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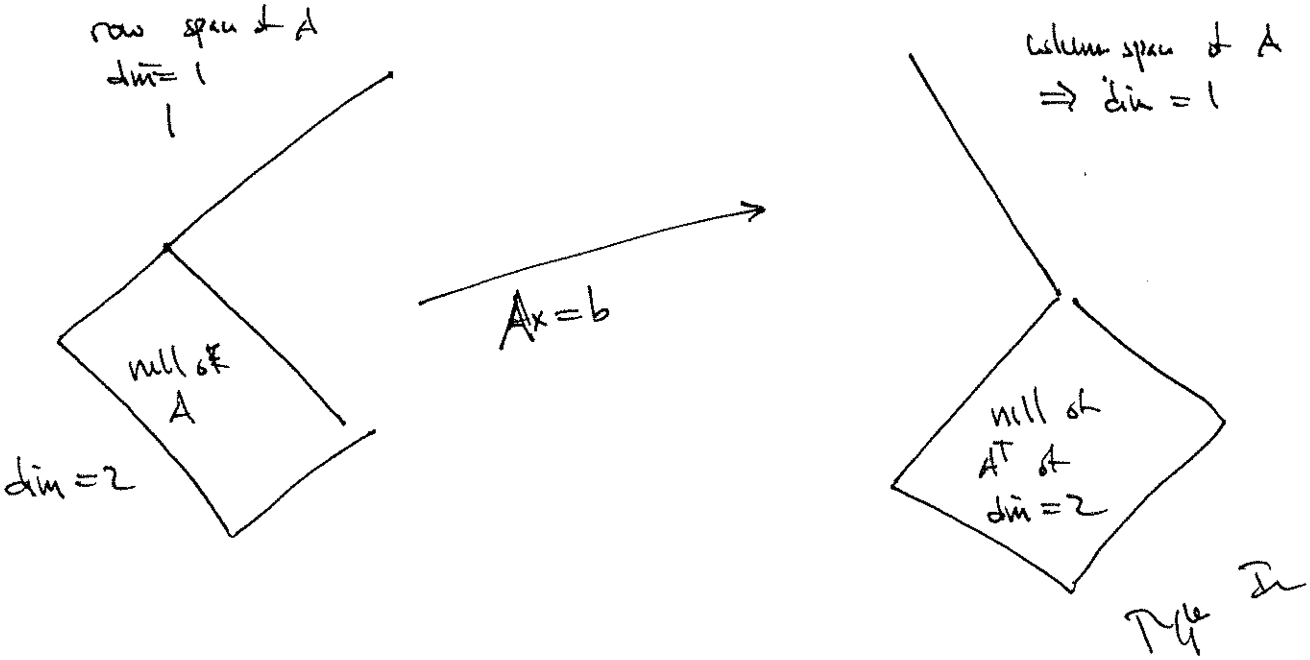
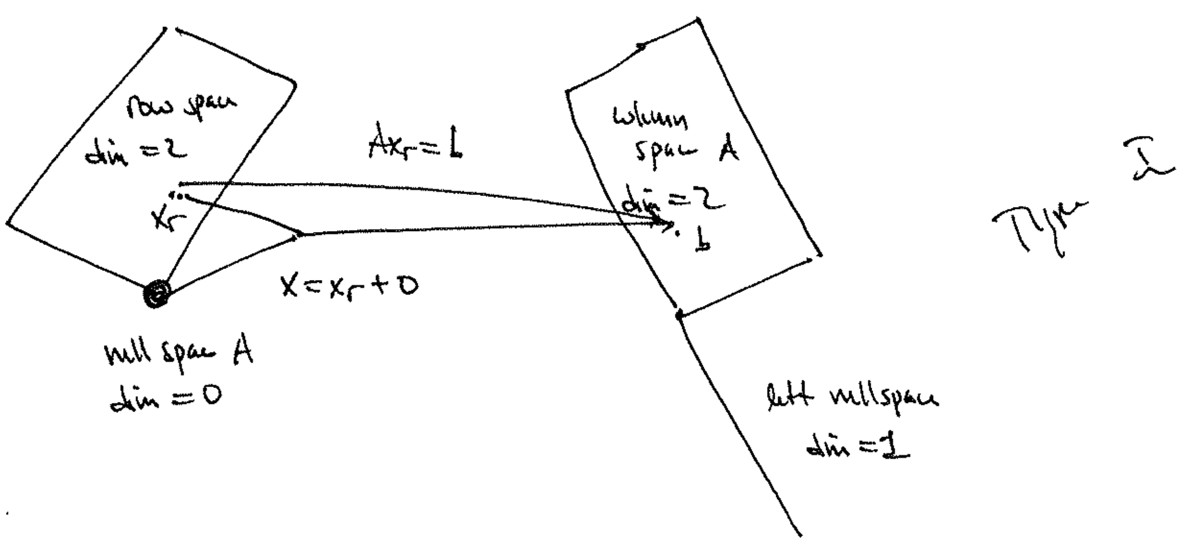
(1) ~~A is 2x3 so $m=2$ + $n=3$ + $r=1$,~~
~~Then row space of A has dimension 1~~
~~" column space of A has dimension 1~~
~~the nullspace of A has size $n-r = 3-1 = 2$~~
~~the left nullspace of A has size $m-r = 2-1 = 1$~~



(2) ~~$m=3$, $n=2$, $r=2$~~
~~dim 1 dim $N(A) = 2-2 = 0$~~
~~dim $N(A^T) = 3-2 = 1$~~



$\boxed{\text{pg 171 Strang}}$

(e) The fact that the columns add to the zero column means that the vector of all ones is in the nullspace of this matrix

Let's see if we can construct a 2x2 example matrix that has the desired properties 1st

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \Rightarrow \begin{matrix} a+b=0 \\ c+d=0 \end{matrix}$$

$$\& \quad \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

~~Thus our system to solve is~~

$$\begin{matrix} a+b=0 \\ c+d=0 \\ a+c=1 \\ b+d=1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

performing row reduction on the augmented matrix we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

since the last two equations contradict each other I conclude this is not possible

Also ~~[illegible]~~, a row of all ones will be in the nullspace but also in the row space, but since ~~a~~ this vector is not zero this is a contradiction & no such matrix exists

(4) $\begin{bmatrix} \cancel{1} & \cancel{1} \\ \cancel{1} & \cancel{-1} \end{bmatrix}$ It is not possible for the ~~null space~~ row space to contain the ~~null space~~ nullspace. ~~It is~~ ~~If~~ let x be a member of both, then from the ~~1st~~ 2nd fundamental theorem of linear algebra that the row space & null space are orthogonal $x^T x = 0$ which is not true ~~unless~~ unless $x = 0$

(5) (a) y is perpendicular to b, since b is in the column space of A + y is in A's left nullspace

(b) ~~If $A^Ty = 0$ then $y^TA = 0$~~ so

~~$y^TAx = y^Tb$~~ If $Ax=b$ has no solution then

b is <u>not</u> in the column space of A therefore $y^Tb \neq 0$

+ y is not perpendicular to b

(6) If $x = x_r + x_n$ Then $Ax = Ax_r + Ax_n = Ax_r + 0 = Ax_r$

~~because Ax_r is a linear~~ It is in the column space

because Ax_r is a linear combination of the columns

(7) For Ax to be in the nullspace of A^T ~~...~~

~~Then~~ ~~$A^TAx = 0$~~ it must be in the left null space of A

~~or~~ ~~the~~ but Ax is in the column space of A + these

two spaces are orthogonal. Because it is in both

it must be the zero vector

(8) (a) Its column space is perpendicular to its left null space but

by the symmetry of A the left nullspace is the same as

its null space

(b) If $Ax = 0$ & $Az = 5z$ then

$z^T A^T = 5 z^T$

or $z^T A x = 5 z^T x$

$\Rightarrow$ since $Ax = 0$ that $5 z^T x = 0$ or $z^T x = 0$.

In terms of subspace x is in the nullspace & left null space of A. z is in the column space of A & $\therefore$ since the column space & ~~nulls~~ left nullspace are $\perp$ we have x & z perpendicular

(9) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Then the row space is given by span $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

The column space is given by span $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

The nullspace is given by span $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

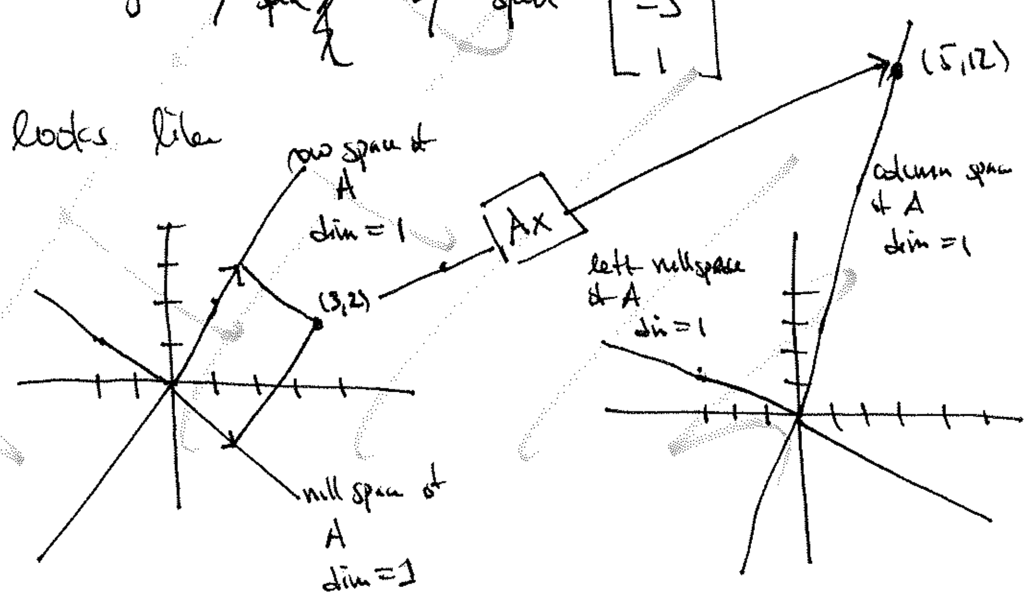
The left nullspace is given ~~by span~~ by span $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Thus Figure 4.2 looks like

$\cancel{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 5 \\ 9+12 \end{bmatrix}} \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

$= \cancel{\begin{bmatrix} 7 \\ 21 \end{bmatrix}}$



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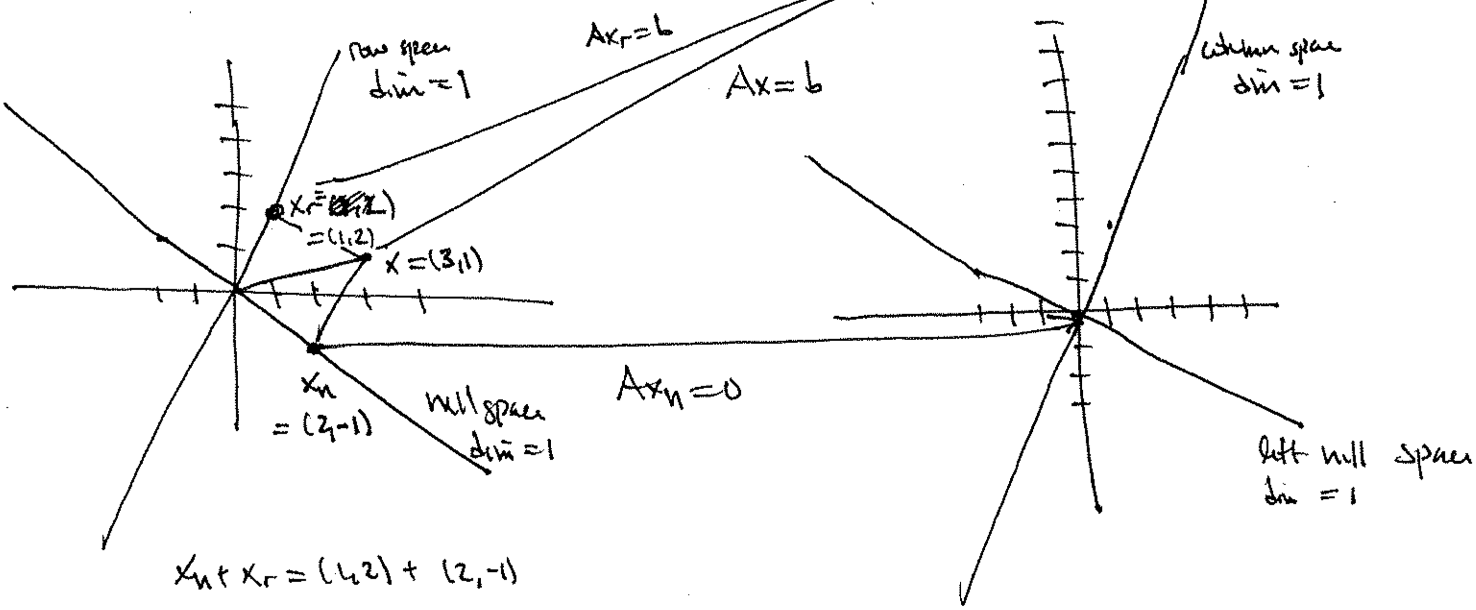
(9) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

has rank = 1 a ~~nullspace given by the span~~ row space given by the span of $(1,2)$ a column space given by the span of $(1,3)$, and a nullspace given by $(-2,1)$, + a left null space given by ~~C-B~~, the span of $(-3,1)$. Then Figure 4,2 would look like the following



$x_n + x_r = (1,2) + (2,-1)$

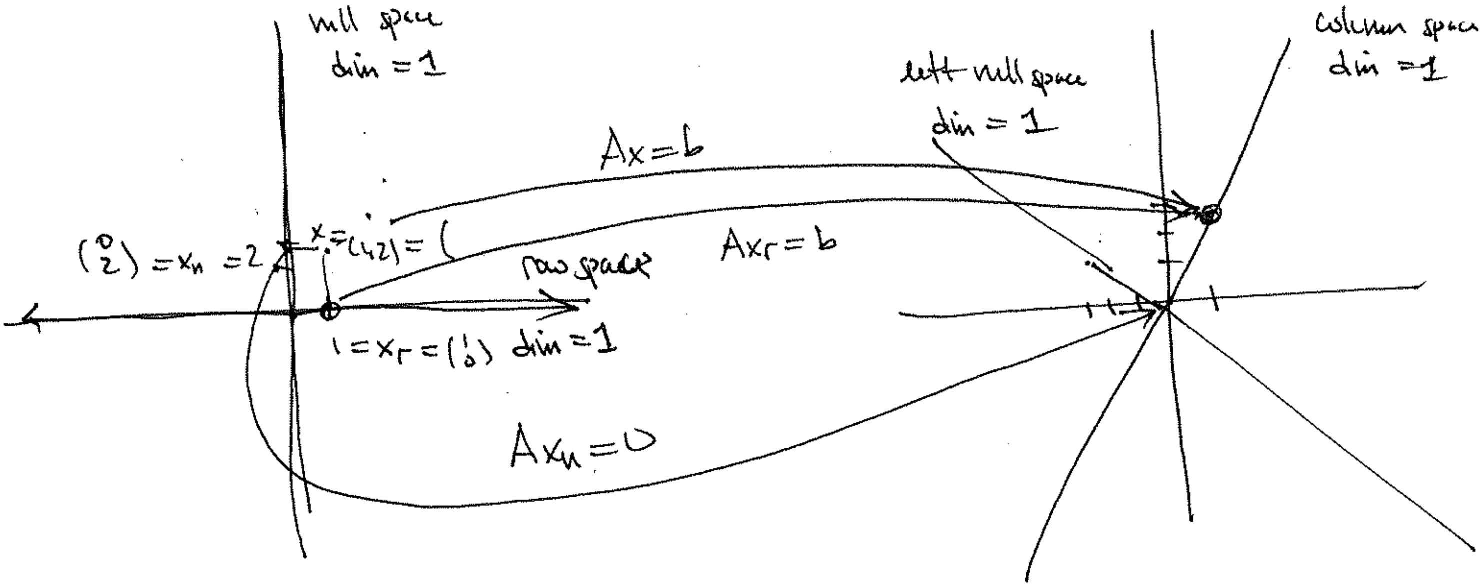
$= (3,1)$

Then $Ax = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 9+6 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$

$Ax_n = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

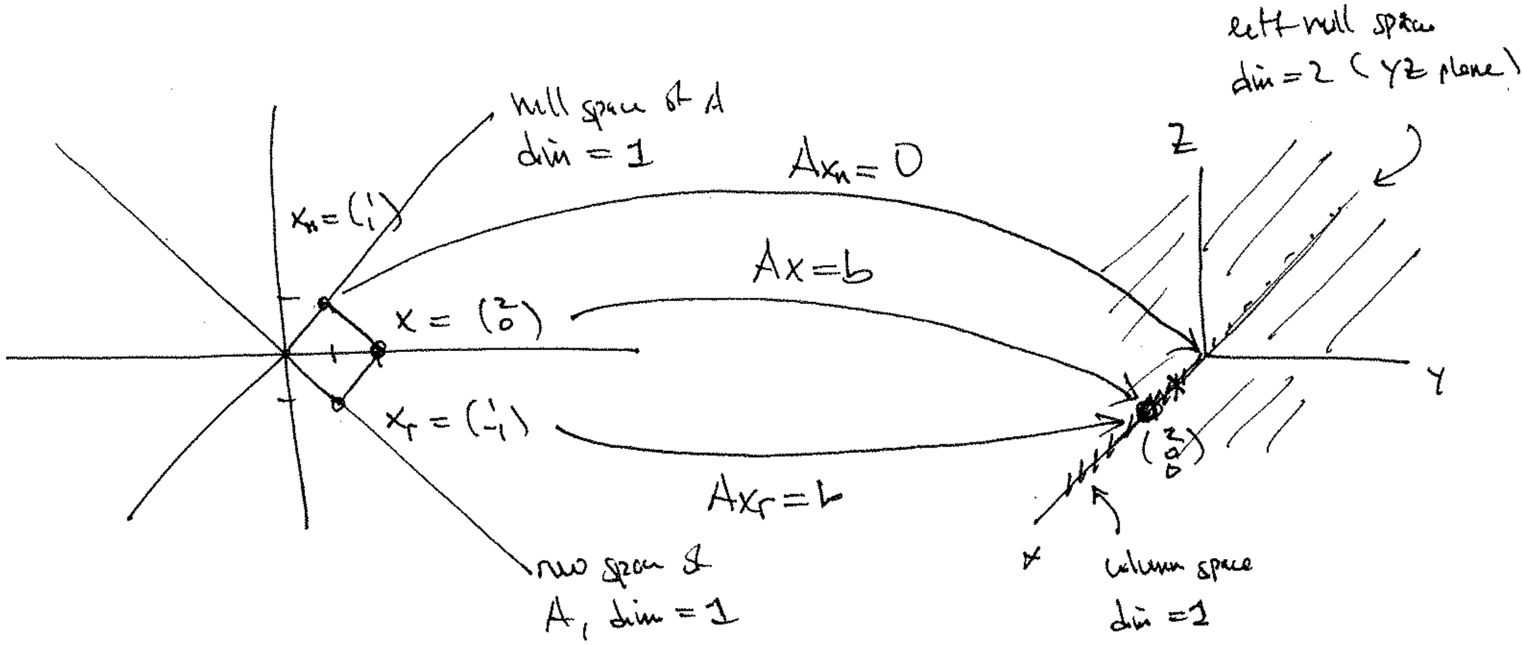
we have a rank of 1, a row space given by the span of (1,0), a column space given by the span of (1,3), A null space given by the span of (0,1), & finally a left null space given by the span of (−3,1). Then Fig 4..2 would look like



Then $Bx = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(10) let $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then A has rank $r=2$, a row space given by the span of $(1,-1)$ a column space given by the span of $(1,0,0)$, a null space given by the span of $(1,1)$ + a left null space given by the span of $(0,1,0)$ + $(0,0,1)$. Then Fig 4.2 will look like



Then $Ax = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(11) let $y \in N(A^T)$

Then $A^T y = 0$

So $y^T A x = \cancel{(x^T A^T y)^T = (x^T(A^T y))^T} \ (y^T A x)^T$

since $y^T A x$ is a scalar. But $(y^T A x)^T = x^T A^T y = x^T \mathbf{0} = 0$

∴ 故 本 y is $\perp$ to Ax.

(12) The Fredholm alternative is, exactly one of these two problems has a solution

(1) $Ax = b$ — b is in the column space of A.

(2) $A^T y = 0$ & $b^T y \neq 0$ — or there exist a ~~???~~ vector in the left null space that is not orthogonal to b.

To make eqn (2) hold

pick $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ & $b = \cancel{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Then $Ax = b$ has no solution. We can show this by considering the augmented matrix $[A\ b] = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

Since the last row is Not all zeros, 故 ∴ $Ax = b$ has No solution

5

For the second part of the Fredholm alternative, find a y such that $A^T y = 0$ + $b^T y \neq 0$

$A^T y$ is given by $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Then $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} c$

so ~~to~~ then $b^T y = 2(-2) + 1(1) = -3 \neq 0$.

$\therefore$ $y = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ will work.

(13) $S = \{0\}$ then $S^\perp = \mathbb{R}^3$.

If $S = \text{span}\{(1,1,1)\}$ then $S^\perp$ = all vectors y such

that $y^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \Rightarrow y_1 + y_2 + y_3 = 0$

~~$S^\perp = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$~~

$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0$

$\therefore$ the two elements in the Nullspace are given by

$y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ + $y_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

If $S = \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$ then $S^{\perp}$ consists of all

vectors y such that $y^T \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 0$ & $y^T \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 0$

So $2y_1 = 0$ & $3y_3 = 0 \Rightarrow y_1 = 0$ & $y_3 = 0$

So $S^{\perp} = \text{span} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(14) $S^{\perp}$ is the null space of $A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

$\therefore$ $S^{\perp}$ is a subspace even if S is not.

(15) $L^{\perp}$ is the plane $\perp$ to this line. Then $(L^{\perp})^{\perp}$ is a line $\perp$ to $L^{\perp}$. so $(L^{\perp})^{\perp}$ is the same line as originally

(16) $V^{\perp}$ contains only the zero vector
Then $(V^{\perp})^{\perp}$ contains all of $\mathbb{R}^4$. so $(V^{\perp})^{\perp}$ is the same as V.

(17)

(17) Suppose S is spanned by $(1,2,2,3)$ + $(1,3,3,2)$. Then S^+ is spanned by the null space to the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

So a nullspace given by $x_3 = 1, x_4 = 0$ + $x_1 = 0$ + $x_2 = -1$

+ $x_3 = 0, x_4 = 1 \therefore x_1 = -5, x_2 = 1$

so it is spanned by

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(18) If P is the plane given then $A = [1\ 1\ 1\ 1]$ has this plane as its Null space. Then $P^\perp$ are the elements of the left Null space of A i.e. the null space of A^T

since $A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ we have a null space

given by the span of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(19) If $S \subset V$ then $S^{\perp} \supset V^{\perp}$

Pr:

let $y \in V^{\perp}$ then $\forall$ element x in V we have $x^T y = 0$.

~~But it~~ But we can also say that $\forall$ element $x \in S$ it ~~is~~ is also in V & $\therefore$ $x^T y = 0$ so $y \in S^{\perp}$. Therefore $V^{\perp} \subset S^{\perp}$

(20) The 1st column ~~of~~ ~~spanned~~ ~~orthogonal~~ of A^{-1} is orthogonal to the span of the 2nd through the ~~end~~ last columns of A

(21) $A^T A$ would be I.

(22)

$$\cancel{A = \begin{bmatrix} 1 & -1 & x & 1 \\ -1 & 1 & 4 & -1 \\ 2 & 1 & 2 & 7 \end{bmatrix}}$$

$-1-1-1 \neq 0$

~~We can easily pick two vector that are orthogonal for example~~

let ~~$A_1 =$~~ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$

~~Then we are looking for a 3rd vector that is orthogonal to these two~~

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(22) ~~Start w/~~ $I = \begin{bmatrix} \cancel{1} & \cancel{0} & 0 \\ \cancel{0} & \cancel{1} & \cancel{0} \\ 0 & 0 & 1 \end{bmatrix}$

How compute a matrix like requested?

$A^T A$ must be diagonal since it represents every ~~row~~ ~~times~~ ~~every~~ ~~column~~ ~~of A. When the rows of~~ ~~to~~

column of A times every column of A. When the two columns are different the result is zero. When they are the same the norm ~~of the~~ squared of that column results

(23) The lines $3x + y = b_1$ & $6x + 2y = b_2$ are parallel

They are the same line if $2b_1 = b_2$. Then (b_1, b_2) is

$\perp$ to the left null space of $A = \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ or

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (check $\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -2b_1 + b_2$

$= -2b_1 + 2b_1 = 0$

The Nullspace of the matrix is the line $3x + y = 0$.

On ~~a~~ vector in that Null space is $(-1, 3)$

24 (a) As discussed in the book if two subspaces are orthogonal then they can only ~~meet~~ meet at the origin. But for these two planes we have many intersections. Solve the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

then (x, y, z) will be on both planes

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

So $z = 0$ & $x + y = 0$ so any vector of the form

$$\underline{x} = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix} = \text{span}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

is in both planes & so

~~cannot~~ ~~be~~ these spaces cannot be orthogonal

(b) The two lines specified are described as the span of the vectors $\begin{bmatrix}2\\4\\5\end{bmatrix}$ + $\begin{bmatrix}1\\-3\\2\end{bmatrix}$ respectively

For their subspaces to be orthogonal, the subspaces generating vectors must be orthogonal. In this case

$$(2\ 4\ 5)^T\begin{pmatrix}1\\-3\\2\end{pmatrix} = 2-12+10 = 0 \quad ✓$$

They are. So the subspaces are orthogonal, ~~that the~~ we still need to show that they are NOT orthogonal complements.

To do so it suffices to find a vector orthogonal to one space that is Not in the other space ~~tht~~. Consider

$A = [\,2\ \ 4\ \ 5\,]$

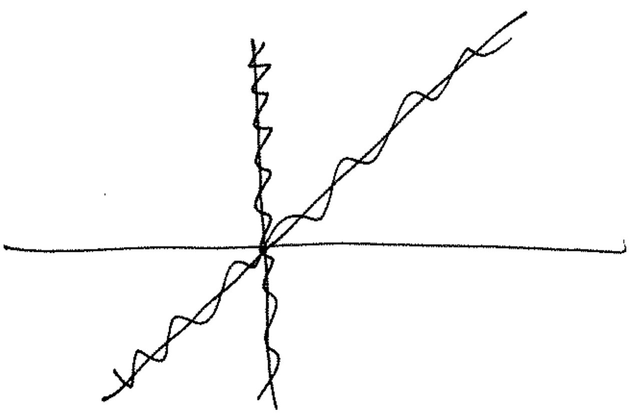
which has a null space given by

$x_2 = 1,\ x_3 = 0 + x_1 = -2 \Rightarrow \quad x = \begin{bmatrix}-2\\1\\0\end{bmatrix}$

$+\ x_2 = 0,\ x_3 = 1 + x_1 = -5/2 \Rightarrow \quad x = \begin{bmatrix}-5/2\\0\\1\end{bmatrix}$

~~the~~ Now consider the vector $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ it is orthogonal to $\begin{bmatrix}2\\4\\8\end{bmatrix}$ & thus is in its orthogonal complement but is Not in the span of $\begin{bmatrix}1\\-3\\2\end{bmatrix}$. Thus the two spaces are Not the orthogonal complements of each other.

(c) consider the subspace span $\begin{bmatrix}0\\1\end{bmatrix}$ & span $\begin{bmatrix}1\\1\end{bmatrix}$



They meet only at the origin but are Not orthogonal

5

(25) let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix}$

has (1 2 3) in both row space & null space

let $B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{bmatrix}$

Then (1, 2, 3) is in the column space &

$$B \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\cdot 0 \\ 3\cdot 0 \end{bmatrix} = \underline{0}$$

v could not be in the row space of A & the nullspace of A

v could not be in the column space of A & the left nullspace of A

But it could be in the ~~[illegible]~~ row space & left null space

or the null space & left nullspace

(26) a basis for the left nullspace of A

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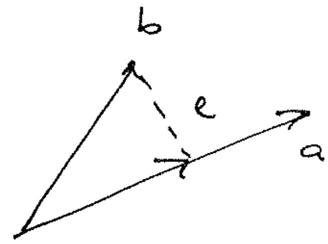
(1)

(a) $\hat{x} = \frac{a^Tb}{a^Ta}$ & $p = a\frac{a^Tb}{a^Ta}$ & $P = \frac{aa^T}{a^Ta}$



Then $\hat{x} = \frac{1+2+2}{1+1+1} = \frac{5}{3}$

so $p = \frac{5}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$

Then $e = b - p = \begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{5}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2/3\\ \frac{6}{3}-\frac{5}{3}\\ \frac{1}{3}\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix}-2\\1\\1\end{bmatrix}$

Is e orthogonal to a?

$e^Ta = \frac{1}{3}(-2+1+1) = 0$ yes

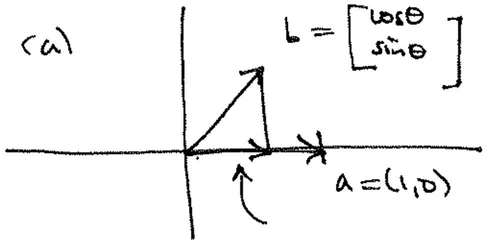
(b) $\hat{x} = \frac{a^Tb}{a^Ta} = \frac{-1-9-1}{1+9+1} = -1$

so $p = \hat{x}a = -a = \begin{bmatrix}1\\3\\1\end{bmatrix}$

The error is then $e = b - p = 0$

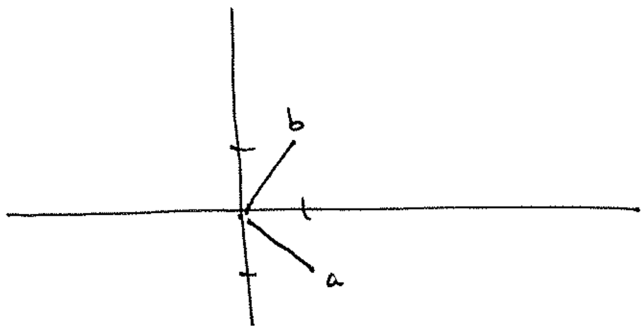
Which is orthogonal to a.

(2) (a)



$$p = \hat{x} a = \frac{a^T b}{a^T a} a = \frac{\cos\theta}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

(b)



The projection of b onto a is zero.

$$p = \hat{x} a = \frac{a^T b}{a^T a} a = \frac{1-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(3) For pt (a)

$$P = \frac{a a^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}{3} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Then

$$P^2 = \frac{1}{9} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = P$$

$$Pb = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

For pt (b) we have

$$P = \frac{\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \end{bmatrix}}{1+9+1} = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

Then $p = Pb = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$$= \frac{1}{11}\begin{bmatrix} 11 \\ 22 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ \cancel{3}\ 2 \\ 1 \end{bmatrix}$$

& $P^2 = \frac{1}{11^2}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \frac{1}{11^2}\begin{bmatrix} 11 & 33 & 11 \\ 33 & 99 & 33 \\ 11 & 33 & 11 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}$

(4) For pt (a) $P_1 = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}}{1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

& For pt (b) $P_2 = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Then $P_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_1$

& $P_2^2 = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = P_2$

Which should be true since the action of one projection will not change when we ~~[illegible]~~ project again.

(5) $$P_1 = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix}-1\\2\\2\end{bmatrix}\begin{bmatrix}-1 & 2 & 2\end{bmatrix}}{1+4+4}$$

$$= \frac{1}{9}\begin{bmatrix}1 & -2 & -2\\ -2 & 4 & 4\\ -2 & 4 & 4\end{bmatrix} \checkmark$$

& $$P_2 = \frac{\begin{bmatrix}2\\2\\-1\end{bmatrix}\begin{bmatrix}2 & 2 & -1\end{bmatrix}}{4+4+1} = \frac{1}{9}\begin{bmatrix}4 & 4 & -2\\ 4 & 4 & -2\\ -2 & -2 & +1\end{bmatrix}$$

So $$P_1 \cdot P_2 = \frac{1}{81}\begin{bmatrix}1 & -2 & -2\\ -2 & 4 & 4\\ -2 & 4 & 4\end{bmatrix}\begin{bmatrix}4 & 4 & -2\\ 4 & 4 & -2\\ -2 & -2 & +1\end{bmatrix} \checkmark$$

$$= \frac{1}{81}\begin{bmatrix}4-8+4 & 4-8+4 & -2+4-2\\ -8+16-8 & -8+16-8 & 4-8+4\\ -8+16-8 & -8+16-8 & 4-8+4\end{bmatrix}$$

$$= \underline{0}$$

Another way to see this is to consider

$$\cancel{P_1 = a_1} \qquad P_1 P_2 = \frac{a_1 a_1^T}{a_1^T a_1} \cdot \frac{a_2 a_2^T}{a_2^T a_2}$$

$$= \frac{1}{a_1^T a_1} \frac{1}{a_2^T a_2} \, a_1 a_1^T a_2 a_2^T$$

$$= (\quad)(\quad) \, a_1 \underbrace{(a_1^T a_2)}_{\equiv \underline{\underline{0}}} a_2^T$$

$$= 0$$

since $a_1^T a_2 = 0$

~~Then~~ This is to be expected since a_1 + a_2 are perpendicular + to project a vector onto a_1 produces a vector that is perpendicular to a_2 + $\therefore$ when projected onto a_2 will then produce the zero vector

(6) From problem 5 we have

$$P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \quad \text{so} \quad P_1\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}1\\-2\\-2\end{bmatrix}$$

$$\& \quad P_2 = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \quad \text{so} \quad P_2\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4\\4\\-2\end{bmatrix}$$

$$P_3 = \frac{a_3 a_3^T}{a_3^T a_3} = \frac{\begin{bmatrix}2\\-1\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 2\end{bmatrix}}{4+1+4} = \frac{1}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

$$\text{so} \quad P_3\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4\\-2\\4\end{bmatrix}$$

$$\text{Then} \quad P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix}1+4+4\\-2+4-2\\-2-2+4\end{bmatrix} = \frac{1}{9}\begin{bmatrix}9\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

We are projecting onto 3 orthogonal axis $a_1, a_2, \& \ a_3$

$$a_3^T a_1 = -2-2+4 = 0$$

$$a_3^T a_2 = 4-2-2 = 0$$

(7) $$P_3 = \frac{a_3 a_3^T}{a_3^T a_3} = \frac{1}{4+1+4}\begin{bmatrix}2\\-1\\2\end{bmatrix}\begin{bmatrix}2 & -1 & 2\end{bmatrix}$$

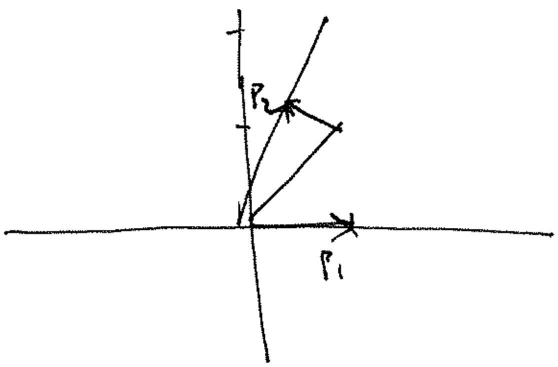
$$= \frac{1}{9}\begin{bmatrix}4 & -2 & 4\\-2 & 1 & -2\\4 & -2 & 4\end{bmatrix}$$

Then $$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix}1+4+4 & -2+4-2 & -2-2+4\\-2+4-2 & 4+4+1 & 4-2-2\\-2-2+4 & 4-2-2 & 4+1+4\end{bmatrix}$$

$$= I$$

(8) $$\hat{x}_1 = \frac{a_1^T b}{a_1^T a_1} = \left(\frac{1}{1}\right)$$ Then $$p_1 = \hat{x}_1 a_1 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\hat{x}_2 = \frac{a_2^T b}{a_2^T a_2} = \frac{3}{5}$$ so $$p_2 = \hat{x}_2 a_2 = \frac{3}{5}\begin{bmatrix}1\\2\end{bmatrix}$$



$$p_1 + p_2 = \begin{bmatrix}1\\0\end{bmatrix} + \frac{3}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}8/5\\6/5\end{bmatrix} = \frac{2}{5}\begin{bmatrix}4\\3\end{bmatrix} \neq$$

(9) The projection onto the plane $a_1 + a_2$ is given by

~~the matrix~~ $\cancel{A(A^TA)^{-1}A^T}$ The full $\mathbb{R}^2$ so a projection

~~W A =~~ matrix is I

Since A is 2×2 with linearly independent columns

A^TA is invertible. $A^TA = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

So $(A^TA)^{-1} = \frac{1}{4}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$

Now $A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I$$ as discussed

(10) $\hat{x} = \cancel{\frac{a_1^T a_2}{a_2^T a_2}} \quad \cancel{a_1^T a_2} / a_2^T a_2 =$

When we project $\underline{b}$ onto a the coefficients are given by $\hat{x} = \frac{a^T b}{a^T a}$ so to project a_1 onto a_2

we have coefficients given by $\hat{x} = \frac{a_2^T a_1}{a_2^T a_2} = \frac{1}{5}$

$$P_1 = \frac{a_2 a_2^T}{a_2^T a_2} = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}1 & 2\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

& the projection is given by $p = \hat{x} a_2 = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}$

Then project this back ~~to~~ onto a_1. We obtain

$$\hat{x} = \frac{p^T a_1}{a_1^T a_1} = \frac{1}{5}\frac{(1)}{1} = \frac{1}{5}$$

& $P_2 = \frac{a_1 a_1^T}{a_1^T a_1} = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$

The $\tilde{p} = \frac{1}{5} a_1 = \frac{1}{5}\begin{bmatrix}1\\0\end{bmatrix}$

So $$P_2 P_1 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \frac{1}{5}\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$$

$$= \cancel{\frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}} = \frac{1}{5}\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$$

I don't think this is ~~a projectio~~ a projection but, since it ~~would~~ would have to be written as proportional to col * row which this can't be

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(11) $A^TA\hat{x} = A^Tb$ + $p = A\hat{x}$

(a) $A^TA = \begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&1\\1&2\end{bmatrix}$

+ $A^Tb = \begin{bmatrix}1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$

so $A^TA\hat{x} = A^Tb$

$\Rightarrow \begin{bmatrix}1&1\\1&2\end{bmatrix}\begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$

$\Rightarrow \begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \frac{1}{1}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}4-5\\-2+5\end{bmatrix} = \begin{bmatrix}-1\\3\end{bmatrix}$

so $p = A\hat{x} = \begin{bmatrix}1&1\\0&1\\0&0\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}-1+3\\3\\0\end{bmatrix} = \begin{bmatrix}2\\3\\0\end{bmatrix}$

Then $e = b - p = \begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}0\\0\\4\end{bmatrix}$

(b) $A^TA = \begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\0&1\end{bmatrix} = \begin{bmatrix}2&2\\2&3\end{bmatrix}$

+ $A^Tb = \begin{bmatrix}1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}4\\4\\6\end{bmatrix} = \begin{bmatrix}8\\14\end{bmatrix}$

So $A^TA\hat{x} = A^Tb$

$\Rightarrow \cancel{\hat{x}} = \begin{bmatrix}2&2\\2&3\end{bmatrix}\begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \begin{bmatrix}8\\14\end{bmatrix}$

$$\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}3 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}8\\ 14\end{bmatrix} = \begin{bmatrix}3 & -2\\ -2 & 2\end{bmatrix}\begin{bmatrix}4\\ 7\end{bmatrix}$$

$$= \begin{bmatrix}12-14\\ -8+14\end{bmatrix} = \begin{bmatrix}-2\\ 6\end{bmatrix}$$

Then $\quad p = A\hat{x} = \begin{bmatrix}1 & 1\\ 1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 6\end{bmatrix} = \begin{bmatrix}4\\ 4\\ 6\end{bmatrix}$

$$e = b - p = \begin{bmatrix}4\\ 4\\ 6\end{bmatrix} - \begin{bmatrix}4\\ 4\\ 6\end{bmatrix} = 0$$

(12) The projection matrices are given by

$$A(A^TA)^{-1}A^T$$

$$P_1 = \begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ -1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Check $\quad P_1^2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix} = P_1$

Form $P_1 b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

For the 2nd part we have

$$P_2 = A(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then $P_2^2 = \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = P_2$$

$$p = P_2 \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

13 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The projection matrix is given by $A(A^TA)^{-1}A^T$

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$0 \quad 0 \quad 0$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so P is 4×4

Then $Pb = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

(14) ~~$b = 2A(:,1)$~~

Since b is in the span of the columns of A the projection will be b itself. $P \neq I$ since for vectors not in the column space of A, their projection is not themselves. As an example let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}$$ then the projection matrix is given by

$$P = A(A^TA)^{-1}A^T$$

Now $$A^TA = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$$

so $$(A^TA)^{-1} = \frac{1}{21}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

& $$A(A^TA)^{-1}A^T = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}\left(\frac{1}{21}\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 & 10 \\ 5 & 8 & -4 \end{bmatrix}$$

$$= \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}$$

so $$p = Pb = \frac{1}{21}\begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 16-16 \\ 34+8 \\ 4+80 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 0 \\ 42 \\ 84 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 0 \\ 14 \\ 28 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = b.$$

(15) The column space of $2A$ is the same as A.

But $\hat{x}$ is not the same for A & $2A$ since $p_A = A\hat{x}$ &

$p_{2A} = 2A\hat{x}$

& $p_A = p_{2A}$ the projection is the same $\longrightarrow$ $A^TA\hat{x}_A = A^T b$

so in fact $\hat{x}_A = 2\hat{x}_{2A}$ $\longrightarrow$ $4A^TA\hat{x}_{2A} = 2A^T b$

or

Which can be seen by writing the equation for $\longrightarrow$ $A^TA\,2\hat{x}_{2A} = A^Tb$

$\hat{x}_A$ & $\hat{x}_{2A}$ in terms of A. For example $\therefore$

the equation for $\hat{x}_A$ is given by

while that for $\hat{x}_{2A}$ is given by

From comparing the two we see that $\hat{x}_A = 2\hat{x}_{2A}$.

(16) Solve for $\hat{x}$ in $A^TA\hat{x} = A^Tb$

with $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$ we have $A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

So $\hat{x} = (A^TA)^{-1}A^Tb$

$$= \begin{bmatrix} 1/6 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

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(17) $(I-P)^2 = (I-P)(I-P) = I - P - P + P^2$

$$= I - 2P + P = I - P$$

$I-P$ projects onto the orthogonal complement of the column space of A or the left null space of A.

(18) (a) $I-P$ is the projection onto the vector spanned by $(-1,1)$

(b) $I-P$ is the projection onto the plane $\perp$ to this line ie $x+y+z=0$. The ~~vector~~ projection matrix is derived from the column of $A = \begin{bmatrix}1\\1\\1\end{bmatrix}$ + has $x+y+z=0$ as its left nullspace

(19) Consider the plane given by $x-y-2z=0$

Setting the free variables equal to a basis $(y=1, z=0)$ + $(y=0, z=1)$

we have the following two vectors in the nullspace

$$\begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}2\\0\\1\end{bmatrix}$$

which are two vectors in the null space of this plane we then have ~~that~~ by defining $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ we get

that $$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

so $$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

& $$P = A(A^T A)^{-1} A^T = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

(20) A vector $\perp$ to the plane $x-y-2z=0$ is

the vector $e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ since $e^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \forall x, y, z$

in the plane. Then the projection onto this vector is

given by $Q = \dfrac{ee^T}{e^Te} = \cancel{\dfrac{[1 \ -1 \ -2]}{1+1+4}} \cancel{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}}$

$$= \cancel{\frac{1}{1+1+4}} \Bigg[$$

$$= \frac{1}{1+1+4} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} [1 \ -1 \ -2]$$

$$= \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & +1 & +2 \\ -2 & 2 & 4 \end{bmatrix}$$

Then the projection onto this plane is given by

$$I - Q = \cancel{\begin{bmatrix} 1-1/6 & -1/6 \end{bmatrix}} \quad \frac{1}{6} \begin{bmatrix} 6-1 & 1 & 2 \\ 1 & 6-1 & -2 \\ 2 & -2 & 6-4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

or the same as obtained in problem #19

(21) If $P = A(A^TA)^{-1}A^T$

then $P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$

$$= A(A^TA)^{-1}A^T = P$$

$P(Pb) = Pb$

since Pb is in the column space of A & $\therefore$ its projection is itself.

(22) If $P = \cancel{A}(A^TA)^{-1}A^T$

then $P^T = \cancel{A}\ (A(A^TA)^{-1}A^T)^T$

$$= A(A^TA)^{-T}A^T$$

$$= A[(A^TA)^T]^{-1}A^T$$

$$= A(A^TA)^{-1}A^T = P$$

(23) When A is invertible the span of its columns is equal to the ~~span~~ entire space from which we are leaving, i.e. $\mathbb{R}^n$. Therefore since b is in ~~this~~ $\mathbb{R}^n$ its projection into $\mathbb{R}^n$ must be itself. The error then is zero

(24) The Nullspace of A^T is perpendicular to the column space $C(A)$ by the 2nd fundamental theorem of Linear algebra.

If $A^T b = 0$ the projection of b onto $C(A)$ will be ~~itself~~ 0. From the expression for the projection matrix we can see that this is so

$$Pb = A(A^TA)^{-1}A^T b = A(A^TA)^{-1} 0 = 0$$

(25) The projection Pb fill the subspace S so S is the basis of P ?

(26) $A^2 = A$ + $\text{rank}(A) = m$ then $A = I$

$$A^2 = A$$

$$\Rightarrow A(A-I) = 0$$

But since rank of A is m, A is invertible $\therefore$ multiply by A^{-1} gives

$A - I = 0$ so $A = I$

(27) Ax is in the null space of A^T. Ax is always in the column space of A. To be in both spaces (which are perpendicular) we must have $Ax = 0$.

(28) $P^T = P$ & $P^2 = P$

let $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Then Px = the 2nd column of P.

Then its length squared is given by $(Px)^T(Px)$

$$= x^T P^T P x = x^T P^2 x = x^T P x = P_{22}$$ element (2,2) in P.

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(1) $b = C + Dt$

The 4 equations are given by

$0 = C + D\cdot 0$

$8 = C + D\cdot 1$

$8 = C + D\cdot 3$

$20 = C + D\cdot 4$

If the measurements are changed to what is given then

$1 = C + D\cdot 0$

$5 = C + D\cdot 1$

$13 = C + D\cdot 3$

$17 = C + D\cdot 4$

$\Rightarrow \quad C = 1 \quad + \quad D = 4$

(2) For the b + t given our matrix A is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad + \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

So the normal eqs are given by

$$A^T A \hat{x} = A^T b$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \hat{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 1+3+4 \\ 1+3+4 & 1+9+16 \end{bmatrix} \hat{x} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ \underbrace{\underbrace{8+24}+80}_{\underbrace{8+104}_{112}} \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

$$104 - 64 = 40$$

$$\begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{104-64} \begin{bmatrix} 26 & -8 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} 36 \\ 112 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 26 \cdot 36 - 8 \cdot 112 \\ -8 \cdot 36 + 4 \cdot 112 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Then $e = b - A\hat{x} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0-1 \\ 8-1-4 \\ 8-1-12 \\ 20-1-16 \end{bmatrix}$

$$= \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$$

The 4 heights are given by $A\hat{x} = \begin{bmatrix} +1 \\ +5 \\ 13 \\ 17 \end{bmatrix}$ $\quad 7 \neq 12$

the error is given by

The smallest possible value of $E = 1 + 9 + 25 + 9 =$

$10 + 25 + 9 = 19 + 25$

$= 44$

(3) From prev problem 2 $p = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$ so

that $e = b - p$ is given by $e = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$

consider $e^T A = \begin{bmatrix} -1 & 3 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$

$= \begin{bmatrix} -1+3-5+3 & 3-15+12 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

The shortest distance is $\|e\| = E = 44$.

(4) $E = \|Ax - b\|^2$

$= ((C + D \cdot 0 - 0)^2 + (C + D \cdot 1 - 8)^2 + (C + D \cdot 3 - 8)^2$

$+ (C + D \cdot 4 - 20)^2$

so that $\frac{\partial E}{\partial C} = 2(C + D \cdot 0) + 2(C + D \cdot 1 - 8) +$

$2(C + D \cdot 3 - 8) + 2(C + D \cdot 4 - 20)$

& $\frac{\partial E}{\partial D} = 2(C + D \cdot 0 - 0) \cdot 0 + 2(C + D \cdot 1 - 8) \cdot 1$

$+ 2(C + D \cdot 3 - 8) \cdot 3 + 2(C + D \cdot 4 - 20) \cdot 4$

Then $\div$ by 2 we have the ~~normal eqs for C + D~~ follow
~~given by~~ $\cancel{C+D\cdot 0 - 0 + C+D\cdot 1 - 8 + C+D\cdot 3 - 8 + C+D\cdot 4 - 20} = 0$

$+ \quad \cancel{((C+D\cdot 0 - 0)\cdot 0 +}$

$\cancel{C+D\cdot 0 =} \; 0$

$$C+D\cdot 0 + C+D\cdot 1 + C+D\cdot 3 + C+D\cdot 4 = 0 + 8 + 8 + 20$$

$$+ \; ((C+D\cdot 0)\cdot 0 + (C+D\cdot 1)\cdot 1 + ((C+D\cdot 3)\cdot 3 + ((C+D\cdot 4)\cdot 4 =$$

$$0\cdot 0 + 8\cdot 1 + 8\cdot 3 + 20\cdot 4$$

$$8 + 24 + 80$$

or ~~in~~ ~~system form~~ $= 104 + 8 = 112$

$\cancel{[1 \; 1 \; 4 \; 1]}x \quad = \cancel{[1 \; 1 \; 1 \; 1]} \begin{bmatrix} \cancel{0} \\ 8 \\ 8 \\ 20 \end{bmatrix}$

Grouping by C + D the following:

$$4\cdot C + 8\cdot D = 36 \quad +$$

$$(1+3+4)C + (1+9+16)D = 112$$

$$\Longleftrightarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

(5) Best horizontal line is given by $y = C$. By least squares the coefficient matrix A is given by

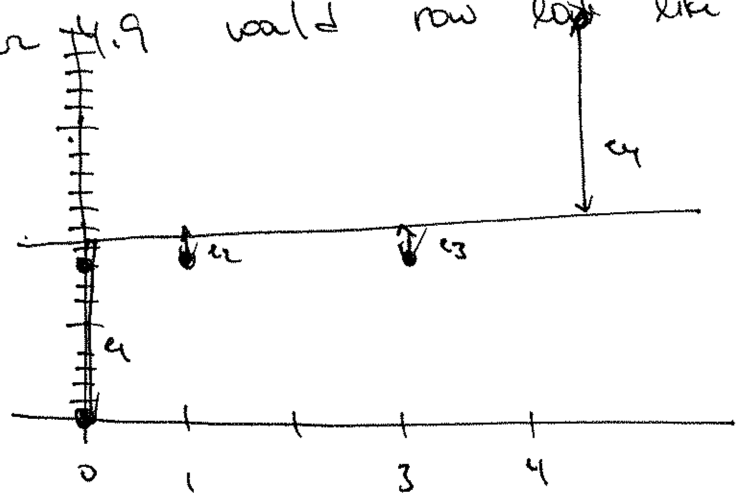
$$A\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} C = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Which has Normal eqs given by

$$A^T A x = A^T b$$

$$4C = 16 + 20 = 36 \Rightarrow C = 9$$

Figure 4.9 would now look like



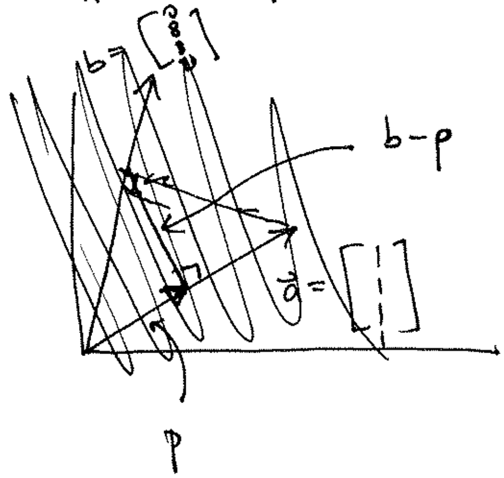
$$e = b - A\hat{x} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} 9 = \begin{bmatrix} -9 \\ -1 \\ -1 \\ 11 \end{bmatrix}$$

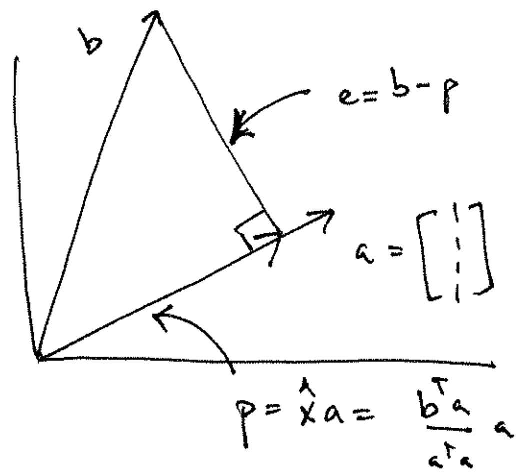
(6) $\hat{x} = \dfrac{a^T b}{a^T a} = \dfrac{(8+8+20)}{4} = 2+2+5 = 9$

Then $p = \hat{x} a = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$ & $e = b - p = \begin{bmatrix} 0-9 \\ 8-9 \\ 8-9 \\ 20-9 \end{bmatrix}$

$e^T a = \begin{bmatrix} -9 & -1 & -1 & +11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$ yes

Then $\|e\| = \|b-p\| = \sqrt{81+1+1+121} = \sqrt{2+202} = \sqrt{204}$





(7) $b = Dt$

In this case our linear system is given by

$A\hat{x} = b$ w/ $A = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$ & $b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$

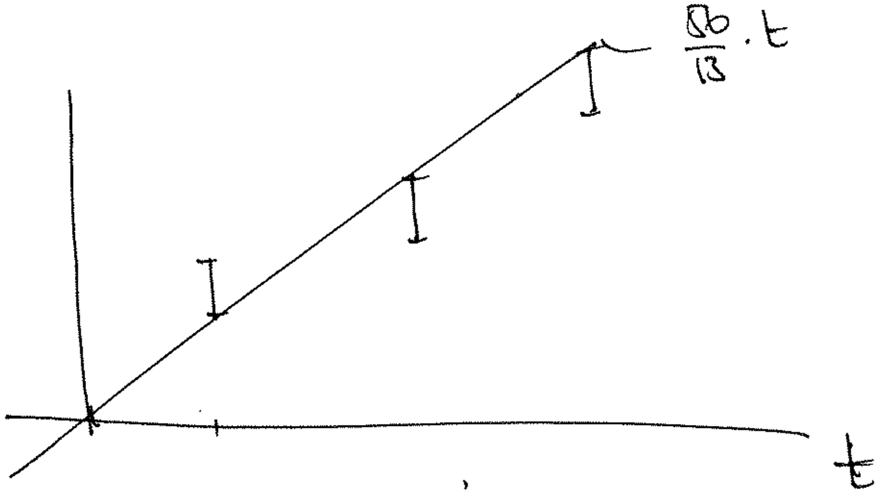
& $\hat{x} = [D]$

Then $A^TA = [1+9+16] = [26]$

& $A^Tb = [0+8+24+80] = [8+104] = [112]$

so $\hat{x} = \frac{112}{26} = \frac{56}{13}$

Then Fig 1.9a looks like



errors are only vertical

(8) $\hat{x} = \frac{a^Tb}{a^Ta} = \frac{0+8+24+80}{1+9+16} = \frac{104+8}{26} = \frac{112}{26} = \frac{56}{13}$

So $p = \frac{56}{13}\begin{bmatrix}0\\1\\3\\4\end{bmatrix} = \cancel{\frac{56}{13}}$

In problems 1–4 the best $(C,D) = (1,4)$
while in problems 5–6 for C &
7–8 for D we have $(C,D) = (9, \frac{56}{13})$

Because $(1,1,1,1)$ & $(0,1,3,4)$ are Not perpendicular

(9) our matrix ~~then~~ ~~is~~ in this case is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \quad \& \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

The Normal equations are given by

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 1+3+4 & 1+9+16 \\ 1+3+4 & 1+9+16 & 1+27+64 \\ 1+9+16 & 1+27+64 & 1+81+256 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}$$

~~16·4 = 48+24 = 64~~

16·~~4~~ = 0

$$\begin{array}{r} 16 \\ \times\, 16 \\ \hline 96 \\ 160 \\ \hline 256 \end{array} \qquad \begin{array}{r} 28 \\ 64 \\ \hline 92 \end{array} \qquad \begin{array}{r} 256 \\ 82 \\ \hline 338 \end{array}$$

$$\& \quad A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix}\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 16+20 \\ 8+24+80 \\ 8+72+320 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}$$

~~16~~ ~~20~~ 320 00 0 ~~392~~ 8

In Figure 4.9 b we are ~~fitting~~ ~~our~~ ~~methods~~ ~~to a~~ ~~plane~~ computing the best fit to ~~8~~ the span of 3 vectors when best is measured in the least squares sense

(10) $$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}$$

So $Ax = b$ has a solution for

$\cancel{A}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 8 \\ 1 & 4 & 16 & 64 & 20 \end{bmatrix}$$

2
24
6
$-144 + 60$
84

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & -84 & \end{bmatrix} \quad \ldots$$

Given $$\hat{x} = \frac{1}{3}\begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix}$$

Then $p = b$ & $e = 0$

(11) (a) The best line is $1+4t$

so $1+4\hat{t} = 1+4(2) = 9 = \hat{b}$

(b) The 1st Normal equation is given by eq 9 in the text

+ is $m \cdot C + \sum t_i \cdot D = \sum b_i$.

Dividing by m gives the requested expression

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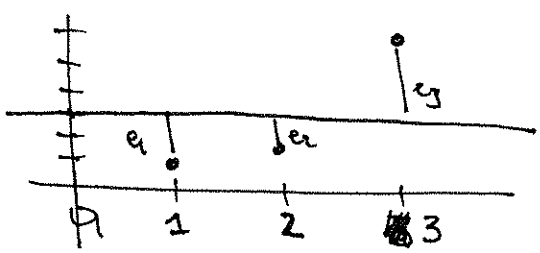
(12) (a) $a^Ta\hat{x} = a^Tb$ is given by

$m\hat{x} = \sum_i b_i$ so $\hat{x} = \frac{1}{m}\sum_i b_i$ or the mean of the b_i's

(b) $e = b - \hat{x}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} = \begin{bmatrix}b_1-\hat{x}\\b_2-\hat{x}\\\vdots\\b_m-\hat{x}\end{bmatrix}$

Then $\|e\|^2 = \sum_{i=1}^{m}(b_i-\hat{x})^2$ so $\|e\| = \sqrt{\sum_{i=1}^{m}(b_i-\hat{x})^2}$

(c) If $b = [1, 2, 6]^T$ then $\hat{x} = \frac{1}{3}(1+2+6) = 3$



$p = 3\begin{bmatrix}1\\1\\1\end{bmatrix}$

$e = \begin{bmatrix}1\\2\\6\end{bmatrix} - \begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}-2\\-1\\3\end{bmatrix}$

Check $p^Te = 3(-2-1+3) = 0$ ✓

$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 1\ 1]}{3}$

$= \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

(13)

I interpret this question as follows. For each instance the residual will be one of the values listed $(\pm 1, \pm 1, \pm 1)$

consider $b - Ax = (\pm 1, \pm 1, \pm 1)$

we have multiplying by $(A^TA)^{-1}A^T$ we have the following

$$(A^TA)^{-1}A^T(b - Ax) = (A^TA)^{-1}A^Tb - (A^TA)^{-1}A^TAx$$

$$= \hat{x} - x$$

If ~~this~~ ~~can~~ ~~equal~~ the residual can equal any of the following vectors

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

So ~~multiply each of these these~~ we 1st note that the average of all these vectors is equal to $\begin{bmatrix}4-4\\4-4\\4-4\end{bmatrix} = \underline{0}$

In the same way the action of $(A^TA)^{-1}A^T$ on each of these vectors would produce

$$\frac{1}{3} * \quad 3, -3, 1, 1, 1, -1, -1, -1$$

which when summed gives ZERO.

(14) Consider $\cancel{E}$ $(b-Ax)(b-Ax)^T$, ~~then~~ & multiply by $(A^TA)^{-1}A^T$ on the left & $A(A^TA)^{-1}$ on the right, to obtain

$$(A^TA)^{-1}A^T(b-Ax)(b-Ax)^T A(A^TA)^{-1}$$

Now since $B^TC = (C^TB)^T$ the above becomes

$$(\hat{x}-x)\left[\left[A(A^TA)^{-1}\right]^T(b-Ax)\right]^T$$

$$= (\hat{x}-x)\left((A^TA)^{-1}A^T(b-Ax)\right)^T$$

$$= (\hat{x}-x)(\hat{x}-x)^T$$

So that if the average of $(b-Ax)(b-Ax)^T$ is $\sigma^2 I$ we have that the average of $(\hat{x}-x)(\hat{x}-x)^T$ is $(A^TA)^{-1}A^T(\sigma^2 I)A(A^TA)^{-1}$ to obtain $\sigma^2 (A^TA)^{-1}A^TA(A^TA)^{-1} = \sigma^2(A^TA)^{-1}$.

(15) Expected error $(\hat{x}-x)^2$ is $\sigma^2(A^TA)^{-1} = \frac{\sigma^2}{m}$.

So the variance drops significantly $O(1/m)$

(16) $$\frac{1}{100} b_{100} + \frac{99}{100}\hat{x}_{99} = \frac{1}{100}\left(\sum_i b_i\right)$$

(17)
$$7 = C + D(-1)$$
$$7 = C + D(1)$$
$$21 = C + D(2)$$

$$\Longleftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} \overset{\text{L.S. sol}}{\Longleftrightarrow} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 14+21 \\ 42 \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{18-4}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 30-12 \\ -10+18 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 18 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Then the line is $b = 9 + 4t$

(18) $p = A\hat{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 9-4 \\ 9+4 \\ 9+8 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$

Gives the values on the closest line. The error vector e is then given by $e = b - p = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$

(19) our matrix A is still given by $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

~~let~~ Now $b = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$ So that

$\hat{x} = (A^T A)^{-1} A^T b = \underline{0}$. Each column of A is perpendicular to the error in the least squares solution and as such has $A^T b = 0$. Thus the projection is zero.

(20) when $b = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$ we have

$\hat{x} = (A^T A)^{-1} A^T b = (A^T A)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} =$ ~~[illegible]~~

$= (A^T A)^{-1} \begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$$\Rightarrow \quad \hat{x} = \frac{1}{14}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

Then the closest line is given by ~~...~~ ~~...~~ $b = 9 + 4t$

& $e = b - A\hat{x} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \underline{0}$

$e = 0$ because this b is in the column space of A

(21) The error e must be perpendicular to the column space of A & $\therefore$ is in the left nullspace of A.

p must be in the column space of A

$\hat{x}$ is in the row space of A

The nullspace of A is the zero vector assuming that the columns of A are linearly independent. This is most generally true if $m > n$.

(22) ~~Step~~ $A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ Form $A^TA\hat{x} = A^Tb$ & solve for $\hat{x}$.

Note also have $\sum t_i = 0$ so

... ~~...~~ ~~...~~ ~~...~~

$$C = \frac{1}{m}\sum_i b_i = \frac{1}{5} 5 = 1$$

$$\& \quad D = \frac{b_1 T_1 + \cdots + b_m T_m}{T_1^2 + \cdots + T_m^2} = \cancel{\ldots}$$

$$D = \frac{4\cdot(-2) + 2(-1) + -1(0) + 0(1) + 0(2)}{4+1+0+1+4} = \cdots$$

Then the least square line is ~~C+D~~ C+Dt.

(23) $P=(x,x,x)$ + $Q=(y,3y,-1)$

Then $\|P-Q\|^2 = (x-y)^2 + (x-3y)^2 + (x+1)^2$

$=$

The to min this set $\frac{\partial \|P-Q\|^2}{\partial x} = 0$

+ $\frac{\partial \|P-Q\|^2}{\partial y} = 0$ + solve for x + y.

(24) e is orthogonal to every thing in the column space of A

So that would be $p = A\hat{x}$.

$\|e\|^2 = (b-p)^T(b-p) = e^T(b-p) = e^T b = (b-p)^T b$

$= b^T b - b^T p$.

(25) Since $\|Ax-b\|^2 = \cancel{\|Ax\|^2 + 2b^TAx +} \|b\|^2$

$(Ax-b)^T(Ax-b) = (Ax)^T(Ax) - (Ax)^T b - b^T Ax$

$+ b^T b$

$= \|Ax\|^2 - 2b^T Ax + \|b\|^2$

So the derivatives of $\|Ax-b\|^2$ are zero when

$$2A^TAx - 2A^Tb = 0 \quad \text{or}$$

$$A^TAx = A^Tb \quad \text{(the normal equations)}$$

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(1)

(a) Check the dot product

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \neq 0$$

& the second vector does not have norm $= 1$, so these vectors are only independent

(b) Check the dot product

$$\begin{bmatrix} .6 & .8 \end{bmatrix} \begin{bmatrix} .4 \\ -.3 \end{bmatrix} = .24 - .24 = 0$$

So they are orthogonal. The norm of each is given by

$$\|v_1\| = \sqrt{.36 + .64} = 1$$

$$\& \quad \|v_2\| = \sqrt{.16 + .09} = \sqrt{.25} = .5$$

so they are not orthonormal, An orthonormal second vector would be given by

$$v_2/.5 = 2v_2 = \begin{bmatrix} .8 \\ -.6 \end{bmatrix}$$

(c) $v_1^T v_2 = -\cos\theta \sin\theta + \sin\theta \cos\theta = 0$

& $\|v_1\| = \|v_2\| = 1$ so these two vectors are orthonormal.

(2) $$q_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$\& \quad q_2 = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

then $Q^TQ = \frac{1}{9}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 9 & -2+4-2 \\ -2+4-2 & 1+4+4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$

+ $QQ^T = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$

$= \frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$

3 (a) A^TA would be the 3×3 idenity matrix times $4^2 = 16$
$3\times4 \; 4\times3$

(b) A^TA would be $\begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

(4) (a) let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

then $QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(b) let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(c) let the ~~bas~~ basis be composed of

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(5) All vectors that lie in the plane must ~~both~~ be in the nullspace of $A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

which has a basis given by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \equiv v_1, v_2$

These vectors are not orthogonal as is

let $w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & let $\cancel{v_2 = w_1 \; v_2 =}$

$\cancel{w_2 = w_2 = w_1^T v}$

$$W_2 = V_2 - \frac{(V_2^T W_1)\, w_1}{\|w_1\|^2}$$

$$= \cancel{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}}(2)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$= \cancel{\begin{bmatrix} -2+\sqrt{2} \\ -\sqrt{2} \\ 1 \end{bmatrix}}$$

Since $V_2^T w_1 = \frac{1}{\sqrt{2}}(2) = \sqrt{2}$

+ $\|w_1\|^2 = 1$

So $\frac{(V_2^T w_1)}{\|w_1\|^2} w_1 = \sqrt{2}\, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

So $W_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

+ $\therefore \quad w_2 = \frac{1}{\|w_2\|}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

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(6) Consider $(Q_1 Q_2)^T (Q_1 Q_2)$ which is equal to

$$Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$$

So $Q_1 Q_2$ is orthogonal

(7) The projection matrix is given by $P = Q(Q^TQ)^{-1}Q^T$

$$= Q I^{-1} Q^T$$

$$= QQ^T$$

So the projection of b will be

$$p = Pb = QQ^Tb = Q\begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_m^T b \end{bmatrix} = (q_1^Tb)q_1 + (q_2^Tb)q_2 + \cdots + (q_m^T b) q_m$$

(8) (a) $Q = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}$

~~.64+.36~~

$$QQ^T = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} .8 & .6 & 0 \\ -.6 & .8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} .64+.36 & 0 & 0 \\ .42-.42 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

so $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P$

(b) $(QQ^T)(\cancel{Q^T}QQ^T) = QQ^TQQ^T = QQ^T$

Then $P = QQ^T = (QQ^T)(QQ^T)$ so that $\overset{=QQ^T}{P}$ is the projection matrix onto the columns of Q.

9 (a) $c_1 q_1 + c_2 q_2 + c_3 q_3 = 0$ taking the dot product with q_1 gives $c_1 q_1^T q_1 = 0 \Rightarrow c_1 = 0$, sim for q_2 & q_3.

Thus the q's are linearly independent.

(b) Defining $Q = [q_1 q_2 q_3]$ then $Qx = 0$

Multiplying by Q^T on both sides gives

$Q^T Q x = 0$

But $Q^T Q = I$ by orthogonality so $x = 0$

(10) TO be in both planes we are looking for variable $\begin{bmatrix} x \\ y \end{bmatrix}$ that

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$$

let $V_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$ so that $V_1 = \dfrac{1}{\sqrt{1+9+16+25+49}} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$

$$= \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{array}{r} 10 \\ 16 \\ 25 \\ 49 \\ \hline 26 \\ 74 \\ \hline 100 \end{array}$$

Then $V_2 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} \left(\dfrac{1}{10^2}\right) \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$

$$= \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \frac{(-6+18+32+56)}{10^2} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{array}{r} 88 \\ 18 \\ \hline 106 \\ = 100 \end{array}$$

$8-20$

$$= \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \cancel{18} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} \cancel{-16} \\ \cancel{-24} \\ \cancel{-32} \\ \cancel{-50} \\ \cancel{-62} \end{bmatrix} \neq \frac{1}{2} \begin{bmatrix} \cancel{8} \\ \cancel{12} \\ \cancel{16} \\ \cancel{25} \\ \cancel{31} \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$ Normalizing we have

$$v_2 = \frac{1}{\sqrt{49+9+16+25+1}} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

(b) The vector closest to $(1,0,0,0,0)$ is given by

$$p = q_1(q_1^T b) + q_2(q_2^T b)$$

$$= \frac{1}{10}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}\frac{1}{10} + \frac{1}{10}\begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}\frac{-7}{10}$$

$$= \frac{1}{100}\left(\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} - 7\begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}\right) = \frac{1}{100}\begin{bmatrix} 1+49 \\ 3-21 \\ 4-28 \\ 5+35 \\ 7-7 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 50 \\ -18 \\ -24 \\ 40 \\ 0 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 25 \\ -9 \\ -12 \\ 20 \\ 0 \end{bmatrix}$$

(11) $(q_1^T b) q_1 + (q_2^T b) q_2$

(12) (a) If ~~q~~ a_i's are orthogonal

$$Ax = b$$

$$\Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} x = b$$ multiplying by A^T (which is the inverse of A)

$$A^T A x = A^T b$$

$$x = \begin{bmatrix} a_1^T b \\ a_2^T b \\ a_3^T b \end{bmatrix}$$

(b) If a's or ~~orthogonal~~ orthogonal then

$$A^T A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & a_1^T a_3 \\ a_2^T a_1 & a_2^T a_2 & a_2^T a_3 \\ a_3^T a_1 & a_3^T a_2 & a_3^T a_3 \end{bmatrix}$$

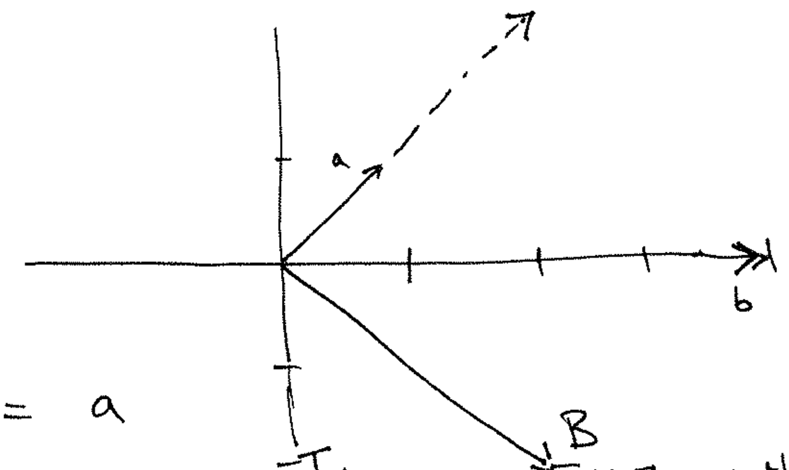
$$= \begin{bmatrix} a_1^T a_1 & 0 & 0 \\ 0 & a_2^T a_2 & 0 \\ 0 & 0 & a_3^T a_3 \end{bmatrix}$$

so $$A^TAx = A^Tb = \begin{bmatrix} a_1^Tb \\ a_2^Tb \\ a_3^Tb \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} \frac{a_1^Tb}{a_1^Ta_1} \\ \frac{a_2^Tb}{a_2^Ta_2} \\ \frac{a_3^Tb}{a_3^Ta_3} \end{bmatrix}$$

(c) a are independent x_1 is the 1st row of A^{-1} times b

(B)



$$A = a$$

$$B = b - \frac{a^Tb}{a^Ta}a = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \left(\frac{4}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

we need to subtract "2" times a to make the result orthogonal to a.

(14) $q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{4+4}}\begin{bmatrix}2\\-2\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\-2\end{bmatrix}$$

$$\frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Then $\begin{bmatrix}1 & 4\\1 & 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & q_1^T b\\0 & 2\sqrt{2}\end{bmatrix}$

with $q_1^T b = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix} = \frac{1}{\sqrt{2}}4 = \frac{4}{\sqrt{2}}$

$$\Rightarrow \begin{bmatrix}1 & 4\\1 & 0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 4/\sqrt{2}\\0 & 2\sqrt{2}\end{bmatrix}$$

We can check by multipling the matrices ~~by~~ together

$$\begin{bmatrix}1 & \frac{4}{2}+2\\1 & \frac{4}{2}-2\end{bmatrix} = \begin{bmatrix}1 & 4\\1 & 0\end{bmatrix} \quad \text{yes.}$$

15 (a) with $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$

let $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ then $q_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

let $b = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

Then $B = b - \frac{a^T b}{a^T a} a = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} - \frac{(1-2-8)}{(1+4+4)} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + \frac{9}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Then $q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{4+1+4}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

Then for q_3 we pick a 3rd vector say $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ that is linearly independent with the rest we then have

$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{c^T a}{a^T a} a - \frac{c^T b}{b^T b} b$

$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{18} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \underbrace{\frac{1}{18}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}}_{\frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \frac{1}{6}\begin{bmatrix} 6-1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

Then $q_3 = \frac{1}{\sqrt{26}}\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

(b) q_3 must be orthogonal to the columns & ∴ is in the left nullspace.

(c) $$p = [1\ 2\ 7]^T \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + [1\ 2\ 7]^T \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

$$= \frac{1}{9}(2+2+14)\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{9}(1+4-14)\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

$$= \cancel{\frac{1}{9}}\ 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + -1\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4-1 \\ 2-2 \\ 4+2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

or solving the normal equations we have

$$A^TA\hat{x} = A^Tb$$

w/
$$A^TA = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 1-2-8 \\ 1-2-8 & 1+1+16 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -9 \\ -9 & 18 \end{bmatrix} = 9\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

&
$$A^Tb = \begin{bmatrix} 1+4-14 \\ 1-2+28 \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \end{bmatrix}$$

Then
$$\frac{1}{9}\frac{1}{(2-1)}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 15 \\ 27 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 19 \\ 14 \end{bmatrix}$$

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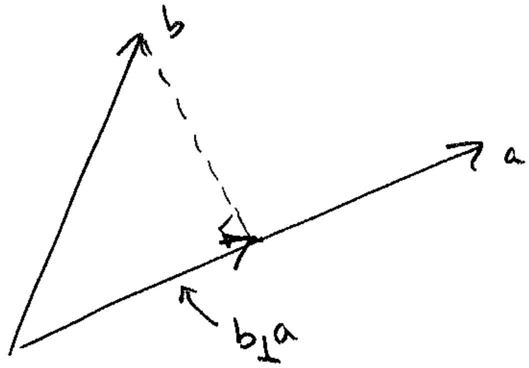
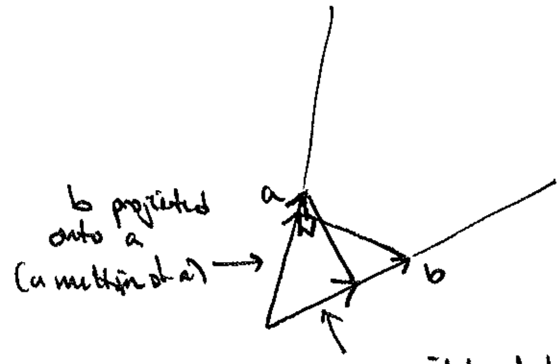
(16) Find the projection of b onto a





$$\hat{x} = \frac{b^T a}{a^T a} = \frac{(4+10)}{16+25+4+4} = \frac{14}{49} = \frac{2}{7}$$

To find orthonormal vectors let

$$q_1 = \frac{1}{\sqrt{16+25+4+4}} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{b^T a}{a^T a} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{14}{49} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{49} \begin{bmatrix} 49 - 56 \\ 98 - 70 \\ -28 \\ -28 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} -7 \\ 28 \\ -28 \\ -28 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

so $q_2 = \dfrac{1}{\sqrt{1+3(16)}} \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} =$

$\begin{array}{r} 8 \\ 16 \\ 25 \\ \hline 1 \\ 16 \\ 25 \\ \hline 41 \\ 41 \\ 8 \\ \hline 49 \end{array}$

$\begin{array}{r} 40 \\ 16 \\ 4 \\ 5 \\ \hline 80 \end{array}$

$\begin{array}{r} 56 \\ 49 \\ \hline 7 \end{array}$

$\begin{array}{r} 16 \\ 3 \\ \hline 30 \\ 18 \end{array}$

12.4
48 = 16.3.16

$$= \frac{1}{4}\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$ ... maybe error check me...

(17) $$p = \frac{b^T a}{a^T a} a = \frac{(1+3+5)}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$e = b - p = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \& \quad q_2 = \frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(18) $A = QR$, then $A^T A = \cancel{R^T R}\ (R^T Q^T)(QR)$

$$= R^T R$$

$$= \text{lower triangular} * \text{upper triangular}$$

$\therefore$ Gram Schmidt on A corresponds to elimination on $A^T A$.

If A is as given then $A^T A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}$

Then $$A^T A \Rightarrow \begin{bmatrix} 3 & 9 \\ 0 & 35-27 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 8 \end{bmatrix}$$

which has pivots equal to $\cancel{\|e\|^2}$ $\|a\|^2$ & $\|e\|^2$ respectively

(19) (a) True since the inverse of ~~an~~ orthogonal matrix is its transpose, ~~[illegible]~~ ~~[illegible]~~

(b) ~~False~~, ~~let~~ $\cancel{Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$ ~~then~~ If Q has orthonormal columns then
Yes

$$\cancel{Qx} = \|Qx\|^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = \|x\|^2$$

(20) ~~A~~ let $q_1 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\cancel{\text{let } B = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix} - \frac{(-2+1+3)}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -8-1 \\ -6-1 \\ 6-1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -9 \\ -7 \\ 5 \\ -1 \end{bmatrix}}$$

$$\cancel{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\cancel{\text{Then } q_2 = \frac{1}{\sqrt{81+49+25+1}}\begin{bmatrix} -9 \\ -7 \\ 5 \\ -1 \end{bmatrix}}$$

$$\cancel{\begin{array}{l} 81 \\ 49 \\ 25 \\ 1 \end{array} \quad \begin{array}{l} 130 \\ 26 \end{array}}$$

~~[illegible]~~

$\div 2$ ~~[illegible]~~

~~2~~

$$\text{let } B = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{(-2+1+3)}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4-1 \\ 0-1 \\ 2-1 \\ 6-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}.$$

Then $\quad q_2 = \dfrac{B}{\|B\|} = \cancel{\dfrac{1}{\sqrt{25+1+1+4}}} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix} =$

$$= \frac{1}{\sqrt{25+1+1+25}}\begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{52}}\begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

Then projecting b onto the column space of A is equivalent to computing

$$p = (q_1^T b)\, q_1 + (q_2^T b)\, q_2$$

$$= \frac{1}{2}(-4-3+3)\,\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{\sqrt{52}}(20+3+3)\,\frac{1}{\sqrt{52}}\begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \frac{26}{52}\begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2-5 \\ -2-1 \\ -2+1 \\ -2+5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -7 \\ -3 \\ -1 \\ 3 \end{bmatrix}$$

$$\frac{26}{52} \Downarrow \frac{13}{26} \Downarrow \frac{1}{2}$$

Then $e = b - p = \frac{1}{2}\begin{bmatrix} -8+7 \\ -6+3 \\ 6+1 \\ -3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ -3 \\ 7 \\ -3 \end{bmatrix}$

so $e^T A(:,1) = \frac{1}{2}(-1-3+7-3) = 0$ ✓

& $e^T A(:,2) = \frac{1}{2}(2+0+7-9) = 0$ ✓

(21) $A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ so $q_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{A^T v}{A^T A} A$$

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1-1)}{A^T A} \cdot A = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$C = v - \frac{A^T v}{A^T A} A - \frac{B^T v}{B^T B} B$$

$$= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \frac{9}{6}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad C = \frac{1}{2}\begin{bmatrix} 2-3-1 \\ 0-3+1 \\ 8-6+0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(22) One could do this by performing ellimination on A^TA as in problem 18 or just simply ~~diagonalize~~ perform gram-schmidt on the columns of the matrix A.. We have

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad q_1 = A$$

~~B~~ with $V =$ ~~B~~ $[2\ 0\ 3]^T$ we have

$$B = V - \frac{V^TA}{A^TA}A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

so $q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Then ~~of~~ it $V = [4\ 5\ 6]^T$ we have a 3rd orthogonal vector C as

$$C = V - \frac{A^TV}{A^TA}A - \frac{B^TV}{B^TB}B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 6\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

so that $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

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(23) (a) The basis for a subspace for the plane given by

$$x_1 + x_2 + x_3 - x_4 = 0$$

consider the Matrix $A = [1 \quad 1 \quad 1 \quad -1]$ then ~~the span of the~~ we need to consider the Nullspace, assign ones to the free variables to ones + zero.

$x_2 = 1, x_3 = 0, x_4 = 0 \Rightarrow x = [-1 \quad 1 \quad 0 \quad 0]^T$

$x_2 = 0, x_3 = 1, x_4 = 0 \Rightarrow x = [-1 \quad 0 \quad 1 \quad 0]^T$

$x_2 = 0, x_3 = 0, x_4 = 1 \Rightarrow x = [1 \quad 0 \quad 0 \quad 1]^T$

This is a basis.

(b) The orthogonal complement to S are all vectors y that ~~are~~ are orthogonal to each component of the Nullspace of A. This is the vector $[1 \quad 1 \quad 1 \quad -1]^T$.

(c) If ~~so~~ $b = (1,1,1,1)^T$

Then to decompose b into $b_1 + b_2$ consider the unit vector of the ~~orthogonal complement~~ vector that spans the orthogonal complement i.e

$q_2 = \frac{1}{2}(1 \quad 1 \quad 1 \quad -1)^T$ Then

$$b_2 = (q_2^T b) q_2 = \frac{1}{2}(2)\,\frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$$

Then $b_1 = b - b_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2-1\\2-1\\2-1\\2+1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\3\end{bmatrix}$

(24) $A = QR$ with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} =$$

$$q_1 = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a \\ c \end{bmatrix}$$

$$B = \begin{bmatrix} b \\ d \end{bmatrix} - \begin{bmatrix} b & d \end{bmatrix} \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a \\ c \end{bmatrix} \cdot \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a \\ c \end{bmatrix}$$

$$= \begin{bmatrix} b \\ d \end{bmatrix} - \frac{1}{(a^2+c^2)} (ab + dc) \begin{bmatrix} a \\ c \end{bmatrix}$$

$$= \begin{bmatrix} b \\ d \end{bmatrix} - \left(\frac{ab+dc}{a^2+c^2}\right) \begin{bmatrix} a \\ c \end{bmatrix} = \frac{1}{a^2+c^2} \begin{bmatrix} b(\cancel{a^2}+c^2) - a(\cancel{ab}+dc) \\ d(a^2+\cancel{c^2}) - c(ab+\cancel{dc}) \end{bmatrix}$$

$$= \frac{1}{a^2+c^2} \begin{bmatrix} bc^2 - adc \\ da^2 - cab \end{bmatrix} = \frac{1}{(a^2+c^2)} \begin{bmatrix} c(bc-ad) \\ a(ad-cb) \end{bmatrix}$$

$$= \frac{(ad-bc)}{(a^2+c^2)} \begin{bmatrix} -c \\ a \end{bmatrix} \quad \text{is orthogonal to} \quad \begin{bmatrix} a \\ c \end{bmatrix}$$

& has a unit vector given by

$$\frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} -c \\ a \end{bmatrix}$$

So the matrix Q in the QR decomposition of A is given

$$Q = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$$

Then R is given by

$$R = \begin{bmatrix} q_1^T A(:,1) & q_1^T A(:,2) \\ & q_2^T A(:,2) \end{bmatrix} = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & -cb+ad \end{bmatrix}$$

So the decomposition is given by

$$A = \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a & -c \\ c & a \end{bmatrix} \frac{1}{\sqrt{a^2+c^2}} \begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & ad-cb \end{bmatrix}$$

If $a,b,c,d = 2,1,1,1$ then we obtain

$$A = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 2+1 \\ 0 & 2-1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$$

If $a,b,c,d = 1,1,1,1$ then we obtain

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$$

From which we see that the (2,2) element of R is zero

(25) Eq 8 is given by $C = C - \frac{A^T C}{A^T A} A - \frac{B^T C}{B^T B} B$

The 1st equation in 12 is given by

$$r_{kj} = \sum_{i=1}^{m} q_{ik} a_{ij}$$ is the expression for the dot product between

The kth column of Q & the ~~column~~ jth column of A.

Then $a_{ij} = a_{ij} - q_{ik} r_{kj}$ subtracts the projection onto the

basis functions

(26) a & b may not be orthogonal so by subtracting ~~the~~ projections

along non orthogonal vectors one maybe double counting.

(27) See Matlab code ~~prob~~ chap4_sect_4_4_prob_27.m

(28) Eq 11 involves m multiplications for the summation

& m divisions for $q_{ik} = \frac{a_{ik}}{r_{kk}}$ $= O(2m)$

Then Eq 12 involves $O(2m)$ multiplications

~~Thus~~ Each of these multiplications are performed multiple times

Thus we have

$$\sum_{k=1}^{n} 2m + \sum_{j=k+1}^{n} 2m = 2mn + \sum_{k=1}^{n} 2m(n-k-1+1)$$

$$= 2mn + 2m\sum_{k=1}^{n}(n-k) = 2mn + 2m\sum_{k=1}^{n-1} k$$

$$= 2mn + 2m\left(\frac{n(n-1)}{2}\right) = 2mn + mn(n-1) = mn^2 - mn + 2mn$$

$$= mn^2 + mn$$ which is the requested # of flops

(29) (a) Check that $Q^TQ = I$, when computing this product we have

$$Q^TQ = c^2 \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$= c^2 \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$ By picking $c = \frac{1}{2}$

(b) $$Q = c\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$ which will be orthogonal if $c = \frac{1}{2}$ as before

$\cancel{Q^T Q_2}$

(30) $\cancel{q_1^T b = \frac{1}{2}(-2) = -1}$ Onto the 1st column of Q we have

$+ \ \cancel{q_2^T b = \frac{1}{2}(-2) = -1}$

$q_1^T b = \frac{1}{2}(-2) = -1$ then we have

$$p = \frac{-1}{2}\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} = \frac{-1}{2}\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}$$

To project onto the 1st 2 columns of the matrix A we give

$(q_1^T b) = -1$

$+ \ q_2^T b = \frac{1}{2}(-2) = -1$

Then $$p = \frac{-1}{2}\begin{bmatrix}+1\\-1\\-1\\-1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\-1\\-1\end{bmatrix} = \frac{-1}{2}\begin{bmatrix}1-1\\-1+1\\-1-1\\-1-1\end{bmatrix} = \frac{-1}{2}\begin{bmatrix}0\\0\\-2\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

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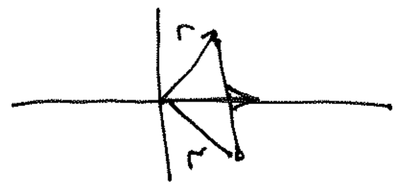
(81) $Q = I - 2uu^T$ is a reflection matrix.

If $u = (0,1)$ then $uu^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

then $Q = I - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



If $r = \begin{bmatrix} x \\ y \end{bmatrix}$ then $Qr = \begin{bmatrix} x \\ -y \end{bmatrix}$

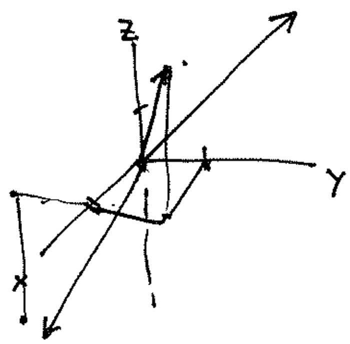
If $u = (0, 1/\sqrt{2}, 1/\sqrt{2})$ then

$$uu^T = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

So $Q = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$

So given (x, y, z) we see that

$$Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -z \\ -y \end{bmatrix}$$



(32) $Q = I - 2uu^T$ where $u^T u = 1$

(a) Then $Qu = u - 2uu^T u = u - 2u = -u$

(b) If $u^T v = 0$ then

$Qv = v - 2uu^T v = v$

(33) The columns of W are orthonormal. The inverse of W is its transpose

$$W^{-1} = W^T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$