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(4) $|z| = 2$ + $|w| = 3$

$$|z \cdot w| = |z| \cdot |w| = 2 \cdot 3 = 6$$

$$|z+w| \le |z| + |w| = 2+3 = 5$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{2}{3}$$

$$|z-w| \le |z| + |w| = 5$$

(11) If $a_{ij} = i-j$ then $A_{3\times3} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

Then $|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & -2 \\ 1 & -\lambda & -1 \\ 2 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & -2 \\ 1 & -\lambda \end{vmatrix}$

$$+ 2 \begin{vmatrix} -1 & -2 \\ -\lambda & -1 \end{vmatrix}$$

$$= -\lambda(\lambda^2 + 1) - 1(\lambda + 2) + 2(1 - 2\lambda)$$

$$= -\lambda^3 - \lambda - \lambda - 2 + 2 - 4\lambda$$

$$= -\lambda^3 - 6\lambda = 0$$

So $\lambda = 0$ + $\lambda^2 = -6$ so $\lambda = \pm\sqrt{6}\,i$

(12) We have

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

(a) If $a=b=d=1$ then from the above we have

$$\lambda = \frac{2 \pm \sqrt{4-4(1-c)}}{2} = \frac{2 \pm \sqrt{4-4+4c}}{2}$$

$$= \frac{2 \pm \sqrt{4c}}{2} = 1 \pm \sqrt{c}$$

which is complex when $c < 0$.

(b) When $ad - bc = 0$ the expression above for λ becomes

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2}}{2} = \frac{a+d \pm (a+d)}{2} = \begin{cases} 0 \\ a+d \end{cases}$$

(c) Conjugates is a notion for complex numbers, not real #'s. ~~λ will be complex iff $(a+d)^2 - 4(ad-bc) < 0$ in which case the plus + minus then do give complex conjugates~~

Thus the eigenvalues can be real + different

(13) Eigenvalues are complex when $(a+d)^2 < 4(ad-bc)$

If ~~bc<0~~ $bc>0$ then the discriminant is given by

$(a+d)^2 - 4(ad-bc) = a^2 + 2ad + d^2 - 4ad + 4bc$

$= a^2 - 2ad + d^2 + 4bc =$ ~~[illegible]~~ $(a-d)^2 + 4bc$

If $bc>0$ then this entire sum is positive & ∴ will have a real squareroot.

(14) The eigenvalues of P are given by $\lambda^4 = 1$ or the 4 - 4th roots of unity.

(15) ~~[illegible]~~ From the Fast Fourier Transform section the eigenvalues of the ~~[illegible]~~ cyclic permutation matrix P is given by

the ~~[illegible]~~ n-th roots of unity so $\lambda^n = 1$ when $n=6$ we have $\lambda^6 = 1$

(16) If $Ax = \lambda x$ then $\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}\begin{bmatrix} x \\ ix \end{bmatrix} = i\lambda\begin{bmatrix} x \\ ix \end{bmatrix}$

Because the left hand side is $\begin{bmatrix} A(ix) \\ -Ax \end{bmatrix}$

Since the transpose of this block matrix is given by

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & A^T \\ -A^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A \\ A & 0 \end{bmatrix} \ne$$

Obviously this matrix is symmetric. So its eigenvalues must be real. So it must be ~~imaginary~~ real & it must be pure imaginary

(17) Skipped

(18) Skipped

(19) Skipped

(20) $e^{3i\theta} = \cos(3\theta) + i\sin(3\theta)$

$$= (\cos\theta + i\sin\theta)^3 = (\cos\theta + i\sin\theta)(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta)$$

$$= \cos^3\theta + 2i\cos^2\theta\sin\theta - \cos\theta\sin^2\theta + i\sin\theta\cos^2\theta - 2\cos\theta\sin^2\theta$$
$$- i\sin^3\theta$$

$$= \cos^3\theta - \cos\theta\sin^2\theta - 2\cos\theta\sin^2\theta$$
$$+ i(2\cos^2\theta\sin\theta + \sin\theta\cos^2\theta - \sin^3\theta)$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

(22) $\bar{z} = \frac{1}{z} \Rightarrow z\cdot\bar{z} = 1 \Rightarrow \|z\|^2 = 1 \Rightarrow z = \cos\theta + i\sin\theta$

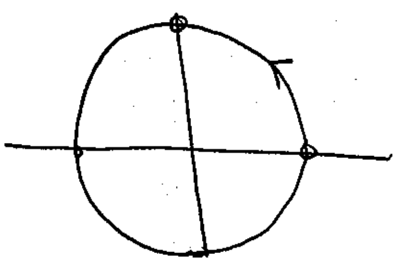
(23) (a) $e^{i} = e^{1i} = \cos(1) + i\sin(1)$ which has magnitude 1.

$i^e = (i)^e$ so ~~the~~ $\|i^e\| = \|i\|^e = 1^e = 1$

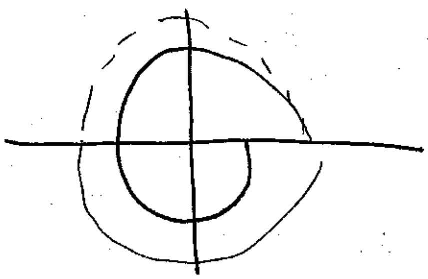
(b) Skipped

(c) They are equal. There are ∞ many representations of any complex number

(24) a)



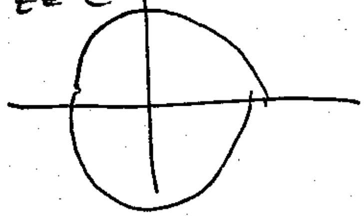
(b)



(c) $(-1)^t = e^{t\pi i} = \cos(t\pi) + i\sin(t\pi)$

when $t=0 \Rightarrow$ to $z=1$

& $t=2\pi \Rightarrow$ to $z = e^{2\pi^2 i} = \cos(2\pi^2) + i\sin(2\pi^2)$



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① If $u = (1+i,\ 1-i,\ 1+2i)$ then $\|u\|^2 = u^H u =$

$$\begin{bmatrix} 1-i & 1+i & 1-2i \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \\ 1+2i \end{bmatrix}$$

$= (1-i)(1+i) + (1+i)(1-i) + (1-2i)(1+2i)$

$= 1-(-1) + 1-(-1) + 1-4(-1)$

$= 2+2+5 \quad = 9$

So $\|u\| = 3$

If $v = (i, i, i)$ then $\|v\|^2 = v^H v = \begin{bmatrix} -i & -i & -i \end{bmatrix} \begin{bmatrix} i \\ i \\ i \end{bmatrix}$

$= -i^2 - i^2 - i^2 \quad = 1+1+1 \quad = 3$

$\therefore \|v\|^2 = 3 \Rightarrow \|v\| = \sqrt{3}$

Now $u^H v = \begin{bmatrix} 1-i & 1+i & 1-2i \end{bmatrix} \begin{bmatrix} i \\ i \\ i \end{bmatrix}$

$= i+1 + i-1 + i+2 \quad = 3i+3$

& $v^H u = \begin{bmatrix} -i & -i & -i \end{bmatrix} \begin{bmatrix} 1+i \\ 1-i \\ 1+2i \end{bmatrix} = -i+1 - i-1 - i+2$

$= -3i+2$

(2) $A^H A$ + AA^H are both Hermitian matrices

$$A^H A = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix} \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} = \begin{bmatrix} 1+1 & i+i & 1+i \\ i-i & 1+1 & i+1 \\ 1-i & -i+1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1+i \\ 0 & 2 & 1+i \\ 1-i & -i+1 & 2 \end{bmatrix}$$

+ $$AA^H = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix} = \begin{bmatrix} 1+1+1 & i-i+1 \\ -i+i+1 & 1+1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

(3) $Az = 0$ is equivalent to

$$\begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix} z = 0 \Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 1 & i & i \end{bmatrix} z = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 2i & i-1 \end{bmatrix} z = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 1 & \frac{i-1}{2i} \end{bmatrix} z = 0$$

Now $\frac{i-1}{2i} = \frac{1}{2}(1+i)$ or

$$\begin{bmatrix} 1 & -i & 1 \\ 0 & 1 & \frac{1}{2}(1+i) \end{bmatrix} z = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1+\frac{i}{2}(1+i) \\ 0 & 1 & \frac{1}{2}(1+i) \end{bmatrix} z = 0$$

$$1 + \frac{i}{2}(1+i) = 1 + \frac{i}{2} - \frac{1}{2} = \frac{1}{2} + \frac{i}{2}$$

Thus z_3 is a free variable. Setting it equal to 1 we have

~~So.~~ $z_1 = -\frac{1}{2} - \frac{i}{2}$

$z_2 = -\frac{1}{2}(1+i)$

$\therefore$ an element of the nullspace is given by

$$z = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2}(1+i) \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1-i \\ -1-i \\ 2 \end{bmatrix}$$

The columns of $A^H = \begin{bmatrix} -i & 1 \\ 1 & -i \\ -i & -i \end{bmatrix}$

So $z^H A^H(:,1) = \begin{bmatrix} -1+i & -1+i & 2 \end{bmatrix} \begin{bmatrix} -i \\ 1 \\ -i \end{bmatrix}$

$= \cancel{i} + 1 - 1 + \cancel{i} - \cancel{2i} = 0$

& $z^H A^H(:,2) = \begin{bmatrix} -1+i & -1+i & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ -i \end{bmatrix}$

$= \cancel{-1} + i + i + \cancel{1} - 2i$

$= 0$

To show that z is Not orthogonal to ~~A~~ the columns of A^T we compute $A^T = \begin{bmatrix} i & 1 \\ 1 & i \\ i & i \end{bmatrix}$ & then

$z^H A^T(:,1) = \begin{bmatrix} -1+i & -1+i & 2 \end{bmatrix} \begin{bmatrix} i \\ 1 \\ i \end{bmatrix} = -\cancel{i} - 1 - 1 + \cancel{i} + 2i$

$= -2 + 2i \neq 0$

$$z^H A^T(:,2) = \begin{bmatrix} -1+i & -1+i & 2 \end{bmatrix} \begin{bmatrix} 1 \\ i \\ i \end{bmatrix} = -1+\cancel{i}\cancel{-i}-1+2i$$

$$= -2+2i \neq 0$$

(4) $C(A)$ + $N(A)$ + $C(A^H)$ + $N(A^H)$. The dimensions of $C(A)$ + $N(A)$ are $r + n-r + r + m-r$

~~$A \in \mathbb{C}^{m\times n}$~~

(5) (a) ~~...~~ compute ~~A~~ the Hermetian of A^HA & we get

$(A^HA)^H = A^HA$ ~~=~~

(b) If $Az=0$ then $A^HAz=0$. If $A^HAz=0$, multiply by z^H on the left to get $z^HA^HAz=0$ which is equivalent to

~~...~~ $(Az)^H(Az)=0 \Rightarrow \|Az\|^2 = 0$ showing that

$\Rightarrow Az=0$.

Thus the nullspace of A & A^HA are the same.

Therefore A^HA is an invertible Hermitian matrix when the nullspace of A has only $z=0$.

6 (a) If A is real matrix then $A+iI$ is invertible.

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A+iI = \begin{bmatrix} a+i & b \\ c & d+i \end{bmatrix}$

which will not be invertible iff $|A+iI| = 0$

$$(a+i)(d+i) - bc = 0$$

$$\Rightarrow ad + (a+d)i - 1 - bc = 0$$

let $a = -d$ then the above ~~term~~ requirement becomes

$$-a^2 - 1 - bc = 0$$

which will be true if $a = 0$ + $b = -1$ + $c = 1$

Thus $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Then $|A| = 0 + (1) = 1$

But $A+iI = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$ so $|A+iI| = -1+1 = 0$

so $A+iI$ is <u>Not</u> invertible

(6) If A is Hermitian matrix then $A+iI$ is invertible

A Hermitian means that $A^H = A$. Consider

$$A = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \quad \text{Then} \quad A+iI = \begin{bmatrix} a+i & b \\ \bar{b} & c+i \end{bmatrix}$$

with a & c real. $A+iI$ will not be invertible iff

$$|A+iI| = (a+i)(c+i) - \bar{b}b = 0$$

$$= ac + i(a+c) - 1 - \bar{b}b = 0$$

we must have $a = -c$ & the above constraint becomes

Ex. $-a^2 - 1 - \bar{b}b = 0$

$\Rightarrow 1 + a^2 + \bar{b}b = 0 \Rightarrow 1 + a^2 + |b|^2 = 0$

But this cannot be made to hold since

a must be real & $\therefore a^2 \geq 0$ & $|b|^2 \geq 0$ so the left

hand side can never sum to 0. Thus ~~Hermetian~~ ~~matrices~~ all 2x2 Hermetian matrices H have $A+iI$ invertible.

Since H is Hermetian it must have real eigenvalues

$\lambda_1, \ldots, \lambda_n$

But then the eigenvalues of $H+iI$ are given by

~~λ~~ $\lambda_1+i, \lambda_2+i, \ldots, \lambda_n+i$. Since to be singular we require that one of these eigenvalues be zero but this cannot happen since each ~~λ~~ λ is real. Thus we conclude that given ~~A~~ A ~~be~~ Hermitian that $A+iI$ is invertible.

(c) If U is a unitary matrix then $U+iI$ is invertible.

False. U being unitary means that $U^H = U^{-1}$ & $\therefore$ $\|Uz\| = \|z\|$ ~~requiring~~ requiring that $|\lambda| = 1$. We will pick a unitary matrix w/ $\lambda = \pm i$ then $U+iI$ will have eigenvalues given by $\pm i + i$ $= 0, 2i$ & will $\therefore$ be singular. Let ~~$U = $~~

$$U = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \qquad U^H U = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so U is unitary & U has eigenvalues given by $\pm i$ as requested. ~~Proving~~ Proving the falseness of this statement

(7) To be Hermetian requires that $A^H = A$. If c is real this is true since $(cA)^H = cA^H = cA$.

If $c = i$ then we have $(cA)^H = A^H (i)^H = A^H(-i)$

$= -iA^H = -(iA)$ so (iA) is skew-Hermetian.

(8) ~~For this problem we have P a right left cyclic shift permutation matrix. Thus $P^{-1} = P^T$ but P is not symmetric so $P^T \neq P$ &~~

~~$\therefore$ P is not orthogonal.~~

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(8) We can recognize P ~~to~~ as a permutation matrix and thus its inverse is given by P^T. Thus P is invertible.

Since $P^T \neq P$ P is Not symmetric. Since $P^{-1} = P^T$ P is orthogonal. P is real + $P^T \neq P$ so $P^H = P^T \neq P$ so P is Not Hermitian. To be unitary P must satisfy $P^H = P^{-1}$ where since $P^H = P^T = P^{-1}$ This is true. ~~[illegible]~~

All matrices have a LU factorization so P has one also.

To be factorable into QR it is sufficient to have ~~enough linear independent eigenvectors. Since we have n = 3 unique eigenvalues + ∴ 3 linearly enough linearly independent eigenvectors Thus A a QR decomposition exists~~ linearly independent columns. Since we have 3 linearly independent columns a QR decomposition exists

(9) Considering $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ we can evaluate powers of P in terms of its eigenvalues + eigenvectors with

$$P = S \Lambda S^{-1}$$

To compute the eigenvectors of P consider $|P - \lambda I| = 0$

Which is given by $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix}$$

$$= -\lambda(-\lambda)^2 + 1 = 0 \Rightarrow \lambda^3 = 1$$

$$\Rightarrow \lambda = e^{\frac{2\pi}{3} i k} \qquad k = 0, 1, 2$$

computing the ~~given~~ eigenvectors can be done in a straightforward manner but it might be easier to simply compute P^2 by direct multiplication (the simplicity of the matrix facilitates the decision)

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now for $P^{100} = P^{99} \cdot P^1 = (P^3)^{33} \cdot P^1$

$$= I^{33} \cdot P^1 = P^1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

(19) From reading the FFT section esp problem 11 the eigenvectors of P are given by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ + $\begin{bmatrix} 1 \\ w^1 \\ w^2 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\frac{2\pi}{3}i} \\ e^{\frac{4\pi}{3}i} \end{bmatrix}$ + $\begin{bmatrix} 1 \\ w^2 \\ w^4 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{\frac{4\pi}{3}i} \\ e^{\frac{8\pi}{3}i} \end{bmatrix}$

w/ $w^1 = e^{\frac{2\pi}{3}i}$ ~~Thus our matrix of eigenvectors is given~~

by $F = \cancel{\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \\ 1 & e^{\frac{4\pi}{3}i} & e^{\frac{8\pi}{3}i} \end{bmatrix}}$ to make them unit vectors we normalize by $w^H w$.

Thus the 1st normalized eigenvector is given by

$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $\quad w_2^H w_2 = \begin{bmatrix} 1 & e^{-\frac{2\pi i}{3}} & e^{-\frac{4\pi i}{3}} \end{bmatrix} \begin{bmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{\frac{4\pi i}{3}} \end{bmatrix}$

$= 3$

So the 2nd normalized eigenvector is given by

$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{\frac{2\pi}{3}i} \\ e^{\frac{4\pi}{3}i} \end{bmatrix}$ The third normalized eigenvector is given

by $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ e^{\frac{4\pi}{3}i} \\ e^{\frac{8\pi}{3}i} \end{bmatrix}$

Then the Fourier matrix F is given by

$$F = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2\pi}{3}i} & e^{\frac{4\pi}{3}i} \\ 1 & e^{\frac{4\pi}{3}i} & e^{\frac{8\pi}{3}i} \end{bmatrix}$$

Since P is Hermitian its eigenvectors must be orthogonal

Note $e^{\frac{8\pi}{3}i} = e^{\frac{6\pi}{3}i + \frac{2\pi}{3}i} = e^{\frac{2\pi}{3}i}$

(11) Given $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

So $C = 2I + 5P + 4P^2$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 5 \\ 5 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 4 \\ 4 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 4 \\ 4 & 2 & 5 \\ 5 & 4 & 2 \end{bmatrix}$$

Since C is computed from ~~composed of~~ all matrices that have the same eigenvectors ~~thus~~ The eigenvalues of C are given by the

$2 + 5\lambda + 4\lambda^2$ with λ the eigenvalues of P

found in problem 9.

(12) To be unitary we must have $U^H = U^{-1}$.

or $U^H U = I$. Consider $(U^{-1})^H U^{-1} = (U^H)^{-1} U^{-1}$

$= (U^{-1})^{-1} U^{-1}$ because U is unitary

$= U \cdot U^{-1} = I$ showing that U^{-1} is unitary

to show UV is unitary consider

$$(UV)^H (UV) = V^H U^H U V = V^H I V = I$$

so UV is unitary if both U + V are

(13) The determinant of a matrix is the product of its eigenvalues. Since a Hermitian matrix has real eigenvalues its determinant must be positive

(14) consider $Z^H A^H A Z = (AZ)^H (AZ)$ + ~~since~~

$$= \|AZ\|^2 \geq 0$$

+ only equal to zero when $AZ = 0$. Since the columns of A are linearly independent $AZ = 0$ iff $Z = 0$

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(15) $A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}$ which is Hermitian $A^H = A$

& $\therefore$ ~~A~~ has a decomposition $A = U \Lambda U^H$ w/ Λ ~~the~~ a diagonal matrix of the eigenvalues of A & U a matrix with the eigenvectors of A as its columns. For the A above the eigenvalues are given by

$$\lambda_1 \cdot \lambda_2 = 0 - (1-i)(1+i)$$
$$= -(1+1) = -2$$

& $\lambda_1 + \lambda_2 = 1$

Thus $\lambda_1 = 1 - \lambda_2$ & $\therefore$ $(1-\lambda_2)\lambda_2 = -2$

so

$$-\lambda_2^2 + \lambda_2 + 2 = 0$$
$$\lambda_2^2 - \lambda_2 - 2 = 0$$
$$(\lambda_2 + 1)(\lambda_2 - 2) = 0 \Rightarrow \lambda_2 = -1 \text{ or } \lambda_2 = 2$$

are the two eigenvalues. The eigenvectors are given by the nullspace for

$$\cancel{\begin{bmatrix} 1 & 1+i \\ i+1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1-i \\ 0 & 0 + -(i+1)(1-i) \end{bmatrix}} = \begin{bmatrix} 1 & 1-i \\ 0 & -2 \end{bmatrix}$$

$$\cancel{\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

~~Thus $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the 1st eigenvector.~~

~~The second eigenvector is given by the nullspace~~ of

$$\begin{bmatrix} 1 & 1-i \\ i+1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ 0 & 2-(i+1)(1-i) \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

Thus $x_1 = \begin{bmatrix} i-1 \\ 1 \end{bmatrix}$ is the 1st eigenvector which when normalized

gives $x_1 = \dfrac{1}{\sqrt{1+(-i-1)(i-1)}} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{1+2}} \begin{bmatrix} i-1 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} i-1 \\ 1 \end{bmatrix}$

For the 2nd eigenvector is given by the null space of the following matrix

$$\begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1-i \\ 0 & 0 \end{bmatrix}$$

$$\left(\tfrac{-1}{2}\right)(1+i)(-1)(1-i) - 1$$

$$= \left(\tfrac{1}{2} + \tfrac{i}{2}\right)(1-i) - 1 = \tfrac{1}{2}(1+i)(1-i) - 1 = 0$$

Thus the 2nd eigenvector is given by $x_2 = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$

which when normalized given by

$$x_2 = \frac{1}{\sqrt{4+2}}\begin{bmatrix} 1-i \\ 2 \end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$$

Then the matrix U is given by $U = \begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$ ✓

So $U^H = \cancel{\begin{bmatrix} \frac{-i-1}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}} \begin{bmatrix} \frac{-i-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$

Then we can check that

$$A = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix} = \cancel{\begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{-i-1}{\sqrt{3}} & \frac{1+i}{} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}}$$

$$= \begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{-i-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1-i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \frac{i+1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}}(1+i) & \frac{4}{\sqrt{6}} \end{bmatrix}$$

$\cancel{(i-1)(i+1) = i^2-1 = -1-1}$

$$= \begin{bmatrix} \frac{-1-1}{3} + \frac{2}{6}(1-i)(1+i) & -\frac{1}{3}(i-1) + \frac{4}{6}(1-i) \\ \frac{1(i+1)}{3} + \frac{4}{6}(1+i) & -\frac{1}{3} + \frac{8}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} + \frac{1}{3}(1+1) & (1-i)\left[\frac{1}{3} + \frac{2}{3}\right] \\ (1+i)\left[\frac{1}{3} + \frac{2}{3}\right] & -\frac{2}{6} + \frac{8}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1-i \\ 1+i & 1 \end{bmatrix}$$ which is correct

(16) When $K = \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$ $K^H =$ ✓

$$K^H = \begin{bmatrix} 0 & 1-i \\ -1-i & -i \end{bmatrix} = -\begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix} = -K$$ ✓

So K is skew-Hermitian matrix. It should have imaginary eigenvectors. To prove this consider

$$|K - \lambda I| = \begin{vmatrix} -\lambda & -1+i \\ 1+i & i-\lambda \end{vmatrix} = 0$$ ✓

$$\Rightarrow \lambda^2 - \lambda i - (-1+i)(1+i) = 0$$ ✓

$$\Rightarrow \lambda^2 - i\lambda + 2 = 0$$ ✓

$$\lambda = \frac{i \pm \sqrt{(-i)^2 - 4(1)(2)}}{2} = \frac{i \pm \sqrt{-1-8}}{2}$$ ✓

$$\Rightarrow \lambda = \frac{i \pm 3i}{2} = \begin{cases} \frac{4i}{2} = 2i \\ \frac{-2i}{2} = -i \end{cases}$$

Which are both imaginary as expected

To Diagonalize we need to compute the eigenvectors. For $\lambda_1 = -i$ we have the eigenvector given by the nullspace of the following operator

$$\begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i+1 \\ 1+i & 2i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i+1 \\ 0 & 2i - (i+1)(i+1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & i+1 \\ 0 & \cancel{2i} - (-1+\cancel{2i}+1) \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & i+1 \\ 0 & 0 \end{bmatrix} \checkmark$$

Thus the 1st eigenvector is given by $\hat{x}_1 = \begin{bmatrix} -(i+1) \\ 1 \end{bmatrix}$

Which when normalized is given by

$$x_1 = \cancel{\frac{1}{\sqrt{1-(i+1)(-i+1)}}\begin{bmatrix} -(i+1) \\ 1 \end{bmatrix}} = \frac{1}{\sqrt{3}}\begin{bmatrix} -(i+1) \\ 1 \end{bmatrix} \checkmark$$

$$\hat{x}^H \hat{x} = \begin{bmatrix} -(-i+1) & 1 \end{bmatrix} \begin{bmatrix} -(i+1) \\ 1 \end{bmatrix} = (-i+1)(i+1) + 1 = 1+1+1 = 3$$

The 2nd eigenvector is given by the nullspace of

$$\begin{bmatrix} -2i & -1+i \\ 1+i & i-2i \end{bmatrix} = \begin{bmatrix} -2i & -1+i \\ 1+i & -i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2}\left(\frac{-1+i}{-i}\right) \\ 1+i & -i \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}(i+1) \\ i+1 & -i \end{bmatrix}$$

$$\left\{ \cancel{\tfrac{1}{2}\left(\tfrac{1}{i}-1\right)} \right. \quad \frac{1}{2}(-i-1)$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2}(i+1) \\ 0 & \underbrace{-i + \frac{(i+1)(i+1)}{2}} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2}(i+1) \\ 0 & 0 \end{bmatrix}$$

$$-i + \frac{1}{2}(\cancel{-1} + 2i \cancel{+1})$$

$$-i + i = 0$$

Thus the unnormalized eigenvector is given by

$$\tilde{x}_2 = \begin{bmatrix} i+1 \\ 2 \end{bmatrix}$$ which when normalized is given by

$$x_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} i+1 \\ 2 \end{bmatrix}$$

So our matrix of eigenvectors is given by

$$T = \begin{bmatrix} -\frac{(i+1)}{\sqrt{3}} & \frac{i+1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad \text{so} \quad T^H = \begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-i+1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

We can check this by computing

$$T \Lambda T^H = \begin{bmatrix} -\frac{(i+1)}{\sqrt{3}} & \frac{i+1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & 2i \end{bmatrix} \begin{bmatrix} \frac{i-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-i+1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(i+1)}{\sqrt{3}} & \frac{i+1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1+i}{\sqrt{3}} & \frac{-i}{\sqrt{3}} \\ \frac{2+2i}{\sqrt{6}} & \frac{4i}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(1+i)^2}{3} + \frac{2(1+i)^2}{6} & \frac{i(i+1)}{3} + \frac{4i(i+1)}{6} \\ \frac{1+i}{3} + \frac{4(1+i)}{6} & \frac{-i}{3} + \frac{8i}{6} \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)^2 \cdot 0 & i(i+1)\left(\frac{1}{3}+\frac{2}{3}\right) \\ (1+i)\left(\frac{1}{3}+\frac{2}{3}\right) & -\frac{i}{3}+\frac{4}{3}i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1+i \\ 1+i & i \end{bmatrix}$$ Which is ~~A~~ k as expected

(17) For $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Q is orthogonal $\therefore$

all eigenvalues have magnitude 1, we can decompose Q as

$Q = U \Lambda U^H$ ~~to~~ ~~do~~ by computing the eigenvalues + eigenvectors

The eigenvalues are given by

$$|Q - \lambda I| = 0 \Rightarrow \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \cos^2\theta - 2\cos\theta\,\lambda + \lambda^2 + \sin^2\theta = 0$$

$$\Rightarrow \lambda^2 - 2\cos\theta\,\lambda + 1 = 0$$

$$\Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm 2\sqrt{-1(1-\cos^2\theta)}}{2} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

For $e^{-i\theta}$

eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} \cos\theta - e^{-i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{-i\theta} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{2}(e^{i\theta}+e^{-i\theta}) - e^{-i\theta} & -\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) & \frac{1}{2}(e^{i\theta}-e^{-i\theta}) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}(e^{i\theta}-e^{-i\theta}) & -\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) & \frac{1}{2}(e^{i\theta}-e^{-i\theta}) \end{bmatrix}$$

From which we see that

$$\tilde{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$ is an unnormalized eigenvector

$$X_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$$ is then the normalized eigenvector.

Note since Q is real the eigenvalues & eigenvectors come complex conjugate pairs &

In a similar way for $e^{+i\theta}$ the eigenvectors are given by :

the nullspace of the following matrix

$$\begin{bmatrix} \frac{1}{2}(e^{i\theta}+e^{-i\theta}) - e^{i\theta} & -\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) & \frac{1}{2}(e^{i\theta}+e^{-i\theta}) - e^{i\theta} \end{bmatrix}$$

another eigenvector is given by $X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{2}(-e^{i\theta}+e^{-i\theta}) & -\frac{1}{2i}(e^{i\theta}-e^{-i\theta}) \\ \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) & \frac{1}{2}(-e^{i\theta}+e^{-i\theta}) \end{bmatrix}$$

From which we see that $\hat{X}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ is an

unnormalized & Normalized we have

$$X_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus our matrix of eigenvectors is given by

$$T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \quad \text{+ so} \quad T^H = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

We can check this result by computing

$$T \Lambda T^H = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{bmatrix} \begin{bmatrix} e^{-i\theta}/\sqrt{2} & -ie^{-i\theta}/\sqrt{2} \\ e^{i\theta}/\sqrt{2} & ie^{i\theta}/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{-i\theta} + \frac{1}{2}e^{i\theta} & \frac{-i}{2}e^{-i\theta} + \frac{ie^{i\theta}}{2} \\ \frac{ie^{-i\theta}}{2} - \frac{ie^{i\theta}}{2} & \frac{e^{-i\theta}}{2} + \frac{e^{i\theta}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\left(\frac{e^{i\theta} + e^{-i\theta}}{2i}\right) \\ +\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ which is θ showing the correctness of the decomposition

(18) For the V given we note that $V^H = V$ so V is Hermitian. Considering $V^H \cdot V = \begin{bmatrix} 1/\sqrt{3} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$

$$= \frac{1}{3} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1+2 & 0 \\ 0 & 1+2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ So V is Unitary.

Thus the eigenvalues must have norm given by 1 i.e. $|\lambda| = 1$.

Computing the eigenvalues we have to solve

$$\begin{vmatrix} \frac{1}{\sqrt{3}} - \lambda & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} - \lambda \end{vmatrix} = 0 \Rightarrow -\left(\frac{1}{\sqrt{3}} - \lambda\right)\left(\frac{1}{\sqrt{3}} + \lambda\right)$$

$$- \frac{1}{3}(1+i)(1-i) = 0$$

$$-\left(\frac{1}{3}-\lambda^2\right)-\frac{1}{3}(1+1) = 0$$

$$\lambda^2-\frac{1}{3}-\frac{2}{3} = 0 \Rightarrow \lambda^2-1=0 \Rightarrow \lambda = \pm 1.$$

Then the eigenvectors are given by (for $\lambda=-1$) the null space of the following matrix

$$\begin{bmatrix} \frac{1}{\sqrt{3}}+1 & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}}+1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1-i}{1+\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1+\sqrt{3}}{\sqrt{3}} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1-i}{1+\sqrt{3}} \\ 0 & \underbrace{\frac{-1+\sqrt{3}}{\sqrt{3}} - \frac{(1-i)}{1+\sqrt{3}}\left(\frac{1+i}{\sqrt{3}}\right)} \end{bmatrix}$$

$$\underbrace{\frac{(-1+\sqrt{3})(1+\sqrt{3})}{\sqrt{3}(1+\sqrt{3})} - \frac{1+1}{\sqrt{3}(1+\sqrt{3})}}$$

$$\frac{-1+3-2}{\sqrt{3}(1+\sqrt{3})} = 0$$

$$\to \begin{bmatrix} 1 & \frac{1-i}{1+\sqrt{3}} \\ 0 & 0 \end{bmatrix}$$

Which ~~has~~ has an eigenvector given by

B $\tilde{X}_1 = \begin{bmatrix} -(1-i) \\ 1+\sqrt{3} \end{bmatrix}$ which is unnormalized. A normalized eigenvector is given by

$$\cancel{\tilde{x}_1^H \tilde{x}_1 = \sqrt{+(1-i)(1+i) + (1+\sqrt{3})^2}}$$

$$\tilde{x}_1^H \tilde{x}_1 = (1-i)(1+i) + (1+\sqrt{3})^2$$

$$= 1+1 + (1+2\sqrt{3}+3) = 6+2\sqrt{3}$$

$$\therefore \quad x_1 = \frac{1}{\sqrt{6+2\sqrt{3}}} \begin{bmatrix} -(1-i) \\ 1+\sqrt{3} \end{bmatrix}$$

~~Because~~ The 2nd eigenvector is given by the nullvector of the following matrix

$$\begin{bmatrix} \frac{1}{\sqrt{3}}-1 & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}}-1 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{3}}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & -\frac{1}{\sqrt{3}}-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1-i}{1-\sqrt{3}} \\ \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}}-1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1-i}{1-\sqrt{3}} \\ \underset{0}{\cancel{\frac{1+i}{\sqrt{3}}}} & \frac{-1}{\sqrt{3}}-1-\left(\frac{1-i}{1-\sqrt{3}}\right)\left(\frac{1+i}{\sqrt{3}}\right) \end{bmatrix}$$

$$-\frac{(1+\sqrt{3})}{\sqrt{3}} - \frac{(1-i)(1+i)}{\sqrt{3}(1-\sqrt{3})}$$

$$= -\frac{(1+\sqrt{3})(1-\sqrt{3})}{\sqrt{3}(1-\sqrt{3})} - \frac{(1-i)(1+i)}{\sqrt{3}(1-\sqrt{3})}$$

$$= \frac{-(1-3)-(1+1)}{\sqrt{3}(1-\sqrt{3})} = \frac{2-2}{\sqrt{3}(1-\sqrt{3})} = 0$$

Thus we have

$$\begin{bmatrix} 1 & \frac{1-i}{1-\sqrt{3}} \\ \cancel{\frac{1+\sqrt{3}}{\sqrt{3}}}\ 0 & 0 \end{bmatrix}$$

so an unnormalized eigenvector is given by

$$\tilde{x}_2 = \begin{bmatrix} -(1-i) \\ 1-\sqrt{3} \end{bmatrix}$$

A normalized eigenvector is given by

$$x_2 = \frac{1}{\sqrt{(1-\sqrt{3})^2 + (1-i)(1+i)}} \begin{bmatrix} -(1-i) \\ 1-\sqrt{3} \end{bmatrix}$$

or $$x_2 = \frac{1}{\sqrt{1-2\sqrt{3}+3+1+1}}\begin{bmatrix} -(1-i) \\ 1-\sqrt{3} \end{bmatrix}$$

$$= \frac{1}{\sqrt{6-2\sqrt{3}}}\begin{bmatrix} -(1-i) \\ 1-\sqrt{3} \end{bmatrix}$$

Then our matrix of eigenvectors is given by

$$V = \begin{bmatrix} \frac{-(1-i)}{c_1} & \frac{-(1-i)}{c_2} \\ \frac{1+\sqrt{3}}{c_1} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix} \quad \text{so} \quad \cancel{V^H = \begin{bmatrix} \frac{-(1+i)}{c_1} & \frac{-(1+i)}{c} \\ \frac{1+\sqrt{3}}{c_1} & \end{bmatrix}}$$

$$V^H = \begin{bmatrix} \frac{-(1+i)}{c_1} & \frac{1+\sqrt{3}}{c_1} \\ \frac{-(1+i)}{c_2} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix} \quad \text{w/} \quad c_1 = \sqrt{6+2\sqrt{3}}$$
$$\& \quad c_2 = \sqrt{6-2\sqrt{3}}$$

Then we can check these results by ~~computing~~ computing

$$V\Lambda V^H = \begin{bmatrix} \frac{-(1-i)}{c_1} & \frac{-(1-i)}{c_2} \\ \frac{1+\sqrt{3}}{c_1} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{-(1+i)}{c_1} & \frac{1+\sqrt{3}}{c_1} \\ \frac{-(1+i)}{c_2} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{(1-i)}{c_1} & -\frac{(1-i)}{c_2} \\[2ex] \frac{1+\sqrt{3}}{c_1} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{c_1} & -\frac{(1+\sqrt{3})}{c_1} \\[2ex] -\frac{(1+i)}{c_2} & \frac{1-\sqrt{3}}{c_2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{(1-i)(1+i)}{c_1^2} + \frac{(1-i)(1+i)}{c_2^2} & \frac{(1-i)(1+\sqrt{3})}{c_1^2} - \frac{(1-i)(1-\sqrt{3})}{c_2^2} \\[2ex] \frac{(1+\sqrt{3})(1+i)}{c_1^2} - \frac{(1-\sqrt{3})(1+i)}{c_2^2} & -\frac{(1+\sqrt{3})^2}{c_1^2} + \frac{(1-\sqrt{3})^2}{c_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 2\left[-\frac{1}{c_1^2} + \frac{1}{c_2^2}\right] & (1-i)\left[\frac{1+\sqrt{3}}{c_1^2} - \frac{(1-\sqrt{3})}{c_2^2}\right] \\[2ex] (1+i)\left[\frac{1+\sqrt{3}}{c_1^2} - \frac{(1-\sqrt{3})}{c_2^2}\right] & -\frac{(1+\sqrt{3})^2}{c_1^2} + \frac{(1-\sqrt{3})^2}{c_2^2} \end{bmatrix}$$

Then evaluating each term gives

$$-\frac{1}{c_1^2} + \frac{1}{c_2^2} = \frac{-1}{6+2\sqrt{3}} + \frac{1}{6-2\sqrt{3}}$$

$$= \frac{-(6-2\sqrt{3}) + 6+2\sqrt{3}}{36-4\cdot 3} = \frac{4\sqrt{3}}{24} = \frac{\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$$

$$\begin{array}{r} 36 \\ 12 \\ \hline 24 \end{array}$$

$$\frac{1+\sqrt{3}}{c_1^2} - \frac{(1-\sqrt{3})}{c_2^2} = \frac{1+\sqrt{3}}{6+2\sqrt{3}} - \frac{(1-\sqrt{3})}{6-2\sqrt{3}}$$

$$= \frac{(1+\sqrt{3})(6-2\sqrt{3}) - (1-\sqrt{3})(6+2\sqrt{3})}{36-4\cdot 3}$$

$$= \frac{\cancel{6}-2\sqrt{3}+6\sqrt{3}-\cancel{6}-(\cancel{6}+2\sqrt{3}-6\sqrt{3}-\cancel{6})}{24}$$

$$= \frac{(-4+6+6)\sqrt{3}}{24}$$

$$= \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

Finally we have

$$\frac{-(1+\sqrt{3})^2}{c_1^2} + \frac{(1-\sqrt{3})^2}{c_2^2} = \frac{-(1+2\sqrt{3}+3)}{6+2\sqrt{3}} + \frac{(1-2\sqrt{3}+3)}{6-2\sqrt{3}}$$

$$= \frac{-4-2\sqrt{3}}{6+2\sqrt{3}} + \frac{4-2\sqrt{3}}{6-2\sqrt{3}}$$

$$= \frac{-(4+2\sqrt{3})(6-2\sqrt{3}) + (4-2\sqrt{3})(6+2\sqrt{3})}{24}$$

$$= \frac{-\cancel{24}+8\sqrt{3}-12\sqrt{3}+\cancel{4}\sqrt{3}+2\cancel{4}+8\sqrt{3}-12\sqrt{3}-\cancel{12}}{24}$$

$$= \frac{16\sqrt{3}-24\sqrt{3}}{24} = \frac{-8\sqrt{3}}{24} = -\frac{\sqrt{3}}{3} = -\frac{1}{\sqrt{3}}$$

given for the product

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 2 & 1-i \\ 1+i & -1 \end{bmatrix}.$$ Showing the correctness

Since $V = V^H$ λ's must be real.

(19) Matrix U w/ $v_1, \ldots, v_n$ an orthogonal basis for C^n is a unitary matrix. And $U^{-1} = U^H$

Thus if c are the coefficients of z with respect to the basis v_i then $Uc = z$ so $c = U^{-1}z = U^H z$

Thus $c = U^H z = \begin{bmatrix} v_1^H z \\ v_2^H z \\ \vdots \\ v_n^H z \end{bmatrix}$ so that $z = Uc = v_1$

$$= (v_1^H z)v_1 + (v_2^H z)v_2 + \cdots + (v_n^H z)v_n$$

(21) $v = (1, i, 1)$

$w = (i, 1, 0)$ $\qquad v^H w = (1 \ \ -i \ \ 1)\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = 0$

$z = ?$

Since z must be orthogonal to v^H & w^H we can require ~~request~~ a z such that it is in the null space of the following matrix

$$\begin{bmatrix} 1 & -i & 1 \\ -i & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 1+1 & +i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 2 & i \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -i & 1 \\ 0 & 1 & i/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & i/2 \end{bmatrix}$$

Thus pick $z = \begin{bmatrix} -1 \\ -i \\ 2 \end{bmatrix}$ & z will be orthogonal to v & w

These 3 vectors are an orthogonal basis for $\mathbb{C}^3$.

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(22) If A is Hermitian then

$$A^H = A$$

But since $A = R + iS$ we have $A^H = R^T + -iS^T$

~~Bo~~ ~~tes~~ assuming R + S are real matrices. Thus

for $A^H = A$ we must have

$$R^T - iS^T = R + iS \quad \text{so} \quad R^T = R \;+\; -S^T = S$$

So R is symmetric + S is skew ~~anti~~ symmetric

(23) The complex dimension for $\mathbb{C}^n$ is n.

A non-real basis for $\mathbb{C}^n$ is given by

$$\begin{pmatrix} i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ i \end{pmatrix}$$

(24) A 1×1 Hermitian matrices must be real since

~~B~~ ~~B~~ $A^H = A \Rightarrow \bar{a} = a$ so a is real

a 2×2 Hermitian matrix must be

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^H = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

So $\bar{a} = a$ & a is real

$\bar{d} = d$ so d is real & the final condition is given

by $\bar{c} = b$ (or $\bar{b} = c$). Thus all 2×2 Hermitian

& ~~BAC~~

matrices look like $A = \begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix}$ w/ a & d real

For a 1×1 matrix to be unitary it must have eigenvalues w/ a magnitude of 1. Thus $|a| = 1 \Rightarrow a = \pm 1$

For a 2×2 matrix to be unitary it must have eigenvalues w/ magnitude of 1. Since the eigenvalues of A above are given

by $\begin{vmatrix} a-\lambda & b \\ \bar{b} & d-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(d-\lambda) - |b|^2 = 0$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - |b|^2 = 0$$

$$\Rightarrow \lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad - |b|^2)}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4|b|^2}}{2}$$

Use the fact that for unitary matrices $T^{-1} = T^H$ to determine another condition on a, b, c

From which we must impose that $|\lambda| = 1$.

... not sure how to get the explicit representation of the Unitary matrix

(25) The eigenvalues of A are determined by

$$\det(A - \lambda I) = 0$$

taking the transpose of both sides produces

$$\det(A^T - \lambda I) = 0$$

& taking the complex conjugate of both sides gives

$$\det(A^H - \bar{\lambda} I) = 0$$

So if λ an eigenvalue of A then $\bar{\lambda}$ is an eigenvalue of A^H.

(26) For $A = 1 - 2uu^H$ to be Hermitian consider

$$A^H = 1 - 2(uu^H)^H = 1 - 2uu^H = A$$

to be unitary we must have ~~$U^{-1} = U^H$~~ $A^{-1} = A^H$

consider $A^H \cdot A = A \cdot A = (I - 2uu^H)(I - 2uu^H)$

$$= I - 2uu^H - 2uu^H + 4uu^H uu^H$$

$$= I - 2uu^H - 2uu^H + 4uu^H = I \quad \therefore \quad A^H = A^{-1}$$

The rank one matrix uu^H is the projection onto the line u in $\mathbb{C}^n$.

(27) If $A+iB$ is unitary then $A^T - iB^T = (A+iB)^{-1}$

$$\cancel{A^H} = \cancel{A^T \ldots} \; iB$$

so $\cancel{A \ldots A}$ + $\cancel{B \ldots} B$

Thus $(A+iB)(A^T - iB^T) = I$ by multiply both sides by ~~...~~ $(A+iB)$ to get

$$AA^T - iAB^T + iBA^T + BB^T = I$$

$$\Rightarrow \quad AA^T + BB^T = I \quad + \quad AB^T - BA^T = 0$$

Now consider the matrix $Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ + compute

$$Q^T Q = \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix}\begin{bmatrix} A & -B \\ B & A \end{bmatrix} = \begin{bmatrix} A^TA + B^TB & -A^TB + B^TA \\ -B^TA + A^TB & B^TB + A^TA \end{bmatrix}$$

$$= \begin{bmatrix} I & -A^TB + B^TA \\ -B^TA + A^TB & I \end{bmatrix}$$

$\cancel{(A+iB)(A^T-iB^T)=I}$ Now considering ~~the~~ an other order of the multiplication inverse we have

$$(A^T - iB^T)(A+iB) = I$$

$$\Rightarrow A^TA + iA^TB - iB^TA + B^TB = I$$

$$\Rightarrow A^TA + B^TB = I \quad + \quad A^TB - B^TA = 0$$

Thus from the 2nd equation above we see that the off diagonal terms are zero giving

$$Q^TQ = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$ showing Q is an orthogonal matrix

(28) If $A+iB$ is Hermitian

$$A^T - iB^T = A + iB \quad \Rightarrow \quad A = A^T \ + \ B = -B^T$$

then consider $M = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ to find that

$$M^T = \begin{bmatrix} A^T & +B^T \\ -B^T & A^T \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} = M$$ so M is symmetric

(29) A Hermitian matrix is one that $A^H = A$

So ~~from~~ $\cancel{A^{-1} \cdot A = I}$ ~~take the Hermitian of~~ both

~~sides we have~~ $\cancel{(A^{-1} \cdot A)^H = I}$

$$\cancel{\Rightarrow A^H (A^{-1})^H = I}$$

$$\cancel{\Rightarrow A^H (A^H)^{-1} = I}$$

$\Rightarrow$ ~~Showing that~~ $\cancel{(A^{-1})^H =}$

consider $(A^{-1})^H = (A^H)^{-1} = A^{-1}$ thus A^{-1} is Hermitian

(30) First compute the eigenvalues & eigenvectors of A.

The eigenvalues are given by

$$\begin{vmatrix} 2-\lambda & 1-i \\ 1+i & 3-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(3-\lambda) - (1-i)(1+i) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 - (1+1) = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, 4$$

them which have to be real since A is Hermitian.

Now consider the eigenvector for $\lambda = 1$. We have to find the null space of

$$\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1-i \\ 0 & 2-(1-i)(1+i) \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}$$

So $\tilde{x}_1 = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ is in the null space & is an eigenvector associated w/ $\lambda = 1$

$$\cancel{x_1 = \frac{1}{\sqrt{1+2}} \begin{bmatrix} 1-i \\ -1 \end{bmatrix} \text{ is a}} \text{ Norm}$$

For $\lambda = 4$ the eigenvector is given by considering

$$\cancel{\begin{bmatrix} 2- \end{bmatrix}} \quad \begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{i-1}{2} \\ 1+i & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{i-1}{2} \\ 0 & -1-\frac{(i-1)(i+1)}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{i-1}{2} \\ 0 & 0 \end{bmatrix}$$

$$-\frac{(-1-1)}{2} = 0$$

So $x_2 = \begin{bmatrix} i-1 \\ -2 \end{bmatrix}$ is a 2nd eigenvector

Then $A = S\Lambda S^{-1} = \begin{bmatrix} 1-i & i-1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} S^{-1}$

(31) If $A = U\Lambda U^H$ then

$$AA^H = U\Lambda U^H U\Lambda^H U^H = U\Lambda\Lambda^H U^H$$

& $$A^H A = U\Lambda^H U^H U\Lambda U^H = U\Lambda^H\Lambda U^H$$

~~But A is Hermitian since Λ is diag~~

But $\Lambda\Lambda^H = \Lambda^H\Lambda$ since Λ is diagonal thus

$A^H A = AA^H$ & A is normal

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(1) Eq 3 is given by

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 0 & -1 & -i & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 0 & -i & -1 & i \end{bmatrix}$$

+ F_4 is given by $F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$

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(2) Eq 3 is given by

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = A B \cdot C$$

Then $F_4^{-1} = C^{-1} B^{-1} A^{-1} = C^T B^{-1} A^{-1}$ $\qquad -1-1=-2$ /

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B^{-1} \cdot A^{-1}$$

Now B ~~tees~~ is block diagonal + so can be inverted easily

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So $$B^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

Finally we require the inverse of A. Given by the following manipulation.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -i & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2i & 0 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & i & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2i} & 0 & -\frac{1}{2i} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & i & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2i} & 0 & -\frac{1}{2i} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2i} & 0 & -\frac{1}{2i} \end{array}\right]$$

Thus $$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix}$$

Check $$A \cdot A^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix}\frac{1}{2}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \frac{1}{2} = I \quad \checkmark$$

So that

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & -i/2 & 0 & i/2 \end{bmatrix}$$ is the fast factorization required

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} =$$

check?

$i^3 = -i$

$i^6 = i^2 = -1$

$i^9 = i$

$$\stackrel{?}{=} \frac{1}{4} \operatorname{conj} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{4} \operatorname{conj} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & +i \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$ which is what we have above.

(3) Eq 3 is given by

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $F_4^T = F_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & i & 0 & -i \end{bmatrix}$

(4) $F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$ $\qquad w = e^{i2\pi/6} = e^{i\pi/3}$

so D is given by $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w^2 \end{bmatrix}$

+ $F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & w^2 & w^4 \\ 1 & w^4 & w^6 \end{bmatrix}$

Thus $\begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & 0 \\ 0 & F_3 \end{bmatrix}$ is given by

$$= \begin{bmatrix} F_3 & DF_3 \\ F_3 & -DF_3 \end{bmatrix}$$

Thus DF_3 is important & is given by

$$DF_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 \\ 1 & \omega^4 & \omega^6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega^3 & \omega^5 \\ \omega^2 & \omega^6 & \omega^8 \end{bmatrix}$$

From the definition of ω_n we have $\omega^6 = 1$ so $\omega^8 = \omega^2$ & DF_3 is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ \omega & \omega^3 & \omega^5 \\ \omega^2 & 1 & \omega^2 \end{bmatrix}$$

The permutation matrix in this case is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus our multiplication to this point is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^4 & \omega^6 & \omega^2 & 1 & \omega^2 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & \omega^2 & \omega^4 & -\omega & -\omega^3 & -\omega^5 \\ 1 & \omega^4 & \omega^6 & -\omega^2 & -1 & -\omega^2 \end{bmatrix} P$$

Since multiplying a matrix A on the left by a permutation matrix selects columns from A according to the rows in P. Thus multiplying by P on the left of the above gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^6 & \omega^2 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\omega & \omega^2 & -\omega^3 & \omega^4 & -\omega^5 \\ 1 & -\omega^2 & \omega^4 & -1 & \omega^6 & -\omega^2 \end{bmatrix}$$

(5) Since F has its 1st row & column equal to all ones

$F \cdot v = w$. Also ~~Fw~~ = in the 4×4 case we have that

$$Fw = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4v.$$

Thus $F^{-1}w = v$ & $F^{-1}v = \frac{1}{4}w$.

(6) The 4×4 Fourier matrix is given by

$$F^2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \cancel{0} & 1-i-1+i & 1+1+1+1 \\ 0 & 1-i-1+i & 4 & 0 \\ 0 & 1+1+1+1 & 1+-1+1-1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Then $F^4 = (F^2)(F^2) = 16\, P \cdot P$ with P the above permutation matrix.

Note that all permutations have as their inverse the transpose of themselves. Since $P = P^T$ or, since this matrix is symmetric $P^2 = P \cdot P^T = I$.

Thus $F^4 = 16 \cdot I$

(7) If $c = (1,0,1,0)$ Then the 3 steps are permutations, two FFTs ~~...~~ on the split data + finally reassembly. For this value of c, we have

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} \Rightarrow \begin{matrix}\begin{bmatrix}1\\1\end{bmatrix}\\ \begin{bmatrix}0\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{matrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}\\ \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{bmatrix}2+0\\0+i0\\2-0\\0-i0\end{bmatrix} = \begin{bmatrix}2\\0\\2\\0\end{bmatrix}$$

For the 2nd value of $c = (0,1,0,1)$ we have the 3 steps

$$\begin{bmatrix}0\\1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}0\\0\\1\\1\end{bmatrix} \Rightarrow \begin{matrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}\\ \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{bmatrix}0+2\\0+i0\\0-2\\0-i0\end{bmatrix} = \begin{bmatrix}2\\0\\-2\\0\end{bmatrix}$$

(8) One multiplication of the FFT matrix can be decomposed into 3 steps 1) The splitting of the vector into two pieces, 2) two FFT's of $\frac{1}{2}$ of the size + then 3) reassembly. Thus for

$c = (1,0,1,0,1,0,1,0)$ we have

$$\begin{bmatrix}1\\0\\1\\0\\1\\0\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix} F_4\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\ F_4\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{matrix} 4\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \\ \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{matrix} 4\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix} & \end{bmatrix}\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \\ \begin{bmatrix}4\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1 & & & \\ & e^{i\pi/4} & & \\ & & e^{i\pi/2} & \\ & & & e^{i3\pi/4}\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\end{matrix}$$

$$w = e^{\frac{i2\pi}{8}} = e^{\frac{i\pi}{4}}$$

which is given by

$$\begin{bmatrix}4\\0\\0\\0\\4\\0\\0\\0\end{bmatrix}$$

For the second ~~set~~ of c we have $c = (0,1,0,1,0,1,0,1)$ we have

$$\begin{bmatrix}0\\1\\0\\1\\0\\1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix} \Rightarrow \begin{matrix} F_4\begin{bmatrix}0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \\ F_4\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix}\end{matrix} \Rightarrow \begin{matrix} \begin{bmatrix}0\\0\\0\\0\end{bmatrix} + D_4\begin{bmatrix}4\\0\\0\\0\end{bmatrix} \\ \begin{bmatrix}0\\0\\0\\0\end{bmatrix} - D_4\begin{bmatrix}4\\0\\0\\0\end{bmatrix}\end{matrix}$$

We have $D_4 \begin{bmatrix}4\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&e^{i\pi/4}&0&0\\0&0&e^{+\pi/2\, i}&0\\0&0&0&e^{\frac{3\pi}{4}i}\end{bmatrix}\begin{bmatrix}4\\0\\0\\0\end{bmatrix} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix}$

giving for the final result of the FFT of c the following

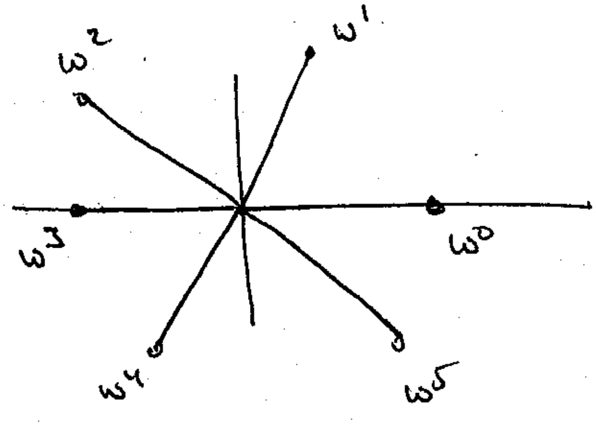
$$\begin{bmatrix}4\\0\\0\\0\\4\\0\\0\\0\end{bmatrix}$$

(9) If $w = e^{\frac{2\pi i}{64}}$ $\cancel{e^{\frac{\pi i}{32}}}$

Then $w^2 = e^{\frac{4\pi i}{64}} = e^{\frac{2\pi i}{32}}$ are among the 32 roots of unity.

& $\sqrt{w} = e^{\frac{2\pi i}{128}}$ are among the 128 roots of unity

(10) (a) $w = e^{i2\pi/6} = e^{i\pi/3}$



So that the 6 - 6 roots of unity are given by

$e^{i\pi/3\,k}$ $\quad k = 0, 1, 2, 3, 4, 5$

Adding the 6 - 6th roots of unity together we have

$1+\omega^1+\omega^2+\omega^3+\omega^4+\omega^5$

~~1 + cos~~(

$= 1+e^{i\pi/3}+e^{\frac{2\pi i}{3}}+ e^{\pi i}+e^{\frac{4\pi i}{3}}+e^{\frac{5\pi i}{3}}$

$= 1+\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})+\cos(\frac{2\pi}{3})+i\sin(\frac{2\pi}{3})+\cos(\pi)+i\sin(\pi)$

$+\cos(\frac{4\pi}{3})+i\sin(\frac{4\pi}{3})+\cos(\frac{5\pi}{3})+i\sin(\frac{5\pi}{3})$

$= \cancel{1}+\cancel{\tfrac{1}{2}}+i\frac{\cancel{\sqrt{3}}}{2}+\cancel{\tfrac{-1}{2}}+i\frac{\cancel{\sqrt{3}}}{2}-\cancel{1}+0-\cancel{\tfrac{1}{2}}-i\frac{\cancel{\sqrt{3}}}{2}+\cancel{\tfrac{1}{2}}-i\frac{\cancel{\sqrt{3}}}{2}=0$

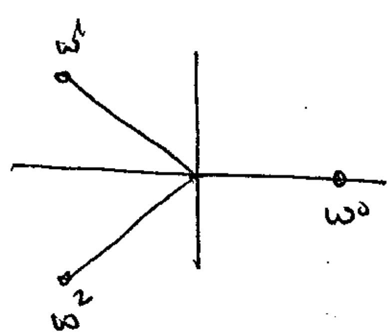
(b) The 3 cube roots of 1 are given by

$\omega^k = e^{\frac{2\pi i}{3}k}$ $k=0,1,2$, which are

$\omega^0=1$; $\omega^1=e^{\frac{2\pi}{3}i}$; $\omega^2=e^{\frac{4\pi}{3}}$ or

$\omega^0=1$; $\omega^1=\cos(\frac{2\pi}{3})+i\sin(\frac{2\pi}{3})$; $\omega^2=\cos(\frac{4\pi}{3})+i\sin(\frac{4\pi}{3})$

or $\omega^0=1$; $\omega^1=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$; $\omega^2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$



Which when we ~~sum~~ sum gives 0.

(11) Multiplying P·F as suggested gives

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

& Multiplying F by the diagonal matrix $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1 & \lambda_2 i & \lambda_3 i^2 & \lambda_4 i^3 \\ \lambda_1 & \lambda_2 i^2 & \lambda_3 i^4 & \lambda_4 i^6 \\ \lambda_1 & \lambda_2 i^3 & \lambda_3 i^6 & \lambda_4 i^9 \end{bmatrix}$$

Thus ~~[illegible]~~ corresponds to $\lambda_1 = 1$; $\lambda_2 = +i$; $\lambda_3 = i^2$; $\lambda_4 = i^3$

(12) $\det(P - \lambda I) = 0$ ~~[illegible]~~ when P is the permutation matrix ~~given~~ in problem 11 is given by

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{vmatrix}$$

$= -\lambda(-\lambda)^3 - 1 = 0$ or $\lambda^4 = 1$ so that the eigenvalues are the 4 - 4th roots of unity.

To have the eigenvalues the cube roots of unity we have P given by $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ which we can check with

$$\det(P - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix}$$

$$= (-\lambda)^3 + 1 = 0$$

$$\Rightarrow \lambda^3 = 1$$

(13) (a) From ~~the~~ problem 11. we see that these eigenvectors correspond to $e_1 = 1 = e^{\frac{2\pi}{4} i \cdot 0}$ & to $e_2 = e^{\frac{2\pi}{4} i \cdot 1} = e^{\frac{\pi}{2} i} = i$

(b) From the representation of C ~~we can see that~~ & the cyclic permutation matrix P we can see that

$$C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3$$

So from $P = F\Lambda F^{-1}$ we have that

$$C = c_0 F F^{-1} + c_1 F\Lambda F^{-1} + c_2 F\Lambda^2 F^{-1} + c_3 F\Lambda^3 F^{-1}$$

$$= F(c_0 I + c_1\Lambda + c_2\Lambda^2 + c_3\Lambda^3)F^{-1} = F E F^{-1}$$

The matrix E is diagonal & contains the eigenvalues of C

(14) If $C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$ $\cancel{= 2I - P - P^3}$ we see that C

has the following decomposition $C = 2I - P - P^3$ so from problem 13 we ~~so~~ ~~have~~ ~~then~~ have

$$C = F(E)F^{-1} \quad \text{with} \quad E = 2I - \Lambda - \Lambda^3 \quad \text{where } \Lambda \text{ is the}$$

~~E~~ ~~the~~ diagonal matrix with ~~eigenvalues~~ ~~values~~ eigenvalues ~~on the diagonal~~ of the cyclic permutation matrix P. For this cyclic permutation matrix P has eigenvalues given by $\lambda_1 = 1;\ \lambda_2 = i;\ \lambda_3 = -1;\ \lambda_4 = -i$

Then $\Lambda = \begin{bmatrix} 1 & & & \\ & i & & \\ & & -1 & \\ & & & -i \end{bmatrix}$ $\quad \Lambda^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$

~~$(\Lambda^3)^2 = P_{ii}$~~

So that E is now given by

$$E = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & +i & & \\ & & -1 & \\ & & & -i \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & -i & & \\ & & -1 & \\ & & & i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & & & \\ & 2-i+i & & \\ & & 2+1+1 & \\ & & & 2+i-i \end{bmatrix} = \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 4 & \\ & & & 2 \end{bmatrix}$$

Thus the eigenvalues of C are given by $\lambda_1 = 0$; $\lambda_2 = 2$; $\lambda_3 = 4$; $\lambda_4 = 2$

(15) Since $C = FEF^{-1}$ to ~~multiply~~ compute Cx we can compute $FEF^{-1}x$. Now to directly compute Cx requires $O(n^2)$ multiplication. Computing $F^{-1}x$ requires $O(\frac{n}{2}\log_2 n)$ calculations since it is an FFT. $EF^{-1}x$ then requires an additional $O(n)$ calculations to multiply the diagonal elements. Then $F(EF^{-1}x)$ requires another $O(\frac{n}{2}\log_2 n)$ terms requiring in total $O(n\log_2 n) + n$ calculations

(16) Given $c_0 + c_2,\ c_0 - c_2,\ c_1 + c_3,\ c_1 - c_3$ to quickly compute

$$Fc = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + ic_1 + i^2c_2 + i^3c_3 \\ c_0 + i^2c_1 + i^4c_2 + i^6c_3 \\ c_0 + i^3c_1 + i^6c_2 + i^9c_3 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + ic_1 - c_2 - ic_3 \\ c_0 - c_1 + c_2 - c_3 \\ c_0 - ic_1 - c_2 + ic_3 \end{bmatrix}$$

let $$\underline{X} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_0 + c_2 \\ c_0 - c_2 \\ c_1 + c_3 \\ c_1 - c_3 \end{bmatrix}$$

Fc is given by $$\begin{bmatrix} x_0 + x_2 \\ x_1 + i x_2 \\ x_0 - x_2 \\ x_1 - i x_3 \end{bmatrix}$$

which we can check ~~with~~ by computing this expression directly to obtain

$$\begin{bmatrix} c_0 + c_2 + c_1 + c_3 \\ c_0 - c_2 + i(c_1 + c_3) \\ c_0 + c_2 - (c_1 + c_3) \\ c_0 - c_2 - i(c_1 - c_3) \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + ic_1 - c_2 + ic_3 \\ c_0 - c_1 + c_2 - c_3 \\ c_0 - ic_1 - c_2 + ic_3 \end{bmatrix}$$