pg 366 Strang

(1)

$$A = \begin{array}{c|ccc|c} & \multicolumn{3}{c|}{\text{node}} & \\ & 1 & 2 & 3 & \\ \hline & -1 & +1 & 0 & 1 \\ & -1 & 0 & 1 & 2 \\ & 0 & -1 & 1 & 3 \end{array} \quad \text{edge}$$

The vectors in its Null space are (c,c,c) for any constant c

It (1,0,0) ~~is not~~ ~~that form~~ were in its row space it would have to be orthogonal to the Null space or to the vector $(1,1,1)$ since $(1,0,0)\circ(1,1,1) = 1 \neq 0$

$(1,0,0)$ cannot be in the null space.

(2) $$A^T = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The vector $(1,-1,1)$ is in the null space of A^T.

In ~~total~~ total ~~1−1+1=1 current is going around the network~~

1 unit of current is going along edge 1, −1 unit of current is going along edge 2 + +1 unit of current is going along edge 3

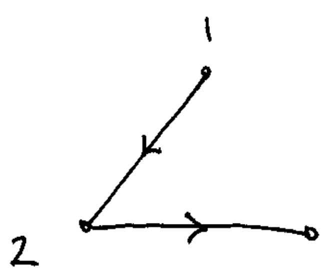
so in summary one unit of current is going around the network

(3) These equations correspond to the incidence matrix for the triangle graph used in problems # 1 + #2. Our augmented matrix is given by (+ row reduction procedures)

$$\begin{bmatrix} -1 & 1 & 0 & b_1 \\ -1 & 0 & 1 & b_2 \\ 0 & -1 & 1 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & -1 & 1 & b_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & b_1 \\ 0 & -1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 + b_1 - b_2 \end{bmatrix}$$

The tree that corresponds to the non zero elements of $\bar{U}$ is shown below



(4) For $Ax=b$ to be solved one must have

$$b_1 - b_2 + b_3 = 0 \quad \text{or} \quad (1,-1,1)\cdot(b_1,b_2,b_3) = 0$$

Pick $\bar{b} = (1,2,1)$ then $Ax=b$ can be solved.

For a $\bar{b}$ that can't be solved pick any b that does not satisfy this condition, ie pick $\bar{b} = (1,2,0)$

The b's that can be solved are orthogonal to (1,−1,1).

(5) The augmented matrix corresponding to $A^T y = f$ will be solvable ~~iff f is orthogonal~~ to ~~the Nullspace of A.~~ Since ~~this was~~ calculated it

$$\begin{bmatrix} -1 & -1 & 0 & f_1 \\ 1 & 0 & -1 & f_2 \\ 0 & 1 & 1 & f_3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 0 & f_1 \\ 0 & -1 & -1 & f_1+f_2 \\ 0 & 1 & 1 & f_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 0 & f_1 \\ 0 & -1 & -1 & f_1+f_2 \\ 0 & 0 & 0 & f_1+f_2+f_3 \end{bmatrix}$$

So $\underline{f}$ must satisfy $f_1+f_2+f_3=0$ or vectors orthogonal to (1,1,1). The equation $A^T y = f$ is ~~Kirchhoff's~~ Kirchhoff's current law

(6) $A^T A$ is given by $\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

~~b~~ Vectors t for which $A^TAx = t$ can be solved must be ~~is the~~ orthogonal to the left nullspace of A^TA. Since this matrix is symmetric the left nullspace of A^TA is equivalent to the nullspace of A^TA which must contain ~~is equivalent~~ ~~in this case~~ to the nullspace of A which we found to be $(1,1,1)$.

Thus any t orthogonal to $(1,1,1)$ will work, pick

$t = \cancel{(1,-1,0)}$ ~~(...)~~ correction then x is given by

$(1,-1,0)$

$\cancel{y = Ax} =$

manipulating the augmented matrix

$$\begin{bmatrix} 2 & -1 & -1 & 1 \\ -1 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & -1 & 1 \\ 0 & 3/2 & -3/2 & -1/2 \\ 0 & -3/2 & 3/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & -2 & \frac{2}{3} \\ 0 & 1 & -1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

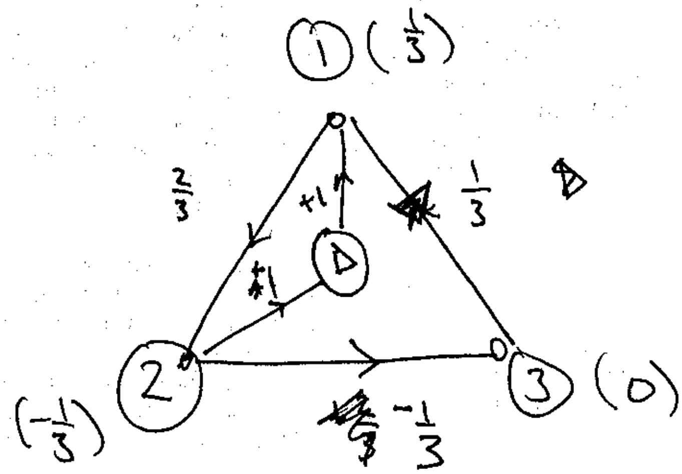
Specify ~~now~~ now x_3 to be grounded i.e to have value 0

$x_3 = 0$ we obtain $x_1 = \frac{1}{3}$ + $x_2 = \frac{-1}{3}$. Then the currents are given by

$$y = -Ax = -\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

$$= -\begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ +\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Placing these values on the graph we have



Note the potentials given $\underline{x}$ can be offset by any constant vector (c,c,c) + the ~~ol~~ current solution will not change

(7) with conductences as given $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

& A^TCA is given by

$$\begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} 1+2 & -1 & -2 \\ -1 & 1+2 & -2 \\ -2 & -2 & 4 \end{bmatrix} \checkmark = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix} \checkmark$$

To find a solution to

$A^TCA\,x = f$ form the augmented matrix

$$\begin{bmatrix} 3 & -1 & -2 & 1 \\ -1 & 3 & -2 & 0 \\ -2 & -2 & 4 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -1 & -2 & 1 \\ 0 & 3-\frac{1}{3} & -2-\frac{2}{3} & \frac{1}{3} \\ 0 & -2-\frac{2}{3} & 4-\frac{4}{3} & -1+\frac{2}{3} \end{bmatrix} \checkmark$$

$$= \begin{bmatrix} 3 & -1 & -2 & 1 \\ 0 & \frac{8}{3} & -\frac{8}{3} & \frac{1}{3} \\ 0 & -\frac{8}{3} & \frac{8}{3} & -\frac{1}{3} \end{bmatrix} \checkmark \Rightarrow$$

$$\begin{bmatrix} 3 & -1 & -2 & 1 \\ 0 & 1 & -1 & 1/8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & -3 & 1+\frac{1}{8} \\ 0 & 1 & -1 & 1/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 0 & -3 & 9/8 \\ 0 & 1 & -1 & 1/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So grounding Node x_3 at Zero we have:

$$x_1 = \frac{3}{8} \quad \& \quad x_2 = \frac{1}{8}$$

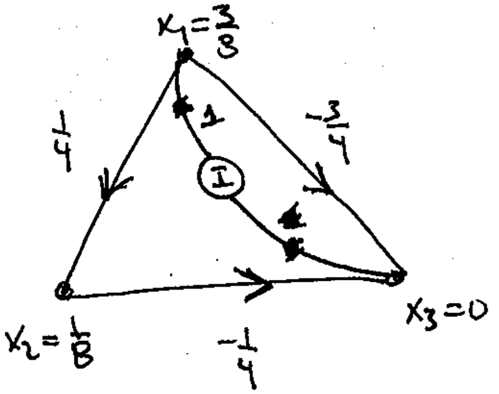
Thus $\underline{X} = \begin{bmatrix} 3/8 \\ 1/8 \\ 0 \end{bmatrix}$ which is

The currents are given by

$$y = -CAx = -\begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 3/8 \\ 1/8 \\ 0 \end{bmatrix} = -\begin{bmatrix} -3/8 + 1/8 \\ -6/8 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 2/8 \\ -3/4 \\ -1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 \\ -3/4 \\ -1/4 \end{bmatrix}$$

On the triangle graph we have



(8)

$$A = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{array}$$

which has a null space given by ~~(c, c, c, c)~~ the span of (1,1,1,1). From the above we have that A^T is given by

$$A^T = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which has rank 3 + $\therefore$ has a nullspace given by $5-3=2$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus the free variables are given by x_3 + x_5, while the pivot variables are given by $x_1, x_2,$ + x_4. To determine the null space assign the free variables to ones + solve for the pivot variables.

let $x_3 = 1$ + $x_5 = 0$ we have

$$\underline{x}_{n_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

let $x_3 = 0$ + $x_5 = 1$ we have

$$\underline{x}_{n_2} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(9) To solve $Ax = b$ we must have b orthogonal to the left nullspace of A. The left nullspace of A is equivalent to the nullspace of A^T thus we must have

$$x_{n_1} \circ (b_1, b_2, b_3, b_4, b_5) = 0$$

$$\rightarrow \quad b_1 - b_2 + b_3 = 0$$

$$+ \quad x_{n_2} \circ (b_1, b_2, b_3, b_4, b_5) = 0$$

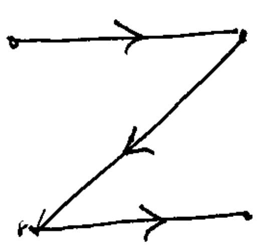
$$\Rightarrow \quad -b_1 - b_2 - b_4 + b_5 = 0$$

Which are Kirchhoff's voltage laws + around two loops in the graph

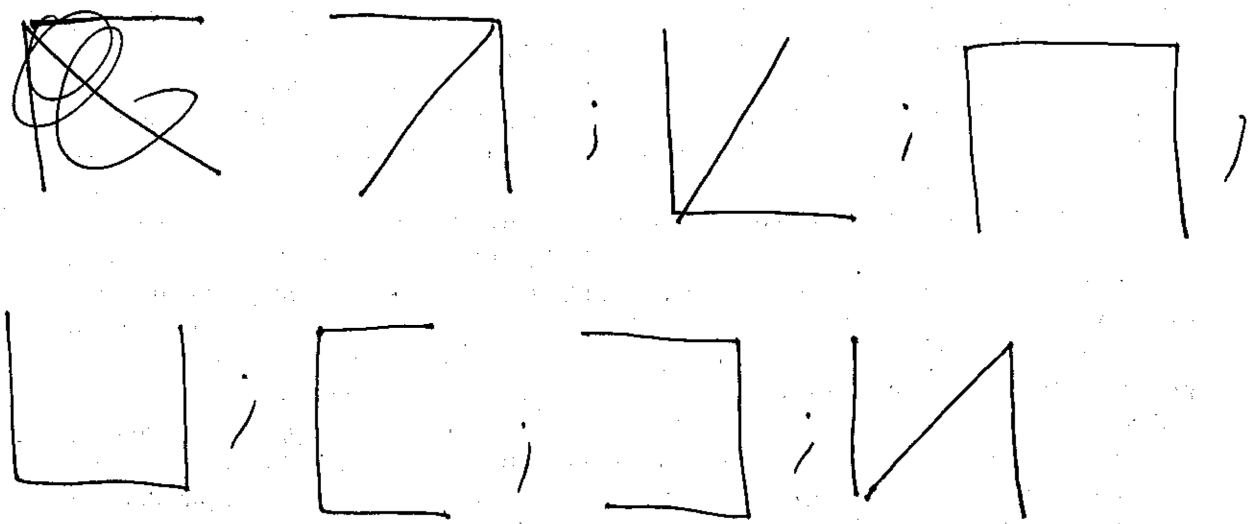
(10) $A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The 3 non zero rows are the following graph.



The other seven graphs are obtained by ~~[illegible]~~ selecting 3 of the 5 edges



Is there a systematic way of deriving these results?

(11) $A^T A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$

$= \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$

(a) The diagonal tells how many edges flow into each node

(b) The off-diagonals -1 or 0 tell which nodes are connected

(12)

(a) True since (1,1,1,1) is in the nullspace of A it is also in the nullspace of A^TA. The rank of A^TA must be less than or equal to that of A equivalently A^T. The dimension ~~rank~~ of A is m = # of edges by n = # of nodes & has rank n-1. Thus the rank of A^TA is ~~less~~. ~~less than or~~ equal to n-1.

~~= n-1~~

(b) True since (1,1,1,1) is in the nullspace

If $y = (1,1,1,1)$ Then

$$y^TA^TAy = 0 \quad \text{but} \quad y \neq 0$$

But $\forall$ other y we have $y^TA^TAy = (Ay)^T(Ay) \geq 0 \ \forall$

since $x^Tx \geq 0 \quad \forall \ x \neq 0$

(c) A^TA is symmetric so its eigenvalues are real & must be ~~positive~~ greater than or equal to zero since A^TA is positive semi definite

(13) $$A^T C A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix}$$

Then forming the augmented matrix A^TCA w/ f we have

$$\begin{bmatrix} 4 & -2 & -2 & 0 & 1 \\ -2 & 8 & -3 & -3 & 0 \\ -2 & -3 & 8 & -3 & 0 \\ 0 & -3 & -3 & 6 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 & -2 & 0 & 1 \\ 0 & 4 & -4 & -3 & 1/2 \\ 0 & -4 & 7 & -3 & 1/2 \\ 0 & -3 & -3 & 6 & -1 \end{bmatrix}$$

~~we~~ ~~have~~ ~~setting~~ $3/4$

$$\Rightarrow \begin{bmatrix} 4 & -2 & -2 & 0 & 1 \\ 0 & 4 & -4 & -3 & 1/2 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & -3-3 & 6-3(+\frac{3}{4}) & -1+\frac{3}{4}(\frac{1}{2}) \end{bmatrix}$$

Using the ~~[illegible]~~ code rref from matlab ~~rref~~ ~~we have~~ ~~SEW~~ see prob_8_1_13.m we have

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 5/12 \\ 0 & 1 & 0 & -1 & 1/6 \\ 0 & 0 & 1 & -1 & 1/6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So our potentials $\underline{x}$ then become (grounding Node x_4)

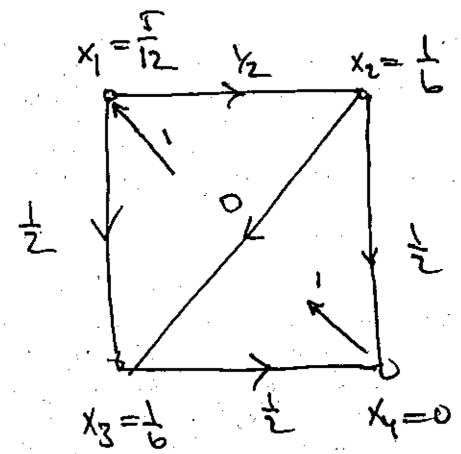
$$\underline{x} = \begin{bmatrix} 5/12 \\ 1/6 \\ 1/6 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Our currents are given by

$$y = -CAx = - \begin{bmatrix} -2 & 2 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 5/12 \\ 1/6 \\ 1/6 \\ 0 \end{bmatrix}$$

$$= - \begin{bmatrix} -\frac{5}{6} + \frac{2}{6} \\ -\frac{5}{6} + \frac{2}{6} \\ -\frac{3}{6} + \frac{3}{6} \\ -\frac{3}{6} \\ -\frac{3}{6} \end{bmatrix} = - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

Drawn at the square graph we have the following



(14) The vector x such that $Ax = 0$ is in the nullspace this is a vector of all ones ~~with~~ To have a solution to

$A^T C A x = f$, f must be orthogonal to the left

Null space of $A^T C A$. this space since $A^T C A$ is symmetric

is the same as the nullspace of A & is the span of the

vector $(1,1,1,1)$ thus $f \circ (1,1,1,1) = 0$ which implies

$f_1 + f_2 + f_3 + f_4 = 0$

(15) ~~A~~ Eulers formula yields

$$7 - 7 + \text{number of loops} = 1$$

$\Rightarrow$ number of loops $= 1$

(16) ~~B~~ Eulers formula is number of nodes $-$ number of edges $+$ number of small loops $= 1$

Which in the 1st case becomes

$$5 - 7 + 3 = 1$$

In the second case we have

$$5 - 8 + 4 = 1$$

(17) (a) ~~The matrix A is~~ A has dimensions $m = \#$ of edges $+$ $n = \#$ of nodes. In this ~~problem~~ problem $m = 12$ + $n = 9$ so The rank is $r = n - 1 = 8 =$ the $\#$ of independent columns

(b) To solve $A^T y = f$, f must be orthogonal to the left null space of A^T or the nullspace of A which includes the vector $x_n =$ ~~(1,1,1,1)~~ $(1,1,1,1,1,1,1,1,1)$ so

$$f \circ x = 0$$

(c) The sum of the diagonal entries is ~~the~~ ~~a~~ ~~count~~ of 2 times the # of edges in the graph

$= 2(12) = 24$

(18) With 6 nodes we must have $\frac{n(n-1)}{2}$ edges or $\binom{n}{2}$ edges

$= \frac{6(5)}{2} = 15.$

A tree ~~with~~ ~~have~~ with 6 nodes will have $6-1=5$ edges

① If $A = \begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix}$ then ~~the eigenvalues are given~~ by

$$\cancel{|A - \lambda I| = \begin{vmatrix} .9-\lambda & .15 \\ .1 & .85-\lambda \end{vmatrix}} = (.9-\lambda)(.85-\lambda) - .015$$

$$= \cancel{.765 - 1.75\lambda + \lambda^2 - .015}$$

$$= \cancel{\lambda^2 - 1.75\lambda + .75}$$

~~Which has~~ since A is a ~~___~~ "Markov matrix" it has $\lambda_1 = 1$ as an eigenvalue. The sum of the eigenvalues must equal the trace of our matrix which is 1.75 so

$\begin{array}{r} .9 \\ .85 \\ \hline 1.75 \end{array}$

$$1 + \lambda_2 = 1.75 \quad \& \text{ therefore } \lambda_2 = .75.$$

The steady state solution is given by the eigenvector for the eigenvalue with $\lambda = 1$. This eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} .9-1 & .15 \\ .1 & .85-1 \end{bmatrix} = \begin{bmatrix} -.1 & .15 \\ .1 & -.15 \end{bmatrix}$$

$\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

which is given by $\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$ thus the steady state eigenvector is given by $\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}\alpha$, where α is a constant, ensuring that this vector sum to 1 i.e. $\alpha = 2/5$ so $v = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$

(2) From problem # 1 the steady state eigenvector (the eigenvector associated with $\lambda=1$) is given by

$x_1 = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. The eigenvector associated with $\lambda = .75$ is given by the nullspace to the following matrix

$$\begin{bmatrix} .9-.75 & .15 \\ .1 & .85-.75 \end{bmatrix} = \begin{bmatrix} .15 & .15 \\ .1 & .1 \end{bmatrix}$$

which is given by the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ thus our matrix of eigenvectors is given by $S = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}$

So $$S^{-1} = \frac{1}{\frac{3}{2}+1}\begin{bmatrix} 1 & +1 \\ -1 & 3/2 \end{bmatrix} = \frac{1}{\frac{5}{2}}\begin{bmatrix} 1 & 1 \\ -1 & 3/2 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$$

So $A = S \Lambda S^{-1}$ or

$$A = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .75 \end{bmatrix}\begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

So $$A^k = \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (.75)^k \end{bmatrix}\begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

Then $A^{\infty} \to \begin{bmatrix} 3/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$

$$= \begin{bmatrix} 3/2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2/5 & 2/5 \\ -2/5 & 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 6/10 & 4/10 \\ 2/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 3/5 & 3/5 \\ 2/5 & 2/5 \end{bmatrix}$$

(3) For $A = \begin{bmatrix} 1 & .2 \\ 0 & .8 \end{bmatrix}$ we have a markov matrix & ∴ one eigenvalue will be equal to 1. The corresponding

A

eigenvector ~~will be~~ also known as the steady state vector is given by the null space to

$\begin{bmatrix} 0 & .2 \\ 0 & -.2 \end{bmatrix}$ or the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Go to $[A]$

For $A = \begin{bmatrix} .2 & 1 \\ .8 & 0 \end{bmatrix}$ we again have a Markov matrix and ∴ an eigenvalue given by $\lambda = 1$. The corresponding eigenvector

B

is known as the steady state vector & is given by the ~~null~~ null space of the following ~~vector~~ matrix

$\begin{bmatrix} -.8 & 1 \\ .8 & -1 \end{bmatrix}$ or the span of $\begin{bmatrix} 1 \\ .8 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 10 \\ 8 \end{bmatrix} \propto \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

go to $[B]$ which if we require sum to 1

For $A = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$ we are again looking for the eigenvector

For the $\lambda = 1$ eigenvalue given by the nullspace of

$$\boxed{U} \quad \begin{bmatrix} -1/2 & 1/4 & 1/4 \\ 1/4 & -1/2 & 1/4 \\ 1/4 & 1/4 & -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$ which has as nullspace

given by the span of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or normalizing by $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ go to $\boxed{C}$

$\boxed{A}$ the 2nd eigenvalue is given by the trace identity in matrices i.e. $1 + \lambda_2 = 1.8 \Rightarrow \lambda_2 = .8$

$\boxed{B}$ The 2nd eigenvalue λ_2 is given by the trace identity in matrices i.e. $1 + \lambda_2 = .2 \Rightarrow \lambda_2 = -.8$

If we require that the steady state sum to 1 we have that

$$V_0 = \begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$$

[c] One other eigenvalue ~~can~~ ~~be~~ of A can be seen to be $\lambda = 1/4$, for then all rows are the same.

A quick way to determine the last & final eigenvalue can be determined by the eigenvalue trace identity, i.e.

$$\cancel{1} + \frac{1}{4} + \lambda_3 = \cancel{\frac{1}{2}} + \frac{1}{2} + \frac{1}{2} \Rightarrow \lambda_3 = \frac{1}{4}.$$

(4) For every 4x4 Markov matrix the eigenvector ~~that~~ of A^T that has $\lambda = 1$ as its eigenvalue is given by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(5) The matrix given ~~has~~ is a Markov matrix & ∴ has 1 as an eigenvalue. ~~By~~ the steady state vector will be the eigenvector of eigenvalue 1. Note 1 we expect the steady state vector to be $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ~~since~~ ~~we~~ ~~are~~ all dead people

Note 2: The presence of zeros in the Markov matrix make it possible for a second eigenvector to be zero & could yield an osscilatory solution (rather than a steady state). Ignoring this for now lets find the $\lambda = 1$ eigenvector, ~~the~~ which requires looking at the null space of

$$\begin{bmatrix} -.02 & 0 & 0 \\ .02 & .03 & 0 \\ 0 & .03 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & .03 & 0 \\ 0 & .03 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ~~which has a~~ the nullspace

of which has a span of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ confirming our expected final

state.

(6) Since $Ax = \lambda x$ Adding together the components of each side ~~we have~~ $\cancel{s}=$(letting s is the sum of the components of Ax

and or x) we have

$$s = \lambda s \Rightarrow (1-\lambda)s = 0 \quad \text{if } \lambda \neq 1 \Rightarrow s = 0$$

(7) If $A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$ we ~~have eigenvectors given by~~ recognize this as a Markov matrix

$$\cancel{\begin{vmatrix} .8-\lambda & .3 \\ .2 & .7-\lambda \end{vmatrix} = 0 \quad \text{or} \quad (.56 - 1.5\lambda + \lambda^2) - .06 = 0}$$

$\cancel{\Rightarrow \lambda^2 - 1.5}$ and know that it must have eigenvalue 1.

the other eigenvalue λ_2 is given by the eigenvalue trace identity

i.e. $1 + \lambda_2 = .7 + .8 = 1.5 \Rightarrow \lambda_2 = .5$

The eigenvector for $\lambda = \cancel{.5}\ 1$ is given by

$\begin{bmatrix} -.2 & .3 \\ .2 & .3 \end{bmatrix}$ or the span of $\begin{bmatrix} .3 \\ .2 \end{bmatrix}$ to ensure this

adds to 1 we have a steady state vector given by

$\begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$. The eigenvector for $\lambda_2 = .5$ is given by

the nullspace to

$\begin{bmatrix} .3 & .3 \\ .2 & .2 \end{bmatrix}$ & is spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Thus to compute $A = S\Lambda S^{-1}$ define $S = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

Then $S^{-1} = \frac{1}{-3-2}\begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$

Using the eigenvector factorization we have

$$A = S\Lambda S^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}\begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$$

so $$A^{16} = S\Lambda^{16} S^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (.5)^{16} \end{bmatrix}\begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & \frac{1}{2^{16}} \\ 2 & -\frac{1}{2^{16}} \end{bmatrix}\begin{bmatrix} 1/5 & 1/5 \\ 2/5 & -3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 + \frac{2}{5}\frac{1}{2^{16}} & \frac{3}{5} - \frac{3}{5}\frac{1}{2^{16}} \\ \frac{2}{5} - \frac{2}{5}\frac{1}{2^{16}} & \frac{2}{5} + \frac{3}{5}\frac{1}{2^{16}} \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 & \frac{3}{5} \\ \frac{2}{5} & \frac{2}{5} \end{bmatrix} + \frac{1}{2^{16}} \begin{bmatrix} 2/5 & -3/5 \\ -2/5 & 3/5 \end{bmatrix}$$

(8) From problem 7 we see that $A^k = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix} + \left(\frac{1}{2}\right)^k \begin{bmatrix} & \end{bmatrix}$

Thus since $\left(\frac{1}{2}\right)^k \to 0$ we have

$$A^{\infty} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$$

All Markov matrices with $\begin{bmatrix} .6 \\ .4 \end{bmatrix}$ as the eigenvector with eigenvalue equal to 1 will produce this steady state thus

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = 1 \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} .6a + .4b &= .6 \\ .6c + .4d &= .4 \end{aligned}$$

But we must also have ~~the~~ each column sum to 1 or $\quad a + c = 1$

$b + d = 1$ Thus we have 4 eqs & 4 unknowns for our matrix A. They are

$$a + c = 1$$
$$b + d = 1$$
$$.6a + .4b = .6$$
$$.6c + .4d = .4$$

which give the system $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & .4 & -.6 & 0 \\ 0 & 0 & .6 & .4 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ .6 & .4 & 0 & 0 \\ 0 & 0 & .6 & .4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ .6 \\ .4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & .4 & -.6 & 0 \\ 0 & 0 & .6 & .4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ .4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -.6 & -.4 \\ 0 & 0 & .6 & .4 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -.4 \\ .4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 1 & 2/3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2/3 \\ 0 \end{bmatrix}$$ ~~with~~

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 2/3 \\ 0 \end{bmatrix}$$ which has rank $= 3$ giving a one

parameter family of solutions, specifically we have letting d be arbitrary that

$$c = \frac{2}{3} - \frac{2}{3}d$$
$$b = 1 - d$$
$$a = \frac{1}{3} + \frac{2}{3}d$$

& thus all Markov matrices with $(.6, .4)$ steady state are given by

$$A = \begin{bmatrix} 1/3 + \frac{2}{3}d & 1-d \\ \frac{2}{3} - \frac{2}{3}d & d \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix} + d \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

The example in the book is chosen with $d = .7$

We also require that each component of our Markov matrix be greater than ~~1~~ 0 & less than 1. This gives the following constraints on d

$$0 \le \frac{1}{3} + \frac{2}{3}d \le 1 \Rightarrow 0 \le 1 + 2d \le 3 \Rightarrow -1 \le 2d \le 2$$

$$\Rightarrow -\frac{1}{2} \le d \le 1$$

$$0 \le \frac{2}{3} - \frac{2}{3}d \le 1 \Rightarrow 0 \le 1 - d \le 3/2 \Rightarrow -1 \le -d \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le d \le 1$$

& $0 \le 1 - d \le 1 \Rightarrow -1 \le -d \le 0 \Rightarrow 0 \le d \le 1$

& $0 \le d \le 1$

Thus our solution with bounds on d is given by

$$A(d) = \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix} + d \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{2}{3} & 1 \end{bmatrix} \qquad 0 \le d \le 1$$

(9) If $v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ we have $v_1 = P v_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

& $v_2 = P v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ & $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ & $v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = v_0$

Note that this matrix has 0's in its elements & ∴ can have multiple eigenvalues such that $|\lambda| = 1$

The eigenvectors of P are given by

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \; -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$\Rightarrow \; -\lambda(-\lambda)^3 - 1(1)\begin{vmatrix} 1 & 0 \\ -\lambda & 1 \end{vmatrix} \Rightarrow \; \lambda^4 - 1 = 0$$ ~~thus~~

which has 4 solutions given by the 4 - 4th roots of unity ie. $\lambda^4 = e^{2\pi k i}$ $\quad k = 0, 1, 2, 3$

$$\Rightarrow \; \lambda = e^{\frac{\pi i k}{2}} \qquad k = 0, 1, 2, 3$$

So $\lambda_0 = 1$, $\lambda_1 = e^{\frac{\pi i}{2}} = i$, $\lambda_2 = e^{\pi i} = -1$, $\lambda_3 = e^{\frac{3\pi i}{2}} = -i$

(10) To be Markov one must have every entry nonnegative & every column adding to one. Since A^2 is A acting on the columns of A, which sum to 1 individually A^2 will be have nonnegative entries & each column will sum to 1

(11) If $A =$ is a Markov matrix then its eigenvalues are

(11) 1 & $a+d-1$. By the trace identity (the eigenvalues must ~~The steady state eigenvector is given~~

~~$ad-bc = \lambda_1\lambda_2 = \lambda_2$~~

add to the trace of the matrix. Similarly by the eigenvalue determinant identity the product of the eigenvalues must equal the

~~The steady state eigenvector is the eigenvector corresponding to $\lambda = 1$ which for this 2x2 problem would be given by the nullspace to~~

determinant which is given by $ad-bc = \lambda_1 \cdot \lambda_2 = 1 \cdot \lambda_2$

so the 2nd eigenvalue is given by $\lambda_2 = ad-bc$

& $\lambda_2 = a+d-1$.

The steady state eigenvector is the eigenvector corresponding to $\lambda = 1$ which for this 2x2 problem would be given by the nullspace to

$$\begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \text{ or } \underline{x} = \begin{bmatrix} -b \\ a-1 \end{bmatrix}$$

taking the 1st row (we know the 2nd row must be a multiple of this one & $\therefore$ will be satisfied by this vector also). Thus we have, when we normalize the steady state eigenvector to a probability that our steady state is given by

$$\underline{X} = \begin{bmatrix} \dfrac{-b}{a-b-1} \\[2ex] \dfrac{a-1}{a-b-1} \end{bmatrix} \Leftarrow$$

(12) To be a Markov matrix the elements must be nonnegative & each column must sum to 1. This latter requirement gives that A is given by

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ .2 & .3 & .5 \end{bmatrix}$$

When A is a symmetric Markov matrix since the columns of A sum to 1 the rows of A^T sum to 1. This means that the vector $(1, \dots, 1)$ is an eigenvector since multiplying by it is equivalent to summing across the rows. Since the rows of A^T sum to one this product is $\lambda(1,1,\dots,1)$ again. Because A is symmetric $A^T = A$ & $\therefore$ $(1,1,\dots,1)$ is an eigenvector for A.

(13) ~~Since the rows/columns of B are linearly dependent $\lambda = 0$ is an eigenvalue. The other eigenvalue can be obtained by the eigenvalue trace theorem or~~

~~$-.2 - .3 = 0 + \lambda_2 \Rightarrow \lambda_2 = -.5$~~

~~Since $\lambda_1 = 0$ when $e^{\lambda t}$ multiplies x_1 ... only a~~

~~multiplication by 1 to the eigenvector~~ x_1. The factor $e^{\lambda_2 t}$ will ~~decay to zero since $\lambda_2 < 0$~~ ∴ the steady state for this ~~ODE is given by the eigenvector x_1 corresponding to $\lambda_1 = 0$, which in this case is given by $\underline{x} = \begin{bmatrix} .3 \\ .2 \end{bmatrix}$ ∴ the solution will approach~~

$9x_1$.

(14) ~~The matrix $B = A - I$ has each column summing to 0. The steady state is the same as that of A~~

(15) ~~$A = \begin{bmatrix} .2 \\ .4 \\ .5 \end{bmatrix}$~~

$$A = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \checkmark$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/2 & 0 \end{bmatrix} \checkmark$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 0 & 1/4 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/8 & 0 \\ 0 & 1/8 \end{bmatrix} \checkmark$$

$$A^5 = A \cdot A^4 = \begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/8 & 0 \\ 0 & 1/8 \end{bmatrix} = \begin{bmatrix} 0 & 1/16 \\ 1/8 & 0 \end{bmatrix} \quad \checkmark$$

Thus our pattern appears to be

$$A^{2k} = \begin{bmatrix} (1/2)^{2k-1} & 0 \\ 0 & (1/2)^{2k-1} \end{bmatrix} \quad \checkmark \qquad k = 1, 2, 3, \ldots$$

$$A^{2k+1} = \cancel{\begin{bmatrix} 0 & (1/2)^{2k} \\ (1/2)^{2k-1} & 0 \end{bmatrix}} \qquad k = 1, 2, 3, \ldots$$

$$\begin{bmatrix} 0 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} (1/2)^{2k-1} & 0 \\ 0 & (1/2)^{2k-1} \end{bmatrix} = \begin{bmatrix} 0 & (1/2)^{2k} \\ (1/2)^{2k-1} & 0 \end{bmatrix} \checkmark$$

Check $k=1$ $\quad A^3 = \begin{bmatrix} 0 & (1/2)^2 \\ (1/2)^1 & 0 \end{bmatrix} \quad \checkmark$

$k=2$ $\quad A^5 = \begin{bmatrix} 0 & (1/2)^4 \\ (1/2)^3 & 0 \end{bmatrix}$ $\quad$ yes

So the total sum of the series is given by

$$I + A + A^2 + A^3 + \cdots = \cancel{\begin{bmatrix} 1 + 1/2 + 1/8 + 1/16 + \cdots \\ 1 + 1/2 + 1/8 + 1/25 + \cdots \end{bmatrix}}$$

$$= \begin{bmatrix} 1 + \tfrac12 + (\tfrac12)^3 + (\tfrac12)^5 + \cdots + (\tfrac12)^{2k-1} + \cdots & \tfrac12 + (\tfrac12)^2 + (\tfrac12)^4 + \cdots + (\tfrac12)^{2k} + \cdots \\ \cancel{0 + 1 + 0 + \tfrac12 + 0 + \cdots} & \\ 1 + \tfrac12 + (\tfrac12)^3 + (\tfrac12)^5 + \cdots + (\tfrac12)^{2k-1} + \cdots & 1 + \tfrac12 + (\tfrac12)^3 + (\tfrac12)^5 + \cdots + (\tfrac12)^{2k-1} + \cdots \end{bmatrix}$$

We evaluate each sum in turn. The ~~diagonal elements~~ sum at position (1,1), (2,1) & (2,2) is ~~are~~ given by

$$1 + \sum_{k=1}^{\infty} \left(\tfrac12\right)^{2k-1} = 1 + \left(\tfrac12\right)^{-1} \sum_{k=1}^{\infty} \left(\tfrac12\right)^{2k} = 1 + 2\sum_{k=1}^{\infty} \left(\tfrac14\right)^{k} = \cancel{1 + 2\left[\frac{1/4}{1 - 1/4}\right]}$$

$$= 1 + 2\left[\sum_{k=0}^{\infty} \left(\tfrac14\right)^{k} - 1\right] = 1 + 2\left[\frac{1}{1-(1/4)} - 1\right] = 1 + 2\left[\frac{1}{3/4} - 1\right]$$

$$= \cancel{1 + 2\left[\frac{1/4}{3/4}\right] = 1 + \frac{2}{3} = \frac{5}{3}} = 1 + 2\left[\frac{4}{3} - 1\right]$$

$$= 1 + 2\left[\frac{1}{3}\right] = \frac{5}{3}$$

The sum in the (1,2) position is given by

$$\cancel{\sum_{k=1}^{\infty}\left(\tfrac12\right)} \quad \frac12\left[1 + \frac12 + \left(\frac12\right)^3 + \cdots + \left(\frac12\right)^{2k-1} + \cdots\right]$$

$$\frac{1}{2}\left[1+\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k-1}\right] = \frac{1}{2}\left[1+2\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k}\right]$$

$$= \frac{1}{2}\left(\frac{5}{3}\right) = \frac{5}{6}$$

Thus the sum $I + A + A^2 + \cdots + A^k$ is given by

$$\begin{bmatrix} 5/3 & 5/6 \\ 5/3 & 5/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 10 & 5 \\ 10 & 10 \end{bmatrix}$$

We can check this result against the inverse of $I-A$

Since $I - A = \begin{bmatrix} 1 & -1/2 \\ -1 & 1 \end{bmatrix}$ which has an inverse

given by $(I-A)^{-1} = \frac{1}{(1+\frac{1}{2})}\begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{6}\begin{bmatrix} 4 & 2 \\ 4 & 4 \end{bmatrix}$$ something must be wrong ...

(17) For $I + A + A^2 + \dots A^n$ to be a nonnegative matrix requires that $\lambda_1 < 1$ where λ_1 is the largest positive eigenvalue of A.

For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ thus the eigenvalue ~~trace eigenvector~~ + determinant

identities give $\lambda_1 \cdot \lambda_2 = 0$ + $\lambda_1 + \lambda_2 = 0$ $\therefore \lambda_1 = \lambda_2 = 0$

So since $\lambda < 1$ $(I - A)^{-1}$ is nonnegative + is equal to the series above.

For $A = \begin{bmatrix} 0 & 4 \\ 1/5 & 0 \end{bmatrix}$ then the eigenvalue Trace + determinant

identities give $\lambda_1 \cdot \lambda_2 = 0 - \frac{4}{5}$ + $\lambda_1 + \lambda_2 = 0$

so $\lambda_1 = -\lambda_2$ so $-\lambda_1^2 = -\frac{4}{5}$ $\rightarrow$ $\lambda_1 = \frac{2}{\sqrt{5}}$ + $\lambda_2 = \frac{-2}{\sqrt{5}}$

Thus $\lambda < 1$ + $(I-A)^{-1}$ exists + is given by the sum above +

is Nonnegative. For $A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$ the eigenvalue Trace + determinant identities give $\lambda_1 \cdot \lambda_2 = -\frac{1}{2}$ + $\lambda_1 + \lambda_2 = \frac{1}{2}$

$\therefore$ $\lambda_1 = 1$ + $\lambda_2 = -\frac{1}{2}$

Thus $(I - A)^{-1}$ does not exist

(18) ~~I~~ For ~~the~~ the 1st A we have

$$(I-A) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{so} \quad (I-A)^{-1} = 1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

so $$p = (I-A)^{-1} y = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

For the second A we have

$$(I-A) = \begin{bmatrix} 1 & -4 \\ -1/5 & 1 \end{bmatrix} \quad \text{so} \quad (I-A)^{-1} = \frac{1}{(1+\frac{4}{5})} \begin{bmatrix} 1 & 4 \\ 1/5 & 1 \end{bmatrix}$$

$$= \frac{5}{9} \begin{bmatrix} 1 & 4 \\ 1/5 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & 20 \\ 1 & 5 \end{bmatrix}$$

$$\text{So } p = (I-A)^{-1} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 10+120 \\ 2+30 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 130 \\ 32 \end{bmatrix}$$

The 3rd matrix has $\lambda = 1$ & $\therefore$ $I-A$ does not exist.

(19) If $$A = \begin{bmatrix} .4 & .2 & .3 \\ .2 & .4 & .3 \\ .4 & .4 & .4 \end{bmatrix}$$

Since the sums of columns is one so the one ~~eigenvect~~ eigenvalue is 1. Adding the 1st & 2nd rows together produces a multiple of the 3rd row so this matrix is singular & $\therefore$ $\lambda = 0$ is an eigenvalue

Finally by inspection subtracting .2 from the diagonal elements will make the 1st & 2nd rows the same. Thus $\lambda = .2$ is an eigenvector. This can also be deduced from the eigenvalue trace theorem $\lambda_1 + \lambda_2 + \lambda_3 = 0 + 1 + \lambda_3 = 1.2$

so $\lambda_3 = .2$ as before.

The eigenvector associated with $\lambda = 1$ is given by the nullspace of the following

$$\begin{bmatrix} -.6 & .2 & .3 \\ .2 & -.6 & .3 \\ .4 & .4 & -.6 \end{bmatrix} \Rightarrow \begin{bmatrix} -.6 & .2 & .3 \\ .2 & -.6 & .3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/3 & -1/2 \\ 1 & -3 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/3 & -1/2 \\ 0 & -3+\frac{1}{3} & \frac{3}{2}+\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/3 & -1/2 \\ 0 & -\frac{8}{3} & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1/3 & -1/2 \\ 0 & 1 & -2\left(\frac{3}{8}\right) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \overset{-3/4}{-\frac{1}{2}-\frac{1}{4}} \\ 0 & 1 & -3/4 \\ 0 & 0 & 0 \end{bmatrix}$$

so the eigenvector is given by

$$\begin{bmatrix} 3/4 \\ 3/4 \\ 1 \end{bmatrix} \propto \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

then to normalize this so that it sums to 1 we have $\begin{bmatrix} .3 \\ .3 \\ .4 \end{bmatrix}$

The eigenvector associated w/ $\lambda = 0$ is given by

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 4 & 3 \\ 4 & 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 & 3/4 \\ 2 & 4 & 3 \\ 4 & 4 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 3 & 3-3/2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 3/4 \\ 0 & 1 & 1/2 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

so $X = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

The eigenvector for $\lambda = .2$ is given by

$$\begin{bmatrix} .2 & .2 & .3 \\ .2 & .2 & .3 \\ .4 & .4 & .2 \end{bmatrix} \Rightarrow \cancel{\begin{bmatrix} 2 & 2 & 3 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}} \Rightarrow \cancel{\begin{bmatrix} 1 & 1 & 3/2 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\Rightarrow \cancel{\begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 0 & \end{bmatrix}} \Rightarrow \begin{bmatrix} 2 & 2 & 3 \\ 4 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 4 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{so } X = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Thus $\begin{bmatrix} 3 & 1 & 1 \\ 3 & 1 & -1 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} =$

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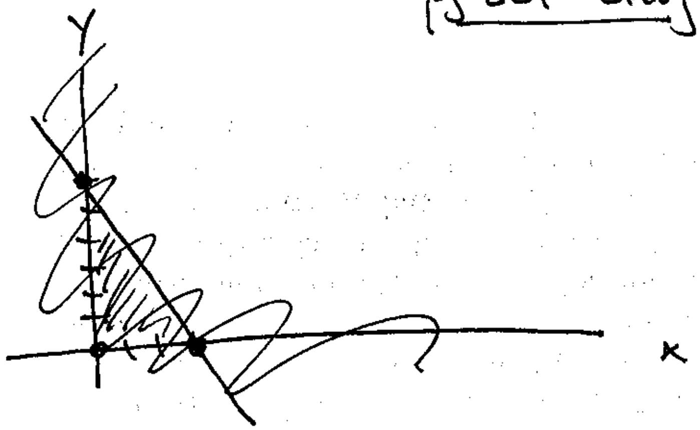
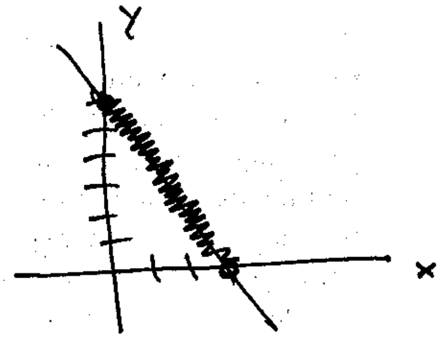
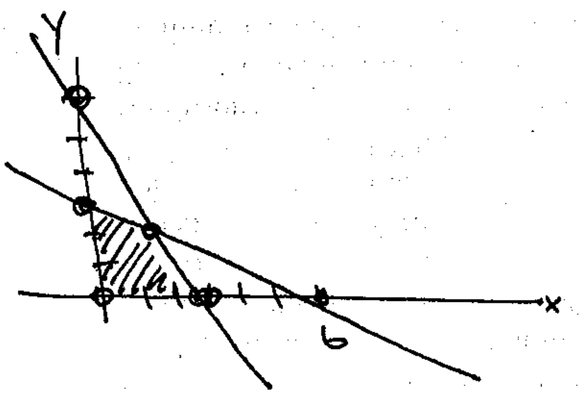
(1)





The possible minimum points are given by (3,0) + (0,6). The cost for each of these points is given by $3+0=3$ + $0+3(6)=18$. Thus the point (3,0) is the minimum solution point.

(2)



The 4 corners are (0,0), (0,3), (3,0) + (2,2)

Evaluating the cost at each point gives

pt	cost
(0,0)	0
(0,3)	-3
(3,0)	6
(2,2)	2

The minimum is at (0,3)

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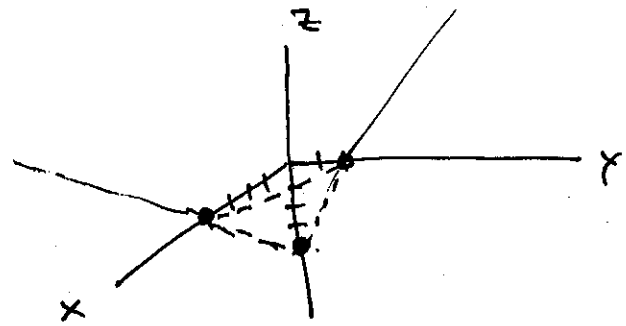
(3) Consider the corners on each of the ~~[illegible]~~ ~~[illegible]~~) coordinate lines

i.e. $(x_1,0,0)$, $(0,x_2,0)$ + $(0,0,x_3)$ we obtain

$(4,0,0)$, $(0,2,0)$, + $(0,0,-4)$



Then the "corners" of this set correspond to the line in the $z=0$ plane of $x_1+2x_2=4$

+ the "walls" corresponding to $x=0$ + $y=0$.

If x_1 + x_3 must be positive the x_1+2x_3 will not be negative

+ I don't see ~~how to do this~~ ~~po~~ how to do what is asked?

(4) With a cost of $5x_1+3x_2+8x_3$ the machine only at $x=(0,0,2)$ solves all 4 problems for 16. Move to $x=(0,1,x_3)$

Then x_3 must solve $x_1+x_2+2x_3=4 \Rightarrow 0+1+2x_3=4 \Rightarrow x_3=3/2$

So we move to $(0,1,3/2)$ + find a cost given by

$3+8(\frac{3}{2}) = 3+12=15$, giving a reduced cost of

$r = \cancel{18-15} = 15-\cancel{18}\,16 = -1$ for the ~~student~~ x_2 variable

(the student). To find the reduced cost for the computer let $x = (1, 0, x_3)$ w/ x_3 computed to satisfy the constraint that $x_1 + x_2 + 2x_3 = 4$ or

$$1 + 0 + 2x_3 = 4 \Rightarrow x_3 = 3/2$$ ~~at~~ Then our

point is $x = (1, 0, 3/2)$ + our cost of this point is

~~θ~~ $c = 5(1) + 0 + 8(3/2) = 5 + 12 = 17$. So the reduced cost is given by $r = 17 - 16 = +1$.

(5) Let $x = (4, 0, 0)$ with a cost coefficients given by $(5, 3, 7)$. The cost of this initial point is then $x^T c = 20$. The simplex method will try to assign some work to 1st the ~~Ph.D.~~ ~~+ then~~ the student + then the computer computing the reduced cost during each step. For example trying the ~~student~~ student we consider the point $x = (x_1, 1, 0)$ which is required to satisfy our constraint of

~~$\cancel{x_1 + x_2 + 2x_3}$~~ = $x_1 + 1 + 0 = 4$ so $x_1 = 3$ $\therefore$ our point is given

by $x = (3, 1, 0)$ to give a cost of

$$c^T x = 15 + 3 = 18$$ + a reduced cost of

$$r = 18 - 20 = -2.$$

Trying the computer/machine we consider the point $x = (x_1, 0, 1)$

subject to the constraint requires that

$$x_1 + 0 + 2\cdot 1 = 4 \quad + \quad x_1 = 2$$ so that the simplex

trial point is given by ~~(3,0,2)~~ (2,0,1) resulting in a cost

of $c^T x = 5\cdot 2 + 7\cdot 1 = 10 + 7 = 17$ + a reduced cost of

$r = 17 - 20 = -3$. Thus the 1st step of the simplex method

would be to include more computer time. To see how much

consider allocating two units of time i.e consider the point

$x = (x_1, 0, 2)$ then the amount of PhD time is given

by requiring the constraint $x_1 + 2\cdot 2 = 4 \Rightarrow x_1 = 0$ Thus

x_1 is the leaving variable + the new point is $x = (0,0,2)$

with a cost of $c^T x = 7\cdot 2 = 14$. To see what step to

next consider adding in some ~~(a~~ Ph.D time i.e consider

$(1,0,x_3)$ to find that $1 + 0 + 2x_3 = 4 \Rightarrow x_3 = 3/2$

to get the point $(1,0,3/2)$ with a cost of $5 + \frac{21}{2} = \frac{31}{2} = 15.5$

to include some student time consider $(0,1,x_3)$ to find

$x_3 = \frac{3}{2}$ ~~&~~ with a cost of $3 + 7\left(\frac{3}{2}\right) = \frac{6}{2} + \frac{21}{3} = \frac{27}{3} = 9$

Thus including the student reduces the cost.

to see how much ~~constr~~ of the student to include

consider the point $(0,2,x_3)$ & solving for x_3 in

the constraint equation gives $0+2+2x_3=4 \to x_3=1$.

Thus x_3 is the vanishing variable & in entirety we can

~~compute~~ consider the point $(0,4,0)$ for a cost of

$c^T x = 3\cdot 4 = 12$ & thus the student is where ~~to~~ the final

solution lies

(b) Since we know from the fundamental theorem of linear programming that the solution must lie at one of the vertices

~~we need to~~ which are at

$P = (4,0,0)$; $Q = (0,4,0)$; & $R = (0,0,2)$

We need a cost function such that the point P is smallest. Change c to $[4,3,8]$ will produce a point with a cost of P equal to that at R.

Changing c to $[3,3,8]$ gives a cost that equals that at Q. Thus any cost w/ $q < 3$ will yield a solution ~~at~~ both at the point P as required. Lets pick a cost matrix given by

$[2,3,8]$

The dual problem is given by:

Minimize $b \cdot y = 4y$ subject to

$$A^T y = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y \le c = \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$$

So the dual problem is given by

minimize $4y$ subject to

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y \le \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix}$$

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(1) consider

$$\int_0^{2\pi} 2\cos(jx)\cos(kx)\,dx = \int_0^{2\pi} \cos((j+k)x) + \cos((j-k)x)\,dx$$

$$= \frac{\sin((j+k)x)}{j+k}\Bigg|_0^{2\pi} + \frac{\sin((j-k)x)}{j-k}\Bigg|_0^{2\pi} \qquad j \neq k \;\&\; j \neq -k$$

$$= \frac{\sin(2\pi(j+k))}{j+k} + \frac{\sin(2\pi(j-k))}{j-k}$$

But if j & k are integers then $\sin(2\pi(j+k)) = 0$ & $\sin(2\pi(j-k)) = 0$

So $\displaystyle\int_0^{2\pi} \cos(jx)\cos(kx)\,dx = 0$ when $j \neq k$

If $j=k$ the above integral becomes

$$\int_0^{2\pi} \cos^2(jx)\,dx = \int_0^{2\pi}\left(\frac{1+\cos(2jx)}{2}\right)dx = \frac{1}{2}(2\pi) + \frac{\sin(2jx)}{2\cdot 2j}\Bigg|_0^{2\pi} = \pi.$$

(2) consider $\int_{-1}^{1} 1(x)\,dx = \left.\frac{x^2}{2}\right|_{-1}^{1} = 0$

$$\int_{-1}^{1} 1(x^2-\tfrac{1}{3})\,dx = \cancel{\int_{-1}^{1}(x^2-\tfrac{2}{3})\,dx} = \left(\frac{x^3}{3} - \frac{1}{3}x\right|_{-1}^{1} = \left(\frac{1}{3}-\frac{1}{3}\right) - \left(\frac{-1}{3}+\frac{1}{3}\right)$$
$$= 0$$

$$\int_{-1}^{1} x(x^2-\tfrac{1}{3})\,dx = \left(\frac{x^4}{4} - \frac{x^2}{6}\right|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} - \frac{1}{6}(1-1) = 0$$

Now $f(x) = 2x^2 = \frac{(f,1)}{(1,1)}\cdot 1 + \frac{(f,x)}{(x,x)}\,x + \frac{(f,x^2-\frac{1}{3})}{(x^2-\frac{1}{3},x^2-\frac{1}{3})}(x^2-\tfrac{1}{3})$

and one could evaluate each inner product above. A simpler way is to recognize that $2x^2$ is 2 times the basis function of x^2 ~~of~~ plus/minus correction

$$2x^2 = 2(x^2-\tfrac{1}{3}) + \text{error}$$

$\Rightarrow$ $2x^2 = 2x^2 - \frac{2}{3}$

So picking the error to be $\frac{2}{3}(1)$ we have

$$2x^2 = \frac{2}{3} + 2(x^2-\tfrac{1}{3})$$

(3) Consider $w = (c, -\frac{1}{2}, -\frac{1}{4}, \dots)$

with c chosen to make ~~the~~ w + v orthogonal since

$$w^T v = c - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{4}\right)^2 - \left(\tfrac{1}{8}\right)^2 - \dots$$

$$= c - \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^4 - \left(\tfrac{1}{2}\right)^6 - \dots$$

$$= c - \sum_{k \geq 1} \left(\frac{1}{2^2}\right)^k = c - \frac{(1/4)}{(1 - \frac{1}{4})}$$

$$= c - \frac{1/4}{3/4} = c - \frac{1}{3}$$

which to be set equal to zero gives $c = \frac{1}{3}$.

Thus $w = (\frac{1}{3}, -\frac{1}{2}, -\frac{1}{4}, \dots)$ + the length of w is

given by $\|w\|^2 = \frac{1}{9} + \sum_{k \geq 1} \left(\frac{1}{2^2}\right)^k = \frac{1}{9} + \frac{1}{3} = \frac{4}{9}$

or $\|w\| = \frac{2}{3}$.

We require c to satisfy

$$(x^3-cx,1)=0=\int_{-1}^{1}(x^3-cx)\,dx=0 \Rightarrow \frac{x^4}{4}-\frac{cx^2}{2}\Big|_{-1}^{1}=0$$

which is true for any c. We also require

$$(x^3-cx,x)=\int_{-1}^{1}(x^4-cx^2)\,dx=2\int_{0}^{1}(x^4-cx^2)\,dx=2\left(\frac{x^5}{5}-\frac{cx^3}{3}\Big|_{0}^{1}\right.$$

$$=2\left(\frac{1}{5}-\frac{c}{3}\right)=0 \Rightarrow c=\frac{3}{5}$$

We need to check that x^3-cx is orthogonal to $x^2-\frac{1}{3}$ also.

for this value of c $(c=3/5)$

$$(x^3-cx,\ x^2-\tfrac{1}{3})=(x^3-cx,x^2)-(x^3-cx,\tfrac{1}{3})=(x^3-cx,x^2)$$

since $(x^3-cx,\frac{1}{3})=0$. Thus we just need to check

$$(x^3-cx,x^2)=\int_{-1}^{1}(x^5-cx^3)\,dx=0$$ since it is an odd

function over a symmetric interval.

(5) The square wave from example 3 is given by

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ +1 & \text{for } 0 < x < \pi \end{cases}$$ extended ~~per~~ with a period of 2π

Thus $$\int_0^{2\pi} f(x)\cos(x)\,dx = \int_0^{\pi} 1\cos(x)\,dx - \int_{\pi}^{2\pi} 1\cos(x)\,dx$$

$$= \sin(x)\Big|_0^{\pi} - \sin(x)\Big|_{\pi}^{2\pi} = 0 - 0 = 0$$

Which gives the Fourier coefficient ~~as~~ a_1. The second integral is given by

$$\int_0^{2\pi} f(x)\sin(x)\,dx = \int_0^{\pi} \sin(x)\,dx - \int_{\pi}^{2\pi} \sin(x)\,dx$$

$$= -\cos(x)\Big|_0^{\pi} + \cos(x)\Big|_{\pi}^{2\pi}$$

$$= -(-1-1) + (1-(-1)) = 2 + 2 = 4$$

Which is proportional to b_1. The 3rd integral is given by

$$\int_0^{2\pi} f(x)\sin(2x)\,dx = \int_0^{\pi} \sin(2x)\,dx - \int_{\pi}^{2\pi} \sin(2x)\,dx$$

$$= -\frac{\cos(2x)}{2}\Big|_0^{\pi} + \frac{\cos(2x)}{2}\Big|_{\pi}^{2\pi}$$

$$= -\frac{1}{2}(\cos(2\pi) - 1) + \frac{1}{2}(\cos(4\pi) - \cos(2\pi))$$

$$= -\frac{1}{2}(1-1) + \frac{1}{2}(1-1) = 0$$ which is proportional to a_2.

(6) $$\|f\|^2 = \int_0^{2\pi} f^2(x)\,dx = \int_0^{2\pi} 1\,dx = 2\pi$$

By equation 6 this is equal to the sum

$$2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + b_2^2 + \cdots)$$

$$= 2\pi(0) + \pi\left(0^2 + \left(\frac{4}{\pi}\right)^2 + 0^2 + \left(\frac{4}{\pi}\cdot\frac{1}{3}\right)^2 + 0 + \left(\frac{4}{\pi}\frac{1}{5}\right)^2 + \cdots\right)$$

$$= \pi\left(\frac{4}{\pi}\right)^2\left[1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{7}\right)^2 + \cdots\right] = \frac{16}{\pi}\sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)^2$$

$$\therefore \sum_{k=0}^{\infty}\left(\frac{1}{2k+1}\right)^2 = \frac{\pi^2}{8}$$

(7) Skipped

(8) $\|v\|^2 = \sum_{k \geq 1} v_k^2$

& By Parseval's equality $\|f\|^2 = (f,f) = \int_0^{2\pi} (1+\sin x)^2 \, dx$

(9) Skipped

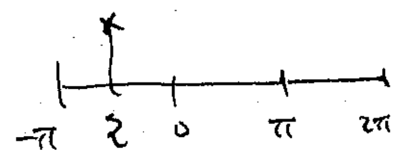
(10) If f has a period of 2π then ~~$f(x+2\pi) = f(x)$~~ ~~or~~ consider ~~$f(x-\pi)$~~

$\cancel{\int_{-\pi}^{\pi} f(x)\,dx}$ ~~let $v = x+\pi$ then $dv = dx$~~ ~~$x = v-\pi$~~

$= \cancel{\int_0^{2\pi} f(v-\pi)\,dv}$

$$\int_{-\pi}^{+\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_0^{\pi} f(x)\,dx$$

Because f is periodic of period 2π



$f(x)$ when $-\pi \leq x \leq 0$ is equivalent to

$f(x+2\pi)$ but for $-\pi \leq x \leq 0$ $x+2\pi$ is in

the range $\pi \leq x \leq 2\pi$ & thus every point in

$-\pi \le x \le 0$ can be mapped to an equivalent point in $\pi \le x \le 2\pi$

Showing $\displaystyle\int_{-\pi}^{0} f(x)\,dx = \int_{\pi}^{2\pi} f(x)\,dx$

$\therefore \displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \int_{0}^{2\pi} f(x)\,dx$

If ~~for~~ $f(-x) = -f(x)$ then $\displaystyle\int_{0}^{2\pi} f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx$

$$= -\int_{0}^{\pi} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx = 0$$

(11) (a) $f = \cos^2 x = \dfrac{1+\cos(2x)}{2} = \frac{1}{2} + \frac{1}{2}\cos(2x)$

So $a_0 = \frac{1}{2}$ & $a_2 = \frac{1}{2}$ all other terms are zero

(b) $f = \cos\left(x + \frac{\pi}{3}\right) = \cos(x)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\sin(x)$

$$= \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

So $a_1 = \frac{1}{2}$ & $b_1 = -\frac{\sqrt{3}}{2}$ all other terms are zero

(12) $f_1 = 1 \quad f_1' = 0$

$f_2 = \cos x \Rightarrow f_2' = \cancel{\sin x} - \sin x$

$f_3 = \sin x \Rightarrow f_3' = \cos x$

$f_4 = \cos(2x) \Rightarrow f_4' = -2\sin(2x)$

$f_5 = \sin(2x) \Rightarrow f_5' = 2\cos(2x)$

So if A function is given in terms of a Fourier series

then $\cancel{f(x)} = \cancel{f(x)} = \cancel{a_0 + \sum}$

$$f(x) = a_0 + a_1\cos x + b_1 \sin x + a_2 \cos(2x) + b_2 \sin(2x) + \cdots$$

Then considering the vector v defined as

$$\underline{v} = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \end{bmatrix} \quad \vdots$$

The derivative operator D would be the matrix producing the Fourier coefficients of $f(x)$ or

$$\underline{v}' = \begin{bmatrix} 0 \\ b_1 \\ -a_1 \\ 2b_2 \\ -2a_2 \end{bmatrix} \quad \vdots$$

or in Matrix form we have

$$\underline{v}' = D\underline{v} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & & \\ 0 & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & 0 & 0 & & & \\ 0 & & & 0 & 0 & & & \\ 0 & 0 & 0 & \boxed{\begin{matrix} 0 & 2 \\ -2 & 0 \end{matrix}} & & & & \\ 0 & 0 & 0 & & & & & \\ 0 & & & & & \boxed{\begin{matrix} 0 & 3 \\ -3 & 0 \end{matrix}} & & \\ \vdots & & & & & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ \vdots \end{bmatrix}$$

(13) The complete solution $y(x)$ is given by

$$y(x) = \sin x + C_1$$

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(1) With at pivoting our matrix gives

$$A = \begin{bmatrix} .001 & 0 \\ 1 & 1000 \end{bmatrix} \Rightarrow \begin{bmatrix} .001 & 0 \\ 0 & 1000-0 \end{bmatrix} \Rightarrow \begin{bmatrix} .001 & 0 \\ 0 & 1000 \end{bmatrix}$$

So the pivots are .001 + 1000. With partial pivoting we exchange the 1st two rows to obtain

$$\begin{bmatrix} 1 & 1000 \\ .001 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1000 \\ 0 & -1 \end{bmatrix}$$ so the two pivots are

~~1/10^3~~ +1 + −1. The LU decomposition will then be

$$\begin{bmatrix} 1 & 0 \\ -.001 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1000 \\ .001 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1000 \\ 0 & -1 \end{bmatrix}$$

So $$\begin{bmatrix} 1 & 1000 \\ .001 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .001 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1000 \\ 0 & -1 \end{bmatrix}$$

With partial pivoting ~~we~~ ~~will select~~ ~~to~~ we will select the largest element below the in the column. ~~If at least one element in this column is greater than one,~~ its ~~repeat i.e the element in l_{ij} will be less than 1~~

So ~~each element of L will be $|l_{ij}| \le 1$. If all the elements of A are less than 1~~

The elements of L will then be given by $\frac{a_{ij}}{\hat{a}_{ii}}$

with $\hat{a}_{ii}$ the largest element in the jth column.

Thus $|l_{ij}| \le 1 \quad \forall \quad \cancel{i \ne j} \quad i > j$

To find a 3x3 matrix with $|a_{ij}| \le 1$ + $|l_{ij}| \le 1$

but with a 3rd pivot = 4 consider assuming such a matrix by giving the U + L.

$$U = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} \quad + \quad L = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then ~~LU =~~

~~LU =~~

$$LU = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 1 & 1/2 & 1/2 \\ 1 & 3/2 & 1 \\ 1 & 3/2 & 5 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 10 \end{bmatrix} = A$$

then each element of A is $|a_{ij}| < 1$ & each element of $|l_{ij}| < 1$ while the 3 pivots are given by $1, 1, \& 4$. The solution given in the back is nice since $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ then

$$A \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

(2) ~~For A given by~~

$$A = \begin{bmatrix} \cancel{1} & \cancel{\tfrac{1}{2}} & \cancel{\tfrac{1}{3}} \\ \cancel{\tfrac{1}{2}} & \cancel{\tfrac{1}{3}} & \cancel{\tfrac{1}{4}} \\ \cancel{\tfrac{1}{3}} & \cancel{\tfrac{1}{4}} & \cancel{\tfrac{1}{5}} \end{bmatrix}$$

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(3) For the A given in problem 2 we compute

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1.833\\1.08\\0.78\end{bmatrix}$$

$$+ \quad A\begin{bmatrix}0\\6\\-3.6\end{bmatrix} = \begin{bmatrix}1.8\\1.1\\0.78\end{bmatrix}$$

Thus the change in b is ~~only~~ ~~about~~ about $\Delta b = \begin{bmatrix}.03\\-.01\\.003\end{bmatrix}$ ~~$\approx O(10^{-2})$~~

~~While the change in Δx is easily $O(1)$~~

So $\|\Delta b\| \sim O(10^{-2})$ while the change in Δx is easily $O(1)$

(4) Since hild(8) is symmetric & ~~at least~~ ~~numerically~~ ~~$\lambda_i > 0$~~ the largest magnitude eigenvalue gives the largest allowed magnification. For this matrix this is given by $\lambda \cong 1.695 \cong 1.7$. Thus $\|A\| \cong 1.7$. In the same way ~~$\|A\|$~~ $\|A^{-1}\| = \frac{1}{\lambda_{min}} = \frac{1}{10^{-9}}$

So $\|A^{-1}\| \cong 10^{10}$. Thus an error in b with magnitude given by $\|\Delta b\| \cong 10^{-16}$ can produce an error in Δx of magnitude $\|A^{-1}\| \cdot \|\Delta b\| = 10^{10} \cdot 10^{-16} = 10^{-6}$

Problem 9

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$$\det(A) = +1\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\left[1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\right]$$

$$-1\left[1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\right]$$

$$= -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

Then $\det(M_{31}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = +1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

So $(A^{-1})_{13} = \frac{1}{(-1)}(1) = -1 \neq 0$

Type In

For $(A^{-1})_{14} = \frac{1}{\det(A)} C_{41} = \frac{1}{\det(A)} (-1)^{4+1} \det(M_{41})$

$$= \det(M_{41})$$

So $(A^{-1})_{14} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

So $(A^{-1})_{14} = 1 \neq 0$

For $(A^{-1})_{24} = \frac{1}{\det(A)} C_{42} = \frac{1}{\det(A)} (-1)^{4+2} \det(M_{42})$

$$= -\det(M_{42}) = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$$

For $(A^{-1})_{31} = \frac{1}{\det(A)} C_{13} = \frac{1}{\det(A)} (-1)^{1+3} \det(M_{13})$

$$= -\det(M_{13}) = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$$

For $(A^{-1})_{41} = \frac{1}{\det(A)} C_{14} = \frac{1}{\det(A)} (-1)^{1+4} \det(M_{14})$

$$= \det(M_{14}) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

For $(A^{-1})_{42} = \frac{1}{\det(A)} C_{24} = \frac{1}{\det(A)} (-1)^{2+4} \det(M_{24})$

$$= -\det(M_{24}) = -\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$$

Note that None of these are zero, even though the corresponding elements in A are zero.

(10) 1st find the LU factorization of $A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix}$ to obtain (this is without partial pivoting which we would want to use since $\varepsilon \ll 1$.

$$A = \begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \varepsilon & 1 \\ 0 & 1 - \frac{1}{\varepsilon} \end{bmatrix}$$ by multiplying by

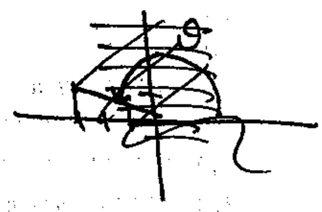
$E_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\varepsilon} & 1 \end{bmatrix}$ Thus we have a direct LU decomposition given by

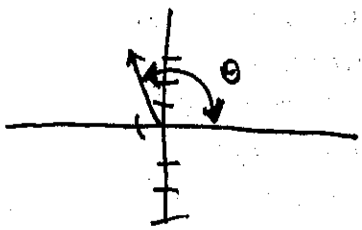
(11) $$Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$$

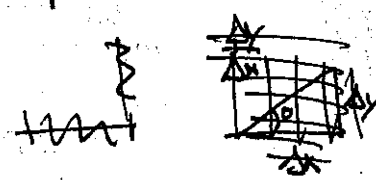
$$= \begin{bmatrix} \cos\theta - 3\sin\theta & -\cos\theta - 5\sin\theta \\ \sin\theta + 3\cos\theta & -\sin\theta + 5\cos\theta \end{bmatrix}$$

to make (2,1) zero we require $\sin\theta + 3\cos\theta = 0$

or $$\frac{\sin\theta}{\cos\theta} = \tan\theta = -3 = -\frac{3}{1} = \frac{3}{(-1)}$$







Which gives $\theta = \dots$

Another way to see what $\sin\theta$ & $\cos\theta$ can be is to consider the expression $\sin\theta + 3\cos\theta = 0$

as $$\frac{1}{\sqrt{1+3^2}}\sin\theta + \frac{3}{\sqrt{1+3^2}}\cos\theta = 0$$

Then pick $\cos\theta = \dfrac{-1}{\sqrt{1+9}}$ & the above will vanish

& $\sin\theta = \dfrac{3}{\sqrt{1+9}}$

Thus $Q_{21} = \begin{bmatrix} \frac{-1}{\sqrt{1+9}} & \frac{-3}{\sqrt{1+9}} \\ \frac{3}{\sqrt{1+9}} & \frac{-1}{\sqrt{1+9}} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix}$

So that $R = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} -1-9 & 1-15 \\ 3-3 & -3-5 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -10 & -14 \\ 0 & -8 \end{bmatrix}$$

(12) For $A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

with $Q_{21} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

we have $Q_{21} A Q_{21}^{-1} = Q_{21} A Q_{21}^{T}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta + \sin\theta & \sin\theta - \cos\theta \\ 3\cos\theta - 5\sin\theta & 3\sin\theta + 5\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin\theta\cos\theta - 3\sin\theta\cos\theta + 5\sin^2\theta & \sin\theta\cos\theta - \cos^2\theta - 3\sin^2\theta - 5\sin\theta\cos\theta \\ \sin\theta\cos\theta + \sin^2\theta + 3\cos^2\theta - 5\sin\theta\cos\theta & \sin^2\theta - \sin\theta\cos\theta + 3\sin\theta\cos\theta + 5\cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + -2\sin\theta\cos\theta + 5\sin^2\theta & -\cos^2\theta - 3\sin^2\theta - 4\sin\theta\cos\theta \\ \sin^2\theta + 3\cos^2\theta - 4\sin\theta\cos\theta & \sin^2\theta + 5\cos^2\theta + 2\sin\theta\cos\theta \end{bmatrix}$$

to make this upper triangle we must have

$$\sin^2\theta + 3\cos^2\theta - 4\sin\theta\cos\theta = 0 \qquad \text{or}$$

$$1 + 2\cos^2\theta - 4\sin\theta\cos\theta = 0$$

if $\theta = \frac{\pi}{4}$ the above becomes

$$1 + 2\left(\tfrac{1}{2}\right) - 4\left(\tfrac{1}{2}\right) = 0 \ \checkmark \qquad \text{which works !!}$$

So the solution to the above is given by $\theta = \frac{\pi}{4}$. Thus

$Q_{21} A Q_{21}^{-1}$ is now given by

$$\begin{bmatrix} \frac{1}{2} - \frac{2}{2} + \frac{5}{2} & -\frac{1}{2} - \frac{3}{2} - \frac{4}{2} \\ 0 & \frac{1}{2} + \frac{5}{2} + \frac{2}{2} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 4 \end{bmatrix} \equiv R$$

Since $Q_{21} A Q_{21}^{-1}$ is a similarity transformation of A the eigenvalues are the <u>same</u> as that of A + the #'s on the diagonal of the R

~~above~~ above or 2, + 4.

(13) since Q_{ij} has $\sin\theta$ + $\cos\theta$ (trigonometric functions)

at positions (i,i), (i,j), (j,i) + (j,j)

The multiplication of Q_{ij} times A modifies rows i + j of A in replacing rows i + j in A.

$$\begin{array}{c c} & \begin{array}{cc} i & j \end{array} \\ \begin{array}{c} i \\ \\ j \end{array} & \begin{bmatrix} \times & \times \\ & \\ \times & \times \end{bmatrix} \end{array} A$$

Thus the entries in rows i + rows j of A are changed.

When $Q_{ij}^{-1} = Q_{ij}^{T}$ is multiplied on the right of ~~A~~ $Q_{ij}A$ the ~~rows~~ columns i + j are replaced with multiples (by trig functions) of the i + jth columns of $(Q_{ij}A)$.

(14) To directly compute $Q_{ij}A$ would require two steps. First multiply row i of A by $\cos\theta$ multiply row j of A by $-\sin\theta$ + Adding these two rows requiring $2n$ multiplications + n additions. Second multiply row i by $\sin\theta$

& adding to $\cos\theta$ multiplied by row j. Again requiring the same # of multiplications & additions as the 1st step. Thus in total we require $4n$ multiplications & $2n$ additions to compute $Q_{ij}A$

(15) From the $Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$

we obtain $= \frac{1}{3} \begin{bmatrix} -\cos\theta - 2\sin\theta & 2\cos\theta + \sin\theta & 2\cos\theta - 2\sin\theta \\ -\sin\theta + 2\cos\theta & 2\sin\theta - \cos\theta & 2\sin\theta + 2\cos\theta \\ 2 & 2 & -1 \end{bmatrix}$

Which to make $(Q_{21}A)_{21} = 0$ requires

$$-\sin\theta + 2\cos\theta = 0$$

pick ~~[illegible]~~ $\cos\theta = \frac{-1}{\sqrt{1+4}} = \frac{-1}{\sqrt{5}}$

& $\sin\theta = \frac{-2}{\sqrt{1+4}} = \frac{-2}{\sqrt{5}}$

& the $(2,1)$ position will vanish

Pg 408 Story

(1) From the text we know that $\|A\| = \lambda_{max}$ when A is positive definite

So when $A = \begin{bmatrix} .5 & 0 \\ 0 & 2 \end{bmatrix}$ A is positive definite

we have eigenvalues given by $\lambda_1 = .5$ + $\lambda_2 = 2$

~~$\begin{vmatrix} .5-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$~~ $\Rightarrow$ ~~$(.5-\lambda)(2-\lambda) = 0$~~ so $\|A\| = 2$

+ $\kappa(A) = \lambda_{max}/\lambda_{min} = 2/.5 = 4$

When $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ we have eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow (2-\lambda) = \pm 1$$

$$\Rightarrow 2-\lambda = \pm 1$$

$$\Rightarrow 2 = \pm 1 + \lambda$$

$$\Rightarrow \lambda = 2 \pm 1$$ so ~~λ~~ each $\lambda > 0$ + A is positive definite.

So $\lambda_{max} = 3$ + $\|A\| = 3$

$\kappa(A) = \lambda_{max}/\lambda_{min} = \frac{3}{1} = 3$

~~When $A =$~~

When $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ we have eigenvalues given by

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(1-\lambda) - 1 = 0$$

or $3 - 3\lambda - \lambda + \lambda^2 - 1 = 0$

$\Rightarrow \quad \lambda^2 - 4\lambda + 2 = 0$

$\Rightarrow \quad \lambda = \dfrac{4 \pm \sqrt{4^2 - 4(2)}}{2} = \dfrac{4 \pm \sqrt{16-8}}{2} = \dfrac{4 \pm \sqrt{8}}{2}$

So $\cancel{\lambda_{max} = \frac{4+\sqrt{6}}{2}}$

& $\cancel{\chi(A) = \frac{\lambda_{max}}{\lambda_{min}} = \frac{\frac{4+\sqrt{6}}{2}}{\left(\frac{4-\sqrt{6}}{2}\right)} = \frac{4+\sqrt{6}}{4-\sqrt{6}}}$

So

$\lambda = \dfrac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$, since $\lambda > 0$ A is positive definite

Thus $\|A\| = 2 + \sqrt{2}$ & $\chi(A) = \dfrac{\lambda_{max}}{\lambda_{min}} = \dfrac{2+\sqrt{2}}{2-\sqrt{2}}$

(2) When A is ~~not~~ ~~symmetric~~ symmetric then $\|A\| = \max|\lambda_i|$

For $A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ one eigenvalue is -2 & A

is not positive definite. Its eigenvalues are ± 2 &

$\therefore \max|\lambda_i| = 2.$

In general the matrix norm is given by the square root of $\lambda_{max}(A^TA)$. From the given A, A^TA is given

by $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ which has eigenvalues of 4

only giving the matrix norm of $\sqrt{4} = 2$, the same as ~~before~~

computed before. For symmetric matrices (like this one) $\varkappa = \frac{|\lambda_{max}|}{|\lambda_{min}|}$
we have $\varkappa(A) = \frac{2}{(\frac{1}{2})} = 4$

For the 2nd matrix we have

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has eigenvalues given by $\lambda = 0$ & $\lambda = 2$

Thus the matrix norm for this A is given by $\|A\| = \sqrt{2}$.

~~For~~ Since in this case A is not positive definite (it is not invertible) we have λ_m or $\lambda_{min} = 0$ so

$$\varkappa(A) = +\infty$$

For the final matrix A we have

$$A^TA = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

so $\lambda_{max}(A^TA) = 2$ + $\|A\| = \sqrt{\lambda_{max}(A^TA)} = \sqrt{2}$.

$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ so the eigenvalues of A^{-1} are

given by $\begin{vmatrix} 1/2-\lambda & -1/2 \\ 1/2 & \frac{1}{2}-\lambda \end{vmatrix} = 0$

$$\left(\tfrac{1}{2}-\lambda\right)^2 + \tfrac{1}{4} = 0$$

$(\lambda - \frac{1}{2}) = \pm\frac{i}{2}$ so $\lambda = \frac{1}{2} \pm \frac{i}{2} = \frac{1}{2}(1 \pm i)$

Then $|\lambda| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

So $\kappa(A) = \dfrac{\sqrt{2}}{\left(\frac{1}{\sqrt{2}}\right)} = 2$

(3) Since ABx can be written $A(Bx)$ + Bx is just another vector from the definition of the norm i.e. equation (2) we have $\|ABx\| \le \|A\| \cdot \|Bx\| \le \|A\| \cdot \|B\| \cdot \|x\|$

$$\Rightarrow \frac{\|ABx\|}{\|x\|} \le \|A\| \cdot \|B\|$$

taking the maximum over all $\|x\| \ne 0$ on the left gives

$$\|AB\| \le \|A\| \cdot \|B\|$$

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(4) Since the condition # is defined as $\kappa(A) = \|A\|\cdot\|A^{-1}\|$.

From $\|A\cdot B\| \le \|A\|\cdot\|B\|$ with $B = A^{-1}$ we have

$$\|I\| \le \|A\|\cdot\|A^{-1}\| = \kappa(A)$$

But $\|I\| = 1$ so $\kappa(A) \ge 1 \quad \forall A$.

(5) To be symmetric implies diagonalizable, and $A = S\Lambda S^{-1}$ becomes $A = Q\Lambda Q^T$. Since every eigenvalue must be 1 $\Lambda = I$ & $A = QQ^T = I$.

So A is actually the identity matrix.

I don't know why $\|A\| = 1$ & $\|A^{-1}\| = 1 \Rightarrow$ A is orthogonal i.e. $A^TA = I$. these matrices are orthogonal

(6) If $A = QR$ then $\|A\| \le \|Q\|\,\|R\| = \|R\|$.

But also $R = Q^TA$ so $\|R\| \le \|Q^T\|\cdot\|A\| = \|A\|$.

thus $\|A\| = \|R\|$.

To find an example of $A = LU$ such that

$$\|A\| < \|L\|\cdot\|U\|$$

~~The eigenvalues of L are all 1's since it is a lower triangular~~ matrix

let $L = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}$ & $U = \begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix}$

Then $L^TL = \begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix} = \begin{bmatrix}5 & -2\\ -2 & 1\end{bmatrix}$

& $U^TU = \begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix} = \begin{bmatrix}4 & 2\\ 2 & 5\end{bmatrix}$

Type In
↓

the eigenvalues of L^TL are given by $\lambda_1 + \lambda_2 = 6$

$\lambda_1 \cdot \lambda_2 = 5+4 = 9$

So $\lambda_1 = 3 = \lambda_2$

The eigenvalues of U^TU are given by

~~$\lambda_1 + \lambda_2 = 9$~~

~~$\lambda_1 \lambda_2 = 20 - 4 =$~~ 16

$$\begin{vmatrix}4-\lambda & 2\\ 2 & 5-\lambda\end{vmatrix} = 0$$

$\Rightarrow (4-\lambda)(5-\lambda) - 4 = 0$

$\Rightarrow 20 - 9\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 9\lambda + 16 = 0$

$$\lambda = \frac{9 \pm \sqrt{81 - 4(16)}}{2} = \frac{9 \pm \sqrt{81-64}}{2} = \frac{9 \pm \sqrt{17}}{2}$$

$$\begin{array}{r} 16 \\ 4 \\ \hline 40 \\ 24 \\ \hline 64 \end{array} \qquad \begin{array}{r} 81 \\ 64 \\ \hline 17 \end{array}$$

Thus $\|L\| = 3^{1/2}$ & $\|U\| = \left(\frac{9\pm\sqrt{17}}{2}\right)^{1/2}$

So $\|L\|\cdot\|U\| = \left[\left(\frac{3}{2}\right)(9+\sqrt{17})\right]^{1/2}$ while

$$A = LU = \begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 0 & 2\end{bmatrix} = \begin{bmatrix}2 & 1\\ -4 & 0\end{bmatrix}$$

which has $A^TA = \begin{bmatrix}2 & -4\\ 1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ -4 & 0\end{bmatrix} = \begin{bmatrix}4+16 & 2\\ 2 & 1\end{bmatrix} = \begin{bmatrix}20 & 2\\ 2 & 1\end{bmatrix}$

~~Thus~~ which has eigenvalues given by

$$\lambda^2 - \text{Trace}(A^TA)\lambda + \det(A^TA) = 0$$

$$\Rightarrow \lambda^2 - 21\lambda + 16 = 0$$

$$\lambda = \frac{21 \pm \sqrt{21^2 - 4(16)}}{2} = \frac{21 \pm \sqrt{441}}{2} =$$

$$\begin{array}{r} 21 \\ 21 \\ \hline 21 \\ 420 \\ \hline 441 \end{array}$$

Thus $\|A\| = \left(\frac{21+\sqrt{441}}{2}\right)^{1/2} =$

(12) If A is singular $|A|=0$ But Ax can be arbitrarily large, i.e. consider $A = 10^6 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Then if $x = (1,1)^T$ $Ax = 10^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

so $\|Ax\| \sim O(10^6)$

But $|A| = 0$ which bears no relation to the size of Ax.

The condition # represents how ~~easily~~ accurately can it is to solve $Ax=b$

Consider the matrix from problem 11 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$

Then $|A| = 10^{-4}$ but infact when solving $Ax=b$ for x errors on the order of $\lambda(A) =$ - - can occur

again $|A|$ bears no relationship to the solvability of the system.

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(15) ~~[illegible]~~

If $x=(1,1,1,1)$ then $\|x\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$

& $\|x\|_1 = |1|+|1|+|1|+|1| = 4$

& $\|x\|_\infty = \max_{1\le i\le n} |x_i| = 1$

If $x=(.1, .7, .3, .4, .5)$ then

$\|x\| = \sqrt{.1^2+.7^2+.3^2+.4^2+.5^2} =$

$\|x\|_1 = |.1|+|.7|+|.3|+|.4|+|.5| = .8+.7+.5 = 1.5+.5=2$

$\|x\|_\infty = \max_{1\le i\le n} |x_i| = .7$

(16) Prove that $\|x\|_\infty \le \|x\| \le \|x_1\|$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| = |x_{i^*}| \le \sqrt{|x_{i^*}|^2} \le \sqrt{|x_1|^2+\cdots+|x_n|^2}$$

$$= \|x\|$$

where i^* is the index ~~[illegible]~~ of the components of $|x_i|$ that is largest

From $\|x\|_\infty \le \|x\| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$

$$= \|x\|_1 \left(\frac{\sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}}{|x_1| + |x_2| + \cdots |x_n|} \right)$$

Thus if we can show that

$$\frac{\sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}}{|x_1| + |x_2| + \cdots + |x_n|} \le 1$$

equivalently by squaring both sides

$$|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \le (|x_1| + |x_2|^2 + \cdots + |x_n|)^2$$

Since the RHS, is equivalent to

$$|x_1|^2 + 2|x_1||x_2| + 2|x_1||x_3| + \cdots + 2|x_1||x_n|$$
$$|x_2|^2 + 2|x_2||x_3| + \cdots \quad 2|x_2||x_n|$$
$$\cdots + |x_n|^2$$

the above ~~simplified~~ simplifies to

$$0 \le 2|x_1||x_2| + 2|x_1||x_3| + \cdots + 2|x_1||x_n| + \cdots + 2|x_{n-1}||x_n|$$

and is true. Thus

the desired inequality is true. i.e. since

$$\frac{\sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2}}{|x_1|+|x_2|+\cdots+|x_n|} \le 1$$

we have $\|x\| \le \|x\|_1$

Consider the ratio of

$$\frac{\|x\|}{\|x\|_\infty} = \frac{\sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2}}{|x_{i^*}|} = \sqrt{\frac{|x_1|^2}{|x_{i^*}|^2}+\frac{|x_2|^2}{|x_{i^*}|^2}+\cdots+\frac{|x_n|^2}{|x_{i^*}|^2}}$$

But since $\frac{|x_i|^2}{|x_{i^*}|^2} \le 1 \quad \forall i$

we have that $\frac{\|x\|}{\|x\|_\infty} \le \sqrt{n}$

The ratio of $\frac{\|x\|_1}{\|x\|}$ is given by

$$\frac{\|x\|_1}{\|x\|} = \frac{|x_1|+|x_2|+\cdots+|x_n|}{\sqrt{|x_1|^2+|x_2|^2+\cdots+|x_n|^2}}$$

From the Schwarz inequality i.e.

$$\sum a_i b_i \le \left(\sum a_i^2\right)^{1/2}\left(\sum b_i^2\right)^{1/2} \quad \text{i.e.} \quad a^T b \le \|a\|\cdot\|b\|$$

Using a vector of absolute values & another vector of ones

we have $\displaystyle\sum_i |x_i| \le \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}\left(\sum_{i=1}^n 1^2\right)^{1/2}$

$$= \|x\|\cdot\sqrt{n}$$

Thus we have $\dfrac{\|x\|_1}{\|x\|} \le \dfrac{\|x\|\cdot\sqrt{n}}{\|x\|} = \sqrt{n}$

~~Vector~~ The vector of all ones has each ratio equal to $\sqrt{n}$.

For example $\dfrac{\|x\|}{\|x\|_\infty} = \dfrac{\sqrt{n}}{1} = \sqrt{n}$

& $\dfrac{\|x\|_1}{\|x\|} = \dfrac{n}{\sqrt{n}} = \sqrt{n}$

5

(17) For the ∞ norm we ~~have~~ have

$$\|x+y\|_\infty = \max_{1\le i\le n} |x_i+y_i| \le \max_{1\le i\le n}(|x_i|+|y_i|) \le \max_{1\le i\le n}|x_i| + \max_{1\le i\le n}|y_i|$$

so $\|x+y\|_\infty \le \|x\|_\infty + \|y\|_\infty$

For the 1 norm we have

$$\|x+y\|_1 = \sum_i |x_i+y_i| \le \sum_i (|x_i|+|y_i|) = \sum_i |x_i| + \sum_i |y_i| = \|x\|_1 + \|y\|_1$$

(18) For $\|x\|_A = |x_1| + 2|x_2|$

Then if ~~$x=(1,1)$~~ then

$$\cancel{\|2x\|_A = |2| + 2|2| = 6 \ne |2|\cdot\|(1,1)\|_A = 2(1+2) = 6}$$

$$\cancel{\|3x\|_A = 3 + 2(3) = 9 \ne 3\|x\|_A = 3(\underbrace{1+2(1)}_{3}) = 3(\cdot 3) = 9}$$

$$\cancel{\|4x\|_A = 4 + 2(4) = 4+8 = 12 \ne 4\|x\|_A} = 4$$

~~If $x=(1,3)$ then~~

$$\cancel{\|2x\|_A = \|(2,6)\|_A = 2 + 2(6) = 2+12} = 14$$

$$\cancel{\ne 2\|x\|_A = 2}$$

let $c \neq 0$ be given the

$$\|cx\|_A = |cx_1| + 2|cx_2| = |c|(|x_1| + 2|x_2|) = |c| \cdot \|x\|_A$$

& $\|x\|_A > 0 \quad \forall \; x \neq 0$

Thus $\|\cdot\|_A$ is a norm

For $\|x\|_B = \min |x_i|$

The if $c \in \mathbb{R}$ is given then

$$\|cx\|_B = \min_i |cx_i| = \min_i |c| \cdot |x_i| = |c| \min_i |x_i| = |c| \cdot \|x\|_B$$

So this property holds, but if $x = (0,1)$ then $x \neq 0$

but $\|x\|_B = 0$ & thus this property is not held.

For $\|x\|_C = \|x\| + \|x\|_\infty$ will be a norm since both

$\|\cdot\|$ & $\|\cdot\|_\infty$ are one.

For $\|x\|_D = \|Ax\|$

The $\|cx\|_D = \|cAx\| = |c| \|Ax\| = |c| \cdot \|x\|_D$ thus this property

is true

But if $x \neq 0$ $\quad \|x\|_D = \|Ax\|$

so $\|x\|_D^2 = \|Ax\|^2 = \cancel{x}\,(Ax)^T(Ax) = x^T A^T A x$

which can equal zero iff $Ax = 0$ when $x \neq 0$ which can happen iff A is not invertible. If A is invertible the $\|x\|_D$ is a norm.

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(1) From $Ax=b$

~~but~~ consider $A = A - I + I = I - (I-A)$

The $Ax = (I-(I-A))x = x - (I-A)x$

Then in the decomposition $A = S - T$ we have $S = I$ &

$T = I - A$ so in the splitting the convergence of the ~~iterat~~ iterative method & $Sx_{k+1} = Tx_k + b$ is determined by $S^{-1}T$ which in this case is $I^{-1}(I-A) = I-A$

(2) If λ is an eigenvalue of A then $1-\lambda$ is an eigenvalue of $B = I - A$. The real eigenvalues of B have absolute value less than 1 if $|1-\lambda| < 1$

$\Leftrightarrow \quad -1 < 1-\lambda < 1$

$\Leftrightarrow \quad -2 < -\lambda < 0$

$\Leftrightarrow \quad 0 < \lambda < 2$

(3) For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ then the convergence of

$x_{k+1} = (I-A)x_k + b$ depends on the ~~the~~ eigenvalues of $I-A$ which in this case is given by

$$\begin{bmatrix} 1-2 & 1 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$. This matrix has eigenvalues

given by $\lambda^2 - \text{Trace}(A)\lambda + \text{Det}(A) = 0$

$\Rightarrow \lambda^2 + 2\lambda + (1-1) = 0 \Rightarrow \lambda = 0$ & $\lambda = -2$

Thus $|\lambda| = 2$ is not less than 1 & therefore will not converge

(4) The Norm of B^k is given by ~~$\|B^k\| \leq$~~

$\|B^k\| = \|B \cdot B^{k-1}\| \leq \|B\| \cdot \|B^{k-1}\| \leq \cdots \leq \|B\|^k$

Thus if $\|B\| < 1$ we can gaurantee that B^k will approach Zero. We know that $|\lambda|_{max} < \|B\| < 1$

(5) If A is singular then all splittings of $A = S - T$ will fail, ~~From~~ From $Ax = 0$ & our splitting we have

$(S-T)x = 0 \rightarrow Sx = Tx \Leftrightarrow S^{-1}Tx = x$

so the matrix $S^{-1}T$ has an eigenvalue $\lambda = 1$ &

Thus $Sx_{k+1} = Tx_k + b$ cannot converge

(6) For the ~~given~~ splitting

$Sx_{k+1} = Tx_k + b$ given by

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_k + b$$

Then the matrix $S^{-1}T = \frac{1}{3} I \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

which has eigenvalues given by

$$\lambda^2 - \text{Trace}(A)\lambda + \text{Det}(A) = 0$$

$$\Rightarrow \lambda^2 - 0\lambda + (0 - \tfrac{1}{9}) = 0 \quad \text{or} \quad \lambda^2 - \tfrac{1}{9} = 0$$

$$\text{or} \quad \lambda = \pm \frac{1}{3}$$

Thus $|\lambda|_{max} = \frac{1}{3}$ + this iteration will converge

(7) For the given Gauss-Seidel method we have

$$S = \begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} \; + \; T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So $S^{-1}T = \frac{1}{9}\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 & 3 \\ 0 & 1 \end{bmatrix}$

So $S^{-1}T$ has eigenvalues given by

$$\lambda^2 - Tr(A)\lambda + Det(A) = 0$$

which has $\lambda^2 - \frac{1}{9}\lambda + 0 = 0$

$\Rightarrow \lambda = 0$ + $\lambda = \frac{1}{9}$

thus ~~the~~ $|\lambda|_{max} = \frac{1}{9}$. Thus we have from problem 6 that $|\lambda|^{GS}_{max} = (|\lambda|^{J}_{max})^2$

(8) For any ~~to~~ 2×2 matrix ~~the~~ ~~have eigenvalues given~~ ~~by~~ $\cancel{\lambda^2 - Tr(A)\lambda + Det(A)} = 0$

~~by~~ $\lambda = \cancel{\frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4Det(A)}}{2}}$

a Jacobi splitting $A = S - T$ is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}}_{S} - \underbrace{\begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}}_{T}$$

then the matrix that determines the convergence is given by $S^{-1}T = \begin{bmatrix} 1/a & 0 \\ 0 & 1/d \end{bmatrix}\begin{bmatrix} 0 & -b \\ -c & 0 \end{bmatrix}$

or $\begin{bmatrix} 0 & -\frac{b}{a} \\ -\frac{c}{d} & 0 \end{bmatrix}$. This matrix has eigenvalues given by

$$\lambda^2 - Tr(A)\lambda + Det(A) = 0$$

$$\Rightarrow \lambda^2 - 0 \cdot \lambda + -\frac{cb}{ad} = 0 \Rightarrow \lambda = \pm\sqrt{\frac{cb}{ad}}$$

For a Gauss Seidel splitting we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} - \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = S - T$$

Thus the matrix that determines convergence is given by $S^{-1}T = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix} = \frac{1}{ad} \begin{bmatrix} d & 0 \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix}$

$$= \frac{1}{ad} \begin{bmatrix} 0 & -bd \\ 0 & cb \end{bmatrix}$$

This matrix has eigenvalues given by

$$\lambda^2 - Trace(A)\lambda + Det(A) = 0$$

$$\Rightarrow \lambda^2 - \frac{cb}{ad}\lambda + D = 0$$

$$\Rightarrow \lambda = 0 \ \ \& \ \ \lambda = \frac{cb}{ad}$$

(9) In the SOR method the splitting $A = S - T$ is given (For a two x two matrix A) by ~~by writing Ax=b~~

~~of~~ ~~$\omega Ax = \omega b$~~ ~~then~~

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ wc & d \end{bmatrix} - \begin{bmatrix} 0 & -b \\ -c+wc & 0 \end{bmatrix}$$

Then $S = \begin{bmatrix} a & 0 \\ wc & d \end{bmatrix}$ + $T = \begin{bmatrix} 0 & -b \\ c(w-1) & 0 \end{bmatrix}$

Then
$$S^{-1}T = \begin{bmatrix} a & 0 \\ wc & d \end{bmatrix}^{-1} \begin{bmatrix} 0 & -b \\ c(w-1) & 0 \end{bmatrix}$$

$$= \frac{1}{ad} \begin{bmatrix} d & 0 \\ -wc & a \end{bmatrix} \begin{bmatrix} 0 & -b \\ c(w-1) & 0 \end{bmatrix}$$

$$= \frac{1}{ad} \begin{bmatrix} 0 & -bd \\ ac(w-1) & bcw \end{bmatrix}$$

From which we see the determinant of $S^{-1}T$ is given by

$$\left(\frac{1}{ad}\right)^2 \left(0 + b \cdot ac(w-1)\right) = \frac{bc}{ad}(w-1)$$

and the trace is equal to $\dfrac{(bc)w}{ad}$

Thus to make two equal eigenvalues let

$$\lambda_1 = \lambda_2 = \lambda \quad \text{then} \quad \lambda^2 = \frac{bc}{ad}(\omega - 1) \quad *$$

$$\text{\&} \quad 2\lambda = \left(\frac{bc}{ad}\right)\omega$$

~~So~~ $\cancel{\omega = 2\left(\frac{ad}{bc}\right)\cdot\lambda}$ ~~which when~~ So $\lambda = \frac{1}{2}\left(\frac{bc}{ad}\right)\omega$

Which when put in eq * gives

$$\frac{1}{4}\left(\frac{bc}{ad}\right)^2 \omega^2 = \left(\frac{bc}{ad}\right)(\omega - 1)$$

$$\Rightarrow \omega^2 - 4\left(\frac{ad}{bc}\right)\omega + 4\left(\frac{ad}{bc}\right) = 0$$

Thus $\omega = \dfrac{4\left(\frac{ad}{bc}\right) \pm \sqrt{16\left(\frac{ad}{bc}\right)^2 - 4\left(4\left(\frac{ad}{bc}\right)\right)}}{2}$

$$= 2\left(\frac{ad}{bc}\right) \pm 2\sqrt{\left(\frac{ad}{bc}\right)^2 - \left(\frac{ad}{bc}\right)}$$

But for the requested problem all we are asked to evaluate the optimal ω for the example given by equation 10

For eq 10 we have a determinant by

$$|S^{-1}T| = (1-\omega)(1-\omega+\tfrac{1}{4}\omega^2) - \tfrac{1}{4}\omega^2(1-\omega)$$

$$= (1-\omega)\left[1-\omega+\cancel{\tfrac{1}{4}\omega^2}-\cancel{\tfrac{1}{4}\omega^2}\right] = (1-\omega)^2$$

Thus we want to pick $\cancel{\lambda=(1-\omega)}$ $\lambda = \pm(1-\omega)$. Thus setting

$$\cancel{\lambda+\lambda = (1-\omega)+(1-\omega) = 2-2\omega = 2-2\omega+\tfrac{1}{4}\omega^2}$$

$$\Rightarrow \cancel{\omega = 0}$$

Thus picking $\lambda = \omega - 1$ we have

$$\lambda+\lambda = 2\omega-2 = 2-2\omega+\tfrac{1}{4}\omega^2$$ ✓

$$\Rightarrow \tfrac{1}{4}\omega^2 - 4\omega - 4 = 0$$

$$\Rightarrow \omega^2 - 16\omega - 16 = 0$$

So $$\omega = \frac{+16 \pm \sqrt{16^2+4(16)}}{2} = \cancel{8 \pm 2\sqrt{164} = 8 \pm \ldots\sqrt{2}}$$

$$= \frac{\cancel{8}\,16 \pm 4\sqrt{16+4}}{2} = 8 \pm 2\sqrt{20}$$

$$= 8 \pm 2\cdot 2\sqrt{5}$$

$$= 8 \pm 4\sqrt{5}$$

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(9) Eq 10 from the book is given by

$$S^{-1}T = \begin{bmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & 1-\omega+\frac{1}{4}\omega^2 \end{bmatrix}$$

~~Equation~~ ~~st~~ it has a determinant given by

$$(1-\omega)^2 + \cancel{\tfrac{1}{4}\omega^2(1-\omega)} - \cancel{\tfrac{1}{4}\omega^2(1-\omega)}$$

$$\Rightarrow \quad (1-\omega)^2$$

to have equal eigenvalues pick values given by $\pm(1-\omega)$.

The trace in equation 10 is given by

$$2(1-\omega) + \tfrac{1}{4}\omega^2.$$

Assuming, as suggested in the text, that our equal eigenvalues is given by $\omega - 1$ we can equate the trace above with $2(\omega-1)$ to obtain

$$2(1-\omega) + \tfrac{1}{4}\omega^2 = 2(\omega-1)$$

$$2 - 2\omega + \tfrac{1}{4}\omega^2 - 2\omega + 2 = 0$$

$$\Rightarrow \quad \tfrac{1}{4}\omega^2 - 4\omega + 4 = 0$$

$$\Rightarrow \quad \omega^2 - 16\omega + 16 = 0$$

2

Then $\omega = \dfrac{16 \pm \sqrt{16^2 - 4(16)}}{2}$

$$\begin{array}{r} 16 \\ \times 16 \\ \hline 96 \\ 160 \\ \hline 256 \end{array} \qquad \begin{array}{r} 4 \\ \times 16 \\ \hline 64 \end{array}$$

~~So $16^2 - 4$~~

So $\omega = \dfrac{16 \pm \sqrt{16(16-4)}}{2}$

$= 8 \pm 2\sqrt{12}$

$= 8 \pm 4\sqrt{3} = \_\_\_, \_\_\_$

To have $0 < \omega < 1$ we should use the minus sign in the choice

(11) If $x_i^{new} = x_i^{old} \equiv x_i$ then the Gauss-Seidel iteration gives

$$\cancel{x_i} = \cancel{x_i} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j - \sum_{j=i}^{n} a_{ij}x_j\right)$$

$\Rightarrow$ $b_i = \sum_{j=1}^{n} a_{ij}x_j$ which is the ith equation in the system $Ax=b$.

~~For Jacobi's method we have the following iteration component~~wise

$$\cancel{x_i^{new} = x_i^{old} + \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{old} - \sum_{j=i+1}^{n} a_{ij}x_j^{old}\right)}$$

~~If $x_i^{new} = x_i^{old}$ ...~~

For Jacobi's method we have the following component-wise iteration.

$A = S - T \qquad Sx_{t+1} = Tx_t + b$

The ith equation then is

$$x_i^{new} = \frac{1}{a_{ii}}\left(-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j^{old} + b_i\right)$$

Then if $x_i^{new} = x_i^{old} = x_i$ the above simplifies to

$$a_{ii}x_i = -\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j + b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} a_{ij}x_j = b_i$$ or which is the ith equation in the system $Ax = b$.

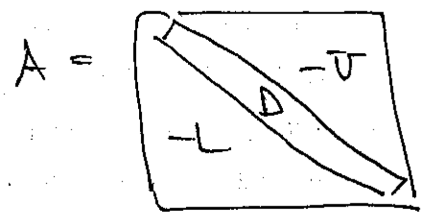
For ~~SOR~~ SOR we perform a splitting ~~$A = S - T = S$~~ ~~with the diagonal terms elements given by Jacobi~~, i.e. that is in some sense part like Jacobi & part like Gauss-Seidel. ~~We use diagon~~ Like Jacobi our splitting will use the main diagonal elements of A

while below the diagonal we will use a weighted version of Gauss-Seidel by weighting the coefficients below the diagonal by ω. This gives the following component wide equation for x_i

Let L denote the ~~negat~~

$\cancel{A = D + \omega L - \omega L + L}$

$\cancel{A = D - L - U}$



negative of the elements below the diagonal of A & U denote the negative of the elements above the diagonal of A. Then we have the splitting $A = D - L - U$, which is adapted in SOR as follows $A = D - \omega L + \omega L - L - U$

$$A = D - L - U$$

Defining $A = S - T$ w/ $S = D - \omega L$ & $T = -\omega L + L + U$

$$= (1-\omega)L + U$$

~~we see~~ That our iteration is given by

$$(D - \omega L)x_{k+1} = ((1-\omega)L + U)x_k + b$$

which in component form is given by (for the ith equation)

$$a_{ii}x_i^{new} + \omega \sum_{j=1}^{i-1} a_{ij}x_j^{new} = -(1-\omega)\sum_{j=1}^{i-1} a_{ij}x_j^{old} - \sum_{j=i+1}^{n} a_{ij}x_j^{old} + b_i$$

or

$$x_i^{new} = \left(-\omega \sum_{j=1}^{i-1} a_{ij} x_j^{new} - (1-\omega) \sum_{j=1}^{i-1} a_{ij} x_j^{old} - \sum_{j=i+1}^{n} a_{ij} x_j^{old} + b_i\right) \Big/ a_{ii}$$

When iterations stop one has that $x_i^{new} = x_i^{old} \equiv x_i$ &

the above scheme ~~[illegible]~~ ~~[illegible]~~ simplifies to

$$a_{ii} x_i = \cancel{-\omega \sum_{j=1}^{i-1} a_{ij} x_j} - (1-\cancel{\omega}) \sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1}^{n} a_{ij} x_j + b_i$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} x_j = b_i$$ which is the ith equation again,

~~which~~ & this shows consistency.

(13) Equation 11 is ~~given~~ ~~by~~

i) $u_k = A^k u_0 = c_1(\lambda_1)^k x_1 + c_2(\lambda_2)^k v_2 + \cdots + c_n(\lambda_n)^k x_n$

$$= \lambda_1^k \left[c_1 x_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k x_3 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x_n \right]$$

Thus the next largest term in the ~~series~~ eigenvector series of u_k is given by $c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x_2 = O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)$

Thus $\frac{u_k}{\lambda_1^k} \longrightarrow c_1 x_1 \propto x_1$ iff $\left(\frac{\lambda_2}{\lambda_1}\right)^k \longrightarrow 0$ as $k \to \infty$.

An orthogonal matrix for which $|\lambda_i| = 1 \quad \forall i$ should not converge ~~when~~ when using the ~~[illegible]~~ power method. Consider

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ Then A is a rotation matrix &

~~[illegible]~~ all eigenvalues of A are one & applications of A simply rotate the vector counter-clockwise around the origin.

(14) with $A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix}$ & the eigenvalues of $\lambda = 1$ & $\lambda = .6$

The eigenvalues of A^{-1} are given by 1 & $\frac{1}{.6} = \frac{1}{\frac{6}{10}} = \frac{10}{6} = \frac{5}{3}$

The inverse matrix $A^{-1} = \frac{1}{(.63 - .03)} \begin{bmatrix} .7 & -.3 \\ -.1 & .9 \end{bmatrix} = \frac{1}{.6} \begin{bmatrix} .7 & -.3 \\ -.1 & .9 \end{bmatrix}$

$$= \frac{1}{6} \begin{bmatrix} 7 & -3 \\ -1 & 9 \end{bmatrix}$$

Then ~~with~~ for the eigenvalue $\lambda = 1$ we have an eigenvector given by the null space of

$$\begin{bmatrix} 7/6 - 1 & -1/2 \\ -1/6 & \frac{9}{6} - 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{1}{2} \end{bmatrix}$$ which has a

nullspace given by the span of $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\frac{1}{2}$

For $\lambda = \frac{5}{3}$ the eigenvector is the span of

$$\begin{bmatrix} 7/6 - \frac{5}{3} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{3}{2} - \frac{5}{3} \end{bmatrix} = \begin{bmatrix} \frac{7}{6} - \frac{10}{6} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{9}{6} - \frac{10}{6} \end{bmatrix} = \begin{bmatrix} \frac{-3}{6} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{-1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{6} & \frac{-1}{6} \end{bmatrix}$$ which is given by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The inverse power rule $v_{-k} = A^{-k} v_0$, selects a v_0 & then finds v_1 such that $Av_1 = v_0$, Then find v_2 such that $Av_2 = v_1$ etc ... Now if

$$v_0 = c_1 x_1 + c_2 x_2$$

Then $(A^{-1})^k (c_1 x_1 + c_2 x_2)$

$$= c_1 (\lambda_1(A^{-1}))^k x_1 + c_2 (\lambda_2(A^{-1}))^k x_2$$

$$= c_1 \cdot 1^k \begin{bmatrix} 6 \\ 2 \end{bmatrix} + c_2 \left(\frac{1}{.6}\right)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus after multiplying by $(.6)^k$ we get

$$(.6)^k v_0 = c_1 (.6)^k \begin{bmatrix} 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(16) Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ & compute the power

method on $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

see the matlab file prob_9_3_16.m

The eigenvalues of A are given by

~~$\lambda_2 = 4$~~

$\lambda^2 - 4\lambda \quad \lambda^2 \cancel{- Tr(A)\lambda + Det(A) = 0}$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)^2 - 1 = 0$

$\Rightarrow \quad 2-\lambda = \pm 1$

$\Rightarrow \quad \lambda - 2 = \pm 1$

$\Rightarrow \quad \lambda = 2 \pm 1 = 1, 3$

with eigenvectors given by

For $\lambda = 1$ the null space of

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{so} \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 3$ the null space of

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \quad \text{so} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So $$A^k v_0 = c_1(1)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(3)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 3^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Thus $\cancel{A^k v_0 \rightarrow}$ for $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So $$\cancel{A^k v_0} \; A^k v_0 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3^k}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \xrightarrow[\text{converges to}]{} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(17) I'm assuming the book means problem 16. Then the inverse power method will converge to the eigenvector of the higher eigenvalue. The two eigenvalues are given $\frac{1}{1}$ + $\frac{1}{3}$ so it will converge to the eigenvector associated to the eigenvalue of 1.

For $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ we have an inverse given by $A^{-1} = \frac{1}{.5}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 2/5 \end{bmatrix}$

Note the inverse power method should be implemented by performing the LU decomposition on A

$A = LU$ in $O(n^3)$ flops Then solving $LU x_{k+1} = x_k$ in $O(n^2)$

Flops