_t + Y_t(dF_t) \\ &= F_t(dY_t) + Y_t(-\alpha^2 F_t dW_t) \\ &= F_t(dY_t) - \alpha Y F dW_t \, . \end{aligned}$$

Thus using this expression Equation 3 above becomes

$$d(Y_t F_t) = r e^{-\alpha W_t - \left(\frac{\alpha^2}{2}\right) t} dt \, .$$

Integrating both sides from t_0 to t gives

$$Y_t F_t - Y_{t_0} F_{t_0} = \int_{t_0}^{t} r e^{-\alpha W_s - \left(\frac{\alpha^2}{2}\right) s} ds$$

so that when we solve for Y_t we find

$$\begin{aligned} Y_t &= F_t^{-1} F_{t_0} Y_{t_0} + r F_t^{-1} \int_{t_0}^{t} e^{-\alpha W_s - \left(\frac{\alpha^2}{2}\right) s} ds \\ &= e^{\alpha W_t + \left(\frac{\alpha^2}{2}\right) t} e^{-\alpha W_{t_0} - \left(\frac{\alpha^2}{2}\right) t_0} Y_{t_0} + r e^{\alpha W_t + \left(\frac{\alpha^2}{2}\right) t} \int_{t_0}^{t} e^{-\alpha W_s - \left(\frac{\alpha^2}{2}\right) s} ds \, . \end{aligned}$$

Since $W_{t_0} = 0$ we get that

$$Y_t = Y_{t_0} e^{\alpha W_t + \left(\frac{\alpha^2}{2}\right)(t - t_0)} + r \int_{s=t_0}^{t} e^{\alpha (W_t - W_s) + \left(\frac{\alpha^2}{2}\right)(t-s)} ds \, ,$$

for the solution.

Problem 6 (solving the Ornstein-Uhlenbeck process)

For the mean reverting Ornstein-Uhlenbeck process given by

$$dX_t = (m - X_t)dt + \sigma dW_t \, , \tag{5}$$

we have by writing the random component on the right hand side

$$dX_t + X_t dt = m dt + \sigma dW_t \, .$$

From this expression an integrating factor for this equation is given by $F = e^t$ and multiplying by this we find the left hand side becomes the perfect differential

$$e^t dX_t + e^t X_t dt = d(e^t X_t) \, .$$

Using this the above becomes

$$d(e^t X_t) = me^t dt + \sigma e^t dW_t \,.$$

Integrating both sides over the limits t to t_0 gives

$$e^t X_t - e^{t_0} X_{t_0} = m(e^t - e^{t_0}) + \sigma \int_{t_0}^{t} e^s dW_s \,,$$

or solving for X_t we obtain

$$\begin{aligned} X_t &= e^{-(t-t_0)} X_{t_0} + m(1 - e^{-(t-t_0)}) + \sigma e^{-t} \int_{t_0}^{t} e^s dW_s \\ &= m + (X_{t_0} - m) e^{-(t-t_0)} + \sigma e^{-t} \int_{t_0}^{t} e^s dW_s \,. \end{aligned} \tag{6}$$

Part (b): Taking the expectation of Equation 6 above we find

$$E[X_t] = m + (X_{t_0} - m) e^{-(t-t_0)} \,, \tag{7}$$

Since $E\left[\int_{t_0}^{t} e^s dW_s\right] = 0$ because dW_s represents a draw from $N(0, ds)$ which has an zero expectation. To compute the variance of X_t or $\text{Var}[X_t]$ first consider subtracting the mean from X_t to get

$$X_t - E[X_t] = \sigma e^{-t} \int_{t_0}^{t} e^s dW_s \,,$$

then squaring this we find

$$(X_t - E[X_t])^2 = \sigma^2 e^{-2t} \int_{u=t_0}^{t} \int_{v=t_0}^{t} e^{u+v} dW_u dW_v \,.$$

Now the expectation of this (which is also the variance of X_t) then is

$$E[(X_t - E[X_t])^2] = \sigma^2 e^{-2t} \int_{u=t_0}^{t} \int_{v=t_0}^{t} e^{u+v} E[dW_u dW_v] \,.$$

Recalling that $E[dW_u dW_v] = E[dW_u^2] \delta(u - v) = du \delta(u - v)$, where $\delta(\cdot)$ is the Dirac delta function. Using this we obtain

$$\begin{aligned} \text{Var}[X_t] &= E[(X_t - E[X_t])^2] \\ &= \sigma^2 e^{-2t} \int_{u=t_0}^{t} e^{2u} du \\ &= \frac{\sigma^2}{2} (1 - e^{-2(t-t_0)}) \,. \end{aligned} \tag{8}$$

Problem 7 (time-dependent Brownian motion)

Part (a): For time-dependent geometric Brownian motion our asset price S_t satisfies

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \,, \tag{9}$$

where now μ and σ are functions of the time variable t. This is the general stochastic differential of the variable S_t with $a_t = \mu_t S_t$ and $b_t = \sigma_t S_t$. Now define a function ϕ such that $\phi = \log(S_t)$, so by Ito's formula we have

$$\begin{aligned} d\phi &= \left(\frac{\partial \phi}{\partial t} + a \frac{\partial \phi}{\partial S} + \frac{b^2}{2} \frac{\partial^2 \phi}{\partial S^2} \right) dt + b \frac{\partial \phi}{\partial S} dW \\ &= \left(\mu_t S_t \left(\frac{1}{S_t} \right) + \frac{\sigma_t^2 S_t^2}{2} \left(-\frac{1}{S_t^2} \right) \right) dt + \sigma_t \frac{S_t}{S_t} dW \\ &= \left(\mu_t - \frac{\sigma_t^2}{2} \right) dt + \sigma_t dW_t \,. \end{aligned}$$

Thus integrating both sides of this expression we have

$$\log(S_t) - \log(S_{t_0}) = \int_{t_0}^{t} \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_{t_0}^{t} \sigma_s dW_s \,, \tag{10}$$

or solving for S_t we have

$$S_t = S_{t_0} \exp \left\{ \int_{t_0}^{t} \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds + \int_{t_0}^{t} \sigma_s dW_s \right\} \,. \tag{11}$$

Taking the expectation of Equation 10 we have

$$E[\log(S_t)] = \log(S_{t_0}) + \int_{t_0}^{t} \left(\mu_s - \frac{\sigma_s^2}{2} \right) ds \,, \tag{12}$$

where we have used the fact that

$$E\left[\int_{t_0}^{t} \sigma_s dW_s \right] = \int_{t_0}^{t} \sigma_s E[dW_s] = 0 \,.$$

Part (b): To compute the variance of $\log(S_t)$ consider its definition

$$\mathrm{Var}[\log(S_t)] = E\left[(\log(S_t) - E[\log(S_t)])^2 \right]$$

$$
\begin{aligned}
&= E\left[\left(\int_{t_0}^{t}\sigma_s dW_s\right)^2\right] = E\left[\int_{u=t_0}^{t}\int_{v=t_0}^{t}\sigma_u\sigma_v dW_u dW_v\right] \\
&= \int_{u=t_0}^{t}\int_{v=t_0}^{t}\sigma_u\sigma_v E[dW_u dW_v] = \int_{u=t_0}^{t}\int_{v=t_0}^{t}\sigma_u\sigma_v 1\delta(u-v)du \\
&= \int_{u=t_0}^{t}\sigma_u^2 du\,,
\end{aligned}
$$

the requested expression.

Problem 8 (the differential of e^{tW_t})

This problem is a direct application of Ito's lemma where one knows the stochastic differential equation followed by the process X_t and we want the stochastic differential equation of a function of X_t and t. That is when we know $dX_t = adt + bdW_t$, some values of a and b, then the function $\phi = \phi(X, t)$ has a differential given by

$$
d\phi = \left(\frac{\partial\phi}{\partial t} + a\frac{\partial\phi}{\partial X} + \frac{b^2}{2}\frac{\partial^2\phi}{\partial X^2}\right)dt + b\frac{\partial\phi}{\partial X}dW\,. \tag{13}
$$

When the functional form form ϕ is $\phi(X, t) = e^{tW}$ we have that the stochastic process followed by X is simply $dX = dW$ and we find the partial derivatives given by

$$
\begin{aligned}
\frac{\partial\phi}{\partial t} &= We^{tW} = W\phi \\
\frac{\partial\phi}{\partial X} &= \frac{\partial\phi}{\partial W} = te^{tW} = t\phi \\
\frac{\partial^2\phi}{\partial X^2} &= \frac{\partial^2\phi}{\partial W^2} = t^2e^{tW} = t^2\phi\,.
\end{aligned}
$$

Using these in the above and recognizing that when $X = W$ we have $a = 0$ and $b = 1$ we find $d\phi$ given by

$$
\begin{aligned}
d\phi &= \left(\phi W_t + \frac{1}{2}t^2\phi\right)dt + t\phi dW_t \\
&= \phi\left(W_t + \frac{1}{2}t^2\right)dt + t\phi dW_t\,,
\end{aligned}
$$

as we were to show.

Problem 9 (the differential of Z_t)

To evaluate the differential of the given $Z_t = \exp\left(\int_{s=0}^t \theta_s dW_s - \frac{1}{2}\int_{s=0}^t \theta_s^2 ds\right)$, observe that Z_t can be considered a function of time only which is in tern a function of only the variables θ_s and W_s. Thus to compute dZ_t we will use the chain rule. Defining a function A_t as the argument of the exponential i.e.

$$A_t = \int_{s=0}^t \theta_s dW_s - \frac{1}{2}\int_{s=0}^t \theta_s^2 ds \,,$$

we have $Z_t = \exp(A_t)$, and $dA_t = \theta_t dW_t - \frac{1}{2}\theta_t^2 dt$ so A_t is stochastic with the given differential. By Ito's lemma we have since $Z_t = \exp(A_t)$ that

$$\begin{aligned}
dZ_t &= \frac{dZ}{dA}dA + \frac{1}{2}\frac{d^2Z}{dA^2}dA^2 \\
&= Z_t dA + \frac{1}{2}Z_t dA^2 \\
&= Z_t\left(\theta_t dW_t - \frac{1}{2}\theta_t^2 dt + \frac{1}{2}\theta_t^2 dW_t^2 - \frac{1}{2}\theta_t^3 dW_t dt + \frac{1}{8}\theta_t^4 dt^2\right) \\
&\approx Z_t\left(\theta_t dW_t - \frac{1}{2}\theta_t^2 dt + \frac{1}{2}\theta_t^2 dt\right) \\
&= Z_t \theta_t dW_t \,.
\end{aligned}$$

Since the product $\theta_t^2 dW_t^2 \to \theta_t^2 dt$ in expectation.

Problem 10 (the function $S_t = S_0 e^{\mu t + \sigma W_t}$)

Part (a): Given $S_t = S_0 \exp(\mu t + \sigma W_t)$ with $W_t \sim N(0,t)$ then Ito's lemma applied to the functional form

$$\phi = S_t = \phi(t, W_t) = S_0 \exp(\mu t + \sigma W_t)\,, \tag{14}$$

Gives

$$\begin{aligned}
d\phi &= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial W}dW + \frac{1}{2}\frac{\partial^2 \phi}{\partial W^2}dW^2 \\
&= \frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial W}dW + \frac{1}{2}\frac{\partial^2 \phi}{\partial W^2}dt \\
&= \mu S_0 e^{\mu t + \sigma W_t} + \sigma S_0 e^{\mu t + \sigma W_t} dW + \frac{1}{2}\sigma^2 S_0 e^{\mu t + \sigma W_t} dt \\
&= \mu \phi dt + \frac{1}{2}\sigma^2 \phi dt + \sigma \phi dW \,.
\end{aligned}$$

So since $\phi = S_t$ we see that

$$dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t \,, \tag{15}$$

as we were to show.
Part (b): Integrating both sides of Equation 15 from t_0 to t we find that

$$S_t - S_{t_0} = \left(\mu + \frac{\sigma^2}{2}\right) \int_{t_0}^{t} S_\tau d\tau + \sigma \int_{t_0}^{t} S_\tau dW_\tau \,.$$

Then taking expectations of the above we find

$$E[S_t] - E[S_{t_0}] = \left(\mu + \frac{\sigma^2}{2}\right) \int_{t_0}^{t} E[S_\tau] d\tau \,, \tag{16}$$

since the expectation of the second term is zero.
Part (c): Note that Equation 16 is an integral relation for $E[S_t]$. To solve for $E[S_t]$ we can convert this into a differential equation for $E[S_t]$ that we can then solve. Taking the derivative of Equation 16 with respect to time we find

$$\frac{dE[S_t]}{dt} = \left(\mu + \frac{\sigma^2}{2}\right) E[S_t] \,.$$

The solution to this equation is

$$E[S_t] = E[S_0] e^{\left(\mu + \frac{\sigma^2}{2}\right) t} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right) t} \,,$$

as we were to show.

Problem 11 (an integration by parts identity)

This problem looks like Ito's product rule in two dimensions. Let $\phi = X_t Y_t$ then from the product rule expression discussed in the book recall that $d\phi$ is given as

$$d\phi = \frac{\partial \phi}{\partial X_1} dX_1 + \frac{\partial \phi}{\partial X_2} dX_2 + \frac{1}{2} E\left[\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 \phi}{\partial X_i \partial X_j} dX_i dX_j\right] \,, \tag{17}$$

from which for when $\phi = X_t Y_t$ we obtain the result

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + E[dX_t dY_t] \,,$$

Then integrating both sides of this expression from t to t_0 gives

$$X_t Y_t - X_{t_0} Y_{t_0} = \int_{s=t_0}^{t} X_s dY_s + \int_{s=t_0}^{t} Y_s dX_s + \int_{s=t_0}^{t} E[dX_s dY_s]\,,$$

which on rearranging gives the desired expression.