# Solutions For Selected Exercises In: Solving ODE's with MATLAB by Lawrence Shampine, Ian Gladwell, and Skip Thompson

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## Problem Solutions

### Chapter 1

#### Problem 1.1

Since the differential equation is *not* in the form

$$y'(t) = f(t, y(t)) \tag{1}$$

one cannot use the simple Lipshitz condition

$$||f(t, u) - f(t, v)|| \leq L||u - v|| \tag{2}$$

to prove existance and uniqueness of this O.D.E. The equation

$$y'^2 + y^2 = 1 \qquad y(0) = 0 \tag{3}$$

can be written as:

$$y'(t) = \pm\sqrt{1 - y^2} \qquad y(0) = 0 \tag{4}$$

which shows that *two* solutions will exist. Thus this result does not violate any of the existance/uniqueness results stated earlier.

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**Problem 1.2**

The equation is

$$y'(t) = \sqrt{|y|} \tag{5}$$

and a question to ask is does this right hand side satisfy a Lipshitz condition? Or mathematically:

$$|\sqrt{|u|} - \sqrt{|v|}| \leq L|u - v| \tag{6}$$

From the text we know that a function $f(t, y)$ will satisfy a Lipshitzs condition if $|\frac{\partial f}{\partial y}|$ is bounded by a constant. Taking derivatives we get

$$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{|u|}}\begin{cases} +1 & u > 0 \\ -1 & u < 0 \end{cases} \tag{7}$$

From the above expression this derivative is *not* bounded as $u \to 0$. By setting $v \equiv 0$ the Lipshitz condition now requires:

$$\sqrt{|u|} \leq L|u| \qquad \forall u \in [-1, +1] \tag{8}$$

this is equivalent to the following:

$$\frac{1}{\sqrt{|u|}} \leq L \tag{9}$$

which can obviously not be true for small enought $u$. In fact for any $|u| < \frac{1}{L^2}$ Thus $f(y) = +\sqrt{|y|}$ is *not* Lipshitz on this domain. To show that this differential equation may have more than one solution assume $y > 0$ for all time and consider the following

$$y'(t) = \sqrt{y} \qquad y(0) = 0 \tag{10}$$

this is equivalent to

$$\frac{dy}{\sqrt{y}} = dt\,. \tag{11}$$

Integrating both sides of the above we get

$$2\sqrt{y} + C = t \tag{12}$$

or

$$y = (\frac{t - c}{2})^2 = \frac{1}{4}(t - c)^2 \tag{13}$$

Imposing the initial condition gives $c = 0$ and $y(t) = \frac{t^2}{4}$. Since indeed the function $y$ is positive for all $t > 0$ our assumptions are valid. To find another solution assume now that $y < 0$ for all time and consider the following

$$y'(t) = \sqrt{-y} \qquad y(0) = 0 \tag{14}$$

this is equivalent to

$$\frac{dy}{\sqrt{-y}} = dt\,. \tag{15}$$

Integrating both sides of the above we get

$$-2\sqrt{-y} + C = t \tag{16}$$

or

$$y = (-\frac{t-c}{2})^2 = -\frac{1}{4}(t-c)^2 \tag{17}$$

Imposing the initial condition gives $c = 0$ and $y(t) = -\frac{t^2}{4}$. Again since indeed the solution $y$ is everywhere negative our manipulations are justified.

By the derivative argument presented above $\sqrt{|y|}$ is not Lipshits when $|y| \leq 1$ but by the same arguments $f(y)$ will be Lipshitz $y \in [\alpha, 1)$ since in this region the first derivative is bounded:

$$|\frac{\partial f}{\partial y}| = |\frac{1}{2\sqrt{y}}| \leq \frac{1}{2\sqrt{\alpha}} \tag{18}$$

**Problem 1.3**

From the differential equation

$$y'(t) = \frac{1}{(t-1)(t-2)} \tag{19}$$

we can expand the right hand side in partial fractions. This is an expansion of the following form:

$$\frac{1}{(t-1)(t-2)} = \frac{A}{t-1} + \frac{B}{t-2} \tag{20}$$

Solving for $A$ and $B$ we get $A = -1$ and $B = 1$. Thus the differential equation is now

$$dy = \frac{-1}{t-1} + \frac{1}{t-2} \tag{21}$$

and both sides can be integrated. This gives:

$$y - y_0 = -\ln(t-1) + \ln(t-2) = \ln(\frac{t-2}{t-1}) \tag{22}$$

or

$$y = y_0 + \ln(\frac{t-2}{t-1}) \tag{23}$$

Imposing the initial condition $y(0) = 1$ we get the equation $1 = y_0 + \ln(2)$ or

$$y(t) = 1 - \ln(2) + \ln(\frac{t-2}{t-1}) \tag{24}$$

From this one can see that one cannot prescribe initial conditions at $t = 1$ or $t = 2$ since these correspond to singularities of the ln function.

Given the following O.D.E.

$$y'(t) = -3y^{\frac{4}{3}}\sin(t) \tag{25}$$

$$-\frac{1}{3}\frac{dy}{y^{\frac{4}{3}}} = \sin(t) \tag{26}$$

integrating both sides gives

$$-\frac{1}{3}\frac{y^{\frac{-1}{3}}}{-\frac{1}{3}} + C = -\cos(t) \tag{27}$$

or after solving for $y(t)$ we obtain

$$y(t) = \frac{1}{(C - \cos(t))^3} \tag{28}$$

Applying the initial condition $y(\frac{\pi}{2}) = 1$ results in

$$y(t) = \frac{1}{(1 - \cos(t))^3} \tag{29}$$

this function has singularities when $t$ is a multiple of $2\pi$.

**Problem 1.4**

WWX: Finish!!!

**Problem 1.5**

The solution to the following differential equation:

$$y' = 5(y - t^2) \qquad y(0) = 0.08 \tag{30}$$

is given by

$$y(t) = t^2 + 0.4t + 0.08 \tag{31}$$

To numerically compute its solution using Euler's method the following difference equation is used

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{32}$$

with a stepsize of $h = 0.1$. In this case the local solution $u$ is the solution to the following D.E.

$$u' - 5u = -5t^2 \qquad \mathrm{with u(t_n) = y_n} \tag{33}$$

The solution to this ODE consists of a homogenous part and a particular part. The homogenous part is given by

$$u(t) = Ce^{5t} \tag{34}$$

while the particular solution maybe found by considering a soltuion of the following form

$$u_p(t) = At^2 + Bt + C \tag{35}$$

Putting this equation into the local solution 33 gives $A = 1$, $B = \frac{2}{5}$, and $C = \frac{2}{25}$. Thus in total the local solution is given by

$$u(t) = t^2 + \frac{2}{5}t + \frac{2}{25} + Ce^{5t} \tag{36}$$

Using the initial condition $u(t_n) = y_n$ gives for the constant $C$ the value

$$C = e^{-5t_n}(y_n - t_n^2 - \frac{2}{5}t_n - \frac{2}{25}) \tag{37}$$

So in total the local solution at each timestep $u(t)$ is given by:

$$u(t) = t^2 + \frac{2}{5}t + \frac{2}{25} + (y_n - t_n^2 - \frac{2}{5}t_n - \frac{2}{25})e^{5(t-t_n)} \tag{38}$$

**Problem 1.6**

Let $y_1 = y$ and $y_2 = y'$. Now

$$(p(x)y')' + q(x)y(x) = r(x) \tag{39}$$

expands to give

$$p'(x)y'(x) + p(x)y''(x) + q(x)y(x) = r(x) \tag{40}$$

or

$$y''(x) = \frac{r(x) - q(x)y(x) - p'(x)y'(x)}{p(x)} \tag{41}$$

therefor the *system* to solve is the following:

$$y_1' = y_2 \tag{42}$$

$$y_2' = \frac{r(x) - q(x)y(x) - p'(x)y'(x)}{p(x)} \tag{43}$$

with initial conditions given by

$$y_1(0) = 0 \qquad \text{and} \qquad \mathrm{y}_2(1) = \frac{2}{\mathrm{p}'(1)} \tag{44}$$

This method of forming a system of ODE's can be contrasted with the following choice. Let $y_1 = y$ and $y_2 = p(x)y'$. Then the system becomes:

$$y_1' = \frac{y_2}{p(x)} \tag{45}$$

$$y_2' = -q(x)y_1(x) + r(x) \tag{46}$$

with initial conditions given by $y_1(0) = 0$ and $y_2(1) = 2$.

**Problem 1.7**

**Part (i):** For the special second order form $y'' = f(t, y)$ then

$$(y'')^2 = \frac{e^{2x}}{y} \tag{47}$$

or

$$y'' = \pm\frac{e^{2x}}{\sqrt{y}} \tag{48}$$

**Part (ii):** Let $y_1(x) = y(x)$ and $y_2(x) = y'(x)$, then our system becomes:

$$y_1'(x) = y_2(x) \tag{49}$$

$$y_2'(x) = \pm\frac{e^{x}}{\sqrt{y}} \tag{50}$$

with initial conditions of $y_1(0) = 0$ and $y_2(0) = 0$.

**Problem 1.8**

Let our individual $y_i$ be the following

$$y_1 = y \tag{51}$$

$$y_2 = y' \tag{52}$$

$$y_3 = y'' \tag{53}$$

$$y_4 = y''' \tag{54}$$

then our given ODE becomes

$$y_1' = y_2 \tag{55}$$

$$y_2' = y_3 \tag{56}$$

$$y_3' = y_4 \tag{57}$$

$$y_4' = -(\Omega + y_1)y_4 - \Omega y_1 y_3 + (2\beta - 1)(y_2 y_3 + \Omega y_2^2) \tag{58}$$

In addition to these equations define $y_5$ and $y_6$ as

$$y_5 = \int_0^\eta (1 - y_2 e^{\Omega\eta})\, d\eta \tag{59}$$

$$y_6 = \int_0^\eta y_2 e^{\Omega\eta}(1 - y_2 e^{\Omega\eta})\, d\eta \tag{60}$$

then the derivatives and initial conditions for $y_5$ and $y_6$ are given by

$$y_5' = 1 - y_2 e^{\Omega\eta} \quad \text{and} \quad \mathrm{y}_5(0) = 0 \tag{61}$$

$$y_6' = y_2 e^{\Omega\eta}(1 - y_2 e^{\Omega\eta}) \quad \text{and} \quad \mathrm{y}_6(0) = 0 \tag{62}$$

Finally, our complete set of differential equations would be

$$
\begin{aligned}
y_1' &= y_2 && (63)\\
y_2' &= y_3 && (64)\\
y_3' &= y_4 && (65)\\
y_4' &= -(\Omega + y_1)y_4 - \Omega y_1 y_3 + (2\beta - 1)(y_2 y_3 + \Omega y_2^2) && (66)\\
y_5' &= 1 - y_2 e^{\Omega\eta} && (67)\\
y_6' &= y_2 e^{\Omega\eta}(1 - y_2 e^{\Omega\eta}), && (68)
\end{aligned}
$$

wit initial/boundary conditions given by

$$
\begin{aligned}
y_1(0) &= 0 && (69)\\
y_2(0) &= 0 && (70)\\
y_2(b) &= e^{-\Omega b} && (71)\\
y_3(b) &= -\Omega e^{-\Omega b} && (72)\\
y_5(0) &= 0 && (73)\\
y_6(0) &= 0 && (74)
\end{aligned}
$$

**Problem 1.9**

Let $y_1 = \mu$ and $y_2 = -\mu'$, then

$$
\begin{aligned}
y_1' &= y_2 && (75)\\
y_2' &= -\omega^2 \left( \frac{1-\alpha^2}{H} \left( \frac{1}{\sqrt{1+\mu^2}} \right) + \alpha^2 ) y_1 \right) && (76)
\end{aligned}
$$

as suggested in the text let $y_3$ be defined by

$$y_3 = \frac{1}{\alpha^2}[1 - (1-\alpha^2) \int_0^\alpha \frac{d\chi}{\sqrt{1 + y_1(\chi)^2}} d\chi] \qquad \text{then} \qquad \mathrm{y}_3(0) = \frac{1}{\alpha^2} \tag{77}$$

The derivative of $y_3$ is then

$$y_3' = -\frac{1-\alpha^2}{\alpha^2} \frac{1}{\sqrt{1+y_1^2}} \tag{78}$$

in addition let $y_4$ be defined by $y_4 = H$, with derivative given by $y_4' = 0$. With these definitions the total system is given by

$$
\begin{aligned}
y_1' &= y_2 && (79)\\
y_2' &= -\omega^2 \left( \frac{1-\alpha^2}{H} \left( \frac{1}{\sqrt{1+\mu^2}} \right) + \alpha^2 ) y_1 \right) && (80)\\
y_3' &= -\frac{1-\alpha^2}{\alpha^2} \frac{1}{\sqrt{1+y_1^2}} && (81)\\
y_4' &= 0 && (82)
\end{aligned}
$$

with initial conditions given by:

$$\begin{aligned} y_1(0) &= \epsilon & (83)\\ y_2(0) &= 0 & (84)\\ y_2(1) &= 0 & (85)\\ y_3(0) &= 1/\alpha^2 & (86)\\ y_4(1) &= \frac{1}{\alpha^2}[1-(1-\alpha^2)y_3(1)] & (87) \end{aligned}$$

**Problem 1.10**

I was not sure how to do this problem.

**Problem 1.11**

Eq. 120 in the book is

$$|y_i(t_n) - y_{n,i}| \leq re|y_i(t_n)| + ae_i \tag{88}$$

with $re = ae_i = \tau$

**Problem 1.12**

The differential equation is

$$y'(t) = \sqrt{1-y^2} \qquad \text{with} y(0) = 0 \tag{89}$$

and this maybe satisfied by $y = \sin(t)$. At $t = \pi/2$ we have $\sin(\pi/2) = 1$ and the expression $1-y^2$ might become negative due to round off. In addition, $f(t, y) = \sqrt{1-y^2}$ is *not* Lipshitz on $0 \leq y \leq 1$, since

$$\frac{\partial f}{\partial y} = \frac{-2y}{\sqrt{1-y^2}} \tag{90}$$

is unbounded as $y \to 1$. Therefore uniqueness may fail to hold as well.

**Problem 1.13**

The differential equation is

$$y'(t) = \left(\frac{2\ln(y)+8}{t} - 5\right) y(t) \qquad \text{with} y(1) = 1\,. \tag{91}$$

Note that $\ln(y)$ is *complex* for $y < 0$. With analytic solution given by

$$y(t) = e^{-t^2+5t-4} \qquad fort \gg 1 \tag{92}$$

we see that $y(t) \ll 1$ and $\lim y(t) = 0$ as $t \to \infty$. Thus numerically, the solution $y(t)$ can fall below zero.

**Problem 1.14**

Writing out a few terms of the given differential equations we obtain:

$$\begin{aligned}
y_1' &= -y_1 \\
y_2' &= y_1 - 2y_2 \\
y_3' &= 2y_2 - 3y_3 \\
&\vdots \\
y_9' &= 8y_8 - 9y_9 \\
y_{10}' &= 9y_9
\end{aligned}$$

We see that by adding each row in the above set of equations we get:

$$\sum_{k=1}^{10} y_k' = 0 \tag{93}$$

Thus if $c$ is a 10 component column vector with entries of all ones, then $c^T y(t)$ is constant for all time and corresponingly this system of ODE's satisfies a conservation law.

**Problem 1.15**

I think there is a typo in the discription of the Volterra predator-prey model. I belive the equations should read:

$$\begin{aligned}
x' &= ax(1-y) \\
y' &= -cy(1-x)
\end{aligned}$$

In this case the derivative of $G \equiv x^{-c}y^{-a}e^{cx+ay}$ with respect to $t$ is:

$$\frac{dG}{dt} = -cx^{-c-1}x'y^{-a}e^{cx+ay} + -ax^{-c}y^{-a-1}y'e^{cx+ay} + x^{-c}y^{-a}(cx' + ay')e^{cx+ay}$$

$$
\begin{aligned}
&= x^{-c}y^{-a}e^{cx+ay}(-cx^{-1}x' - ay^{-1}y' + cx' + ay') \\
&= x^{-c}y^{-a}e^{cx+ay}(-cx^{-1}(ax(1-y)) - ay^{-1}(-cy(1-x)) + cx' + ay') \\
&= x^{-c}y^{-a}e^{cx+ay}(-ca(1-y) + ac(1-x) + acx(1-y) - acy(1-x)) \\
&= x^{-c}y^{-a}e^{cx+ay}(-ca + acy + ac - acx + acx - acxy - acy + acxy) \\
&= 0
\end{aligned}
$$

Using the Volterra equations above. Eulers method for $y' = f(t, y)$ (with constant stepsize $h$) is given by

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{94}$$

For the problem at hand this becomes:

$$
\begin{aligned}
x_{n+1} &= x_n + hax_n(1 - y_n) \\
y_{n+1} &= y_n - hcx_n(1 - y_n)
\end{aligned}
$$

For an implementation of this, see the matlab code prob_1_15.m.

# Chapter 2

## Problem 2.1

The consistency condition we need is

$$\frac{1}{k} = \sum_{j=1}^{s} A_j \alpha_j^{k-1} \qquad \text{fork} = 1, 2, \ldots, \text{p} \tag{95}$$

For the first example we have

$$
\begin{aligned}
\int_a^b f(x)\,dx &\approx \frac{b-a}{6}\{f(a + 0(b-a)) + 4f(a + \frac{b-a}{2}) + f(a + (b-a))\} && (96) \\
&= \frac{h}{6}\{f(a + 0h) + 4f(a + \frac{h}{2}) + f(a + h)\} && (97)
\end{aligned}
$$

with $h = b - a$. Then we see that $s = 3$, $A_1 = 1/6$, $A_2 = 2/3$, $A_3 = 1/6$, $\alpha_1 = 0$, $\alpha_2 = 1/2$, and $\alpha_3 = 1$. So with $k = 1$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^0 = \sum_{j=1}^{3} A_j = 1 \tag{98}$$

for $k = 2$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^1 = \frac{1}{6}0 + \frac{2}{3}\frac{1}{2} + \frac{1}{6}1 = \frac{1}{2} \tag{99}$$

for $k = 3$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^2 = \frac{1}{6}0^2 + \frac{2}{3}\frac{1}{4} + \frac{1}{6}1 = \frac{1}{3} \tag{100}$$

for $k = 4$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^3 = \frac{1}{6} 0^3 + \frac{2}{3}\frac{1}{8} + \frac{1}{6} 1 = \frac{1}{4} \tag{101}$$

however for $k = 5$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^4 = \frac{1}{6} 0^4 + \frac{2}{3}\frac{1}{16} + \frac{1}{6} 1 = \frac{1}{24} + \frac{1}{6} = \frac{1}{24} + \frac{4}{24} = \frac{5}{24} \neq \frac{1}{5} \tag{102}$$

therefor the given method has local error of $O(h^5)$. From the discussion on the top of page 46, the global error is then $O(h^4)$. For the second example we have

$$\int_a^b f(x)\, dx \quad \approx \quad \frac{h}{2}[f(a + \frac{1}{2}(1 - \frac{1}{\sqrt{3}})h) + f(a + \frac{1}{2})(1 + \frac{1}{\sqrt{3}})h)] \tag{103}$$

with $h = b - a$. Then we see that $s = 4$, $A_1 = 0$, $A_2 = 1/2$, $A_3 = 1/2$, $A_4 = 0$, $\alpha_1 = 0$, $\alpha_2 = 1/2(1 - 1/\sqrt{3})$, $\alpha_3 = 1/2(1 + 1/\sqrt{3})$, and $\alpha_4 = 0$. So with $k = 1$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^0 = \sum_{j=1}^{3} A_j = 1 \tag{104}$$

for $k = 2$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^1 = \frac{1}{2}\frac{1}{2}(1 - \frac{1}{\sqrt{3}}) + \frac{1}{2}\frac{1}{2}(1 + \frac{1}{\sqrt{3}}) = \frac{1}{2} \tag{105}$$

for $k = 3$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^2 = \frac{1}{2}\frac{1}{4}(1 - \frac{1}{\sqrt{3}})^2 + \frac{1}{2}\frac{1}{4}(1 + \frac{1}{\sqrt{3}})^2 = \frac{1}{3} \tag{106}$$

for $k = 4$ the right hand side of Eq. 95 is

$$\sum_{j=1}^{3} A_j \alpha_j^3 = \frac{1}{2}\frac{1}{8}(1 - \frac{1}{\sqrt{3}})^3 + \frac{1}{2}\frac{1}{8}(1 + \frac{1}{\sqrt{3}})^3 = ??? \tag{107}$$

**Problem 2.2**

Explicitly the Runge-Kutta formulas have the following form

$$y_{n+1} = y_n + h \sum_{j=1}^{s} A_j f(t_{n,j}, y_{n,j}) \tag{108}$$

with $y_{n,j}$ given explicity interms of previous $y_{n,j}$'s i.e.

$$y_{n,j} = y_n + h \sum_{k=1}^{j-1} \beta_{j,k} f_{n,k} \tag{109}$$

with $f_{n,j} = f(t_n + \alpha_j h_n, y_{n,j})$ using the midpoint rule to evaluate the explicit Runge-Kutta expression gives

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_{n,1/2}) \tag{110}$$

with $y_{n,1/2} = y(t_n + h/2)$ using Euler's method to evaluate $y_{n,1/2}$ gives

$$y_{n,1/2} = y_n + \frac{h}{2} f(t_n, y_n) \tag{111}$$

thus the entire update step is given by

$$y_{n,1/2} = y_n + \frac{h}{2} f(t_n, y_n) \tag{112}$$

$$y_{n+1} = y_n + hf(t_n + \frac{h}{2}, y_{n,1/2}) \tag{113}$$

**Problem 2.3**

The equation of condition for a Runge-Kutta code of order $p$ are given on Page 51 of the book. They are given by

$$\frac{1}{k} = \sum_{j=1}^{s} \gamma_j \alpha_j^{k-1} \quad \text{for} \quad \text{k} = 1, 2, 3, \cdots, \text{p} \tag{114}$$

Then for the given method to be second order we require

$$y_{n,1} = y_n \tag{115}$$

$$f_{n,1} = f(t_n, y_{n,1}) \tag{116}$$

$$y_{n,j} = y_n + h_n \sum_{k=1}^{j-1} \beta_{j,k} f_{n,k} \tag{117}$$

$$f_{n,j} = f(t_n + \alpha_j h_n, y_{n,j}) \tag{118}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} \gamma_j f_{n,j} \tag{119}$$

have

$$\frac{1}{1} = \sum_{j=1}^{2} \gamma_j \alpha_j^0 = \sum_{j=1}^{2} \gamma_j \tag{120}$$

$$\frac{1}{2} = \sum_{j=1}^{2} \gamma_j \alpha_j^1 = \gamma_1 \cdot 0 + \gamma_2 \alpha_1 \tag{121}$$

$$y_{n+1} = y_n + h(\gamma_1 f_{n,1} + \gamma_2 f_{n,2}) \tag{122}$$

$$f_{n,1} = f(t_n, y_n) \tag{123}$$

$$f_{n,2} = f(t_n + \alpha_1 h, y_n + h\beta_{1,0} f_{n,1}) \tag{124}$$

giving

$$y_{n+1} = y_n + h(\gamma_1 f(t_n, y_n) + \gamma_2 f(t_n + \alpha_1 h, y_n + h\beta_{1,0} f(t_n, y_n))) \tag{125}$$

with $f(t, u)$ a scalar, we have

$$u(t_{n+1}) \approx y_{n+1} \tag{126}$$

$$u(t_{n+1}) = u(t_n) + hu'(t_n) + \frac{h^2}{2}u''(t_n) + \frac{h^3}{6}u'''(t_n) + O(h^4) \tag{127}$$

since $u(t_n) = y_n$, and $u'(t_n) = f(t_n, y_n)$ gives

$$u(t_{n+1}) = y_n + hf(t_n, y_n) + \frac{h^2}{2}u''(t_n) + \frac{h^3}{6}u'''(t_n) + O(h^4) \tag{128}$$

and

$$y_{n+1} = y_n + h\gamma_1 f(t_n, y_n) + h\gamma_2 f(t_n + \alpha_1 h, y_n -$$

$$y_n + h\gamma_1 f(t_n, y_n) + h\gamma_2 f(t_n, y_n + h\beta_{1,0} f(t_n, y_n))$$

$$h\gamma_2(f(t_n, y_n) + \alpha_1 h f_t(t_n, y_n) + h\beta_{1,0} f_y(t_n, y_n) f(t_n, y_n) + O(h^2)$$

therefore

$$y_{n+1} = y_n + (h\gamma_1 + h\gamma_2) f(t_n, y_n) + h^2\gamma_2\alpha_1 f_t(t_n, y_n) + h^2\gamma_2\beta_{1,0} f_y(t_n, y_n) f(t_n, y_n) + O(h^3) \tag{132}$$

so we have

$$u(t_{n_1}) - y_{n+1} = h(1 - \gamma_1 - \gamma_2) f(t_n, y_n) + \frac{h^2}{2}u''(t_n) - h^2\gamma_2\alpha_1 f_t(t_n, y_n) - h^2\gamma_2\beta 1, 0 f_y(t_n, y_n) f(t_n, y_n) + O(h^3) \tag{133}$$

But $u''(t_n) =$ with $u'(t) = f(t, u)$, so $u''(t) = f_t + f_u u' = f_t + f_u f$, so $u''(t_n) = f_t(t_n, y_n) + f_u(t_n, y_n) f(t_n, y_n)$ therfore

$$u(t_{n+1}) - y_{n+1} = h(1 - \gamma_1 - \gamma_2) f(t_n, y_n) + h^2(\frac{1}{2} f_t(t_n, y_n) + \frac{1}{2} f_u(t_n, y_n) f(,) - \gamma_2\alpha_1 f_t(,) - \gamma_2\beta 1, 0 f_u(,)) + O(h^3) \tag{134}$$

therefore the equations of consistency becomes

$$1 = \gamma_1 + \gamma_2 \tag{135}$$

$$\frac{1}{2} = \gamma_2\alpha_1 \tag{136}$$

$$\frac{1}{2} = \gamma_2\beta_{1,0} \tag{137}$$

$$P_{\text{BDF2}}(t) = \frac{y_{n+1}(t - t_n)(t - t_{n-1})}{h_n(h_n + h_{n-1})} \tag{138}$$

$$+ \frac{y_{n-1}(t - t_{n+1})(t - t_{n-1})}{-h_n(h_{n-1})} \tag{139}$$

or

$$P_{\text{BDF2}}(t) = A(t - t_n)(t - t_{n-1}) + B(t - t_{n+1})(t - t_{n-1}) + C(t - t_{n+1})(t - t_n) \tag{140}$$

so

$$P'_{\text{BDF2}}(t) = A(t-t_{n-1}) + A(t-t_n) + B(t-t_{n-1}) + B(t-t_{n+1}) + C(t-t_n) + C(t-t_{n+1}) \quad (141)$$

evaluating this at $t = t_{n+1}$ gives

$$P'_{\text{BDF2}}(t_{n+1}) = \frac{y_{n+1}}{h_n} + \frac{y_{n+1}}{(h_n + h_{n+1}} + \frac{y_n(h_n + h_{n-1})}{(-h_n)h_{n-1}} + \frac{y_{n-1}h_n}{(h_n + h_{n-1})h_{n-1}} \quad (142)$$

plus the collocation requirement that

$$P'_{\text{BDF1}}(t_{n+1}, \{\ldots\}) = f(t_{n+1}, y_{n+1}) \quad (143)$$

so

$$P'_{\text{BDF1}}(t; y_n, y_{n-1}) = -\frac{y_{n-1}}{h_{n-1}}(t - t_n) + \frac{y_n}{h_n}(t - t_n) \quad (144)$$

so

$$P'_{\text{BDF1}}(t; \{\ldots\}) = -\frac{y_{n-1}}{h_{n-1}} + \frac{y_n}{h_n} \quad (145)$$

$$P'(t_{n+1}) = f(t_{n+1}, P(t_{n+1})) \quad (146)$$

$$P_{\text{BDF1}}(t_{n+1}, \{\ldots\}) = y_{n+1} + P'_{\text{BDF1}}(t_{n+1}, \{\ldots\}) = f(t_{n+1}, y_{n+1}) \quad (147)$$

$$P_{\text{BDF1}}(t_{n+1}, \{\ldots\}) = \frac{(t - t_{n+1})}{(t_n - t_{n+1})}y_n + \frac{(t - t_n)}{(t_{n+1} - t_n)}y_{n+1} \quad (148)$$

$$P'_{\text{BDF1}}(t_{n+1}, \{\ldots\}) = \frac{y_n}{-h_n} + \frac{y_{n+1}}{h_n} = \frac{y_{n+1} - y_n}{h_n} = f(t_{n+1}, y_{n+1}) \quad (149)$$

For the BDF2 we have

$$\begin{aligned} P_{\text{BDF2}}(t_{n+1}, \{\ldots\}) &= y_{n+1}\frac{(t - t_n)(t - t_{n-1})}{(t_{n+1} - t_n)(t_{n+1} - t_{n-1})} && (150) \\ &+ y_n\frac{(t - t_{n+1})(t - t_{n-1})}{(t_n - t_{n+1})(t_n - t_{n-1})} && (151) \\ &+ y_{n-1}\frac{(t - t_{n+1})(t - t_n)}{(t_{n-1} - t_{n+1})(t_{n-1} - t_n)} && (152) \end{aligned}$$

For BDF2 (rathar than approximating $f$ at previous mesh points we approximate $y(t)$ at $y_{n-j}$ for $j \geq 0$, with the requirement that the polynomial satitifsy the ODE at $t = t_{n+1}$ i.e. it collocates the ODE at $t_{n+1}$ or

$$P'(t_{n+1}) = f(t_{n+1}, P(t_{n+1})) = f(t_{n+1}, y_{n+1}) \quad (153)$$

$$y_{n+1} = y_n + h_n \int_{t_n}^{t_n + h_n} f(t', u(t'))dt' \approx y_n + h_n \quad (154)$$

so

$$y_{n+1} = P(t_{n+1}, \{y_{n-j}\}) \quad (155)$$

so

$$P'(t_{n+1}; \{y_{n-j}\}) = f(t_{n+1}, y_{n+1}) \quad (156)$$

so first order extrapolation of $y_{n+1}$ gives

$$y_{n+1} \approx \frac{(t-t_n)}{(t_{n-1}-t_n)} y_{n-1} + \frac{(t-t_{n-1})}{t_n - t_{n-1}} y_n = P(t;\{y_n, y_{n-1}\}) \tag{157}$$

this may come before that other section ...

so AB2 is

$$\begin{aligned}
y_{n+1} &= y_n + \int_{t_n}^{t_n+h_n} \frac{(t-t_{n-1})}{(t_n-t_{n-1})} f_n + \frac{(t-t_n)}{(t_{n-1}-t_n)} f_{n-1} dt \\
&= y_n + \frac{f_n}{h_{n-1}} \left. \frac{(t-t_{n-1})^2}{2} \right|_{t_n}^{t_n+h_n} + \frac{f_{n-1}}{(-h_{n-1})} \left. \frac{(t-t_n)^2}{2} \right|_{t_n}^{t_n+h_n} \\
&= y_n + \frac{f_n}{2h_{n-1}} (h_{n-1}+h_n)^2 - \frac{f_n}{2h_{n-1}} h_{n-1}^2 - \frac{f_{n-1}}{2h_{n-1}} h_n^2 \\
&= y_n + \frac{f_n}{2h_{n-1}} (h_{n-1}^2 + 2h_{n-1}h_n + h_n^2 - h_{n-1}^2) - \frac{f_{n-1}h_n^2}{2h_{n-1}} \\
&= y_n + h_n \left[ \left(1 + \frac{1}{2}\frac{h_n}{h_{n-1}}\right) f_n - \left(\frac{1}{2}\left(\frac{h_n}{h_{n-1}}\right)\right) f_{n-1} \right]
\end{aligned} \tag{158}$$

define $r = \frac{h_n}{h_{n-1}}$ then we have

**Problem 2.5**

AB2

$$\int_{t_n}^{t_n+h} f(x, u(x)) dx \tag{159}$$

Take $t_{n,j} = t_{n-j}$ for $j \geq 1$ Adams Bashforth with $u' = f(t,u)$ with $u(t_n) = y_n$.

$$u(t_n + h_n) = y_n + \int_{t_n}^{t_n+h_n} f(x, u(x)) dx \tag{160}$$

The Adams Bashforth first order method (AB1) is defind by interpolating $f$ at $t_n$ using the assumed inital condition $f(t_n, y_n) = f_n$. For the Adams Bashforth second order method (AB2) we interpolate $f$ at $t_n$ and $t_{n-1}$ i.e. interpolate betweeen the points $(t_n, f(t_n, y_n))$, and $(t_{n-1}, f(t_{n-1}, y_{n-1}))$. To simplify the notation we define $f_n = f(t_n, y_n)$ and $f_{n-1} = f(t_{n-1}, y_{n-1})$ as the two previous points. Lagrange interpolation to these two previous points then gives

$$f(t,u) \approx \frac{(t-t_{n-1})}{(t_{n-1}-t_n)} f_n + \frac{(t-t_n)}{t_{n-1}-t_n} f_{n-1} \tag{161}$$

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$$y_{n+1} = h\gamma f(t_{n+1}, y_{n+1}) + \Psi \tag{162}$$

$$y_{n+1}^{[m+1]} = \Psi + h\gamma(f(t_{n+1}, y_{n+1}^{[m]}) + J(y_{n+1}^{[m+1]} - y_{n+1}^{[m]})) \tag{163}$$

let $y_{n+1}^{[m+1]} = y_{n+1}^{[m]} + \Delta_m$ then the above becomes

$$y_{n+1}^{[m]} + \Delta_m = \Psi + h\gamma(f(t_{n+1}, y_{n+1}^{[m]}) + J(\Delta_m)) \tag{164}$$

$$(I - h\gamma J)\Delta_m = \Psi + h\gamma f(t_{n+1}, y_{n+1}^{[m]} - y_{n+1}^{[m]}) \tag{165}$$

with $J \equiv \frac{\partial f}{\partial y}(t_{n+1}, y_{n_1}^{[m]}$, this is Shampine Eq. 2.36.

$$|\mathrm{est}| \leq \tau_\mathrm{r}|\mathrm{y}^*_{\mathrm{n+1}}| + \tau_\mathrm{a} \tag{166}$$

with $h_\text{new} = \sigma h$ we get

$$\begin{aligned}
\text{est} &= h\left(\frac{2}{9}f_{n,1} + \left(\frac{1}{3} - 1\right)f_{n,2} + \frac{4}{9}f_{n,3}\right) && (167)\\
&= \frac{h}{9}\left(2f_{n,1} - 6f_{n,2} + 4f_{n,3}\right) && (168)\\
&= \frac{2h}{9}\left(f_{n,1} - 3f_{n,2} + 2f_{n,3}\right) && (169)
\end{aligned}$$

Define

$$\begin{aligned}
y_{n,1} &= y_n && (170)\\
f_{n,1} &= f(t_n, y_{n,1}) && (171)\\
y_{n,2} &= y_n + h\frac{1}{2}f_{n,1} && (172)\\
f_{n,2} &= f(t_n + \frac{h}{2}, y_{n,2}) && (173)\\
y_{n,3} &= y_n + h\frac{3}{4}f_{n,2} && (174)\\
f_{n,3} &= f(t_n + \frac{3}{4}h, y_{n,3}) && (175)
\end{aligned}$$

Finally we have

$$y_{n+1} = y_n + h\left(\frac{2}{9}f_{n,1} + \frac{3}{9}f_{n,2} + \frac{4}{9}f_{n,3}\right) \tag{176}$$

so we have

$$\alpha = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{3}{4} \end{pmatrix} \quad \gamma = \begin{pmatrix} \frac{2}{9} \\ \frac{3}{9} \\ \frac{4}{9} \end{pmatrix} \quad \beta = \begin{pmatrix} 0 & \\ \frac{1}{2} & \\ 0 & \frac{3}{4} \end{pmatrix} \tag{177}$$

Butcher's tableu is:

Euler-Huen is

$$y_{n,1} = y_n \tag{178}$$
$$f_{n,1} = f(t_n, y_{n,1}) \tag{179}$$
$$y_{n,2} = y_n + h\beta_2 f_{n,1} \tag{180}$$
$$f_{n,2} = f(t_n + \alpha_2 h, y_{n,2}) \tag{181}$$

and finally we have

$$y_{n+1} = y_n + h\sum_{j=1}^{2}\gamma_j f_{n,j} \tag{182}$$

$$y_{n,1} = y_n \tag{183}$$
$$f_{n,1} = f(t_n, y_{n,1}) \tag{184}$$
$$y_{n,2} = y_n + hf_{n,1} \tag{185}$$
$$f_{n,2} = f(t_n + \alpha_2 h, y_{n,2}) \tag{186}$$
$$y_{n+1} = y_n + h\left(\frac{1}{2}f_{n,1} + \frac{1}{2}f_{n,2}\right) \tag{187}$$

**Problem 2.4**

$$y_{n+1} = y_n + hf_{n,2} \tag{188}$$
$$y^*_{n+1} = y_n + \frac{h}{9}(2f_{n,1} + 3f_{n,2} + 4f_{n,3}) \tag{189}$$

$f_{n,1} = 2$. The Euler-Heun Runge-Kutta method is

$$y_{n,1} = y_n + hf(t_n, y_n) \tag{190}$$
$$y_{n+1} = y_n + h\left(\frac{1}{2}f(t_n, y_n) + \frac{1}{2}f(t_{n+1}, y_{n,1})\right) \tag{191}$$

Here $\bar{\alpha} = (0, 1)$, and $\bar{\gamma} = (\frac{1}{2}, \frac{1}{2})$, and $\beta = (0, 1)$. so

$$y_{n,1} = y_n \tag{192}$$
$$f_{n,1} = f(t_n, y_{n,1}) \tag{193}$$
$$y_{n,j} = y_n + h_n\sum_{k=1}^{j-1}\beta_{j,k}f_{n,k} \quad \text{and} \tag{194}$$
$$f_{n,j} = f(t_n + \alpha_j h_n, y_{n,j}) \quad \text{for} \quad \text{j} = 2, 3, 4, \ldots, \text{s} \tag{195}$$
$$y_{n+1} = y_n + h_n\sum_{j=1}^{s}\gamma_j f_{n,j} \tag{196}$$