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$x = N_1(\xi) x_{2i-1} + N_2(\xi) x_{2i} + N_3(\xi) x_{2i+1}$ how know only possible form? or only solution?

If $x_{2i} = \frac{1}{2}(x_{2i-1} + x_{2i+1})$

$$\Rightarrow \quad x = \frac{1}{2}(\xi - 1)\xi x_{2i-1} + \frac{1}{2}(1-\xi^2)(x_{2i-1} + x_{2i+1}) + \frac{1}{2}\xi(\xi+1) x_{2i+1}$$

$$= \frac{1}{2}\left[(\cancel{\xi^2} - \xi) x_{2i-1} + (1 - \cancel{\xi^2}) x_{2i-1} + (1 - \cancel{\xi^2}) x_{2i+1} + (\cancel{\xi^2} + \xi) x_{2i+1}\right]$$

$$= \frac{1}{2}\left[(1-\xi) x_{2i-1} + (1+\xi) x_{2i+1}\right]$$

$$= \frac{1}{2}(1-\xi) x_{2i-1} + \frac{1}{2}(1+\xi) x_{2i+1} \qquad \text{eq 3.65.}$$

$$T = \sum_{\text{elements}} T_j N_j$$

$$-k \int_{e_k} W \frac{d^2 T}{dx^2} dx$$

$$= -k \left[W \frac{dT}{dx} \Bigg|_{x_{2k-1}}^{x_{2k+1}} - \int_{x_{2k-1}}^{x_{2k+1}} \frac{dW}{dx} \frac{dT}{dx} dx \right]$$

k denotes the element.

$$= k \int_{x_{2k-1}}^{x_{2k+1}} \frac{dN_i^{(e_k)}}{dx} \frac{dN_j^{(e_k)}}{dx} T_j \, dx + \left(-k N_i^{(e_k)} \frac{dT}{dx}\right) \Bigg|_{x_{2k-1}}^{x_{2k+1}} = 0$$

$$J = (-2\xi)\frac{L}{2} + \frac{1}{2}(2\xi+1)L = -\xi L + L\xi + \frac{L}{2} = \frac{L}{2}$$

$$\frac{dN_i^{(e)}}{dx} = J^{-1}\frac{dN_i}{d\xi} = \frac{2}{L}\frac{dN_i}{d\xi}$$

$$\Rightarrow \begin{bmatrix} \frac{dN_1^{(e)}}{dx} \\ \frac{dN_2^{(e)}}{dx} \\ \frac{dN_3^{(e)}}{dx} \end{bmatrix} = \frac{2}{L}\begin{bmatrix} \frac{1}{2}(2\xi-1) \\ -2\xi \\ \frac{1}{2}(2\xi+1) \end{bmatrix} = \frac{1}{L}\begin{bmatrix} -1+2\xi \\ -4\xi \\ 1+2\xi \end{bmatrix}$$ eq 3.79

$$k\int_{x_{2k-1}}^{x_{2k+1}} \begin{bmatrix} \frac{dN_1^{(e)}}{dx} \\ \frac{dN_2^{(e)}}{dx} \\ \frac{dN_3^{(e)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_1^{(e)}}{dx} & \frac{dN_2^{(e)}}{dx} & \frac{dN_3^{(e)}}{dx} \end{bmatrix} dx = \quad |J|\,d\xi$$

$$= \frac{kL}{2}\cdot\left(\frac{1}{L}\right)^2 \int_{-1}^{1} \begin{pmatrix} -1+2\xi \\ -4\xi \\ 1+2\xi \end{pmatrix} \begin{pmatrix} -1+2\xi & -4\xi & 1+2\xi \end{pmatrix} d\xi$$

$$= \frac{k}{2L}\int_{-1}^{1}$$

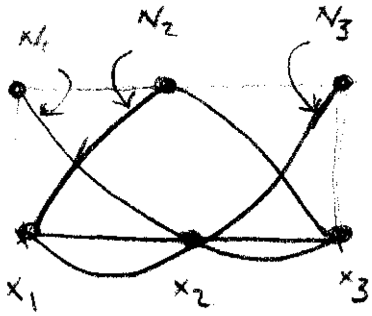
eq 3.80 ✓ed w/ MMA.

Pg 36 Append Heat Cond.

From F.E.D for 1 element:

$$\left[k\int_{x_{2k-1}}^{x_{2k+1}} \frac{dN_i^{(k)}}{dx}\,\frac{dN_j^{(k)}}{dx}\,dx\right]T_j + \left(-k\frac{dT}{dx}N_i\right|_{x=x_{2k-1}}^{x=x_{2k+1}} = 0 \qquad i=1,2,3.$$

$i=1$



w/ $-k\frac{dT}{dx}\Big|_{\xi=-1} = q$ (*)

$$\Leftrightarrow\; k\int_{x_{2k-1}}^{x_{2k+1}}\Big[\quad\Big]\Big[\qquad\Big]\,dx - q\begin{bmatrix}1\\0\\0\end{bmatrix} = 0$$

$$= k\int_{x_{2k-1}}^{x_{2k+1}}\begin{bmatrix}\frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_1}{dx}\frac{dN_2}{dx} & \frac{dN_1}{dx}\frac{dN_3}{dx}\\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} & \frac{dN_2}{dx}\frac{dN_3}{dx}\\ \frac{dN_3}{dx}\frac{dN_1}{dx} & \frac{dN_3}{dx}\frac{dN_2}{dx} & \frac{dN_3}{dx}\frac{dN_3}{dx}\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix}$$ } encapsulate all test fns.

Then All Boundary terms vanish but one at $\xi = -1$,

(*) Testing w/ N_1, N_2, N_3 gives

$N_1(x_{2k+1}) = N_1(L) = 0$ } only x

$N_1(x_{2k-1}) = N_1(0) = 1$

$N_2(x_{2k+1}) = N_2(L) = 0$

$N_2(x_{2k-1}) = N_2(0) = 0$

$N_3(x_{2k+1}) = N_3(L) = 0$

$N_3(x_{2k-1}) = N_3(0) = 0$

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$$Q\int_{x_k}^{x_{k+1}} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \end{bmatrix} dx = Q\int_{-1}^{1}$$

let: to evaluate the integral

$$x(\xi) = x_k \frac{1}{2}(1-\xi) + x_{k+1}\frac{1}{2}(1+\xi) \qquad \begin{cases} x(-1) = x_k \\ x(1) = x_{k+1} \end{cases}$$

$$dx = \left(-\frac{1}{2}x_k + \frac{x_{k+1}}{2}\right)d\xi = \frac{L}{2}d\xi$$

$$= \frac{LQ}{2}\int_{-1}^{1}\begin{bmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{bmatrix} d\xi = \frac{LQ}{2}\cdot\frac{1}{2}\begin{bmatrix} \frac{-(1-\xi)^2}{2} \\ \frac{(1+\xi)^2}{2} \end{bmatrix}\Bigg|_{-1}^{1} = \frac{LQ}{8}\begin{bmatrix} 0+2^2 \\ 4-0 \end{bmatrix} = \frac{LQ}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quadratic elements:

using 3.74

$$Q\int_{x_{2i-1}}^{x_{2i+1}} \begin{bmatrix} N_1^{(e)} \\ N_2^{(e)} \\ N_3^{(e)} \end{bmatrix} dx \qquad \begin{cases} J = \frac{dx}{d\xi} = x_{2i-1}\frac{1}{2}(2\xi-1) + x_{2i}(-2\xi) + x_{2i+1}\frac{1}{2}(2\xi+1) \\ \quad = \frac{1}{2}(2\xi-1)x_{2i-1} - 2\xi x_{2i} + \frac{1}{2}(2\xi+1)x_{2i+1} \end{cases}$$

$$= Q\frac{L}{2}\int_{-1}^{1}\begin{bmatrix} \frac{1}{2}\xi(\xi-1) \\ 1-\xi^2 \\ \frac{1}{2}\xi(\xi+1) \end{bmatrix} d\xi \qquad J = \frac{dx}{d\xi} = \frac{L}{2}$$

$$d\xi = J^{-1}dx$$

$$= \frac{QL}{2}\int_{-1}^{1}\begin{bmatrix} \frac{1}{2}(\xi^2-\xi) \\ 1-\xi^2 \\ \frac{1}{2}(\xi^2+\xi) \end{bmatrix} d\xi = \frac{QL}{2}\begin{bmatrix} \frac{1}{2}\left(\frac{\xi^3}{3}-\frac{\xi^2}{2}\right) \\ \xi-\xi^3/3 \\ \frac{1}{2}\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right) \end{bmatrix}\Bigg|_{\xi=-1}^{\xi=+1}$$

$$= \frac{QL}{2}\begin{bmatrix} \frac{1}{2}\left(\frac{1}{3}-\frac{1}{2}\right)-\frac{1}{2}\left(-\frac{1}{3}-\frac{1}{2}\right) \\ 1-\frac{1}{3}-\left(-1+\frac{1}{3}\right) \\ \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right)-\frac{1}{2}\left(-\frac{1}{3}+\frac{1}{2}\right) \end{bmatrix} = \frac{QL}{2}\begin{bmatrix} \frac{1}{2}\left[\frac{1}{3}-\cancel{\frac{1}{2}}+\frac{1}{3}+\cancel{\frac{1}{2}}\right] \\ 2\left(1-\frac{1}{3}\right) \\ \frac{1}{2}\left[\frac{1}{3}+\cancel{\frac{1}{2}}+\frac{1}{3}-\cancel{\frac{1}{2}}\right] \end{bmatrix}$$

$$= \frac{QL}{2}\begin{bmatrix} \frac{1}{2}\,\frac{2}{3} \\ 2\,{}^{2}\!/_{3} \\ \frac{1}{2}\,\frac{2}{3} \end{bmatrix} = \frac{QL}{2}\begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} = \frac{QL}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \quad \text{eq } 3.83$$

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Example 3.5

$$\sum_{j=1}^{3}\left(\int_0^h \frac{dN_i}{dx} k(x) \frac{dN_j}{dx}\,dx\right) T_j + N_i^{(e)} h (T - T_\infty)\Big|_{x=0} = 0 \qquad i = 1,2,3$$

to natural coordinates: $x(\xi) = N_1(\xi)\, x_{2i-1} + N_2(\xi) x_{2i} + N_3(\xi) x_{2i+1}$

w/ x_{2i-1} = left end of element = 0 $\quad \frac{dx}{d\xi} = J = \frac{L}{2}$

x_{2i} = middle of element = $L/2$

x_{2i+1} = right end of element = $L = 0.1$

$$\Rightarrow \int_0^h \underbrace{\begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \\ \frac{dN_3}{dx} \end{bmatrix} \begin{bmatrix} N_1(x) & N_2 & N_3 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}}_{} \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} dx + \underbrace{\begin{bmatrix} h(T_1 - T_\infty) \\ 0 \\ 0 \end{bmatrix}_{x=0}}_{} = \bar{0}$$

decomposition of a fn $f(x)$ is done by

$$f(x) \cong \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} -hT_\infty \\ 0 \\ 0 \end{bmatrix}}_{}$$

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \end{bmatrix} + \begin{bmatrix} -40\,000 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{2}{L}\frac{d}{d\xi}$$

$$\Rightarrow \int_{-1}^{1} \begin{bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \\ \frac{dN_3}{d\xi} \end{bmatrix} \left(\frac{2}{L}\right) \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} \begin{bmatrix} \frac{dN_1}{d\xi} & \frac{dN_2}{d\xi} & \frac{dN_3}{d\xi} \end{bmatrix} \left(\frac{2}{\cancel{L}}\right) \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \frac{\cancel{L}}{2}\, d\xi$$

$L = \frac{1}{10}$ meter is the correct value

$$+ 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \bar{0}$$

$$= \frac{2}{L} \int_{-1}^{1} \underbrace{\begin{bmatrix} \frac{1}{2}(2\xi - 1) \\ -2\xi \\ \frac{1}{2}(2\xi + 1) \end{bmatrix} \begin{bmatrix} -\frac{1}{2}\xi(\xi-1) & 1-\xi^2 & \frac{1}{2}\xi(\xi+1) \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(2\xi-1) & -2\xi & \frac{1}{2}(2\xi+1) \end{bmatrix}}_{3\times 3} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} d\xi$$

$$+ \quad 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 40000 \\ 0 \\ 0 \end{bmatrix}$$

$$= \Bigg\{ 20 \int_{-1}^{1} \begin{bmatrix} \frac{1}{4}\xi(\xi-1)(2\xi-1) & \frac{1}{2}(1-\xi^2)(2\xi-1) & \frac{1}{4}\xi(\xi+1)(2\xi-1) \\ -\xi^2(\xi-1) & -2\xi(1-\xi^2) & -\xi^2(\xi+1) \\ \frac{1}{4}(2\xi+1)\xi(\xi-1) & \frac{1}{2}(2\xi+1)(1-\xi^2) & \frac{1}{4}(2\xi+1)\xi(\xi+1) \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 20(2\xi-1) & -80\xi & 20(2\xi+1) \\ 25(2\xi-1) & -100\xi & 25(2\xi+1) \\ 30(2\xi-1) & -120\xi & 30(2\xi+1) \end{bmatrix} d\xi + 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \quad "$$

$$= 20 \int_{-1}^{1} \Big((2\xi-1)^2 \big(5\xi(\xi-1) + 10(1-\xi^2) + \tfrac{15}{2}\xi(\xi+1) \big) \cdots \qquad \text{Finish} \ldots$$

...

$$= \Bigg\{ \frac{\cancel{20}5}{\cancel{2}} \int_{-1}^{1} (5+\xi) \begin{bmatrix} (2\xi-1)^2 \\ -4\xi(1+2\xi) \\ (1+2\xi)(\end{bmatrix} d\xi + 100 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= 100 \begin{pmatrix} \frac{34}{3} & -12 & \frac{5}{3} \\ -12 & \frac{80}{3} & -\frac{44}{3} \\ \frac{5}{3} & -\frac{44}{3} & 13 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 40000 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 34 & -36 & 5 \\ -36 & 80 & -44 \\ 5 & -44 & 39 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 3.400 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 68 & -72 & 10 \\ -72 & 160 & -88 \\ 10 & -88 & 78 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 2400 \\ 0 \\ 0 \end{pmatrix} \qquad \text{eq } 3.35$$

imposing B.C at T_3

$$\Rightarrow \begin{pmatrix} 68 & -72 & 10 \\ -72 & 160 & -88 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} 2400 \\ 0 \\ 39.18 \end{pmatrix}$$

Solving
$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{matrix} 99.9924 \\ 66.54 \\ 39.18 \end{matrix}$$

Quadratic w/ 1 element gives relative error = $8 \cdot 10^{-5}$

$h \neq .1$ $h = .05$ as in linear case h = distance between nodes

Linear elements w/ $h = .05$ give relative error = $\frac{|99.822 - 100|}{100}$

$= 1.78 \cdot 10^{-3}$

Check eq 3.36

$|e(x)| \le Ch^{n+1}$

Quadratic

$|e(x)| \le C(.05)^3 = 1.25 \cdot 10^{-4}$

Linear

$|e(x)| \le C(.05)^2 = C\,.0025 = C\,2.5 \cdot 10^{-3}$

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$$\int_0^L W(x)\left(\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2}\right)dx = 0$$

$$= \int_0^L W(x)\frac{\partial T}{\partial t} - \alpha \int_0^L W(x)\frac{\partial^2 T}{\partial x^2}\,dx = 0$$

$$-\alpha\left[W(x)\frac{\partial T}{\partial x}\Big|_0^L - \int_0^L W'(x)\frac{\partial T}{\partial x}\,dx\right] = 0$$

$$= \int_0^L \left(W(x)\frac{\partial T}{\partial t} + \alpha W'(x)\frac{\partial T}{\partial x}\right)dx - \alpha W(x)\frac{\partial T}{\partial x}\Big|_0^L$$

$$T(x,t) = \sum_{i=1}^{n+1} T_i N_i(x)$$

$$= \int_0^L \left\{ N_i(x)\sum_{j=1}^{n+1}\dot{T}_j(t)N_j(x) + \alpha \frac{dN_i}{dx}\sum_{j=1}^{n+1} N_j'(x)T_j\right\}dx - \alpha N_i(x)\frac{\partial T}{\partial x}\Big|_0^L = 0$$

eq 3.95

$$= \int_0^L N_i(x)N_j(x)\dot{T}_j\,dx + \alpha \frac{dN_i}{dx}\frac{dN_j}{dx}T_j + \left(N_i(x)\left(-\alpha\frac{\partial T}{\partial x}\right)\right)\Big|_0^L = 0$$

eq 3.97

$$\left[\int_0^L N_i(x)N_j(x)\,dx\right]\dot{T}_j + \left[\alpha\int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx}\right]T_j + \left[N_i(x)\left(-\alpha\frac{\partial T}{\partial x}\right)\right]\Big|_0^L = 0$$

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$$\int_0^L N_i N_j \, dx \left(\frac{T_j^{n+1} - T_j^n}{\Delta t} \right) + \left[\alpha \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx \right] \left(\Theta T_j^{n+1} + (1-\Theta) T_j^n \right)$$

$$+ \cancel{N_i \left(-\alpha \frac{\partial T}{\partial x} \right)} \Big|_0^L$$

$$+ \left[N_i(x) \left(-\alpha \left(\Theta \frac{\partial T^{n+1}}{\partial x} + (1-\Theta) \frac{\partial T^n}{\partial x} \right) \right) \right] \Bigg|_{x=0}^{x=L} = 0$$

$$\Rightarrow \left(\frac{1}{\Delta t} \int_0^L N_i N_j \, dx \right) T_j^{n+1} - \left(\frac{1}{\Delta t} \int_0^L N_i N_j \, dx \right) T_j^n + \left(\alpha \Theta \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx \right) T_j^{n+1}$$

$$+ \left(\alpha (1-\Theta) \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx \right) T_j^n + \Theta N_i(x) \left(-\alpha \frac{\partial T^{n+1}}{\partial x} \Big|_{x=0}^{x=L} \right) + (1-\Theta) N_i(x) \left(-\alpha \frac{\partial T^n}{\partial x} \Big|_{x=0}^{x=L} \right)$$

$$= 0$$

$$\Rightarrow \left[\int_0^L N_i N_j \, dx + \Delta t \, \alpha \Theta \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx \right] T_j^{n+1}$$

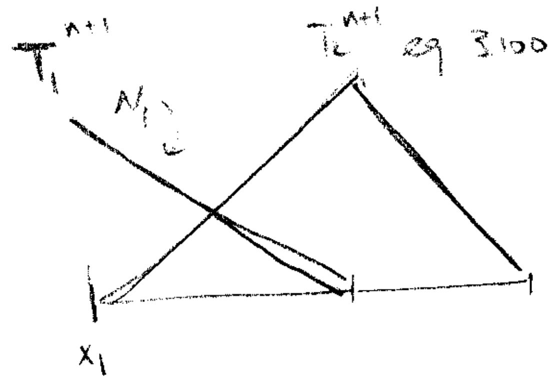
$$+ \left[- \int_0^L N_i N_j \, dx + \alpha \Delta t (1-\Theta) \int_0^L \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \, dx \right] T_j^n + \Delta t \, \Theta N_i(x) \left(-\alpha \frac{\partial T^{n+1}}{\partial x} \Big|_{x=0}^{x=L} \right)$$

$$+ \Delta t (1-\Theta) N_i(x) \left(-\alpha \frac{\partial T^n}{\partial x} \Big|_{x=0}^{x=L} \right) = 0$$

$$\Rightarrow \left[\int_0^L N_i N_j \, dx + \Delta t \, \alpha \, \theta \int_0^L \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \, dx \right] T_j^{n+1}$$

$$+ \Delta t \, \theta N_i(x) \left(-\alpha \frac{\partial T^{n+1}}{\partial x} \right) \Bigg|_{x=0}^{x=L}$$

$$= \left[\int_0^L N_i N_j \, dx + \alpha \Delta t (1-\theta) \int_0^L \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \, dx \right] T_j^{n} - \Delta t (1-\theta) N_i(x) \left(-\alpha \frac{\partial T^{n}}{\partial x} \right) \Bigg|_{x=0}^{L}$$



$\theta = 0$ explicit 1st order Euler

FEM Eqs For 2 linear elements

To 1st element: from eq 3.100

$N_1(x)$ $N_2(x)$ $i = 1, 2$

$i = 1$:

$$\left[\int_0^L N_1(x) N_j(x) \, dx + \Delta t \, \alpha \, \theta \int_0^L \frac{dN_1}{dx} \cdot \frac{dN_j}{dx} \, dx \right] T_j^{n+1} + \Delta t \, \theta N_1(x) \left(-\alpha \frac{\partial T^{n+1}}{\partial x} \right) \Bigg|_{x=0}^{x=L}$$

$$= \left\{ \int_0^L \begin{bmatrix} N_1^{(e)}(x) \\ N_2^{(e)}(x) \end{bmatrix} \left[N_1^{(e)}(x) \quad N_2^{(e)}(x) \right] dx + \Delta t \, \alpha \, \theta \int_0^L \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_2}{dx} \end{bmatrix} \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \right] dx \right\}$$

$$\cdot \begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix}$$

elemental matrix for linear

$$\left\{ \alpha = \frac{k}{\rho c_p} \right\} \Rightarrow \frac{\alpha}{k} = \frac{1}{\rho c_p}$$

$$\int_0^{L/2} N_i(x) N_j(x)\,dx = \frac{h^{(e)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\int_0^{L/2} \frac{dN_i}{dx}\cdot\frac{dN_j}{dx}\,dx = \frac{1}{h^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\left. + \quad -k\frac{\partial T}{\partial x} = -h(T - T_\infty)\right|_{(0,t)} = \begin{bmatrix} -h & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + hT_\infty\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

on 1st element

$\div k$ mult by α

Then eq 3.100 becomes replacing integrals w/ elemental matrices

$$\left(\frac{L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2\alpha\Delta t\Theta}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix} + \Theta\Delta t\left[\left(\frac{\alpha}{k}\right)\begin{bmatrix} +h & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix} + \left(\frac{\alpha}{k}\right)hT_\infty\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]\cdot$$

$$= \left(\frac{L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2\alpha\Delta t(\Theta-1)}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} T_1^n \\ T_2^n \end{bmatrix}$$

$$+(\Theta-1)\Delta t\left(-\begin{bmatrix} -h & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1^n \\ T_2^n \end{bmatrix} - hT_\infty\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\cdot\left(\frac{\alpha}{k}\right)$$

$$\Rightarrow \left[\frac{L}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2\alpha\Delta t\Theta}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{h\Theta\Delta t}{\rho c_p}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix}$$

$$= \frac{h\Theta\Delta t}{\rho c_p}\begin{bmatrix} 1 \\ 0 \end{bmatrix}T_\infty - \frac{h(\Theta-1)\Delta t}{\rho c_p}\begin{bmatrix} 1 \\ 0 \end{bmatrix}T_\infty$$

$$+\left(\frac{L}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\alpha\Delta t(\theta-1)}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{(\theta-1)\Delta t h}{\rho c_p}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)\begin{Bmatrix}T_1^n\\ T_2^n\end{Bmatrix}$$

w/ $\bar{h} = \frac{h}{\rho c_p}$

$$\Rightarrow \left\{\frac{L}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\alpha\Delta t\theta}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \bar{h}\theta\Delta t\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}\begin{Bmatrix}T_1^{n+1}\\ T_2^{n+1}\end{Bmatrix}$$

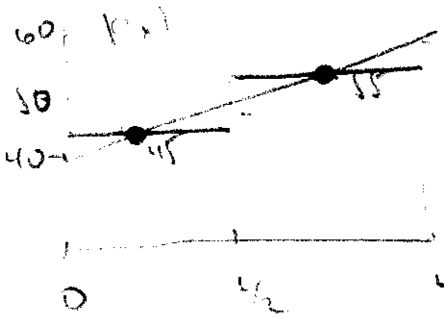
$$= \left\{\frac{L}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\alpha\Delta t(\theta-1)}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + (\theta-1)\Delta t\bar{h}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right\}\begin{Bmatrix}T_1^n\\ T_2^n\end{Bmatrix}$$

$$+ \bar{h}\Delta t T_\infty\begin{Bmatrix}1\\ 0\end{Bmatrix} \qquad \text{eq } 3.101$$

For 2nd element (All fluxes drop away) we obtain
$\bar{h} = 0$

$$\left\{\frac{L}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\alpha\Delta t\theta}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\}\begin{Bmatrix}T_2^{n+1}\\ T_3^{n+1}\end{Bmatrix} = \left\{\frac{L}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} + \frac{2\alpha\Delta t(\theta-1)}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\}\begin{Bmatrix}T_2^n\\ T_3^n\end{Bmatrix}$$

eq 3.102



$\rho c_p = 4 \cdot 10^6 \frac{W \cdot s}{m^3 C}$

$\alpha = \frac{k}{\rho c_p}$ $\quad \alpha_{(1)} = \frac{45}{4 \cdot 10^6 \frac{W \cdot s}{m^3 C}} = \frac{45}{4} \cdot 10^{-6} \frac{m^3 C}{W \cdot s}$

$\alpha_{(e_1)} = 11.25 \; 10^{-6} \frac{m^3 C}{W s}$

$\frac{45}{4} = \boxed{11.25} = 1125 \cdot 10^{-5} \frac{m^3 C}{W s}$

$\alpha_{(2)} = \frac{55}{4 \cdot 10^6 \frac{W s}{m^3 C}} = \begin{array}{r} 12.5 \\ +1.25 \\ \hline 13.75 \end{array} \; 10^{-6} = 1375 \cdot 10^{-5} \frac{m^3 C}{W \cdot s}$

$[W] = [F] m = \frac{m}{s^2} kg \cdot m = \frac{m^2 kg}{s^2}$ What are absolute units of W ? or C ?

$$+\ \bar{h} = \frac{h}{\rho c_p} = \frac{100\ W/m^2 C}{4 \cdot 10^6 \frac{W s}{m^3 C}} = 25 \cdot 10^{-6}\ m/s = 2.5 \cdot 10^{-5}\ m/s$$

Then w/ $\theta = 1$ element eqs become

$$\left\{\frac{.1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2(1.25 \cdot 10^{-5})\Delta t}{(.1)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + (2.5 \cdot 10^{-5})\Delta t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix}$$

$$= \left\{\frac{.1}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right\}\begin{bmatrix} T_1^n \\ T_2^n \end{bmatrix} + 2.5 \cdot 10^{-5}\ m/s\ \Delta t\ 400C \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

mult by 120

$$\Rightarrow \left\{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + .027\ \Delta t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + .003\ \Delta t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} T_1^n \\ T_2^* \end{bmatrix} + 1.2\ \Delta t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 + .03\Delta t & 1 - .027\Delta t \\ 1 - .027\Delta t & 2 + .027\Delta t \end{bmatrix}\begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} T_1^n \\ T_2^n \end{bmatrix} + \begin{bmatrix} 1.2\Delta t \\ 0 \end{bmatrix}$$

+ Element e_2:

$$\left[\frac{(.1)}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{2(1.375 \cdot 10^{-5})\Delta t}{.1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right]\begin{bmatrix} T_2^{n+1} \\ T_3^{n+1} \end{bmatrix}$$

$$= \frac{(.1)}{12}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} T_2^n \\ T_3^n \end{bmatrix}$$

If $\Delta t = 100\,s$ & $T_0(x) = 39.18C$

$$\Rightarrow \begin{bmatrix} 5 & -1.7 & 0 \\ -1.7 & 10 & -2.3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 39.18 \\ 39.18 \\ 39.18 \end{bmatrix} + \begin{bmatrix} 120 \\ 0 \\ 39.18 \end{bmatrix}$$

$$= \begin{bmatrix} 237.54 \\ 235.08 \\ 39.18 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \overbrace{\frac{120 \cdot 2}{.1}(1.37(\cdot 10^{-5}))}^{\approx .033} \Delta t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \begin{bmatrix} T_2^{n+1} \\ T_3^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} T_2^{n} \\ T_3^{n} \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 + .033\Delta t & 1 - .033\Delta t \\ 1 - .033\Delta t & 2 + .033\Delta t \end{pmatrix} \begin{pmatrix} T_2^{n+1} \\ T_3^{n+1} \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} T_2^{n} \\ T_3^{n} \end{bmatrix}$$

Assembling gives

$$\begin{pmatrix} 2 + .03\Delta t & 1 - .027\Delta t & 0 \\ 1 - .027\Delta t & 4 + .06\Delta t & 1 - .033\Delta t \\ 0 & 1 - .033\Delta t & 2 + .033\Delta t \end{pmatrix} \begin{pmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \end{pmatrix} =$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} T_1^{n} \\ T_2^{n} \\ T_3^{n} \end{bmatrix} + \begin{pmatrix} 1.2\Delta t \\ 0 \\ 0 \end{pmatrix} \qquad \text{eq } 3.103a$$

R.H.S. B.C $\Rightarrow$

$$\begin{pmatrix} 2 + .03\Delta t & 1 - .027\Delta t & 0 \\ 1 - .027\Delta t & 4 + .06\Delta t & 1 - .033\Delta t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_1^{n} \\ T_2^{n} \\ T_3^{n} \end{pmatrix} + \begin{pmatrix} 1.2\Delta t \\ 0 \\ 37.13 \end{pmatrix}$$

eq 3.103b

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3.93 $\frac{\partial T}{\partial t} \approx \frac{T^{n+1} - T^{n}}{\Delta t}$

3.94 $T = \Theta T^{n+1} + (1-\Theta)T^{n}$ $\quad t_n < t < t_{n+1}$

$$M\left(\frac{T^{n+1} - T^{n}}{\Delta t}\right) + k\left(\Theta T^{n+1} + (1-\Theta)T^{n}\right) = \Theta F^{n+1} + (1-\Theta)F^{n}$$

$$MT^{n+1} + \Delta t\, k\Theta T^{n+1} - MT^{n} + \Delta t\, k(1-\Theta)T^{n} = \Delta t\left(\Theta F^{n+1} + (1-\Theta)F^{n}\right)$$

$$(M + \Theta \Delta t\, k)T^{n+1} = (M + (\Theta-1)\Delta t\, k)T^{n} + \Delta t\left(\Theta F^{n+1} + (1-\Theta)F^{n}\right) \quad \text{eq } 3.107$$

Pg 117 Peyret [illegible]

$$\phi_i^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} \phi_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} \phi_j^{(k)}$$

$$\Rightarrow \quad a_{ii}\phi_i^{(k+1)} + \sum_{j=1}^{i-1} a_{ij}\phi_j^{(k+1)} = b_i - \sum_{j=i+1}^{n} a_{ij}\phi_j^{(k)}$$

$$\Rightarrow \quad \sum_{j=1}^{i} a_{ij}\phi_j^{(k+1)} = b_i - \sum_{j=i+1}^{n} a_{ij}\phi_j^{(k)}$$

$$\begin{array}{c} \\ \rightarrow \\ \\ \end{array} \begin{pmatrix} a_{11} & & & & \\ \vdots & a_{i-1,i-1} & 0 & & \\ a_{i1}\,a_{i2} & a_{i,i-1} & a_{ii} & 0 & \cdots \\ & & & \ddots & \\ & & & & a_{nn} \end{pmatrix} \begin{pmatrix} \phi_1^{(k+1)} \\ \phi_2^{(k+1)} \\ \vdots \\ \phi_n^{(k+1)} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} - \begin{array}{c} i \\ \downarrow \\ \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & a_{i,i+1} \cdots a_{in} \\ & & & \ddots \end{pmatrix} \end{array} \begin{pmatrix} \phi_1^{(k)} \\ \phi_2^{(k)} \\ \vdots \\ \phi_n^{(k)} \end{pmatrix} \leftarrow i$$

$$\uparrow$$
$$i$$

$$\Rightarrow \quad (L+D)\phi = b - U\phi \qquad \text{eq } 3.119$$

or all $A = L + D + U$

Then $A\phi = b \Rightarrow (L+D)\phi + U\phi = b$

$$(L+D)\phi = b - U\phi$$

Then eq 3.117 becomes

$$(L+D)\phi^{(k+1)} = b - U\phi^{(k)}$$

$$D\phi^{(k+1)} + L\phi^{(k+1)} = b - U\phi^{(k)}$$

$$\phi^{(k+1)} = -D^{-1} L \phi^{(k+1)} + D^{-1} b - D^{-1} U \phi^{(k)} \qquad \text{eq } 3.123$$

$$D\phi^{(k+1)} + L\phi^{(k+1)} = b - U\phi^{(k)}$$

$$\phi^{(k+1)} = (D+L)^{-1}(b - U\phi^{(k)}) \qquad \text{eq } 3.124$$

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$A\phi - B = 0$

If $A = LU$

$LU\phi - B = 0$ let $y = U\phi$

$Ly = B$ Solve for y Then solve

$y = U\phi$ for ϕ.

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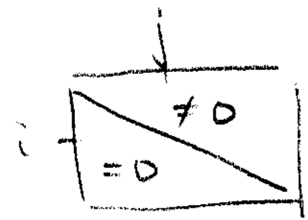
$A = U^T U$ Cholesky decomposition requires



A symmetric positive definite

U upper Δ

$\Rightarrow u_{ij} = 0$ iff $i > j$

$u_{ij} \neq 0$ iff $i \leq j$

Then

$$a_{ij} = \sum_{k=1}^{n} (U^T)_{ik} u_{kj}$$

$$= \sum_{k=1}^{n} u_{ki} u_{kj} = \sum_{k=1}^{\min(i,j)} u_{ki} u_{kj}$$

Let $j = i$ $\min(i,j) = i$

$$a_{ii} = \sum_{k=1}^{i} u_{ki}^2 = u_{ii}^2 + \sum_{k=1}^{i-1} u_{ki}^2 \Rightarrow u_{ii} = \pm\sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2}$$

sum of all elements above u_{ii}.

eq 3.132

Let $i < j$ Then $\min(i,j) = i$

$$a_{ij} = \sum_{k=1}^{i} u_{ki} u_{kj} = u_{ii} u_{ij} + \sum_{k=1}^{i-1} u_{ki} u_{kj}$$

$$\Rightarrow u_{ij} = \frac{1}{u_{ii}}\left(a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj}\right)$$

eq 3.133

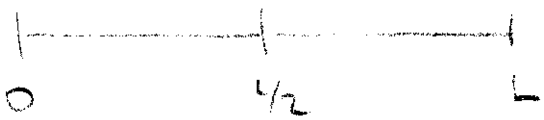
Thus the elements of U are filled as follows

$$U = \begin{pmatrix} \overset{1}{u_{11}} & \overset{2}{u_{12}} & \overset{4}{u_{13}} & \overset{7}{\cdot} & \cdots & u_{1n} \\ & \overset{3}{u_{22}} & \overset{5}{u_{23}} & \overset{8}{\cdot} & \cdots & \\ & & \overset{6}{u_{33}} & \overset{9}{\cdot} & \cdots & \\ & & & \overset{10}{\cdot} & & \\ & & & & \ddots & \\ & & & & & u_{nn} \end{pmatrix}$$

$u_{11}, u_{12}, u_{22}, u_{13}, u_{23}, u_{33}$

3.1

Use two linear elements to discretize $[0, L]$

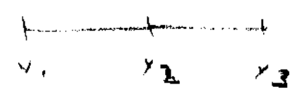


Then In element 1 $x \in (0, L/2)$ $T_1(x) = \alpha_1^1 + \alpha_2^1 x$

" " 2 $x \in (L/2, L)$ $T_2(x) = \alpha_1^2 + \alpha_2^2 x$

& we require that

$$\left.\begin{aligned} T_1(0) &= T_1 \\ T_1(L/2) &= T_2 \\ T_2(L/2) &= T_2 \\ T_2(L) &= T_3 \end{aligned}\right\} \text{4 eqs \& 4 unknowns } \alpha_1^1, \alpha_2^1, \alpha_1^2, \alpha_2^2$$



As on pg 17 $\alpha_1^1 = \dfrac{T_1 x_2 - T_2 x_1}{h_1}$ $\alpha_2^1 = \dfrac{T_2 - T_1}{h_1}$

& $\alpha_1^2 = \dfrac{T_2 x_3 - T_3 x_2}{h_2}$ $\alpha_2^2 = \dfrac{T_3 - T_2}{h_2}$

$$\left.\begin{aligned} \therefore\ T_1(x) &= \left(\frac{x_2 - x}{h_1}\right) T_1 + \left(\frac{x - x_1}{h_1}\right) T_2 \qquad x \in (0, L/2) \\ T_2(x) &= \left(\frac{x_3 - x}{h_2}\right) T_2 + \left(\frac{x - x_2}{h_2}\right) T_3 \qquad x \in (L/2, L) \end{aligned}\right\} \text{elemental shape fns}$$

Then global shape fns are

$$T_1(x) = \left(\frac{x_2 - x}{h_1}\right) T_1 \quad x \in (0, L/2), \qquad = 0 \quad x \in (L/2, L)$$

$$T_2(x) = \begin{cases} \left(\dfrac{x - x_1}{h_1}\right) T_2 & x \in (0, L/2) \\ \left(\dfrac{x_3 - x}{h_2}\right) T_2 & x \in (L/2, L) \end{cases}$$

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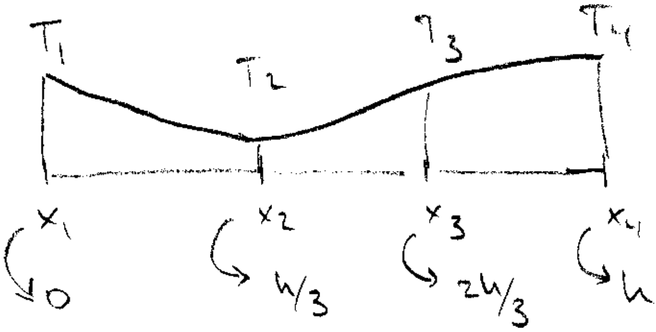
$$T_3(x) = \begin{cases} 0 & x \in (0, L/2) \\ \left(\frac{x - x_2}{h_2}\right) T_3 & x \in (L/2, L) \end{cases}$$

Expressions that would be shown 3.2 - 3.6 produce the same things.

(3.2) $T^{(e)}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$



Then $T^{(e)}(x_1) = T_1$, $T^{(e)}(x_2) = T_2$, $T^{(e)}(x_3) = T_3$, $T^{(e)}(x_4) = T_4$

$T^{(e)}(0) = T_1$, $T(h/3) = T_2$, $T(2h/3) = T_3$, $T(h) = T_4$

These 4 equations then become:

$$\left.\begin{aligned} \alpha_1 &= T_1 \\ \alpha_1 + \alpha_2 \frac{h}{3} + \alpha_3 \frac{h^2}{9} + \alpha_4 \frac{h^3}{27} &= T_2 \\ \alpha_1 + \alpha_2 \frac{2h}{3} + \alpha_3 \frac{4h^2}{9} + \alpha_4 \frac{8h^3}{27} &= T_3 \\ \alpha_1 + \alpha_2 h + \alpha_3 h^2 + \alpha_4 h^3 &= T_4 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & h/3 & h^2/9 & h^3/27 \\ 1 & 2h/3 & 4h^2/9 & 8h^3/27 \\ 1 & h & h^2 & h^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

$\Rightarrow$ Solving w/ MMA gives

$\alpha_1 = T_1$

$$\alpha_2 = \frac{-11T_1 + 18T_2 - 9T_3 + 2T_4}{2h}$$

$$\alpha_3 = \frac{9}{2h^2}(2T_1 - 5T_2 + 4T_3 - T_4)$$

$$\alpha_4 = -\frac{9}{2h^3}(T_1 - 3T_2 + 3T_3 - T_4)$$

Then

$$T(x) = T_1 + \frac{x}{2h}(-11T_1 + 18T_2 - 9T_3 + 2T_4) + \frac{9x^2}{2h^2}(2T_1 - 5T_2 + 4(T_3 - T_4))$$

$$- \frac{9x^3}{2h^3}(T_1 - 3T_2 + 3T_3 - T_4)$$

$$= T_1\left[1 - \frac{11x}{2h} + \frac{9x^2}{2h^2}\cdot 2 - \frac{9x^3}{2h^3}\right]$$

$$+ T_2\left[\frac{18x}{2h} - \frac{45x^2}{2h^2} + \frac{27x^3}{2h^3}\right]$$

$$+ T_3\left[-\frac{9x}{2h} + \frac{36x^2}{2h^2} - \frac{27x^3}{2h^3}\right]$$

$$+ T_4\left[\frac{2x}{2h} - \frac{9x^2}{2h^2} + \frac{9x^3}{2h^3}\right]$$

look for the factors of

x, $x - \frac{h}{3}$, $x - \frac{2h}{3}$,

$x - h$

As these are zeros.

Try to factor $x - h$ out of $1 - \frac{11x}{2h} + \frac{9x^2}{h^2} - \frac{9x^3}{2h^3}$

$$\begin{array}{r|rrrr}
h & -\frac{9}{2h^3} & \frac{9}{h^2} & -\frac{11}{2h} & 1 \\
+ & & -\frac{9}{2h^2} & \frac{9}{2h} & -1 \\
\hline
& -\frac{9}{2h^3} & \frac{9}{2h^2} & -\frac{1}{h} & \boxed{0}
\end{array}$$

$a + bx + cx^2 + dx^3 = a + x(b + x(dx + c))$ = used in polynomial factoring or algorithm used above.

$$T(x) = T_1 (x-h)\left(-\frac{9}{2h^3}x^2 + \frac{9}{2h^2}x - \frac{1}{h}\right)$$

$$+ \quad + \frac{T_2 x}{2h}\left[18 - \frac{45x}{h} + \frac{27x^2}{h^2}\right]$$

$$+ T_3 \frac{x}{2h}\left[-9 + \frac{36x}{h} - \frac{27x^2}{h^2}\right]$$

$$+ T_4 \frac{x}{2h}\left[2 - \frac{9x}{h} + \frac{9x^2}{h^2}\right]$$

$$\Rightarrow T(x) = T_1 \frac{(x-h)}{2h}\left(-\frac{9x^2}{h^2} + \frac{9x}{h} - 2\right)$$

$$+ \frac{x}{2h} T_2 \left(3 - 3\frac{x}{h}\right)\left(6 - 9\frac{x}{h}\right)$$

$$- \frac{x}{2h} T_3 \left(3 - 3\frac{x}{h}\right)\left(3 - 9\frac{x}{h}\right)$$

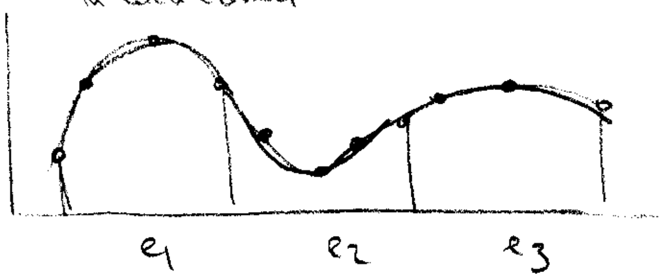
$$+ \frac{x}{2h} T_4 \left(1 - \frac{3x}{h}\right)\left(2 - \frac{3x}{h}\right)$$

$$\Rightarrow T(x) = T_1 \left(\frac{x-h}{2h}\right)\left(-\frac{3x}{h} + 1\right)\left(\frac{3x}{h} - 2\right) = T_1 \left(\frac{x}{h} - 1\right)\left(1 - \frac{3x}{h}\right)\left(\frac{3x}{2h} - 1\right)$$

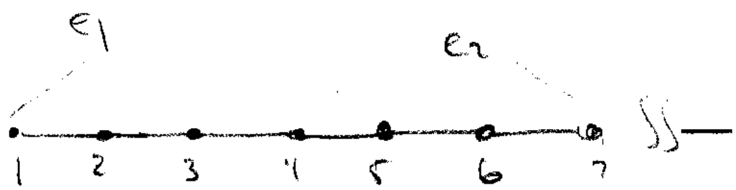
$$+ T_2 \frac{9x}{h}\left(1 - \frac{x}{h}\right)\left(1 - \frac{3x}{2h}\right)$$

$$- T_3 \frac{9x}{2h}\left(1 - \frac{x}{h}\right)\left(1 - \frac{3x}{h}\right) + T_4 \frac{x}{h}\left(1 - \frac{3x}{h}\right)\left(1 - \frac{3x}{2h}\right)$$

eq 3.25



Plotting local Basis fns results in see MMA File
FEM-BasicConceptsAndApps



<u>Node #'s</u>

<u>Local</u>	<u>Global</u>
1	$4(i-1)+1$
2	$4(i-1)+2$
3	$4(i-1)+3$
4	$4(i-1)+4$

Function values:
<u>Local</u> Global

$T_1^{(i)} \to T_{4(i-1)+1}$

$T_2^{(i)} \to T_{4(i-1)+2}$

$T_3^{(i)} \to T_{4(i-1)+3}$

$T_4^{(i)} \to T_{4(i-1)+4}$

element e_i has Global nodes $4(i-1), 4(i-1)+1, 4(i-1)+2, 4(i-1)+3$

$$T^{(e)}(x) = T_1^{(e)} N_1^{(e)} + T_2^{(e)} N_2^{(e)} + T_3^{(e)} N_3^{(e)} + T_4^{(e)} N_4$$

$$= \begin{bmatrix} N_1, & N_2 & N_3 & N_4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}$$

$$\frac{dT^{(1)}(x)}{dx} = \left[\frac{dN_1^{(1)}}{dx} \quad \frac{dN_2^{(1)}}{dx} \quad \frac{dN_3^{(1)}}{dx} \quad \frac{dN_4}{dx} \right]$$

$$\frac{dN_1^{(1)}}{dx} = -\frac{3}{h}\left(1 - \frac{3x}{2h}\right)\left(1 - \frac{x}{h}\right) - \frac{3}{2h}\left(1 - \frac{3x}{h}\right)\left(1 - \frac{x}{h}\right)$$
$$+ \left(1 - \frac{3x}{h}\right)\left(1 - \frac{3x}{2h}\right)\left(-\frac{1}{h}\right)$$

If $\quad f(x) = \prod_i (c_i - a_i x)^{p_i}$

$$\log f(x) = \sum_i p_i \log(c_i - a_i x)$$

$\frac{d}{dx} \rightarrow$

$$\frac{1}{f(x)} f'(x) = \sum_i \frac{p_i(-a_i)}{(c_i - a_i x)}$$

$$\Rightarrow \quad f'(x) = f(x) \sum_i \frac{-a_i p_i}{(c_i - a_i x)}$$

$$\therefore \quad \frac{dN_1}{dx} = N_1(x) \left[\frac{-3/h}{\left(1 - \frac{3x}{h}\right)} + \frac{-3/2h}{\left(1 - \frac{3x}{2h}\right)} + \frac{-1/h}{\left(1 - x/h\right)} \right]$$

$$= -\frac{N_1(x)}{h} \left[\frac{3}{\left(1 - \frac{3x}{h}\right)} + \frac{3}{\left(1 - \frac{3x}{2h}\right)} + \frac{1}{\left(1 - x/h\right)} \right]$$

$$=$$

$$\frac{dN_2}{dx} = N_2(x)\left[\frac{9/h}{\frac{9x}{h}} + \frac{-3/2h}{\left(1-\frac{3}{2h}x\right)} + \frac{-1/h}{\left(1-\frac{x}{h}\right)} \right]$$

$$= N_2(x)\left[1 - \frac{3}{2h}\frac{1}{\left(1-\frac{3}{2}h\right)} - \frac{1}{h}\frac{1}{\left(1-\frac{x}{h}\right)} \right]$$

$$=$$

$$\frac{dN_3}{dx} = N_3(x)\left[\frac{-9/2h}{-9/2h} + \frac{-3/h}{\left(1-\frac{3x}{h}\right)} + \frac{-1/h}{\left(1-x/h\right)} \right]$$

$$= +N_3(x)\left[1 - \frac{3}{h}\frac{1}{\left(1-\frac{3x}{h}\right)} - \frac{1}{h\left(1-x/h\right)} \right]$$

$$=$$

$$\frac{dN_4}{dx} = N_4(x)\left[1 + \frac{-3/h}{\left(1-\frac{3x}{h}\right)} + \frac{-3/2h}{\left(1-\frac{3x}{2h}\right)} \right]$$

$$= N_4(x)\left[1 - \frac{3}{h\left(1-\frac{3x}{h}\right)} - \frac{3}{2h\left(1-\frac{3}{2h}x\right)} \right]$$

$$=$$

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(3.3)

$$T(x) = \sin x \cong \sum_{e=1}^{2} T^{(e)}(x)$$

$$T^{(e)}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$= N_1^{(e)}(x) T_1 + N_2^{(e)} T_2 + N_3^{(e)} T_3$$

$T(0) = 0$

$T(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$T(\frac{\pi}{2}) = 1$

$T(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$

$T(\pi) = 0$

w/ $N_1^{(e)} = (1 - \frac{2x}{h})(1 - \frac{x}{h})$

$N_2^{(e)} = 4\frac{x}{h}(1 - \frac{x}{h})$ $\quad h = \frac{\pi}{2}$

$N_3^{(e)} = \frac{x}{h}(\frac{2x}{h} - 1)$ $\quad$ **

Then $T^{(e_1)}(x) = 0 \cdot N_1^{(e)}(x) + \frac{1}{\sqrt{2}} N_2^{(e_1)}(x) + N_3^{(e_1)}(x)$

$T^{(e_2)}(x) = 1 N_1^{(e_2)}(x) + \frac{1}{\sqrt{2}} N_2^{(e_2)}(x) + 0 N_3^{(e_2)}(x)$

$$\Rightarrow T^{(e_1)}(x) = \frac{1}{\sqrt{2}} \cdot \frac{8x}{\pi}\left(1 - \frac{2x}{\pi}\right) + \frac{2x}{\pi}\left(\frac{4x}{\pi} - 1\right)$$

$$T^{(e_2)}(x) = \left(1 - \frac{4x}{\pi}\right)\left(1 - \frac{2x}{\pi}\right) + \frac{1}{\sqrt{2}} \cdot \frac{8x}{\pi}\left(1 - \frac{2x}{\pi}\right)$$

X Not correct. Above formulas ** only work for intervals centered from (0, h) to work in an interval centered (x_1, x_2)

$$\Rightarrow T^{(e_1)}(x) = \frac{8}{\pi\sqrt{2}}x - \frac{16x^2}{\pi^2\sqrt{2}} + \frac{8x^2}{\pi^2} - \frac{2x}{\pi}$$

$$T^{(e_2)}(x) = 1 - \cancel{\frac{2x}{\pi}} - \frac{4x}{\pi} + \cancel{\frac{8x^2}{\pi^2}} + \frac{8x}{\pi\sqrt{2}} - \frac{16x^2}{\pi^2\sqrt{2}}$$

$$\Rightarrow T^{(e_1)}(x) = \frac{2}{\pi}\left(\frac{4}{\sqrt{2}} - 1\right)x + \frac{8}{\pi^2}\left(1 - \frac{2}{\sqrt{2}}\right)x^2$$

$$T^{(e_2)}(x) = 1 + \frac{2}{\pi}\left(\cancel{-3} + \frac{4}{\sqrt{2}}\right)x + \frac{8}{\pi}\left(\cancel{1} - \frac{2}{\sqrt{2}}\right)x^2$$

For 2nd element approximation map $\xi: (\frac{\pi}{2}, \pi)$ to $(0, h)$

w/ $\xi(x) = h\dfrac{(x - \frac{\pi}{2})}{\frac{\pi}{2}} = (\frac{2}{\pi}x - 1)h$

Check:

$\xi(\frac{\pi}{2}) = 0$ & $\xi(\pi) = h$ ✓

Then $N_i^{(e)} [0,h] \longrightarrow [0,1]$

$N_i^{(e)} \rule{1em}{0.4pt} [\frac{\pi}{2}, \pi] \longrightarrow [0,1]$

$\xi [\frac{\pi}{2}, \pi] \underset{\text{maps to}}{\longrightarrow} [0,h]$

$$N_i^{(e)}(\xi(\bullet)) \; [\tfrac{\pi}{2}, \pi] \rightarrow [0,1]$$

$$\Rightarrow N_1^{(2)}(x) = N_1^{(e)}(\xi(x)) = \left(1 - \frac{2\xi}{h}\right)\left(1 - \frac{\xi}{h}\right)$$

$$= \left(1 - \frac{2}{\cancel{h}}\cancel{h}\left(\frac{2}{\pi}x - 1\right)\right)\left(1 - \frac{1}{\cancel{h}}\cancel{h}\left(\frac{2}{\pi}x - 1\right)\right)$$

$$= \left(1 - \frac{4}{\pi}x + 2\right)\left(1 - \frac{2}{\pi}x + 1\right)$$

$$= \left(3 - \frac{4}{\pi}x\right)\left(2 - \frac{2}{\pi}x\right)$$

$$= 2\left(1 - \frac{x}{\pi}\right)\left(3 - 4\frac{x}{\pi}\right)$$

Check:

$$N_1^{(e2)}\left(\frac{\pi}{2}\right) = 2\left(1-\frac{1}{2}\right)(3-2) = 1$$

$$N_1^{(e2)}\left(\frac{3\pi}{4}\right) = 0 \quad \checkmark$$

$$N_1^{(e2)}(\pi) = 0 \quad \checkmark$$

$$N_2^{(e2)}(x) = \tilde{N}_1^{(e2)}(\xi(x)) = \frac{4}{\cancel{k}}\cancel{k}\left(\frac{2}{\pi}x-1\right)\left(1-\left(\frac{2}{\pi}x-1\right)\right)$$

$$= 4\left(\frac{2}{\pi}x-1\right)\left(2-\frac{2}{\pi}x\right)$$

$$= 8\left(\frac{2}{\pi}x-1\right)\left(1-\frac{x}{\pi}\right)$$

Check

$$N_2^{(e2)}\left(\frac{\pi}{2}\right) = 0$$

$$N_2^{(e2)}\left(\frac{3\pi}{4}\right) = 8\left(\frac{3}{2}-1\right)\left(1-\frac{3}{4}\right) = 4\left(\frac{1}{4}\right) = 1 \quad \checkmark$$

$$N_2^{(e2)}(\pi) = 0$$

$$N_3^{(e2)}(x) = \tilde{N}_3^{(e2)}(\xi(x)) = \frac{1}{\cancel{k}}\cancel{k}\left(\frac{2}{\pi}x-1\right)\left(\frac{2}{\cancel{k}}\cancel{k}\left(\frac{2}{\pi}x-1\right)-1\right)$$

$$= \left(\frac{2}{\pi}x-1\right)\left(\frac{4}{\pi}x-2-1\right)$$

$$= \left(\frac{2}{\pi}x-1\right)\left(\frac{4}{\pi}x-3\right)$$

Check

$$N_3^{(e_2)}\left(\frac{\pi}{2}\right) = 0$$

$$N_3^{(e_2)}\left(\frac{3\pi}{4}\right) = \left(\frac{3}{2}-1\right)(0) = 0$$

$$N_3^{(e_2)}(\pi) = 1(1) = 1$$

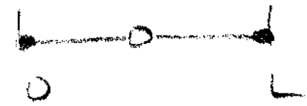
Then

$$T^{(e_2)}(x) = 1 \cdot 2\left(1-\frac{x}{\pi}\right)\left(3-\frac{4x}{\pi}\right) + \frac{8}{\sqrt{2}}\left(\frac{2}{\pi}x-1\right)\left(1-\frac{x}{\pi}\right)$$

$$= \ldots$$

$$\frac{dT^{(e)}(x)}{dx} = \frac{2}{\pi}\left(\frac{4}{\sqrt{2}}-1\right) + 16 \quad \ldots$$

(3.4) $T(x) = \alpha_1 + \alpha_2 x \qquad x \in (0, L)$



(a) Approximate T w/ quadratic element.

Approximation to $T(x)$

$$T_2^{(e)}(x) = N_1^{(e)}(x)\,T_1 + N_2^{(e)}(x)\,T_2 + N_3^{(e)}(x)\,T_3$$

$\underset{\text{quad}}{2}$; $T_1 = T(0) = \alpha_1$; $T_2 = T(L/2) = \alpha_1 + \alpha_2 L/2$; $T_3 = T(L) = \alpha_1 + \alpha_2 L$

$$= N_1(x)\alpha_1 + N_2(x)\alpha_1 + \tfrac{L}{2} N_2(x)\alpha_2 + N_3(x)\alpha_1 + L N_3(x)\alpha_2$$

$$= \alpha_1\left[N_1 + N_2 + N_3\right] + \alpha_2 L\left[\frac{N_2}{2} + N_3\right]$$

$$= \alpha_1\left[1 - \cancel{\frac{3x}{L}} + \cancel{\frac{2x^2}{L^2}} + \cancel{\frac{4x}{L}} - \cancel{\frac{4x^2}{L^2}} + \cancel{\frac{2x^2}{L^2}} - \cancel{\frac{x}{L}}\right]$$

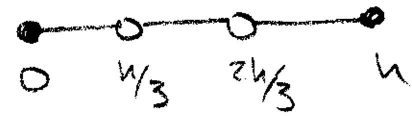
$$+ \alpha_2 L\left[\frac{2x}{L} - \cancel{\frac{2x^2}{L^2}} + \cancel{\frac{2x^2}{L^2}} - \frac{x}{L}\right]$$

$$= \alpha_1\left[1\right] + \alpha_2 x \equiv T(x).$$

(b) If $T(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \qquad x \in (0, L)$

Then Approximate $T(x)$ w/ cubic polynomials over one element.

$$T(x) \cong T_1 N_1(x) + T_2 N_2(x) + T_3 N_3(x) + T_4 N_4(x)$$



For cubic polynomials one needs 4 pts

w/ $T_1 = \alpha_1$

$$T_2 = T(h/3) = \alpha_1 + \alpha_2 \frac{h}{3} + \alpha_3 \frac{h^2}{9}$$

$$T_3 = T(2h/3) = \alpha_1 + \alpha_2 \frac{2h}{3} + \alpha_3 \frac{4h^2}{9}$$

$$T_4 = T(h) = \alpha_1 + \alpha_2 h + \alpha_3 h^2$$

Then w/ def on pg 20

$T(x)$ Approximation

$$= \alpha_1 \left(1 - \frac{3x}{h}\right)\left(1 - \frac{3x}{2h}\right)\left(1 - \frac{x}{h}\right)$$

$$+ \left(\alpha_1 + \alpha_2 \frac{h}{3} + \alpha_3 \frac{h^2}{9}\right)\left(\frac{9x}{h}\right)\left(1 - \frac{3x}{2h}\right)\left(1 - \frac{x}{h}\right)$$

$$+ \left(\alpha_1 + \frac{2h}{3}\alpha_2 + \frac{4h^2}{9}\alpha_3\right)\left(-\frac{9x}{2h}\right)\left(1 - \frac{3x}{h}\right)\left(1 - \frac{x}{h}\right)$$

$$+ \left(\alpha_1 + \alpha_2 h + \alpha_3 h^2\right)\left(\frac{x}{h}\right)\left(1 - \frac{3x}{h}\right)\left(1 - \frac{3x}{2h}\right)$$

Then coefficients of α_1 is 1 By MMA

α_2 is x

α_3 is x^2

$$= \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

(c)

let $T(x) \equiv 1$ then $T_i = 1 \quad \forall i$

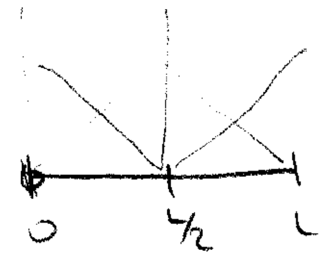
+ $$\sum_{i=1}^{3} N_i^{(e)}(x) = 1$$ By result for quadratic element

$$\sum_{i=1}^{4} N_i(x) = 1$$ By result for cubic element

For linear elements, this result follows from expressions 3.14 + 3.15

Then taking the derivative of both sides one gets

$$\sum_{i=1}^{k} \frac{dN_i}{dx} = 0$$



(3.5) (a) Derive eq 3.36 see notes pg 23

Solving 3.38 gives

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{1}{(2-1)}\frac{L}{2k}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left[\frac{QL}{4}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} + \frac{2k}{L}T_L\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left[\frac{QL^2}{8k}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{L}{2k}\begin{bmatrix} q \\ 0 \end{bmatrix} + T_L\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{QL^2}{8k}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{L}{2k}\begin{bmatrix} 2q \\ q \end{bmatrix} + T_L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$T_1 = \frac{QL^2}{2k} + \frac{Lq}{k} + T_L$$

$$T_2 = \frac{3QL^2}{8k} + \frac{L}{2k}q + T_L$$

exact sol to 3.27 is $T(x) = T_L + \frac{q}{k}(L-x) + \frac{Q}{2k}(L^2 - x^2)$

$T_1 \cong T(x=0) = T_L + \frac{qL}{k} + \frac{QL^2}{2k}$ exactly the same!!

$T_2 \cong T(x=\frac{L}{2}) = T_L + \frac{qL}{2k} + \frac{Q}{2k}\left(\frac{3L^2}{4}\right)$ "

No surprise since we are ~~using~~ quadratic elements that we can exactly solve for a quadratic input fn.

Not true!! Here we are using two linear elements

Solving eq 3.43

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{1}{(14\cdot 32 - 16^2)}\left(\frac{6L}{k}\right)\begin{bmatrix} 32 & 16 \\ 16 & 14 \end{bmatrix}\left[\frac{QL}{6}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} + \frac{k}{6L}T_L\begin{bmatrix} -2 \\ 16 \end{bmatrix}\right]$$

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{1}{192}\begin{bmatrix} 32 & 16 \\ 16 & 14 \end{bmatrix}\left[\frac{QL^2}{k}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{bL}{k}\begin{bmatrix} 9 \\ 0 \end{bmatrix} + T_L\begin{bmatrix} -2 \\ 16 \end{bmatrix}\right]$$

$$= \frac{1}{192}\left(\frac{QL^2}{k}\begin{bmatrix} 32+4\cdot16 \\ 16+4\cdot14 \end{bmatrix} + \frac{bL}{k}\begin{bmatrix} 32\cdot9 \\ 16\cdot9 \end{bmatrix} + T_L\begin{bmatrix} -32\cdot2+16^2 \\ -2\cdot16+14\cdot16 \end{bmatrix}\right)$$

$$= \frac{1}{192}\left[\frac{QL^2}{k}\begin{bmatrix} 96 \\ 72 \end{bmatrix} + \frac{bL}{k}\cdot16\begin{bmatrix} 2\cdot9 \\ 9 \end{bmatrix} + T_L\begin{bmatrix} 192 \\ 192 \end{bmatrix}\right]$$

$$= \frac{QL^2}{k}\begin{bmatrix} 1/2 \\ 72/192 \end{bmatrix} + \overset{\frac{9bL}{k}}{\frac{L}{2k}9\begin{bmatrix} 2 \\ 1 \end{bmatrix}} + T_L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{72}{192} = \frac{36}{96} = \frac{18}{48} = \frac{9}{24} = \frac{3}{8}$$

$$= \frac{QL^2}{k}\begin{bmatrix} 1/2 \\ 3/8 \end{bmatrix} + \quad ''$$

$$= \frac{QL^2}{8k}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{L9}{2k}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + T_L\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Same as sol in case of 2.

Thus both two linear elements & one quadratic elements give the same exact correct values at the nodal points x_1, x_2 & x_3. A comparison plot gives see MMA

Two linear elements

$$T(x) = \sum_{n=1}^{2} T = T_1 N_1^{(e_1)}(x) + T_2 N_2^{(e_1)}(x) + T_2 N_1^{(e_2)}(x) + T_2 N_2^{(e_2)}(x)$$

One quadratic element:

$$T(x) = \sum_{n=1}^{1} T_e(x) = T_1 N_1(x) + T_2 N_2 + T_3 N_3$$

Two linear elements & One quadratic element have:

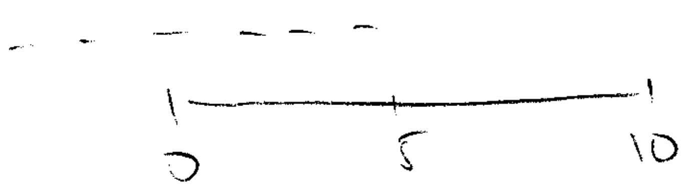
$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \frac{(1.5\ \cancel{cal}/cm^3\cancel{s})(10\ cm)^2}{8(1.5\ cal/cm\,s\,C)}\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \frac{(10cm)(.15\ cal/cm^2 s)}{2(.15\ cal/cm^2 s\,C)}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$+\ 100\,C \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{100}{8}\ cm\,C \begin{bmatrix} 4 \\ 3 \end{bmatrix} + 5\ cm\,C \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 100\,C \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left\{ \left[\frac{QL^2}{k}\right] = \frac{\cancel{cal}}{cm^3\cancel{s}}\ cm^2\ \frac{cm\,\cancel{s}\,C}{\cancel{cal}} = cm\,C \quad \text{How get rid of cm?} \right\}$$

$$\Rightarrow \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 160 \\ 142 \end{bmatrix}$$



Linear:

$N_1(x) = 1 - \frac{x}{h}$ & $h = 5cm$

$N_2(x) = \frac{x}{h}$

$[5,10] \longrightarrow [0,h] = [0,5]$

$\therefore (x-5)$

$h = 5cm$

$N_1^{(e_1)}(x) = 1 - \frac{x}{5}$ $\qquad N_1^{(e_2)}(x) = 1 - \frac{1}{5}(x-5)$

$N_2^{(e_1)}(x) = \frac{x}{5}$ $\qquad N_2^{(e_2)}(x) = \frac{1}{5}(x-5)$

Then

$$T_{linear}(x) \cong T_1\left(1-\frac{x}{5}\right)+T_2\left(\frac{x}{5}\right) \qquad x<5$$

$$= T_2\left(1-\frac{1}{5}(x-5)\right)+\frac{T_3}{5}(x-5) \qquad x>5$$

$$= 160\left(1-\frac{x}{5}\right)+142\,\frac{x}{5} \qquad x<5$$

$$142\left(1-\frac{1}{5}(x-5)\right)+\frac{100}{5}(x-5) \qquad x>5$$

<u>Quadratic</u>.

$$N_1(x) = \left(1-\frac{2x}{h}\right)\left(1-\frac{x}{h}\right) \qquad h=10\,\text{cm}$$

$$N_2(x) = 4\frac{x}{h}\left(1-\frac{x}{h}\right)$$

$$N_3(x) = \frac{x}{h}\left(2\frac{x}{h}-1\right)$$

$$T_{quad}(x) \cong \left(1-\frac{2x}{10}\right)\left(1-\frac{x}{10}\right)T_1+\frac{4x}{10}\left(1-\frac{x}{10}\right)T_2+\frac{x}{10}\left(\frac{2x}{10}-1\right)T_3$$

$$= \left(1-\frac{2x}{10}\right)\left(1-\frac{x}{10}\right)160+\frac{4x}{10}\left(1-\frac{x}{10}\right)142+\frac{x}{10}\left(\frac{2x}{10}-1\right)100$$

(3.6) eq 3.38 is

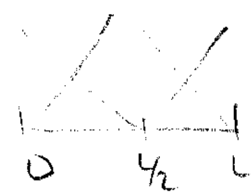
$$\frac{2k}{L}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}\begin{pmatrix}T_1\\T_2\end{pmatrix} = \frac{QL}{4}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2k}{L}T_L\begin{bmatrix}0\\1\end{bmatrix} + \begin{pmatrix}q\\0\end{pmatrix}$$

$$\begin{pmatrix}T_1\\T_2\end{pmatrix} = \frac{1}{2-1}\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\left(\frac{L}{2k}\left(\frac{QL}{4}\begin{pmatrix}1\\2\end{pmatrix} + \frac{2k}{L}T_L\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}q\\0\end{pmatrix}\right)\right)$$

$$\begin{pmatrix}T_1\\T_2\end{pmatrix} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\left(\frac{QL^2}{8k}\begin{pmatrix}1\\2\end{pmatrix} + T_L\begin{pmatrix}0\\1\end{pmatrix} + \frac{L}{2k}\begin{bmatrix}q\\0\end{bmatrix}\right)$$

$$= \frac{QL^2}{8k}\begin{pmatrix}4\\3\end{pmatrix} + T_L\begin{bmatrix}1\\1\end{bmatrix} + \frac{L}{2k}\begin{bmatrix}2q\\q\end{bmatrix}$$

Then sol w/ 2 linear elements becomes



$$T(x) \simeq T_1 N_1 + T_2 N_2 + T_3 N_3$$

$$\left.\frac{dT}{dx}\right|_{x=0} = T_1\frac{dN_1}{dx} + T_2\frac{dN_2}{dx} = T_1\left(-\frac{1}{(L/2)}\right) + T_2\left(\frac{1}{L/2}\right)$$

$$= -\frac{2}{L}T_1 + \frac{2}{L}T_2$$

$$= -\frac{QL}{4k}(4) - \cancel{\frac{2T_L}{L}} - \frac{1}{k}(2q) + \frac{QL}{4k}(3) + \cancel{\frac{2T_L}{L}} + \frac{1}{k}(q)$$

$$= -\frac{QL}{4k} - \frac{q}{k}$$

$$-k\left.\frac{dT}{dx}\right|_{x=0} = -\frac{QL}{4} + q \neq q \text{ by } -\frac{QL}{4}$$

In this example $L = 2h_1^{(e)}$

$$\Rightarrow \quad -k\frac{dT}{dx}\bigg|_{x=0} = -\frac{Qh_1^{(e)}}{2} + q \xrightarrow{h \to 0} q$$

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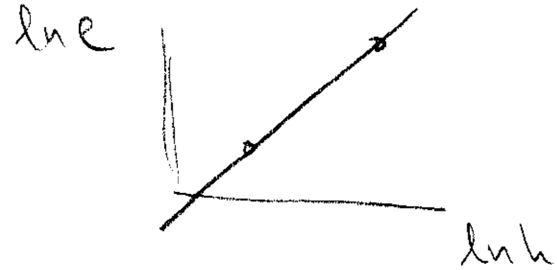
(3.7) $e = Ch^p$

$\log e = \log C + p \log h$



plot log e v.s. log h

Exact $T_1^{ex} = 100$

One linear element: $T_1 = 99.317$, $e = .683$ $h = .1$

two linear elements: $T_1 = 99.822$, $e = .178$ $h = .05$

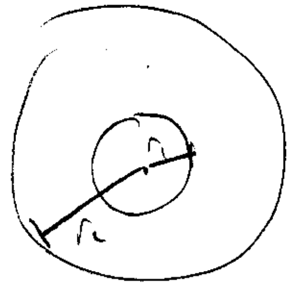
	One linear elmt	two linear elts
T_1 (Approx)	99.317	99.822
e	.638	.178
h	.1	.05
$\ln e$	-3.3126	-1.7259
$\ln h$	-2.302	-2.9957

$$m = \text{slope} \quad \frac{\ln e_1 - \ln e_2}{\ln h_1 - \ln h_2} = \frac{-3.3126 - (-1.7259)}{-2.302 - (-2.9957)} = 1.9400\ldots$$

(3.8) eqs computing is

$$-\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0 \qquad r_1 < r < r_2 \qquad k \text{ constant independent of } x.$$

w/ $-k\frac{dT}{dr} + h(T - T_\infty) = 0 \qquad r = r_1$



$r_1 < r_2$

$T = T_L \qquad r = r_2$

Then $kr\frac{dT}{dr} = C_1 \qquad C_1$ constant ✓

$$\frac{dT}{dr} = \frac{C_1}{kr} \quad ✓$$

$$T = \frac{C_1}{k}\ln r + C_2 \quad ✓$$

Then B.C at $r = r_1$ gives $-k\left(\frac{C_1}{kr_1}\right) + h\left(\frac{C_1 \ln r_1}{k} + C_2 - T_\infty\right) = 0$ ✓

+ B.C at $r = r_2$ gives

$$\frac{C_1 \ln r_2}{k} + C_2 = T_L \quad ✓$$

∴ solving for $C_1 + C_2$

$$\left(\frac{-1}{r_1} + \frac{h}{k}\ln r_1\right)C_1 + hC_2 = hT_\infty \quad ✓$$

$$\frac{\ln r_2}{k}C_1 + C_2 = T_L$$

Then $$C_1 = \frac{1}{\left(\frac{-1}{r_1} + \frac{h \ln r_1}{k} - \frac{h \ln r_2}{k}\right)} \begin{vmatrix} hT_\infty & h \\ T_L & 1 \end{vmatrix}$$

$$= \frac{h(T_\infty - T_L)}{\frac{-1}{r_1} + \frac{h}{k}\ln(r_1/r_2)} \cdot \frac{r_1 k}{r_1 k} \qquad \left[\frac{h}{k}\right] = \frac{1}{L} \checkmark$$

$$= \frac{r_1 k (T_\infty - T_L) h}{h r_1 \ln(r_1/r_2) - k}$$

& $$C_2 = \frac{1}{\left(\frac{-1}{r_1} + \frac{h}{k}\ln(r_1/r_2)\right)} \begin{vmatrix} \frac{-1}{r_1} + \frac{h}{k}\ln r_1 & hT_\infty \\ \frac{\ln r_2}{k} & T_L \end{vmatrix}$$

$$= \frac{-\frac{T_L}{r_1} + \frac{h}{k}(T_L \ln r_1 - T_\infty \ln r_2)}{h r_1 \ln(r_1/r_2) - k} \cdot \frac{k r_1}{k r_1}$$

$$= \frac{-T_L k + h r_1 (T_L \ln r_1 - T_\infty \ln r_2)}{h r_1 \ln(r_1/r_2) - k}$$

$$= \frac{-T_L k + T_L(h r_1 \ln r_1 - h r_1 \ln r_2) + T_L h r_1 \ln r_2 - h r_1 T_\infty \ln r_2}{h r_1 \ln(r_1/r_2) - k}$$

$$\Rightarrow C_2 = \frac{T_L\left(-k + hr_1 \ln(r_1/r_2)\right) + T_L h r_1 \ln r_2 - h r_1 T_\infty \ln r_2}{h r_1 \ln(r_1/r_2) - k}$$

$$= T_L + \frac{(T_L - T_\infty) h r_1 \ln r_2}{h r_1 \ln(r_1/r_2) - k}$$

$\therefore$

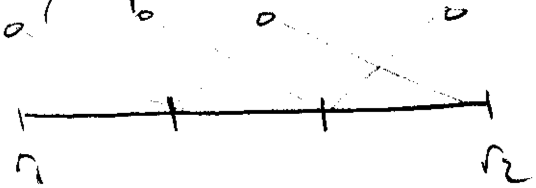
$$T(r) = \frac{r_1 h (T_\infty - T_L)}{h r_1 \ln(r_1/r_2) - k} \ln r + \frac{(T_L - T_\infty) h r_1 \ln r_2}{h r_1 \ln(r_1/r_2) - k} + T_L$$

$$= \frac{h r_1 (T_L - T_\infty) \ln(r_2/r)}{h r_1 \ln(r_1/r_2) - k} + T_L$$

$$= \frac{h r_1 (T_L - T_\infty) \ln(r/r_2)}{k - h r_1 \ln(r_1/r_2)} + T_L \quad \text{As stated}$$

$$T(r_1) = \frac{h r_1 (T_L - T_\infty) \ln(r_1/r_2)}{k - h r_1 \ln(r_1/r_2)} + T_L \overset{\text{exactly intern.}}{=} 100$$

3 equally spaced elements (linear)



Flux prescribed $\qquad$ T known (Dirichlet condition)

4 global nodes $=$ 4x4 system

$$-\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$ mult by W + integrate over an element by parts

$$\int_{r_1^{(ei)}}^{r_2^{(ei)}} W\frac{d}{dr}\left(kr\frac{dT}{dr}\right)dr = 0 \qquad \int a\,db = ab - \int b\,da$$

$\forall$ elements except 1st & last respectively

$$= +\int_{r_1^{(ei)}}^{r_2^{(ei)}} kr\frac{dW}{dr}\cdot\frac{dT}{dr}\,dr - \cancelto{0}{Wkr\frac{dT}{dr}}\Bigg|_{r_1^{(ei)}}^{r_2^{(ei)}} = 0$$

For internal element, $T \cong$ 1st order polynomial (linear)

$r \cong$ 1st order polynomial linear or Approx by value at mid point.

$$= K^{(ei)}\frac{\left(r_1^{(ei)}+r_2^{(ei)}\right)}{2}\underbrace{\int_{r_1^{(ei)}}^{r_2^{(ei)}}\begin{bmatrix}\frac{dN_1^{(ei)}}{dr}\\ \frac{dN_2^{(ei)}}{dr}\end{bmatrix}\begin{bmatrix}\frac{dN_1^{(ei)}}{dr} & \frac{dN_2^{(ei)}}{dr}\end{bmatrix}dr}\begin{bmatrix}T_1^{(ei)}\\ T_2^{(ei)}\end{bmatrix} = 0$$

$$\int_{r_1^{(ei)}}^{r_2^{(ei)}}\underbrace{\frac{1}{(\Delta r)}\begin{bmatrix}-1\\+1\end{bmatrix}\begin{bmatrix}-1 & +1\end{bmatrix}\frac{1}{\Delta r}}_{\text{constant}}\,dr\begin{bmatrix}T_1^{(ei)}\\ T_2^{(ei)}\end{bmatrix} = 0$$

5

$$= k^{(e_1)} \frac{\left(r_1^{(e_1)} + r_2^{(e_1)}\right)}{2h^{(e_1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix} = 0$$

For e_1 the term $-wkr \left.\dfrac{dT}{dr}\right|_{r=r_1^{(e_1)}}^{r_2^{(e_1)}} =$

From BC,

need Additional r here BC's are

$$-kr_1 \left.\frac{dT}{dr}\right|_{r_1} = -h(T - T_\infty)\big|_{r_1} r \qquad \underbrace{-k\left.\frac{dT}{dr}\right|_{r_1}} + h(T-T_\infty)\big|_{r_1} = 0$$

$$\therefore \quad -wkr\left.\frac{dT}{dr}\right|_{r=r_1^{(e_1)}}^{r_2^{(e_1)}} = 0 - \left(-wkr_1 \left.\frac{dT}{dr}\right|_{r_1^{(e_1)}}\right)$$

$$= -\left(-wh(T - T_\infty)\right)r\Big|_{r=r_1^{(e_1)}} = \begin{bmatrix} N_1^{(e_1)} \\ N_2^{(e_1)} \end{bmatrix} h \begin{bmatrix} N_1^{(e_1)} & N_2^{(e_1)} \end{bmatrix} \begin{bmatrix} T_1^{(e_1)} - T_\infty \\ T_2^{(e_1)} - T_\infty \end{bmatrix} r \Bigg|_{r=r_1^{(e_1)}}$$

$$= \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} h \begin{bmatrix} 1 & 0 \end{bmatrix}} \begin{bmatrix} T_1^{(e_1)} - T_\infty \\ T_2^{(e_1)} - T_\infty \end{bmatrix} r_1$$

$$r_1 h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{(e_1)} - T_\infty \\ T_2^{(e_1)} - T_\infty \end{bmatrix}$$

elemental matrix for element e_3 would have an additional

term

$$-wkr\frac{dT}{dr}\Big|_{r=r_1^{(e_3)}}^{r_2^{(e_3)}} = -wkr\frac{dT}{dr}\Big|_{r=r_2^{(e_3)}} =$$

other term is zero

BC at $r=r_2$ is $T=T_L$ wr test fun w must satisfy

the homogeneous B.C. at this Boundary $\Rightarrow w(r_2^{(e_3)})=0$

$\Rightarrow$ this term is not present on this end

Thus local elemental matrices are:

<u>e_1</u>

$$\frac{k^{(e_1)}(r_1^{(e_1)}+r_2^{(e_1)})}{2h^{(e_1)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1^{e_1}\\ T_2^{e_1}\end{bmatrix} + r_1h\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}T_1^{e_1}\\ T_2^{e_1}\end{bmatrix} = r_1hT_\infty\begin{bmatrix}1\\ 0\end{bmatrix}$$

<u>e_2</u>

$$\frac{k^{(e_2)}(r_1^{e_2}+r_2^{e_2})}{2h^{(e_2)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1^{e_2}\\ T_2^{e_2}\end{bmatrix} = 0$$

<u>e_3</u>

$$\frac{k^{(e_3)}(r_1^{e_3}+r_2^{e_2})}{2h^{e_3}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1^{e_3}\\ T_2^{e_3}\end{bmatrix} = 0$$

Then these become for 3 equally spaced linear elements.

1 — e_1 — 2 — e_2 — 3 — e_3 — 4

20 cm 23.$\bar{3}$ cm 26.$\bar{6}$ cm 30 cm

$k = 200 \, W/mC$

$h^{(e_1)} = h = 3\tfrac{1}{3}\text{ cm} = \tfrac{10}{3}\text{ cm} = 3.33$

$\underline{e_1}$
$$\frac{20(20+23.33)}{2(3.33)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix}$$
$$(.2)50\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = (.2)\cdot 50\cdot 400\begin{bmatrix}1\\ 0\end{bmatrix}$$

$\underline{e_2}$
$$\frac{20(23.33+26.66)}{2(3.33)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_2\\ T_3\end{bmatrix} = \bar{0}$$

$\underline{e_3}$
$$\frac{20(26.66+30)}{2(3.33)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_3\\ T_4\end{bmatrix} = 0$$

$\Rightarrow$ $\underline{e_1}$
$$130.12\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} + \begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = 4000\begin{bmatrix}1\\ 0\end{bmatrix}$$

$\underline{e_2}$
$$150.12\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_2\\ T_3\end{bmatrix} = \bar{0}$$

$\underline{e_3}$
$$170.15\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_3\\ T_4\end{bmatrix} = \bar{0}$$

Now Assembling elemental matrix results in 4x4 system

$$\begin{bmatrix}130.12+10 & -130.12 & 0 & 0\\ -130.12 & 130.12+150.12 & -150.12 & 0\\ 0 & -150.12 & 150.12+170.15 & -170.15\\ 0 & 0 & -170.5 & 170.15\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix} = \begin{bmatrix}4000\\ 0\\ 0\\ 0\end{bmatrix}$$

Imposing known condition that $T_4 = T_L = 39.13\,C$ gives

$$\Rightarrow \begin{bmatrix} 140.12 & -130.12 & 0 & 0 \\ -130.12 & 280.24 & -150.12 & 0 \\ 0 & -150.12 & 320.27 & -170.15 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 4000 \\ 0 \\ 0 \\ 39.13 \end{bmatrix}$$

$$\Rightarrow \quad T_1 = 99.8762$$
$$T_2 = 76.811$$
$$T_3 = 56.818$$
$$T_4 = 39.13$$

For 3 non equally spaced elements

e_1 e_2 e_3

20cm 22cm 25cm 30cm

Then elemental matrices in this case become:

$\underline{e_1}$

$$\frac{20(20+22)}{2(2)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 4000 \\ 0 \end{bmatrix}$$

$\underline{e_2}$

$$\frac{20(22+25)}{2(3)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \bar{0}$$

$\underline{e_3}$

$$\frac{20(25+30)}{2(5)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \bar{0}$$

$\Rightarrow$ $\underline{e_1}$ $210\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} + \begin{bmatrix}10 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = \begin{bmatrix}4000\\ 0\end{bmatrix}$

$\underline{e_2}$ $156.66\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_2\\ T_3\end{bmatrix} = \bar{0}$

$\underline{e_3}$ $110\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_3\\ T_4\end{bmatrix} = \bar{0}$

Assembly then gives

$$\begin{bmatrix}210+10 & -210 & 0 & 0\\ -210 & 210+156.66 & -156.66 & 0\\ 0 & -156.66 & 156.66+110 & -110\\ 0 & 0 & -110 & 110\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix} = \begin{bmatrix}4000\\ 0\\ 0\\ 0\end{bmatrix}$$

Enforcing $T_4 = 39.18$ gives

$$= \begin{bmatrix}220 & -210 & 0 & 0\\ -210 & 366.66 & -156.66 & 0\\ 0 & -156.66 & 266.66 & -110\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix} = \begin{bmatrix}4000\\ 0\\ 0\\ 39.18\end{bmatrix}$$

$\Rightarrow$ $T_1 = 96.1992$
$T_2 = 81.73$
$T_3 = 62.34.$
$T_4 = 39.18$

(3.9) (a) see notes pg 34.

(b) elemental shape fns w/ Nodes $x_1 = 0, x_2 = \frac{1}{3}, x_3 = +1$

$$N_1(\xi) = 3(\xi - \tfrac{1}{3})(\xi - 1)$$

$\frac{2}{9}$

$$N_2(\xi) = \tfrac{9}{2}\xi(\xi - 1)$$

$$N_3(\xi) = \tfrac{3}{2}\xi(\xi - \tfrac{1}{3})$$

(3.10) Shape fns now defined on $(-1,+1)$ to be 1 at $-1, -\frac{1}{3}, +\frac{1}{3}, +1$

Thus the local coordinates are

$$N_1(\xi) = \frac{-9}{16}(\xi+\tfrac{1}{3})(\xi-\tfrac{1}{3})(\xi-1) \qquad (-\tfrac{2}{3})(-\tfrac{4}{3})(-2) = -\frac{16}{9}$$

$$N_2(\xi) = \frac{27}{16}(\xi+1)(\xi-\tfrac{1}{3})(\xi-1) \qquad (\tfrac{2}{3})(-\tfrac{2}{3})(-\tfrac{4}{3}) = \frac{16}{27}$$

$$N_3(\xi) = -\frac{27}{16}(\xi+1)(\xi+\tfrac{1}{3})(\xi-1) \qquad \tfrac{4}{3}\,\tfrac{2}{3}(-\tfrac{2}{3}) = \frac{-16}{27}$$

$$N_4(\xi) = \frac{9}{16}(\xi+1)(\xi+\tfrac{1}{3})(\xi-\tfrac{1}{3}) \qquad 2(\tfrac{4}{3})(\tfrac{2}{3}) = \frac{16}{9}$$

+ Transformation is

$$x(\xi) = x_1 N_1(\xi) + x_2 N_2(\xi) + x_3 N_3(\xi) + x_4 N_4(\xi)$$

If $\quad x_2 = \dfrac{(2x_1+x_4)}{3} \quad + \quad x_3 = \dfrac{(x_1+2x_4)}{3}$

$$x(\xi) = -\frac{9x_1}{16}(\xi+\tfrac{1}{3})(\xi-\tfrac{1}{3})(\xi-1) + \frac{9}{16}(\xi+1)(\xi-\tfrac{1}{3})(\xi-1)(2x_1+x_4)$$

$$-\frac{9}{16}(\xi+1)(\xi+\tfrac{1}{3})(\xi-1)(x_1+2x_4) + \frac{9x_4}{16}(\xi+1)(\xi+\tfrac{1}{3})(\xi-\tfrac{1}{3})$$

$$\frac{16}{9} X(\xi) = x_1(\xi-1)\left[-\left(\xi+\tfrac{1}{3}\right)\left(\xi-\tfrac{1}{3}\right) + 2(\xi+1)\left(\xi-\tfrac{1}{3}\right) - (\xi+1)\left(\xi+\tfrac{1}{3}\right)\right]$$

$$+ x_4(\xi+1)\left[\left(\xi-\tfrac{1}{3}\right)(\xi-1) - 2\left(\xi+\tfrac{1}{3}\right)(\xi-1) + \left(\xi+\tfrac{1}{3}\right)\left(\xi-\tfrac{1}{3}\right)\right]$$

$$= x_1(\xi-1)\left[-\cancel{\xi^2} + \tfrac{1}{9} + \cancel{2\xi^2} + \cancel{\tfrac{4}{3}\xi} - \tfrac{2}{3} - \cancel{\xi^2} - \cancel{\tfrac{4}{3}\xi} - \tfrac{1}{3}\right]$$

$$-1 = -\tfrac{9}{9}$$

$$+ x_4(\xi+1)\left[\cancel{\xi^2} - \cancel{\tfrac{4}{3}\xi} + \tfrac{1}{3} - \cancel{2\xi^2} + \cancel{\tfrac{4}{3}\xi} + \tfrac{2}{3} + \cancel{\xi^2} - \tfrac{1}{9}\right]$$

$$= x_1(\xi-1)\left[-\tfrac{8}{9}\right] + x_4(\xi+1)\left(+\tfrac{8}{9}\right)$$

$$\Rightarrow \quad x(\xi) = -\tfrac{1}{2}(\xi-1)x_1 + \tfrac{1}{2}(\xi+1)x_4 \qquad \text{eq } 3.65$$

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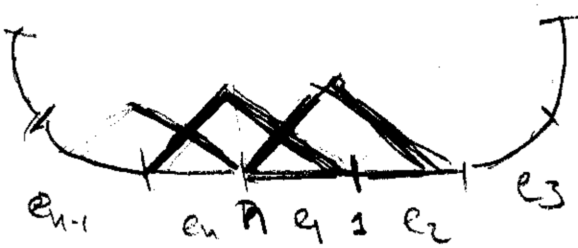
(3.11) $-k\frac{d^2T}{dx^2} = Q$

n linear elements

Boundary conditions for heat conduction at e_n, e_1 boundary are

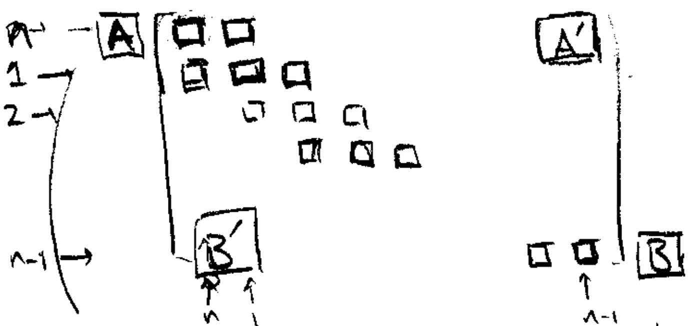
$$T(0^-) = T(0^+)$$

$$-k(0^-)\frac{dT}{dx}(0^-) = -k(0^+)\frac{dT}{dx}(0^+)$$



These are the same conditions that must hold at all interelement Boundaries. Thus the elemental eqs

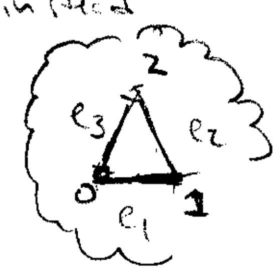
Global matrix



elemental eq for e_1:

$$\frac{k}{h}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}T_1^{(1)}\\ T_2^{(1)}\end{bmatrix} = \frac{Qh}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

The nonzero # that would go at location A is represented by the interaction of Global tent fn $N_0(x)$ + $N_{n-1}(x)$ & would be found instead at location $(0, n-1)$



Similarly the nonzero element which should be found at location B & represents the interaction of global basis fn $N_{n-1}(x)$ + $N_0(x)$ & would be found at location B'

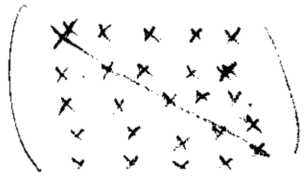
Thus the Global matrix structure is that of a tridiagonal matrix w/ the upper Right & lower Left elements filled in also

For $n = 20$ using as a definition of band width that

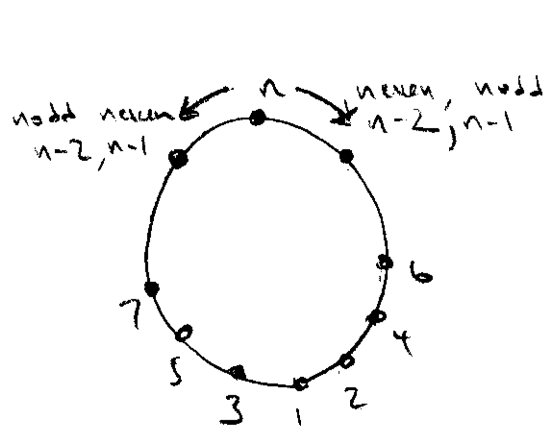
$A_{ij} = 0$ if $i > j + \ell$ & $A_{ij} = 0$ if $i < j - \ell$

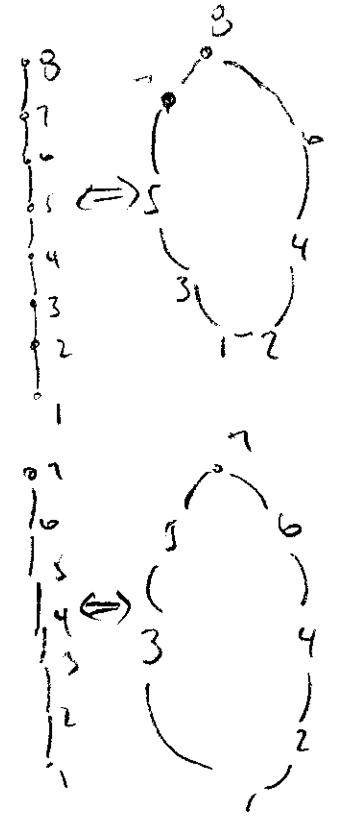
$\Rightarrow A_{ij} = 0$ if $\begin{matrix} i-j > \ell \\ i-j < -\ell \end{matrix}$ $\Rightarrow |i-j| > \ell$

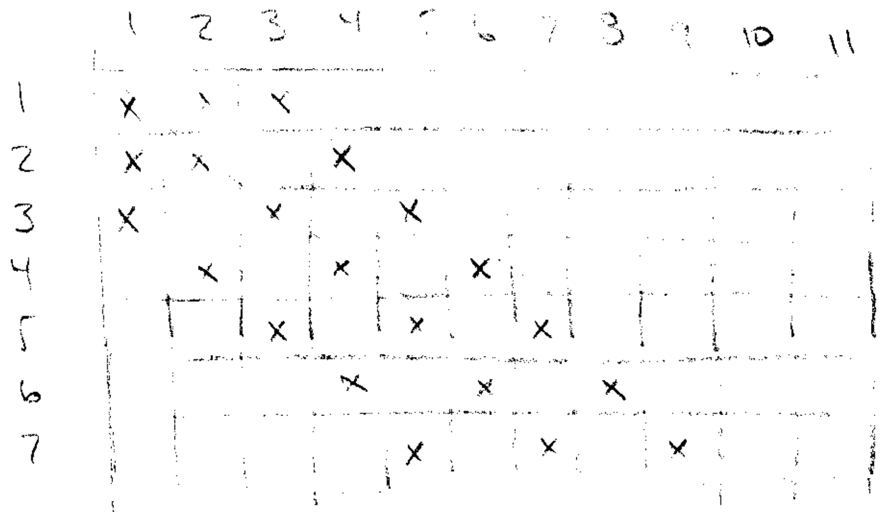
Then one would strictly require a band width of **20** full band width

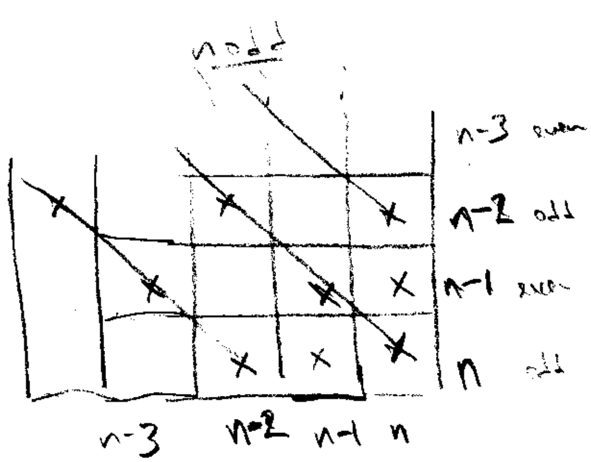


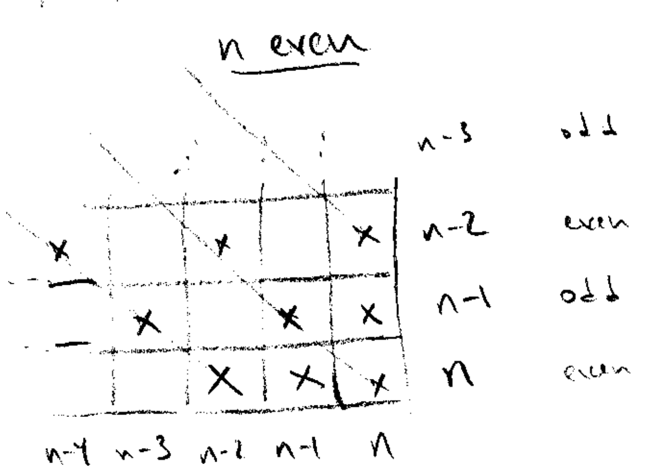
w/ Node #'s (C)









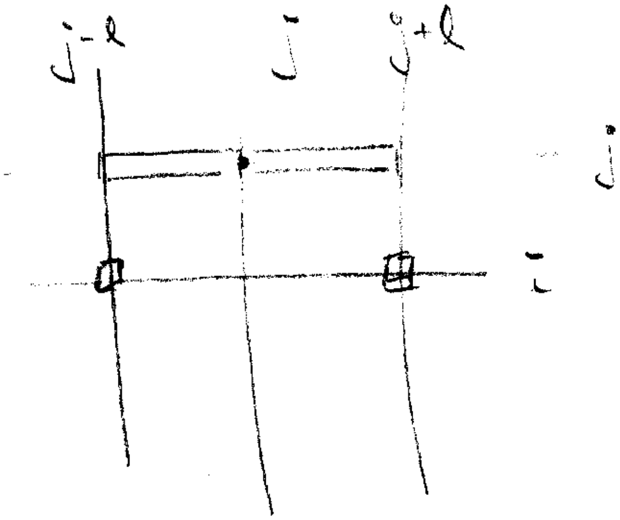


Thus w/ $n=20$ there are still only a bandwidth of $\underline{\underline{1}}$,

A significant improvement!

$A_{ij} = 0 \quad i > j + l$

$A_{ij} = 0 \quad i < j - l$



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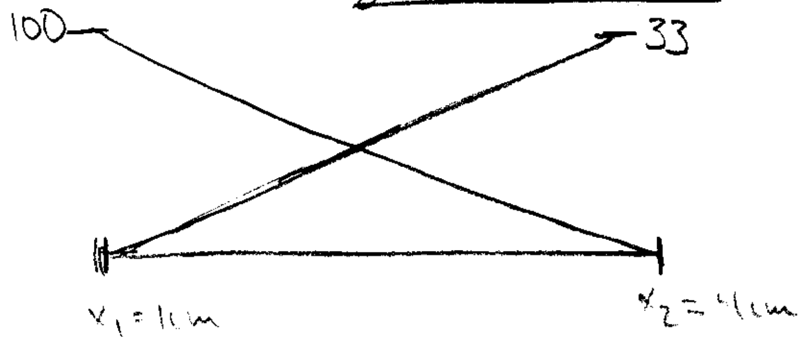
(3.12)



$$N_1(\xi) = \frac{1}{2}(1-\xi)$$

$$N_2(\xi) = \frac{1}{2}(1+\xi)$$

$$T(x) \cong 100\,N_1(\xi) + 33\,N_2(\xi) \qquad \xi : x \longrightarrow [-1,1]$$

$$\xi(x) = \frac{(-1-1)}{1-4}(x-1) - 1 \qquad \Rightarrow \xi(1cm) = -1$$

$$\xi(4cm) = +1$$

$$= \frac{-2}{-3}(x-1) - 1 = \frac{2}{3}(x-1) - 1$$

Then $T(x) \cong 100\,N_1(\xi(x)) + 33\,N_2(\xi(x))$

$$T(2.5cm) = 100\,N_1(0) + 33\,N_2(0)$$

$$= 100\left(\frac{1}{2}\right) + 33\left(\frac{1}{2}\right) = \frac{1}{2}\,133 = 65.5$$

$$T'(x) = 100\,\frac{N_1}{d\xi}(\xi)\,\frac{d\xi}{dx} + 33\,\frac{dN_2}{d\xi}\,\frac{d\xi}{dx}$$

$$= 100\left(\frac{-1}{\cancel{2}}\right)\frac{\cancel{2}}{3} + 33\left(\frac{1}{2}\right)\frac{2}{3} = -\frac{100}{3} + 11 = -22.\overline{3}$$

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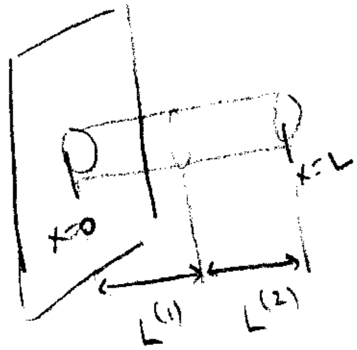
(3.13)



Eqs:

$$-k\frac{d^2T}{dx^2} = \cancel{Q}^{\,0} \qquad 0<x<L$$

$$-k\frac{dT}{dx} + h(T-T_\infty) = 0 \quad \text{at } x=L$$

$$T(0) = T_w \text{ given.}$$

Then for 2 linear elements; weighted residual becomes:
temperature prescribed force testing fns to satisfy Homogeneous boundary condition

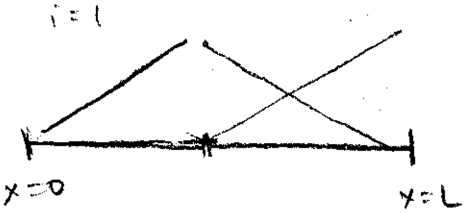
$$k\int_0^L \frac{dw}{dx}\frac{dT}{dx} - \left[w\cancel{\left(-k\frac{dT}{dx}\right)}^{\,0}\right|_{x=0} + \left[w\left(-k\frac{dT}{dx}\right)\right|_{x=L} = 0$$

$$\Rightarrow k\int_0^L \frac{dw}{dx}\frac{dT}{dx} - w\left(-h(T-T_\infty)\right)\Big|_{x=L} = 0$$

$$\Rightarrow k\int_0^L \frac{dw}{dx}\cdot\frac{dT}{dx} + wh(T-T_\infty)\Big|_{x=L} = 0$$

Approximate $T(x)$ as $T(x) = \sum_{i=1}^{2} N_i(x)T_i$ N_i are global node fns

$+ 2 = \#$ elements T is given here

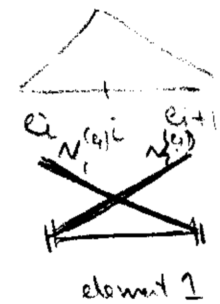


$$\Rightarrow k\int_0^L \frac{dN_i(x)}{dx}\left(\sum_{j=1}^{n+1}\frac{dN_j(x)}{dx}T_j\right)dx + N_i(x)h(T-T_\infty)\Big|_{x=L} = 0$$

$$\Rightarrow \sum_{j=1}^{n+1} k \int_0^L \frac{dN_i}{dx}\cdot\frac{dN_j}{dx}\cdot T_j + N_i(x)h(T-T_\infty)\Big|_{x=L} = 0$$ i-th row of matrix

Then for $i=1$ **1st element** $\frac{dN_i}{dx} = 0$ outside this element

$$k\int_0^{L^{(1)}} \begin{bmatrix} \frac{dN_1^{(e_1)}}{dx} \\ \frac{dN_2^{(e_1)}}{dx} \end{bmatrix} \begin{bmatrix} \frac{dN_1^{(e_1)}}{dx} & \frac{dN_2^{(e_1)}}{dx} \end{bmatrix} dx \begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix} \qquad N_i(x) = \begin{cases} N_1^{(e_i)} \\ N_2^{(e_{i+1})} \end{cases}$$



$$+ 0 = 0$$

$$\Rightarrow \frac{k}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_1)} \\ T_2^{(e_1)} \end{bmatrix} = \vec{0}$$

Then for 2nd element $\quad (T-T_\infty)^{(e_2)}(x) = \begin{bmatrix} N_1^{(e_2)} & N_2^{(e_2)} \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} - T_\infty \\ T_2^{(e_2)} - T_\infty \end{bmatrix}$

$$\frac{k}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} \\ T_2^{(e_2)} \end{bmatrix} + h\begin{bmatrix} N_1^{(e_2)} \\ N_2^{(e_2)} \end{bmatrix}\begin{bmatrix} N_1^{(e_2)} & N_2^{(e_2)} \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} - T_\infty \\ T_1^{(e_2)} - T_\infty \end{bmatrix}\Bigg|_{x=L} = 0$$

why is this representation of $N(x)$ a column vector & the one in $(T-T_\infty)^{(e_2)}(x)$ a row vector?

$$\Rightarrow \frac{k}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} \\ T_2^{(e_2)} \end{bmatrix} + h\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} - T_\infty \\ T_1^{(e_2)} - T_\infty \end{bmatrix} = 0$$

$$\Rightarrow \quad \frac{k}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} \\ T_2^{(e_2)} \end{bmatrix} + h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} - T_\infty \\ T_2^{(e_2)} - T_\infty \end{bmatrix} = 0$$

$$\Rightarrow \quad \frac{k}{L_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} \\ T_2^{(e_2)} \end{bmatrix} + \underbrace{h\begin{bmatrix} 0 \\ T_2^{(e_2)} \end{bmatrix}} = hT_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$h\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} T_1^{(e_2)} \\ T_2^{(e_1)} \end{bmatrix} = hT_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then Assembling gives

$$k\begin{bmatrix} 1/L_1 & -1/L_1 & 0 \\ -1/L_1 & 1/L_1 + 1/L_2 & -\frac{1}{L_2} \\ 0 & -1/L_2 & 1/L_2 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

$$= hT_\infty \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} k/L_1 & -k/L_1 & 0 \\ -k/L_1 & k/L_1 + k/L_2 & -k/L_2 \\ 0 & -k/L_2 & k/L_2 + h \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = hT_\infty \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$[h] = \frac{W}{m^2 C}$

$[k] = \frac{W}{m C}$

But $T_1 \equiv T_w$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ -k/L_1 & k/L_1 + k/L_2 & -k/L_2 \\ 0 & -k/L_2 & k/L_2 + h \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} T_w \\ 0 \\ hT_\infty \end{bmatrix}$$

Solving for T_2 + T_3 gives

$T_1 = 150$, $T_2 = 149.294$, $T_3 = 148.235$ check w/ other means

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(3.14)

$$EI \frac{d^2y}{dx^2} + M = 0 \quad (*) \qquad y(0) = y(L) = 0$$

$M = M(x)$ Mult eq $(*)$ by testing fn $W(x)$ + integrate from 0 to L

$$\underbrace{EI \int_0^L W(x) \frac{d^2y}{dx^2}} + \int_0^L M(x) W(x)\, dx = 0$$

integrate by parts

$$EI \left[W(x) \frac{dy}{dx} \Big|_0^L - \int_0^L \frac{dW}{dx} \cdot \frac{dy}{dx}\, dx \right] + \int_0^L M(x) W(x)\, dx = 0$$

When temperature is given at a boundary point we set the heat flux to zero there from bottom of pg 9.

$$\Rightarrow \quad -EI \int_0^L \frac{dW}{dx} \cdot \frac{dy}{dx}\, dx + \int_0^L M(x) W(x)\, dx = 0 \qquad \text{weak statement}$$

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3.15 For example 3.1 elemental eqs are 3.52 for e_1 & 3.53 for e_i $i \neq 1$

Then for 6 linear elements

$$\begin{array}{ccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 \\ & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \frac{0}{3} & \frac{5}{3} & \frac{10}{3} & \frac{15}{3}=5 & \frac{20}{3} & \frac{25}{3} & \frac{30}{3}\text{cm} \end{array}$$

$\frac{10}{6} = \frac{5}{3}$

Then elemental matrices then become:

$k(x) = 40 + 20\left(\frac{x}{10\text{cm}}\right)$ x cm

$= 40 + 20\frac{x}{10 \cdot 10^{-2}}$ x in meters

$k(x) = 40 + 200x$

$h^{(e)} = \frac{5}{3} \cdot 10^{-2}$ m.

$k(x) = 40 + 2x$ x in cm

e_1: $\frac{1}{2\left(\frac{5}{3}\right)}\left(40+40+\frac{10}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$

$+\begin{bmatrix} 10^{-2}100 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = 400\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$

e_2: $\frac{1}{\frac{10}{3}}\left(40+\frac{10}{3}+40+\frac{20}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \bar{0}$

e_3: $\frac{3}{10}\left(40+\frac{20}{3}+40+\frac{30}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_3 \\ T_4 \end{Bmatrix} = \bar{0}$

e_4: $\frac{3}{10}\left(40+\frac{30}{3}+40+\frac{40}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_4 \\ T_5 \end{Bmatrix} = \bar{0}$

e_5: $\frac{3}{10}\left(40+\frac{40}{3}+40+\frac{50}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_5 \\ T_6 \end{Bmatrix} = \bar{0}$

$\underline{e_6}$: $\frac{3}{10}\left(40 + \frac{50}{3} + 40 + \frac{60}{3}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_6 \\ T_7 \end{bmatrix} = 0$

$\underline{e_1}$:

$\rightarrow$ $\frac{1}{10}(250)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = 400\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{e_2}$: $\frac{1}{10}(270)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \vec{0}$

$\underline{e_3}$

$\frac{1}{10}(290)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_3 \\ T_4 \end{bmatrix} = \vec{0}$

$\underline{e_4}$

$\frac{1}{10}(310)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_4 \\ T_5 \end{bmatrix} = \vec{0}$

$\underline{e_5}$

$\frac{1}{10}(330)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_5 \\ T_6 \end{bmatrix} = \vec{0}$

$\underline{e_6}$

$\frac{1}{10}(350)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_6 \\ T_7 \end{bmatrix} = \vec{0}$

Assembling gives: a 7×7 matrix

$$\begin{bmatrix} 25+1 & -25 & & & & & \\ -25 & 25+27 & -27 & & & & \\ & -27 & 27+29 & -29 & & & \\ & & -29 & 29+31 & -31 & & \\ & & & -31 & 31+33 & -33 & \\ & & & & -33 & 33+35 & -35 \\ & & & & & -35 & 35 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{bmatrix} = \begin{bmatrix} 400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 26 & -25 & & & & & \\ -25 & 52 & -27 & & & & \\ & -27 & 56 & -29 & & & \\ & & -29 & 60 & -31 & & \\ & & & -31 & 64 & -33 & \\ & & & & -33 & 68 & -35 \\ & & & & & -35 & 35 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Then imposing BC $T_7 \equiv 39.18$

$\Rightarrow$ last row of matrix becomes $0 \cdots 1 \quad 39.18$.

See MMA output for inverse of this matrix

Pg 53 Pepper/Heinrich :

(3.16) Discretize $\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = \bar{Q}$ $\quad \alpha = \frac{k}{\rho c_p}$, $\bar{Q} = \frac{Q}{\rho c_p}$

Data from example 3.1

$L = 10cm$

$k(x) = 40 + 2x$ $\quad x$ in cm $\quad x \in (0, 10)$

$h = 100 \, W/m^2C$

$T_\infty = 400C$

$T_L = 39.18 \quad x = L$

0 $\qquad$ $x = L$ Dirchlet B.C.'s

$-k\frac{dT}{dx} + h(T - T_\infty) = 0$ at $x = 0$

Then weighted residual formulation of the problem then is

$$\int_0^L W(x)\left(\frac{\partial T}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2} - \bar{Q} \right) dx = 0$$

Approximating $T(x,t) = \sum_{i=1}^{n+1} N_i(x) T_i(t)$ Then $\frac{\partial T}{\partial x} = \sum_{i=1}^{n+1} \frac{\partial N_i}{\partial x}(x)\, T_i(t) = \left[\frac{\partial N_i}{\partial x}\right][T_i]$

eq 3.108 is $M\dot{T} + kT = F$ globally spacially descritized system

+ 3.109 is $(M + \Theta\Delta t k)T^{n+1} = (M + (\Theta - 1)\Delta t k)T^n + \Delta t(\Theta F^{n+1} + (1-\Theta)F^n)$
global spacial + temporally descritized system.

Following same procedure as in the book we would end up w/
eq 3.97 w/ an additional $\bar{Q}$ term i.e

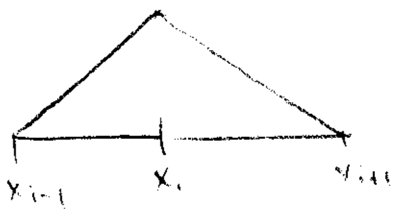
$$\left[\int_0^L N_i N_j \, dx\right]\dot{T}_j + \left[\alpha\int_0^L \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx\right]T_j + \left[N_i\left(-\alpha\frac{\partial T}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L} - \bar{Q}\int_0^L N_i(x)\,dx = 0$$

Now : $$\int_{x_i}^{x_{i+1}} N_i(x) N_j(x)\, dx = \frac{h^{ei}}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\int_{x_i}^{x_{i+1}} \frac{dN_i}{dx}\frac{dN_j}{dx}\,dx = \frac{1}{h^{(e_i)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

& the integral of $\displaystyle\int_0^L N_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} N_i(x)\,dx = \underbrace{\int_{x_{i-1}}^{x_i} N_i^{(1)}(x)\,dx}_{} + \underbrace{\int_{x_i}^{x_{i+1}} N_i^{2}\,dx}_{}$

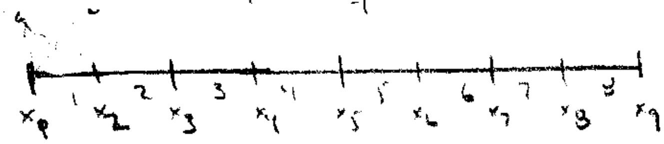
$\parallel$ two elemental integrals



$$\frac{h^{e_{i-1}}}{2}\int_{-1}^{+1}\tfrac{1}{2}(1-\xi)\,d\xi + \frac{h^{e_i}}{2}\int_{-1}^{+1}\tfrac{1}{2}(1+\xi)\,d\xi$$

$$= \frac{h^{e_{i-1}}}{4}\left(\xi-\frac{\xi^2}{2}\right)\Big|_{-1}^{+1} + \frac{h^{e_i}}{4}\left(\xi+\frac{\xi^2}{2}\right)\Big|_{-1}^{1} = \frac{h^{e_{i-1}}}{4}\left(\tfrac{1}{2}-\left(-1-\tfrac{1}{2}\right)\right) + \frac{h^{e_i}}{4}\left(\tfrac{3}{2}-\left(-1+\tfrac{1}{2}\right)\right)$$

Then for 8 elements



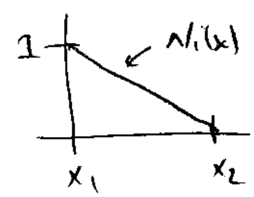
$$= \frac{h^{e_{i-1}}}{4}\left(\tfrac{4}{2}\right) + \frac{h^{e_i}}{4}\left(\tfrac{3}{2}+\tfrac{1}{2}\right) = \underbrace{\frac{h^{e_{i-1}}}{2} + \frac{h^{e_i}}{2}}_{\text{two elemental contributions.}}$$

elemental matrix e_1

$$\int_{x_1}^{x_2}$$

$$= \frac{h^{e_1}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{bmatrix} + \frac{\alpha}{h^{e_1}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} + \text{see below} - \bar{Q}\frac{h^{e_1}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

For element e_1 boundary flux term becomes: $N_i = N_1(x)$



$$\left[N_i\left(-\alpha\frac{\partial T}{\partial x}\right)\right|_{x=0}^{x=L} = 0 - N_i\left(-\alpha\frac{\partial T}{\partial x}\right)\Big|_{x=0}$$

$$= -N_i\left(-h(T-T_\infty)\right)\Big|_{x=0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} h \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} T_1^{e_1} - T_\infty \\ T_2^{e_1} - T_\infty \end{bmatrix}$$

$$= h \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{e_1} - T_\infty \\ T_2^{e_1} - T_\infty \end{bmatrix} = h \begin{bmatrix} T_1^{e_1} - T_\infty \\ 0 \end{bmatrix} = h \begin{bmatrix} T_1 - T_\infty \\ 0 \end{bmatrix}$$

element e_2:

$$\frac{h^{e_2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} + \frac{\alpha}{h^{e_2}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} - \frac{\bar{Q} h^{e_2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$$

e_3:

$$\frac{h^{e_3}}{6} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \dot{T}_3 \\ \dot{T}_4 \end{bmatrix} + \frac{\alpha}{h^{e_3}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_3 \\ T_4 \end{bmatrix} - \frac{\bar{Q} h^{e_3}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$$

e_4

same $\begin{bmatrix} \dot{T}_4 \\ \dot{T}_5 \end{bmatrix}$

e_5

same $\begin{bmatrix} \dot{T}_5 \\ \dot{T}_6 \end{bmatrix}$ e_6 same $\begin{bmatrix} \dot{T}_6 \\ \dot{T}_7 \end{bmatrix}$ e_7 $\begin{bmatrix} \dot{T}_7 \\ \dot{T}_8 \end{bmatrix}$

e_8

$$\frac{h^{e_8}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \dot{T}_8 \\ \dot{T}_9 \end{bmatrix} + \frac{\alpha}{h^{e_8}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_8 \\ T_9 \end{bmatrix} + \cancel{\vec{0}}$$

$$- \frac{\bar{Q} h^{e_8}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0}$$

? $\vec{0}$ Here the weight for $N_9(x)$ is taken to be zero at
hom Dirichlet BC involve
Homogeneous condition & thus
we enforce them strongly.

Assembling into full matrix, gives. 9x9 system Assuming all elements of equal lengths h & α constant.

$$\frac{h}{6}\begin{bmatrix} 2 & 1 & & & & & & & \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & 1 & 4 & 1 & & & \\ & & & & 1 & 4 & 1 & & \\ & & & & & 1 & 4 & 1 & \\ & & & & & & 1 & 4 & 1 \\ & & & & & & & 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \dot{T}_6 \\ \dot{T}_7 \\ \dot{T}_8 \\ \dot{T}_9 \end{bmatrix} +$$

$$\frac{\alpha}{h}\begin{bmatrix} 1 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 2 & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} -$$

$$\frac{\bar{Q}h}{2}\begin{bmatrix}1\\2\\2\\2\\2\\2\\2\\2\\1\end{bmatrix} + \begin{bmatrix}\tilde{h}T_1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} - \tilde{h}T_\infty\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} = \vec{0}$$

$$\begin{bmatrix}\tilde{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & & & & & & & & \\ 0 & & & & & & & & \\ 0 & & & & & & & & \\ 0 & & & & 0 & & & & \\ 0 & & & & & & & & \\ 0 & & & & & & & & \\ 0 & & & & & & & & \\ 0 & & & & & & & & \end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\\T_4\\T_5\\T_6\\T_7\\T_8\\T_9\end{bmatrix}$$

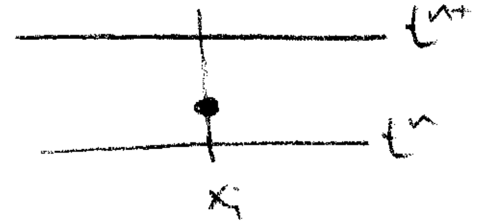
Then $M = \frac{h^e}{6}\begin{bmatrix}2+\tilde{h} & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & & \ddots & \end{bmatrix}$ $\qquad \vec{F} = -\frac{\bar{Q}h}{2}\begin{bmatrix}1\\2\\2\\\vdots\\1\end{bmatrix} + \begin{bmatrix}\tilde{h}T_\infty\\0\\\vdots\\0\end{bmatrix}$

$K = \frac{\alpha}{h}\begin{bmatrix}1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \end{bmatrix}$

TO solve in time we must descretize these discrete eqs.

Using Θ method

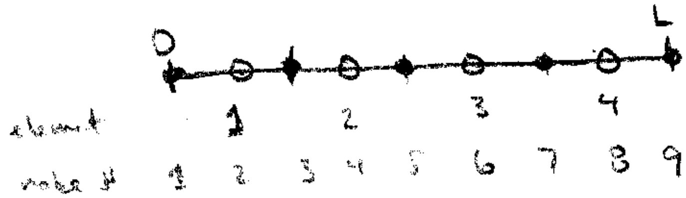
$$\frac{\partial T}{\partial t} = \frac{T^{n+1} - T^n}{\Delta t} \quad + \quad T = \Theta T^{n+1} + (1-\Theta) T^n$$



we obtain

(b) Use matrices defined above to evaluate 3.109.

(c) 4 quadratic elements N_i is different depending if one is an "interior" node or an exterior node



Get same Gerlerkin Form as before but now evaluations of integrals is different.

$\Rightarrow$ we require

$$\int_{x_{2i-1}}^{x_{2i+1}} N_i N_j \, dx =$$

let $x = N_1(\xi) x_{2i-1} + N_2(\xi) x_{2i} + N_3(\xi) x_{2i+1}$

$$dx = \left(N_1'(\xi) x_{2i-1} + N_2'(\xi) x_{2i} + N_3'(\xi) x_{2i+1}\right) d\xi$$

$$= \left(\underbrace{\tfrac{1}{2}(2\xi-1) x_{2i-1} + (-2\xi) x_{2i} + \tfrac{1}{2}(2\xi+1) x_{2i+1}}\right) d\xi$$

$$= \frac{h}{2} \int_{-1}^{+1} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} d\xi \qquad = \quad \xi\underbrace{(x_{2i-1} - 2x_{2i} + x_{2i+1})}_{=0 \text{ if } x_{2i} = \frac{1}{2}(x_{2i+1} + x_{2i-1})} + \frac{1}{2}\underbrace{(-x_{2i-1} + x_{2i+1})}_{= \frac{h^e}{2}}$$

$$= \frac{h}{2}\int_{-1}^{+1} \begin{bmatrix} \frac{1}{2}\xi(\xi-1) \\ 1-\xi^2 \\ \frac{1}{2}\xi(\xi+1) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\xi(\xi-1) & 1-\xi^2 & \frac{1}{2}\xi(\xi+1) \end{bmatrix} d\xi$$

By MMA

$$= \frac{h}{15\cdot 2}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} = \frac{h}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

Then elemental matrices look like:

$\underline{e_1}$:

$$\frac{h^{e_1}}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \end{bmatrix} + \frac{\alpha}{2h^{e_1}}\begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} +$$

$$+ \underbrace{0 - \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}(-h(T-T_\infty))}\Bigg|_{x=0} - \frac{\bar{Q}h}{6}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \vec{0}$$

$$\underbrace{+h\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}\begin{bmatrix} T_1 - T_\infty \\ T_2 - T_\infty \\ T_3 - T_\infty \end{bmatrix}}\Bigg|_{x=0}$$

$$h\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 - T_\infty \\ T_2 - T_\infty \\ T_3 - T_\infty \end{bmatrix} = h\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 - T_\infty \\ T_2 - T_\infty \\ T_3 - T_\infty \end{bmatrix}$$

$$= h\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - hT_\infty\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\underline{e_2}$:

$$\frac{h^{e_2}}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}\begin{bmatrix}\dot{T}_3\\ \dot{T}_4\\ \dot{T}_5\end{bmatrix} + \frac{\alpha}{2h^{e_2}}\begin{bmatrix}14 & -16 & 2\\ -16 & 32 & -16\\ 2 & -16 & 14\end{bmatrix}\begin{bmatrix}T_3\\ T_4\\ T_5\end{bmatrix} - \frac{\bar{Q}h}{6}\begin{bmatrix}1\\ 4\\ 1\end{bmatrix} = \bar{0}$$

$\underline{e_3}$:

$$\frac{h^{e_3}}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}\begin{bmatrix}\dot{T}_5\\ \dot{T}_6\\ \dot{T}_7\end{bmatrix} + \frac{\alpha}{2h^{e_3}}\begin{bmatrix}14 & -16 & 2\\ -16 & 32 & -16\\ 2 & -16 & 14\end{bmatrix}\begin{bmatrix}T_5\\ T_6\\ T_7\end{bmatrix} - \frac{\bar{Q}h}{6}\begin{bmatrix}1\\ 4\\ 1\end{bmatrix} = \bar{0}$$

$\underline{e_4}$:

$$\frac{h^{e_4}}{30}\begin{bmatrix}4 & 2 & -1\\ 2 & 16 & 2\\ -1 & 2 & 4\end{bmatrix}\begin{bmatrix}\dot{T}_7\\ \dot{T}_8\\ \dot{T}_9\end{bmatrix} + \frac{\alpha}{2h^{e_4}}\begin{pmatrix}14 & -16 & 2\\ -16 & 32 & -16\\ 2 & -16 & 14\end{pmatrix}\begin{bmatrix}T_7\\ T_8\\ T_9\end{bmatrix} + 0 - \bar{Q}\frac{h^{e_4}}{6}\begin{bmatrix}1\\ 4\\ 1\end{bmatrix} = 0$$

Assembling into global matrix of size 9×9 + all of equal length

$$\Rightarrow \frac{h}{30}\begin{bmatrix}4 & 2 & -1 & & & & & & \\ 2 & 16 & 2 & & & & & & \\ -1 & 2 & 8 & 2 & -1 & & & & \\ & & 2 & 16 & 2 & & & & \\ & & -1 & 2 & 8 & 2 & -1 & & \\ & & & & 2 & 16 & 2 & & \\ & & & & -1 & 2 & 8 & 2 & -1\\ & & & & & & 2 & 16 & 2\\ & & & & & & -1 & 2 & 4\end{bmatrix}\begin{bmatrix}\dot{T}_1\\ \dot{T}_2\\ \dot{T}_3\\ \dot{T}_4\\ \dot{T}_5\\ \dot{T}_6\\ \dot{T}_7\\ \dot{T}_8\\ \dot{T}_9\end{bmatrix}$$

$$+\ \frac{\alpha}{2h}\begin{bmatrix} 14 & -16 & 2 & & & & & & \\ -16 & 32 & -16 & & & & & & \\ 2 & -16 & 28 & -16 & 2 & & & & \\ & & -16 & 32 & -16 & & & & \\ & & 2 & -16 & 28 & -16 & 2 & & \\ & & & & -16 & 32 & -16 & & \\ & & & & 2 & -16 & 28 & -16 & 2 \\ & & & & & & -16 & 32 & -16 \\ & & & & & & 2 & -16 & 14 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}$$

$$+\ h\begin{bmatrix} 1 & 0 & — \\ 0 & 0 & \\ 0 & & \\ & & 0 \end{bmatrix}_{9\times 9}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_9 \end{Bmatrix}$$

$$-\ \frac{\bar{Q}h}{6}\begin{bmatrix} 1 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 2 \\ 4 \\ 1 \end{bmatrix} - hT_\infty\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

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$$\begin{aligned}\phi_1 &= \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1\\ \phi_2 &= \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2\\ \phi_3 &= \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3\end{aligned} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}$$

Find inverse of $\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & x_1 & y_1 & 1 & 0 & 0 \\ 1 & x_2 & y_2 & 0 & 1 & 0 \\ 1 & x_3 & y_3 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & x_1 & y_1 & 1 & 0 & 0 \\ 0 & x_2-x_1 & y_2-y_1 & -1 & 1 & 0 \\ 0 & x_3-x_1 & y_3-y_1 & -1 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & x_1 & y_1 & 1 & 0 & 0 \\ 0 & 1 & \frac{y_2-y_1}{x_2-x_1} & \frac{-1}{x_2-x_1} & \frac{1}{x_2-x_1} & 0 \\ 0 & x_3-x_1 & y_3-y_1 & -1 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & y_1 - x_1\left(\frac{y_2-y_1}{x_2-x_1}\right) & 1+\frac{x_1}{x_2-x_1} & \frac{-x_1}{x_2-x_1} & 0 \\ 0 & 1 & \frac{y_2-y_1}{x_2-x_1} & \frac{-1}{x_2-x_1} & \frac{1}{y_2-x_1} & 0 \\ 0 & 0 & y_3-y_1 - (x_3-x_1)\left(\frac{y_2-y_1}{x_2-x_1}\right) & -1+\frac{x_3-x_1}{x_2-x_1} & \frac{-(x_3-x_1)}{x_2-x_1} & 1 \end{array}\right)$$

$$\overset{?}{\Rightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & \frac{x_2y_1 - \cancel{x_1y_1} - x_1y_2 + \cancel{x_1y_1}}{x_2-x_1} & \frac{x_2}{x_2-x_1} & \frac{-x_1}{x_2-x_1} & 0 \\ 0 & 1 & \frac{y_2-y_1}{x_2-x_1} & \frac{-1}{x_2-x_1} & \frac{1}{y_2-x_1} & 0 \\ 0 & 0 & \begin{array}{c} x_2y_3 - y_1x_2 - x_1y_3 + \cancel{x_1y_1} + \\ -x_3y_2 + x_3y_1 + x_1y_2 - \cancel{x_1y_1} \end{array} & \frac{-x_2 + \cancel{x_1} + x_3 - \cancel{x_1}}{x_2-x_1} & \frac{-(x_3-x_1)}{x_2-x_1} & 1 \end{array}\right)$$

2

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & \dfrac{x_2y_1 - x_1y_2}{x_2 - x_1} & \dfrac{x_2}{x_2-x_1} & \dfrac{-x_1}{x_2-x_1} & 0 \\ 0 & 1 & \dfrac{y_2-y_1}{x_2-x_1} & \dfrac{-1}{x_2-x_1} & \dfrac{1}{x_2-x_1} & 0 \\ 0 & 0 & \underbrace{x_2y_3 - y_1x_2 - x_1y_3 - x_3y_2 + x_3y_1 + x_1y_2} & -\dfrac{(x_3-x_2)}{x_2-x_1} & -\dfrac{(x_3-x_1)}{x_2-x_1} & 1 \end{array}\right)$$

$$(x_1y_2 - x_2y_1) + (x_3y_1 - y_3x_1)$$
$$+ (x_2y_3 - x_3y_2) \equiv 2A$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & \dfrac{x_2y_1 - x_1y_2}{x_2-x_1} & \dfrac{x_2}{x_2-x_1} & \dfrac{-x_1}{x_2-x_1} & 0 \\ 0 & 1 & \dfrac{y_2-y_1}{x_2-x_1} & \dfrac{-1}{x_2-x_1} & \dfrac{1}{x_2-x_1} & 0 \\ 0 & 0 & 1 & -\dfrac{(x_3-x_2)}{2A(x_2-x_1)} & -\dfrac{(x_3-x_1)}{2A(x_2-x_1)} & \dfrac{1}{2A} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \dfrac{x_2}{x_2-x_1} + \left(\dfrac{x_2y_1 - x_1y_2}{x_2-x_1}\right)\left(\dfrac{x_3-x_2}{2A(x_2-x_1)}\right) & \dfrac{-x_1}{x_2-x_1} + \dfrac{(x_3-x_1)}{2A(x_2-x_1)}\dfrac{(x_2y_1 - x_1y_2)}{(x_2-x_1)} & -\dfrac{(x_2y_1 - x_1y_2)}{x_2-x_1}\left(\dfrac{1}{2A}\right) \\ 0 & 1 & 0 & \dfrac{-1}{x_2-x_1} + \dfrac{(y_2-y_1)}{(x_2-x_1)}\dfrac{(x_3-x_2)}{2A(x_2-x_1)} & \dfrac{1}{x_2-x_1} + \dfrac{(y_2-y_1)}{(x_2-x_1)}\dfrac{(x_3-x_1)}{2A(x_2-x_1)} & -\dfrac{(y_2-y_1)}{x_2-x_1}\left(\dfrac{1}{2A}\right) \\ 0 & 0 & 1 & -\dfrac{(x_3-x_2)}{2A(x_2-x_1)} & -\dfrac{(x_3-x_1)}{2A(x_2-x_1)} & \dfrac{1}{2A} \end{array}\right)$$

Individual terms on like element 11

$\doteq$

$$\frac{1}{2A(x_2-x_1)}\left[2Ax_2 + \frac{(x_2y_1 - x_1y_2)(x_3-x_2)}{x_2-x_1}\right]$$

$$= \frac{1}{2A(x_2-x_1)^2}\left[\underbrace{2Ax_2(x_2-x_1) + (x_2y_1 - x_1y_2)(x_3-x_2)}\right]$$

$$= (x_1y_2 - x_2y_1)(x_2-x_1) + (x_3y_1 - y_3x_1)(x_2-x_1) + (x_2y_3 - x_3y_2)(x_2-x_1)$$

$$+ (x_2y_1 - x_1y_2)(x_3-x_2)$$

Don't expand All the way ...

$$= \frac{1}{2A(x_2-x_1)^2}\Big[x_1x_2y_2 - x_2^2y_1 - x_1^2y_2 + x_1x_2y_1 + x_2x_3y_1 - x_1x_2y_3$$
$$- x_1x_3y_1 + x_1^2y_3 + x_2^2y_3 - x_2x_3y_2 - x_1x_2y_3 + \cancel{x_1x_3}y_2$$
$$+ x_2x_3y_1 - \cancel{x_1x_3}y_2 \quad - x_2^2y_2 + x_1x_2y_2\Big]$$

Use the fact that term $(x_2y_1 - x_1y_2)$ Appears in the area calculation

$=$

$\therefore$

$$2Ax_2(x_2-x_1) + (x_2y_1 - x_1y_2)(x_3-x_2)$$

$$= \underline{(x_1y_2 - x_2y_1)}(x_2-x_1) + (x_3y_1 - y_3x_1)(x_2-x_1) + (x_2y_3 - x_3y_2)(x_2-x_1)$$

$$+ \underline{(x_2y_1 - x_1y_2)}(x_3-x_2)$$

$$= (x_1y_2 - x_2y_1)(x_2 - x_1 - x_3 + x_2)$$

element 21

$$\frac{1}{2A(y_2-x_1)^2}\left[-(x_2-x_1)+(y_2-y_1)(x_3-x_2)\right]$$

$\vdots$

By MMA

$$\alpha_1 = -\frac{(x_3y_2-x_2y_3)\phi_1+(x_1y_3-x_3y_1)\phi_2+(x_2y_1-x_1y_2)\phi_3}{(x_1y_2-x_2y_1)+(x_3y_1-x_1y_3)+(x_2y_3-x_3y_2)}$$

$$\alpha_2 = -\frac{(y_3-y_2)\phi_1+(y_1-y_3)\phi_2+(y_2-y_1)\phi_3}{2A}$$

$$\alpha_3 = -\frac{(\cancel{x_1}-x_3)\phi_1+\phi_2(x_3-x_1)+\phi_1(-\cancel{x_1}+x_2)+\phi_3(x_1-x_2)}{-(\cancel{x_1y_1}-x_1y_2-x_3y_1+x_3y_2)+\cancel{x_1y_1}-x_1y_3-x_2y_1+x_2y_3}$$

$$\Rightarrow \alpha_1 = \frac{1}{2A}\left[(x_2y_3-x_3y_2)\phi_1+(x_3y_1-x_1y_3)\phi_2+(x_1y_2-x_2y_1)\phi_3\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(y_2-y_3)\phi_1+(y_3-y_1)\phi_2+(y_1-y_2)\phi_3\right]$$

$$\alpha_3 = \frac{(x_3-x_2)\phi_1+(x_1-x_3)\phi_2+(x_2-x_1)\phi_3}{2A}$$

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Given α_1, α_2 + α_3 on pg 58

$$\phi = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$\Rightarrow 2A\phi = (x_2y_3 - x_3y_2)\phi_1 + (x_3y_1 - x_1y_3)\phi_2 + (x_1y_2 - x_2y_1)\phi_3$$

$$+ (y_2 - y_3)x\phi_1 + (y_3 - y_1)x\phi_2 + (y_1 - y_2)x\phi_3$$

$$+ (x_3 - x_2)y\phi_1 + (x_1 - x_3)y\phi_2 + (x_2 - x_1)y\phi_3$$

$$\Rightarrow 2A\phi = \left((x_2y_3 - x_3y_2) + x(y_2 - y_3) + y(x_3 - x_2)\right)\phi_1$$

$$+ \left((x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right)\phi_2$$

$$+ \left((x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right)\phi_3$$

$$\phi = \frac{1}{2A}(\qquad)$$

define $N_1 = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$

$$N_2 = \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$N_3 = \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

$$N_1(x_1,y_1) = \frac{1}{2A^{(e)}}\left[(x_2y_3 - x_3y_2) + x_1y_2 - x_1y_3 + y_1x_3 - x_2y_1\right]$$

$$= 1$$

$$N_1(x_2,y_2) = \frac{1}{2A^{(e)}}\left[\, x_2\cancel{y_3} - x_3\cancel{y_2} + \cancel{x_2}y_2 - \cancel{x_2}y_3 + \cancel{y_2}x_3 - \cancel{x_2}y_2 \right] = 0$$

$$N_1(x_3,y_3) = \frac{1}{2A}\left[\, x_2\cancel{y_3} - x_3\cancel{y_2} + x_3\cancel{y_2} - x_2\cancel{y_3} + x_3y_3 - \cancel{x_3}y_3 \right] = 0$$

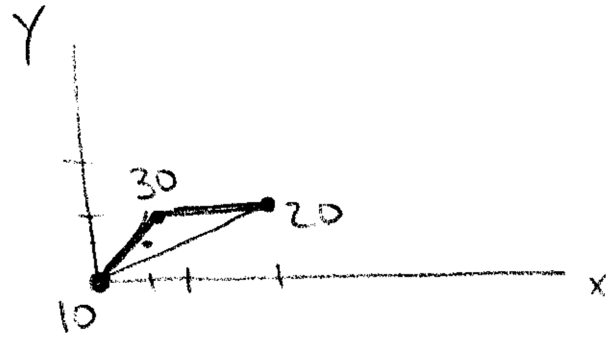
line passing through (x_2,y_2) & (x_3,y_3) can be parameterized

by
$$x(t) = x_2 + (x_3 - x_2)t$$
$$y(t) = y_2 + (y_3 - y_2)t$$

$$N_1(x(t),y(t)) = \frac{1}{2A}\Big[\, x_2y_3 - x_3y_2 + (y_2-y_3)(x_2 + (x_3-x_2)t)$$
$$+ (x_3 - x_2)(y_2 + (y_3-y_2)t) \Big]$$

$$= \frac{1}{2A}\Big[\, x_2\cancel{y_3} - x_3\cancel{y_2} + x_3\cancel{y_2} - x_2\cancel{y_3} + \cancel{(y_2-y_3)(x_3-x_2)t}$$
$$+ x_3\cancel{y_2} - x_2\cancel{y_2} + \cancel{(x_3-x_2)(y_3-y_2)t} \Big]$$

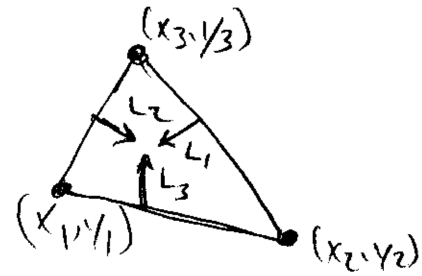
$$= 0$$

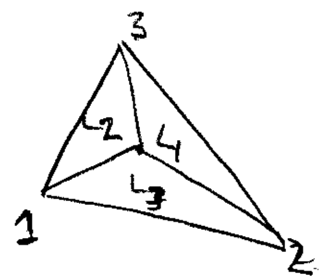


$$T = T_1 N_1(x,y) + T_2 N_2(x,y) + T_3 N_3(x,y)$$

$$=$$

$$N_1(x,y) = \frac{1}{2A}\{(2\cdot 2 - 1\cdot 1) + ($$





$N_1 = L_1$

$N_2 = L_2$

$N_3 = L_3$

given $(L_1, L_2 \& L_3)$ how does on get (x,y)?

$$L_1 = \frac{1}{2A}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$L_2 = \frac{1}{2A}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$L_3 = \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

$$L_1 + L_2 + L_3 = 1$$

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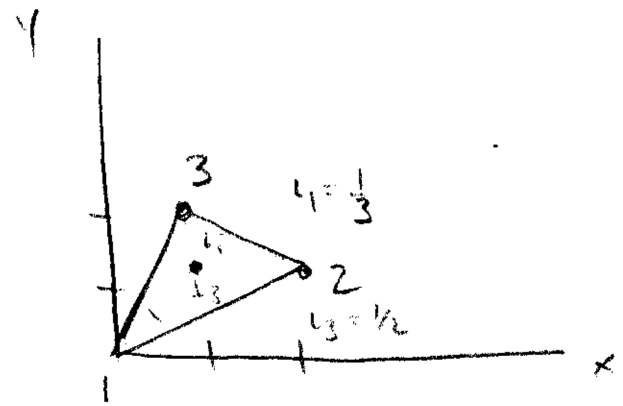


$L_2 = 1 - \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$

$T_{point} = L_1T_1 + L_2T_2 + L_3T_3 = \frac{1}{3}(10) + \frac{1}{6}(20) + \frac{1}{2}(3) =$

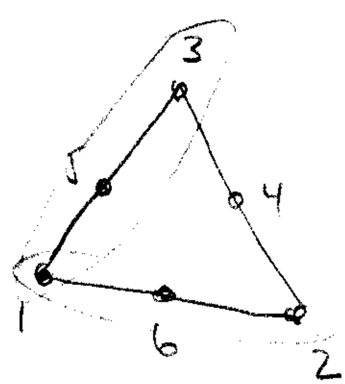
Shape fns given in terms of Area coordinates

$N_1 = 0$ when pt P is on line 2-3 & line 5-6 & 1 when full Δtriangle.

$= L_1(2L_1 - 1)$

$N_2 = L_2(2L_2 - 1)$

$N_3 = L_3(2L_3 - 1)$



$N_4 =$ Zero when at nodes 1, 6, 2 & 1, 5, 3 & 1 when at node 4. to be on node 1, 6, 2

$\Rightarrow L_3 = 0$ & to be on 1, 5, 3 $\Rightarrow L_2 = 0$

$\Rightarrow N_4 \propto L_3 L_2$

At pt 4 $L_3 = 1/2$ + $L_2 = 1/2$

$N_4 = A L_3 L_4$

$N_4(L_3 = 1/2, L_2 = 1/2) = \frac{A}{4} = 1 \Rightarrow A = 4.$

$N_5 = 4 L_1 L_3$

$N_6 = 4 L_2 L_1$

$J = \frac{\partial(x,y)}{\partial(L_1,L_2)}$

Find $\frac{\partial N_5}{\partial x}$ $\quad N_5 = 4 L_1 L_3$

$\frac{\partial N_5}{\partial y}$

$\begin{vmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{vmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -2 \end{bmatrix}$

$x = L_1 x_1 + L_2 x_2 + L_3 x_3 = 3L_2 + L_3$

$y = L_1 y_1 + L_2 y_2 + L_3 y_3 = L_2 + 3L_3$

$\frac{\partial x}{\partial L_1} = \frac{\partial x}{\partial L_1} - \frac{\partial x}{\partial L_3} = -1$ $\quad \frac{\partial y}{\partial L_1} = \frac{\partial y}{\partial L_1} - \frac{\partial y}{\partial L_3} = 0 - 3 = -3$

$\frac{\partial x}{\partial L_2} = \frac{\partial x}{\partial L_2} - \frac{\partial x}{\partial L_3} = 3 - 1 = 2$ $\quad \frac{\partial y}{\partial L_2} = \frac{\partial y}{\partial L_2} - \frac{\partial y}{\partial L_3} = 1 - 3 = -2$