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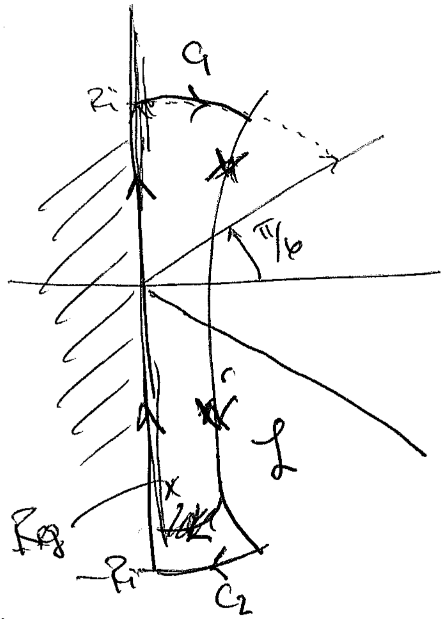

$$Ai(\cancel{z}) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp\left\{\frac{v^3}{3} - xv\right\} dv$$

$\exp\{—\}$ is holomorphic in Region R

$$\cancel{Ai(z)} = \quad \cancel{Ai(z)} = \frac{1}{2\pi i}\int_{-Ri}^{Ri} \exp\{\quad\} dv + \cancel{\int_{C_1}}^{\,0} + \int_{\mathcal{L}} + \cancel{\int_{C_2}}^{\,0} = 0$$

By Cauchys Thm: Both C_1 + C_2 go to zero

As $R \to \infty$

$$\Rightarrow \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \exp\left\{\frac{v^3}{3} - xv\right\} dv = \frac{1}{2\pi i}\int_{\bar{\mathcal{L}}} \exp\{—\} dv$$

($\bar{\mathcal{L}}$ opposite path to original $\mathcal{L}$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{z-1} dt$$

let $v = (3t)^{1/3} e^{\pm \pi i/3} = 3^{1/3} t^{1/3} e^{\pm i\pi/3}$

$$dv = 3^{-2/3} t^{-2/3} e^{\pm \pi i/3}$$

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(1.01) is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad \text{let} \quad v = 3^{1/3} t^{1/3} e^{\pm \pi i/3} \Rightarrow t^{1/3} = \frac{v e^{\mp \pi i/3}}{3^{1/3}}$$

$$dv = 3^{-2/3} t^{-2/3} e^{\pm i\pi/3}\, dt \qquad t = \frac{v^3 e^{\mp \pi i}}{3}$$

$$\Rightarrow \int_0^{\infty e^{\pm \pi i/3}} e^{-\frac{v^3 e^{\mp \pi i}}{3}} \frac{1}{3^{z-1}} v^{3z-3} (e^{\mp \pi i})^{z-1}\, dv\, 3^{2/3} t^{2/3} e^{\mp i\pi/3}$$

$$t^{2/3} \downarrow v^2 \qquad \Downarrow \quad \frac{v^3}{3} = t e^{\pm \pi i}$$

Given $\displaystyle\int_0^{\infty e^{\pm i\pi/3}} v^s \exp(v^3/3)\, dv = ?$

$$\text{let } v = (3t)^{1/3} e^{\pm \pi i/3}; \quad dv = 3^{-2/3} t^{-2/3} e^{\pm i\pi/3}\, dt$$

$$= \int_0^\infty (3t)^{s/3} (e^{\pm \pi i/3})^s \exp\{t e^{\pm \pi i}\}\, 3^{-2/3} t^{-2/3} e^{\pm i\pi/3}\, dt$$

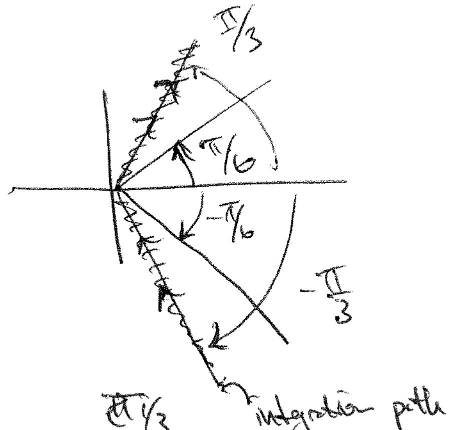
$$= e^{\pm i\pi s/3}\, 3^{s/3} \int_0^\infty t^{s/3} t^{-2/3} e^{-t}\, dt \cdot 3^{-2/3} e^{\pm i\pi/3}$$

$$= 3^{(s-2)/3} \exp\left\{\pm (s+1)\frac{\pi i}{3}\right\} \int_0^\infty t^{\frac{s-2}{3}} e^{-t}\, dt$$

$$= 3^{(s-2)/3} e^{\pm (s+1)\pi i/3} \underbrace{\Gamma\left(\frac{s-2}{3}+1\right)}_{\Gamma\left(\frac{s+1}{3}\right)}$$

$$Ai(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \exp\left\{\frac{v^3}{3} - zv\right\} dv$$

take $\mathcal{L} = \mathrm{ph}\, v = \pm\frac{\pi}{3}$



Then

$$Ai(z)\cdot 2\pi i = \int_{-\infty e^{-\pi i/3}}^{0} \cdots dv + \int_{0}^{\infty e^{\pi i/3}} \cdots dv$$

~~let $v = p-v$~~

1st $\quad e^{-zv} = \sum_{k=0}^{\infty} \frac{(-1)^k z^k v^k}{k!}$

$$\therefore Ai(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \exp\left\{\frac{v^3}{3}\right\} \sum_{k=0}^{\infty} \frac{(-1)^k z^k v^k}{k!}\, dv$$

By Thm 8.1

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k \int_{\mathcal{L}} v^k \exp\left\{\frac{v^3}{3}\right\} dv$$

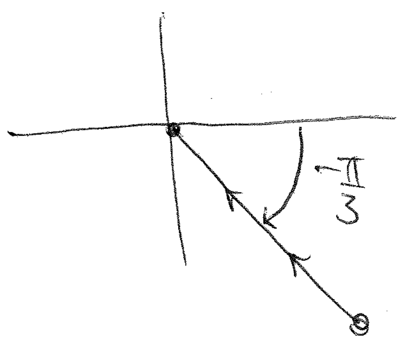
Now: $\displaystyle\int_{\Gamma} r^k \exp\left\{\frac{r^3}{3}\right\} dr = \int_{-\infty e^{-\pi i/3}}^{0} dr + \int_{0}^{\infty e^{i\pi/3}} dr$

But let $p = -r$

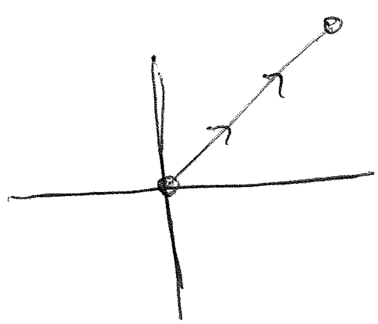
$dp = -dr$

not correct ...

$$\int_{-\infty e^{-\pi i/3}}^{0} \cdots dr = \int_{\infty e^{-\pi i/3}}^{0} (-1)^k \exp\left(-\frac{p^3}{3}\right)(-dp) = (-1)^k \int_{0}^{\infty e^{-\pi i/3}} \exp\left(-\frac{p^3}{3}\right) dp.$$



$\Rightarrow z = re^{-\pi/3\, i} \quad -\infty < r < 0$

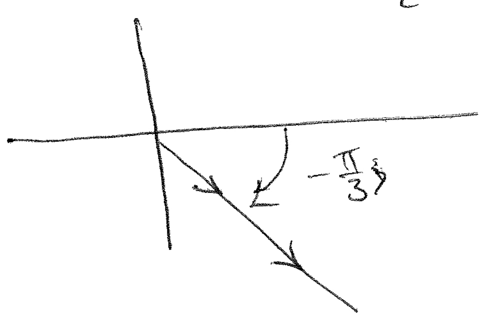


$z = re^{+\pi/3\, i} \quad 0 < r < \infty$

Thus

$z = re^{-\pi/3\, i} \quad 0 < r < \infty$

$$\int_{-\infty e^{-\pi i/3}}^{0} \cdots dr = -\int_{0}^{\infty e^{-i\pi/3}} \cdots dr$$



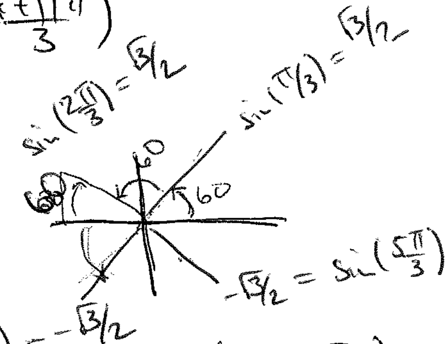
As contour is just reversed $\therefore$

$$Ai(z) = \frac{1}{2\pi i}\int$$

$$\therefore \; Ai(z) = \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\left[\underbrace{\int_0^{\infty e^{i\pi/3}} r^k \exp\left\{\frac{r^3}{3}\right\}dr - \int_0^{\infty e^{-i\pi/3}} \cdots \, dr}_{3^{(k-2)/3}e^{+(k+1)\pi i/3}\Gamma\left(\frac{k+1}{3}\right) \; - \; 3^{(k-2)/3}e^{-(k+1)\pi i/3}\Gamma\left(\frac{k+1}{3}\right)}\right]$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\,3^{(k-2)/3}\,\Gamma(k+1/3)\,\Big(\underbrace{e^{(k+1)\pi/3\, i} - e^{-(k+1)\pi/3\, i}}_{2i\sin\frac{(k+1)\pi}{3}}\Big)$$

$$= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{k!}\,3^{(k-2)/3}\,\Gamma\left(\frac{k+1}{3}\right)\sin\left(\frac{(k+1)\pi}{3}\right)$$



k	$\sin((k+1)\pi/3)$
0	$\sqrt{3}/2$
1	$\sqrt{3}/2$
2	0
3	$-\sqrt{3}/2$
4	$-\sqrt{3}/2$
5	0

mistake here !!

$$\Rightarrow \sin((k+1)\pi/3)$$

$= \frac{\sqrt{3}}{2}$ if $k = 0, 6, 12, \cdots 6n$

$= \frac{\sqrt{3}}{2}$ if $k = 1, 7, 13, \cdots 6n+1$

$= 0$ if $k = 2 + 6n \quad n = 0, 1,$

$= -\frac{\sqrt{3}}{2}$ if $k = 3 + 6n \quad n = 0, 1,$

$= -\frac{\sqrt{3}}{2}$ if $k = 4 + 6n, \ldots$

$\therefore$

$$Ai(z) = \frac{1}{\pi}\sum_{k=6n,\, n=0,1,2,\ldots} + \frac{1}{\pi}\sum_{\substack{k=6n+1\\ n=0,1,2,\ldots}} + \cancelto{0}{\frac{1}{\pi}\sum_{\substack{k=6n+2\\ n=0,1,2,\ldots}}} + \frac{1}{\pi}\sum_{\substack{k=6n+3\\ n=0,1,2,\ldots}}$$

$$+ \frac{1}{\pi}\sum_{\substack{k=6n+4\\ n=0,1,2,\ldots}} + \cancelto{0}{\frac{1}{\pi}\sum_{\substack{k=6n+5\\ n=0,1,2,\ldots}}}$$

$$= \frac{1}{\pi}\frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\left\{ \frac{(-1)^{6n} z^{6n}}{6n!} 3^{(6n-2)/3}\,\Gamma\!\left(\frac{6n+1}{3}\right) + \frac{(-1)^{6n+1} z^{6n+1}}{(6n+1)!} 3^{(6n+1-2)/3}\,\Gamma\!\left(\frac{6n+2}{3}\right)\right\}$$

$$+ \frac{-\sqrt{3}}{2\pi}\sum_{n=0}^{\infty}\left\{ \frac{(-1)^{6n+3} z^{6n+3}}{(6n+3)!} 3^{(6n+1)/3}\,\Gamma\!\left(\frac{6n+4}{3}\right) + \frac{(-1)^{6n+4} z^{6n+4}}{(6n+4)!} 3^{(6n+2)/3}\,\Gamma\!\left(\frac{6n+5}{3}\right)\right\}$$

$$= \frac{\sqrt{3}}{2\pi}\sum_{n=0}^{\infty}\left\{ \frac{z^{6n}}{6n!} 3^{2n-2/3}\,\Gamma(2n+1/3) - \frac{z^{6n+1}\, 3^{(6n-1)/3}}{(6n+1)!}\,\Gamma(2n+2/3)\right\}$$

$$- \frac{\sqrt{3}}{2\pi}\sum_{n=0}^{\infty}\left\{ \frac{-z^{6n+3}}{(6n+3)!} 3^{2n+1/3}\,\Gamma(2n+4/3) + \frac{z^{6n+4}}{(6n+4)!} 3^{2n+2/3}\,\Gamma(2n+5/3)\right\}$$

$$= \frac{\sqrt{3}}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{z^{6n}}{6n!} 3^{2n-2/3} \Gamma(2n+1/3) + \frac{z^{6n+3}}{(6n+3)!} 3^{2n+1/3} \Gamma(\underbrace{2n+4/3}_{2n+1+1/3}) \right\}$$

$$- \frac{\sqrt{3}}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{z^{6n+1}}{(6n+1)!} 3^{2n-1/3} \Gamma(2n+2/3) + \frac{z^{6n+4}}{(6n+4)!} 3^{2n+2/3} \Gamma(\underbrace{2n+5/3}_{2n+1+2/3}) \right\}.$$

$$= \frac{\sqrt{3}}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{z^{6n}}{6n!} 3^{2n-2/3} \underbrace{(2n-1+1/3)(2n-2+1/3)(2n-3+1/3)\cdots(1+1/3)}_{2n \text{ terms}}(1/3)\Gamma(1/3) \right.$$

$$\left. + \frac{z^{6n+3}}{(6n+3)!} 3^{2n+1/3} \underbrace{(2n+1/3)(2n-1+1/3)(2n-2+1/3)\cdots(1+1/3)}_{2n+1 \text{ terms}} 1/3\,\Gamma(1/3) \right\}$$

$$- \frac{\sqrt{3}}{2\pi} \sum_{n=0}^{\infty} \left\{ \frac{z^{6n+1}}{(6n+1)!} 3^{2n-1/3} \underbrace{(2n-1+2/3)(2n-2+2/3)\cdots(1+2/3)(2/3)}_{2n \text{ terms}} \Gamma(2/3) \right.$$

$$\left. + \frac{z^{6n+4}}{(6n+4)!} 3^{2n+2/3} \underbrace{(2n+2/3)(2n-1+2/3)\cdots(1+2/3)(2/3)}_{2n+1 \text{ terms}} \Gamma(2/3) \right\}$$

$$= \frac{\Gamma(1/3)\sqrt{3}}{2\pi} 3^{-2/3} \sum_{n=0}^{\infty} \left\{ \frac{z^{6n}}{6n!} (6n-3+1)(6n-6+1)(6n-9+1)\cdots(3+1)1 \right.$$

$$\left. + \frac{z^{6n+3}}{(6n+3)!} (6n+1)(6n-3+1)(6n-6+1)\cdots(3+1)(1) \right\}$$

$$-\frac{\sqrt{3}}{2\pi}\, 3^{-1/3}\,\Gamma(2/3) \sum_{n=0}^{\infty} \Big\{ \frac{z^{6n+1}}{(6n+1)!}\, \cancel{3^{2n}} (6n-3+2)(6n-6+2)\cdots(3+2)(2)$$

$$+ \frac{z^{6n+4}}{(6n+4)!} (6n+2)(6n-3+2)\cdots(3+2)2 \Big\}$$

$= \ldots$ ~~Ko~~ defining? $Ai(0) = \dfrac{\Gamma(1/3)\, 3^{1/2}\, 3^{-2/3}}{2\pi} = \dfrac{\Gamma(1/3)\, 3^{\cancel{\ } 3/6 - 4/6}}{2\pi}$

$= \dfrac{\Gamma(1/3)}{2\pi\, 3^{1/6}}$ Also from reciprocation formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\therefore \Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin \pi/3} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$$

$$\Rightarrow \Gamma(1/3) = \frac{2\pi}{\Gamma(2/3)\sqrt{3}}$$

$$\therefore Ai(0) = \frac{1}{\Gamma(2/3)\, 3^{1/6}\, 3^{1/2}} = \frac{1}{\Gamma(2/3)\, 3^{1/6}\, 3^{3/6}} = \frac{1}{\Gamma(2/3)\, 3^{2/3}}$$

$$\& \ Ai'(0) = -\frac{\sqrt{3}\, 3^{-1/3}\, \Gamma(2/3)}{2\pi} = -\frac{3^{1/2}\, 3^{-1/3}\, \Gamma(2/3)}{2\pi} = -\frac{\Gamma(2/3)\, 3^{1/6}}{2\pi}$$

$$\& = -\Gamma(1/3)^{-1} \frac{3^{1/6}}{\sqrt{3}} = \frac{-1}{\Gamma(1/3)}\, 3^{-1/3} = \frac{-1}{3^{1/3}\, \Gamma(1/3)}$$

Releasing terms...

$$= Ai(0)\Big\{ 1 + \frac{z^6}{6!}4\cdot 1 + \frac{z^{12}}{12!}\underbrace{(12-3+1)}_{10}\underbrace{(12-6+1)}_{7}4\cdot 1 + \cdots$$

$$+ \frac{z^3}{3!}1 + \frac{z^9}{9!}7\cdot 4\cdot 1 + \cdots \Big\}$$

$$+ Ai'(0)\Big\{ z + \frac{z^7}{7!}5\cdot 2\cdot 1 + O(z^{13}) + \cdots$$

$$+ \frac{z^4}{4!}2 + \frac{z^{10}}{10!}8\cdot 5\cdot 2 + \cdots \Big\}$$

$$= Ai(0)\Big\{ 1 + \frac{z^3}{3!} + \frac{4\cdot 1}{6!}z^6 + \frac{7\cdot 4\cdot 1}{9!}z^9 + \cdots \Big\}$$

$$+ Ai'(0)\Big\{ z + \frac{2z^4}{4!} + \frac{5\cdot 2\cdot 1}{7!}z^7 + \frac{8\cdot 5\cdot 2\, z^{10}}{10!} + \cdots \Big\}$$

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$$Ai'(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} (-v)\exp\left\{\frac{v^3}{3} - zv\right\} dv$$

$$Ai''(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} (-v)^2 \exp\left\{\frac{v^3}{3} - zv\right\} dv$$

$$\therefore\ Ai''(z) - zAi(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} (v^2 - z)\exp\left\{\frac{v^3}{3} - zv\right\} dv = \frac{1}{2\pi i}\exp\left\{\frac{v^3}{3} - zv\right\}\Big|_{\mathcal{L}}$$

If $\mathcal{L}$ is in sector ph v ≠ $-\frac{\pi}{2} < \text{ph } v < -\frac{\pi}{6}$

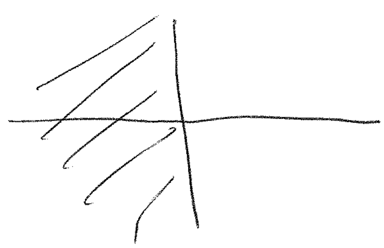
$v = re^{i\theta}$ w/ $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{6}$

$$\exp\left\{\frac{r^3 e^{3\theta i}}{3} - zre^{i\theta}\right\}$$

If $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{6}$

$\times 3 \Rightarrow -\frac{3\pi}{2} \le 3\theta \le -\frac{\pi}{2}$ Then $e^{3\theta i} = \cos 3\theta + i\sin 3\theta$

$\cos 3\theta < 0$



$\Rightarrow \exp\{\cdots\} \longrightarrow 0.$

let $v = ze^{\pm 2i\pi/3}$

$dv = dz\, e^{\pm 2i\pi/3} \Rightarrow dz = e^{\mp 2\pi i/3} dv$

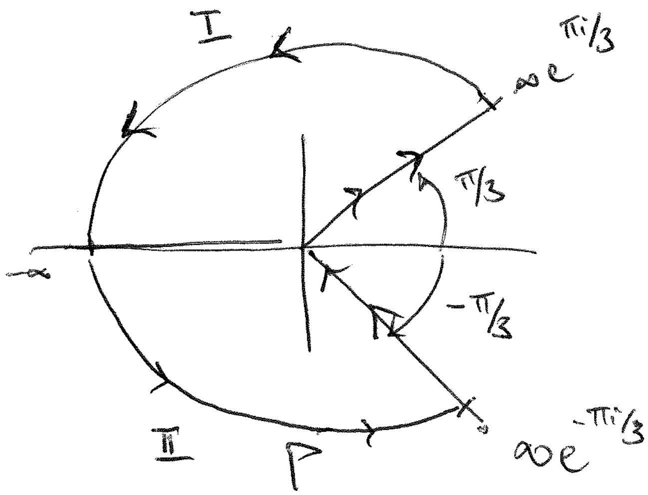
$\Rightarrow$ 8.05

$$\frac{d^2 w}{(e^{\mp 2\pi i/3})^2 dv^2} = v e^{\mp 2\pi i/3} w$$

$$\Rightarrow \frac{d^2 w}{dv^2} = vw\, e^{\mp 2\pi i/3} e^{\mp 4\pi i/3} = vw\, e^{\mp 2\pi i} = vw$$

$$\int_P \exp\left\{\frac{v^3}{3} - zv\right\} dv$$

$= 0$ By Cauchy's Thm



$$\Rightarrow \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left\{\frac{v^3}{3} - zv\right\} dv + \int \exp\{\cdots\} dv + \int$$

By Cauchy I can pick the following path w.o.l.o.g. $\rightarrow$ $v = Re^{i\theta}$, $\frac{\pi}{3} \leq \theta \leq \pi$

$v = Re^{i\theta}$, $\pi \leq \theta \leq \frac{5\pi}{3}$ $\quad 2\pi - \frac{\pi}{3} \quad =$

For 1st integral ... let $w = Re^{?}$

want $\theta = \frac{\pi}{3} \Rightarrow \theta' = -\frac{\pi}{3}$

& $\theta = \pi \Rightarrow \theta' = \frac{\pi}{3}$

& $\theta' - \frac{\pi}{3} = \frac{\frac{\pi}{3} - (-\frac{\pi}{3})}{(\pi - \frac{\pi}{3})}(\theta - \pi)$

$$\Rightarrow \theta' - \frac{\pi}{3} = \frac{\frac{2\pi}{3}}{\frac{2\pi}{3}}(\theta - \pi) = \theta - \pi. \rightarrow \quad \theta' = \theta - \pi + \frac{\pi}{3} = \theta - \frac{2\pi}{3}$$

Then let $w = Re^{\theta'(\theta)i} = Re^{(\theta - \frac{2\pi}{3})i} = Re^{\theta i} \cdot e^{-2\pi/3 i} = ve^{-2\pi/3 i}$

$$dw = Re^{-\frac{2\pi}{3}i} e^{i\theta}\, i\,d\theta = dv\, e^{-2\pi/3 i} \quad \Downarrow$$

$$v = we^{2\pi/3 i}$$

$$\therefore \int_{w\theta' = Re^{-\frac{\pi}{3}i}}^{Re^{\frac{\pi}{3}i}} \exp\left\{\frac{w^3 e^{2\pi i}}{3} - ze^{2\pi/3 i} w\right\} e^{2\pi/3 i}\, dw$$

$$= e^{2\pi/3 i} \int_{w = Re^{-\pi/3 i}}^{Re^{\pi/3 i}} \exp\left\{\frac{w^3}{3} - (ze^{2\pi/3 i})w\right\} dw = \boxed{e^{2\pi/3 i} Ai(ze^{2\pi/3 i})}$$

For 2nd integral. $v = \text{"}\infty\text{"} e^{i\theta}$, $\pi \le \theta \le \frac{5\pi}{3}$ $\quad \lim_{R \to \infty} \int_{Re}$

want $\theta' = \theta'(\theta)$ +

$\theta = \pi \qquad \theta' = -\frac{\pi}{3} \quad \Rightarrow \quad \theta' - \frac{\pi}{3} = \dfrac{-\frac{\pi}{3} - \frac{\pi}{3}}{\pi - \frac{5\pi}{3}}\left(\theta - \frac{5\pi}{3}\right)$

$\theta = \frac{5\pi}{3} \qquad \theta' = \frac{\pi}{3}$

$$\theta' - \frac{\pi}{3} = \frac{-\frac{2\pi}{3}}{\frac{\pi}{3}}\left(\theta - \frac{5\pi}{3}\right)$$

$$\theta' - \frac{\pi}{3} = -2\left(\theta - \frac{5\pi}{3}\right)$$

$$\theta' = -2\theta + \frac{10\pi}{3} + \frac{\pi}{3} = -2\theta + \frac{11\pi}{3}$$

let $w = Re^{i\theta'} = Re^{i\left(-2\theta + \frac{11\pi}{3}\right)} = Re^{-2i\theta} e^{i\pi\frac{11}{3}}$ $\quad \frac{12}{3} = 4\pi$

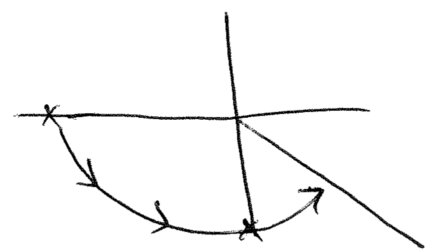
$= Re^{-2i\theta} e^{+\frac{\pi}{3}i}$

$\cancel{=\theta}\; re^{-\frac{\pi}{3}i}$ w/ θ, $\le \theta \le \frac{\pi}{2}$

$$\therefore \int_{\text{II}} = \int_{\theta=\pi}^{5\pi/3} \exp\left\{\frac{R^3 e^{3i\theta}}{3} - zRe^{i\theta}\right\} Re^{i\theta} i\, d\theta$$

Got stuck? How fix??

Going the other way.



$\theta = -\pi$ to $\theta = -\frac{\pi}{3}$

$\theta' = -\frac{\pi}{3}$ to $\theta' = \frac{\pi}{3}$

Then $\theta' + \frac{\pi}{3} = \dfrac{\frac{\pi}{3} - \frac{-\pi}{3}}{-\pi + \frac{\pi}{3}}(\theta + \pi) = \dfrac{-\frac{2\pi}{3}}{-\frac{2\pi}{3}}(\theta + \pi) = +(\theta + \pi)$

$\Rightarrow \theta' = +\theta + \pi - \frac{\pi}{3} = +\theta + \frac{2\pi}{3}$. $\Rightarrow$ $\cancel{\theta = \pm\theta' - \left(\theta + \frac{4\pi}{3}\right)} = \theta$

in original integral $v = Re^{i\theta}$ w/ $-\pi \le \theta \le -\frac{\pi}{3}$

$dv = Re^{i\theta} i\, d\theta$

$\cancel{v = Re^{-i\theta' - \frac{4\pi}{3}i}}$

$\cancel{dv = R(-i)e^{-\frac{4\pi}{3}i} e^{-i\theta'}}$ $\quad \cancel{-\frac{\pi}{3} \le \theta' \le \frac{\pi}{3}}$

The $\displaystyle \int_{\text{II}} \cancel{= \int_{\theta=-\pi}^{-\pi/3} \exp\left\{\frac{v^3}{3} - zv\right\} dv} \neq \int_{-\frac{\pi}{3}} \int_{\theta=-\pi}^{-\pi/3} \exp\left\{\frac{v^3}{3} - zv\right\} dv$

$$= \int_{\theta = -\pi}^{-\pi/3} \exp\left\{ \frac{R^3}{3} e^{i3\theta} - zRe^{i\theta} \right\} Re^{i\theta}\, i\, d\theta$$

let $\theta' = +\theta + \frac{2\pi}{3}$ $\qquad \theta' = -\frac{3\pi}{3} + \frac{2\pi}{3} = -\frac{\pi}{3}$ ✓

$\to \theta = \theta' - \frac{2\pi}{3}$ $\qquad \theta' = -\frac{\pi}{3} + \frac{2\pi}{3} = +\frac{\pi}{3}$ ✓

$$= \int_{\theta' = -\frac{\pi}{3}}^{\pi/3} \exp\left\{ \frac{R^3}{3} e^{3i(\theta' - \frac{2\pi}{3})} - zRe^{i\theta'} e^{-\frac{2i\pi}{3}} \right\} Re^{i\theta'} e^{-\frac{i2\pi}{3}}\, i\, d\theta'$$

$$= e^{-2\pi i/3} \int_{\theta' = -\frac{\pi}{3}}^{\pi/3} \exp\left\{ \frac{R^3}{3} e^{3i\theta'} - (ze^{-2\pi i/3}) e^{i\theta'} \right\} Re^{i\theta'}\, i\, d\theta'$$

$$= e^{-2\pi i/3} Ai(ze^{-2\pi i/3}) \quad \therefore$$

$$Ai(z) + e^{2\pi i/3} Ai(ze^{\frac{2\pi}{3}i}) + e^{-2\pi i/3} Ai(ze^{-2\pi i/3}) = 0$$

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Ex 8.1

$w = Ai^2$

$w' = 2 Ai\, Ai'$

$w'' = 2Ai' Ai' + 2 Ai Ai''$

$w''' = 2Ai'' Ai' + 2Ai' Ai'' + 2Ai' Ai'' + 2 Ai\, Ai'''$

$= 6 Ai' Ai'' + 2 Ai\, Ai'''$

$\cancel{w''' = 4zw'}$ But $Ai'' = z Ai$ By 8.05

$\Rightarrow Ai''' = Ai + z Ai'$

$\Rightarrow w''' = 6 Ai'(z Ai) + 2Ai(Ai + z Ai')$

$= 3z(2Ai' Ai) + z 2 Ai Ai' + 2 Ai^2$

$= 3zw' + zw' + 2w$

$\Rightarrow w''' - 4zw' - 2w = 0$

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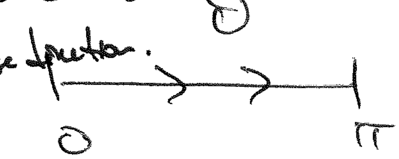
$$J_n(z) = \underbrace{\frac{1}{2\pi}\int_0^{\pi} \exp\{in\theta - iz\sin\theta\}\, d\theta}_{} + \frac{1}{2\pi}\int_0^{\pi} \exp\{-in\theta + iz\sin\theta\}\, d\theta$$

$\cancel{\theta = \pi}$

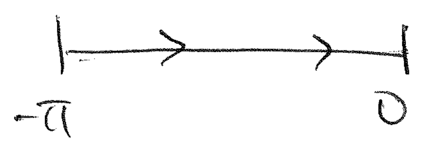
$v \cancel{= -\pi + \theta}$ or $\cancel{v = \pi - \theta}$

$\cancel{dv = d\theta}$ $\quad \cancel{dv = -d\theta}$

Note: we want to "shift the range of integration from $(0, \pi)$ $\leftarrow$ $(-\pi, 0)$ & reverse the direction of the integration. No don't want to reverse direction.

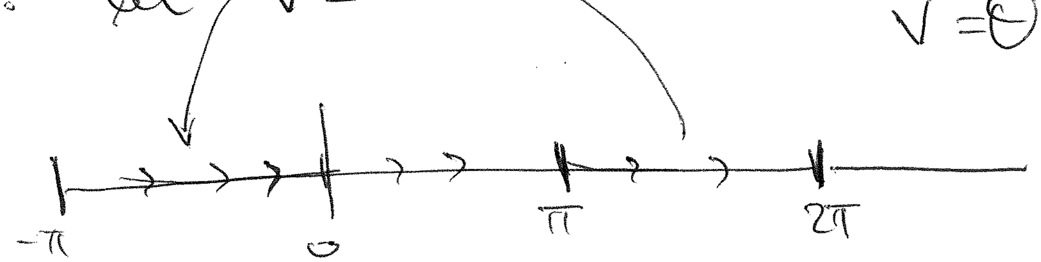


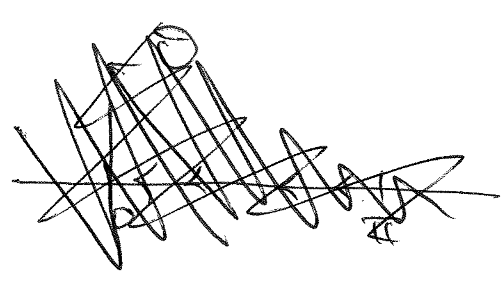
$v = \theta - \pi$ $\qquad$ $dv = d\theta$

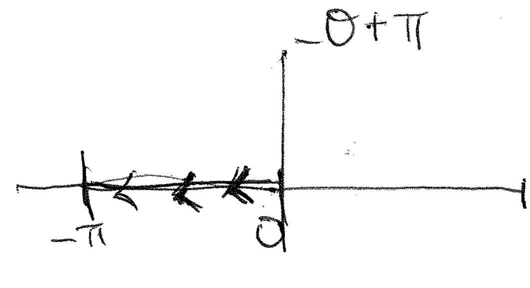


But Note that integrand is 2π periodic...

$\therefore$ let $v =$ $\qquad$ $v = \theta + \pi$







~~from θ to~~ let ~~$\theta = f(\theta)\ f(v)$~~ $v = f(\theta)$

~~$d\theta = f'(\theta)\,d\theta$~~ $dv = f'(\theta)\,d\theta$

$$\cancel{\int_{f(0)}^{f(\pi)} \exp\{in f^{-1}(v) - iz\sin f^{-1}(v)\} \frac{dv}{f'(f^{-1}(v))}}$$

$$\boxed{\begin{aligned} \text{let} \quad \theta &= -v \\ d\theta &= -dv \end{aligned}}$$

$$= \frac{1}{2\pi}\int_0^{\pi} \exp\{-inv + iz\sin v\}(-dv) = \frac{1}{2\pi}\int_{-\pi}^{0} \exp\{-inv + iz\sin v\}\,dv$$

$$\Rightarrow J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-in\theta + iz\sin\theta)\,d\theta$$

let $h = e^{i\theta}$ $\quad dh = e^{i\theta} i\,d\theta \Rightarrow d\theta = \dfrac{dh\,e^{-i\theta}}{i}$

$$\Rightarrow J_n(z) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} (e^{-i\theta})^n \exp\{iz\underbrace{\sin\theta}_{= \frac{e^{i\theta} - e^{-i\theta}}{2i}}\}\,d\theta$$

$$= \frac{1}{2\pi i}\int_C (h^{-n}) \exp\{\cancel{i}\frac{z}{2\cancel{i}}(h - h^{-1})\}\,dh\,h^{-1}$$

$$= \frac{1}{2\pi i}\int_C \exp\left\{\underbrace{\frac{z}{2}(h-h^{-1})}_{\frac{z}{2}\left(\frac{h^2-1}{h}\right)}\right\}\frac{dh}{h^{n+1}}$$

$$J_n^{(s)}(0) = \frac{1}{2\pi i}\int_C \left(\frac{1}{2}(h-h^{-1})\right)^s \exp\left\{\frac{z}{2}(h-h^{-1})\right\}\Big|_{z=0}\frac{dh}{h^{n+1}}$$

$$= \frac{1}{2\pi i}\int_C \left\{\frac{1}{2}(h-h^{-1})\right\}^s h^{-n-1}\,dh$$

~~If s,~~ $\cancel{0 \le s \le n-1}$ Now the integrand is ...

$$\frac{1}{2^s}\left(\frac{(h^2-1)^s}{h^s}\right)h^{-n-1} = \frac{(h^2-1)^s h^{-n-s-1}}{2^s}$$

~~If~~ $\cancel{0 \le s \le n-1}$

~~then~~ $\cancel{h^{-n-s-1}}$ ~~does not ha~~

$$= 2^{-s}h^{-n-s-1}(h^2-1)^s = 2^{-s}h^{-n-s-1}\sum_{k=0}^{s}\binom{s}{k}(h^2)^k(-1)^{s-k}$$

$$= 2^{-s}h^{-n-s-1}(-1)^s(1-h^2)^s$$

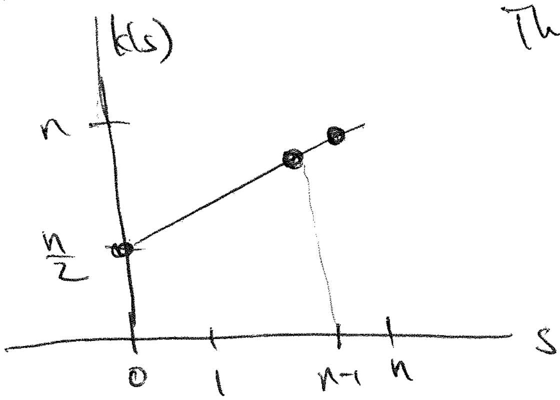
$$= 2^{-s}h^{-n-s-1}(-1)^s\sum_{k=0}^{s}\binom{s}{k}(-h^2)^k = 2^{-s}(-1)^s\sum_{k=0}^{s}\binom{s}{k}(-1)^k h^{2k-n-s-1}$$

To evaluate this integral we want the residue $\Rightarrow O(h^{-1})$ term.

$2k - n - s = 0 \Rightarrow k = \frac{n+s}{2}$ Now k must be an integer...

$0 \le k \le s$

$\Rightarrow$ if $s = 0$



Thus if $0 \le s \le n-1$

$\Leftrightarrow$ ~~2k-~~ $2k - s = n$ fixed

& $0 \le k \le s$

$\therefore 0 \le 2k \le 2s$

$\Rightarrow -s \le 2k - s \le s$ Thus if $s \le n-1$

As $2k - s \le s \Rightarrow 2k - s \le n-1 \ne n \Rightarrow$ residue is 0.

As $s \ge 0 \Rightarrow 0 \le s \le n-1$ residue is 0.

Now if $s = n + p$ w/ $1 \le p \le \infty$

Then we get $k = \frac{n+n+p}{2} = n + \frac{p}{2}$

But ~~p must~~ k must be an integer so p must be even or else the residue is 0.

If $p = 2m$ The

$$k = n + \frac{2m}{2} = n+m$$

+ Residue is $2^{-s} (-1)^s \binom{s}{k^*} (-1)^{k^*}$

$$= 2^{-(n+2m)} (-1)^{(n+2m)} \binom{n+2m}{\underline{n+m}} (-1)^{n+m} .$$

$$= \frac{(-1)^m}{2^{n+2m}} \binom{n+2m}{n+m} = \frac{(-1)^m}{2^{n+2m}} \frac{(n+2m)!}{(n+m)!\,(\cancel{n}+2m-\cancel{n}-m)}$$

$$= \frac{(-1)^m}{2^{n+2m}} \frac{(n+2m)!}{(n+m)!\,(m!)} = \frac{(-1)^m}{2^{n+2m}} \binom{n+2m}{m}$$ replacing m by s

$$\Rightarrow \cancel{J_n(z) = \sum_{s=0}^{\infty} \frac{1}{2\pi i} 2\pi i \frac{(-1)^s}{2^{n+2s}} \binom{n+2s}{s}}$$

$\therefore$ By Taylor

$$J_n(z) = \sum_{s=0}^{\infty} z^s \frac{J_n^{(s)}(0)}{s!} = \sum_{s=n}^{\infty} \frac{z^s J_n^{(s)}(0)}{s!}$$

All terms up to n vanish.

$$= \sum_{s=n,n+2,n+4,\ldots}^{\infty} \frac{z^s}{s!} J_n^{(s)}(0) = \sum_{s=0}^{\infty} \frac{z^{n+2s}}{(2s+n)!} J_n^{(n+2s)}(0)$$

All "odd" terms vanish

$$= z^n \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s+n)!} \frac{(-1)^s}{2^{n+2s}} \binom{n+2s}{s}$$

$$= \left(\frac{z}{2}\right)^n \sum_{s=0}^{\infty} \frac{z^{2s}(-1)^s}{\cancel{(2s+n)!}} \frac{\cancel{(n+2s)!}}{(n+s)!\, s!\; 2^{2s}} = \left(\frac{z}{2}\right)^n \sum_{s=0}^{\infty} \frac{(-1)^s (z^2/4)^s}{s!\,(n+s)!}$$

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta \qquad \text{let } v = \pi - \theta \Rightarrow \theta = \pi - v$$
$$dv = -d\theta$$

$$= \frac{1}{\pi}\int_{\pi}^{0} \cos\big(n(\pi - v) - z\underbrace{\sin(\pi - v)}_{\sin\pi\cos(v) - \cos\pi\sin v \,=\, \sin v}\big)(-dv)$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos(+nv - z\sin(v) - n\pi)\,dv = \frac{1}{\pi}\int_0^{\pi} \cos(nv - z\sin(v) - n\pi)\,dv$$

plus minus minus

$$= \frac{1}{\pi}\int_0^{\pi} \cos(nv - z\sin(v))\,\cancel{\sin(n\pi)}\,\cos(n\pi)\,dv = \frac{(-1)^n}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta$$

$$= (-1)^n J_n(z) \quad ✓$$

$$\cancel{\frac{1}{\pi}\int_0^{\pi} \sin\Theta \sin \Theta \, d\Theta = -\frac{\sin\Theta}{\pi} \cos\Theta \Big|_0^{\pi}}$$

or

$$+\int_0^{\pi} \cos\Theta \, d\left(\sin\theta \frac{d\theta}{d\Theta}\right)$$

$$= \int_0^{\pi} \cos\Theta \left[\cos\theta \, d\theta \frac{d\theta}{d\Theta} + \sin\theta \, d\left(\frac{d\theta}{d\Theta}\right)\right]$$

$$\{Z J_n'\}' = \frac{-Z}{\pi}\int_0^{\pi} \sin^2\theta \cos\Theta \, d\theta + \frac{1}{\pi}\int_0^{\pi} \sin\theta \sin\Theta \, d\theta$$

$$= \frac{-Z}{\pi}\int_0^{\pi} (1-\cos^2\theta)\cos\Theta \, d\theta + \frac{1}{\pi}\int_{"0"}^{"\pi"} \underbrace{\sin\theta(\Theta)}_{\text{pull at } 0} \sin\Theta \, \underbrace{\frac{d\theta}{d\Theta}}_{\text{pull at}} d\Theta$$

$$+ \frac{1}{\pi} \sin\theta \frac{d\theta}{d\Theta}(-\cos\Theta \Big|_0^{\pi}$$

$$- \int_0^{\pi} (-\cos\Theta) \frac{d}{d\Theta}\left(\sin\theta \frac{d\theta}{d\Theta}\right) d\Theta$$

$$= -\frac{Z}{\pi}\int_0^{\pi} \sin^2\theta \cos\Theta \, d\theta + \int_0^{\pi} \cos\Theta \left\{ \cos\theta \frac{d\theta}{d\Theta}\frac{d\theta}{d\Theta} + \sin\theta \frac{d^2\theta}{d\Theta^2} \right\} d\Theta$$

But $\Theta = n\theta - Z\sin\theta \Rightarrow 1 = n\frac{d\theta}{d\Theta} - Z\cos\theta\frac{d\theta}{d\Theta}$

$$\frac{d\Theta}{d\theta} = n - Z\cos\theta \qquad \Rightarrow \frac{d\theta}{d\Theta} = \frac{1}{n - Z\cos\theta}$$

$$\frac{d^2\Theta}{d\theta^2} = Z\sin\theta \qquad \therefore \text{ Above becomes}$$

$$= -\frac{Z}{\pi}\int_0^{\pi} \sin^2\theta\cos\Theta \, d\theta + \int_0^{\pi}$$

$$\exp\left\{\frac{z}{2}(h-h^{-1})\right\} = \sum_{n=-\infty}^{\infty} C_n h^n$$

w/ $\underbrace{C_n = \frac{1}{2\pi i}\int_C \frac{\exp\left\{\frac{z}{2}(h-h^{-1})\right\}}{h^{n+1}}\,dh}$ By Laurent's thm:

$= J_n(z)$ By 9.03.

I7 $\quad \Theta = n\theta - z\sin\theta$

$$J_n'(z) = \frac{1}{\pi}\int_0^\pi -\sin\Theta\,(-\sin\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \sin\theta\sin\Theta\,d\theta$$

$$\{zJ_n'(z)\}' = \frac{1}{\pi}\int_0^\pi \frac{d}{dz}\{z\sin\theta\sin\Theta\}\,d\theta = \frac{1}{\pi}\int_0^\pi (\sin\theta\sin\Theta + z\sin\theta\cos\Theta \cdot -\sin\theta)\,d\theta$$

$$= \underbrace{\frac{1}{\pi}\int_0^\pi \sin\theta\sin\Theta\,d\theta}_{\text{I.B.P's.}} - \frac{z}{\pi}\int_0^\pi \sin^2\theta\cos\Theta\,d\theta$$

? $\rightarrow$ $\frac{d\Theta}{d\theta}\sin\Theta\,d\theta$

$\cancel{n\frac{d\Theta}{d\theta}} - z\cos\theta$

$$= -\frac{z}{\pi}\int_0^\pi \sin^2\theta\cos\Theta\,d\theta + \frac{1}{\pi}\Big[$$

$n - z\cos\theta = \frac{d\Theta}{d\theta}$

$\rightarrow \cos\theta$

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$$z\{zJ_n'(z)\}' = \underbrace{-\frac{z^2}{\pi}\int_0^\pi \cos\Theta\, d\theta}_{-z^2 J_n(z)} + \frac{nz}{\pi}\int_0^\pi \cos\theta\cos\Theta\, d\theta$$

$$z\{zJ_n'(z)\}' + z^2 J_n(z) = \frac{nz}{\pi}\int_0^\pi \cos\theta\cos\Theta\, d\theta$$

$$\Rightarrow z\{zJ_n'(z)\}' + (z^2-n^2)J_n(z) = \frac{nz}{\pi}\int_0^\pi \cos\theta\cos\Theta\, d\theta - \frac{n^2}{\pi}\int_0^\pi \cos\Theta\, d\theta$$

$$= \frac{n}{\pi}\int_0^\pi \underbrace{(z\cos\theta - n)}_{-\frac{d\Theta}{d\theta}}\cos\Theta\, d\theta$$

$$= -\frac{n}{\pi}\int_0^\pi \cos\Theta\, d\Theta = -\frac{n}{\pi}\sin\Theta\Big|_{\theta=0}^{\pi}$$

$$= -\frac{n}{\pi}(\sin(n\pi) - 0) = 0$$

$$\Rightarrow z^2 J_n'' + zJ_n' + (z^2-n^2)J_n = 0$$

If ν is non integer

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{4}z^2\right)^s}{s!\,\Gamma(\nu+s+1)}$$

If ν is neg integer say $-n$

$$J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z^2}{4}\right)^s}{s!\,\Gamma(-n+s+1)}$$

Now $\Gamma(-n) = \infty$ for n integer

$\therefore\ \Gamma(-n+s+1) = \infty \quad s = 0, 1, 2, \cdots n-1$

As All other terms vanish

$$\therefore\ J_{-n}(x) = \left(\frac{x}{2}\right)^{-n} \sum_{s=n}^{\infty} \frac{(-1)^s \left(\frac{z^2}{4}\right)^s}{s!\,\Gamma(-n+s+1)}$$

$$= \left(\frac{x}{2}\right)^{-n} \sum_{s=0}^{\infty} \frac{(-1)^{s+n} \left(\frac{z^2}{4}\right)^{s+n}}{(s+n)!\,\Gamma(s+1)} = (-1)^n \left(\frac{x}{2}\right)^{-n} \left(\frac{x^2}{4}\right)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z^2}{4}\right)^s}{\Gamma(s+n+1)\,s!}$$

$$= (-1)^n \left(\frac{x}{2}\right)^{-n} \left(\frac{x}{2}\right)^{2n} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x^2}{4}\right)^s}{\Gamma(s+n+1)\,s!} = (-1)^n \underbrace{\left(\frac{x}{2}\right)^n \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x^2}{4}\right)^s}{\Gamma(s+n+1)\,s!}}_{J_n(x)}$$

$$J_\nu(z e^{m\pi i}) = \frac{(z)^\nu e^{\nu m\pi i}}{2^\nu} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{4}\right)^s e^{m\pi i 2s}\, z^{2s}}{s!\,\Gamma(s+\nu+1)}$$

$$= e^{\nu m\pi i} \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{2}\right)^{2s}}{s!\,\Gamma(s+\nu+1)}\, 1 = e^{\nu m\pi i} J_\nu(z)$$

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$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^t t^{-z}\,dt$$

$$J_\nu(z) = (z/2)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s (z^2/4)^s}{s!\,\Gamma(\nu+s+1)} = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty} \frac{(-1)^s (z^2/4)^s}{s!} \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^t t^{-(\nu+s+1)}\,dt$$

$$= \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} (z^2/4)^s \int_{-\infty}^{(0^+)} e^t t^{-\nu-s-1}\,dt$$

$$= \left(\frac{z}{2}\right)^\nu \frac{1}{2\pi i} \int_{-\infty}^{(0^+)} \left(\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{(z^2/4)^s}{t^s}\right) e^t \frac{1}{t^{\nu+1}}\,dt$$

$$= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int^{(0^+)} \exp\left\{\frac{-z^2}{4t}\right\} e^t \frac{dt}{t^{\nu+1}}$$

$$= \frac{(z/2)^\nu}{2\pi i}\int_{-\infty}^{(0^+)} \exp\left\{t - \frac{z^2}{4t}\right\} \frac{dt}{t^{\nu+1}}$$ let $z > 0$ & $t = \frac{z}{2}h$

$$= \frac{(z/2)^\nu}{2\pi i}\int_{-\infty}^{(0^+)} \exp\left\{\frac{z}{2}h - \frac{z^2}{4}\frac{1}{(z/2)h}\right\} \frac{\cancel{z/2}\,dh}{\cancel{(z/2)^{\nu+1}}\,h^{\nu+1}}$$ $dt = \frac{z}{2}\,dh$

$$= \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} \exp\left\{\frac{z}{2}(h - h^{-1})\right\} \frac{dh}{h^{\nu+1}}$$ let $h = e^\tau$, $dh = e^\tau d\tau$

$$= \frac{1}{2\pi i}\int \exp\left\{\frac{z}{2}(e^{\tau}-e^{-\tau})\right\}\frac{e^{\tau}\,d\tau}{e^{(\nu+1)\tau}}$$

$$h = e^{\tau} = e^{re^{i\theta}}$$

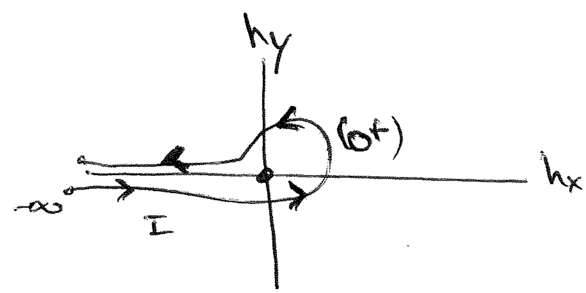
$$= e^{r(\cos\theta + i\sin\theta)}$$

$$= e^{r\cos\theta}\cdot e^{ir\sin\theta}$$

$$= e^{r\cos\theta}\left\{\underline{\cos(r\sin\theta)} + i\sin(r\sin\theta)\right\}$$

to go to $-\infty$ τ

$$= \frac{1}{2\pi i}\int \exp\left\{\frac{z}{2}(e^{\tau}-e^{-\tau})\right\}\frac{d\tau}{e^{\nu\tau}}$$



$$= \frac{1}{2\pi i}\int e^{z\sinh\tau - \nu\tau}\,d\tau$$

$h = e^{\tau}$

$\tau = \log h$

$= \log|h| + i\operatorname{Arg} h$

$=$

Now get limits.

At $-\infty$ $h \to -\infty$. $\operatorname{Arg} h = -\pi$

~~Around Zero~~.

$\Rightarrow \tau \to +\infty - \pi i$

Around 0 h wraps around Zero $\longrightarrow$ (over)

$\Rightarrow \tau$ phh changes from $-\pi$ to π.

At $-\infty$ $h \to -\infty$ $\operatorname{Arg} h = \pi$

$$\Rightarrow \quad J_\nu(z) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} \exp\{z\sinh\tau - \nu\tau\}\, d\tau$$

$$\frac{1}{2} z J_{\nu-1}(z) + \frac{z}{2} J_{\nu+1}(z) - \nu J_\nu(z)$$

$$= \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi}\left(\frac{z}{2}e^{z\sinh\tau-(\nu-1)\tau} + \frac{z}{2}e^{z\sinh\tau-(\nu+1)\tau} - \nu e^{z\sinh\tau-\nu\tau}\right)d\tau$$

$$= \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi}\left(z\left(\frac{e^{+\tau}}{2}+\frac{e^{-\tau}}{2}\right)e^{z\sinh\tau-\nu\tau} - \nu e^{z\sinh\tau-\nu\tau}\right)d\tau$$

$$= \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi}\left(z\cosh\tau - \nu\right)e^{z\sinh\tau-\nu\tau}\, d\tau$$

$$= \cancel{\frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi}\frac{d}{d\tau}\left(z\cosh\tau - \nu\right)} = \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi}\frac{d}{d\tau}\left(z\sinh\tau-\nu\tau\right)e^{z\sinh\tau-\nu\tau}\, d\tau$$

$$= \frac{1}{2\pi i}\left(e^{z\sinh\tau-\nu\tau}\Big|_{\infty-i\pi}^{\infty+i\pi}\right.$$

Now $\exp\left\{ z\sinh\tau - \nu\tau \right\}\Big|_{\tau=\infty \pm i\pi}$

$$= \exp\left\{ z\frac{\left(e^{\tau}-e^{-\tau}\right)}{2} - \nu\tau \right\}\Bigg|_{\tau=\infty \pm i\pi}$$

$$\lim_{r\to\infty} \exp\left\{ \frac{z}{2}\left(e^{r\pm i\pi} - e^{-r\mp i\pi}\right) - \nu(r\pm i\pi) \right\}$$

$$= \lim_{r\to\infty} \exp\left\{ \mp\nu i\pi \right\} \cdot \exp\left\{ \frac{z}{2}\left(e^{r\pm\pi i} - e^{-r\mp i\pi}\right) - \nu r \right\}$$

$$\left\{ \frac{z}{2}\Big(\underbrace{e^{\mp\pi i}e^{r}}_{(-1)e^{r}} - \underbrace{e^{\mp i\pi}e^{-r}}_{(-1)e^{-r}}\Big) - \nu r \right\}$$

$$= e^{\mp\nu i\pi} \lim_{r\to\infty} \exp\left\{ \frac{z}{2}\left(-e^{r} + e^{-r}\right) - \nu r \right\}.$$

$$= 0. \quad \checkmark$$

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu$$

$$J_\nu' = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+i\pi} \sinh\tau\, e^{z\sinh\tau - \nu\tau}\, d\tau$$

$$J_{\nu-1} - J_{\nu+1} = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+i\pi} e^{z\sinh\tau-\nu\tau}\left(e^{+\tau} - e^{-\tau}\right) d\tau$$

$$= \frac{1}{2\pi i}\int_{\infty-i\pi}^{\infty+i\pi} 2\sinh\tau\, e^{z\sinh\tau-\nu\tau}\, d\tau$$

$$= 2J_\nu'$$

1st in 9.16 elliminate $J_{\nu-1}$ (From 9.15)

$$J_{\nu-1} = 2J_\nu' + J_{\nu+1}$$

in 9.14

$$2J_\nu' + J_{\nu+1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu$$

$$J_{\nu+1} + J_\nu' = \frac{\nu}{z} J_\nu$$

2nd eq in 9.16 gives. elliminating $J_{\nu+1}$ from Both eqs.

$$J_{\nu+1} = J_{\nu-1} - 2J_\nu' \quad \text{From eq 9.15.}$$

$$\therefore J_{\nu-1} + J_{\nu-1} - 2J_\nu' = \frac{2\nu}{z} J_\nu$$

$$J_{\nu-1} - J_\nu' = \frac{\nu}{z} J_\nu$$

eq 9.16 $\Rightarrow$ $J_1(z) = 0 - J_0'(z)$

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$$J_\nu(iz) = \frac{i^\nu z^\nu}{2^\nu} \sum_{s=0}^{\infty} \frac{\cancel{(-1)^s} \cancel{(-1)^s} (z^2/4)^s}{s!\,\Gamma(\nu+s+1)} = \underbrace{e^{i\pi\nu/2} (z/2)^\nu \sum_{s=0}^{\infty} \frac{(z^2/4)^s}{s!\,\Gamma(\nu+s+1)}}_{I_\nu(z)}$$

From 9.09

$$I_\nu(z) = e^{-i\pi\nu/2} J_\nu(iz)$$

As $J_\nu(z)$ satisfies $\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)w = 0$

$$J_{\pm\nu}(z) = J_{\pm\nu}(v/i)$$

Transforming to $v = iz$

$dv = i\,dz$

$$\Rightarrow -\frac{d^2w}{dv^2} + \frac{1}{(v/i)}\frac{dw}{(dv/i)} + \left(1 - \nu^2/(v/i)^2\right)w = 0$$

$$\Rightarrow \frac{d^2w}{dv^2} + \frac{1}{v}\frac{dw}{dv} - \left(1 + \frac{\nu^2}{v^2}\right)w = 0$$

Thus ~~$J_{\pm\nu}(iz)$~~ ~~satisfy~~

Try $v = -iz$

$dv = -i\,dz$

$$-\frac{d^2w}{dv^2} + \frac{1}{(v/-i)}\frac{dw}{(dv/-i)} + \left(1 - \frac{\nu^2}{(v/-i)^2}\right)w = 0$$

$$\Rightarrow \frac{d^2w}{dv^2} + \frac{1}{v}\frac{dw}{dv} - \left(1 + \frac{\nu^2}{v^2}\right)w = 0.$$

Thus

$J_{\pm\nu}(z)$ is sat. by

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}+\left(1-\frac{\nu^2}{z^2}\right)w=0 \qquad w=J_{\pm\nu}(z)$$

Thus

$$\frac{d^2w}{d\cancel{z}^2}+\frac{1}{\cancel{z}}\frac{dw}{d\cancel{z}}-\left(1+\frac{\nu^2}{\cancel{z}^2}\right)w=0 \quad \text{sat. by}$$

$$w=J_{\pm\nu}(\cancel{z}/-i)$$

$$=J_{\pm\nu}(iv)$$

$$\frac{d^2w}{dz^2}+\frac{1}{z}\frac{dw}{dz}+\left(1-\frac{\nu^2}{z^2}\right)w=0$$

let $z=\underline{f(v)}$ $\qquad w(v)=J_{\pm\nu}(f(v))$

$dz=f'(v)\,dv$ $\qquad \frac{dw}{d\cancel{z}}$

$$\frac{d}{dz}=\frac{df}{d\cancel{v}}\frac{d}{df}$$

$$\frac{d^2}{dz^2}=\frac{df}{d\cancel{v}}\frac{d}{df}\left(\frac{df}{d\cancel{v}}\frac{d}{df}\right)=$$

If ~~f(v)~~ = $f(v)=av+b$ $\qquad \therefore \frac{d^2}{dz^2}=a^2\frac{d^2}{df^2}$

$$\frac{df}{dv}=a$$

$$\Rightarrow \quad a^2 \frac{d^2 w}{df^2} + \frac{a}{(a\cancel{x}+b)} \frac{dw}{df} + \left(1 - \frac{\nu^2}{(ax+b)^2}\right) w = 0$$

$$\Rightarrow \quad \frac{d^2 w}{df^2} + \frac{1}{a(ax+b)} \frac{dw}{df} + \frac{1}{a^2}\left(1 - \frac{\nu^2}{(ax+b)^2}\right) w = 0$$

Sol to This orig eq is $J_{\pm\nu}(f(z))$

From 9.14 $\quad J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu$ one gets.

$$J_{\nu-1}(iz) + J_{\nu+1}(iz) = \frac{2\nu}{iz} J_\nu(iz)$$

$$\Rightarrow e^{(\nu-1)\pi i/2} I_{\nu-1}(z) + e^{(\nu+1)\pi i/2} I_{\nu+1}(z) = \frac{2\nu}{iz} e^{\nu\pi i/2} I_\nu(z)$$

$$\Rightarrow e^{-\pi i/2} I_{\nu-1}(z) + e^{\pi i/2} I_{\nu+1}(z) = \frac{2\nu}{iz} I_\nu(z)$$

$$I_{\nu-1} + e^{\pi i} I_{\nu+1} = \frac{e^{\pi i/2}}{i} \frac{2\nu}{z} I_\nu(z)$$

$$I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{z} I_\nu$$

From 9.15 we get

$$e^{(\cancel{\nu}-1)\pi i/2} I_{\nu-1} - e^{(\cancel{\nu}+1)\pi i/2} I_{\nu+1} = 2\cancel{e^{\nu\pi i/2}} I_\nu' /i$$

$$\left\{ J_\nu(z) = e^{\nu\pi i/2} I_\nu(z/i) \right\} \Rightarrow I_{\nu-1} e^{-\pi i/2} - e^{\pi i/2} I_{\nu+1} = 2 I_\nu' /i$$

$$J_\nu' = e^{\nu\pi i/2} I_\nu'(z/i)/i \Rightarrow J_\nu'(iz) = e^{\nu\pi i/2} I_\nu'(z)/i$$

$$\Rightarrow I_{\nu-1} + I_{\nu+1} = 2 I_\nu'$$ 2nd eq in 10.04.

From 9.16 we get

$$\cancel{e^{(\nu+1)\pi i/2}} I_{\nu+1}(z) = \frac{\nu}{iz} \cancel{e^{\nu\pi i/2}} I_\nu(z) - \cancel{e^{\nu\pi i/2}} I_\nu'(z)/i$$

$$\Rightarrow I_{\nu+1} = \frac{(-1)\nu}{z} I_\nu + I_\nu'(z)$$ 1st eq in 10.05

From 9.16 2nd eq we get

$$e^{(\cancel{\nu}-1)\pi i/2} I_{\nu-1}(z) = \frac{\nu}{iz} e^{\cancel{\nu}\pi i/2} I_\nu(z) + \cancel{e^{\nu\pi i/2}} I_\nu'(z)/i$$

$$I_{\nu-1} = \frac{\nu}{z} I_\nu + I_\nu'$$

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Consider $\displaystyle\int_0^\infty e^{-st} t^{z-1}\,dt$ let $v = st$ =
$dv = s\,dt$

$$= \int_0^\infty e^{-v} \frac{v^{z-1}}{s^{z-1}} \frac{dv}{s} = \frac{1}{s^z} \underbrace{\int_0^\infty e^{-v} v^{z-1}\,dv}_{\Gamma(z)}$$

$$\therefore \quad \frac{1}{s^z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-st} t^{z-1}\,dt$$

$$\cancel{\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty} \qquad \zeta(z) = \sum_{s=1}^{\infty} \frac{1}{s^z} = \sum_{s=\phi}^{\infty} \frac{1}{\Gamma(z)} \int_0^\infty e^{-st} t^{z-1}\,dt$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} \underbrace{\sum_{s=1}^{\infty} e^{-ts}}_{\underbrace{\frac{(e^{-t})^\infty - (e^{-t})^1}{(e^{-t}-1)}}_{\frac{e^{-t}}{1-e^{-t}}}} dt \qquad \left\{ \sum_{k=n_1}^{n_2} r^k = \frac{r^{n_2+1} - r^{n_1}}{r-1} \right.$$

$$\frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\,dt$$

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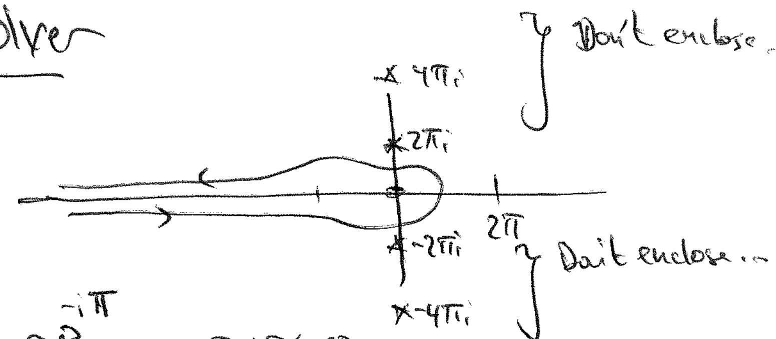
$$I(z) = \int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1}\,dt$$



= on Bottom path $t = +re^{-i\pi}$ $\quad 0<r<\infty$

✓ $dt = dre^{-i\pi}$; on top path $t = re^{i\pi}$

$dt = e^{i\pi}dr$

$$= \int_{r=\infty}^{0} \frac{r^{z-1}e^{-i\pi(z-1)}}{\exp\{-re^{-i\pi}\}-1} e^{-i\pi}dr + \int_0^\infty \frac{r^{z-1}e^{i\pi(z-1)} e^{i\pi}dr}{\exp\{-re^{i\pi}\}-1}$$

$$= e^{-i\pi(z-1)}\int_0^\infty \frac{r^{z-1}dr}{e^r-1} - e^{i\pi(z-1)}\int_0^\infty \frac{r^{z-1}dr}{e^r-1}$$ + integral around curves vanishes

$$= \cancel{\cos(\pi(z-1))}\ \underbrace{-2i\sin(\pi(z-1))}\int_0^\infty \frac{r^{z-1}dr}{e^r-1}$$

$$-2i\left[\sin(\pi z)\cos\pi + 0\right]\int_0^\infty \cdots$$

$$= 2i\sin(\pi z)\int_0^\infty \frac{r^{z-1}dr}{e^r-1} = 2i\sin(\pi z)\Gamma(z)h(z)$$

But $\sin(\pi z)\Gamma(z) = \dfrac{\pi}{\Gamma(1-z)}$

$$\therefore I(z) = \frac{2\pi i}{\Gamma(1-z)}h(z) \Rightarrow h(z) = \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{(0^+)}\frac{t^{z-1}}{e^{-t}-1}\,dt$$

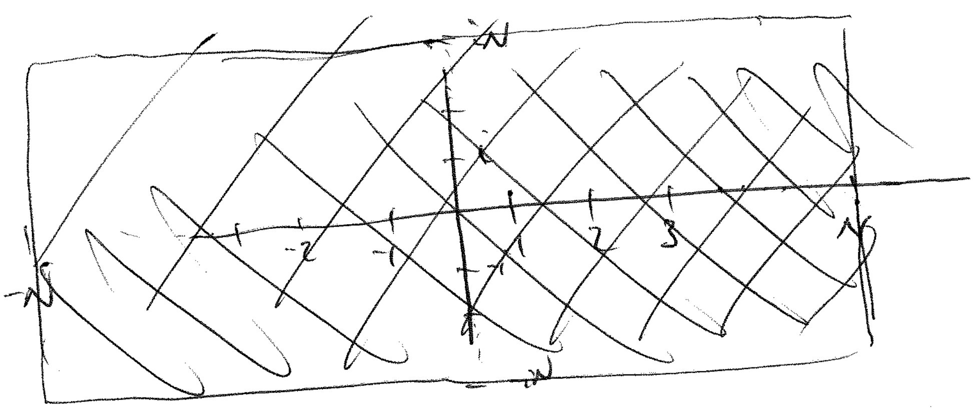
$$\int_{-\infty}^{(0^+)} \frac{dt}{e^{-t}-1}$$

At $t=0$ simple pole? $\quad \frac{1}{\phi(t)} = \frac{1}{e^{-t}-1}$

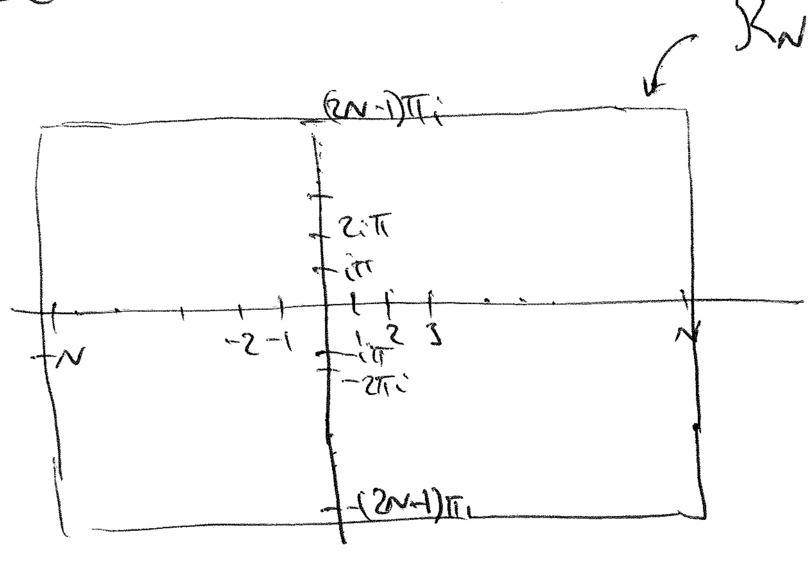
$\phi(t) = e^{-t}-1$

$\phi(0)=0 \qquad \phi'(0) = -e^{-t}\Big|_{t=0} = -1 \neq 0.$ yes simple pole

$$= \int_{-\infty}^{(0^+)} \frac{dt}{e^{-t}-1} = 2\pi i \lim_{t\to 0} \frac{t}{e^{-t}-1} = 2\pi i \frac{1}{-e^{-t}\Big|_{t=0}} = -2\pi i$$



consider $\displaystyle\int_{R_N} \frac{t^{z-1}}{e^{-t}-1}\,dt$



on R_N ~~=~~ Any fn must Achieve min/max out in R_N actually on R_N

$$|e^{-t}-1| = |e^{-(N+\eta i)}-1| \qquad \pi(2N-1) \le \eta \le (2N-1)\pi$$

If $\quad |e^{-t}-1| \geq 1-e^{-N}$

Then $$\left|\int_{R_N} \frac{t^{z-1}}{e^{-t}-1}\,dt\right| \le \frac{1}{(1-e^{-N})}\int_{R_N} |t^{z-1}|\,dt$$

Find Residue of

$$\frac{t^{z-1}}{e^{-t}-1} \qquad \text{At } t = \pm 2\pi s i$$

This is a simple pole $\Rightarrow$ $\mathrm{Res}(\pm 2\pi s i, f(t))$

$$= \left.\frac{t^{z-1}}{-e^{-t}}\right|_{\pm 2\pi i s} = \frac{(\pm 2\pi i s)^{z-1}}{-(e^{2\pi i s})} = -(\pm 2\pi s i)^{z-1}$$

$$\zeta(z) = \Gamma(1-z)\left\{\sum_{s=1}^{\infty} -(+2\pi s i)^{z-1} + \sum_{s=1}^{\infty} -(-2\pi s i)^{z-1}\right\}$$

$$h(z) = \Gamma(1-z)\left\{ \sum_{s=1}^{\infty} 2^{z-1} s^{z-1} \pi^{z-1} \underbrace{\left((i)^{z-1} + (-i)^{z-1}\right)} \right\}$$

$$\left(e^{\frac{\pi}{2}i}\right)^{z-1} + \left(e^{-\frac{\pi}{2}i}\right)^{z-1}$$

$$= e^{\frac{\pi}{2}(z-1)i} + e^{-\frac{\pi}{2}(z-1)i}$$

$$= 2\cos\left(\frac{\pi}{2}(z-1)\right)$$

$$h(z) = \Gamma(1-z)\left\{ \sum_{s=1}^{\infty} 2^{z} s^{z-1} \pi^{z-1} \cos\left(\frac{\pi}{2}(z-1)\right) \right\}$$

$$= \Gamma(1-z) 2^{z} \pi^{z-1} \cos\left(\frac{\pi}{2}(z-1)\right) \sum_{s=1}^{\infty} s^{z-1}$$

$$\cancel{h(z-1)} \sum_{s=1}^{\infty} s^{-(1-z)} = h(1-z)$$

$$h(z) = \Gamma(1-z) 2^{z} \pi^{z-1} \cos\left(\frac{\pi}{2}(z-1)\right) h(1-z)$$

$$h(1-z) = \frac{h(z)}{\Gamma(1-z) 2^{z} \pi^{z-1}\left(\cos\left(\frac{\pi}{2}z\right)\cancel{\cos\frac{\pi}{2}}_{0} + \sin\left(\frac{\pi}{2}z\right)\sin\frac{\pi}{2}\right)}$$

$$= \frac{h(z)}{\Gamma(1-z) 2^{z} \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)} = \frac{h(z)}{\dfrac{\pi^{z} 2^{z} \sin\left(\frac{\pi}{2}z\right)}{\sin \pi z\, \Gamma(z)}}$$

$$\Rightarrow \quad \zeta(1-z) = \pi^{-z} 2^{-z} \zeta(z) \frac{\sin \pi z}{\sin\left(\frac{\pi z}{2}\right)} \Gamma(z)$$

$$= \pi^{-z} 2^{-z+1} \zeta(z) \cos\left(\frac{\pi}{2} z\right) \Gamma(z)$$

From 11.05

let $z = -2m$

$$\underbrace{\zeta(1+2m)}_{\text{finite}} = 2^{1+2m} \pi^{+2m} \cos(\pi m) \underbrace{\Gamma(-2m)}_{\infty} \underbrace{\zeta(-2m)}_{0.}$$

$$\zeta(-2m) = 0.$$

let $z = 2m$

$$\zeta(1-2m) = 2^{1-2m} \pi^{-2m} \underbrace{\cos\left(\frac{\pi}{2} 2m\right)}_{(-1)^m} \Gamma(2m) \zeta(2m)$$

$$= (-1)^m 2^{1-2m} \pi^{-2m} (2m-1)! \zeta(2m)$$

let $\underline{z = 1}$

$$\zeta(0) = \pi^{-1} \cos\left(\frac{\pi}{2}\right) \Gamma(1) \zeta(1)$$

$\underline{z = 0}$

$$\zeta(1) = 2^1 (1) \Gamma(0) \zeta(0)$$

$$\zeta(0) = \frac{1}{2} \frac{\zeta(1)}{\Gamma(0)}$$

But to evaluate to get $\zeta(0) = ?$

consider 11.03

$$\frac{\zeta(z)}{\Gamma(1-z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} \frac{t^{z-1}}{e^{-t}-1}\,dt$$

The $\cancel{\frac{\zeta(0)}{\Gamma(1)}}$ $\frac{\zeta(1)}{\Gamma(0)} = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} \frac{dt}{e^{-t}-1} = \frac{-2\pi i}{2\pi i} = -1$

$$\Rightarrow \quad \zeta(0) = -\frac{1}{2}$$

---

consider $\quad 2^{-z}\zeta(z) = 2^{-z}\sum_{s=1}^{\infty} s^{-z} = \sum_{s=1}^{\infty}(2s)^{-z}$

$\therefore \quad \zeta(z) - 2^{-z}\zeta(z)$

$$\zeta(z)(1-2^{-z}) = \sum_{s=1}^{\infty} s^{-z} - \underbrace{\sum_{s=1}^{\infty}(2s)^{-z}}_{\text{removes even terms.}} = \frac{1}{1^z}+\frac{1}{3^z}+\frac{1}{5^z}+\frac{1}{7^z}+\dots$$

Then

$$\zeta(z)(1-2^{-z}) - 3^{-z}(1-2^{-z})\zeta(z)$$

$$= \sum_{s=0}^{\infty}(2s+1)^{-z} - \sum_{s=0}^{\infty}(3(2s+1))^{-z} = \sum_{s=0}^{\infty}(2s+1)^{-z} - \sum_{s=0}^{\infty}(6s+3)^{-z}$$

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$$\left|\sum_{\substack{s\neq 1\\ s\neq \omega_1,\omega_2,\dots,\omega_n}} \frac{1}{s^z}\right| \leq \sum_{s=\omega_n+1}^{\infty} \frac{1}{|s|^z} = \sum_{s=\omega_n+1}^{\infty} \frac{1}{s^{\Re z}}.$$

Now $|s|^z = |e^{z\log s}| = |e^{(z_r+iz_i)\log s}| = e^{z_r \log s} = s^{z_r} = s^{\Re(z)}$.

$$\zeta(z)\prod_{s=1}^{\infty}(1-\omega_n^{-z}) = 1$$

Check $\displaystyle\sum_{s=2}^{\infty}|s^{-z}| = \sum_{s=2}^{\infty}\frac{1}{|s|^z} < \infty \quad \Re(z)>1$

$\Rightarrow$ product is Absolutely convergent.

$$\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\left(\frac{\pi z}{2}\right)\Gamma(z)\zeta(z) \quad \text{reflection formula}$$

$$\Rightarrow \prod_{s=0}^{\infty}(1-\omega_s^{-z}) \neq 0$$

$$\underbrace{\prod_{s=0}^{\infty}(1-\omega_s^{-z})}\zeta(1-z) = 2^{1-z}\pi^{-z}\cos\left(\frac{\pi z}{2}\right)\Gamma(z)$$

Abs convergent $\forall z \ni \Re(z)>1$ RHS vanish at $z = 1, 3, 5, \dots 2n+1$

$\Rightarrow \underbrace{\zeta(0)}, \zeta(-2), \zeta(-4), \zeta(-6), \dots, \zeta(-2n)$

Not sure well $\Re(z) \not\geq 1$ + for $z=1$ $\Re(z) = 1$.

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$$\Gamma(\alpha,z) = \int_z^\infty e^{-t} t^{\alpha-1}\,dt = e^{-t}\frac{t^{\alpha}}{\alpha}\Big|_z^\infty - \frac{1}{\alpha}\int_z^\infty (-e^{-t})\,t^{\alpha}\,dt$$

or $$-t^{\alpha-1}e^{-t}\Big|_z^\infty - \int_z^\infty (\alpha-1)t^{\alpha-2}(-e^{-t})\,dt$$

$$= z^{\alpha-1}e^{-z} + \int_z^\infty (\alpha-1)t^{\alpha-2}e^{-t}\,dt$$

$$= z^{\alpha-1}e^{-z} + (\alpha-1)\int_z^\infty t^{\alpha-2}e^{-t}\,dt$$

$$= z^{\alpha-1}e^{-z} + (\alpha-1)\Gamma(\alpha-1,z) \qquad (\alpha-1)e^{-z}z^{\alpha-2}$$

$$= z^{\alpha-1}e^{-z} + (\alpha-1)\left[-e^{-t}t^{\alpha-2}\Big|_z^{\infty} + \int_z^\infty e^{-t}(\alpha-2)t^{\alpha-3}\,dt\right]$$

$$= z^{\alpha-1}e^{-z} + (\alpha-1)z^{\alpha-2}e^{-z} + (\alpha-1)(\alpha-2)\int_z^\infty t^{\alpha-3}e^{-t}\,dt$$

$$= z^{\alpha-1}e^{-z}\left\{1+\frac{\alpha-1}{z}\right\} + (\alpha-1)(\alpha-2)\left[\underbrace{-t^{\alpha-3}e^{-t}\Big|_z^\infty}_{z^{\alpha-3}e^{-z}} + \int_z^\infty (\alpha-3)t^{\alpha-4}e^{-t}\,dt\right]$$

$$z^{\alpha-3}e^{-z} + (\alpha-1)(\alpha-2)(\alpha-3)\int_z^\infty t^{\alpha-4}e^{-t}\,dt$$

$$= z^{\alpha-1} e^{-z} \left\{ 1 + \frac{(\alpha-1)}{z} + \frac{(\alpha-1)(\alpha-2)}{z^2} + \frac{(\alpha-1)(\alpha-2)(\alpha-3)}{z^3} + \cdots \right.$$

$$\left. \frac{(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n+1)}{z^{n-1}} \right\} + \underbrace{(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n) \int_z^{\infty} t^{\alpha-n-1} e^{-t}\, dt}_{\epsilon_n(z)}$$

If $t^{\alpha-n-1} \le z^{\alpha-n-1}$ $\quad t \in (z, \infty)$

$$\therefore \ |\epsilon_n(z)| \le |(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-n)| \int_z^{\infty} |t^{\alpha-n-1} e^{-t}|\, dt$$

$$= |(\alpha-1)(\alpha-2)\cdots(\alpha-n)|\, z^{\alpha-n-1} \left(-e^{-t}\right)\Big|_z^{\infty}$$

$$= |(\alpha-1)(\alpha-2)\cdots(\alpha-n)|\, z^{\alpha-n-1} e^{-z}$$

Check $\quad \epsilon_n(z) = O(z^{-n})$ $\quad$ check

$$\therefore \ P(\alpha, x) \quad \lim_{z\to\infty} \frac{\epsilon_n(z)}{1/z^n} = \lim_{z\to\infty} |(\alpha-1)(\alpha-2)\cdots(\alpha-n)|\, z^{\alpha-1} e^{-z}$$

$$= 0 \le 1 \text{ say. } \checkmark$$

$$\therefore \Gamma(\alpha, x) \sim e^{-x} x^{\alpha-1} \sum_{s=0}^{\infty} \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-s)}{z^{s}}$$

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$$\varepsilon_n(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-n)\underbrace{\int_x^\infty e^{-t}t^{\alpha-n-1}\,dt}_{\text{always positive.}}$$

& n+1 st term of the series is

$$\frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{x^n} = a_{n+1}(x)$$

$\therefore$ $\varepsilon_n(x)$ & $a_{n+1}(x)$ are of the same sign.

$$\therefore |\varepsilon_n(x)| \le |(\alpha-1)(\alpha-2)\cdots(\alpha-n)|\, e^{-x}x^{\alpha-n-1} \quad \text{by } 1.03$$

$$\le |(\alpha-1)(\alpha-2)\cdots(\alpha-n)|\, \underbrace{x^{\alpha-1}e^{-x}}\, x^{-n}$$

$$\le \frac{|(\alpha-1)(\alpha-2)\cdots(\alpha-n)|}{}$$

$x^{\alpha-n-1} < x^{-n}$ when

$x^{\alpha-1} < 1$? How get $n \geq \alpha - 1$?

$$\Gamma(\alpha,x) = \underbrace{\int_z^\infty e^{-t}t^{\alpha-1}\,dt}_{\text{Always positive}} \qquad \Gamma(\alpha,x) = e^{-x}x^{\alpha-1} + \varepsilon_1(x)$$

& $\varepsilon_1(x)$ is the same sign as

$$a_1(x) = \frac{\alpha-1}{x} \gtrless 0 \quad \text{or} \quad \cancel{\varepsilon_1(x)} \; 0$$

< 0 if $\alpha < 1$

$$\therefore \Gamma(\alpha,x) \le e^{-x}x^{\alpha-1} \qquad \Rightarrow \varepsilon_1(x) < 0$$

$$\varepsilon_n(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-n)\int_x^\infty e^{-t}t^{\alpha-n-1}\,dt$$

$$= (\alpha-1)(\alpha-2)\cdots(\alpha-n)\left[t^{\alpha-n-1}(-e^{-t}\Big|_x^\infty - \int_x^\infty(-e^{-t})(\alpha-n-1)t^{\alpha-n-2}\,dt\right]$$

$$= (\alpha-1)(\alpha-2)\cdots(\alpha-n)\left[e^{-x}x^{\alpha-n-1} + \int_x^\infty(\alpha-n-1)e^{-t}t^{\alpha-n-2}\,dt\right]$$

$$= (\alpha-1)(\alpha-2)\cdots\underline{(\alpha-n)}\,e^{-x}x^{\alpha-n-1} + (\alpha-1)(\alpha-2)\cdots(\alpha-n)(\alpha-n-1)\int_x^\infty e^{-t}t^{\alpha-n-2}\,dt$$

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$$\int_0^\infty |e^{-xt} q(t)\,dt| \le \cdots$$

$$I(x) = \int_0^\infty e^{-xt} q(t)\,dt = \frac{q(t)e^{-xt}}{-x}\Big|_0^\infty - \frac{1}{(-x)}\int_0^\infty q'(t)e^{-xt}\,dt$$

$$= \frac{q(0)}{x} + \frac{1}{x}\int_0^\infty q'(t)e^{-xt}\,dt = \frac{q(0)}{x} + \frac{1}{x}\left[\frac{q'(t)e^{-xt}}{-x}\Big|_0^\infty - \int_0^\infty \frac{1}{(-x)} q''(t)e^{-xt}\,dt\right]$$

$$= \frac{q(0)}{x} + \frac{1}{x}\left[\frac{q'(0)}{x} + \frac{1}{x}\int_0^\infty q''(t)e^{-xt}\,dt\right]$$

$$= \frac{q(0)}{x} + \frac{q'(0)}{x^2} + \frac{1}{x^2}\int_0^\infty q''(t)e^{-xt}\,dt = \cdots$$

$$= \frac{q(0)}{x} + \frac{q'(0)}{x^2} + \frac{q''(0)}{x^3} + \cdots + \frac{q^{(n-1)}(0)}{x^n} + \underbrace{\frac{1}{x^n}\int_0^\infty q^{(n)}(t)e^{-xt}\,dt}_{E_n(x)}$$

If $q^{(n)}(t) = O(e^{\sigma t}) \quad 0 \le t < \infty$

$$E_n(x) = \frac{1}{x^n}\int_0^\infty e^{-xt} O(e^{\sigma t})\,dt = \frac{1}{x^n}\int_0^\infty O(e^{(\sigma-x)t})\,dt = \frac{1}{x^n} O\left(\frac{e^{(\sigma-x)t}}{(\sigma-x)}\right)\Big|_0^\infty$$

$$= \cancel{O(} \; \frac{1}{x^n(\sigma - x)} O(1) = O\left(\frac{1}{x^n(\sigma-x)}\right) = O\left(\frac{1}{x^n(x-\sigma)}\right)$$

$$= O\left(\frac{1}{x^{n+1}}\right) = O\left(\frac{1}{x^n}\right)$$

$$I(x) \sim \sum_{s=0}^{\infty} \frac{q^{(s)}(0)}{x^{s+1}}$$

$$\varepsilon_n(x) = \frac{1}{x^n}\int_0^\infty e^{-xt} q^{(n)}(t)\,dt$$

$$\therefore |\varepsilon_n(x)| \leq \frac{1}{x^n}\int_0^\infty e^{-xt}|q^{(n)}(0)|\,dt \leq \frac{|q^{(n)}(0)|}{x^n}\frac{e^{-xt}}{(-x}\Big|_0^\infty$$

$$= \frac{|q^{(n)}(0)|}{x^{n+1}}$$

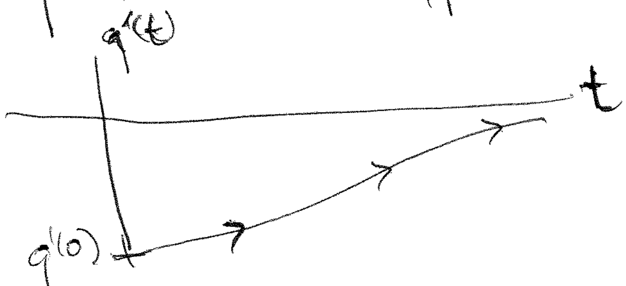
If $q(t)$ is an alternating fn...

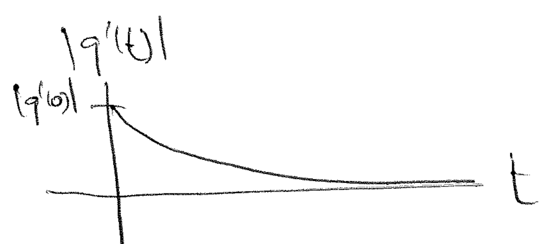
$q^{(0)}(t) = q(t) > 0$ Thus $q(t)$ is decreasing for

$q'(t) < 0 \Rightarrow q(t)$ is decreasing for $t \geq 0$. $q(t) \leq q(0)$

$q''(t) > 0 \Rightarrow q'(t)$ is increasing. $q'(t) \geq q'(0)$.

But $q'(t) < 0 \Rightarrow |q'(t)| > 0$





$\therefore$ ~~$|q'(0)| \leq$~~ $|q'(t)| \leq |q'(0)|$

Assume $|q^{(n-1)}(t)| \le |q^{(n-1)}(0)|$

& pv for $q^{(n)}(t)$

Now ~~$q^{(n)}(t)$~~ Case A: n even

$$q^{(n)}(t) \geq 0 \quad q^{(n+1)}(t) \le 0 \Rightarrow q^{(n)}(t) \text{ is decreasing}$$

$$\Rightarrow q^{(n)}(t) \le q^{(n)}(0) \quad \text{done}$$

Case B: n odd

$$q^{(n)}(t) \le 0 \quad q^{(n+1)}(t) \geq 0 \Rightarrow q^{(n)} \text{ is increasing}$$

$$\Rightarrow |q^{(n)}(t)| \text{ is } = -q^{(n)}(t) \text{ (when}$$

$q^{(n)}(t) < 0$) is decreasing.

$$\Rightarrow |q^{(n)}(t)| \le |q^{(n)}(0)| \quad \text{done} \checkmark.$$

$$\mathcal{E}_n(x) - \mathcal{E}_{n+1}(x) = \frac{1}{x^n}\int_0^\infty e^{-xt} q^{(n)}(t)\,dt - \frac{1}{x^{n+1}}\int_0^\infty e^{-xt} q^{(n+1)}(t)\,dt$$

$$= \frac{1}{x^{n+1}}\left[\int_0^\infty \left(xe^{-xt} q^{(n)}(t) - e^{-xt} q^{(n+1)}(t)\right) dt\right]$$

$$= \frac{1}{x^{n+1}}\left[\int_0^\infty \cancel{e^{-xt}(xq^{(n)}} \left(-\frac{d}{dt}\left(e^{-xt} q^{(n)}(t)\right)\right) dt\right.$$

$$= \frac{-1}{x^{n+1}} \int_0^\infty \frac{d}{dt}\left(e^{-xt} q^{(n)}(t)\right) dt$$

$$= \frac{-1}{x^{n+1}} e^{-xt} q^{(n)}(t) \Big|_0^\infty = \frac{-1}{x^{n+1}}\left(0 - q^{(n)}(0)\right) = \frac{q^{(n)}(0)}{x^{n+1}}$$

$\mathcal{E}_n$ & $\mathcal{E}_{n+1}$ opposite signs $\quad x-y=R$

$x, y \in \mathbb{R}^+$.

If $\mathcal{E}_n$ & $\mathcal{E}_{n+1}$ have opp. signs then $q^{(n)}(0)$ has the same sign as $\mathcal{E}_n(x)$ & $-\mathcal{E}_{n+1}$ Thus these 2 $\mathcal{E}$'s "sum" to $q^{(n)}(0)x^{-n-1}$ & Both must be less than $q^{(n)}(0)x^{-n-1}$

$$\mathcal{E}_n(x) = \frac{1}{x^n}\left[q^{(n)}(t)\frac{e^{-xt}}{-x}\Big|_0^\infty + \frac{1}{x}\int_0^\infty q^{(n+1)}(t)e^{-xt}\,dt \right]$$

$$= \frac{1}{x^{n+1}}\left(q^{n}(0) + \int_0^\infty q^{(n+1)}(t)e^{-xt}\,dt\right)$$

$$= \frac{1}{x^{n+1}}\left(q^{(n)}(0) + \frac{1}{x}q^{(n+1)}(0) + \frac{1}{x}\int_0^\infty q^{(n+2)} e^{-xt}\,dt\right)$$

$$= \frac{1}{x^{n+1}}\left(q^{(n)}(0) + \frac{1}{x}q^{(n+1)}(0) + \frac{1}{x}\left[\frac{q^{(n+2)}(0)}{x} + \frac{1}{x}\underbrace{\int_0^\infty q^{(n+3)}e^{-xt}\,dt}_{?\ \text{if}\ O(1)}\right]\right.$$

$$= \frac{q^{(n)}(0)}{x^{n+1}} + \frac{q^{(n+1)}(0)}{x^{n+2}} + \underbrace{\frac{q^{(n+2)}(0)}{x^{n+3}} + \mathcal{O}(x^{-n-3})}_{\mathcal{O}(x^{-n-3})}$$

$$|\mathcal{E}_n(x)| = x^{-n}\int_0^\infty e^{-xt}|q^{(n)}(t)|\,dt \leq C_n x^{-n}\int_0^\infty e^{-xt}\,dt$$

$$= C_n x^{-n} \left.\frac{e^{-xt}}{-x}\right|_0^\infty = C_n x^{-n-1} \quad \text{w/} \quad C_n = \sup_{(0,\infty)}|q^{(n)}(t)|$$

Alte $|q^{(n)}(t)| \leq |q^{(n)}(0)|e^{\Gamma_n t}$ (depends on t now...)

Then 204 $\Rightarrow$ $|\mathcal{E}_n(x)| \leq \frac{1}{x^n}\int_0^\infty e^{-xt}|q^{(n)}(0)|e^{\Gamma_n t}\,dt$

$$= \frac{|q^{(n)}(0)|}{x^n}\int_0^\infty e^{t(\Gamma_n - x)}\,dt = \frac{|q^{(n)}(0)|}{x^n}\left.\frac{e^{t(\Gamma_n-x)}}{(\Gamma_n-x)}\right|_0^\infty$$

$$= 0 + \frac{|q^{(n)}(0)|}{x^n}\frac{e^{t(x-\Gamma_n)}}{(x-\Gamma_n)} \qquad x > \max(\Gamma_n, 0)$$

$$2.08 \Rightarrow \quad \frac{|q^{(n)}(t)|}{|q^{(n)}(0)|} \le e^{\sigma_n t}$$

$$\log\left(\frac{|q^{(n)}(t)|}{|q^{(n)}(0)|}\right) \le \sigma_n \cancel{\log}\, t$$

$$\sigma_n \geq \frac{1}{t}\log\Big(\qquad\qquad \Rightarrow \quad 2.10$$

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$$\Gamma(\alpha,x) = \int_{x}^{\infty} e^{-t} t^{\alpha-1} dt$$

For complementary incomplete Γ fn

$$\mathcal{E}_n(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-n)\int_x^{\infty} e^{-t} t^{\alpha-n-1} dt$$

let $t = x(1+\tau)$ change var to get limits from $0,\infty$.

& $q(\tau) = (1+\tau)^{\alpha-1}$

Then $\mathcal{E}_n(x) = (\alpha-1)(\alpha-2)\cdots(\alpha-n)\int_{\tau=0}^{\infty} e^{-x(1+\tau)} (x(1+\tau))^{\alpha-n-1} d\tau\, x$

$$= (\alpha-1)(\alpha-2)\cdots(\alpha-n)\, e^{-x} x^{\alpha-n-1} \int_0^{\infty} e^{-x\tau} (1+\tau)^{\alpha-n-1} d\tau\, x$$

$$\Rightarrow \mathcal{E}_n(x) e^{x} x^{-\alpha} = x\, x^{-n-1} \int_0^{\infty} e^{-x\tau} \frac{d}{d\tau}\left((1+\tau)^{\alpha-n} (\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\right) d\tau$$

$$= x^{-n} \int_0^{\infty} e^{-x\tau} \frac{d^{(n)}}{d\tau^{(n)}} (1+\tau)^{\alpha-1}\, d\tau$$

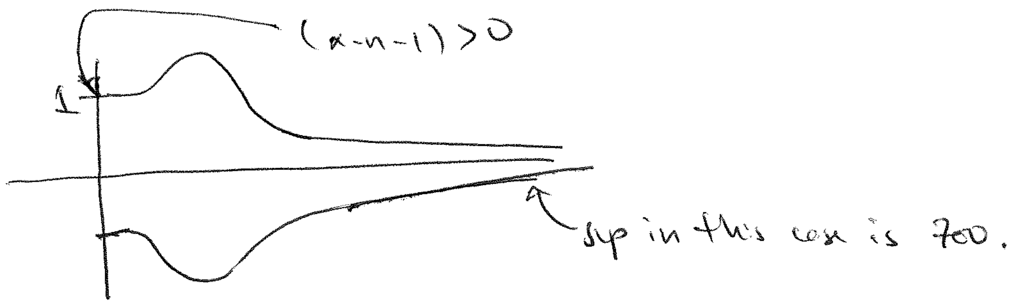
2.10 ⇒

$$f_n = \sup_{(0,\infty)} \left\{ \frac{1}{\tau} \ln\left| \frac{\cancel{(\alpha-1)}\cancel{(\alpha-2)}\cdots\cancel{(\alpha-n)}\,(1+\tau)^{\alpha-n-1}}{\cancel{(\alpha-1)}\cancel{(\alpha-2)}\cdots\cancel{(\alpha-n)}\,(1)} \right| \right\}$$

$$= \sup_{(0,\infty)} \left\{ \frac{1}{\tau} \ln\left| (1+\tau)^{\alpha-n-1} \right| \right\}$$

$$= \sup_{(0,\infty)} \left\{ \frac{(\alpha-n-1)}{\tau} \ln(1+\tau) \right\}$$

$$\frac{\ln(1+\tau)}{\tau}$$



At $\tau = 0$ unknown

$$\frac{\frac{1}{1+\tau}}{1} = 1$$

If $f_n = 0$ ⇒ B_1 2.09 $|E_n(x)| \leq \dfrac{|g^{(n)}(0)|}{x^{n+1}}$ Same as before...

---- If $\alpha-n-1>0$ $f(\tau) = \dfrac{\ln(1+\tau)}{\tau}$ has max on $\tau \in (0,\infty)$ from

$$f'(\tau) = \cancel{\frac{\frac{1}{1+\tau}}{\tau}} - \cancel{\frac{\ln(1+\tau)}{\tau^2}}$$

Now $\lim_{\tau\to 0} f(\tau) = 1$ (see above) & τ grows more quickly than $\ln(1+\tau)$

$$\Rightarrow \frac{\ln(1+\tau)}{\tau} \leq 1 = \lim_{\tau\to 0} \frac{\ln(1+\tau)}{\tau}$$

$\Rightarrow \sup_{(0,\infty)} \left\{ \left| \frac{1}{t} \right. \right.$

$\sigma_n = \alpha - n - 1$

$(2.09) \Rightarrow \quad |\varepsilon_n(x)| \le \dfrac{|q^{(n)}(0)|}{x^n(x-(\alpha-n-1))}$

why won't this work?

$$= \frac{\dfrac{|(\alpha-1)(\alpha-2)\cdots(\alpha-n)(1)|}{x^n}}{}$$

If $\quad |q^{(n)}(\tau)| \le |q^{(n)}(0)| e^{\sigma_n t}$

$\Rightarrow \quad q^{(n)}$ in 2.11

$$|e^x x^{-\alpha} \varepsilon_n(x)| = \left| x^{-n} \int_0^\infty e^{-xt} q^{(n)}(\tau)\, d\tau \right| \le x^{-n} |q^{(n)}(0)| \int_0^\infty e^{(\sigma_n - x)t}\, dt$$

$$= x^{-n} |q^{(n)}(0)| \left(\frac{e^{(\sigma_n - x)t}}{\sigma_n - x} \right|_0^\infty = \frac{x^{-n} |q^{(n)}(0)|}{x - \sigma_n}$$

$$\Rightarrow |\varepsilon_n(x)| \le \frac{x^{-n+\alpha} (\alpha-1)(\alpha-2)\cdots(\alpha-n) e^{-x}}{x - (\alpha-n-1)}$$

$$\Rightarrow |\varepsilon_n(x)| \le \frac{(\alpha-1)(\alpha-2)\cdots(\alpha-n) e^{-x} x^{\alpha-n}}{x - \alpha + n + 1}$$

$$\Gamma(\alpha,x) = a_0(x) + \epsilon_1(x)$$

$$\Gamma(\alpha,x) - \underbrace{a_0(x)}_{e^{-x}x^{\alpha-1}} = \epsilon_1(x)$$

$$\Gamma(\alpha,x) \leq e^{-x}x^{\alpha-1} + \underbrace{\frac{e^{-x}x^{\alpha-1}}{x-\alpha+2}}(\alpha-1)$$

taking positive. $|\epsilon_n| \leq \cdots$

$$\leq e^{-x}x^{\alpha-1}\left[\underbrace{\frac{x-\cancel{\alpha}+2+\cancel{\alpha}-1}{x-\alpha+2}}_{\frac{x-1}{x-\alpha+2}}\right]$$

or

$$\Gamma(\alpha,x) = \epsilon_0(x) \leq \frac{e^{-x}x^{\alpha}}{x-\alpha+1}$$

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$$q(t) = \sum_{n=0}^{\infty} q^{(n)}(0)\frac{t^n}{n!}$$

$$I(x) = \int_0^{\infty} e^{-xt} q(t)\,dt = \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!} \int_0^{\infty} e^{-xt} t^n\,dt$$

let $v = xt$
$dv = x\,dt$

$$= \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!} \int_0^{\infty} e^{-v} \frac{v^n}{x^{n+1}}\,dv$$

$$= \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!\,x^{n+1}} \underbrace{\int_0^{\infty} e^{-v} x^{(n+1)-1}\,dv}_{\Gamma(n+1) = n!} = \sum_{n=0}^{\infty} \frac{q^{(n)}(0)}{n!}$$

Given $q(t) \sim \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu}$ can one get 3.03 from 3.02 & integrating this expression term by term?

$$\int_0^{\infty} q(t) e^{-xt}\,dt \overset{?}{\sim} \int_0^{\infty} \sum_{s=0}^{\infty} a_s t^{(s+\lambda-\mu)/\mu} e^{-xt}\,dt$$

$$\sim \sum_{s=0}^{\infty} a_s \int_0^{\infty} e^{-xt} t^{(s+\lambda-\mu)/\mu}\,dt.$$

$v = xt$ Assume $x > 0$

$dv = x\,dt$

$$\sim \sum_{s=0}^{\infty} a_s \int_0^{\infty} e^{-v} \frac{v^{(s+\lambda-\mu)/\mu}}{x^{(s+\lambda-\mu)/\mu}} \frac{dv}{x} = \sum_{s=0}^{\infty} \frac{a_s}{x^{(s+\lambda)/\mu}} \int_0^{\infty} e^{-v} v^{(s+\lambda-\mu)/\mu} dv$$

$$= \sum_{s=0}^{\infty} \frac{a_s}{x^{(s+\lambda)/\mu}} \underbrace{\int_0^{\infty} e^{-v} v^{\frac{(s+\lambda)}{\mu}-1} dv}_{\Gamma\left(\frac{s+\lambda}{\mu}\right)} = \sum_{s=0}^{\infty} \frac{a_s}{x^{(s+\lambda)/\mu}} \Gamma\left(\frac{s+\lambda}{\mu}\right)$$

same as 303

$$\phi_n(t) = q(t) - \sum_{s=0}^{n-1} a_s t^{(s+\lambda-\mu)/\mu}$$

$$e^{-xt}\phi_n(t) = e^{-xt} q(t) - \sum_{s=0}^{n-1} a_s e^{-xt} t^{(s+\lambda-\mu)/\mu}$$

$$\int_0^{\infty} e^{-xt}\phi_n(t)\,dt = \int_0^{\infty} e^{-xt} q(t)\,dt - \sum_{s=0}^{n-1} a_s \underbrace{\int_0^{\infty} e^{-xt} t^{(s+\lambda-\mu)/\mu}\,dt}_{\frac{1}{x^{(s+\lambda)/\mu}}\Gamma\left(\frac{s+\lambda}{\mu}\right)}$$

From Above

$$\Rightarrow \int_0^\infty e^{-xt} q(t)\,dt = \sum_{s=0}^{n-1} \frac{a_s}{x^{(s+1)/\mu}} \Gamma\left(\frac{s+1}{\mu}\right) + \int_0^\infty e^{-xt} \phi_n(t)\,dt$$

As $t \to 0$ $\quad \phi_n(t) = \mathcal{O}\left(t^{(n+1-\mu)/\mu}\right)$ By def of Asym. exp.

$$\Rightarrow \exists\, k_n \in \mathbb{R}^+ \ni \quad |\phi_n(t)| \le k_n t^{(n+1-\mu)/\mu}$$

Because $\quad \lim\limits_{t \to 0} \dfrac{|\phi_n(t)|}{t^{(n+1-\mu)/\mu}} = k_n$.

$$\Rightarrow \exists\, k_n \quad |\phi_n(t)| \le k_n t^{(n+1-\mu)/\mu} \qquad 0 < t \le k_n$$

$$\therefore \left| \int_0^{k_n} e^{-xt} \phi_n(t)\,dt \right| \le k_n \int_0^{k_n} e^{-xt}\, t^{(n+1-\mu)/\mu}\,dt$$

$$\le k_n \int_0^\infty e^{-xt}\, t^{(n+1-\mu)/\mu}\,dt$$

let $\quad v = xt$
$\quad dv = x\,dt$

$$= k_n \int_0^\infty e^{-v} \frac{v^{(n+1-\mu)/\mu}}{x^{(n+1-\mu)/\mu}} \frac{dv}{x}$$

$$= \frac{k_n}{x^{(n+1)}} \underbrace{\int_0^\infty e^{-v}\, v^{(n+1)/\mu - 1}\,dv}_{\Gamma\left(\frac{n+1}{\mu}\right)}$$

$$\therefore \left| \int_0^{k_n} e^{-xt} \phi_n(t)\,dt \right| \le \Gamma\left(\frac{n+1}{\mu}\right) \frac{k_n}{x^{(n+1)/\mu}} .$$

(1) $\Rightarrow$ $\underbrace{\int_0^{\infty} e^{-xt} q(t)\,dt}_{\text{exists } \forall x \text{ suff. large}} = \sum_{s=0}^{n-1} \Gamma\left(\frac{s+1}{\mu}\right) \frac{a_s}{x^{(s+1)/\mu}} + \int_0^{\infty} e^{-xt} \phi_n(t)\,dt$

$\Rightarrow \int_0^{\infty} e^{-xt} \phi_n(t)\,dt$ exists $\forall x$ suff large

Let $\bar{X}$ be real.

$$\int_{k_n}^{\infty} e^{-xt} \phi_n(t)\,dt = \int_{k_n}^{\infty} e^{-(x-\bar{x})t} e^{-\bar{x}t} \phi_n(t)\,dt$$

Note:

$$\frac{d}{dt} \Phi_n(t) = e^{-\bar{x}t} \phi_n(t)$$

$$\therefore \text{Above} = \int_{k_n}^{\infty} e^{-(x-\bar{x})t} \frac{d}{dt} \Phi_n(t)\,dt$$

$$= e^{-(x-\bar{X})t} \Phi_n(t) \Big|_{k_n}^{\infty} + (x-\bar{X}) \int_{k_n}^{\infty} e^{-(x-\bar{X})t} \Phi_n(t)\, dt$$

$$= 0 - 0 + (x-\bar{X}) \int_{k_n}^{\infty} e^{-(x-\bar{X})t} \Phi_n(t)\, dt$$

$$\therefore \left| \int_{k_n}^{\infty} e^{-xt} \phi_n(t)\, dt \right| \le (x-\bar{X}) L_n \int_{k_n}^{\infty} e^{-(x-\bar{X})t}\, dt$$

$$= (x-\bar{X}) L_n \left(\frac{e^{-(x-\bar{X})t}}{-(x-\bar{X})} \right|_{k_n}^{\infty}$$

$$= L_n e^{-(x-\bar{X})k_n}$$

$$\therefore \left| \int_0^{\infty} e^{-xt} \phi_n(t)\, dt \right| \le \left| \int_0^{k_n} \cdots dt \right| + \left| \int_{k_n}^{\infty} \cdots dt \right|$$

$$\le \Gamma\left(\frac{n+1}{u}\right) \frac{k_n}{x^{(n+1)/u}} + L_n e^{-(x-\bar{X})k_n}$$

$\Rightarrow$ can get $\displaystyle\int_0^{\infty} e^{-xt} \phi_n(t)\, dt = O\left(x^{-(n+1)/u}\right) \quad x \to \infty$

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$$\int_0^\infty \frac{e^{-xit}}{t^\delta}\,dt \qquad 0<\delta<1$$

Obvibusly $\displaystyle\int_0^\infty \frac{dt}{t^\delta} \not< \infty$ if $0<\delta<1$ then.

what does it mean converge uniformly at 0 & ∞.

$$|q(t)-q(t_s)| < \frac{\epsilon}{2(b-a)} \qquad t_{s-1} \le t \le t_s$$

for $s = 1,2,3,\ldots,n$

$$\int_a^b e^{ixt} q(t)\,dt =$$

$$= \sum_{\delta=1}^{n} \int_{t_{s-1}}^{t_s} q(t) e^{ixt}\,dt$$

$$= \sum_{s=1}^{n} q(t_s) \int_{t_{s-1}}^{t_s} e^{ixt}\,dt + \sum_{s=1}^{n} \int_{t_{s-1}}^{t_s} e^{ixt}\left[q(t)-q(t_s)\right] dt$$

$$\left|\int_\alpha^\beta e^{ixt}\,dt\right| = \left| \frac{e^{ixt}}{ix}\Big|_\alpha^\beta \right| = \left| \frac{e^{ix\beta} - e^{ix\alpha}}{ix} \right| \le \frac{2}{x}$$

$$\left|\int_a^b e^{ixt} q(t)\,dt\right| \le \sum_{s=1}^{n} |q(t_s)| \left|\int_{t_{s-1}}^{t_s} e^{ixt}\,dt\right| + \left|\sum_{s=1}^{n} \int_{t_{s-1}}^{t_s} e^{ixt} \{q(t)-q(t_s)\}\,dt\right|$$

$$\le Q\frac{2}{x}n + \sum_{s=1}^{n} \int_{t_{s-1}}^{t_s} |e^{ixt}|\,|q(t)-q(t_s)|\,dt$$

$$\le \frac{2Qn}{x} + \frac{\epsilon}{2(b-a)} \sum_{s=1}^{n} (\underbrace{t_s - t_{s-1}}_{\Delta t_{s-1}})$$

from $|q(t)-q(t_s)| < \frac{\epsilon}{2(b-a)}$

$$= \frac{2Qn}{x} + \frac{\epsilon}{2(b-a)} \left(t_{s-1} \Big|_1^{n+1} \right)$$

$$= \frac{2Qn}{x} + \frac{\epsilon}{2\cancel{(b-a)}} \underbrace{(t_n - t_0)}_{\cancel{b-a}} = \frac{2Qn}{x} + \frac{\epsilon}{2} < \epsilon$$

If $\frac{2Qn}{x} < \frac{\epsilon}{2}$

$$x > \frac{4Qn}{\epsilon}$$

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$$I(x) = \int_a^b e^{ixt} q(t)\,dt$$

$$= q(t)\frac{e^{ixt}}{ix}\Big|_a^b - \frac{1}{ix}\int_a^b q'(t)e^{ixt}\,dt$$

$$= \frac{+i}{x}\left\{ \cancel{q(b)}\; q(a)e^{ixa} - q(b)e^{ixb} \right\} + \underbrace{\frac{i}{x}\int_a^b q'(t)e^{ixt}\,dt}_{E_1(x)}$$

$$= \frac{i}{x}\left\{ q(a)e^{ixa} - q(b)e^{ixb} \right\}$$

$$+ \frac{i}{x}\left[q'(t)\frac{e^{ixt}}{ix}\Big|_a^b + \frac{i}{x}\int_a^b q''(t)e^{ixt}\,dt \right]$$

$$= \frac{i}{x}\left[q(a)e^{ixa} - q(b)e^{ixb} \right] + \frac{i}{x}\left[\frac{i}{x}\left(q'(a)e^{ixa} - q'(b)e^{ixb} \right) \right.$$

$$\left. + \frac{i}{x}\int_a^b q''(t)e^{ixt}\,dt \right]$$

$$= \left(\frac{i}{x}\right)\left[q(a)e^{ixa} - q(b)e^{ixb} \right] + \left(\frac{i}{x}\right)^2\left[q'(a)e^{ixa} - q'(b)e^{ixb} \right]$$

$$+ \left(\frac{i}{x}\right)^2 \int_a^b q''(t)e^{ixt}\,dt$$

$$= \sum_{s=0}^{n-1} \left(\frac{i}{x}\right)^{s+1} \left\{ g^{(s)}(a)e^{ixa} - g^{(s)}(b)e^{ixb} \right\}$$

$$+ \left(\frac{i}{x}\right)^{n} \int_a^b g^{(n)}(t)\, e^{ixt}\, dt$$