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(1) let ~~w~~ $w = -v$, then

$$T(v+w) = T(v) + T(w) \quad \text{becomes by the substitution choice}$$

$$T(0) = T(v) + T(-v) = T(v) - T(v) = 0$$

From requirement (b) we have $T(cv) = cT(v)$
with $c = 0$ we have $T(0) = 0 \cdot T(v) = 0$.

(2) By addition $T(cv) + T(dw) = cT(v) + dT(w)$

$$\Rightarrow \quad T(cv+dw) = cT(v) + dT(w)$$

To expand $T(cv + dw + eu)$ we can use the same expansion as before giving

$$cT(v) + dT(w) + eT(u)$$

(3) (a) Is linear
(b) Is linear
(c) Is linear
(d) $T(v) = (0,1)$ is Not linear since

$$T((v_1+w_1, v_2+w_2)) = T((v_1,v_2)) + T(w_1,w_2)$$

$$(0,1) \neq (0,1) + (0,1) = (0,2)$$

(4) If S + T are linear transformations then
$S(T(v))$ is a linear transformation.

(a) If $S(v) = v$ + $T(v) = v$ then $S(T(v)) = S(v) = v$

(b) $S(T(v_1+v_2)) = S(T(v_1) + T(v_2)) = S(T(v_1)) + S(T(v_2))$

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6 (a) $T(v) = \frac{v}{\|v\|}$ satisfies neither of the given identities

(b) $T(v) = v_1 + v_2 + v_3$ satisfies both identities

(c) $T(v) = (v_1, 2v_2, 3v_3)$ satisfy $T(cv) = cT(v)$

(d) $T(v) =$ largest component of v satisfies neither of them, c negative

5 If $T(v) = v$ but $T(0, v_2) = (0,0)$

~~Pf~~ ~~$T(cv) = cv$ if $cv_1 \neq 0$~~

+ ~~$T(cv) = (0,0)$ if $cv_1 = 0$~~

+ ~~$T(v) = v$ if $v_1 \neq 0$~~

+ ~~$T(v) = (0,0)$ if $v_1 = 0$~~

If $c = 0$ then $T(cv) = T(0) = (0,0) = 0 \cdot T(v)$

If $c \neq 0$ then $T(cv) = T((cv_1, cv_2)) = \begin{cases} (cv_1, cv_2) & v_1 \neq 0 \\ (0,0) & v_1 = 0 \end{cases}$

$$= \begin{cases} c(v_1, v_2) & v_1 \neq 0 \\ c(0,0) & v_1 = 0 \end{cases}$$

$$= \begin{cases} cT(v) & v_1 \neq 0 \\ c \cdot T(v) & v_1 = 0 \end{cases}$$

So in all cases $T(cv) = cT(v)$

Now To show $T(v+w) \neq T(v)+T(w)$ it suffices to show this equality is not true for some v & w

~~To~~ let $v=(1,2)$ & $w=(-1,2)$

Then $v+w = (0,4)$

so $T(v+w) = (0,0)$ while

$T(v)+T(w) = (1,2)+(-1,2) = (0,4) \neq T(v+w)$.

---

(7) (a) ~~T~~ $T(T(v)) = -(-v) = v$. which is linear

(b) $T(v) = v+(1,1)$ would have

$T(T(v)) = T(v+(1,1)) = T((v_1+1, v_2+1)) = (v_1+1, v_2+1) + (1,1)$

$= (v_1, v_2) + (2,2)$

This is not linear.

(c) $T(v) = (-v_2, v_1)$

Then $T(T(v)) = T((-v_2, v_1)) = (-v_1, -v_2) = \cancel{-v}\ -v$

~~T~~ which is linear

(d) $T(v) = \left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right) = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$

then $T(T(v)) = \frac{1}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}^2\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}$

So Φ which is liner

(8)
(a) If ~~####~~ $T((v_1,v_2)) = (v_2,v_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Then the range of T is $\mathbb{R}^2$ + the kernel of T is the point $(0,0)$

(b) $T((v_1,v_2,v_3)) = (v_1,v_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

So the Range of T is $\mathbb{R}^2$ + the kernel of T is $(0,0,v_3)$

(c) If $T(v_1,v_2) = (0,0)$ then the Range of T is $(0,0)$ while the kernel of T is $\mathbb{R}^2$

(d) $T(v_1,v_2) = (v_1,v_1)$ The range of T is $v_1\begin{pmatrix}1\\1\end{pmatrix} = \text{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$ while the kernel of T is on all vectors w/ $v_1 = 0$ or span of $\begin{pmatrix}0\\1\end{pmatrix}$.

(9) $T((v_1,v_2,v_3)) = (v_2,v_3,v_1)$

$T(T(v)) = (v_3,v_2,v_1)$

$T^3(v) = (v_1,v_2,v_3) = v$

$T^{100}(v) = T^{99+1}(v) = T^1(v) = (v_2,v_3,v_1)$

(10) (a) T has ~~a~~ a range not all of $\mathbb{R}^2$ but only span$\begin{pmatrix}1\\1\end{pmatrix}$

& a kernel given by the span of $\begin{pmatrix}1\\1\end{pmatrix}$ which is not (0,0)

(b) T(v) has a range that is given by the range of

$\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$ or the span of $\begin{pmatrix}1\\0\\1\end{pmatrix}+\begin{pmatrix}0\\1\\1\end{pmatrix}$. Since this is

of dimension $2 \neq$ dimension 3 the range of T is not all of W.

The ~~Also~~ ~~the~~ kernel of T is given by (0,0). &

(c) T has a range given by all of $\mathbb{R}^1$ but a

kernel that ~~includes~~ is the span of $\begin{pmatrix}0\\1\end{pmatrix}$ which is larger

than (0,0)

(A)

(11) (a) $V = \mathbb{R}^n$ & $W = \mathbb{R}^m$

(b) The range of T = span$\{Ax \mid \forall x\}$ = column space of A

(c) The kernel of $T = \{x \mid Ax = 0\}$ = nullspace of A

(12) (a) $T(2,2) = 2T(1,1) = 2(2,2) = (4,4)$

(b) $T(3,1) = T((1,1) + (2,0)) = T(1,1) + T(2,0) = (2,2) + (0,0) = (2,2)$

(c) $T(-1,1) = T((1,1) - (2,0)) = T(1,1) - T(2,0) = (2,2)$

(d) $T(a,b) = T(\;b(1,1) + \frac{a-b}{2}(2,0)\;)$

$= bT(1,1) + (\frac{a-b}{2})T(2,0)$

$(a,0) \quad (0,b)$

~~(a,b) = (b,b) + (a-b,0)~~

$= b(2,2) + (\frac{a-b}{2})(0,0) = b(2,2)$

(13) Scaling by a constant (when multiplied)

+ Distributivity

(14) (1) If $AM = 0$ then since $|A| = 5 - 6 = -1 \neq 0$

A is invertible + $M = 0$ (the zero matrix)

(2) Since $AM = B \Rightarrow M = A^{-1}B$

where A^{-1} is applied columnwise to B

(15) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

Then to have the identity matrix in the range of A would require

$AM = I \Rightarrow \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ which require that the vector $\begin{pmatrix}1\\0\end{pmatrix}$ + $\begin{pmatrix}0\\1\end{pmatrix}$

are in the column space of A. Since the column space of A

consists of only vectors $\propto \begin{pmatrix}1\\3\end{pmatrix}$ neither of these vectors is possible

To ~~find~~ require a non zero M such that AM is zero

consider $M = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$

Then $AM = \begin{bmatrix} -2+2 & -4+4 \\ -6+6 & -12+12 \end{bmatrix} = 0$

(16) We 1st show that no matrix will do it. Assume there existed a matrix M such that $AM = M^T$ then ~~$M^T A^T = M$~~

or ~~$AM(A^T) = M$~~ considering the identity for M we have that

$\Rightarrow$ ~~$AI =$~~ since $M^T = M$ that

$A \cdot I = I \Rightarrow A = I$ but ~~...~~ this ~~...~~ transformation matrix ~~could~~ ~~will~~ then implies that $I \cdot M = M^T$ which is <u>NOT</u> true for non symmetric matrices. Thus NO matrix exists that performs this test (when written with matrix notation)

This does not mean that we have a linear transformation that does not come from a matrix only that attempting to write the ~~input~~ input & output spaces ~~of~~ as formatted in a matrix is not the correct way to formulate the problem, they should be formulated as column vectors. Thus reformat the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ into a 4×1 vector $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

Then the linear transformation are given by 4×4 matrices.

(17) (a) $T^2 = I$ yes

(b) True

c) True

(d) False $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an example where $T(M) = -M$

(18) ~~Suppose~~ If $T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$$

Thus pick $b \neq 0$. or ~~let~~ $M = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ~~with~~ is an example

The kernel of T is given by all matrices $\begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$

the range of T is given by the span of $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(19) ~~If~~ ~~A≠0~~ For $M = I$ to fail we need ~~three~~ two matrices A & B such that $A \cdot B = 0$ let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

& $B = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

to find a M such that $AMB \neq 0$ let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ &

expand the product AMB.

(20) If $T(M) = AMB$

Then $T^{-1}(M) = \cancel{A^{-1}M}$ must be such that

$$T(T^{-1}(M)) = M \quad \& \quad T(T^{-1}(M)) = M$$

Thus let $T^{-1}(M) = A^{-1} \cdot M \cdot B^{-1}$

(21) ~~TTAA~~

(a) A diagonal matrix A will scale the x & y direction only. horizontal & vertical lines don't change.

(b) A rank one matrix will scale along one direction only sending the orthogonal direction to zero, Extreme shear. or onto a line

(c) ~~Accxxxx~~ triangle matrix will have vertical lines ~~staying~~ that stay vertical $\begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$

(22) See Matlab code ...

(23) (a) The x coordinate of each house point cannot change, therefore.

~~AB~~ $A = \begin{bmatrix} a & 0 \\ b & d \end{bmatrix}$

(b) $A = 3 \cdot I$

(c) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some θ

(24) (a) $|A| = 0$

(b) $|A| > 0$, $|A| < 0$ means that our "house" is reflected

(c) $|A| = 1$

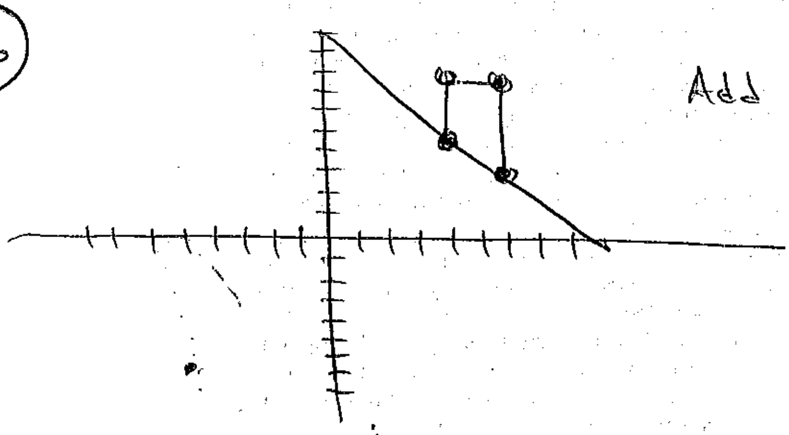
If one ~~xxxx~~ side of the house stays in place then both end points must not move ie ~~A~~ $T p_1 = p_1$

& $T p_2 = p_2$

Assuming p_1 + p_2 are not ~~[illegible]~~ colinear $\Rightarrow T = I$

(25) ~~[illegible]~~ ~~[illegible]~~ the "$-x$" operation flips the house through the origin, while (1,0) shifts the center of the base to (1,0) or one unit to the right

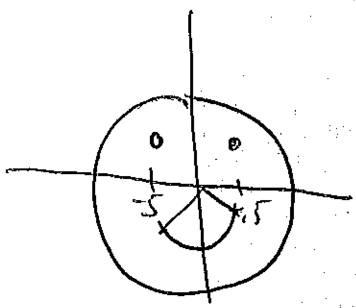
(26)



Add these points, see problem...

(27) See matlab...

(28)



theta2 = [linspace($-\frac{3\pi}{4}$, $-\frac{\pi}{4}$);

~~[illegible]~~

smile = [cos(theta2); sin(theta2)];

eye1 = [-.5, +.5]

eye2 = [+.5; +.5];

plot(eye1( ), eye1( ), 'o');

plot(eye2( ), —( ), 'o');

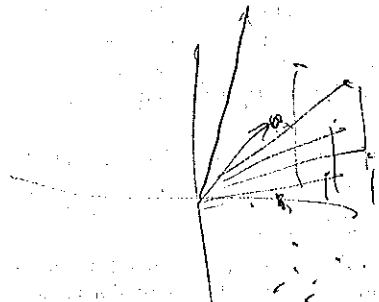
See matlab

(29) see methods

(30) $\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix}$ Shrinks the y-component ~~[illegible]~~ to 1/10th its original size.

$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ collapses the house along the line (1, 1)

$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$



Not sure

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ~~keeps~~ ~~horizontal~~ ~~lines~~ sheers the house?

(31)

$A_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $A_{12} = I$ $A_{13} = \begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix}$

$A_{21} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

... Not entirely ~~[illegible]~~ sure how to do this problem

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(1) with v_1, v_2, v_3 + v_4 as $1, x, x^2, x^3$

we have
$$Sv_1 = \frac{d^2}{dx^2}(1) = 0 = 0\cdot v_1$$
$$Sv_2 = \frac{d^2}{dx^2}(x) = 0 = 0\cdot v_1$$
$$Sv_3 = \frac{d^2(x^2)}{dx^2} = 2 = 2\cdot v_1$$
$$Sv_4 = \frac{d^2}{dx^2}(x^3) = 6x = 6\cdot v_2$$

Then a 4×4 matrix B for S is given by

$$B = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In general, apply our linear opperation to each input basis vector. Write the basis vector in terms of the output vectors. ~~Then~~ The coefficient of these decompositions become the columns of our ~~trans~~ matrix representation of our transformation.

(2) Functions that have $v'' = 0$ will be in the kernel of S, or the nullspace of the matrix B. For the matrix B ~~of~~ ~~B~~ we have x_3 + x_4 as pivot variables + x_1 + x_2 as free variables we have a nullspace spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ which translates into any function

given by $\begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}$ or equivalently $a+bx$.

(3) Adding a zero ~~at~~ ~~rows~~ the last row of A gives

$$A' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then the input space is given by $1, x, x^2, + x^3$ and this is also the same for the output space

Then $A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is the same as B from problem 1

For $B = A^2$ we want the output basis equal to the input basis (then $m=n$)

(4) AB is given by

(a)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) since B^2 represents the 4th derivative, but since the basis vectors only contain polynomials of maximum degree x^3 (which has a zero 4th derivative)

(5) The matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(6) (a) If the input is $v_1 + v_2 + v_3$ then

$$T(v_1+v_2+v_3) = T(v_1) + T(v_2) + T(v_3)$$
$$= w_2 + 2w_1 + 2w_3 = 2w_1 + w_2 + 2w_3$$

(b) $$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}$$ the coefficients of the above

(7) If $T(v_2) = T(v_3)$ then $T(v_2 - v_3) = 0$. Thus the vectors proportional to

$$\begin{bmatrix}0\\1\\-1\end{bmatrix}$$ are in the nullspace of A.

Thus all solutions to $T(v) = w_2$ are given by

$\cancel{v_1+v_2}$ $v_1 + \alpha(v_2 - v_3)$ $\forall \alpha$ or in the basis for the input space this is given by

$$\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

(8) Since $A = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 0\\0 & 1 & 1\end{bmatrix}$ a vector NOT in the column space is given by the orthogonal complement of the column space or the left null space of A

equivalently the nullspace of $A^T = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\1 & 0 & 1\end{bmatrix}$

or $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

Thus the combination of $w_1 - w_2$ is Not in the range of T

(9) We don't know ~~the~~ what $T(w_i)$ is or given by. unless the w's are the same as the v's. Thus If $w = v$'s then T^2 is given by A^2

(10) A has rank 2. Which is Not the dimension of the output space W (which is 3). 2 is the dimension of the range of T

(11) The matrix for T is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

we are looking for a vector v such that $T(v) = w_1$ or

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

pick $a=1$ then $b=-1$ then $c=0$ so if

$V = V_1 - V_2$ we have $T(v) = w_1$. To check

$$T(v) = T(v_1) - T(v_2)$$

$$= w_1 + w_2 + w_3 - w_2 - w_3 = w_1 \quad \text{Yes}$$

(12) For the A given in problem 11 we have

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Thus $T^{-1}(w_1) = v_1 - v_2$

$$T^{-1}(w_2) = v_2 - v_3$$

$$T^{-1}(w_3) = \cancel{w_3}\ v_3$$

to find all v's that solve $T(v)=0$ we are looking for the nullspace of A since A is invertible the only v that has $T(v)=0$ is when $v=0$.

(13) (a) Is true if T represents the matrix of the linear transformation

(b) Is true if T is the linear operator

(c) w is not necessarily in the domain of T & ∴ $T(w)$ may not make any sense

(14) (a) we are looking for a matrix A such that $A^2=I$.

let $A = $

$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so

$$A^2=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & cb+d^2 \end{bmatrix}$$

so we want $a^2+bc=1 \quad ab+bd=0$

$ac+cd=0 \quad cb+d^2=1$

$\Rightarrow$ ~~b=0 or~~ ~~a+b~~ ~~a+d=0~~ ~~let a=+1~~ + ~~then d=-1~~

so $1^2 \cancel{+bc=1}$ $\quad c\cdot\cancel{b+1=0}$

$\cancel{1c+c(-1)=0}$

$\cancel{bc=0}$ + $\cancel{cb=1}$

The (1,1) position requires $a^2 + bc = 1$

(1,2) $(a+d)b = 0$

(2,1) $(a+d)c = 0$

(2,2) $cb + d^2 = 1$

let $a=1$ + $d=-1$ then the (1,2) + (2,1) will be satisfied

also these relations require that $bc = 0$ + $cb = 0$

so pick $b = 0$ + $c = 1$ we get for $A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

Then the transformation T this describes is given by

$T(v_1) = v_1 + v_2$

+ $T(v_2) = -v_2$

(b) This requires that $A^2 = A$

$\cancel{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}$

let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then the transformation T requires

$T(v_1) = \frac{1}{\sqrt{2}} v_1 + \frac{1}{\sqrt{2}} v_2$ + $T(v_2) = \frac{1}{\sqrt{2}} v_1 + \frac{1}{\sqrt{2}} v_2$

(C). The same T would require a matrix representation that satisfied

$A^2 = I$ and $A^2 = A$ which is impossible ~~at all~~ unless $A = I$

(15) (a) $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

(b) A^{-1} with A the matrix above in part a

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

(c) To transform (2,6) to (1,0) + (1,3) to (0,1)

the matrix would have to be the inverse of the matrix transform

$A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ but this matrix has no inverse so

this mapping is not possible

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(16) (a) The matrix that transforms (1,0) + (0,1) to (r,t) + (s,u) is given by $M = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$

(b) The matrix that transforms (a,c) + (b,d) into (1,0) + (0,1) will be the inverse of the one that takes (1,0) + (0,1) into (a,c) + (b,d) or

$$N^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{so} \quad N = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(c) To make N possible we must have $ad - bc \neq 0$

(17) (a) From problem 16 N transforms (a,c) into (1,0) from which M transforms (1,0) into (r,t). Also N transforms (b,d) into (0,1) from which M transforms (0,1) into (s,u) which is the transformation desired. Thus the transformation is given by

$$M \cdot N = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{(ad-bc)} \begin{bmatrix} r & s \\ t & u \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(b) For this part $a=2, c=5, r=1, t=1, b=1, d=3, s=0, + u=2$

+ we get $M \cdot N = \frac{1}{(2\cdot 3 - 1\cdot 5)} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$= \frac{1}{1} \begin{bmatrix} 3 & -1 \\ 3-10 & -1+4 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -7 & 3 \end{bmatrix}$$

(18) If you keep the same basis vectors but put them in a different order the change of basis matrix M is a permutation matrix. If you keep the basis but change the length, M is a diagonal matrix

(19) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + b \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ which will be solved by

~~let~~ $a = \cos\theta$ + $b = -\sin\theta$

(20) $M = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$

The combination of $(1,4)$ + $(1,5)$ that equals $(1,0)$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 5 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

the new ~~coor~~ coordinates of $(1,0)$ is given by $M^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(21) (a) $w_2 = -(x-1)(x+1)$

(b) $w_3 = \frac{1}{2}x(x-1)$ $\qquad\qquad \cancel{-1(-1-1)} = 2$

(c) $y(x) = Aw_1(x) + Bw_2(x) + Cw_3(x)$

$$y(1) = Aw_1(1) + Bw_2(1) + Cw_3(1) = A = 4$$
$$y(0) = Aw_1(0) + Bw_2(0) + Cw_3(0) = B = 5$$
$$y(-1) = Aw_1(-1) + Bw_2(-1) + Cw_3(-1) = C = 6$$

$\therefore$ $y(x) = 4w_1 + 5w_2 + 6w_3$ which could be expanded into functions of x if needed

(22) From the w to the v's we have

$$w_1 = \frac{1}{2}(x^2+x) = \frac{1}{2}x + \frac{1}{2}x^2$$
$$w_2 = -(x-1)(x+1) = -(x^2-1) = 1 - x^2$$
$$w_3 = \frac{1}{2}x(x-1) = \frac{1}{2}x^2 - \frac{1}{2}x = -\frac{1}{2}x + \frac{1}{2}x^2$$

So the transformation matrix is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

The mapping from v's to the w's is given by A^{-1}.

(23) If $Y = A + Bx + Cx^2$

$$4 = A + Ba + Ca^2$$
$$5 = A + Bb + Cb^2$$
$$6 = A + Bc + Cc^2$$

So $A, B,$ & C are given by $$\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

It will be possible to invert this matrix if $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

(24) We require that the change of basis matrix be invertible i.e.

$$M = \begin{bmatrix} m_1 & m_4 & m_7 \\ m_2 & m_5 & m_8 \\ m_3 & m_6 & m_9 \end{bmatrix}$$ so $|M| \neq 0$.

(25) $\cancel{A=QR}$

The change of basis matrix from a_1, a_2, a_3 to q_1, q_2, q_3 we transform a_1, a_2, a_3 into the basis q_1, q_2 & q_3. The coefficients of these transformations is ~~given by~~ go into the coefficients of the change of basis matrix. The matrix eq $A=QR$ when written in terms of the columns of A & Q

$$a_1 = r_{11} q_1$$
$$a_2 = r_{12} q_1 + r_{22} q_2$$
$$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$$

Thus the change of basis matrix is then given by

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}$$

(26) When $A = LU$ row 2 of A is the combination of the 1st & 2nd row of U. Writing $A^T = U^T L^T$
The 2nd column of A^T is a linear combination of the 1st two columns of U^T. So the change of basis matrix is given by L^T.
(in the same way as in problem 25)
We have bases provided the ~~matrix~~ matrices are invertible.

(27) The matrix of T when v_i is the input & output basis is given by $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

(28) Let v_i be the input basis & $w_i = T(v_i)$ Then the transformation would be I. ~~T must be invertible~~ I don't see why T must be invertible ...

(29) (a) $A = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

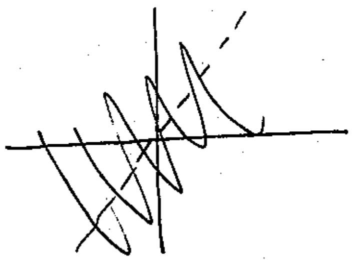
3D $T((x,y)) = (x,-y)$

$S((x,y)) = (-x,y)$

$S(T(v)) = S((x,-y)) = (-x,-y)$ so the transformation $ST(v) = -v$

$+ \cancel{T(S(v)) = T((-x,y)) = (-x,-y)}$

RS1



(31) The line at 45° is the vector $p=\begin{bmatrix}1\\1\end{bmatrix}$ & the vector that is perpendicular to the 1st is given by $n=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$. ~~Thus~~ The matrix that projects through the $\begin{bmatrix}1\\1\end{bmatrix}$ line is given by $I-\frac{2nn^T}{n^Tn} = I-2nn^T$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\frac{2}{2}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

$$=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

This is the matrix T. The matrix S is the same thing but $p=\begin{bmatrix}0\\1\end{bmatrix}$ so n in this case is given by $n=\begin{bmatrix}1\\0\end{bmatrix}$ so

$$S = I-\frac{2nn^T}{n^Tn} = \begin{bmatrix}1&0\\0&1\end{bmatrix}-2\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}2&0\\0&0\end{bmatrix}=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

Which can be checked by considering the action of S on a simple vector $\begin{bmatrix}1\\2\end{bmatrix}$

we have $S\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$ which is the correct reflection.

If $v=(2,1)$ then $T(v)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$

Now $S(T(v)) = \begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$

while $T(S(x)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

so from which we see that $TS = -ST$ but in general $TS \neq ST$

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(32) A reflection through the direction $p = \frac{1}{\sqrt{p_x^2+p_y^2}}\begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is

given by (with $n = \frac{1}{\sqrt{p_x^2+p_y^2}}\begin{bmatrix} -p_y \\ p_x \end{bmatrix}$)

$I - 2nn^T$. Defining $p = \begin{bmatrix} \frac{p_x}{\sqrt{p_x^2+p_y^2}} \\ \frac{p_y}{\sqrt{p_x^2+p_y^2}} \end{bmatrix} \equiv \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

we have n in terms of θ given by $n = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

then our reflection is given by (in terms of θ)

$$I - 2nn^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1-2\sin^2\theta & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 1-2\cos^2\theta \end{bmatrix}$$

Now using the facts that $\sin^2\theta = 1-\cos^2\theta$ + $2\sin\theta\cos\theta = \sin(2\theta)$

we have that ~~the~~ ~~above~~ ~~becomes~~ $1-2\sin^2\theta = 1-2+2\cos^2\theta = -1+2\cos^2\theta$

$$= \begin{bmatrix} -1+2\cos^2\theta & \sin(2\theta) \\ \sin(2\theta) & 1-2\cos^2\theta \end{bmatrix}$$

with $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ we have

$+1 - 2\cos^2\theta = +1 - 1 - \cos(2\theta) = -\cos(2\theta)$ & the above matrix is given

by $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$ which is given in the text.

The application of two reflections is then given by

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta)\cos(2\alpha) + \sin(2\theta)\sin(2\alpha) & \sin(2\alpha)\cos(2\theta) - \sin(2\theta)\cos(2\alpha) \\ \sin(2\theta)\cos(2\alpha) - \cos(2\theta)\sin(2\alpha) & \sin(2\theta)\sin(2\alpha) + \cos(2\theta)\cos(2\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta - 2\alpha) & -\sin(2\theta - 2\alpha) \\ \sin(2\theta - 2\alpha) & \cos(2\theta - 2\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta-\alpha)) & -\sin(2(\theta-\alpha)) \\ \sin(2(\theta-\alpha)) & \cos(2(\theta-\alpha)) \end{bmatrix}$$

From which we see the rotation angle is given by $2(\theta-\alpha)$

(33) False ~~+~~ will not be true if the vectors are not linearly independent. (I'm assuming that $T(\cdot)$ is linear).

This will be true if the n nonzero vectors are linearly independent i.e. form a basis of $\mathbb{R}^n$

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(1) As in Example #1 ~~our~~ for each vector the components in the wavelet basis are given by for $e = (1,0,0,0)$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathrm{diag}\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \mathrm{diag}(\longrightarrow) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

For $v = (1,-1,1,-1)$ we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathrm{diag}\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right) \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \mathrm{diag}\left(\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}\right) \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

(4)

(2) For $(7,5,3,1)$ we 1st average every two blocks to get

$\frac{7+5}{2} = 6$ & $\frac{3+1}{2} = 2$ to get the averages vector

$(6,6,2,2)$. The difference vector is obtained by subtracting

$7-5 = 2$ & dividing by 2 to get 1. Also for $3+1$

we have $3-1 = 2$ divided by 2 to get 1. Thus the

difference divided by two gives the following vector $(1,-1,1,-1)$

Recursing we write $(6,6,2,2)$ as an overall average i.e. $\frac{1}{4}(6+6+2+2)$

$= \frac{1}{4}(16) = 4$ i.e. $(4,4,4,4)$ and a difference given by

$6-2=4$ divided by 2 to obtain 2. & the vector

$(2,2,-2,-2)$. Thus our original vector is given by

$$\begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

which can be checked by assembling the right hand side as

$$\begin{pmatrix} 4+2+1 \\ 4+2-1 \\ 4-2+1 \\ 4-2-1 \end{pmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \end{bmatrix}$$ and is correct.

(3) The eight vectors in the wavelet basis for $\mathbb{R}^8$ are given by

$$W = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(4) From the factorization $W^{-1} = W_2^{-1} W_1^{-1}$, we ~~will~~ need compute W_1^{-1} & W_2^{-1}. Since ~~each~~ the columns in W_1 & W_2 are all orthogonal W^{-1} is almost W^T but requires dividing by the magnitudes of the internal dot products i.e.

$$W_1^{-1} = (W_1^T W_1)^{-1} W_1^T$$

$$= \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \end{bmatrix}$$

also $\quad W_2^{-1} = (W_2^T W_2)^{-1} W_2^T$

$$= \left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\quad W_2^{-1} W_1^{-1} \begin{bmatrix} 6 \\ 4 \\ 8 \\ 1 \end{bmatrix} = W_2^{-1} \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 8 \\ 1 \end{bmatrix}$

$$= W_2^{-1} \left(\frac{1}{2}\right) \begin{bmatrix} 10 \\ 6 \\ 2 \\ 4 \end{bmatrix} = W_2^{-1} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

the same as before

(5) From the given H we recognize that its columns are orthogonal

So $H^{-1} = (H^T H)^{-1} H^T$

We 1st compute $H^T H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

So $H^{-1} = \frac{1}{4} H^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

To write the given v in terms of the columns of H we would compute

$H^{-1} v$.

(6) To compute b from c we see that

$b = V^{-1} W c$ so the change of basis matrix

is given by $V^{-1} W$

(7) The original basis vectors when put into the columns of a matrix W, have an inverse matrix W^{-1} that satisfies

$W \cdot W^{-1} = I$. taking the transpose of this eq gives

$(W^{-1})^T W^T = I$ or $W^* W^T = I$ ~~or $W^* (W^*)$~~

Thus the dual of the dual would be the ~~columns~~ rows of matrix W^T i.e. the columns of the matrix W or the w_i's themselves

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(1) For $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ with the SVD given by

$$A = U\Sigma V^T$$

$$A^TA = V\Sigma U^T U \Sigma V^T$$

So the eigenvectors of A^TA are the columns of the matrix V.

$$= V\Sigma^2 V^T$$

$$\& \ Av_i = \sigma_i u_i$$

Now $A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

the eigenvalues of A^TA ~~(the singular values squared)~~ are given by $\lambda_2 = 0$ & $\lambda_1 = 50$. Then the eigenvectors are given by the nullspace of

$$\begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 \\ 1 & -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

so $v_1 \propto \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

& v_2 is given by $v_2 \propto \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ so $v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Then $V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ Now

~~$A v_1 =$~~ The only singular value is given by $\sigma_1 = \sqrt{50}$

(2) (a) Now if $A = V^T \Sigma U$ then $AA^T = U \Sigma^2 U^T$

& $AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}$

which has eigenvalues given by $\lambda_1 = \sigma_1^2 = 50$ & $\lambda_2 = 0$

them with eigenvectors given by (for $\lambda_1 = 50$) the nullspace of

$$\begin{bmatrix} -45 & 15 \\ 15 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/3 \\ 1 & -1/3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

so $u_1 \propto \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ so $u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & we must

have $Av_1 = \sigma_1 u_1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \stackrel{?}{=} \frac{\sqrt{50}}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\frac{1}{\sqrt{5}} \begin{bmatrix} 5 \\ 15 \end{bmatrix} = \sqrt{5} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ yes.

The second eigenvector is given by $u_2 \propto \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

so $u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Again we must have $Av_2 = \cancel{\sigma_2}\, 0 u_2 = 0$

Then the SVD in this case is given by

$$A = U \Sigma V^T$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}^T$$

which we can check by multiplying out the matrices on the right

$$= \frac{\sqrt{50}}{\sqrt{50}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$ yes ✓

③ For the matrix A. The basis for the 4 fundamental spaces are given by $(A = U\Sigma V^T) \Leftrightarrow$ ~~AV=~~ $AV = U\Sigma$

So V must span the row space + the null space while U must span the column space of A + the left nullspace of A.

So

$$V_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ is a basis for the row space

$$V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$ is a basis for the null space of A

$u_1 = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$ is a basis for the column space of A

& $u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\end{bmatrix}$ is a basis for the left nullspace of A.

(4) ~~We can change the signs of the eigenvectors but require that~~ $\cancel{Av_i = \sigma_i u_i}$ ~~so the~~ signs of v_i & u_i must change together

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(4) ~~We can change the sign of~~ ~~require~~ since $A = U\Sigma V^T$

$A^TA = V\Sigma U^TU\Sigma V^T = V\Sigma^2 V^T$ so v_i must be the ~~eigenvector~~ eigenvectors (normalized of A^TA) For this there are two sign of each vector. ~~Since~~ ~~$Av_i = \sigma_i u_i$~~ Now $AA^T = U\Sigma V^T V\Sigma U^T$
$= U\Sigma^2 U^T$ so U must hold the eigenvectors of AA^T. Since $Av_i = \sigma_i u_i$. When $\sigma_i > 0$ v_i & u_i have correlated signs. Thus From A^TA $v_1 = \frac{\pm}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$ & $v_2 = \pm\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$

So we have 4 total choices for all the v's. From ~~A^TA~~ AA^T the eigenvectors are given by $u_1 = \frac{\pm 1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$ & $u_2 = \pm\frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}$

But since $Av_i = \sigma_i u_i$ ~~the~~ ~~signs of~~ ~~v_i~~ u. we have for $i=1$

$$\begin{bmatrix}1&2\\3&6\end{bmatrix}\left(\frac{\pm 1}{\sqrt{5}}\right)\begin{bmatrix}1\\2\end{bmatrix} = \frac{\pm 1}{\sqrt{5}}\begin{bmatrix}5\\15\end{bmatrix} = \pm\sqrt{5}\begin{bmatrix}1\\3\end{bmatrix} = \pm\frac{\sqrt{50}}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$$

with the plus & minus signs must agree

For $i=2$

$$\begin{bmatrix}1&2\\3&6\end{bmatrix}\left(\frac{\pm 1}{\sqrt{5}}\right)\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\cdot\pm\frac{1}{\sqrt{10}}\begin{bmatrix}3\\-1\end{bmatrix}$$

with either sign for the left & ~~eigthe~~ sign for the right vector

~~thus one all possible metrices can be given by one~~

~~taking the plus sign on~~

Thus all the possible metrices can be enumerated with

sign of V_1	sign of V_2	sign of U_1	sign of U_2
+	+	same as V_1	+
+	+		−
+	−		+
+	−		−
−	+		+
−	+		−
−	−		+
−	−		−

Thus we have 8 different metrices that ~~all have the same~~ could be used to decompose A.

5 With $U = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}$; $V = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$

+ $\Sigma = \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}$ a ~~QR~~ QH decomposition of A

can be obtained from the SVD as $A = U\Sigma V^T$

with $A = UV^T(V\Sigma V^T) = Q(V\Sigma V^T)$

so $Q = UV^T = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} =$ ✓

$$= \begin{bmatrix} \frac{1}{\sqrt{50}} + \frac{6}{\sqrt{50}} & \frac{2}{\sqrt{50}} - \frac{3}{\sqrt{50}} \\ \frac{3}{\sqrt{50}} - \frac{2}{\sqrt{50}} & \frac{6}{\sqrt{50}} + \frac{1}{\sqrt{50}} \end{bmatrix} = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}$$

+ $H = V\Sigma V^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\frac{1}{\sqrt{5}}$

$$= \frac{\sqrt{50}}{5}\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \frac{\sqrt{50}}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$= \sqrt{2}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Also H is singular ~~cannot~~ ~~#~~ since $h_{11} = \sqrt{2} > 0$ it is semi definite because 0 is an eigenvalue. To check the decomposition we compute QH with

$$\frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix}\sqrt{2}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & \overset{10}{\cancel{18}} \\ 15 & 30 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

which is A

(6) The pseudoinverse ~~is~~ ~~given~~ ~~to~~ with $A = U\Sigma V^T$ is given by

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$= \left(\frac{1}{\sqrt{50}}\right)^2\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$$

$$= \cancel{\frac{1}{5\sqrt{2}}}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\frac{1}{50}$$

~~Since $A^+ \propto A$ the 4 subspaces to A are the same~~ as ~~to A~~. ~~Also~~

$$\cancel{A^+A = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 7 & 21 \\ 14 & \end{bmatrix}}$$

From the Decomposition $A^+ = V\Sigma^+ U^T$ we have $A^+ U = V\Sigma^+$

So U spans the row space & null space of A^+ while

V spans the column space & left nullspace of A^+. Specifically

$u_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ~~spans~~ is a basis for the row space of A^+

$u_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ ~~spans~~ is a basis for the null space of A^+

$v_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis for the column space of A^+

$v_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a basis for the left nullspace of A^+

Now $A^+A = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

& $AA^+ = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\left(\frac{1}{50}\right)\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 5 & 15 \\ \cancel{21}\ 15 & 45 \end{bmatrix}$

~~9+36=45~~ $$= \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

(7) From $A = \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$ then $A^T A = \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 10 & 8 \\ 8 & 10 \end{bmatrix}$ which has an eigenvalue given by $\lambda_2 = 2$ & the other then by $\lambda_1 = 20 - 2 = 18$.

The eigenvectors v_1 & v_2 are given by for λ

$\begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix}$ or $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

& for $\lambda_2 = 2$ $v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Thus the singular values $\sigma_1 = \sqrt{18}$ + $\sigma_2 = \sqrt{2}$.

(8) For AA^T we have

$$\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$$

Which has ~~eigenvecto~~ eigenvalues by 18 + 2 w/ eigenvectors given by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ + $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Then the SVD becomes

$$A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

(9) From problem 8 we have

$$A = \begin{bmatrix} u_1 & u_2 \end{bmatrix}\begin{bmatrix} \sigma_1 v_1^T \\ \sigma_2 v_2^T \end{bmatrix} = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T$$

$$1\times 2 \cdot 2\times 1 = 1\times 1$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

Which using the vectors found in problems 7 & 8 gives

$$A = \sqrt{18}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= 3\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = A \checkmark$$

The proof that every matrix is the sum of rank 1 matrices is given by the same logic ~~as~~ given earlier

(10) $$Q = UV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\& \; K = U\Sigma U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Then from the SVD of $A = U\Sigma V^T = \cancel{UV^T(V\Sigma V^T)}$

$$= U\Sigma U^T(UV^T)$$

$$= \begin{bmatrix} \sqrt{18} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & -1 \end{bmatrix} = A \checkmark$$

(11) The pseudoinverse of A is the same as the inverse of A because A is invertible.

(12) For $A = [3\ 4\ 0]$ we have

$$A^T A = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} [3\ 4\ 0] = \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which has eigenvalues given by

$$\begin{vmatrix} 9-\lambda & 12 & 0 \\ 12 & 16-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda \begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = -\lambda\left[(9-\lambda)(16-\lambda) - 144\right]$$

But from $\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$ we see that $\lambda = 0$ must be an eigenvalue for this matrix & thus the other eigenvalue is

$\lambda = 9 + 16 = 25$

So our eigenvalues are given by $0, 0, 25$

So the singular value of A is $\sigma_1 = \sqrt{25} = 5$.

From AA^T we have $\begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = 9+16 = 25$

which has the same singular value $= 25$

The eigenvectors of A^TA are given by

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -4 \\ +3 \\ 0 \end{bmatrix}$$ for $\lambda = 0$ & for $\lambda = 25$ we ~~have~~ need to consider

the nullspace of $A - 25I = \begin{bmatrix} -16 & 12 & 0 \\ 12 & -9 & 0 \\ 0 & 0 & -25 \end{bmatrix}$

which is given by $\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$

Normalizing everything we have

$$v_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}; \quad v_2 = \frac{1}{5}\begin{bmatrix} -4 \\ +3 \\ 0 \end{bmatrix}; \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The SVD of A then becomes

$$A = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & +3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

The pseudoinverse of A is given by

$$A^+ = \begin{bmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/5 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \left(\frac{1}{25}\right)$$

Then $$AA^+ = \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} \frac{1}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \frac{1}{25}(25) = 1$$

& $$A^+A = \frac{1}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 9 & 12 & 0 \\ 12 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(14) This would be the 2x3 zero matrix. Then from the SVD

$A = U\Sigma V^T$ U must span the "range" of our operator $\therefore U = I_{2\times2}$

& V must span the domain of A & $\therefore V = I_{3\times3}$. Then

$\underset{2\times3}{A} = (U)_{2\times2}\underset{2\times3}{\Sigma}(V^T)_{3\times3}$ so Σ is all zeros of size 2x3

Now $A^+ = V_{3\times3}\Sigma^+(U^T)_{2\times2}$

w/ Σ^+ zero matrix of size 3x2 thus A^+ is 3x2 where

A was 2x3

(15) If $\det(A) = 0$ then from the SVD of $A = U\Sigma V^T$

one of the diagonal elements of Σ must be zero since

$|A| = |U|\cdot|\Sigma|\cdot|V^T| = |\Sigma|$ since $|U| = 1 = |V^T|$

But from the definition of A^+ we have

$A^+ = V\Sigma^+U^T$ where Σ^+ is a transpose with the non zero

diagonal elements inverted. The zero diagonal elements of Σ do not change

~~Now~~ Now $|A^+| = |\Sigma^+| = 0$ since the zero elements in Σ have

not changed.

(16) When are the factors in $U\Sigma V^T$ the same as $Q\Lambda Q^T$

We must have positive eigenvalues since the nonzero ~~Elts~~ diagonal elements of Σ are positive. Then A must be symmetric + positive definite

(17) (a) $A^TA\hat{x} = A^Tb$ has many solutions since

A^TA is singular

(b) $x^+ = A^+b$ is given by computing

$A^+ = V\Sigma^+U^T$

From A^TA we compute the matrix V

$A^TA = .5\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ so ~~$v_1 = [\ ]$~~ we have eigenvalues of

$$\begin{vmatrix} .5-\lambda & .5 \\ .5 & .5-\lambda \end{vmatrix} = 0 \Rightarrow (.5-\lambda)^2 - .25 = 0$$

$$\Rightarrow \cancel{.25} - \lambda + \lambda^2 - \cancel{.25} = 0$$

$$+ \lambda(\lambda-1) = 0 \Rightarrow \lambda = 0 + \lambda = 1$$

So we have ~~[illegible]~~ v_1 given by the nullspace of

$\begin{bmatrix} -.5 & .5 \\ .5 & -.5 \end{bmatrix}$ or $v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} +1 \\ +1 \end{bmatrix}$ + v_2 is given by

$$v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

~~Then~~ $Av_i = \sigma_i u_i$

$$Av_1 = \begin{bmatrix}.2 & .2\\ .1 & .1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}.4\\.2\end{bmatrix} = (1)\frac{1}{\sqrt{2}}\begin{bmatrix}.4\\.2\end{bmatrix}$$

Thus $u_1 = \frac{2}{10\sqrt{2}}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{5\sqrt{2}}\begin{bmatrix}2\\1\end{bmatrix}$

Is this normalized? $\|u_1\| = \frac{1}{5\sqrt{2}}\sqrt{4+1} \neq 1$ No

~~why is this not correct??~~

From AA^T we can compute the vectors u_i from the eigenvectors of

$AA^T = \begin{bmatrix}.8 & .4\\ .4 & .2\end{bmatrix}$ which gives for $\lambda_1 = 1$ the following matrix

$\begin{bmatrix}-.2 & .4\\ .4 & -.8\end{bmatrix}$ which has $u_1 \propto \begin{bmatrix}2\\1\end{bmatrix}$ as the eigenvector

normalizing we have $u_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}$

For $\lambda_2 = 0$ we have $u_2 \propto \begin{bmatrix}1\\-2\end{bmatrix}$ so $u_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\-2\end{bmatrix}$

Now from the problem as given ~~then~~ A + A^T don't make sense. A consistent problem will be come from assuming that

$$A = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad + \quad A^T = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{then} \quad AA^T + A^TA$$

are as given. We can recompute A from its SVD as

$$A = U\Sigma V^T = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2/\sqrt{5} & 0 \\ 1/\sqrt{5} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \qquad \text{yes}$$

Then $A^+ = V\Sigma^+U^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}^T$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Then $x^+ = A^+ b = \frac{1}{\sqrt{10}}(2b_1 + b_2, 2b_1 + b_2)$.

Does this solve $A^TA x^+ = A^T b$?

We can check this equation is

$$\frac{1}{10}\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x_1^+ \\ x_2^+ \end{bmatrix} \stackrel{?}{=} \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Putting in x^+ given by the above we get

$$\frac{1}{10}\frac{1}{\sqrt{10}}\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \stackrel{?}{=}$$

$$= \frac{5}{10\sqrt{10}}\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \stackrel{?}{=}$$

$$= \frac{1}{2\sqrt{10}}\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$ yes the same!! so this x^+ does solve

$$A^TA\hat{x} = A^Tb$$

(c) Now $AA^+ = U\Sigma V^T(V\Sigma^+U^T) = U\Sigma\Sigma^+U^T$

+ since from $A = U\Sigma V^T \Rightarrow AV = U\Sigma$ we see that the vectors in U span the column space of A + the left nullspace of A.

Thus $U\Sigma\Sigma^+U^T$ will be a vector in the column space of A since Σ + Σ^+ will kill any vector in the left nullspace of A

therefore AA^+ projects onto the columns ~~s~~ space of A.

Therefore $I - AA^+$ projects onto the nullspace of A^T, or in otherwords, $A^T(I-AA^+)b = 0$ which is the same as

$$A^T b = A^T A A^+ b$$

defining $x^+ = A^+ b$ we see that the above is equivalent to

$$A^T A x^+ = A^T b$$ ~~or the least squares solution~~

which implies that $\hat{x}$ the solution to $A^T A \hat{x} = A^T b$ can be

(18) given by $x^+ = A^+ b$.

(13) ~~$\|\hat{x}\|^2 = x^T \hat{x}$~~ ~~If x^+ is perpendicular to $\hat{x} - x^+$~~

Equalities like this can be shown if it can be shown that x^+ & $\hat{x} - x^+$ are perpendicular which can sometimes be obtained from the 2nd fundamental theorem of linear algebra. (the theorems dealing with the orthogonality of the 4 fundamental spaces). Now $x^+ = A^+ b$ ~~is~~ & $A^+ = V \Sigma^+ U^T$ so $A^+ b$ is an element that spans the 1st r ~~nonzero~~ columns of V that correspond to the nonzero singular values & $\therefore$ x^+ is an element of the row space (the space spanned by these 1st r columns) of A.

The vector $\hat{x} - x^+$ is in the nullspace of A^TA equivalently the nullspace of A. Since the nullspace of A & the row space of A are orthogonal the given equality holds by ~~pythag~~ pythagoras right triangle identity.

(19) $AA^+p = \cancel{AA^+}\ \cancel{A}p$ since p is in the column space & AA^+ projects into that ~~column~~ space.

$AA^+e = 0$ since e is orthogonal to the column space

$A^+Ax_r = \cancel{A^+(Ax_r)}\ x_r$ since A brings x_r into the column space & A^+ on an element in the column space is that element back again

$A^+Ax_n = A^+ \cdot 0 = 0.$

(20) If the SVD is as given for

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$$

Then $A^+ = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \end{bmatrix} \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix}^T$

$$= \frac{1}{5}[1 \;\; 0]\begin{bmatrix} .6 & .8 \\ -.8 & .6 \end{bmatrix} = \frac{1}{5}[.6 \;\; .8] = \frac{1}{50}[6 \;\; 8]$$

$$= \frac{1}{25}[3 \;\; 4]$$

Then $$A^{+}A = \frac{1}{25}[3 \;\; 4]\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{25}(9+16) = 1$$

& $$AA^{+} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}\frac{1}{25}(3 \;\; 4)$$

$$= \frac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

(21) In the ~~fa~~ LU factorization L has 1 unknown the L_{21} element & U has 3 unknowns the elements of U_{11}, U_{12}, & U_{22} giving a total of 4 unknowns.

(22) In the LDU factorization, again L has 1 unknown the L_{21} element, D has 2 unknowns D_{11} & D_{22} & U then has one unknown the element of U_{21}.

In the QR decomposition Q is an orthogonal matrix & ∴ has only one unknown while R has R_{11}, R_{12}, & R_{22} unknowns to determine it.

in the $U\Sigma V^T$ decomposition U + V are orthogonal + ∴ require only one number to specify them, while Σ requires two #'s Σ_{11} + Σ_{22}.

The decomposition $S\Lambda S^{-1}$ requires 2 #'s to specify Λ i.e. Λ_{11} + Λ_{22} + S requires 1 # ~~each~~ for the specification of an eigen direction i.e. the direction of the eigenvector. In all cases the total unknown count is 4.

(22) For LDL^T L requires the specification of 1 # L_{12} + D the specification of two #'s D_{11} + D_{22}

For QAQ^T Q requires the specification of one # (the rotation angle) + Λ requires the specification of 2 #'s Λ_{11} + Λ_{22}. This is correct because in these cases A is symmetric

(23) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = E \cdot H$ w/ E lower unit triangular + H symmetric

so $A = \begin{bmatrix} 1 & 0 \\ e & 1 \end{bmatrix}\begin{bmatrix} g & h \\ h & k \end{bmatrix} = \begin{bmatrix} g & h \\ ge+h & eh+k \end{bmatrix} \underset{\substack{\uparrow \\ \text{set}}}{=} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Thus in our factorization

$g = a$ $h = b$

& $a \cdot e + b = c \Rightarrow e = \frac{c-b}{a}$

& $\left[\frac{c-b}{a}\right] \cdot b + k = d \Rightarrow k = d - \frac{(c-b)b}{a}$

Thus our factorization is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c-b}{a} & 0 \end{bmatrix} \begin{bmatrix} a & b \\ b & d - \frac{(c-b)}{a} \cdot b \end{bmatrix}$$

This will obviously not be possible if $a = 0$.

In general ~~...~~ for $m \times m$ matrices A the factorization

$A = LDU^T = (LU^{-T})(U^T D U)$ gives a triangular

matrix times a symmetric matrix

(24) Since we are only given ~~the~~ basis for the row space (the v's) & the column space (the U's) we can construct an ~~invertible~~ square matrix

$A_{r \times r} = U \Sigma V^T$ by selecting r ~~...~~ positive nonzero σ's

for ~~...~~ the diagonal of Σ. We can create a non invertible but square A

by allowing some of the diagonal (the latter ones to be zero)

In addition we can create non square matrices A by "growing" the basis $v_1, \ldots, v_r$ and or $u_1, \ldots, u_r$ to include n + m elements respectively (where $m > r$ + $n > r$). This ~~is technically not possible since~~ assumes that the v vectors are inside $\mathbb{R}^n$ + the u vectors are inside $\mathbb{R}^m$. When we complete the basis for $\mathbb{R}^n$ + $\mathbb{R}^m$ using the r-provided vectors $v_1, \ldots, v_r$ + $u_1, \ldots, u_r$ we obtain $n-r$ + $m-r$ extra vectors + then can reassemble A via

$$A = U \Sigma V^T.$$

(25) ~~If~~ let $\cancel{B = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}}$ ~~then~~

$$\cancel{B \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} Av \\ A^T \end{bmatrix}} \qquad \cancel{B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}}$$ then

$$\cancel{B \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} Au \\ A^T v \end{bmatrix}} = \Big[$$

$$B = \begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} \text{ then } B \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} Av \\ A^T u \end{bmatrix} = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix}$$

$$= \sigma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}$$

So $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} B \begin{bmatrix} V \\ U \end{bmatrix} = \sigma \begin{bmatrix} V \\ U \end{bmatrix}$

Now $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}^{-1} = -1 \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

Check $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ ✓

So our matrix is given by

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix} = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$$

For this matrix

$$\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} V \\ U \end{bmatrix} = \begin{bmatrix} A^T U \\ AV \end{bmatrix} = \begin{bmatrix} \sigma V \\ \sigma U \end{bmatrix} = \sigma \begin{bmatrix} V \\ U \end{bmatrix}$$

So the symmetric matrix is $\begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}$ + the eigenvalue is σ.