Notes and Solutions to Selected Problems In: Applications of Linear Algebra by Chris Rorres and Howard Anton.

John L. Weatherwax*

May 20, 1993

*wax@alum.mit.edu

Chapter 1 (Constructing Curves and Surfaces Through Specified Points)

Problem Solutions

Problem 1.1 (lines in the plane)

Part (a): Since our line in the plane can be written as

$$c_1x + c_2y + c_3 = 0\,, \tag{1}$$

for the given two points we have

$$\begin{vmatrix} x & y & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0\,.$$

Cofactor expanding this determinant about the first row gives

$$x \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} + y \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} = 0\,.$$

Continuing the evaluation of the determinants above we finally find

$$-3x + y + 4 = 0\,.$$

Part (b): For the given points we need to evaluate

$$\begin{vmatrix} x & y & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0\,.$$

Expanding this expression as in the first case gives

$$2x + y - 1 = 0\,.$$

Problem 1.2 (the equation for circles in the plane)

The equation for a circle is given by

$$c_1(x^2 + y^2) + c_2x + c_3y + c_4 = 0\,.$$

Part (a): For the given three points we need to evaluate

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 4 + 36 & 2 & 6 & 1 \\ 4 & 2 & 0 & 1 \\ 25 + 9 & 5 & 3 & 1 \end{vmatrix} = 0\,.$$

Part (b): For the given three points we need to evaluate

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 8 & 2 & -2 & 1 \\ 34 & 3 & 5 & 1 \\ 56 & -4 & 6 & 1 \end{vmatrix} = 0\,.$$

Problem 1.3 (the equation for conic sections)

A conic section is given by the equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0\,.$$

For any of the non-zero values of c_i we can divide by this value to derive an equation involving only five unknown coefficients. Given five points in the plane we can thus determine the arbitrary conic section that goes through them. For the given points in this problem this conic section is given by expanding

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & -10 & 25 & 2 & -5 & 1 \\ 16 & -4 & 1 & 4 & -1 & 1 \end{vmatrix} = 0\,.$$

Problem 1.4 (planes in three space)

The equation for a plane in three-space is given by

$$c_1x + c_2y + c_3z + c_4 = 0\,.$$

Part (a): For the three points given the equation of the plane must satisfy

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & -3 & 1 \\ 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 1 \end{vmatrix} = 0\,.$$

Part (b): For the given three points the equation of the plane must satisfy

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 3 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 1 & 2 & 1 & 1 \end{vmatrix} = 0\,.$$

Problem 1.6 (deriving the determinant expression for a conic section)

To show the given determinant expression holds for the points on a conic section recall that the general equation for a conic section in the plane is given by

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6 = 0\,. \tag{2}$$

Since we can divide this expression by any non-zero coefficient c_i to produce an equation with only five unknowns, to completely specify these five unknowns requires us to specify five points that are in the plane and on the conic say (x_i, y_i) for $i = 1, 2, \ldots, 5$. Evaluating Equation 2 at these five points gives the five equations

$$c_1x_i^2 + c_2x_iy_i + c_3y_i^2 + c_4x_i + c_5y_i + c_6 = 0\,,$$

for $i = 1, 2, \ldots, 5$. Listing these five equations together with Equation 2 as one system gives an overdetermined linear system for the six coefficients c_i. Thus we have a homogeneous system of six unknowns with six equations. Since all of the c_i are not equal to zero this system will only have the trivial solution unless it is *nonsingular*. Thus the determinant of the system must be zero. This is the books equation 1.10.

Problem 1.7 (deriving the determinant expression for a plane)

Since the expression for a plane in three space can be written as Equation 1 where not all $c_i = 0$. This expression, coupled with the arguments of problem 1.6 above and three points on the plane gives equation 1.11.

Problem 1.8 (deriving the determinant expression for a sphere in three space)

An expression for a sphere in three-space can be written as

$$c_1(x^2 + y^2 + z^2) + c_2x + c_3y + c_4z + c_5 = 0\,,$$

with not all $c_i = 0$. This expression, coupled with four points on the sphere and the arguments from problem 1.6 give the books equation 1.12.

Problem 1.9 (deriving the determinant expression for a sphere in three space)

Following problem 1.6 the determinant expression required would be

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0\,,$$

assuming that (x_i, y_i) are three points on the parabola.

Chapter 2 (Graph Theory)

Problem Solutions

Problem 2.1 (constructing vertex matrices)

Using the definition of a vertex matrix for the given graphs we find

Part (a):

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Part (b):

$$M = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

Part (c):

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} .$$

Problem 2.3 (the number of r-step connections)

Part (b): Theorem 2.1 (the number of r-step connections) states that when M is the vertex matrix of a directed graph and $M_{ij}^{(r)}$ is the (i, j)-th element of M^r then $M_{ij}^{(r)}$ is the number of r-step connections from vertex P_i to vertex P_j. From the given vertex matrix M we compute

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} ,$$

and

$$M^3 = \begin{bmatrix} 2 & 3 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}.$$

Thus we see that in this vertex matrix we have the following connections

- one, one-step connection from P_1 to P_2 given by $P_1 \to P_2$.
- two, two-step connection from P_1 to P_2 given by
  - $P_1 \to P_4 \to P_2$.
  - $P_1 \to P_3 \to P_2$.
- three, three-step connection from P_1 to P_2 given by
  - $P_1 \to P_2 \to P_1 \to P_2$.
  - $P_1 \to P_4 \to P_3 \to P_2$.
  - $P_1 \to P_3 \to P_4 \to P_2$.

Part (c): Looking at the various matrices M, M^2, and M^3 above we see that we have the following connections

- one, one-step connection from P_1 to P_4 given by $P_1 \to P_4$.
- one, two-step connection from P_1 to P_4 given by
  - $P_1 \to P_3 \to P_4$.
- two, three-step connection from P_1 to P_4 given by
  - $P_1 \to P_4 \to P_3 \to P_4$.
  - $P_1 \to P_2 \to P_1 \to P_4$.

Problem 2.4 (finding cliques in the given directed graphs)

The cliques in these graphs are given by the sets

Part (a): $\{P_1, P_2, P_3\}$.

Part (b): $\{P_4, P_5, P_3\}$.

Part (c): $\{P_2, P_4, P_6, P_8\}$ and $\{P_6, P_4, P\}$.

Problem 2.5 (finding cliques in the given vertex matrices)

For this problem we need to recall Theorem 2.2. Theorem 2.2 (membership in a clique) states that after forming the matrix S composed of elements

$$s_{ij} = \begin{cases} 1 & P_i \leftrightarrow P_j \\ 0 & \text{otherwise} \end{cases},$$

then the vertex P_i belongs to the some cliques if and only if $s_{ij}^{(3)} \neq 0$ where $s_{ij}^{(3)}$ is the (i,j)-th element of S^3.

Part (a): For this matrix we form S as the "symmetric part" of M as

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

So that S^2 and S^3 are given by

$$S^2 = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad S^3 = \begin{bmatrix} 0 & 3 & 1 & 3 & 2 \\ 3 & 0 & 3 & 1 & 2 \\ 1 & 3 & 1 & 2 & 4 \\ 3 & 1 & 2 & 1 & 4 \\ 2 & 2 & 4 & 4 & 5 \end{bmatrix}.$$

See the Matlab/Octave script `chap_2_prob_5.m` where we calculate these matrices.

Problem 2.6 (the power of a dominance directed graph)

We find the vertex matrix for the given graph to be

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The *power* of a vertex in a dominance directed graph is the total number of one-step and two-step connections from the given vertex to the other vertexes. Equivalently, the power of the i-th vertex is the sum of the elements in the i-th row of the matrix $M + M^2$. For M above we find

$$M + M^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The row sums of this matrix are given by

$$P_1 : 5\,, P_2 : 3\,, P_3 : 4\,, P_4 : 2\,.$$

Problem 2.7 (dominance in baseball)

The verbal description of the ordering of the baseball teams translates into the following vertex matrix

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

From which we see that M^2 is given by

$$M^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \end{bmatrix}.$$

Thus the matrix $M + M^2$ is given by

$$M + M^2 = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}.$$

Using this expression, the power of each vertex is seen to be

$$P_1 = A : 8\,, P_2 = B : 6\,, P_3 = C : 5\,, P_4 = D : 3\,, P_5 = E : 6\,.$$

These can then be sorted in rank order.

See the Matlab/Octave script `chap_2_prob_7.m` where we calculate these matrices.

Chapter 3 (Theory of Games)

Problem Solutions

Problem 3.1 (a given games expected payoff)

Part (a): For the game given we find

$$E(p,q) = pAq = \frac{1}{4}p \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix} = -\frac{5}{8}\,.$$

The fact that $E(p,q)$ is negative means that R pays C the amount $\frac{5}{8}$.

Part (b): Assuming that C keeps the strategy $q_f = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (here f is for *fixed*) the expected payoff is

$$E(p,q_f) = pAq_f = \frac{1}{4}p \begin{bmatrix} -1 \\ 9 \\ -4 \end{bmatrix} = \frac{1}{4}(-p_1 + 9p_2 - 5p_3)\,.$$

For R to maximize this expression R should choose the strategy $p = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

Part (c): Now player R keeps his strategy fixed at $p_f = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$ (again "f" is for fixed) so the expected payoff is given by

$$E(p_f,q) = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} Aq = \frac{1}{2}\begin{bmatrix} -12 & 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}\,.$$

Thus we see that C should pick the strategy where $q_1 = 1$ and all other q values are zero.

Problem 3.2 (nonuniqueness of a saddle point)

Consider the trivial payoff matrix $\begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$.

Problem 3.3 (examples of strictly determined games)

We are looking for an element a_{rs} of our payoff matrix such that a_{rs} is the smallest entry in its row *and* a_{rs} is the largest entry in its column.

Part (a): Take $r = 2$ and $s = 2$ thus the optimal strategy is $p^* = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $q^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This game then has the value of 3.

Part (b): Take $r = 2$ and $s = 1$ thus the optimal strategies are $p^* = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $q^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This game has the value 2.

Part (c): Take $r = 3$ and $s = 2$ thus the optimal strategies are $p^* = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ and $q^* = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. This game has the value 2.

Part (d): Take $r = 2$ and $s = 1$ thus the optimal strategies are $p^* = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$ and $q^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. This game has the value -2.

Problem 3.4 (optimal strategies for 2×2 games)

When given the payoff matrix for a 2×2 game we first look to see if the game is strictly determined (i.e. has a saddle point). This means that we look to see if there is an element a_{rs} that is the smallest element in its row and the largest element in its column. If the game is not strictly determined then we apply Theorem 3.2 from the book to compute the optimal strategies p^*, q^*, and the games value. Rather than do these calculations by hand each time these calculations are done in the python code `optimal_mixed_strategy_2by2_game.py`. The pieces of this problem are done in the python code `chapter_3_problems.py`.

Problem 3.5 (a simple card game payoff)

For this problem we would have the payoff matrix given by $\begin{bmatrix} +3 & -4 \\ -6 & +7 \end{bmatrix}$ where the first row corresponds to R playing his black ace and the second row corresponds R playing his red four. The first column corresponds to C playing his black two and the second column corresponds to C playing his red three. We see from the given payoff matrix that there is no pure strategy. The optimal randomized strategy is given by $p^* = \begin{bmatrix} 0.65 & 0.35 \end{bmatrix}$ and $q^* = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$ and the game has a value of $v = -0.15$.

Problem 3.6 (properties of pure strategies)

Let p^* be a row vector of all zeros but with a single 1 in the rth spot and q^* be a column vector of all zeros but with a one in the sth spot. Then note that the product p^*A selects the rth row from A. That is

$$p^*A = \left[\begin{array}{cccccc} a_{r1} & a_{r2} & a_{r3} & \cdots & a_{r,n-1} & a_{r,n} \end{array}\right]$$

In the same way Aq^* selects the sth column from A. That is

$$Aq^* = \left[\begin{array}{c} a_{1s} \\ a_{2s} \\ a_{3s} \\ \vdots \\ a_{m-1,s} \\ a_{m,s} \end{array}\right].$$

From either of these expressions (multiplying as $p^*(Aq^*)$ or $(p^*A)q^*$) we see that $p^*Aq^* = a_{rs}$. Now note that

$$\begin{aligned} E(p^*, q) &= \left[\begin{array}{cccccc} a_{r1} & a_{r2} & a_{r3} & \cdots & a_{r,n-1} & a_{r,n} \end{array}\right] \left[\begin{array}{c} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n-1} \\ q_n \end{array}\right] \\ &= a_{r1}q_1 + a_{r2}q_2 + a_{r3}q_3 + \cdots + a_{r,n-1}q_{n-1} + a_{r,n}q_n\,. \end{aligned}$$

Since $a_{rj} \geq a_{rs}$ we can write the above as

$$E(p^*, q) \geq a_{rs}\left(\sum_{j=1}^{n} q_j\right) = a_{rs}\,,$$

since the sum of the q_j's is one. In the same way we have

$$E(p, q^*) = p(Aq^*) = p\left[\begin{array}{c} a_{1s} \\ a_{2s} \\ a_{3s} \\ \vdots \\ a_{m-1,s} \\ a_{m,s} \end{array}\right] = \sum_{i=1}^{m} p_i a_{is}\,.$$

Since a_{rs} is the largest value in this column $a_{is} \leq a_{rs}$ for all $1 \leq i \leq m$. Thus

$$E(p, q^*) \leq \left(\sum_{i=1}^{m} p_i\right) a_{rs} = a_{rs}\,. \tag{3}$$

Problem 3.7 (the optimal strategies are probabilities)

We will work with the elements of p^*. The elements of q^* are similar. To begin note that the elements of p^* sum to 1. Thus we only need to show that $0 < p_1^* < 1$ and we have shown that both elements in p^* are in the desired range. Recall that p_1^* is given by

$$p_1^* = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{a_{22} - a_{21}}{a_{11} - a_{12} + (a_{22} - a_{21})} = \frac{1}{1 + \left(\frac{a_{11} - a_{12}}{a_{22} - a_{21}}\right)}.$$

In the same way for q_1^* we have

$$q_1^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{21} - a_{12}} = \frac{1}{1 + \left(\frac{a_{11} - a_{21}}{a_{22} - a_{12}}\right)}.$$

For p_1^* if we can show that $\frac{a_{11} - a_{12}}{a_{22} - a_{21}} > 0$ then we would have $0 < p_1^* < 1$. The given fraction will be positive in two cases. For each case we argue that the fact that our matrix A is not strictly determined means that the given inequalities hold. The fraction will be positive when

- Both $a_{11} - a_{12} > 0$ and $a_{22} - a_{21} > 0$. If $a_{22} > a_{21}$ then a_{21} is the smallest element in its row. Since A is not strictly determined means that a_{21} cannot be the largest element in its column. This means that $a_{21} < a_{11}$ or $a_{11} - a_{12} > 0$. Thus $\frac{a_{11} - a_{12}}{a_{22} - a_{21}} > 0$.
- Both $a_{11} - a_{12} < 0$ and $a_{22} - a_{21} < 0$. If $a_{22} < a_{21}$ then a_{22} is the smallest element in its row. Since A is not strictly determined means that a_{22} cannot be the largest element in its column or $a_{22} < a_{12}$ or $a_{22} - a_{21} < 0$. Thus $\frac{a_{11} - a_{12}}{a_{22} - a_{21}} > 0$.

Chapter 4 (Markov Chains)

Problem Solutions

Most of the computation for this chapter is done in the python script `chapter_4_problems.py`.

Problem 4.1

Part (a): We iterate $x^{(n)} = Px^{(n-1)}$ from $x^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Part (b): P is regular since it is a transition matrix (has nonnegative entries and the columns sum to 1). It is regular since it has a power (here the first) that has all positive entries. The steady state is given by the probability vector that is the solution to $(P - I)q = 0$, or

$$\begin{bmatrix} -0.6 & 0.5 \\ 0.6 & -0.5 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0\,.$$

Thus we have $q_1 = \frac{5}{6} q_2$ or $\mathbf{q} = \frac{q_2}{6} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. To make this a probability vector we take $\mathbf{q} = \begin{bmatrix} 5/11 \\ 6/11 \end{bmatrix}$. Note $x^{(n)}$ converges to this vector in Part (a).

Problem 4.2

Part (a): We iterate $x^{(n)} = Px^{(n-1)}$ from $x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Part (b): P is regular since it is a transition matrix (has nonnegative entries and the columns sum to 1). It is regular since it has a power (here the first) that has all positive entries. The steady state is given by the probability vector that is the solution to $(P - I)q = 0$, or we can compute P^n to a large value of n and read off a column in the P^n matrix (they are all the same vector).

Problem 4.3 (computing steady state probability vectors)

We can find the steady state vector $\mathbf{q}$ by either computing P^n for a large power of n or finding a probability vector that satisfies $(P - I)q = 0$.

Part (c): Note that this matrix P is regular since the second power of P has all positive elements. Computing P to a high power we get a steady state vector $\mathbf{q} = \begin{bmatrix} 0.15789474 \\ 0.21052632 \\ 0.63157894 \end{bmatrix}$.

Problem 4.4 (a non-regular P)

Part (a): The matrix P is a transition matrix since it has nonnegative elements and its columns sum to 1. Computing powers of P we see that this matrix limits to the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and the element at position $(1, 2)$ is always zero thus this matrix cannot be regular.

Part (b): From the limiting matrix P^n computed above for any probability vector $x^{(0)}$ we see that $P^n x^{(0)}$ would equal the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Part (c): Not all of the elements of the limiting vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are positive.

Problem 4.5 (the uniform matrix)

A steady-state vector $\mathbf{q}$ is one for which $P\mathbf{q} = \mathbf{q}$. Since we are told P and $\mathbf{q}$ we can just verify that the previous expression is true. We find

$$\begin{bmatrix} \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} \\ \vdots & & & & \vdots \\ \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} \end{bmatrix} \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \\ \frac{1}{k} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{k} \frac{1}{k^2} \\ \sum_{i=1}^{k} \frac{1}{k^2} \\ \vdots \\ \sum_{i=1}^{k} \frac{1}{k^2} \end{bmatrix} = \begin{bmatrix} \frac{k}{k^2} \\ \frac{k}{k^2} \\ \vdots \\ \frac{k}{k^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \\ \vdots \\ \frac{1}{k} \end{bmatrix} = \mathbf{q}\,.$$

Problem 4.6 (almost the uniform matrix)

Note that P is a transition matrix (all nonnegative entries with columns that sum to 1). Also note that P^2 has all positive entries and thus this transition matrix is regular. When we look at powers of P^n we see that we eventually limit to the matrix

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Extracting any column (they are all the same) we get that the limiting steady-state vector $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

Problem 4.7 (a happy of a sad John)

Let the first component of our probability vector $\mathbf{x}$ be the probability that John is happy. Then the second component must be the probability that he is sad. From the given statement of the problem the transition matrix for John's happiness is given by

$$P = \left[\begin{array}{cc} 4/5 & 2/3 \\ 1/5 & 1/3 \end{array} \right] .$$

We are looking for the steady-state solution to this Markov chain which we do by solving the the probability vector $\mathbf{q}$ such that $(P - I)\mathbf{q} = 0$. The matrix $P - I$ is $\left[\begin{array}{cc} -1/5 & 2/3 \\ 1/5 & -2/3 \end{array} \right]$. Thus the components of $\mathbf{q}$ must satisfy $\frac{1}{5}q_1 = \frac{2}{3}q_2$. Thus

$$\mathbf{q} \propto \left[\begin{array}{c} 10 \\ 3 \end{array} \right] \quad \text{and normalizing gives} \quad \mathbf{q} = \frac{1}{13} \left[\begin{array}{c} 10 \\ 3 \end{array} \right] .$$

Thus $10/13 = 0.7692308$ of the time John is happy while $3/13 = 0.2307692$ of the time John is sad.

Problem 4.8 (demographic regions)

If we let the first, second, and third components of our state vector $\mathbf{x}$ represent the proportion of the people that live in regions I, II, and III respectively. Then from the problem description the transition matrix P is given by

$$P = \left[\begin{array}{ccc} 0.9 & 0.15 & 0.1 \\ 0.05 & 0.75 & 0.05 \\ 0.05 & 0.1 & 0.85 \end{array} \right] .$$

Finding the steady-state vector $\mathbf{q}$ by computing P^n for large values of n (and extracting any column) we find $\mathbf{q} = \left[\begin{array}{c} 0.54166488 \\ 0.16666668 \\ 0.29166844 \end{array} \right]$. We can decide if n is large enough and we have a good enough approximation to the steady-state vector $\mathbf{q}$ if the columns of P^n are sufficiently similar in numerical value.

Chapter 5 (Leontief Economic Models)

Problem Solutions

The computations for this chapter are done in the python code `chapter_5_problems.py`.

Problem 5.1

The equilibrium condition 5.3 is to find a nonnegative price vector **p** such that $(I - E)p = 0$.

Part (a): We have $I - E = \begin{bmatrix} 1/2 & -1/3 \\ -1/2 & 1/3 \end{bmatrix}$ The vector $\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ must then satisfy $\frac{1}{2}p_1 = \frac{1}{3}p_2$ and we get

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}p_2 \\ p_2 \end{bmatrix} = \frac{p_2}{3} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

If we take any $p_2 > 0$ we have a nonnegative multiple of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Part (b): We have

$$I - E = \begin{bmatrix} 1/2 & 0 & -1/2 \\ -1/3 & 1 & -1/2 \\ -1/6 & -1 & 1 \end{bmatrix}.$$

Thus we seek a nonnegative price vector $\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ that satisfies

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ -1/3 & 1 & -1/2 \\ -1/6 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0.$$

Performing the steps of Gaussian elimination on the leading coefficient matrix we can transform the coefficient matrix into the matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -5/6 \\ 0 & 1 & -5/6 \end{bmatrix}.$$

Which indicate that $p_1 = p_3$ and $p_2 = \frac{5}{6}p_3$. Thus our vector **p** is given by

$$\mathbf{p} = \begin{bmatrix} p_3 \\ \frac{5}{6}p_3 \\ p_3 \end{bmatrix} = \frac{p_3}{6} \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}.$$

For any value of $p_3 > 0$ we will have a nonnegative price vector **p** proportional to $\begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix}$.

Problem 5.2 (are these consumption matrices productive?)

Theorem 5.3 is the statement that a consumption matrix C is productive if there exists a production vector $\mathbf{x} \geq 0$ such that $C\mathbf{x} < \mathbf{x}$. Some corollaries of this theorem are that the consumption matrix C will be productive if all of its rows (or columns) sum to less than one.

Part (a): Since each of the rows of C sum to less than one, this consumption matrix is productive.

Part (b): Since each of the columns of C sum to less than one, this consumption matrix is productive.

Part (c): This consumption matrix does not have all of its rows sum to less than one or all of its columns sum to less than one. Thus we cannot use these two methods to determine if C is productive instead we will look for a vector $\mathbf{x}$ such that $C\mathbf{x} < \mathbf{x}$. We do this by noting that if a given consumption matrix C has an eigenvalue that is less than one and the eigenvector associated with this eigenvalue has all positive elements then this eigenvector is an example of a vector $\mathbf{x}$ that satisfies $C\mathbf{x} < \mathbf{x}$ and C is productive. For this problem, the eigenvector with an eigenvalue less than one is $\begin{bmatrix} 0.8406 & 0.379332 & 0.38660 \end{bmatrix}^T$.

Problem 5.3 (a solution to $(I - E)p = 0$)

Theorem 5.2 states that there is exactly one solution to $Ep = p$ if all the entries of the matrix E^m are positive for some value of m. For the exchange matrix given we find that $E^2 > 0$ and thus there is exactly one solution to $Ep = p$.

Problem 5.4 (neighbors growing vegetables)

The exchange matrix (where rows represent neighbors A, B, and C and columns represent tomatoes, corn, and lettuce) for this problem looks like

$$E = \begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/3 & 1/4 \\ 1/6 & 1/3 & 1/2 \end{bmatrix}.$$

Note that the columns of this matrix sum to one as they should. We seek a pricing vector $\mathbf{p}$ such that $E\mathbf{p} = \mathbf{p}$ and thus perform Gaussian elimination on the matrix

$$I - E = \begin{bmatrix} 1/2 & -1/3 & -1/4 \\ -1/3 & 2/3 & -1/4 \\ -1/6 & -1/3 & 1/2 \end{bmatrix}.$$

This reduces to the matrix

$$\begin{bmatrix} 1 & 0 & -9/8 \\ 0 & 1 & -15/16 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we see that the components of $\mathbf{p}$ satisfy $p_1 = \frac{9}{8}p_3$ and $p_2 = \frac{15}{16}p_3$. Thus $\mathbf{p}$ looks like

$$\mathbf{p} = \begin{bmatrix} \frac{9}{8}p_3 \\ \frac{15}{16}p_3 \\ p_3 \end{bmatrix} = \frac{p_3}{16} \begin{bmatrix} 18 \\ 15 \\ 16 \end{bmatrix}.$$

Any positive value for p_3 will make the above vector positive. To make sure that the lowest priced crop has a price of 100 we will set $\frac{15}{16}p_3 = 100$ or $p_3 = 106.666666667$. Thus the vector

$$\mathbf{p} = \begin{bmatrix} 120 \\ 100 \\ 106.67 \end{bmatrix}.$$

Problem 5.5 (three engineers)

Let the ordering of the variable be CE, EE, and ME. Then the consumption matrix for this problem is

$$C = \begin{bmatrix} 0 & 0.2 & 0.3 \\ 0.1 & 0 & 0.4 \\ 0.3 & 0.4 & 0 \end{bmatrix}$$

Following the example in the book of the town with three industries we have that we need to solve $(I - C)\mathbf{x} = d$ for a demand vector of $d = \begin{bmatrix} 500 \\ 700 \\ 600 \end{bmatrix}$. When I solve the given system for $\mathbf{x}$ I find

$$\mathbf{x} = \begin{bmatrix} 1256.48 \\ 1448.13 \\ 1556.20 \end{bmatrix}.$$

Warning: Note these numbers are different from what the back of the book has. If anyone sees anything wrong with this (or agrees with it) please contact me.

Problem 5.6 (column sums of an exchange matrix)

Since the column sums of an exchange matrix E are one, when form the matrix $I - E$ the sums of each column now must be 1 minus the column sum of E which is one. Thus the column sums of $I - E$ must be zero. This means that the rows of the matrix $I - E$ are linearly dependent (a nontrivial linear combination of them combines to zero). Thus the matrix $I - E$ must be singular and must have a zero determinant. Thus $(I - E)p = 0$ must have a nontrivial solution vector p.

Problem 5.7 (C is productive if its columns sum to less than one)

Assume C is a consumption matrix where all of its columns sum to less than one and that C is *not* productive. That is that $(I - C)^{-1}$ does not exist or that $(I - C)^{-1} < 0$. We will show that this leads to a contradiction. Since the columns of C sum to less than one we know that all the rows of C^T sum to less than one. Then from Corollary 5.1 we have that C^T is productive so that $(I - C^T)^{-1}$ exists and that $(I - C^T)^{-1} \geq 0$. This however means that

$$(I - C^T)^{-1} = ((I - C)^T)^{-1} = ((I - C)^{-1})^T,$$

exists and that $((I - C)^{-1})^T \geq 0$. The transpose on these expressions don't change the existence and the positive nature thus we have that $(I - C)^{-1}$ exists and that $(I - C)^{-1} \geq 0$ so that C is in fact productive.

Problem 5.8 (C is productive if $x > Cx$)

Part I: Since we are to assume that C is productive we know that $(I - C)^{-1}$ exists and $(I - C)^{-1} \geq 0$. Consider the multiplication of the matrix $(I - C)^{-1}$ and a positive vector $\mathbf{x} > 0$. Let the result of this multiplication be called $\mathbf{v}$. That is

$$(I - C)^{-1}\mathbf{x} = \mathbf{v}.$$

Since $(I - C)^{-1} \geq 0$ and $\mathbf{x} > 0$ we know that $\mathbf{v} \geq 0$. Solving for $\mathbf{x}$ we have

$$\mathbf{x} = (I - C)\mathbf{v} > 0,$$

since $\mathbf{x}$ was taken to be $\mathbf{x} > 0$. This last expression means that a $\mathbf{v} \geq 0$ exists such that $\mathbf{v} - C\mathbf{v} > 0$ or $C\mathbf{v} < \mathbf{v}$ as we were to show.

Part II: For this part of the problem we want to show that if there exists a vector $\mathbf{x} \geq 0$ such that $\mathbf{x} > C\mathbf{x}$ then C is productive.

We start with a vector $\mathbf{x}^* \geq 0$ such that $C\mathbf{x}^* < \mathbf{x}^*$. Then since C has only nonnegative entries we know that $C\mathbf{x}^* \geq 0$ so $\mathbf{x}^* \geq 0$.

Since we start with $C\mathbf{x}^* < \mathbf{x}^*$, by continuity there must be a value of $\lambda < 1$ for which $C\mathbf{x}^* < \lambda\mathbf{x}^*$. In other words, if we computed the product $C\mathbf{x}^*$ we know we get some vector that is less than $\mathbf{x}^*$. Whatever this product vector equals to we could "shrink" $\mathbf{x}^*$ by a bit (by multiplying by λ) and still keep it greater than $C\mathbf{x}^*$.

From $C\mathbf{x}^* < \lambda\mathbf{x}^*$ by repeatedly multiplying by C and using the same inequality we get $C^n\mathbf{x}^* < \lambda^n\mathbf{x}^*$.

Now since $\lambda^n\mathbf{x}^* \to 0$ as $n \to \infty$ and since $C^n\mathbf{x}^*$ is less than this we have that $C^n\mathbf{x}^* \to 0$. This could happen if C^n limits to a matrix and $\mathbf{x}^*$ is in its null space or $C^n \to 0$. The first condition cannot happen since C has all nonnegative entries so C^n has nonnegative entries and $\mathbf{x}^* > 0$. Thus it must be that $C^n \to 0$ as $n \to \infty$.

Direct multiplication will show the given algebraic identity.

Letting $n \to \infty$ we get

$$(I - C)(I + C + C^2 + C^3 + \cdots) = I,$$

Thus defining

$$S \equiv I + C + C^2 + C^3 + \cdots ,$$

we see that this is the inverse of $I - C$ and since $C > 0$ the sum of all the terms in S are positive so $S \geq 0$. Since this is the definition of a productive consumption matrix we have proven the desired statement.

Chapter 6 (Forest Management)

Problem Solutions

Problem 6.1

For this problem we have $p_2 = 30$, $p_3 = 60$, $g_1 = \frac{1}{2}$, and $g_2 = \frac{2}{3}$. We then look at the optimal yeilds where we get

$$\begin{aligned}
\mathrm{Yld}_2 &= \frac{p_2 s}{\frac{1}{g_1}} = \frac{30s}{2} = 15s \\
\mathrm{Yld}_3 &= \frac{p_3 s}{\frac{1}{g_1} + \frac{1}{g_2}} = \frac{60s}{2 + 1.5} = 17.14s\,.
\end{aligned}$$

Since $\mathrm{Yld}_3 > \mathrm{Yld}_2$ we would harvest the third type of trees. When $s = 1000$ this optimal yield is given by $\mathrm{Yld}_3 = 17140.0$. This is a different result than in the book. If anyone sees anything wrong with this (or agrees with it) please contact me.

Problem 6.2

For this problem we have

$$\mathrm{Yld}_5 = \frac{p_5 s}{0.28^{-1} + 0.31^{-1} + 0.25^{-1} + 0.23^{-1}} = \frac{p_5 s}{15.15} > 14.7s\,.$$

This means that $p_5 > 222.705 = 223.0$

Problem 6.3

When the prices as are yet unspecified we would have

$$\begin{aligned}
\mathrm{Yld}_2 &= \frac{p_2 s}{0.28^{-1}} = \frac{p_2 s}{3.57} \\
\mathrm{Yld}_3 &= \frac{p_3 s}{0.28^{-1} + 0.31^{-1}} = \frac{p_2 s}{6.79} \\
\mathrm{Yld}_4 &= \frac{p_4 s}{10.79} \\
\mathrm{Yld}_5 &= \frac{p_5 s}{15.14} \\
\mathrm{Yld}_6 &= \frac{p_6 s}{17.84}\,.
\end{aligned}$$

To have all of the yields be the same the prices must cancil the denominator (to give just the s) thus we have

$$3.57 : 6.79 : 10.79 : 15.14 : 17.84\,.$$

Dividing by 3.57 to normalize the first price to 1 we get

$$1 : 1.90 : 3.02 : 4.24 : 4.99\,.$$

Problem 6.4

Once we decide on a level k to harvest we have x_1 give by Eq. 6.17 or

$$x_1 = \frac{s}{g_1\left(\frac{1}{g_1} + \frac{1}{g_2} + \cdots + \frac{1}{g_{k-2}} + \frac{1}{g_{k-1}}\right)}\,. \tag{4}$$

Then $x_2, x_3, \cdots x_{k-2}, x_{k-1}$ are given by Eq. 6.16 or

$$\begin{aligned}
x_2 &= \frac{g_1}{g_2}x_1 = \frac{s}{g_2\left(\frac{1}{g_1} + \frac{1}{g_2} + \cdots + \frac{1}{g_{k-2}} + \frac{1}{g_{k-1}}\right)} \\
x_3 &= \frac{g_1}{g_3}x_1 = \frac{s}{g_3\left(\frac{1}{g_1} + \frac{1}{g_2} + \cdots + \frac{1}{g_{k-2}} + \frac{1}{g_{k-1}}\right)} \\
&\vdots \\
x_{k-1} &= \frac{g_1}{g_{k-1}}x_1 = \frac{s}{g_{k-1}\left(\frac{1}{g_1} + \frac{1}{g_2} + \cdots + \frac{1}{g_{k-2}} + \frac{1}{g_{k-1}}\right)}\,.
\end{aligned}$$

These with $x_k = x_{k+1} = \cdots = x_n = 0$ by the books Eq. 6.13. When we put all of these components into a vector we get the requested result.

Problem 6.5

In optimal sustainable harvesting we remove only y_k trees all other y_j are zero. Then we have our y_k given by

$$y_k = g_1 x_1 = \frac{g_1 s}{1 + \frac{g_1}{g_2} + \frac{g_1}{g_3} + \cdots + \frac{g_1}{g_{k-2}} + \frac{g_1}{g_{k-1}}}\,. = \frac{s}{\frac{1}{g_1} + \frac{1}{g_2} + \frac{1}{g_3} + \cdots + \frac{1}{g_{k-2}} + \frac{1}{g_{k-1}}}\,.$$

Problem 6.6 (all equal growth)

We would need to evaluate the yields at each value of k. Note that

$$\begin{aligned}
\text{Yld}_2 &= \frac{p_2 s}{\frac{1}{g}} \\
\text{Yld}_3 &= \frac{p_3 s}{\frac{2}{g}} \\
\text{Yld}_4 &= \frac{p_4 s}{\frac{3}{g}} \\
&\vdots \\
\text{Yld}_k &= \frac{p_k s}{\frac{k-1}{g}}\,.
\end{aligned}$$

In order that any sustainable harvesting policy be an option requires $\text{Yld}_k = \text{Yld}_j$ or

$$\frac{p_k s}{\frac{k-1}{g}} = \frac{p_j s}{\frac{j-1}{g}} \quad \text{or} \quad \frac{p_k}{k-1} = \frac{p_j}{j-1}\,.$$

Thus iterating a few of these we would have $\frac{p_3}{2} = \frac{p_2}{1}$ or $p_3 = 2p_2$. Another one that we have is that $\frac{p_4}{3} = \frac{p_3}{2}$ or $p_4 = 4p_2$. In general, the result is $p_k = kp_2$. Thus the ratios are $1 : 2 : 3 : \cdots : k$.

Chapter 7 (Temperature Distributions)

Problem Solutions

Problem 7.1

Part (a): The discrete mean value property requires the equations

$$\begin{aligned}
t_1 &= \frac{1}{4}(0 + 0 + t_3 + t_2) \\
t_2 &= \frac{1}{4}(1 + t_1 + t_4 + 1) \\
t_3 &= \frac{1}{4}(t_1 + 0 + 0 + t_4) \\
t_4 &= \frac{1}{4}(t_2 + t_3 + 1 + 1)\,.
\end{aligned}$$

Thus if we let the vector $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix}$ then the vector $\mathbf{t}$ satisfies the matrix equation $\mathbf{t} = M\mathbf{t}+\mathbf{b}$ which is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 0 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Part (b): We could solve $(I - M)\mathbf{t} = \mathbf{b}$ to compute $\mathbf{t}$.

Part (c): We would iterate $\mathbf{t}^{(n)} = M\mathbf{t}^{(n-1)} + \mathbf{b}$ starting from $\mathbf{t}^{(0)} = 0$.

Problem 7.2

The value of t in the center is the average temperature value on the circle or

$$\frac{1}{2\pi}(1\pi + 0\pi) = \frac{1}{2}\,.$$

Chapter 8 (Genetic Applications)

Problem Solutions

The computations for this chapter are done in the python code `chapter_8_problems.py`.

Problem 8.2 (repeated fertilization with Aa)

In this case for the fraction of the population that has genotypes AA, Aa, and aa in the next generation, given the fractions, in the current generation is given by

$$\begin{aligned}
a_n &= \frac{1}{2}a_{n-1} + \frac{1}{4}b_{n-1} \\
b_n &= \frac{1}{2}a_{n-1} + \frac{1}{2}b_{n-1} + \frac{1}{2}c_{n-1} \\
c_n &= \frac{1}{4}b_{n-1} + \frac{1}{2}c_{n-1} \,.
\end{aligned}$$

This can be written as the matrix system

$$\left[\begin{array}{c} a_n \\ b_n \\ c_n \end{array}\right] = \left[\begin{array}{ccc} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{array}\right] \left[\begin{array}{c} a_{n-1} \\ b_{n-1} \\ c_{n-1} \end{array}\right] .$$

Defining $\mathbf{x}^{(n)} \equiv \left[\begin{array}{c} a_n \\ b_n \\ c_n \end{array}\right]$ we can write this as $\mathbf{x}^{(n)} = M_{Aa}\mathbf{x}^{(n-1)}$, where we have defined the matrix M_{Aa} as

$$M_{Aa} \equiv \left[\begin{array}{ccc} 1/2 & 1/4 & 0 \\ 1/2 & 1/2 & 1/2 \\ 0 & 1/4 & 1/2 \end{array}\right] . \tag{5}$$

We find the eigenvalues of M_{Aa} to be given by solving

$$\begin{aligned}
|M - \lambda I| &= \left|\begin{array}{ccc} \frac{1}{2} - \lambda & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} - \lambda & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} - \lambda \end{array}\right| = \left(\frac{1}{2} - \lambda\right)\left[\left(\frac{1}{2} - \lambda\right)^2 - \frac{1}{8}\right] - \frac{1}{2}\left[\frac{1}{4}\left(\frac{1}{2} - \lambda\right)\right] \\
&= \left(\frac{1}{2} - \lambda\right)\lambda(\lambda - 1) = 0 \,.
\end{aligned}$$

From which we see that our eigenvalues are $\{1, \frac{1}{2}, 0\}$. The corresponding eigenvector for $\lambda_1 = 1$ is given by finding a vector in the null space of the matrix

$$M_{Aa} - I = \left[\begin{array}{ccc} -\frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{2} \end{array}\right] .$$

Performing Gaussian elimination on this coefficient matrix we get that it transforms into

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

From which we see that the components of this eigenvector are $v_1 = v_3$, $v_2 = 2v_3$, and v_3 is arbitrary. Thus the eigenvector for $\lambda_1 = 1$ is given by $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. In the same way as with $\lambda = 1$ we find that the eigenvector for $\lambda_2 = \frac{1}{2}$ is given by $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and the eigenvector for $\lambda_3 = 0$ is given by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Using these we have the decomposition of M_{Aa} of

$$M_{Aa} = PDP^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

We have then that

$$M_{Aa}^n = PD^nP^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

When we multiply these matrices we get

$$M_{Aa}^n = \begin{bmatrix} \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} & \frac{1}{4} & \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} \end{bmatrix}.$$

Using this expression we can get the value of $\mathbf{x}^{(n)}$ from $M_{Aa}^n \mathbf{x}^{(0)}$. We find

$$\mathbf{x}^{(n)} = M_{Aa}^n \mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1} (a_0 - c_0) \\ \frac{1}{2} \\ \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1} (a_0 - c_0) \end{bmatrix}.$$

As $n \to \infty$ from the above we see that the limiting form of the matrix M_{Aa} is given by

$$M_{Aa}^n \to \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

with a limiting form for $\mathbf{x}^{(n)}$ of $\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}$.

Problem 8.3 (alternative fertilizations as AA, Aa, AA, Aa etc.)

As shown in the book every fertilization with genotype AA involves a transition of the fraction vector $\mathbf{x}^{(n)} = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$ of the form $\mathbf{x}^{(n)} = M_{AA}\mathbf{x}^{(n-1)}$ where the matrix M_{AA} is given by

$$M_{AA} = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

Every fertilization with genotype Aa involves a transformation of the form $\mathbf{x}^{(n)} = M_{Aa}\mathbf{x}^{(n-1)}$ with M_{Aa} given by Equation 5. These alternative fertilization's give rise to the transitions

$$\begin{aligned}
\mathbf{x}^{(1)} &= M_{AA}\mathbf{x}^{(0)} \\
\mathbf{x}^{(2)} &= M_{Aa}\mathbf{x}^{(1)} = M_{Aa}M_{AA}\mathbf{x}^{(0)} \\
\mathbf{x}^{(3)} &= M_{AA}\mathbf{x}^{(2)} = M_{AA}M_{Aa}M_{AA}\mathbf{x}^{(0)} \\
\mathbf{x}^{(4)} &= M_{Aa}\mathbf{x}^{(3)} = M_{Aa}M_{AA}M_{Aa}M_{AA}\mathbf{x}^{(0)} \\
&\vdots \\
\mathbf{x}^{(2n)} &= (M_{Aa}M_{AA})^n\mathbf{x}^{(0)} \\
\mathbf{x}^{(2n+1)} &= M_{AA}(M_{Aa}M_{AA})^n\mathbf{x}^{(0)}\,.
\end{aligned}$$

We find the product $M_{Aa}M_{AA}$ when we multiply given by

$$\begin{bmatrix} 0.5 & 0.375 & 0.25 \\ 0.5 & 0.5 & 0.5 \\ 0 & 0.125 & 0.25 \end{bmatrix}.$$

To compute powers of this matrix we would need to perform an eigenvalue-eigenvector decomposition in the same way as was done for Problem 2 on Page 25.

Problem 8.4 (autosomal recessive diseases)

For the M given in the section on autosomal recessive diseases the eigenvalues of our matrix M are given by solving

$$|M - \lambda I| = (1-\lambda)\left(\frac{1}{2} - \lambda\right) = 0\,,$$

Thus $\lambda = 1$ and $\lambda = \frac{1}{2}$. For $\lambda = 1$ the eigenvector is given by the null space of the matrix

$$M - I = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and for $\lambda = \frac{1}{2}$ the eigenvector is given by the null space of the matrix

$$M - \frac{1}{2}I = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Using these eigenvectors and eigenvalues we can write M as

$$M = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

With this we easily have powers of M or

$$M^n = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

So that $\mathbf{x}^{(n)}$ is given by

$$\begin{aligned} \mathbf{x}^{(n)} = M^{(n)}\mathbf{x}^{(0)} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -\left(\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1-\left(\frac{1}{2}\right)^n \\ 0 & \left(\frac{1}{2}\right)^n \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_0 + \left(1-\left(\frac{1}{2}\right)^n\right) b_0 \\ \left(\frac{1}{2}\right)^n b_0 \end{bmatrix} = \begin{bmatrix} 1-\left(\frac{1}{2}\right)^n b_0 \\ \left(\frac{1}{2}\right)^n b_0 \end{bmatrix}, \end{aligned}$$

for $n \geq 1$. In the previous manipulations we have used the fact that the initial proportion of genotypes was normalized i.e. $a_0 + b_0 = 1$.

Problem 8.5 (the time until the percentage of carriers drops)

For the proportions of various genotypes under random mating Eq 8.9 is

$$b_n = \frac{b_{n-1}}{1 + \frac{1}{2}b_{n-1}} \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{6}$$

In the controlled mating policy than the proportion of carries is given by Eq. 8.8 which is

$$b_n = \frac{1}{2}b_{n-1} \quad \text{for} \quad n = 1, 2, 3, \ldots$$

If we start with an genotype distribution where the initial percentage of carriers is $b_0 = 0.25$ then iterating the above two expressions (for $n \geq 1$) we get the values for n and b_n under both models

n	random mating	controlled mating
0	0.250000	0.250000
1	0.222222	0.125000
2	0.200000	0.062500
3	0.181818	0.031250
4	0.166667	0.015625
5	0.153846	0.007812
6	0.142857	0.003906
7	0.133333	0.001953
8	0.125000	0.000977
9	0.117647	0.000488
10	0.111111	0.000244
11	0.105263	0.000122
12	0.100000	0.000061
13	0.095238	0.000031

Thus we see that under random mating we need $n \geq 13$ before $b_n < 0.1$ or the number of carriers is less than 10%. With controlled mating we only need $n \geq 2$ for $b_n < 0.1$. Thus

Problem 8.6 (the limiting probability distributions of X linked trains)

The fact that the initial parents are equally likely to be any of the six possible genotype parents means that $x^{(0)} = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ we would calculate $x^{(n)}$ using the books Eq. 8.12 using matrix vector multiplication starting with this value of $\mathbf{x}^{(0)}$.

Problem 8.7 (X-linked inheritance with inbreeding)

The books Eq. 8.13 shows the limiting $n \to \infty$ distribution of genotypes under X-linked inheritance with inbreeding. From the expression we see that the probability that the limiting sibling-pairs will be of the genotype (A, AA) is given by

$$a_0 + \frac{2}{3}b_0 + \frac{1}{3}c_0 + \frac{2}{3}d_0 + \frac{1}{3}e_0 \,.$$

The proportion of A genes in the initial population where the $\mathbf{x}$ vector is ordered as

$$(A, AA)\,, (A, Aa)\,, (A, aa)\,, (a, AA)\,, (a, Aa)\,, (a, aa)\,,$$

is given by $1a_0$, $\frac{2}{3}b_0$, $\frac{1}{3}c_0$, $\frac{2}{3}d_0$, and $\frac{1}{3}e_0$ respectively. Adding these up we get the same expression as the limiting probability above.

Problem 8.8 (X-linked inheritance with no surviving females)

We can find the given transition matrix using the transition matrix M where all genotypes survive by simply removing all rows and columns corresponding to the genotypes (A, Aa) and (a, Aa). This gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 8.9 (inbreeding in X-linked inheritance)

For this problem to derive the given matrix M recall with X-linked inheritance *males* posses only one gene say A or a, while females posses *two* genes say AA, Aa, or aa. Any male offspring receive one gene from the mother while any females offspring receive one gene from the father (his only gene) and one random gene from their mother.

Given this inheritance pattern to derive our matrix M, recall that multiplying M by a column vector of all zeros but with a single one located at the jth row selects the jth column from M. A column vector only a single one in the jth row corresponds to a sibling-pair populated with only one type of genotype. For example, the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ corresponds to the sibling-pair with genotype (A, AA).

Column 1: Consider the $(n-1)$st generation where the sibling-pair is of type (A, AA). All male offspring will receive an A gene and all female offspring will receive an A gene from their father and an A gene from their mother. Thus all sibling-pair offspring must be of genotype (A, AA).

Column 3: Consider the $(n-1)$st generation where the sibling-pair is of type (A, aa). All male offspring will receive an a gene from their mother and all the female offspring receive an A from their father and an a from their mother to get the genotype (a, Aa).

Column 4: The $(n-1)$st generation has the sibling-pair of (a, AA). Male offspring gets an A gene while female offspring ends with the genotype Aa so all sibling-pair offspring have genotype (A, Aa).

Column 5: The $(n-1)$st generation has a sibling-pair of (a, Aa). All male offspring can get either an A or an a from their mother. All female offspring gets A or an a with equal probability from their mother and an a gene from their father. Thus we could end with sibling-pairs of the form (A, Aa), (A, aa), (a, Aa), or (a, aa) each with equal or $1/4$ probability.

Column 6: The $(n-1)$st generation has a sibling-pair of (a, aa). The male gets the a gene from his mother and the female gets a from father and a from mother. Thus we can only get sibling-pair genotypes like (a, aa).

Chapter 9 (Age-Specific Population Growth)

Notes on the Text

Some python code written for this chapter include

- `Leslie_matrix.py` computes a Leslie matrix given data for a_i and b_i.

Notes on verifying the equality $Lx_1 = \lambda_1 x_1$

If x_1 is given by

$$x_1 = \begin{bmatrix} 1 & \frac{b_1}{\lambda_1} & \frac{b_1 b_2}{\lambda_1^2} & \cdots & \frac{b_1 b_2 \cdots b_{n-3}}{\lambda_1^{n-3}} & \frac{b_1 b_2 \cdots b_{n-2}}{\lambda_1^{n-2}} & \frac{b_1 b_2 \cdots b_{n-1}}{\lambda_1^{n-1}} \end{bmatrix}^T , \tag{7}$$

where the jth component of the vector x_1 is $\frac{b_1 b_2 \cdots b_{j-2} b_{j-1}}{\lambda_1^{j-1}}$. Then Lx_1 formally is the product

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & b_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{b_1}{\lambda_1} \\ \frac{b_1 b_2}{\lambda_1^2} \\ \vdots \\ \frac{b_1 b_2 \cdots b_{n-3}}{\lambda_1^{n-3}} \\ \frac{b_1 b_2 \cdots b_{n-2}}{\lambda_1^{n-2}} \\ \frac{b_1 b_2 \cdots b_{n-1}}{\lambda_1^{n-1}} \end{bmatrix} ,$$

which equals

$$\begin{bmatrix} a_1 + \frac{a_2 b_1}{\lambda_1} + \frac{a_3 b_1 b_2}{\lambda_1^2} + \cdots + \frac{a_{n-2} b_1 b_2 \cdots b_{n-4} b_{n-3}}{\lambda_1^{n-3}} + \frac{a_{n-1} b_1 b_2 \cdots b_{n-3} b_{n-2}}{\lambda_1^{n-2}} + \frac{a_n b_1 b_2 \cdots b_{n-2} b_{n-1}}{\lambda_1^{n-1}} \\ b_1 \\ \frac{b_1 b_2}{\lambda_1} \\ \vdots \\ \frac{b_1 b_2 \cdots b_{n-4} b_{n-3}}{\lambda_1^{n-4}} \\ \frac{b_1 b_2 \cdots b_{n-3} b_{n-2}}{\lambda_1^{n-3}} \\ \frac{b_1 b_2 \cdots b_{n-2} b_{n-1}}{\lambda_1^{n-2}} \end{bmatrix} .$$

The first row is $\lambda_1 q(\lambda_1)$. Since λ_1 is an eigenvalue of L is must satisfy $q(\lambda_1) = 1$, and thus $\lambda_1 q(\lambda_1) = \lambda_1$. This gives us that the product Lx_1 becomes

$$\lambda_1 \left[\begin{array}{c} 1 \\ \frac{b_1}{\lambda_1} \\ \frac{b_1 b_2}{\lambda_1^2} \\ \vdots \\ \frac{b_1 b_2 \cdots b_{n-4} b_{n-3}}{\lambda_1^{n-3}} \\ \frac{b_1 b_2 \cdots b_{n-3} b_{n-2}}{\lambda_1^{n-2}} \\ \frac{b_1 b_2 \cdots b_{n-2} b_{n-1}}{\lambda_1^{n-1}} \end{array} \right] = \lambda_1 x_1 \,,$$

as we were to show.

Problem Solutions

Problem 9.1 (a Leslie matrix)

Part (a): Using $|L - \lambda I| = 0$ we compute $\lambda \in \{-\frac{1}{2}, +\frac{3}{2}\}$. for the positive eigenvalue $\lambda = \frac{3}{2}$ we find the eigenvector given by $\mathbf{x}_1 = \left[\begin{array}{c} 3 \\ 1 \end{array} \right]$.

Part (b): In the python code `chapter_9_problems.py` we iterate $x^{(n)} = L^n x^{(0)}$, using the known L. We find when we print the iteration index, the vector $x^{(n)}$, and the sum of the elements (total female population size) that we get

```
0 [[100    0]] [[100]]
1 [[ 100.    50.]] [[ 150.]]
2 [[ 175.    50.]] [[ 225.]]
3 [[ 250.    88.]] [[ 338.]]
4 [[ 381.   125.]] [[ 506.]]
5 [[ 569.   191.]] [[ 760.]]
```

Part (c): We find $x^{(6)} = Lx^{(5)} = \left[\begin{array}{cc} 855 & 284 \end{array} \right]^T$ and $x^{(6)} \approx \lambda_1 x^{(5)} = \frac{3}{2} x^{(5)} = \left[\begin{array}{cc} 853 & 286 \end{array} \right]^T$.

Problem 9.2 (the characteristic polynomial for a Leslie matrix)

To compute the characteristic polynomial of a Leslie matrix we need to evaluate

$$
p(\lambda) = |\lambda I - L| = \left|\begin{array}{ccccccc}
\lambda - a_1 & -a_2 & -a_3 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\
-b_1 & \lambda & 0 & \cdots & 0 & 0 & 0 \\
0 & -b_2 & \lambda & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \lambda & 0 & 0 \\
0 & 0 & 0 & \cdots & -b_{n-2} & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & -b_{n-1} & \lambda
\end{array}\right| .
$$

If we expand this determinant about the first row we get

$$
p(\lambda) = (\lambda - a_1) \left|\begin{array}{ccccccc}
\lambda & 0 & 0 & \cdots & 0 & 0 & 0 \\
-b_2 & \lambda & 0 & \cdots & 0 & 0 & 0 \\
0 & -b_3 & \lambda & \ddots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ddots & \lambda & 0 & 0 \\
0 & 0 & 0 & \cdots & -b_{n-2} & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & -b_{n-1} & \lambda
\end{array}\right| + b_1 \left|\begin{array}{ccccccc}
-a_2 & -a_3 & -a_4 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\
-b_2 & \lambda & 0 & \cdots & 0 & 0 & 0 \\
0 & -b_3 & \lambda & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & \lambda & 0 & 0 \\
0 & 0 & 0 & \cdots & -b_{n-2} & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & -b_{n-1} & \lambda
\end{array}\right| .
$$

This last determinant of size $(n-1) \times (n-1)$ has its first row of $n-1$ elements given by

$$
-a_2 \,, -a_3 \,, -a_4 \,, \cdots \,, -a_{n-1} \,, -a_n \,,
$$

its $n-1$ diagonal elements given by

$$
-a_2 \,, \lambda \,, \lambda \,, \cdots \,, \lambda \,, \lambda
$$

and $n-2$ subdiagonal diagonal elements given by

$$
-b_2 \,, -b_3 \,, -b_4 \,, \cdots \,, -b_{n-2} \,, -b_{n-1} \,,
$$

Lets denote the value of this determinant by $D_{2,n}$ to denote that the indices on $-a_n$ and $-b_n$ start with $n = 2$ and go down to n. A smaller matrix of this same form will appear in in next computation. If we take the previous expression for $p(\lambda)$ and in the second determinant expand about the first column we will get for $p(\lambda)$

$$
p(\lambda) = (\lambda - a_1)\lambda^{n-1} + b_1 \left[-a_2\lambda^{n-2} + b_2 \left|\begin{array}{ccccccc}
-a_3 & -a_4 & -a_5 & \cdots & -a_{n-2} & -a_{n-1} & -a_n \\
-b_3 & \lambda & 0 & \cdots & 0 & 0 & 0 \\
0 & -b_4 & \lambda & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 & 0 & 0 & \cdots & \lambda & 0 & 0 \\
0 & 0 & 0 & \cdots & -b_{n-2} & \lambda & 0 \\
0 & 0 & 0 & \cdots & 0 & -b_{n-1} & \lambda
\end{array}\right| \right] .
$$

This last determinant is of the same form as we have seen before and denoted $D_{3,n}$. Thus it looks like these determinants $D_{k,n}$ satisfy

$$
D_{k,n} = -a_k\lambda^{n-k} + b_k D_{k+1,n} \,.
$$

When $k = n - 1$ one of our last determinants we get

$$D_{n-1,n} = \begin{vmatrix} -a_{n-1} & -a_n \\ -b_{n-1} & \lambda \end{vmatrix} = -a_{n-1}\lambda + b_{n-1}a_n \,.$$

Expanding what we have shown thus far we get

$$\begin{aligned} p(\lambda) &= (\lambda - a_1)\lambda^{n-1} + b_1 D_{2,n} \\ &= \lambda^n - a_1\lambda^{n-1} - b_1 a_2 \lambda^{n-2} + b_1 b_2 D_{3,n} \\ &= \lambda^n - a_1\lambda^{n-1} - b_1 a_2 \lambda^{n-2} - b_1 b_2 a_3 \lambda^{n-3} + b_1 b_2 b_3 D_{4,n} \\ &\vdots \\ &= \lambda^n - a_1\lambda^{n-1} - b_1 a_2 \lambda^{n-2} - b_1 b_2 a_3 \lambda^{n-3} - \cdots - b_1 b_2 \cdots b_{n-3} b_{n-2} a_{n-1} \lambda + b_1 b_2 \cdots b_{n-2} b_{n-1} a_n \,, \end{aligned}$$

which is the result we desired to show.

Problem 9.3 (the positive eigenvalue of a Leslie matrix is always positive)

Following the hint if $q'(\lambda_1) \neq 0$ we have that λ_1 is simple. Given the expression for $q(\lambda)$ of

$$q(\lambda) = \frac{a_1}{\lambda} + \frac{a_2 b_1}{\lambda^2} + \frac{a_3 b_1 b_2}{\lambda^3} + \cdots + \frac{a_n b_1 b_2 \cdots b_{n-1}}{\lambda^n} \,, \tag{8}$$

so that

$$q'(\lambda) = -\frac{a_1}{\lambda^2} - \frac{2a_2 b_1}{\lambda^3} - \frac{3a_3 b_1 b_2}{\lambda^4} - \cdots - \frac{n a_n b_1 b_2 \cdots b_{n-1}}{\lambda^{n+1}} \,,$$

which is the sum of all negative terms and thus $q'(\lambda) < 0$ and cannot be equal to zero.

Problem 9.4 (evaluating $\lim_{k\to\infty} x^{(k)}$)

Let the product of $P^{-1}x^{(0)}$ be the vector $\begin{bmatrix} c \\ v_2 \\ v_3 \\ \vdots \\ v_{n-1} \\ v_n \end{bmatrix}$. Then multiplying this vector by the matrix of all zeros except for a single 1 at the $(1,1)$ location gives the vector $\begin{bmatrix} c \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$.

If this vector is the multiplied by P we get

$$cP \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = c(\text{first column of } P) = cx_1 \, .$$

Problem 9.5 (the net reproduction rate)

The definition of the net reproduction rate is

$$R = a_1 + a_2 b_1 + a_3 b_2 b_1 + \cdots + a_n b_1 b_2 \cdots b_{n-2} b_{n-1} \, , \tag{9}$$

where we recall a_i and b_i are defined as

- a_i is the average number of daughter born to a single female during the time she is in the ith age class for $1 \leq i \leq n$.
- b_i is the fraction of females in the ith class that can be expected to survive and pass into the $(i+1)$th age class for $1 \leq i \leq n-1$.

Now consider a single female, and let D be the random variable representing the number of daughters born to this female during her lifetime. Let A_i be the event that this female lives to age i for $1 \leq i \leq n$. Then by conditional expectation we have

$$E[D] = \sum_{i=1}^{n} E[D|A_i] P(A_i) \, .$$

Now

$$\begin{aligned} P(A_1) &= 1 \\ P(A_2) &= P(A_2|A_1)P(A_1) = b_1 \\ P(A_3) &= P(A_3|A_2)P(A_2) = b_2 b_1 \\ &\vdots \\ P(A_i) &= b_{i-1} b_{i-2} \cdots b_2 b_1 \, , \end{aligned}$$

and $E[D|A_i] = a_i$ by its definition. Combining these expressions gives the net reproduction rate Equation 9 as we were to show.

Problem 9.6 (an eventually decreasing population)

The total number of females in the population $p^{(n)}$ at time n is the vector product

$$p^{(n)} = \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right] x^{(n)} \approx \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}\right] c\lambda_1^n x_1 \,.$$

Thus our population will eventually be decreasing if $\lambda_1 < 1$ or eventually increasing if $\lambda_1 > 1$. Here λ_1 is the largest eigenvalue of the Leslie matrix L. The eigenvalues of L must satisfy $q(\lambda) = 1$ where $q(\lambda)$ is given by Equation 8. To start, lets assume that $1 < \lambda_1$. Then since $q(\lambda)$ is an decreasing function of its argument λ we have that

$$q(1) > q(\lambda_1) = 1 \,.$$

From Equation 8 we see that $q(1) = R$ where R is given by Equation 9. Thus we have shown that $R > 1$. If we assume the opposite or that that $\lambda_1 < 1$ then again since $q(\lambda)$ is a decreasing function of λ we get that

$$1 = q(\lambda_1) > q(1) = R \,,$$

and thus $R < 1$.

Problem 9.7 (the next reproductive rate)

From the given example in the book we have $a_1 = 0$, $a_2 = 4$, $a_3 = 3$, $b_1 = 1/2$, and $b_2 = 1/4$ so that Equation 9 gives

$$R = 0 + 4(1/2) + 3(1/2)(1/4) = 2.375 > 1 \,,$$

thus the population eventually increases.

Chapter 10 (Harvesting of Animal Populations)

Notes on the Text

Some python code written for this chapter include

- `Leslie_matrix.py` computes a Leslie matrix given data for a_i and b_i.

Problem Solutions

Some the calculations for this problem are done in the python code `chap_10_problems.py`.

Problem 10.1 (harvesting policies)

Part (a): For a uniform harvest we have $h_1 = h_2 = h_3 = h$ where $h = 1 - \frac{1}{\lambda_1}$ with λ_1 is the largest eigenvalue of L. For the matrix given here $\lambda_1 = 1.5$ and with this we compute $h = 0.3333$ as the fraction harvested from each age group (and it is also the yield). Then x_1 the age distribution after each harvest is given by

$$x_1 = \begin{bmatrix} 1 \\ \frac{b_1}{\lambda_1} \\ \frac{b_1 b_2}{\lambda_1^2} \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.3333 \\ 0.0555 \end{bmatrix} .$$

Part (b): If the youngest class is harvested then $h_1 = h$ and $h_2 = h_3 = 0$ where

$$h = 1 - \frac{1}{R} = 1 - \frac{1}{2.375} = 0.5789 .$$

The age distribution after each harvest is given by

$$x_1 = \begin{bmatrix} 1 \\ b_1 \\ b_1 b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ (1/2)(1/4) \end{bmatrix} .$$

The age distribution before each harvest is given by

$$Lx_1 = \begin{bmatrix} 2.375 \\ 0.5 \\ 0.125 \end{bmatrix} .$$

Thus the yield from this harvesting policy is $\frac{h(2.375)}{2.375+0.5+0.125} = 0.4583$.

Problem 10.2 (the optimal sustainable yield for the sheep population)

We are told that for the sheep population the optimal harvesting policy has allocations $h_1 = 0.522$ and $h_9 = 1.0$ with all other $h_i = 0$. Then from equation 10.5 in the book that the age distribution after each harvest (or x_1) is given by

$$x_1 = \begin{bmatrix} 1 \\ b_1 \\ b_1 b_2 \\ b_1 b_2 b_3 \\ b_1 b_2 b_3 b_4 \\ b_1 b_2 b_3 b_4 b_5 \\ b_1 b_2 b_3 b_4 b_5 b_6 \\ b_1 b_2 b_3 b_4 b_5 b_6 b_7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.845 \\ 0.823875 \\ 0.79503937 \\ 0.75528741 \\ 0.69939614 \\ 0.62595954 \\ 0.53206561 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

This is computed in the python code `chap_10_problems.py`. Next in that code we compute the Leslie matrix L and the product Lx_1 which is the distribution of the population after the growth period. In this case we get for Lx_1 the following vector

$$Lx_1 = \begin{bmatrix} 2.08984108 \\ 0.845 \\ 0.823875 \\ 0.79503937 \\ 0.75528741 \\ 0.69939614 \\ 0.62595954 \\ 0.53206561 \\ 0.41820357 \\ 0. \\ 0. \\ 0. \end{bmatrix} .$$

Now since the vector Lx_1 is the distribution of the population after the growth period and x_1 is the distribution of the population after the harvest is x_1 then the ratio of the sum of the elements of x_1 divided by the sum of the elements of Lx_1 the proportion of the population still around after the harvest. Thus one minus this number is the fraction of the population harvested. When we compute this number we get 0.19882804 as we were to show.

Problem 10.3 (only the first age class is harvested)

Eq. 10.10 is the age distribution after each harvest in this case

$$x_1 = \begin{bmatrix} 1 \\ b_1 \\ b_1 b_2 \\ \vdots \\ b_1 b_2 \cdots b_{n-4} b_{n-3} \\ b_1 b_2 \cdots b_{n-3} b_{n-2} \\ b_1 b_2 \cdots b_{n-2} b_{n-1} \end{bmatrix} . \tag{10}$$

With this vector we find the product Lx_1 given by

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ b_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b_2 & 0 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & b_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_1 b_2 \\ \vdots \\ b_1 b_2 \cdots b_{n-4} b_{n-3} \\ b_1 b_2 \cdots b_{n-3} b_{n-2} \\ b_1 b_2 \cdots b_{n-2} b_{n-1} \end{bmatrix} ,$$

or when we multiply the vector

$$\begin{bmatrix} a_1 + a_2 b_1 + a_3 b_1 b_2 + \cdots + a_{n-1} b_1 b_2 \cdots b_{n-3} + a_{n-1} b_1 b_2 \cdots b_{n-2} + a_n b_1 b_2 \cdots b_{n-1} \\ b_1 \\ b_1 b_2 \\ \vdots \\ b_1 b_2 \cdots b_{n-4} b_{n-3} \\ b_1 b_2 \cdots b_{n-3} b_{n-2} \\ b_1 b_2 \cdots b_{n-2} b_{n-1} \end{bmatrix}$$

, or recalling the definition of R given by Equation 9 we get

$$\begin{bmatrix} R \\ b_1 \\ b_1 b_2 \\ \vdots \\ b_1 b_2 \cdots b_{n-4} b_{n-3} \\ b_1 b_2 \cdots b_{n-3} b_{n-2} \\ b_1 b_2 \cdots b_{n-2} b_{n-1} \end{bmatrix} .$$

Using this we see that $Lx_1 - x_1$ is the vector specified.

Problem 10.4 (harvesting the Ith class)

For this problem we have $h_1 = h_2 = \cdots = h_{I-1} = h_{I+1} = \cdots = h_n = 0$ and h_I is unknown. Then Eq 10.4 in the book is

$$\begin{aligned}(1-h_1)[a_1 &+ a_2b_1(1-h_2) + a_3b_1b_2(1-h_2)(1-h_3) + a_4b_1b_2b_3(1-h_2)(1-h_3)(1-h_4) + \cdots \\ &+ a_nb_1b_2\cdots b_{n-2}b_{n-1}(1-h_2)(1-h_3)\cdots(1-h_n)] = 1\,. \end{aligned} \tag{11}$$

or in a more compact notation which introduces cumulative products

$$(1-h_1)\sum_{i=1}^{n} a_i \left(\prod_{j=1}^{i-1} b_j\right)\left(\prod_{k=1}^{i}(1-h_k)\right) = 1\,.$$

When we restrict this expression to the case considered in this problem we get

$$\begin{aligned}a_1 &+ a_2b_1 + a_3b_1b_2 + a_4b_1b_2b_3 + \cdots + a_Ib_1b_2\cdots b_{I-2}b_{I-1}(1-h_I) + a_{I+1}b_1b_2\cdots b_{I-1}b_I(1-h_I) \\ &+ \cdots + a_nb_1b_2\cdots b_{n-1}(1-h_I) = 1\,.\end{aligned}$$

In the above we recognize the expression for R given by Equation 9 and we get

$$R - (a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_nb_1b_2\cdots b_{n-1})\,h_I = 1\,.$$

When we solve for h_I we get

$$h_I = \frac{R-1}{a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_nb_1b_2\cdots b_{n-1}}\,.$$

Problem 10.5 (harvesting all of the Jth class and some of the Ith class)

Note we must have $1 \le I < J < n$ with $h_J = 1$, h_I is unknown, and $h_i = 0$ otherwise. Then Equation 11 becomes in this situation

$$\begin{aligned}a_1 &+ a_2b_1 + a_3b_1b_2 + \cdots + a_Ib_1b_2\cdots b_{I-2}b_{I-1}(1-h_I) + a_{I+1}b_1b_2\cdots b_{I-1}b_I(1-h_I) + \cdots \\ &+ a_{J-1}b_1b_2\cdots b_{J-3}b_{J-2}(1-h_I) = 1\,.\end{aligned}$$

This becomes when we expand some of the products

$$\begin{aligned}a_1 &+ a_2b_1 + a_3b_1b_2 + \cdots + a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_{J-1}b_1b_2\cdots b_{J-3}b_{J-2} \\ &- h_I(a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_{J-1}b_1b_2\cdots b_{J-3}b_{J-2}) = 1\,.\end{aligned}$$

The above can be solved for h_I where we get

$$h_I = \frac{a_1 + a_2b_1 + a_3b_1b_2 + \cdots + a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_{J-1}b_1b_2\cdots b_{J-3}b_{J-2} - 1}{a_Ib_1b_2\cdots b_{I-2}b_{I-1} + a_{I+1}b_1b_2\cdots b_{I-1}b_I + \cdots + a_{J-1}b_1b_2\cdots b_{J-3}b_{J-2}}\,.$$

Chapter 11 (Least Squares)

Problem Solutions

Problem 11.5 (independent columns of M means $M^T M$ is invertible)

For this problem we assume that $m > n$ (which is the case encountered in practice). Since M has n linearly independent columns and our assumption is that $m > n$ the matrix M has a rank of n. Now to show that $M^T M$ is invertible we will show that the rank of this product matrix is equal to that of its dimension which is n. To show this we recall two "rank preserving product identities" and use the one that is applicable in this case. These identities are stated in the general case as follows.

- If A is a $m \times n$ matrix and B is a $n \times k$ matrix of rank n then

$$\operatorname{rank}(AB) = \operatorname{rank}(A)\,. \tag{12}$$

- If A is a $m \times n$ matrix and C is a $l \times m$ matrix of rank m then

$$\operatorname{rank}(CA) = \operatorname{rank}(A)\,. \tag{13}$$

In the case considered here we want to multiply the matrix M on the left by M^T. Since M^T has a rank of n (as M does) using Equation 13 we see that

$$\operatorname{rank}(M^T M) = \operatorname{rank}(M) = n\,.$$

But the dimension of the matrix $M^T M$ is n. As $M^T M$ is a square matrix with a rank equal to its dimension it is invertible.

Problem 11.6

Recall that the matrix in 11.2 is $M = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$.

If no two x_i and x_j are are different then they must all be the same number and the second column of M is a multiple of the first. Thus the columns of M are not independent and M is of rank 1. Since by the rank contraction property of matrix multiplication

$$\operatorname{rank}(M^T M) \le \min(\operatorname{rank}(M^T), \operatorname{rank}(M)) = \operatorname{rank}(M) = 1\,,$$

the matrix $M^T M$ is not invertible. The statement that no two x_i and x_j are are different means that they are all the same and all the data points lie on a vertical line.

Problem 11.7 (for a Vandermonde matrix we need at least $m+1$ distinct numbers)

Recall that the matrix 11.5 from the book is

$$M = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}.$$

We will show the contraposative of this statement that is if we have less than $m+1$ distinct numbers than the columns are *not* linearly independent. We will argue this statement in an induction form. If we assume that there is only one distinct number then the second column of the above is a scalar multiple of the first column and the columns of M are not lineally independent. If we two distinct numbers we now argue that the first three columns of M will not be lineally independent. By changing the order of the samples and since we assume that we only have two unique numbers, with out loss of generality we can write the first three columns of the matrix M as

$$\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{or equivalently} \quad \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\,.$$

From this we can perform row reduction to get

$$\begin{bmatrix} 1 & 0 & -ab \\ 0 & 1 & a+b \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\,.$$

Thus if we take $x = ab$ and $y = -a - b$ and $z = 1$ then we have a nonzero vector in the null space of the first three columns. Thus the first three columns are not linearly independent. We can continue this procedure, showing that with only three, four, five, etc. unique numbers the submatrix from M with the first four, five, six, etc. columns are linearly independent. When we have only m unique numbers the submatrix with $m+1$ rows from M (or the entire matrix M) will be linearly independent. If we have more than m unique numbers (or at least $m+1$ unique numbers) the matrix M will have $m+1$ linearly independent columns and by problem 10.5 the matrix $M^T M$ will be invertible.

Problem 11.8

If the conditions for this problem are true then the conditions for problem 11.7 hold true and from that problem the columns of M must be linearly independent. If the columns of M are linearly dependent then by problem 11.5 $M^T M$ is invertible.

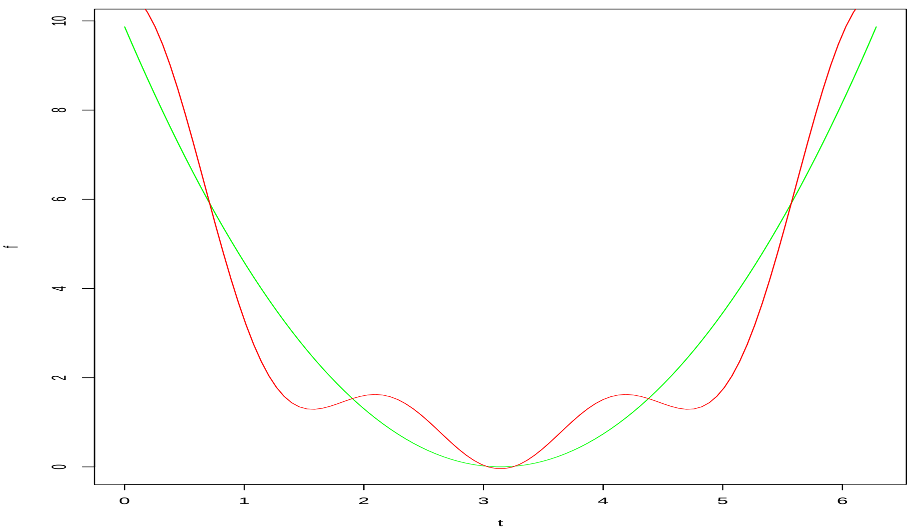

Figure 1: A plot of $f(t)$ and its trigonometric approximation $g(t)$ for Problem 1.

Chapter 12 (A LS Model for Human Hearing)

Problem Solutions

Problem 12.1 (a trigonometric polynomial for $f(t) = (t-\pi)^2$)

We want an approximation to $f(t)$ of the form

$$g(t) = \frac{1}{2}a_0 + a_1\cos(t) + a_2\cos(2t) + a_3\cos(3t) + b_1\sin(t) + b_2\sin(2t) + b_3\sin(3t)\,.$$

The coefficients in the above expression are given from the formulas given in the book. For example,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(t)dt = \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2 dt = \frac{2\pi^2}{3} \\ a_k &= \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2\cos(kt)dt = \frac{4}{k^2} \\ b_k &= \frac{1}{\pi}\int_0^{2\pi}(t-\pi)^2\sin(kt)dt = 0\,. \end{aligned}$$

Thus we get

$$g(t) = \frac{\pi^2}{3} + 4\cos(t) + 2\cos(2t) + \frac{4}{3}\cos(3t)\,.$$

When we plot this approximation with the function $f(t)$ we get the plot shown in Figure 1.

Problem 12.2 (a trigonometric polynomial for $f(t) = t^2$)

First we need to compute a_k. For $k = 0$ this is

$$a_0 = \frac{2}{T}\int_0^T f(t)dt = \frac{2}{T}\int_0^T t^2 dt = \frac{2}{T}\cdot\frac{T^3}{3} = \frac{2T^2}{3}.$$

For $k \geq 1$ using integration by parts twice we find

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi k}{T}t\right)dt = \frac{2}{T}\int_0^T t^2\cos\left(\frac{2\pi k}{T}t\right)dt \\
&= \frac{2}{T}\left[\left.\frac{t^2\sin\left(\frac{2\pi k}{T}t\right)}{\left(\frac{2\pi k}{T}\right)}\right|_0^T - \frac{2T}{2\pi k}\int_0^T t\sin\left(\frac{2\pi k}{T}t\right)dt\right] = -\frac{2}{\pi k}\int_0^T t\sin\left(\frac{2\pi k}{T}t\right)dt \\
&= -\frac{2}{\pi k}\left[-\left.\frac{t\cos\left(\frac{2\pi k}{T}t\right)}{\left(\frac{2\pi k}{T}\right)}\right|_0^T + \frac{T}{2\pi k}\int_0^T \cos\left(\frac{2\pi k}{T}t\right)dt\right] \\
&= \frac{2}{\pi k}\left(\frac{T}{2\pi k}\right)(T\cos(2\pi k)) - \frac{2}{\pi k}\left(\frac{T}{2\pi k}\right)\left.\frac{\sin\left(\frac{2\pi k}{T}t\right)}{\left(\frac{2\pi k}{T}\right)}\right|_0^T = \frac{T^2}{\pi^2 k^2}.
\end{aligned}$$

For b_k we have more integration by parts

$$\begin{aligned}
b_k &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi k}{T}t\right)dt = \frac{2}{T}\int_0^T t^2\sin\left(\frac{2\pi k}{T}t\right)dt \\
&= \frac{2}{T}\left[-\left.\frac{t^2\cos\left(\frac{2\pi k}{T}t\right)}{\left(\frac{2\pi k}{T}\right)}\right|_0^T + \frac{2T}{2\pi k}\int_0^T t\cos\left(\frac{2\pi k}{T}t\right)dt\right] \\
&= -\frac{T^2}{\pi k} + \frac{2}{\pi k}\int_0^T t\cos\left(\frac{2\pi k}{T}t\right)dt \\
&= -\frac{T^2}{\pi k} + \frac{2}{\pi k}\left[\left.\frac{t\sin\left(\frac{2\pi k}{T}t\right)}{\left(\frac{2\pi k}{T}\right)}\right|_0^T - \frac{T}{2\pi k}\int_0^T \sin\left(\frac{2\pi k}{T}t\right)dt\right] \\
&= -\frac{T^2}{\pi k} + \frac{2}{\pi k}\frac{T^2}{(2\pi k)^2}\left[\left.\cos\left(\frac{2\pi k}{T}t\right)\right|_0^T\right] = -\frac{T^2}{\pi k}.
\end{aligned}$$

Thus for this problem we get

$$
\begin{aligned}
g(t) \approx\ & \frac{1}{2}a_0 \\
& + a_1 \cos\left(\frac{2\pi}{T}t\right) + a_2 \cos\left(\frac{4\pi}{T}t\right) + a_3 \cos\left(\frac{6\pi}{T}t\right) + a_4 \cos\left(\frac{8\pi}{T}t\right) \\
& + b_1 \sin\left(\frac{2\pi}{T}t\right) + b_2 \sin\left(\frac{4\pi}{T}t\right) + b_3 \sin\left(\frac{6\pi}{T}t\right) + b_4 \sin\left(\frac{8\pi}{T}t\right) \\
& = \frac{T^2}{3} \\
& + \frac{T^2}{\pi^2}\left(\cos\left(\frac{2\pi}{T}t\right) + \frac{1}{4}\cos\left(\frac{4\pi}{T}t\right) + \frac{1}{9}\cos\left(\frac{6\pi}{T}t\right) + \frac{1}{16}\cos\left(\frac{8\pi}{T}t\right)\right) \\
& - \frac{T^2}{\pi^2}\left(\sin\left(\frac{2\pi}{T}t\right) + \frac{1}{2}\sin\left(\frac{4\pi}{T}t\right) + \frac{1}{3}\sin\left(\frac{6\pi}{T}t\right) + \frac{1}{4}\sin\left(\frac{8\pi}{T}t\right)\right) .
\end{aligned}
$$

Note that I got a negative sign in the above expression for the sin terms which the solution in the book does not have. If anyone sees anything wrong with what I have done for this problem (or agrees with me) please contact me.

Problem 12.3 (a trigonometric polynomial for half the sin function)

For the given f we need to compute

$$
\begin{aligned}
a_k &= \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)dt \quad \text{for} \quad k \geq 0 \\
b_k &= \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(kt)dt \quad \text{for} \quad k \geq 1 .
\end{aligned}
$$

For $k = 0$ we have

$$
a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(t)dt = -\left.\frac{\cos(t)}{\pi}\right|_0^{\pi} = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi} .
$$

For $k \geq 1$ we need to evaluate the integral

$$
a_k = \frac{1}{\pi}\int_0^{\pi} \sin(t)\cos(kt)dt .
$$

To evaluate this we will add two identities

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) ,
\end{aligned}
$$

to get the sin-cos product sum identity

$$
\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)) . \tag{14}
$$

Using this expression we have that a_k can be evaluated (when $k \neq 1$) as

$$\begin{aligned}
a_k &= \frac{1}{2\pi} \int_0^\pi (\sin((k+1)t) + \sin((1-k)t))dt \\
&= \frac{1}{2\pi} \left[- \left. \frac{\cos((k+1)t)}{k+1} \right|_0^\pi - \left. \frac{\cos((1-k)t)}{1-k} \right|_0^\pi \right] \\
&= \frac{1}{2\pi} \left[-\frac{(-1)^{k+1} - 1}{k+1} + \frac{(-1)^{k-1} - 1}{k-1} \right] .
\end{aligned}$$

If k is even we have $(-1)^{k+1} = -1$ and $(-1)^{k-1} = -1$ so the above becomes

$$a_k = \frac{1}{2\pi} \left[-\frac{1}{k+1}(-2) + \frac{1}{k-1}(-2) \right] = -\frac{2}{\pi(k^2-1)} .$$

If k is odd we have $(-1)^{k+1} = 1$ and $(-1)^{k-1} = 1$ and we get

$$a_k = \frac{1}{2\pi} \left[-\frac{1}{k+1}(0) + \frac{1}{k-1}(0) \right] = 0 .$$

In the case $k = 1$ we can use Equation 14 to find

$$a_1 = \frac{1}{\pi} \int_0^\pi \sin(t)\cos(t)dt = \frac{1}{2\pi} \int_0^\pi \sin(2\pi)dt = \frac{1}{2\pi} \left(- \left. \frac{\cos(2t)}{2} \right|_0^\pi \right. = -\frac{1}{4\pi}(0-0) = 0 .$$

Now to compute b_k we need to compute

$$b_k = \frac{1}{\pi} \int_0^\pi \sin(t)\sin(kt)dt .$$

To evaluate this we will use the sin-sin product sum identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) , \tag{15}$$

to write the integral expression for b_k as

$$b_k = \frac{1}{2\pi} \int_0^\pi (\cos((1-k)t) - \cos((1+k)t)dt .$$

We can evaluate this (when $k \neq 1$) as

$$b_k = \frac{1}{2\pi} \left[\left. \frac{\sin((k-1)t)}{k-1} \right|_0^\pi - \left. \frac{\sin((k+1)t)}{k+1} \right|_0^\pi \right] = \frac{1}{2\pi}(0-0) = 0 .$$

If $k = 1$ using Equation 15 we have

$$b_1 = \frac{1}{\pi} \int_0^\pi \sin^2(t)dt = \frac{1}{2\pi} \int_0^\pi (1 - \cos(2t))dt = \frac{1}{2\pi} \left[\pi - \left. \frac{1}{2}\sin(2t) \right|_0^\pi \right] = \frac{1}{2} .$$

Using these expressions we get for $g(x)$ the following

$$g(x) = \frac{1}{\pi} + a_2\cos(2t) + a_4\cos(4t) + b_1\sin(t) = \frac{1}{\pi} - \frac{2}{3\pi}\cos(2t) - \frac{2}{15\pi}\cos(4t) + \frac{1}{2}\sin(t) .$$

Problem 12.4 (the trigonometric polynomial for $\sin(\frac{1}{2}t)$)

We have

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)dt = \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{1}{2}t\right)\cos(kt)dt\,.$$

For $k = 0$ we find

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{1}{2}t\right)dt = -\frac{1}{\pi}\left.\frac{\cos\left(\frac{1}{2}t\right)}{\frac{1}{2}}\right|_0^{2\pi} = -\frac{2}{\pi}(\cos(\pi)-1) = \frac{4}{\pi}\,.$$

Now using $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta)+\sin(\alpha-\beta))$ we find a_k can be written as

$$\begin{aligned}
a_k &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sin\left(\left(k+\frac{1}{2}\right)t\right)+\sin\left(\left(\frac{1}{2}-k\right)t\right)\right)\\
&= \frac{1}{2\pi}\left[-\frac{\cos((k+\frac{1}{2})t)}{k+\frac{1}{2}} - \frac{\cos((\frac{1}{2}-k)t)}{\frac{1}{2}-k}\right|_0^{2\pi}\\
&= -\frac{1}{2\pi}\left[\frac{\cos(2\pi(k+\frac{1}{2}))-1}{k+\frac{1}{2}} + \frac{\cos(2\pi(k-\frac{1}{2}))-1}{\frac{1}{2}-k}\right]\,.
\end{aligned}$$

Now

$$\cos(2\pi(k+\frac{1}{2})) = \cos(2\pi k+\pi) = \cos(2\pi k)\cos(\pi) = -1\,,$$

and

$$\cos(2\pi(k-\frac{1}{2})) = \cos(2\pi k-\pi) = -1\,,$$

so a_k becomes

$$a_k = -\frac{1}{2\pi}\left[-\frac{2}{k+\frac{1}{2}} - \frac{2}{\frac{1}{2}-k}\right] = \frac{1}{\pi}\left[\frac{1}{k+\frac{1}{2}}+\frac{1}{\frac{1}{2}-k}\right] = \frac{4}{\pi(1-4k^2)}\,.$$

For b_k the integral we need to evaluate is

$$b_k = \frac{1}{\pi}\int_0^{2\pi}\sin\left(\frac{1}{2}t\right)\sin(kt)dt\,.$$

Using the sin-sin product sum identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta))\,,$$

we have

$$\begin{aligned}
b_k &= \frac{1}{2\pi}\int_0^{2\pi}(\cos\left((\frac{1}{2}-k)t\right) - \cos((\frac{1}{2}+k)t))dt\\
&= \frac{1}{2\pi}\left[\frac{\sin((\frac{1}{2}-k)t)}{\frac{1}{2}-k} - \frac{\sin((\frac{1}{2}+k)t)}{\frac{1}{2}+k}\right|_0^{2\pi}\\
&= \frac{1}{2\pi}\left[\frac{\sin((\frac{1}{2}-k)2\pi)}{\frac{1}{2}-k} - \frac{\sin((\frac{1}{2}+k)2\pi)}{\frac{1}{2}+k}\right]\,.
\end{aligned}$$

Now

$$\sin((\frac{1}{2}-k)2\pi) = \sin(\pi - 2\pi k) = \sin(\pi)\cos(2\pi k) - \cos(\pi)\sin(2\pi k) = 0\,,$$

and

$$\sin((\frac{1}{2}+k)2\pi) = 0\,.$$

Because of this $b_k = 0$. Thus the fourth order trigonometric polynomial is given by

$$g(x) \approx \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{3}\cos(t) + \frac{1}{15}\cos(2t) + \frac{1}{35}\cos(3t) + \frac{1}{63}\cos(4t)\right].$$

Problem 12.5 (a trigonometric approximation)

We will use

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi k}{T}t\right)dt \quad \text{for} \quad k \geq 0 \\
b_k &= \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi k}{T}t\right)dt \quad \text{for} \quad k \geq 1\,.
\end{aligned}$$

For $k = 0$ we find a_0 given by

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^{T/2} t dt + \frac{2}{T}\int_{T/2}^T (T-t)dt = \frac{2}{T}\left(\left.\frac{t^2}{2}\right|_0^{T/2} - \frac{2}{T}\left(\left.\frac{(T-t)^2}{2}\right|_{T/2}^T\right.\right. \\
&= \frac{1}{T}\left(\frac{T^2}{4}\right) - \frac{1}{T}\left(0 - \frac{T^2}{4}\right) = \frac{T}{4} + \frac{T}{4} = \frac{T}{2}\,.
\end{aligned}$$

For a_k when $k \geq 1$ we have

$$\begin{aligned}
a_k &= \frac{2}{T}\int_0^{T/2} t\cos\left(\frac{2\pi k}{T}t\right)dt + \frac{2}{T}\int_{T/2}^T (T-t)\cos\left(\frac{2\pi k}{T}t\right)dt \\
&= \frac{2}{T}\left[\left.\frac{t\sin\left(\frac{2\pi k}{T}t\right)}{\frac{2\pi k}{T}}\right|_0^{T/2} - \frac{T}{2\pi k}\int_0^{T/2}\sin\left(\frac{2\pi k}{T}t\right)dt\right] \\
&+ \frac{2}{T}\left[\left.\frac{(T-t)\sin\left(\frac{2\pi k}{T}t\right)}{\frac{2\pi k}{T}}\right|_{T/2}^T + \frac{T}{2\pi k}\int_{T/2}^T \sin\left(\frac{2\pi k}{T}t\right)dt\right] \\
&= \frac{2}{T}\left[\frac{T}{2\pi k}\cdot\frac{T}{2}\sin\left(\frac{2\pi k}{T}\frac{T}{2}\right) + \frac{T}{2\pi k}\cdot\frac{T}{2\pi k}\cos\left.\left(\frac{2\pi k}{T}\frac{T}{2}\right)\right|_0^{T/2}\right] \\
&+ \frac{2}{T}\left[-\frac{T}{2\pi k}\cdot\frac{T}{2}\sin(\pi k) + \frac{T^2}{(2\pi k)^2}\left(-\cos\left.\left(\frac{2\pi k}{T}t\right)\right|_{T/2}^T\right.\right] \\
&= \frac{2}{T}\left[\frac{T^2}{(2\pi k)^2}(\cos(\pi k) - 1)\right] + \frac{2}{T}\left[\frac{T^2}{(2\pi k)^2}(-\cos(2\pi k) + \cos(\pi k))\right] \\
&= \frac{T}{2\pi^2 k^2}((-1)^k - 1) + \frac{T}{2\pi^2 k^2}(-1 + (-1)^k)\,.
\end{aligned}$$

If k is even then $(-1)^k = +1$ so $a_k = 0$. If k is odd then $(-1)^k = -1$ so

$$a_k = -\frac{2T}{\pi^2 k^2}\,.$$

For b_k we have

$$\begin{aligned}
b_k &= \frac{2}{T}\int_0^{T/2} t\sin\left(\frac{2\pi k}{T}t\right)dt + \frac{2}{T}\int_{T/2}^{T}(T-t)\sin\left(\frac{2\pi k}{T}t\right)dt \\
&= \frac{2}{T}\left[-\frac{t\cos\left(\frac{2\pi k}{T}t\right)}{\frac{2\pi k}{T}}\Bigg|_0^{T/2} + \frac{T}{2\pi k}\int_0^{T/2}\cos\left(\frac{2\pi k}{T}t\right)dt\right] \\
&+ \frac{2}{T}\left[-\frac{(T-t)\cos\left(\frac{2\pi k}{T}t\right)}{\frac{2\pi k}{T}}\Bigg|_{T/2}^{T} - \frac{T}{2\pi k}\int_{T/2}^{T}\cos\left(\frac{2\pi k}{T}t\right)dt\right] \\
&= \frac{2}{T}\left[-\frac{T}{2\pi k}\left(\frac{T}{2}\cos\left(\frac{2\pi k}{T}\frac{T}{2}\right) - 0\right) + \frac{T}{2\pi k}\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}t\right)\Bigg|_0^{T/2}\right] \\
&+ \frac{2}{T}\left[-\frac{T}{2\pi k}\left(0 - \frac{T}{2}\cos(\pi k)\right) - \frac{T}{2\pi k}\frac{T}{2\pi k}\sin\left(\frac{2\pi k}{T}t\right)\Bigg|_{T/2}^{T}\right] \\
&= \frac{2}{T}\left[-\frac{T}{2\pi k}\frac{T}{2}(-1)^k\right] + \frac{2}{T}\left[\frac{T}{2}\frac{T}{2\pi k}(-1)^k\right] = 0\,.
\end{aligned}$$

Thus we get

$$\begin{aligned}
g(x) &= \frac{T}{4} - \frac{2T}{\pi^2}\sum_{k=1,3,5,\cdots}^{\infty}\frac{1}{k^2}\cos\left(\frac{2\pi k}{T}t\right) \\
&= \frac{T}{4} - \frac{2T}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\cos\left(\frac{2\pi(2k+1)}{T}t\right) \\
&= \frac{T}{4} - \frac{2T}{\pi^2}\left[\cos\left(\frac{2\pi t}{T}\right) + \frac{1}{9}\cos\left(\frac{6\pi t}{T}\right) + \frac{1}{25}\cos\left(\frac{10\pi t}{T}\right) + \cdots\right].
\end{aligned}$$

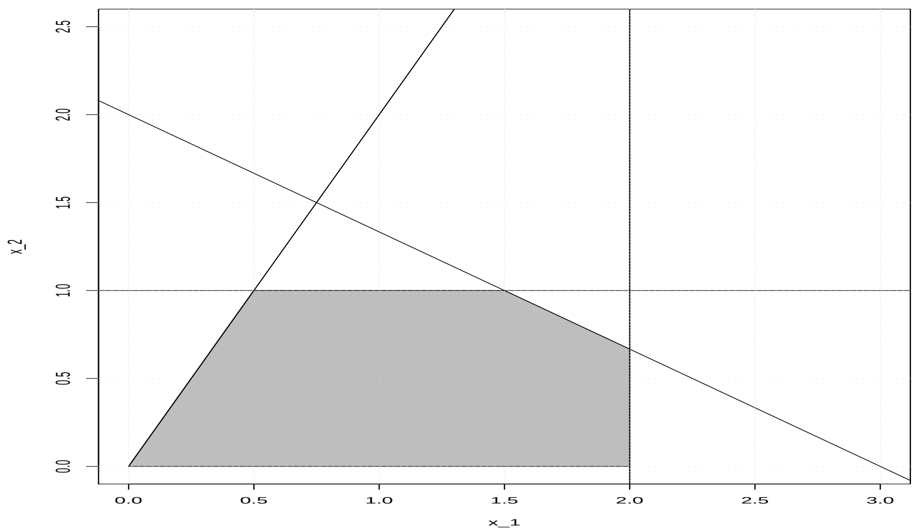

Figure 2: The feasible region for Problem 1.

Chapter 13 (Linear Programming I)

Notes on Example 13.3

Since x_1 is in units of $\frac{1}{2}$-cup units we have that $\frac{1}{2}x_1$ is the amount of milk in 1-cup units. That is if $x_1 = 1$ then we have $\frac{1}{2}x_1 = \frac{1}{2}$ cups of milk. If we want to restrict ourselves to mixtures that contain between 1-3 oz of corn flakes per *cup* of milk we must have

$$\frac{x_2}{\frac{1}{2}x_1} \geq 1 \quad \text{or} \quad x_2 \geq \frac{1}{2}x_1 \quad \text{or} \quad x_1 - 2x_2 \leq 0\,,$$

and

$$\frac{x_2}{\frac{1}{2}x_1} \leq 3 \quad \text{or} \quad x_2 \leq \frac{3}{2}x_1 \quad \text{or} \quad 3x_1 - 2x_2 \geq 0\,.$$

These are the two inequality expressions quoted in the book.

Problem Solutions

Problem 13.1 (a linear programming problem)

In Figure 2 we plot the feasible region for this problem in gray. The possible maximum must happen at an extreme point. From the above diagram this region has extreme points given by

$$(0,0)\,, \quad (2,0)\,, \quad (2,2/3)\,, \quad (3/2,1)\,, \quad (1/2,1)\,.$$

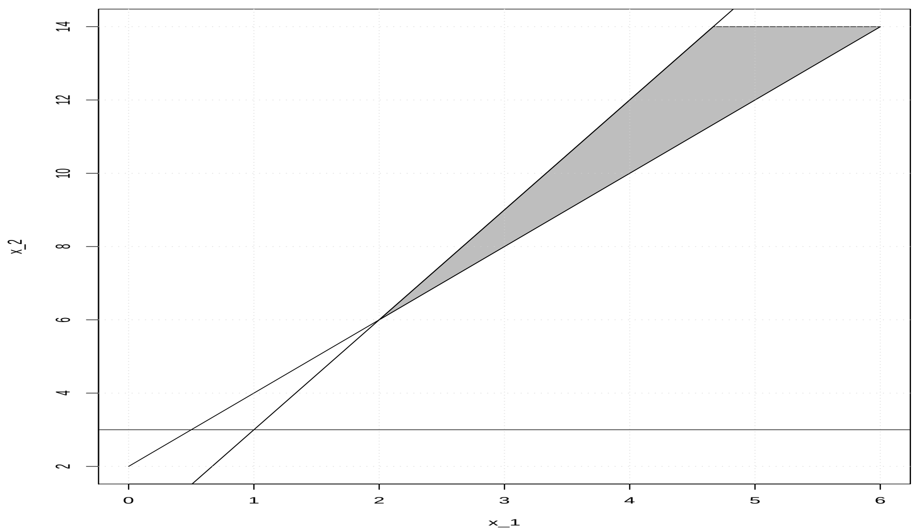

Figure 3: The feasible region for Problem 2.

Evaluating the objective function z at each of these points gives the values

$$0\,, \quad 6\,, \quad 7.333\,, \quad 6.5\,, \quad 3.5\,.$$

The largest value of the the objective function happens at the value of $(x_1, x_2) = (2, 2/3)$ which is our optimal solution.

Problem 13.2 (another linear programming problem)

In Figure 3 we plot the feasible region for this problem in gray. From the above diagram we see that the feasible region is unbounded. Thus this maximization problem has no solution.

Problem 13.3 (another linear programming problem)

In Figure 4 we plot the feasible region for this problem in gray. We see that in the feasible region x_1 can increase without bounds. Thus the objective function $z = -3x_1 + 2x_2$ can be made infinitely small. Thus there is no solution.

Problem 13.4 (example 13.2)

In Figure 5 we plot the feasible region for this problem in gray. From this plot we see that the extreme points (with the value of the objective function at them) are given by

$$(2000, 2000, 340)\,, \quad (6000, 2000, 740)\,, \quad (6000, 4000, 880)\,, \quad (5000, 5000, 850)\,.$$

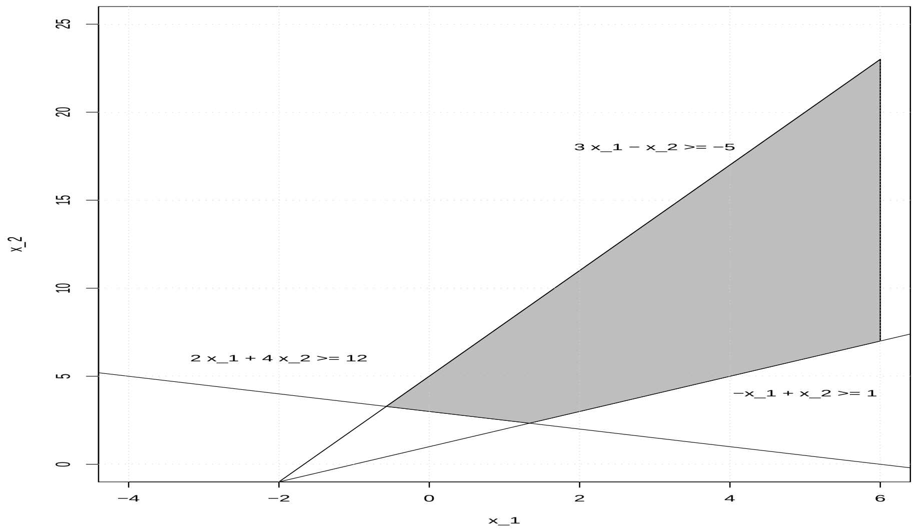

Figure 4: The feasible region for Problem 3.

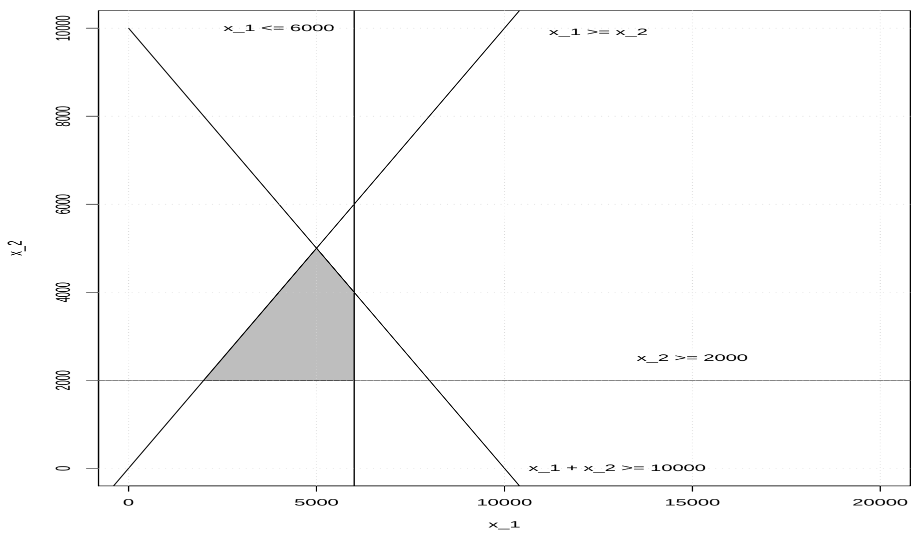

Figure 5: The feasible region for Problem 4.

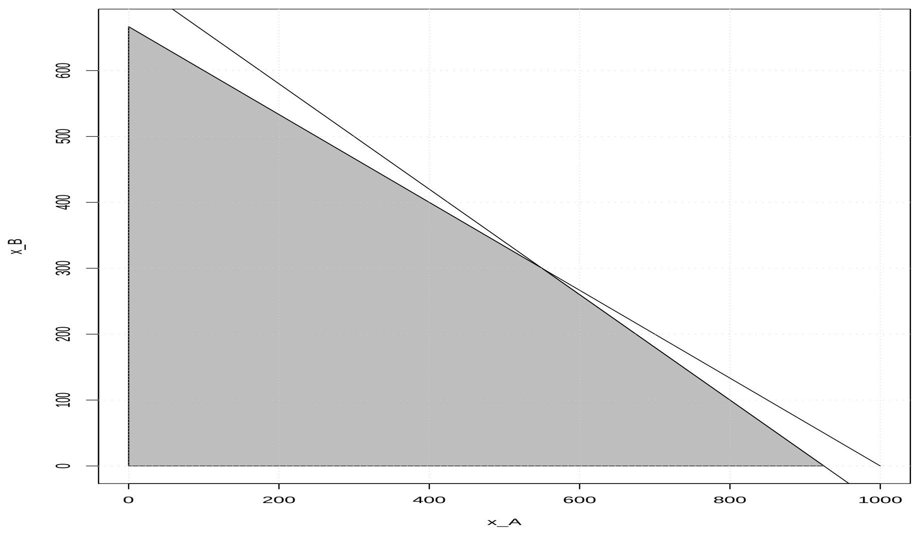

Figure 6: The feasible region for Problem 6.

Thus the maximum is given at the point $(x_1, x_2) = (6000, 4000)$ with a value of 880.

Problem 13.6 (maximizing the shipping costs)

Let x_A and x_B be the number of containers of A and B shipped in the truck. For this problem, we want to maximize the shipping charge, z, which is given in terms of the values of x_A and x_B by

$$z = 2.2x_A + 3.0x_B .$$

The constraints on x_A and x_B (related to the allowed weight and volume in the truck) are given by

$$\begin{aligned} 40x_A + 50x_B &\leq 37000 \\ 2x_A + 3x_B &\leq 2000 . \end{aligned}$$

The the above constraints we also have the nonnegativity constraints $x_A \geq 0$ and $x_B \geq 0$. This feasible region is drawn in Figure 6. The extreme points for this problem are given by

$$(0, 0) , \quad (3700/4, 0) , \quad (550, 300) , \quad (0, 2000/3) .$$

These have values of the objective function give by

$$0 , \quad 2035 , \quad 2110 , \quad 2000 .$$

Thus the largest of these happens at the point $(x_A, x_B) = (550, 300)$.

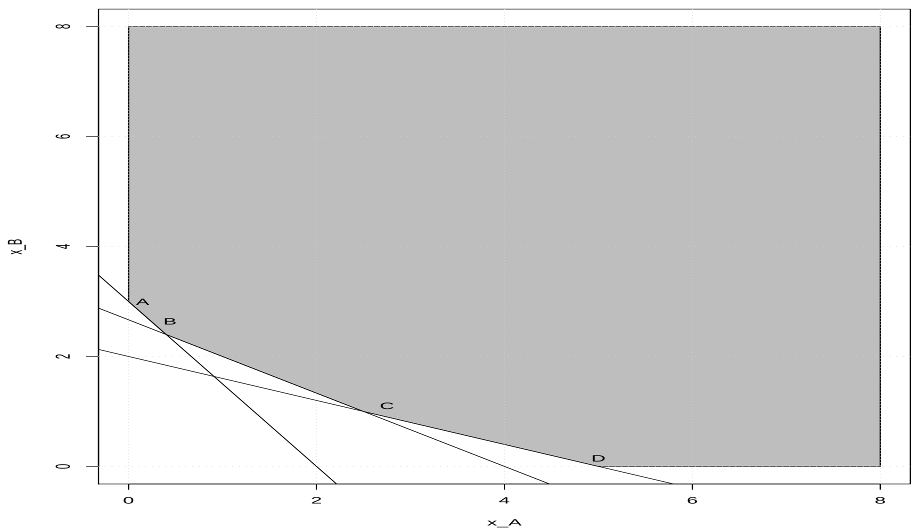

Figure 7: The feasible region for Problem 8.

Problem 13.7 (a new objective function for shipping costs)

In this case the objective function becomes $z = 2.5x_A + 3.0x_B$. The extreme points don't change when only the objective function changes. Thus we need to only evaluate this new objective function at the same extreme points found before. We find

$$0.0\,, \quad 2312.5\,, \quad 2275.0\,, \quad 2000.0\,.$$

Thus the optimum under this new objective function is $(x_A, x_B) = (3700/4, 0)$.

Problem 13.8 (the diet problem)

Let x_A and x_B be the amounts (in pounds) of ingredient A and B used to make the feed mixture. We have to satisfy the following constraints on the nutritional content N_1, N_2, and N_3 given by

$$\begin{aligned} 2x_A + 5x_B &\geq 10 \\ 2x_A + 3x_B &\geq 8 \\ 6x_A + 4x_B &\geq 12\,, \end{aligned}$$

while at the same time minimizing the objective function $z = 0.08x_A + 0.09x_B$. This feasible region is drawn in Figure 7. The extreme points for this region and the objective function at these points are found to be

```
  x_A x_B objective
D 5.0 0.0     0.400
```

A	0.0	3.0	0.270
B	0.4	2.4	0.248
C	2.5	1.0	0.290

Here the rows are denoted with the same label A-D that is found in Figure 7. The objective function is minimized at the point $(x_A, x_B) = (0.4, 2.4)$.

Problem 13.9 (an infinite number of solutions)

Assume that our objective function is of the form $z = c_1x_1 + c_2x_2$ and that at the two points (x_1', x_2') and (x_1'', x_2'') have the same value of the the objective function z^*. Then on the streight line between them we have the objective function given by

$$\begin{aligned} c_1x_1 + c_2x_2 &= c_1(tx_1' + (1-t)x_1'') + c_2(tx_2' + (1-t)x_2'') \\ &= t(c_1x_1' + c_2x_2') + (1-t)(c_1x_1'' + c_2x_2'') \\ &= tz^* + (1-t)z^* = z^*\,, \end{aligned}$$

the same value of the objective function at either of the two points (x_1', x_2') and (x_1'', x_2'').

Problem 13.10 (example 13.8)

To maximize the objective function $z = -5x_1 + x_2$ we want to move from any given point in the direction of the gradient of z. That is in the direction of the vector $(-5, +1)$. Looking at the feasible region drawn in Figure 13.5 we see that moving in this direction will place us at the point $(1, 6)$.

Chapter 14 (Linear Programming II)

Problem Solutions

Problem 14.1 (converting to the standard form for linear programming)

The standard form for linear programming is defined by

- Converting a minimization problem into a maximization problem by taking the negative of the objective function if needed
- Converting all "greater than" inequalities into "less than" inequalities
- Add *positive* slack variables to turn all "less than" inequalities into true equalities

The first two steps for the problem given here are to maximize

$$z = -2x_1 - 5x_2\,,$$

subject to

$$\begin{aligned} 3x_1 - 6x_2 &\leq 2 \\ x_1 + x_2 &\leq 3 \\ -x_1 &\leq -6 \\ x_2 &\leq 5 \\ x_1, x_2 &\geq 0\,. \end{aligned}$$

For the third step we introduce the slack variables $x_3, x_4, x_5, x_6 \geq 0$ to turn the less than inequalities into equalities we get

$$\begin{aligned} 3x_1 - 6x_2 + x_3 &= 2 \\ x_1 + x_2 + x_4 &= 3 \\ -x_1 + x_5 &= -6 \\ x_2 + x_6 &= 5\,. \end{aligned}$$

Thus in total, our standard form for the given linear program is for us to maximize

$$z = -2x_1 - 5x_2\,,$$

subject to

$$\begin{aligned} 3x_1 - 6x_2 + x_3 &= 2 \\ x_1 + x_2 + x_4 &= 3 \\ -x_1 + x_5 &= -6 \\ x_2 + x_6 &= 5 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0\,. \end{aligned}$$

Problem 14.2 (more standard form)

Following the steps outlined in the book and in the previous problem we would maximize

$$z = -3x_1 + x_2 + x_3 \,,$$

subject to

$$\begin{aligned} 3x_1 - 5x_2 + x_3 &= 3 \\ -2x_1 - x_2 + x_4 &= 2 \\ -x_1 + x_5 &= -5 \\ -x_2 + x_6 &= -2 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \,. \end{aligned}$$

Problem 14.3

Part (a): Note that from $x_1 + x_2 + 2x_3 = 2$ and the fact that $x_i \geq 0$ (so that each term in the previous sum is nonnegative) we then must have that

$$\begin{aligned} x_1 &\leq 2 \\ x_2 &\leq 2 \\ 2x_3 &\leq 2 \,. \end{aligned}$$

Thus x_1, x_2 and x_3 are at least bounded above by 2. From the constraint $2x_1 + 4x_3 + x_4 = 1$ using the same logic as above we can conclude that each term in the linear constraint must be less than 1 and in particular that $x_4 \leq 1$. Thus each x_i is bounded above and as $||x||^2$ is the sum of squared terms each of which is bounded the norm must also be bounded.

Part (b): Here $A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}$ so $m = 2$ and $n = 4$. Recall that basic solutions are ones where we set $n - m = 2$ values of x_1, x_2, x_3 or x_4 to zero and solve for the other $m = 2$ variables. We need to pick two variables to set equal to zero and then solve. If $x_1 = x_2 = 0$ we have the system

$$\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} .$$

If $x_1 = x_3 = 0$ we have the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

If $x_1 = x_4 = 0$ we have the system

$$\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.25 \end{bmatrix} .$$

If $x_2 = x_3 = 0$ we have the system

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} .$$

If $x_2 = x_4 = 0$ we have the system

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \text{is a singular system} .$$

If $x_3 = x_4 = 0$ we have the system

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix} .$$

If we use these results with the zero values assigned we get for the basic solutions to this problem

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \end{bmatrix} , \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1.5 \\ 0.25 \\ 1 \end{bmatrix} , \begin{bmatrix} 2 \\ 0 \\ 0 \\ -3 \end{bmatrix} , \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \\ 0 \end{bmatrix} .$$

Part (c): To be a basic *feasible* solution one must be a basic solution that satisfies the nonnegativity constraints. From the basic solutions above the only ones that satisfy these nonnegativity requirement are

$$\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1.5 \\ 0.25 \\ 1 \end{bmatrix} , \begin{bmatrix} 0.5 \\ 1.5 \\ 0 \\ 0 \end{bmatrix} .$$

Part (d): We can evaluate the optimization objective for this problem at each of the basis feasible solutions and then the optimum is the basic feasible solution that has the largest objective function. For the basic feasible solutions found for this problem the value of the objective function are 1.0, -1.25, and -0.5. Thus $\begin{bmatrix} 0 & 2 & 0 & 1 \end{bmatrix}^T$ gives the optimal solution to this problem. Some of the computations for this problem can be found in the Octave function `chap_14_prob_3.m`.

Problem 14.4 (an example with no basis feasible solutions)

Part (a): For this problem we have $A = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 2 & 4 \end{bmatrix}$ so $m = 2$ and $n = 3$ thus we need to set $n - m = 1$ variables equal to zero and then solve for the $m = 2$ nonzero variables. If $x_1 = 0$ we have

$$\begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.07 \\ 0.78 \end{bmatrix} .$$

If $x_2 = 0$ we have

$$\begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.14 \\ 0.71 \end{bmatrix}.$$

If $x_3 = 0$ we have

$$\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1.57 \\ 0.71 \end{bmatrix}.$$

Part (b): As none of the basis solutions have all positive entries there are no *feasible* solutions to this problem and the feasible set for this problem is empty. Some of the computations for this problem can be found in the Octave function `chap_14_prob_4.m`.

Problem 14.5

Part (a): For this problem we need to introduce two slack variables $x_4 \geq 0$ and $x_5 \geq 0$ such that the inequalities become equalities. This means the linear constraints we have to satisfy are given by

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + x_4 &= 4 \\ x_1 + 2x_2 + 3x_3 + x_5 &= 5\,, \end{aligned}$$

with $x_1, x_2, x_3, x_4, x_5 \geq 0$.

Part (b): For this problem $A = \begin{bmatrix} 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & 1 \end{bmatrix}$. Here $m = 2$ and $n = 5$ so we must set $n - m = 3$ variables equal to zero, in other words select two variables which will be nonzero. We find

$$\begin{bmatrix} -7 \\ 6 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.4 \\ 0 \\ 1.2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2.5 \\ 0 \\ -3.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1.333 \\ 0 \\ 0 \\ 2.333 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 \\ 0 \\ 1.666 \\ 2.333 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}.$$

Part (c): To be feasible means that all elements of our basic solution must be nonzero. Only the second, fourth, fifth, seventh, eighth, and tenth basic solutions above have this property. Evaluating the objective function at each basic feasible solution gives the values

$$3\,, \quad 6\,, \quad 1\,, \quad 2.666\,, \quad -1.666\,, \quad 0.0\,.$$

The largest value is for the basic feasible solution of $\begin{bmatrix} 2 & 0 & 0 & 0 & 3 \end{bmatrix}^T$

Part (f): The original solution is $x_1 = 2$, $x_2 = 0$, $x_3 = 0$ with an objective function value of $z = 6$.

Some of the computations for this problem can be found in the Octave function `chap_14_prob_5.m`.

Problem 14.6 (solving a linear program)

Part (a): For this problem we first need to convert the minimization problem into a maximization problem i.e. our objective function is now to maximize

$$z = -2x_1 + 3x_2 - 3x_3 \,.$$

Note that the solution in the back of the book has to maximize the objective function $z = -2x_1 - 3x_2 - 3x_3$. I think this is a typo since this is not the negative of the initial expression for z. In addition to converting the problem from a minimization problem to a maximization problem we need to introduce one slack variables $x_4 \geq 0$ such that the inequality becomes an equality. This means the linear constraints we have to satisfy is given by

$$x_1 - 2x_2 + 3x_3 + x_4 = 5 \,.$$

with $x_1, x_2, x_3, x_4 \geq 0$.

Part (b): For this problem we have $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -2 & 0 \end{bmatrix}$, thus $m = 2$ and $n = 4$. To find the basic solutions we need to set $m - n = 2$ variables to zero and solve for the other $m = 2$ variables. If $x_1 = x_2 = 0$ we have the system

$$\begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \end{bmatrix} .$$

If $x_1 = x_3 = 0$ we have the system

$$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix} .$$

If $x_1 = x_4 = 0$ we have the system

$$\begin{bmatrix} -2 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -16 \\ -9 \end{bmatrix} .$$

If $x_2 = x_3 = 0$ we have the system

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} .$$

If $x_2 = x_4 = 0$ we have the system

$$\begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

If $x_3 = x_4 = 0$ we have the system

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.8 \\ -1.6 \end{bmatrix}.$$

Part (c): To be feasible means that our solutions must satisfy the nonnegativity constraints. From the above the only solutions that do this are the second, fourth, and fifth.

Part (d): We can evaluate our objective function $z = -2x_1 + 3x_2 - x_3 + 0x_4$ on each basic feasible solution to determine the basic feasible solution that is maximal. Evaluating our objective function on each of the above basic feasible solution gives the values of 6, -2, -5. The largest is the value of 6 which happens with $x_1 = 2$, $x_2 = 9$, $x_3 = 0$, and $x_4 = 0$.

Part (f): In terms of the original problem our solution is given by $x_1 = 2$, $x_2 = 9$, and $x_3 = 0$ with an objective function value of 6.

Problem 14.7 (the feasible set is a convex set)

Part (a): Note that

$$Ax = tAx_1 + (1-t)Ax_2 = tb + (1-t)b = b\,.$$

Part (b): If $x_1 \geq 0$ and $x_2 \geq 0$ then $tx_1 \geq 0$ and $(1-t)x_2 \geq 0$ so their sum is nonnegative (when $0 \leq t \leq 1$).

Problem 14.8 (the feasible set of example 14.2)

We want to find the line that is the intersection of the two planes $3x_1 + x_2 + x_3 = 10$ and $2x_1 - x_2 + 2x_3 = 10$. We will do this by writing these two equations in a matrix form and performing elementary row operations to derive the row reduced echelon form for this system. With the resulting relationship we will be able to more easily express the variables x_1 and x_2 in terms of x_3, which with the nonnegativity constraints on x_1, x_2, and x_3 will enable us to evaluate the limiting points on this line that still give feasible solutions. To derive the row reduced echelon form we compute

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 10 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & -5/3 & 4/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 10/3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & -4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/3 \\ -2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & -4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

From this last expression we see that we can write x_1 and x_2 in terms of x_3 as

$$\begin{aligned} x_1 &= -\frac{3}{5}x_3 + 4 \\ x_2 &= \frac{4}{5}x_3 - 2\,. \end{aligned}$$

If we let $x_3 = 5t$ we get $x_1 = -3t + 4$ and $x_2 = 4t - 2$. As we vary t we trace out the line that is the intersection of the two planes. We can now apply the nonnegativity constraints on x_1, x_2, and x_3 to determine feasible values of t. To have $x_3 \geq 0$ we must have $t \geq 0$. To have $x_1 \geq 0$ we must have

$$-3t + 4 \geq 0 \quad \text{so} \quad t \leq \frac{4}{3}\,.$$

To have $x_2 \geq 0$ we must have

$$4t - 2 \geq 0 \quad \text{so} \quad t \geq \frac{1}{2}\,.$$

The intersection (over t) of all of the valid regions means that we must have $\frac{1}{2} \leq t \leq \frac{4}{3}$. The two end points of this line are then given by the points (x_1, x_2, x_3) when we take t equal to the values $\frac{1}{2}$ and $\frac{4}{3}$. We find

$$\begin{aligned} t = \frac{1}{2} &\Rightarrow (x_1, x_2, x_3) = \left(\frac{5}{2}, 0, \frac{5}{2}\right) \\ t = \frac{4}{3} &\Rightarrow (x_1, x_2, x_3) = \left(0, \frac{10}{3}, \frac{20}{3}\right)\,. \end{aligned}$$

Problem 14.9 (show the feasible set is bounded)

Ex 14.5 is the problem of maximizing $z = 2x_1 + 3x_2 - x_3 + 4x_5 + x_6$ subject to

$$\begin{aligned} 2x_1 + x_3 + 4x_4 + 2x_5 &= 20 \\ x_1 + x_2 - x_3 + x_4 + x_5 &= 10 \\ x_1, x_2, x_3, x_4, x_5 &\geq 0\,. \end{aligned}$$

Part (a): Given that $2x_1 + x_3 + 4x_4 + 2x_5 = 20$ and that all $x_i \geq 0$ that

$$\begin{aligned} 2x_1 &\leq 20 \\ x_3 &\leq 20 \\ 4x_4 &\leq 20 \\ 2x_5 &\leq 20\,. \end{aligned}$$

All of these imply that $x_i \leq 20$ for $i = 1, 3, 4, 5$.

Part (b): The other constraint of $x_1+x_2-x_3+x_4+x_5 = 10$ when added to the first constraint to remove the x_3 term gives a linear equation with all positive terms i.e. everything is added together. We get

$$3x_1 + x_2 + 5x_4 + 3x_5 = 30\,.$$

Using the same logic as before we have

$$\begin{aligned}
3x_1 &\leq 30 \\
x_2 &\leq 30 \\
5x_4 &\leq 30 \\
3x_5 &\leq 30\,,
\end{aligned}$$

which implies that $x_2 \leq 30$.

Part (c): As $||x||^2$ is the sum of the square of terms that are bounded the total expression is bounded.

Chapter 14 (Linear Programming III: The Simplex Method)

Problem Solutions

Problem 14.1

In standard form this problem is to maximize

$$z = 3x_1 + 4x_2 + 0x_3 + 0x_4 \,.$$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 7 \\ 5x_1 + 2x_2 + x_4 &= 3 \,, \end{aligned}$$

and $x_1, x_2, x_3, x_4 \geq 0$. As a tableau this problem is

$$\begin{array}{c} \left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \\ 2 & 3 & 1 & 0 & 0 & 7 \\ 5 & 2 & 0 & 1 & 0 & 3 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ = x_3 \\ = x_4 \\ = z \end{array}$$

Picking the most negative element in the objective row gives the second column as the pivot column and the variable x_2 is the entering variable. To determine the pivot row consider the ratios:

$$\text{1st row: } \frac{7}{3} = 2.3 \,, \quad \text{2nd row: } \frac{3}{2} = 1.5 \,.$$

The smaller of these corresponds to row two and row two is our pivot row. We now seek to transform the pivot entry from the value of two to the value of one by

- Divide the second row by the value of 2
- Add -3 times the second row to the fist row
- Add 4 times the second row to the third row

When we do this we get the new tableau of

$$\begin{array}{c} \left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & z & \\ -\frac{11}{2} & 0 & 1 & -\frac{3}{2} & 0 & \frac{5}{2} \\ \frac{5}{2} & 1 & 0 & \frac{1}{2} & 0 & \frac{3}{2} \\ \hline 7 & 0 & 0 & 2 & 1 & 6 \end{array}\right] \end{array} \begin{array}{l} \\ = x_3 \\ = x_2 \\ = z \end{array}$$

As there are no negative elements in the objective row the simplex method has found the the solution of $x_1 = 0$, $x_2 = \frac{3}{2}$ and $z = 6$. The numerics for this problem are worked in the Octave file `chap_15_prob_1.m`.

Problem 14.2

In standard form this problem is to maximize

$$z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5\,.$$

subject to the constraints

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_4 &= 3 \\ 2x_1 + x_5 &= 3\,, \end{aligned}$$

and $x_1, x_2, x_3, x_4, x_5 \geq 0$. As a tableau this problem is

$$\begin{array}{cc}
\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \end{array} & \\
\left[\begin{array}{cccccc|c}
3 & 2 & 1 & 0 & 0 & 0 & 4 \\
3 & 1 & 0 & 1 & 0 & 0 & 3 \\
2 & 0 & 0 & 0 & 1 & 0 & 3 \\
\hline
-2 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right] & \begin{array}{l} = x_3 \\ = x_4 \\ = x_5 \\ = z \end{array}
\end{array}$$

The most negative element in the objective row is -2, in the column for x_1. Thus x_1 is the entering variable and x_1 is the pivot column. To determine the pivot row we consider the ratios

$$\text{1st row: } \frac{4}{3}\,, \quad \text{2nd row: } \frac{3}{3}\,, \quad \text{3rd row: } \frac{3}{2}\,.$$

WE pick the smallest ratio which corresponds to row 2 with a value of 1. Thus row two is our pivot row. We now transform our pivot column of $\begin{bmatrix} 3 & 3 & 2 & -2 \end{bmatrix}^T$ into $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$. We do this by

- Dividing the second row by 3
- Adding -3 times the second row to the first row
- Adding -2 times the second row to the third row
- Adding 2 times the second row to the fourth row

This gives the tableau

$$\begin{array}{cc}
\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \end{array} & \\
\left[\begin{array}{cccccc|c}
0 & 1 & 1 & -1 & 0 & 0 & 1 \\
1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 1 \\
0 & -\frac{2}{3} & 0 & -\frac{2}{3} & 1 & 0 & 1 \\
\hline
0 & -\frac{1}{3} & 0 & \frac{2}{3} & 0 & 1 & 2
\end{array}\right] & \begin{array}{l} = x_3 \\ = x_1 \\ = x_5 \\ = z \end{array}
\end{array}$$

The most negative entry in the objective row is $-1/3$ so our pivot column is x_2. Our pivot row is the smallest of

$$\text{1st row: } 1\,, \quad \text{2nd row: } \frac{1}{\frac{1}{3}} = 3\,.$$

Note that we do not divide by the number in row three since we would have to divide by $-\frac{2}{3}$ but that term is negative. The smallest between the two choices above is for row 1 and we want to transform our pivot column vector $\left[\begin{array}{cccc} 1 & 1/3 & -2/3 & -1/3 \end{array}\right]^T$ into $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]^T$. We can do this by

- Adding $-1/3$ times the first row to the second row
- Adding $2/3$ times the first row to the third row
- Adding $1/3$ times the first row to the fourth row

This given the tableau

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \frac{2}{3} & -\frac{4}{3} & 1 & 0 & \frac{5}{3} \\ \hline 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 1 & \frac{7}{3} \end{array}\right] \begin{array}{l} \\ = x_2 \\ = x_1 \\ = x_5 \\ = z \end{array}$$

As all elements of the objective row are positive the optimal solution has been found and it is

$$x_1 = \frac{2}{3}, \quad x_2 = 1, \quad z = \frac{7}{3}.$$

The numerics for this problem are worked in the Octave file `chap_15_prob_2.m`.

Problem 14.3

This problem is to maximize

$$z = 3x_1 - 2x_2 + 6x_4 + 0x_4 + 0x_5,$$

subject to the constraints that

$$\begin{aligned} 2x_1 - 5x_2 + x_3 + x_4 &= 2 \\ x_1 + x_2 + x_3 + x_5 &= 5, \end{aligned}$$

with $x_1, x_2, x_3, x_4, x_5 \geq 0$. The initial tableau for this problem is

$$\left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ 2 & -5 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 & 0 & 5 \\ \hline -3 & 2 & -6 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ = x_4 \\ = x_5 \\ = z \end{array}$$

The pivot column corresponds to the variable x_3 and the pivot row is 1. We thus want to change $\left[\begin{array}{cccc} 1 & 1/3 & -2/3 & -1/3 \end{array}\right]^T$ into $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]^T$. We can do this by

- Dividing the second row by 6

- Add five times the second row to the first row
- Add 28 times the second row the third row

This given the tableau

$$\begin{array}{c} \left[\begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & z & \\ \frac{7}{6} & 0 & 1 & \frac{1}{6} & \frac{5}{6} & 0 & \frac{9}{2} \\ -\frac{1}{6} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \\ \hline \frac{13}{3} & 0 & 0 & \frac{4}{3} & \frac{14}{3} & 1 & 26 \end{array}\right] \end{array} \begin{array}{l} \\ = x_3 \\ = x_2 \\ = z \end{array}$$

The objective row has all positive elements so we have found the optimal solution. In this case it is given by

$$x_1 = 0\,, \quad x_2 = \frac{1}{2}\,, \quad x_3 = \frac{9}{2}\,, \quad \text{with} \quad z = 26\,.$$

The numerics for this problem are worked in the Octave file `chap_15_prob_3.m`.

Problem 14.4

This problem is to maximize

$$z = 2x_1 + x_2 - x_3 + 0x_4 + 0x_5 + 0x_6\,,$$

subject to the constraints that

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + x_4 &= 2 \\ x_1 + 5x_2 - 2x_3 + x_5 &= 4 \\ 2x_1 - 4x_2 - x_3 + x_6 &= 3\,, \end{aligned}$$

with $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. The initial tableau for this problem is

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & \\ 2 & -3 & 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 5 & -2 & 0 & 1 & 0 & 0 & 4 \\ 2 & -4 & -1 & 0 & 0 & 1 & 0 & 3 \\ \hline -2 & -1 & +1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ = x_4 \\ = x_5 \\ = x_6 \\ = z \end{array}$$

From the above, the first column is the pivot column and the pivot row is the smallest value of

$$\text{1st row: } \frac{2}{2}\,, \quad \text{2nd row: } \frac{4}{1} = 4\,, \quad \text{3rd row: } \frac{3}{2}\,.$$

showing that the first row is the pivot row. Continuing in the same way as the previous problems we finish this problem in the Octave file `chap_15_prob_4.m` where we find the solution

$$x_1 = \frac{22}{13}\,, \quad x_2 = \frac{6}{13}\,, \quad x_3 = 0\,, \quad \text{with} \quad z = \frac{50}{13}\,.$$

Problem 14.5

This problem is to maximize

$$z = 3x_1 - 2x_2 - x_3 + x_4 + 0x_5 + 0x_6\,,$$

subject to the constraints that

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - x_4 + x_5 &= 6 \\ x_1 + 2x_2 - x_3 + 2x_4 + x_6 &= 4\,, \end{aligned}$$

with $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. The initial tableau for this problem is

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & z & \\ 2 & -3 & 1 & -1 & 1 & 0 & 0 & 6 \\ 1 & 2 & -1 & 2 & 0 & 1 & 0 & 4 \\ \hline -3 & 2 & 1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} \\ = x_5 \\ = x_6 \\ = z \end{array}$$

For this problem, in the Octave file `chap_15_prob_5.m` we find the optimal solution using the simplex method to be

$$x_1 = 3.2\,, x_2 = 0\,, x_3 = 0\,, x_4 = 0.4\,, \quad \text{with} \quad z = 10\,.$$

Problem 14.6-10

Rather than repeat these same calculations for the rest of the problem we will code the simplex method in an Octave function in the file `simplex.m` and then solve each problem using this code. This code will be given the initial tableau and will print the tableau at each stage of the calculation (as done above) and then finally end when the optimal solution has been obtained. Here I'll just print the final tableau. See the Octave file `chap_15_prob_6_10.m` for the set up for each of the given problems.

Problem 14.11

This statement is just the fact that all of the constraint rows do not include the objective function in their expression. That is, recall that a constraint row (with slack variables) looks like

$$\sum_{j=1}^{n+m} a_{ij} x_j = d_i\,.$$

Notice that z does not appear in this expression.