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(1) if $\det(A) = 2$

Then $\det(2A) = 2^4 \det(A) = 2^4 2 = 2^5 = 32$

$\det(-A) = (-1)^4 \det(A) = 2$

$\det(A^2) = \det(A)^2 = 4$

$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{2}$

(2) $\det(A) = -3$

$\det(\frac{1}{2}A) = (\frac{1}{2})^3 \det(A) = \frac{-3}{8}$

$\det(-A) = (-1)^3 \det A = -(-3) = 3$

$\det(A^2) = (\det A)^2 = 9$

$\det(A^{-1}) = \frac{1}{\det A} = -\frac{1}{3}$

(3) (a) False If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

then $\det(A) = -2$ + $I + A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

so $\det(I + A) = 10 - 6 = 4$

so + $1 + \det(A) = 1 - 2 = -1$

so the two expressions are Not equal

(b) True

(c) True

(d) False let $A = I$ then

$$4A = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

So $\det(4A) = 16 \neq 4\det(A) = 4\det(I) = 4$

(4) If $J_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then J_3 is obtained from I by

exchanging rows 1 + 3 from the 3x3 Identity matrix

If $J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ Is obtained from the 4x4 identity

by exchanging the 2nd + 3rd row + the 1st + 4th row.

(5) For $n = 5$ the   pg 214 Strang

Identity matrix is given by

$$I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

+ the reverse identity matrix is given by

$$J_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has 2 row exchanges row 1 + row 5

rows 2 + row 4

So the determint of J_5 is given by $(-1)^2 = +1$

For $n = 6$ has the identity given by

$$I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

+ the reverse identity matrix is given by

$$J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

has 3 row exchanges

rows 1 + rows 6

rows 2 + rows 5

row 3 + rows 4

So the determinate is given by $(-1)^3 = -1$

$$I_7 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

will have 3 exchanges ~~so~~ to obtain the reverse identity relationship. & so the determinant J is ~~equal~~ equal to $(-1)^3 = -1$

& so our summary is given by

n	# of row exchanges
3	1
4	2
5	2
6	3
7	3

Thus the general rule is that the # of exchanges for transforming the $n \times n$ identity matrix to the $n \times n$ reverse identity matrix ~~involves~~ involves $\left\lfloor \frac{n}{2} \right\rfloor$ row exchanges

So to produce J_{101} we have $\left\lfloor \frac{101}{2} \right\rfloor = 50$ row exchanges

So the determinant is given by $\det(J_{101}) = (-1)^{50} = 1$

(6) If a matrix has a row of all zero, we can replace that row with a ~~zero multiplier~~ row of ~~all~~ non-zero #'s. ~~ones~~ times a multiplier t ~~a~~ which is zero i.e. take $t=0$. Then part of rule #3 says that

~~By multiplying a row by~~ t The determinant of this matrix is equal to t times the determinant of ~~this matr~~ the matrix ~~without~~ with those non zero row. Since $t=0$ times anything the original determinant must be zero.

(7) An orthonormal matrix has the property that

$Q^T Q = I$. taking the determinant of both sides we obtain

$|Q^T||Q| = 1$ but $|Q^T| = |Q|$ so we have

$|Q|^2 = 1 \Rightarrow |Q| = \pm 1$

From the text $|Q^T| = |Q|$ & for orthogonal matrices

$Q^T = Q^{-1}$ so $|Q^T| = |Q^{-1}| = |Q|$

(8) If Q is a 2d rotation we have

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ then $|Q| = \cos^2\theta + \sin^2\theta = +1$

For a reflection $Q = \begin{bmatrix} 1-2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1-2\sin^2\theta \end{bmatrix}$

The $|Q| = (1-2\cos^2\theta)(1-2\sin^2\theta) - 4\cos^2\theta\sin^2\theta$

$$= 1 - 2\sin^2\theta - 2\cos^2\theta + 4\cos^2\cancel{\theta}\sin^2\theta - 4\cos^2\theta\sin^2\theta$$

$$= 1 - 2(\cos^2 + \sin^2\theta) = 1-2 = -1$$

(9) If $A = QR$ then $A^T = R^T Q^T$ so

$|A^T| = |R^T||Q^T|$ & since R is upper triangular ~~R'~~

$|R^T| = |R|$ since both ~~is~~ ~~the~~ is the product of the diagonal elements in each matrix. Also from the problem above $|Q^T| = |Q|$ for an orthonormal matrix, thus

$$|A^T| = |R^T||Q^T| = |R||Q| = \cancel{RQ}\ \cancel{QR} = |Q||R| = |QR| = |A|$$

(10) If the entries of every row of A add to zero, then ~~we have~~ from the rule that $|A^T| = |A|$, & the fact that ~~by~~ subtracting a multiple of one row from another leaves the determinant unchang'd. So ~~so~~ ~~since~~ subtracting a ~~column~~ ~~of~~ multiple of ~~a~~ a column from another column leaves the determinant unchanged we see that

by ~~repeatin~~ repeatedly ~~subtracting~~ adding ~~x+y+z~~
a multiple (1) of each ~~row~~ column to each other
~~to~~ ~~to~~ say accumulating the sum in the 1st column we will
obtain a column of zeros + thus the determinant is zero.

If every row adds to 1 prove that ~~det~~ $\det(A-I)=0$
can be shown by recognizing that if every row of A adds to 1
then every row of $A-I$ adds to zero + correspondingly the
determinant must be zero. Does this give $\det(A)=1$. is No
since let $A = \cancel{\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}}$ $\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ + every row
adds to 1 but $\det(A) = 2 \neq 1$

(11) If $CD = -DC$
Then the determinant of ~~both~~ ~~sides~~ ~~is~~ ~~given~~ ~~by~~ the left hand
~~$|C|\cdot|D| = |D|\cdot|C|$ there~~ side is given by
$|CD| = |C|\cdot|D|$ + the determinant of the right hand side is
given by $|-DC| = (-1)^n |DC| = (-1)^n |D|\cdot|C|$

So we have $(1-(-1)^n)\,|D|\cdot|C| = 0$

$k+l$

So $|D| = 0$ or $|C| = 0$ or $1-(-1)^n = 0$ i.e. n is even

(12) $\det A^{-1} = \det\left(\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{(ad-bc)^2}\det\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

A

The correct calculation is given by the following

$$= \frac{1}{(ad-bc)^2}(ad-cb) = \frac{1}{(ad-bc)}$$

(13) $$\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 5 \end{bmatrix} \cancel{\Leftrightarrow} = \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 3 & 3 \\ 0 & 2 & 0 & 5 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1\cdot 2\cdot 3\cdot 4 = 24$$

The second example is given by

$$\det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$-1\left(\frac{2}{3}\right)+2 = \frac{-2}{3}+\frac{6}{3}=\frac{4}{3}$$

$$= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

$$-1\left(\frac{3}{4}\right)+2$$

$$-\frac{3}{4}+\frac{8}{4}=\frac{5}{4}$$

$$= 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right) = 5$$

(14) $$\det\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & \end{bmatrix}$$

$$\cancel{-\frac{(b^2-a^2)}{b-a}}(c-a)$$

= so the ~~3x3~~ & (3,3) element is given by $\cancel{\frac{(c-a)}{(b-a)}}$

$$(c^2-a^2) - \frac{(c-a)(b^2-a^2)}{(b-a)} = c^2-a^2-(c-a)(b+a)$$

$$= c^2 - \cancel{a^2} - (cb + ca - ab - \cancel{a^2})$$

$$= c^2 - cb - ca + ab = c(c-a) + b(a-c)$$

$$= (c-b)(c-a)$$

So we have

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-b)(c-a) \end{bmatrix}$$

$$= (b-a)(c-b)(c-a)$$

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(15) For A we know its determinant must equal zero since it will be a 3x3 but of rank II & ∴ will not be invertible. Because it is not invertible its determinant must be zero. Another way to see this is to recognize that this matrix can easily be reduced via elementary row operations to a matrix with a row of zeros

For $K^T = -K$ then $|K^T| = |K|$ from proposition 10 in this section of the book. & $|-K| = (-1)^3 |K|$ since K is a 3x3 matrix.

Thus the determinant of K must satisfy

$$|K| = (-1)^3 |K| = -|K| \Rightarrow 2|K| = 0 \text{ or } |K| = 0$$

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(16) From problem 15 ~~&~~ ~~then~~ ~~&~~ we have shown that for K skew symmetric + ~~of~~ with m odd we have

$|K| = 0$. But m can be even + hence a non zero

determinant. For a 4x4 example consider

$$\cancel{K = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}}$$

then Consider the matrix

$$K = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

$$\cancel{|K| = (-1) \begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}}$$

$$\cancel{= (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = (-1)\,1 \cdot 1\,(-2)(-1)} = -2$$

$K = \begin{bmatrix} \\ \\ \end{bmatrix}$ ~~Then $|K| =$~~

$$|H| = (-1)\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -2 & -1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

By exchanging rows #3 & #4

$$= (-1)(1)(-1)(1) = +1$$

(17) $\det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}$ ~~Multiply the 1st row by 1~~ & ~~subtract from the second~~

& subtract the 2nd row from the 3rd

$$= \det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 1 & 1 & 1 \end{bmatrix}$$

Now subtract the 1st row from the second

$$= \det\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

~~Thus~~ & From this matrix since it has two ~~rows~~ rows equivalent will have a determinant zero.

For the second problem

$$\det\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix} = \cancel{\det\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \end{bmatrix}}$$

$$= \det\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix}$$

~~If $t=1$ this determinant is zero, since we have a row of all zeros~~

~~Assuming t ≠ 1 then we have~~

$$\det\begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^4-t(t-t^3) \end{bmatrix}$$

The 3,3 element ~~the~~ then becomes $1-\cancel{t^4}-t^2+\cancel{t^4}$

$= 1-t^2$ & the determinant becomes

$1\cdot(1-t^2)(1-t^2)$ so if $t=\pm 1$ this determinant will be zero

(18) For the 1st U given by $U=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

we have $|U| = 1\cdot 4\cdot 6 = 24$

& $|U^{-1}| = \dfrac{1}{|U|} = \dfrac{1}{24}$

$|U^2| = |U|\cdot|U| = |U|^2 = 24^2 = 416$

$$\begin{array}{r} 20 \\ 20 \\ \hline 00 \\ 400 \end{array}$$

For the second U where $U=\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$

we have $|U| = ad$

$|U^{-1}| = \dfrac{1}{|U|} = \dfrac{1}{ad}$ & $|U^2| = |U|^2 = a^2d^2$

(19) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $|A| = \cancel{\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = \left|\begin{bmatrix} a & b \\ c-la & \end{bmatrix}\right|}$

No. The correct manipulations are given by

$$|A| = \left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right| = \left|\begin{bmatrix} a & b \\ c-la & d-lb \end{bmatrix}\right| = \left|\begin{bmatrix} a-L(c-la) & b-L(d-lb) \\ c-la & d-lb \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} a-Lc+Lla & b-Ld+Llb \\ c-la & d-lb \end{bmatrix}\right|$$

Another way to show that the two determinants are not equal is to compute the second one. Which is given

$$(a-Lc)(d-lb) - (b-Ld)(c-la)$$

$$= ad - a\cancel{l}b - L\cancel{c}d + Llcb - [bc - \cancel{lb}a - L\cancel{d}c + Llad]$$

$$= ad - bc + Llcb - Llad$$

$$= ad - bc - Ll(ad-cb) = (ad-bc)(1-Ll)$$

(20) Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

we desire to evaluate

$$\det(A) = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & b \\ c+a & d+b \end{bmatrix}$$

$$= \det\begin{bmatrix} -c & -d \\ c+a & d+b \end{bmatrix} = \det\begin{bmatrix} -c & -d \\ a & b \end{bmatrix}$$

$$= (-1)\det\begin{bmatrix} c & d \\ a & b \end{bmatrix} = (-1)\det B$$

Where we have used the ~~rules~~ rules that subtracting a multiple of one row from another & the ~~multiplier~~ multiplier of a row, multiplies the determinant rule.

(21) $|A| = 4 - 1 = 3$

$$|A^{-1}| = \frac{1}{3^2}(4-1) = \frac{1}{3}$$

$$|A - \lambda I| = (2-\lambda)^2 - 1$$

for $|A - \lambda I| = 0$ we must have

$$(2-\lambda) = \pm 1 \quad \text{or} \quad 2 \mp 1 = \lambda$$

or $\lambda = 1, 3$

If $\lambda = 1$, $A - \lambda I$ is given by

$$A - \lambda I = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If $\lambda = 3$, $A - \lambda I$ is given by

$$A - \lambda I = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

(22) If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 16+2 & 7 \\ 8+6 & 2+9 \end{bmatrix} = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$

so $|A| = 12 - 2 = 10$ & $|A^2| = 18 \cdot 11 - 7 \cdot 14 = \ldots$

$$A^{-1} = \frac{1}{12-2} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$$

$$|A^{-1}| = \frac{1}{10^2}(12-2) = \frac{1}{10}$$

& $\cancel{A - \lambda I} = \cancel{\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}}$ so $\cancel{|A - \lambda I| = (2-\lambda)^2 - 1}$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} \quad \text{so} \quad |A - \lambda I| = (4-\lambda)(3-\lambda) - 2$$

So $|A - \lambda I| = 0$ if $\cancel{(4-\lambda)(}$

$$12 - 7\lambda + \lambda^2 - 2 = 0$$

$$10 - 7\lambda + \lambda^2 = 0$$

$$(\lambda - 2)(\lambda - 5) = 0 \quad \text{so } \lambda = 2 \text{ or } \lambda = 5$$

(23) so $|L| = 1$, $|U| = 3(2)(-1) = -6$

$|A| = |L|\cdot|U| = -6$

~~$|U^{-1}|$~~ ~~$|U^{-1}|$~~ =

Then since $A = LU$ we have $A^{-1} = U^{-1}L^{-1}$

so $|A^{-1}| = |U^{-1}|\cdot|L^{-1}| = \frac{1}{|U|}\cdot\frac{1}{|L|} = 1\cdot\frac{1}{(-6)} = -\frac{1}{6}$

$U^{-1}L^{-1}A = I$

So $|U^{-1}L^{-1}A| = 1$

(24) If $A_{ij} = i\cdot j$ then if A is $m\times m$ then A is given by

$$A = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ m \end{bmatrix} \begin{bmatrix} 1 & 2 & \cdots & m \end{bmatrix}$$

which is a rank 1 matrix & ∴ has determinant equal to zero.

Multiple rows are multiples of a single row

(25) from If $A_{ij} = i+j$ then $\det A = 0$

when A is 3×3 it looks like

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

Consider the 1st, 2nd + 3rd row of A (assumed to be of dimension m)

~~then~~ these rows are given by

$$\begin{matrix} 1+1 & 1+2 & 1+3 & 1+4 & \cdots & 1+m \\ 2+1 & 2+2 & 2+3 & 2+4 & \cdots & 2+m \\ 3+1 & 3+2 & 3+3 & 3+4 & \cdots & 3+m \end{matrix}$$

then the determinant is unchanged by subtracting the ~~1st~~ ~~2nd~~ 2nd row from the 3rd. Doing this gives

$$\begin{matrix} 1+1 & 1+2 & 1+3 & 1+4 & \cdots & 1+m \\ 2+1 & 2+2 & 2+3 & 2+4 & \cdots & 2+m \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{matrix}$$

Then subtracting the 1st row from the second row gives

$$\begin{matrix} 1+1 & 1+2 & 1+3 & 1+4 & \cdots & 1+m \\ 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{matrix}$$

Which has two repeated rows. With two repeated rows the determinant must be zero

(26) If $A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}$

Then $\det(A) = \det \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} = \det \begin{bmatrix} c & 0 & 0 \\ 0 & 0 & b \\ 0 & a & 0 \end{bmatrix} (-1)$

$= \det \begin{bmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} (-1)^2 = a \cdot b \cdot c$

If $B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix}$ Then $\det(B)$ is given by

$\det(B) = (-1) \det \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & a & 0 & 0 \end{bmatrix} = (-1)^2 \det \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & b & 0 \end{bmatrix}$

$= (-1)^3 \det \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} = -abcd$

Finally for C we have

$$\det C = \det\begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} = \det\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix}$$

$$= \det\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-\cancel{a}-(b-\cancel{a}) \end{bmatrix} = \det\begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix}$$

$$= a(b-a)(c-b)$$

(27) (a) ~~G~~ True we know that $\text{rank}(AB) \le \text{rank}(A)$ from a previous problem & since $\text{rank}(A) < m$ the product must have ~~rank~~ that $\text{rank}(AB) \le \text{rank}(A) < m$ & $\therefore$ ~~so~~ AB cannot be invertible

(b) True since elementary row operations change A into U & the determinant of U is the product of the pivots

(c) False, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then $A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $\det(A-B) = 1$ but $\det(A) - \det(B) = 4 - 1 = 3$

(d) True it ~~AB~~ the product of A & B is defined in that way

(28) If $f(A) = \ln(\det A)$

Then for a 2x2 system or f is given by

$f(A) = \ln(ad - bc)$ define $\Delta = ad - bc$

So $\frac{\partial f}{\partial a} = \frac{1}{\Delta} d$; $\frac{\partial f}{\partial b} = \frac{-1}{\Delta} c$; $\frac{\partial f}{\partial c} = \frac{1}{\Delta}(-b)$; $\frac{\partial f}{\partial d} = \frac{1}{\Delta} a$

So

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \begin{bmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ or } A^{-T}.$$

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1

(1) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\quad |A| = \sum \pm a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$

Then $|A| = 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$

$= 1(-1) - 2(-1) + 3(1) = -1+2+3 = 4 \neq 0$

with $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ we have

$|A| = 1\begin{vmatrix} 4 & 4 \\ 6 & 7 \end{vmatrix} - 2\begin{vmatrix} 4 & 4 \\ 5 & 7 \end{vmatrix} + 3\begin{vmatrix} 4 & 4 \\ 5 & 6 \end{vmatrix}$

$= 1(28-24) - 2(28-20) + 3(24-20)$

$= 4 - 16 + 3(4) = 0$

So these columns are not independent for B but for A they are

(2) For $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

then $|A| = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & \cancel{1} \\ 0 & 1 \end{vmatrix} + 0 = -1 - 1 = -2 \neq 0$

For $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ ~~then~~ ~~we know that this matrix~~ is ~~singular from problem 25.~~

$$|B| = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= (45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 - 2(-6) + 3(-3)$$

$$= -3 + 12 - 9 = 0$$

$$\begin{array}{r} 32 \\ 16 \\ \hline 48 \end{array}$$

$6 \cdot 8$
$4 \cdot 8 +$

So A's columns are independent while B's are not

(3) $$|A| = x \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0$$

The rank of A is ~~two~~. at most two, column two has no pivot

(4) (a) Since the rank of A is at most two + ∴ there can only be two linearly independent rows + as such must have a zero determinant

(b) Formula 7 is given by

$$\det A = \sum (\det P)\, a_{1\alpha}\, a_{2\beta} \cdots a_{n\omega}$$

Every term will be zero because ~~in row #1 when we select~~ ~~or 1st column if we pick columns 1, 2, or 3~~ when we select columns we eventually have to select a zero in the 3×3 block in the lower left. This zero in the multiplication is what makes every term zero.

(5) For A we can expand the determinant about the 1st row giving

$$|A| = 1\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 1\left[1\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right]$$

$$-1\left[-1\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right]$$

$$= -1(1) - 1\left[-1\right] = -1 + 1 = 0$$

Expanding $|A|$ about the ~~last~~ last row of A gives

$$|A| = -1\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -1(1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -(-1) + -1 \;=\; 1 - 1 = 0$$

For B we can expand in the same way as with A.

So ~~|B|~~ = expanding about the 1st row gives

$$|B| = 1\begin{vmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 3 & 4 \\ 5 & 4 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

then expanding each determinant by ~~[illegible]~~ along the bottom row we obtain

$$|B| = 1(1)\begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} - 2(2)\begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix}$$

$$= -16 - 4(-16) = -16\left[1 - 4\right] = 48$$

(b) ~~A= It is better to work with the row reduced echelon form of an matrix A since this stores the "essence" of A with out much clutter. The minimal # of zeros must occur if the bottom row of A contains all zeros, i.e. we have~~

~~$R = \begin{bmatrix} x & x & x & x \\ 0 & & & \\ & & & \end{bmatrix}$~~

By creating a matrix with NO zeros we have certainly used the smallest. let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

the $\det(A) = 0$

in which we have no zeros (obviously the smallest possible)

To create a matrix with as many zeros as possible & still have $\det(A) \neq 0$ consider the diagonal matrices

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

w/ $a, b, c, d \neq 0$. This matrix is certainly not singular but by setting any of a, b, c, or d equal to zero a singular matrix would result

7 (a) Since $|A| = \sum (\pm) a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$

with $a_{11} = a_{22} = a_{33} = 0$

All permutations w/ a_{11} in them i.e. $(1,2,3)$ $(1,3,2)$ ~~$(2,3)$~~ ~~$(3,2)$~~

will have a zero in them. All permutations w/ a_{22} in them will

be zero i.e. $(1,2,3)$, $(3,2,1)$. But the 1st permutation is

~~already~~ already used up in the consideration of when

we select a_{11} for the 1st term in the $(1,2,3)$ permutation

The $a_{33} = 0$ will cause the two permutations

$(1,2,3)$ & $(2,1,3)$ to be zero.

Thus 4 terms will be zero from the determinant.

(b) For the 4x4 case, the following column permutations will

have zero products

~~$(1,2,3,4)$~~ ~~$(1,x,x,x)$~~ or ~~$3! = 6$ choices~~

~~$(x,2,x,x)$ but~~

$(1,i,j,k)$ or when $i,j,$ & k take on all possible

values we have $3 \cdot 2 \cdot 1 = 6 = 3! = 6$

possible combinations

The following permutations will be zero due to the term a_{22}

$(i,2,j,k)$ but if $i=1$ we will have already counted

these $2! = 2$ permutations

in our calculation of the terms that vanish when ~~a_{11}~~ due to $a_{11} = 0$. Thus we now have

~~$6+2$~~

looking for all permutations that have ~~a~~, a one in the 1st position or a 2 in the 2nd column or a 3 in the 3rd column or a 4 in the 4th column, these permutations will result in a zero term in our determinant calculation.

There are ~~6~~ $\overset{3 \cdot 2 \cdot 1}{6}$ permutations with a one in the 1st column + 6 permutations with a 2 in the 2nd column, ~~this gives~~ but of these 6 permutations ~~2~~ 2 are already counted for 2.

... Not sure exactly how to count

(8) To have $\det(P) = +1$ we must have an even # of row exchanges. The total # of $\underset{5 \times 5}{\wedge}$ permutation matrices is given by $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 5 \cdot 4 \cdot 6 = 5 \cdot 24 = 120$

Half of this number are odd + ½ of this number are even so 60 permutation matrices have $\det(P) = +1$ + 60 have $\det(P) = -1$.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

will require 4 exchanges to obtain the identity

Then $P_1 =$ through row exchanges

1 exchanges 1st + last row

$$P \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

2 exchanges 2nd + last row

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

3 exchanges 3rd + last row

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

4 exchanges 4th + 5th row

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(9) $\det(A) \neq 0$ then say $a_{1\alpha}\, a_{2\beta} \cdots a_{n\omega} \neq 0$

for some specification of $(\alpha, \beta, \ldots, \omega)$ a permutation.

Construct the permutation that takes $(\alpha, \beta, \ldots, \omega) \to (1, 2, 3, \ldots, n)$

i.e. the inverse permutation then $A \cdot P$ ~~PA~~ will have $a_{1\alpha}$ in

position $(1,1)$, $a_{2\beta}$ in position $(2,2)$, $a_{3\cdot}$ in position $(3,3)$

+ finally $a_{n,\omega}$ in position (n,n).

This is because $A \cdot P$ permutes the columns of A + will move

$a_{1,\alpha}$ to $1,1$ etc.

(10) (a) The total # of permutations of $\{1,2,3,4\}$ is $4! = 24$
half of them are even i.e. $\frac{24}{2} = 12$ are even. They are

~~(1,2,3,4)~~, ~~(...)~~,
~~(1,2,3,4)~~, ~~(2,1,4,3)~~, ~~(3,4,1,2)~~, ~~(4,3,2,1)~~
~~(1,4,3,2)~~, ~~(2,3,1,4)~~, ~~(1,4,2,3)~~

My systematic way to do this problem will be to enumerate all the possible permutations & seperate them into positive & negative permutations

1,2,3,4 +	3,1,2,4 +
1,2,4,3 −	3,1,4,2 −
1,3,2,4 −	3,2,1,4 −
1,3,4,2 +	3,2,4,1 +
1,4,2,3 +	3,4,1,2 +
1,4,3,2 −	3,4,2,1 −
2,1,3,4 −	4,1,2,3 −
2,1,4,3 +	4,1,3,2 +
2,3,1,4 +	4,2,3,1 −
2,3,4,1 −	4,2,1,3 +
2,4,1,3 −	4,3,2,1 +
2,4,3,1 +	4,3,1,2 −

(b) An odd permutation times an odd permutation is a even permutation matrices

(11) For $A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$ we have

$C_{11} = +6$, $C_{12} = -3$; $C_{21} = -1$; $C_{22} = 2$

So $C = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix}$

For $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$ we have

$C_{11} = +\begin{vmatrix} 5 & 6 \\ 0 & 0 \end{vmatrix} = 0$; $C_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 0 \end{vmatrix} = +42$; $C_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 0 \end{vmatrix} = -35$

$C_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} = 0$; $C_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} = -21$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 0 \end{vmatrix} = 14$

$C_{31} = +\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12-15 = -3$

$C_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -(6-12) = 6$

$$C_{33} = -\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -(5-8) = 3$$

Thus $C = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & 3 \end{bmatrix}$

The det of B is given by (expanding about the 3rd row)

$$\det(B) = +7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 7(12-15) = 7(-3) = -21$$

(12) For $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

So $C_{11} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$

$$C_{12} = (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = (-1)(-2) = 2$$

$$C_{13} = +1\begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$$

$$C_{21} = (-1)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = (-1)(-2) = 2$$

$$C_{22} = 1\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$$

$$C_{23} = (-1)\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = (-1)(-2) = 2$$

$$C_{31} = 1\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$$

$$C_{32} = (-1)\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = (-1)(-2) = 2$$

$$C_{33} = +1\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$$

So $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ so $C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Now $C^T A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$= \Big[\cancel{4} \quad \cancel{-3+4-1} = 2$$

$$= \begin{bmatrix} 6-2 & -3+4-1 & -2+2 \\ 4-4 & -2+8-2 & -4+4 \\ 2-2 & -1+4-3 & -2+6 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$= 4 \cdot I$. But Note that

$$\det(A) = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2)$$

$$= 2(3) - 2 = 4$$

So $A^{-1} = \frac{1}{\det(A)} \cdot C^T$

(13) As suggested expanding $|B_4|$ using cofactors of the last row of B_4 we have

$$|B_4| = +2 \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix}$$

$$= \cancel{2|B_3| + (-1)} \begin{vmatrix} \cancel{1} & \cancel{-1} & 0 \\ \cancel{0} & \cancel{-1} & \cancel{-1} \\ -1 & 2 & 0 \end{vmatrix}$$

$$= 2|B_3| + (-1) \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 2|B_3| - |B_2|$$

Continuing our expansion we have that $|B_2| = 2 - 1 = 1$

& $$|B_3| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 1 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$$

$$2(4-1) + -2 = 1$$

So $|B_4| = 2(1) - 1 = 1$

(14) $C_1 = |0| = 0$

$$C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

$$C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

(b) $$|C_n| = \begin{vmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 1 & 0 & 1 & \cdots & & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 \cdots & 0 \\ & & & & & \end{vmatrix}$$

By expanding about the 1st row we have

$$|C_n| = (-1)\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & & \\ 0 & 0 & 1 & 0 & 1 & \\ \vdots & & & & & \\ 0 & & & & & \end{vmatrix}$$

$$\Rightarrow |C_n| = (-1)(1)\begin{vmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & & \\ 0 & 1 & 0 & 1 & & \\ \vdots & & & & \ddots & \end{vmatrix}$$

$$= (-1)\,|C_{n-2}|$$

So $|C_1| = 0$ gives $|C_3|, |C_5|, |C_7|, \ldots$ are all zero

$|C_2| = -1$ gives $|C_4|, |C_6|, \ldots,$ [illegible]

As $|C_4| = 1$, $|C_8| = 1$, $|C_{12}| = 1$, $\ldots$

& $|C_6| = -1$, $|C_{10}| = -1$, $|C_{12}| = -1$, $\ldots$

Thus $|C_{10}| = -1$

(15) From prob 14 we have shown the desired relationships.
I don't know what they mean by going down the matrix.

(16) $|E_1| = 1$

$|E_2| = 0$

$$|E_3| = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 1(1) = -1$$

$\vdots$

to derive a recursion relationship consider

$$|E_n| = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \vdots & & & & & \end{vmatrix}$$

expanding about the 1st row gives

$$|E_n| = +1 \begin{vmatrix} 1 & 1 & 0 & \cdots & & \\ 1 & 1 & 1 & 0 & - & \\ 0 & 1 & 1 & 1 & 0 & - \\ \vdots & & & & & \end{vmatrix} - 1 \times \cancel{\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}}$$

$$\neq -1 \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \end{vmatrix}$$

$$= |E_{n-1}| - \begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{vmatrix}$$

$$= |E_{n-1}| - |E_{n-2}|$$ as requested to be shown

(b) with $E_1 = 1$ + $E_2 = 0$

$$E_3 = E_2 - E_1 = -1$$

$$E_4 = E_3 - E_2 = -1 - 0 = -1$$

$$E_5 = E_4 - E_3 = -1 - (-1) = 0$$

$$E_6 = E_5 - E_4 = 0 - (-1) = 1$$

$$E_7 = E_6 - E_5 = 1 - 0 = 1$$

$$E_8 = E_7 - E_6 = 1 - 1 = 0$$

$$E_9 = E_8 - E_7 = 0 - 1 = -1$$

$$\vdots$$

The pattern looks like

$E_{2,5,8,\ldots} = 0$ or $E_{3n+2} = 0 \quad n = 0, 1, 2 \cdots$

$E_{3,4,9,10,15,16,\ldots} = \cancel{0}-1$ or $\cancel{E_{3n} = 0 - 1 \quad n = 1, 2, 3, \ldots}$

$+ \cancel{E_{3n+1} = 0 - 1 \quad n = 1, 2, 3, \ldots}$

$\cancel{E_{1,7,12,13,18,19,\ldots} = 1}$ or $E_{3+\cancel{8n}\,6n} = -1 \quad n = 0, 1, 2 \ldots$

$E_{4+6n} = -1 \quad n = 0, 1, 2 \cdots$

+ $E_{6,7,12,13,18,19,\ldots} = 1$ or $E_{6n} = 1 \quad n = 1, 2 \cdots$

$E_{6n+1} = 1 \quad n = 1, 2, 3, \ldots$

Then E_{100} can be written $E_{16\cdot 6+4}$

$$\cancel{\frac{100}{6} = 10 + \frac{40}{6} = 10 + 6 + \frac{4}{6} = 16 + \frac{2}{3}}$$

$$\begin{array}{r} 16 \\ \times 6 \\ \hline 60 \\ 36 \end{array}$$

So looking back we see that $E_{6n+4} = -1$ so

E_{100} is $= -1$.

(17) $\cancel{E_8} =$

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & & \\ 1 & 1 & -1 & 0 & & \\ 0 & 1 & 1 & -1 & & \\ 0 & 0 & 1 & 1 & -1 & \\ 0 & 0 & 0 & 1 & 1 & -1 \end{vmatrix}$$

So

$$F_n = 1 \begin{vmatrix} 1 & -1 & 0 & \\ 1 & 1 & -1 & \\ 0 & 1 & 1 & -1 \\ & & & \ddots \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 0 & & \\ 0 & 1 & -1 & & \\ 0 & 1 & 1 & -1 & \\ 0 & 0 & 1 & 1 & -1 \end{vmatrix}$$

$$= F_{n-1} + F_{n-2}$$

(18) Thus linearity gives

$$|B_n| = |A_n| - \cancel{|B_n|}\,|A_{n-1}| =$$

$$= x+1 - (x-1+1) \quad \text{from the discussion in this section}$$

$$= 1$$

(19) The 4x4 Vandermonde determinant contains x^3 & not x^4 because as a polynomial requires 4 points to fit to. Thus a nxn ~~of~~ degree

Vandermonde determinate will have x^{n-1}.

This determinant is zero if $x = a, b,$ or c. The cofactor of x^3 is given by

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= 1\begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - a\begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} + a^2\begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix}$$

$$= bc^2 - cb^2 - a(c^2 - b^2) + a^2(c-b)$$

$$= bc(c-b) - a(c-b)(c+b) + a^2(c-b)$$

$$= (c-b)\left[bc - ac - ab + a^2\right] = (c-b)\left[b(c-a) - a(c-a)\right]$$

$$= (c-b)(c-a)(b-a)$$

Thus ~~V_4~~ since V_4 is a polynomial with roots $a, b, + c$ + the cofactor of x^3 represents the leading coefficient of ~~V_4~~ the x^3 term in the total determinant. Thus

$$V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$$

(20) $$G_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Then $$|G_4| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix}$$

$(-\frac{1}{2})$

$\frac{1}{2} - 2 = -\frac{3}{2}$

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -3/2 \end{vmatrix}$$

$= (-1)(1)(1)(-2)(-3/2) = -3$

$$\det G_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$\det G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ \phi & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= (-1)(-1) + 1(1) = 2$$

So by ~~hypothesis~~ ~~to~~ an induction hypothesis

$$\det G_n = (-1)^{n-1}(n-1)$$

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(21) (a) The 1st statement is true since by applying elementary row operations to the matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ the pivots obtained will be determined from the matrices A + D only. Since the determinant is the product of the pivots it is equal to the product of the pivots from A + from D.

(b) Let our block matrix be

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$$ which has block matrices given by

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}; \quad D = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}.$$

with determinants
$|A| = 1$ $\quad |B| = 5$ $\quad |C| = \cancel{0}1$ $\quad |D| = \cancel{4-8} = -5$

The determinant of the block matrix is given by 15

while the product $|A|\cdot|D| = 1\cdot(-5) \neq 15$

In addition the ~~terms~~ expression

$$|A||D| - |C|\cdot|B| = 1\cdot(-5) - \overset{1\cdot(5)}{\cancel{0}} = \overset{-10}{\cancel{-5}}$$ is Not equal

to the true determinant (15) either

(c) computing $AD - CB$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 5 \\ 7 & 8 \end{bmatrix} = \cancel{\begin{bmatrix} -6 & -3 \\ -3 & -4 \end{bmatrix}}$$

~~which has determinant given by~~ $\cancel{24 - 9 = 15}$

$$= \begin{bmatrix} -6 & -3 \\ -5 & -7 \end{bmatrix}$$

which has determinant given by $42 - 15 = \cancel{15+12}$
$= 27$

which does Not equal the true value either

(22) (a) Since the matrix L_k is lower triangular & constructed to have ones on its diagonal $|L_k| = 1$ for $k = 1, 2, + 3$

I'm assuming that ~~[illegible]~~ the index k refers to how many rows + columns the matrix L_k / U_k assumes i.e. L_1/U_1 or 1×1, L_2/U_2 or 2×2, etc

The determinant of U_k will then be

$|U_1| = 2 \quad |U_2| = 2 \cdot 3 = 6 \quad |U_3| = 2(3)(-1) = -6$

In the same way $|A_k| = |U_k|$

(16) If ~~A_k~~ $A_1, A_2, + A_3$ have determinants given by $2, 3, + -1$ the pivots are given by

$$P_1 = 2\ ,\ P_2 = \frac{3}{2}\ +\ P_3 = \frac{-1}{2\left(\frac{3}{2}\right)} = \frac{-1}{3}$$

(23) Taking the determinant of the left hand side + using the determinant rule that row operations don't change the value of the determinant, or the fact that this matrix ~~has~~ is lower triangular w/ ones on the diagonal $\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$, we have

$$\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = \left|\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = \left|\begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}\right|$$

$= |A| \cdot |D - CA^{-1}B|$ if A^{-1} exists

$= |AD - ACA^{-1}B|$ by distributing $|A|$ into the ~~addition~~ $|D - CA^{-1}B|$ term

$= |AD - CAA^{-1}B|$ if $AC = CA$

$= |AD - CB|$

(24) (b) $\det M = \det \begin{bmatrix} AB & A \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$

But $\cancel{\det}$ $= \det\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \cdot \det\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$

But $\det\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} = 1$ so the above is

$$= \det\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} = \det(AB) \quad \text{from problem 21}$$

If A is a single row + B is a single column then AB is a scalar + equals its own determinant. So

$\det M = AB$. For a 3×3 example let

$A = [1\ 2\ \cancel{3}]$ + $B = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$

Then $M = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} [1\ 2]\begin{bmatrix} 1 \\ 1 \end{bmatrix} & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

So $\det(M) = (-1)\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + 2\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$

$= (-1)(-1) + 2(1) = 3$

& $AB = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3$ in agreement.

If A is a column & B is a row then AB has rank 1

& ∴ $\det(AB) = 0$. Example of this case is skipped

(25) (a) From eq (9) we see that

$$\frac{d \det A}{d a_{11}} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \det M_{11}, \text{ or the}$$

cofactor of the ~~...~~ cofactor C_{11}

(b) $$\frac{d}{da_{11}} \ln(\det(A)) = \frac{1}{\det(A)} \frac{d(\det(A))}{da_{11}}$$

$$= \frac{M_{11}}{\det(A)} \quad \text{which is}$$

the $(1,1)$ entry in A^{-1}.

(26) $$D = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

will not be invertible ~~since~~ ~~subtracting the 1st row from the 2nd row gives~~ since subtracting the 2nd row from the 3rd row gives

$$\cancel{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 9 & 0 & \end{bmatrix}} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{bmatrix}$$

then subtracting the 1st row from the 2nd gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

Since this matrix has two identical rows & these row operations don't change the determinant, we see that the determinant should be zero. The expression given in the text is derived from

$$\left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 4 & 5 & 6 & 4 & 5 \\ 7 & 8 & 9 & 7 & 8 \end{array}\right|$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$= 1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8 - 3\cdot5\cdot7 - 1\cdot6\cdot8 - 2\cdot4\cdot9$$

$$= 45 + 84 + 96 - 105 - 48 - 72$$

$$= 0$$

(27) In problem 16 $E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$

to find the 5 nonzero terms in E_4 we pick the terms in eq 7 that will be non zero, for example

$a_{11}a_{22}a_{33}a_{44}$, $a_{11}a_{22}a_{34}a_{43}$, $a_{11}a_{23}a_{32}a_{44}$,

$a_{12}a_{21}a_{33}a_{44}$, $a_{12}a_{21}a_{34}a_{43}$

which each equal 1 in magnitude + when combined with the correct signs from the given permutations gives

$$+1-1-1-1+1 = -1 \quad \text{as expected}$$

(28) For the given matrix at the beginning of the section we have w/

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

So the non zero terms are given by $a_{11}a_{22}a_{33}a_{44}$, $a_{11}a_{22}a_{34}a_{43}$, $a_{11}a_{23}a_{32}a_{44}$, $a_{12}a_{21}a_{33}a_{44}$, $a_{12}a_{21}a_{34}a_{43}$

which have values given by (+ corresponding signs due to the permutation

$$2(2)(2)(2) - 2(2)(-1)(-1) - 2(-1)(-1)(2) - (-1)(-1)(2)(2)$$

$$+ (-1)(-1)(-1)(-1)$$

$$= 16-4-4-4+1 = 4+1 = 5 \quad \text{as expected}$$

(29) $|S_1| = 3$

$|S_2| = 9 - 1 = 8$

$$|S_3| = 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix}$$

$$= 3(9-1) - (3) = 24 - 3 = 21$$

Noticing that the Fibonacci numbers are given by

$F_0, F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9$

$1, 1, 2, 3, 5, 8, 13, 21, \cancel{3}4, \cancel{65}\ 55,$

we see that $|S_1| = F_3$

$|S_2| = F_5$

$|S_3| = F_7$

thus we guess that $|S_4| = \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = F_9 = 55$

Expanding the determinant $|S_4|$ we have

$$|S_4| = 3\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 3|S_3| - \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 3|S_3| - |S_2|$$

$$= 3(21) - 8 = 63 - 8 = 55 \quad \text{yes}$$

(30) From Problem 29 we have shown that

$$|S_n| = 3|S_{n-1}| - |S_{n-2}|$$

to show that a solution to this recurrence is given by

$|S_n| = F_{2n+2}$ consider the right hand side of this expression if

$|S_n| = F_{2n+2}$. We would then have

$$3F_{2(n-1)+2} - F_{2(n-2)+2} = 3F_{2n} - F_{2n-2}$$

But using the Fibonacci relationship $F_n = F_{n-1} + F_{n-2}$ so

$F_{n-2} = F_n - F_{n-1}$ so the above becomes replacing F_{2n-2}

$$3F_{2n} - (F_{2n} - F_{2n-1}) = 2F_{2n} + F_{2n-1}$$

recognizing $F_{2n} + F_{2n-1}$ as F_{2n+1} the above becomes

$$F_{2n} + F_{2n} + F_{2n-1} = F_{2n} + F_{2n+1}$$ which is equals

F_{2n+2} or the hypothetical value of $|S_n|$ showing ~~the equivalence~~

the solution is indeed F_{2n+2}.

(31) Consider a ~~different~~ definition of $|\tilde{S}_n|$ that given by

$$|\tilde{S}_n| = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & \cdots & 0 \\ & & & \ddots & & \end{vmatrix}$$

$$= 2 \begin{vmatrix} 3 & 1 & 0 & & \\ 1 & 3 & 1 & & \\ 0 & 1 & 3 & 1 & \cdots \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 3 & 1 & 0 & \cdots \\ 0 & 1 & 3 & 1 & \cdots \\ 0 & 0 & 1 & 3 & 1 & \cdots \end{vmatrix}$$

$$= 2|S_{n-1}| - 1 \begin{vmatrix} 3 & 1 & 0 & \cdots \\ 1 & 3 & 1 & \cdots \\ 0 & 1 & 3 & 1 & \cdots \end{vmatrix} = 2|S_{n-1}| - |S_{n-2}|$$

But since $|S_n| = F_{2n+2}$ we see that $|\tilde{S}_n|$ is given by

from problem 30

$$|\tilde{S}_n| = \cancel{2F_{2n+2} - F_{2(n-2)+2}} \quad 2F_{2(n-1)+2} - F_{2(n-2)+2}$$

$$= \cancel{2F_{2n+2} - F_{2n-2}} \quad 2F_{2n} - F_{2n-2}$$

~~using the expression~~ $\cancel{F_{2n+2} = F_{2n} + F_{2n+1}}$ ~~we have~~

$$\cancel{|\tilde{S}_n| = 2(F_{2n} + F_{2n+1}) - F_{2n-2}} \qquad \cancel{F_{n-2} = F_n - F_{n-1}}$$

$$= \cancel{2F_{2n+1} + 2F_{2n} - F_{2n-2}}$$

$$= 2F_{2n+1} \cancel{+ 2F_{2n} - (F_{2n} - F_{2n-1})}$$

$$= \cancel{2F_{2n+1} + F_{2n} + F_{2n-1}} \; 3\cancel{F_{2n+1}}$$

See Next page for continued derivation.

Check: write the determinant $|\tilde{S}_n|$ as follows

$$|\tilde{S}_n| = \begin{vmatrix} 3 & 1 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 \cdots & 0 \\ 0 & 1 & 3 & 1 \cdots & 0 \\ & & \ddots & & \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 & \cdots & & \\ 0 & 3 & 1 & 0 & \cdots & \\ 0 & 1 & 3 & 1 & 0 \cdots & \\ 0 & 0 & 1 & 3 & 1 & 0 \cdots \end{vmatrix}$$

$$= |S_n| + (-1)|S_{n-1}|$$

$$= F_{2n+2} - F_{2(n-1)+2} = F_{2n+2} - F_{2n}$$

$$= F_{2n} + F_{2n+1} - F_{2n} = F_{2n+1} \quad \text{as expected}$$

Using the fact that $F_{2n} = F_{2n+1} - F_{2n-1}$ we have

$$|\tilde{\mathfrak{S}}_n| = F_{2n+1} - F_{2n-1} + F_{2n} - F_{2n-2}$$

$$= F_{2n+1} + 0 = F_{2n+1}$$ as expected

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① (a) our system is given by

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

then cramer's rule is given by

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad + \quad x_2 = \frac{\det(B_2)}{\det(A)}$$

$$\text{w/ } B_1 = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} \quad + \quad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

so $\det(A) = 8 - 3 = 5$; $\det(B_1) = 4 + 6 = \cancel{2}\,10$; $\det(B_2) = -4 - 1 = \cancel{3}\,-5$

$\therefore\ x_1 = \frac{+10}{5} = 2 + x_2 = \frac{-\cancel{3}}{5} = -1$

(b) our system is given by

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \{ \underline{x} \text{ is the 1st column of the inverse matrix of A}$$

cramer's rule is given by

$$x_1 = \frac{\det(B_1)}{\det(A)} \ ; \ x_2 = \frac{\det(B_2)}{\det(A)} \ ; \ x_3 = \frac{\det(B_3)}{\det(A)}$$

$$\text{w/ } B_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} ; \ B_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} ; \ B_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

So $\det(A) = 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2(4-1) - 1(2)$

$= 6 - 2 = 4$

& $\det(B_1) = 1\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$

$\det(B_2) = 2\begin{vmatrix} 0 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = -2$

$\det(B_3) = -1\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 1$

$\therefore \quad x_1 = \frac{3}{4}\ ; \quad x_2 = \frac{-2}{4} = \frac{-1}{2}\ ; \quad x_3 = \frac{1}{4}$

(2) (a) $y = \dfrac{\left|\begin{bmatrix} a & 1 \\ c & 0 \end{bmatrix}\right|}{\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|} = \cancel{\dfrac{-c}{D}} = \dfrac{-c}{ad-bc}$

(b) $y = \dfrac{\left|\begin{bmatrix} a & 1 & c \\ d & 0 & f \\ g & 0 & i \end{bmatrix}\right|}{D} = \dfrac{(-1)\begin{vmatrix} d & f \\ g & i \end{vmatrix}}{D}$

$= \dfrac{(-1)(id - fg)}{D} = \dfrac{fg - id}{D}$

③ (a) This system has No solution & Cramer's rule gives for x_1 & x_2

$$x_1 = \frac{\det(B_1)}{\det(A)} \quad \& \quad x_2 = \frac{\det(B_2)}{\det(A)}$$

Now $\det(A) = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$!!

$\det(B_1) = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 6 - 3 = 3$

$\det(B_2) = \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 2 - 4 = -2$

So $x_1 = \frac{3}{0}$ & $x_2 = \frac{-2}{0}$ No solution

For (b) $\det(A) = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0$

$\det(B_1) = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$; $\det B_2 = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

thus $x_1 = \frac{0}{0}$ & $x_2 = \frac{0}{0}$ both indetermined

④ Consider $|b \;\; a_2 \;\; a_3| = |\sum a_j x_j \;\; a_2 \;\; a_3|$

Since the determinant is a linear function of the first column

we have

$$= \sum_{j=1}^{3} x_j |a_j \; a_2 \; a_3|$$

But when columns repeat the determinant is zero so the above becomes

$$x_1 |a_1 \; a_2 \; a_3| = x_1 \det(A) \Rightarrow x_1 = \frac{\det(B_1)}{\det(A)}$$ is the

~~the~~ ~~part~~ answer to part (a)

(b) Is shown in the proof above

(8) If the 1st column of A is equivalent to the right hand side b then we know before beginning that $\underline{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. To see this with Cramers rule consider

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{\det(A)}{\det(A)} = 1$$ since B_1, which is A with ~~the~~ its 1st column replaced by the column vector b, which is the 1st column of A is again A.

$$x_2 = \frac{\det(B_2)}{\det(A)} = 0$$ since B_2 has two copies of the same column the 1st + 2nd columns are the same. Sim.

$$x_3 = \frac{\det(B_3)}{\det(A)} = 0$$ again because B_3 has two repeated columns this time columns #1 + #3

(b) Remembering $A^{-1} = \dfrac{C^T}{\det(A)}$

in part (a) we have

$C_{11} = +1\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3$; $C_{12} = -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$; $C_{13} = +\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0$

$C_{21} = -\begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} = -(2) = -2$; $C_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$; $C_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = -4$

$C_{31} = +\begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix} = 0$; $C_{32} = -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$; $C_{33} = +\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = +3$

So $C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix}$ $\therefore$ $C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$

~~So~~ ~~det~~ so since $\det(A) = +1\begin{vmatrix} 3 & 0 \\ 4 & 1 \end{vmatrix} = 3$

Thus $A^{-1} = \dfrac{C^T}{\det(A)} = \dfrac{1}{3}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$

check $A^{-1} \cdot A = \dfrac{1}{3}\begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = I$ ✓

(b) Since $A^T = A$ we expect A^{-1} to be symmetric as seen by the following simple proof.

$A(A^{-1}) = I$

$(A^{-1})^T A^T = I$

$(A^{-1})^T A = I$ $\therefore$ $(A^{-1})^T = A^{-1}$ so A^{-1} is symmetric, thus when we compute A^{-1} we need only compute the upper triangular portion ~~of~~ ~~of~~ A^{-1} or equivalently the upper triangular / lower triangular portion of C. Thus to ~~com~~ fully compute C we only need

$C_{11} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3;$ $C_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = -(-2) = 2$

$C_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$; $\cancel{C_{23}} =$ $C_{21} = C_{12}$; $C_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$

$C_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = -(-2) = 2$; $C_{31} = \cancel{C_{13}}\ C_{13}$; $C_{32} = C_{23}$

$C_{33} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-1 = 3$ $\therefore$

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ so } A^{-1} = \frac{1}{\det(A)} C^T = \frac{1}{\det(A)} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Since $\det(A) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2(4-1) + 1(-2)$
$= 6-2 = 4$

We hence that

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Which we can check with

$$\frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2+8-2 & 0 \\ 0 & -1+4-3 & -2+6 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = I \checkmark$$

7) If All cofactors are zero then the determinant is zero since the determinant of A can be written in terms of the cofactors C_{ij} using the cofactor formula i.e.

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

so if $C_{ij} = 0 \ \forall \ i+j$ $\det(A) = 0$ + our matrix is not invertable

If all of the cofactors are non zero is A sure to have an ~~OAN~~ inverse? No. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ Then $C_{ij} = \pm 1 \ \forall \ i+j$

But A does not have an inverse

(8) For A given by $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$ we have

$$C_{11} = +\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 6; \quad C_{12} = -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -3; \quad C_{13} = 0$$

$$C_{21} = \overset{C_{12}}{\cancel{-\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = -3}}; \quad C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 4; \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$C_{31} = \underset{C_{13}}{\cancel{+\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0}}; \quad C_{32} = \underset{C_{23}}{\cancel{-\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1}}; \quad C_{33} = +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

Using ~~By~~ the fact that A^{-1} is symmetric

$$\therefore \quad C = \begin{bmatrix} 6 & -3 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

So $$AC^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 & -3 & 0 \\ -3 & 4 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\therefore \det(A) = +3$

(9) If $\det(A) = +1$ & we know C_{ij} $\forall\, i,j$

Then $A^{-1} = \dfrac{1}{\det(A)} C^T$ $\therefore$ to determine A compute the inverse of the ~~[illegible]~~ of A^{-1} using cofactors

Thus we ~~[illegible]~~ compute the cofactors of the cofactor matrix to obtain the inverse of A^{-1} or A.

(10) $AC^T = \det(A) \cdot I$

then taking the determinant of both sides we have

$\det(A) \cdot \det(C^T) = \det(A)^n$.

$\Rightarrow \det(C^T) = \det(A)^{n-1}$

~~(11) Given all the cofactors of A ~~use~~ ~~Cramer's rule to~~ ~~compute~~ 1st use~~

~~$\frac{\cancel{1}}{\det(A)}$. the result from problem 10 to determine $\det(A) = (\det(C^T))^{1/n-1}$~~

~~Then since $A^{-1} = \frac{1}{\det(A)} C^T$ which we can compute without~~

~~knowledge of the cofactors of A we can ~~go~~ compute the~~

~~inverse of this matrix using cofactor expansions~~

~~If we consider the solution to $A^{-1}x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$~~

~~Then x would be the 1st column of A. The above is equivalent~~

~~to $\frac{1}{\det(A)} C^T x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ which by Crammers rule gives~~

$$\cancel{x_1 = \frac{\det(B_1)}{\det\left(\frac{C^T}{\det(A)}\right)} = \frac{\det(B_1)}{\frac{\det(C^T)}{\det(A)}}}$$

~~From problem 10 we have that~~

~~$\det(C^T) = \det(A)^{n-1}$ so x_1 is given by~~

$$\cancel{x_1 = \frac{\det(B_1)}{\det(A)}}$$

$$\cancel{\text{or} \quad C^T x = \begin{bmatrix} \det(A) \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

~~Then by cramers rule we have~~

$$\cancel{x_i = \frac{\det(B_i)}{\det(C^T)} = \det \begin{bmatrix} \det(A) & C_{21} & C_{31} & C_{41} \\ 0 & C_{22} & C & \\ 0 & C_{23} & & \\ 0 & C_{24} & & \end{bmatrix}}$$

— — — —

From problem 10 we have that

$$\det(A) = (\det(C^T))^{1/n-1} = (\det(C))^{1/n-1}$$ which can

be computed using the known cofactors. Then since $A^{-1} = \frac{1}{\det(A)} C^T$

we can explicitly construct A^{-1} using the known cofactors. This

matrix we can then invert to find A.

(12) If the entries of A are all integers then the cofactors are all integers since they are derived from the elements of A by multiplication + ~~subtraction~~ additions. Since the inverse of A is given by $\frac{C^T}{\det(A)}$ + $\det(A) = \pm 1$ our inverse matrix ~~is~~ consists of only integers. Consider

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$ then $|A| = 6 - 5 = +1$ so

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

(13) If A + A^{-1} have all entries given by integers

Then since $A \cdot A^{-1} = I$ taking the determinants of both sides

gives $|A| \cdot |A^{-1}| = 1$ but since $|A|$ + $|A^{-1}|$ must be integers, the only possible integers they could be is given by

~~$|A| = \pm 1$~~ $|A| = +1$ $|A^{-1}| = +1$ or

$|A| = -1$ $|A^{-1}| = -1$

(14) In example 5 we had

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad C_{11} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

so $C_{13} = + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \qquad C_{23} = - \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -(-2) = 2$

$C_{21} = - \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = +3 \qquad C_{31} = + \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$

$C_{32} = - \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = +3 \qquad C_{33} = + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$\therefore C = \begin{bmatrix} -1 & 2 & 1 \\ 3 & -6 & 2 \\ 1 & 3 & -1 \end{bmatrix} \quad \therefore C^T = \begin{bmatrix} -1 & 3 & 1 \\ 2 & -6 & 3 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{5} C^T$$

(15) Given that L^{-1} is lower triangular also the entries
(a)

$(L^{-1})_{12} = (L^{-1})_{13} = (L^{-1})_{23} = 0$

so from $L^{-1} = \frac{1}{\det(L)} C^T$ this implies that

$$C_{21} = C_{31} = C_{32} = 0$$

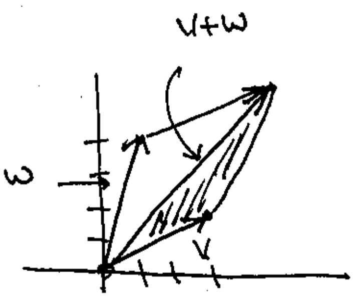
(b) Since S is symmetric S^{-1} will also be symmetric +

$C_{12}=C_{21}$ + $C_{13}=C_{31}$ + $C_{23}=C_{32}$

(16) For $n=5$ the matrix C contains 25 cofactors (5^2) and each 4×4 cofactor contains $4!$ terms + each term ~~needs~~ has 4 elements in the product + needs 3 multiplications to evaluate. Thus to compute A^{-1} using cofactors requires

$$25\cdot(4!)\cdot 3 = 75\cdot 24 = 1800 \text{ multiplications}$$

~~21~~
~~75~~
~~24~~
300
~~1800~~
1800



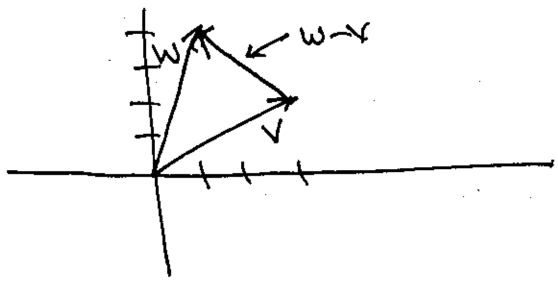
(17) (a) $\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 12-2 = 10$

(b) $\frac{1}{2}\begin{vmatrix} 3 & 2 \\ 3+1 & 2+4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix}$

$= \frac{1}{2}(18-8) = 5$

(c) $w - v = (1-3, 4-2) = (-2, 2)$

so plotting $v, w,$ + $w-v$ we have



$$\frac{1}{2}\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = \frac{1}{2}|12-2| = 5$$

(18) $$V = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= 3(9-1) - (3-1) + (1-3)$$

$$= 24 - 2 - 2 = 20$$

The area of each face is given by

$\|u \times v\|$ for the various combinations of u + v. For example

let $u = (3,1,1)$ + $v = (1,3,1)$ then

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= \hat{i}(1-3) - \hat{j}(\overset{3-1}{\cancel{9-1}}) + \hat{k}(9-1)$$

15

$= -2\hat{\imath} - \overset{2}{\cancel{8}}\hat{\jmath} + 8\hat{k}$

so $\|u \times v\| = \sqrt{\cancel{4+64+64}} = \sqrt{\cancel{4+128}} = \cancel{\sqrt{132}}$

$\cancel{2\sqrt{132}} = \quad = \sqrt{4+4+64} = \sqrt{72}$

$2\cancel{\sqrt{132}} = \cancel{80+15+1} \cancel{= 66}$

$2\cancel{\sqrt{66} = 33} \quad 2\cancel{\sqrt{74}} = \cancel{35+2} \cancel{= 37}$

$\underline{4 \cdot 33} \quad 2\cancel{\sqrt{72} = 35+1 = 36}$

$\therefore \quad \cancel{72 = 4 \cdot 18 = 4 \cdot 9 \cdot 2}$

$\therefore \quad \|u \times v\| = \sqrt{2 \cdot 4 \cdot 9} = 2 \cdot 3\sqrt{2} = 6\sqrt{2}$

If we let $u = (3,1,1)$ + $v = (1,1,3)$ then

$$u \times v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \hat{\imath}(3-1) - \hat{\jmath}(9-1) + \hat{k}(3-1)$$

$$= 2\hat{\imath} - 8\hat{\jmath} + 2\hat{k}$$

so As before $\|u \times v\| = 6\sqrt{2}$

Finally if $u = (1,3,1)$ + $v = (1,1,3)$

Then $u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}$

$= \hat{i}(9-1) - \hat{j}(3-1) + \hat{k}(1-3) = 8\hat{i} - 2\hat{j} - 2\hat{k}$

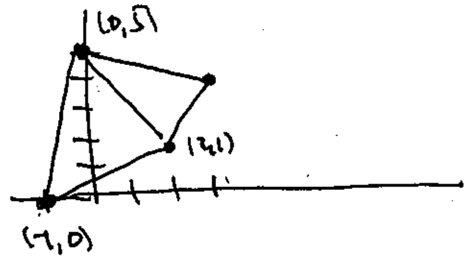
so $\|u \times v\| = 6\sqrt{2}$

19 (a) $\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix}$

$= \frac{1}{2}\left[2\begin{vmatrix} 4 & 1 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix}\right] = \frac{1}{2}\left[2(4-5) - 3(1-5)\right]$

$= \frac{1}{2}(-2+12) = \frac{10}{2} = 5$

(b) Add to the area of the 1st triangle the area of the second newly added triangle i.e



$\frac{1}{2}\begin{vmatrix} 0 & 5 & 1 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \frac{1}{2}\left[-2\begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix}\right] = \frac{1}{2}\left[-2(5) - 1(5-1)\right]$

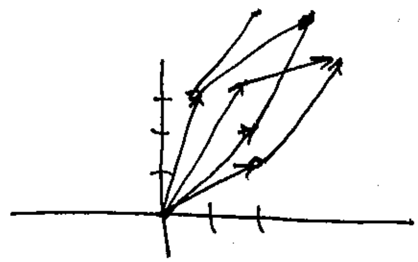
$= \frac{1}{2}(-10-4) = -7$ the reason for the negative sign is that we didn't encircle our triangle's vertices in a ~~clockwise~~ consistent manner, i.e. all counterclockwise or all ~~cou~~ clockwise

When we permute rows to obtain a consistent order i.e. one way is to exchange the 2nd + 3rd rows the introduced sign then ~~corrects~~ produces a positive one giving for the total area

$$5+7=12$$

(20) The area of the 1st parallelogram is given by

$$A_{I} = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2 = 4$$



The area of the 2nd parallelogram is given by

$$A_{II} = \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2 = 4$$

They are equal because the matrices holding the two sets elements are transposes of each other.

(21) Let $A = [c_1\ c_2\ c_3]$ be the matrix with 3 columns with lengths $L_1, L_2,$ + L_3 then consider the determinant of A

$$|A| = \left| [c_1\ c_2\ c_3] \right| = \left| [L_1\hat{c}_1\ L_2\hat{c}_2\ L_3\hat{c}_3] \right|$$ where $\hat{c}_i$ is the unit vector for c. Then using the linearity of each column, we have

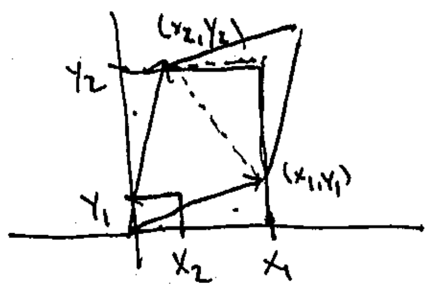
$$|A| = L_1 \left| [\hat{c}_1\ L_2\hat{c}_2\ L_3\hat{c}_3] \right| = L_1 \cdot L_2 \cdot L_3 \left| [\hat{c}_1\ \hat{c}_2\ \hat{c}_3] \right|$$

Recognizing that the determinant is a measure of volume the largest volume we can obtain is given when each of the colums/rows is orthogonal (then there are no trig functions depending on the angles between vectors. In that case the volume is given by the product of the lengths of each of the sides or $L_1 \cdot L_2 \cdot L_3$.

(b) No, as stated above the maximal value of the determinate is given by $L_1 \cdot L_2 \cdot L_3$. which when $|a_{ij}|=1$ we have

$L_1 = (\sum |a_{i1}|^2)^{1/2} = \sqrt{3}$, $L_2 = \sqrt{3}$, $L_3 = \sqrt{3}$ so the maximal determinant is given by $3^{3/2} < 6$

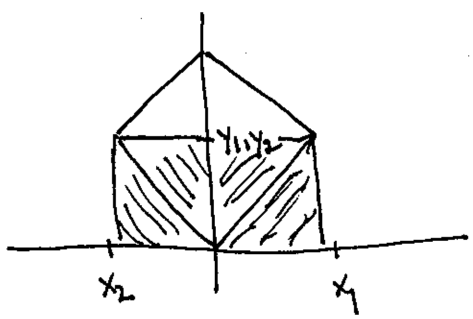
(22) Let x_1, y_1, x_2, y_2 all be positive. Then the point (x_1, y_1) + (x_2, y_2) could look something like



Then the area of our parallelogram is given by

$$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$= (x_1 y_2 - x_2 y_1)$



(23) The matrix A^TA is diagonal with $\|a\|^2, \|b\|^2, \|c\|^2$ on the diagonal. Then $|A^TA| = \prod_{i=1}^{3} d_{ii} = \|a\|^2\|b\|^2\|c\|^2$

But also $|A^TA| = |A|^2$ so

$$|A| = \|a\|\cdot\|b\|\cdot\|c\|$$

(24) Has height given by 4 since w has an $\hat{i} + \hat{j}$ component ~~tot~~ ~~only~~ + the only differing component is given by the $\hat{k}$ component or 4.

The volume equals Area $\cdot$ height $= 1\cdot 4 = 4$. The matrix with this as its volume is given by,

$$\cancel{\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{vmatrix}} = 1\cancel{\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix}} = 2$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4$$

$\hat{i} \times \hat{j} = \hat{k}$ & $\hat{k} \cdot w = 4$

(25) A n-dimensional cube has corners specified by a vector with n-components & with components given by 0's or 1's. Thus counting the # of vectors like this is easy it is 2^n.

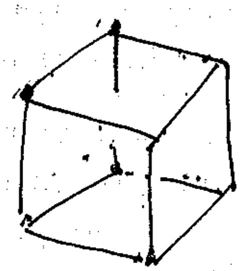
An edge results in selecting two vertices that only differ in one component. For example if $n=4$ the vertices $(0,0,0,0)$ + $(1,0,0,0)$ span an edge. ~~Thus~~ Now every vertices is connected to n other vertices by changing one of its n components. ~~Thus~~ Now counting the # of edges that each vertices connects to (for every vertex) will double count the # of edges. Thus the correct # of edges is given by

$$\cancel{2^n(2^n-1)} \quad \frac{2^n \cdot n}{2} = n \cdot 2^{n-1}$$

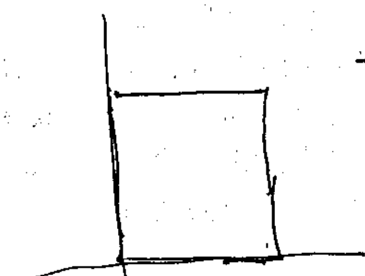
Check: $n=2$ this formula gives $\cancel{2^{4-1}=8}$

$2 \cdot 2^1 = 4$ yes ✓

$n=3$ $\quad 3 \cdot 2^2 = 3 \cdot 4 = 12$ yes ✓



The # of $n-1$ dimensional faces are given by holding one of the coordinates fixed + then letting all the other components take on all their possible values. For example if $n=2$



the "faces" are given by fixing the 1st component at 0 + letting the 2nd component be both 0 + 1.

$(0,\{0,1\}),\ (1,\{0,1\}),\ (\{0,1\},0),\ (\{0,1\},1)$

Where $\{0,1\}$ is to be understood to consist of both 0 + 1 elements. Thus ~~when~~ in general n dimension we pick a component of which there are n & fix a value for that component of which there are two possibilities. Thus there are

2n faces. Check if $n=2$ this is 4 yes
if $n=3$ this is 6 yes

A cube whose edges are the rows $2I$ has volume $\cancel{2\cdot 2\cdot 2=8}$ 2^n

(26) The triangle has area given by $\frac{1}{2}\begin{vmatrix}1&0\\0&1\end{vmatrix} = \frac{1}{2}$

The pyramid with 4 corners as given has volume given by

$$\cancel{\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}} = 1 \qquad \cancel{\tfrac{1}{2}\,\text{Area}\cdot\text{height}}$$

~~The volume would be 1 (still) for now~~

I don't see how to compute the normalizing factors in these volumes

(27) $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

The columns are orthogonal & the length are given by $+1$ & r

$\therefore J = 1 \cdot r = r$

(28) $x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Then $J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$ or

$$J = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi \overset{\sin\theta}{\cancel{\cos\theta}} & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$

which when expanded about the 3rd row gives

$$J = \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$+ \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi \overset{\sin\theta}{\cancel{\cos\theta}} & \rho\sin\phi\cos\theta \end{vmatrix}$$

$$\Rightarrow J = \rho^2 \cos^2\phi \sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$+ \rho^2 \sin^2\phi \sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$= \rho^2 \cos^2\phi \sin\phi \,(1) + \rho^2 \sin^2\phi \sin\phi \,(1)$$

$$= \rho^2 \sin\phi$$

Then $dV = dx\,dy\,dz = \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$

(29) J from problem 27 is given by

$$J = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

So $$J^{-1} = \frac{1}{(r\cos^2\theta + r\sin^2\theta)} \begin{bmatrix} r\cos\theta & +r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

Which has a determinant given by $|J^{-1}| = \frac{1}{|J|} = \frac{1}{r}$

(29) (30) The triangle w/ corners (0,0) (6,0) + (1,4) has area given by

$$\frac{1}{2}\begin{vmatrix} 6 & 0 \\ 1 & 4 \end{vmatrix} = 12$$

rotated by 60° it should still have area 12. The determinant of the rotation matrix better ~~have~~ be 1. to check

$$J = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{3}{4} = 1 \;\checkmark$$

(31)

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(31) If $u = (2,4,0)$, $v = (-1,3,0)$ + $w = (1,2,2)$

Then base area is $\|u \times v\|$ ~~and~~

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & 0 \\ -1 & 3 & 0 \end{vmatrix} = \hat{i}\begin{vmatrix} 4 & 0 \\ 3 & 0 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 0 \\ -1 & 0 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix}$$

$$= \hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}(6+4) = 10\hat{k}$$

So $\|u \times v\| = 10$.

Its perpendicular height is given by $\|w\| \cos\theta$ where θ is the angle made by w + the plane spanned by u + v i.e. The length of the ~~orthogonal~~ ~~complement~~ error vector when we compute the projection of w onto the span of u + v. This projection is given by solving $A^T A\hat{x} = A^T b$ ~~or~~ for $\hat{x}$ with $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}$

So $$A^T A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 10 \end{bmatrix} = 10\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

+ $$A^T b = \begin{bmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

So $$\hat{x} = \frac{1}{10}\frac{1}{(2-1)}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 10 \\ 5 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then the projection is given by

$$p = A\hat{x} = \begin{bmatrix} 2 & -1 \\ 4 & 3 \\ 0 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

So the error is $w - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

which has magnitude given by $\|w-p\| = 2$.

Then the volume equals $\|u \times v\| \cdot \|w\| \cos\theta$

$= 10 \cdot 2 = 20$

(32) $V = \begin{vmatrix} 2 & 4 & 0 \\ -1 & 3 & 0 \\ 1 & 2 & 2 \end{vmatrix} = \cancel{2\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 2(-2-3) = -10}$

$$= 2\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 2(6+4) = 20$$

(33) The determinant in equation 13 is given by

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - w_1v_2)$$

$$= u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - w_1v_2)$$

This is the dot product of $\underline{u}$ with the vector $v \times w$.

(84) Expressing each triple product as a determinant we see that

$$(u \times w) \circ v = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$(w \times u) \circ v = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$(v \times w) \circ u = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

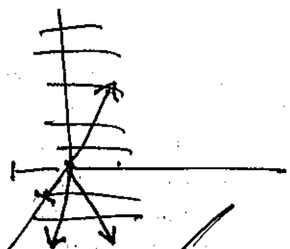
& finally ~~to~~

$$(u \times v) \circ w = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Thus using the fact that exchanging rows of a determinant causes a ~~sign~~ negation we see that

$$(u \times v) \circ w = -(u \times w) \circ v = (w \times u) \circ v = (v \times w) \circ u$$

(35) let ~~$S = P + Q + R = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$~~



let $S = P + Q + R = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ then S is in the plane spanned by the previous 3 vectors. This parallelogram will have area given by

$$\|P \times Q\| = \left\| \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{vmatrix} \right\| = \left\| \hat{\imath} \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \right\|$$

$$= \| \hat{\imath}(1) - \hat{\jmath}(1+1) + \hat{k}(1) \|$$

$$= \| \hat{\imath} - 2\hat{\jmath} + \hat{k} \| = \sqrt{1+2+1} = 2$$

To select $T, U,$ & V

Q: Don't understand how to do this problem

(36) The following determinant is zero

$$\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

since the 3 vectors lie in the same plane.

This determinant is given by

$$x\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - y\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + z\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(1) - y(1) + \cancel{z(2)}\, z(2-1) = 0$$

$$\Rightarrow \quad x - y + z = 0$$

(37) The following determinant is zero since the volume occupied by these 3 vectors is zero

$$D = \begin{vmatrix} x & y & z \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

Expanding this determinant we have

$$x\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} - y\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + z\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(9-2) - y(6-1) + z(4-3) = 0$$

or $7x - 5y + z = 0$

(38) (a) by doubling the length of every vector the volume is increased by 2^n.

(b) only size 1 (scalar) matrices). For 2x2 matrices let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad + \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Then $\det(A) = 1$; $\det(B) = 4$ so $\det(A) + \det(B) = 5$

but $\det(A+B) = 9 \neq \det(A) + \det(B)$