Solution Manual for: Linear Algebra by Gilbert Strang

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Introduction

A Note on Notation

In these notes, I use the symbol $\Rightarrow$ to denote the results of elementary elimination matrices used to transform a given matrix into its reduced row echelon form. Thus when looking for the eigenvectors for a matrix like

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

rather than say, multiplying A on the left by

$$E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

produces

$$E_{33}A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we will use the much more compact notation

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

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Derivation of the decomposition (Page 170)

Combining the basis for the row space and the basis for the nullspace into a common matrix to assemble a general right hand side $x = \begin{bmatrix} a & b & c & d \end{bmatrix}^T$ from some set of components $c = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^T$ we must have

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Inverting the coefficient matrix A by using the teaching code `elim.m` or augmentation and inversion by hand gives

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

So the coefficients of c_1, c_2, c_3, and c_4 are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a+c \\ b+d \\ a-c \\ b-d \end{bmatrix}$$

As verified by what is given in the book.

Chapter 1 (Introduction to Vectors)

Section 1.1 (Vectors and Linear Combinations)

Problem 16 (dimensions of a cube in four dimensions)

We can generalize Problem 15 by stating that the corners of a cube in *four* dimensions are given by

$$n(1,0,0,0) + m(0,1,0,0) + l(0,0,1,0) + p(0,0,0,1),$$

for indices n, m, l, p taken from $\{0, 1\}$. Since the indices n, m, l, p can take two possible values each the total number of such vectors (i.e. the number of corners of a four dimensional cube) is given by $2^4 = 16$.

To count the number of faces in a four dimensional cube we again generalize the notion of a face from three dimensions. In three dimensions the vertices of a face is defined by a configuration of n, m, l where *one* component is specified. For example, the top face is specified by $(n, m, 1)$ and the bottom face by $(n, m, 0)$, where m and n are allowed to take all possible values from $\{0, 1\}$. Generalizing to our four dimensional problem, in counting faces we see that each face corresponds to first selecting a component (either n, m, l, or p) setting it equal to 0 or 1 and then letting the other components take on all possible values. The component n, m, l, or p can be chosen in one of four ways, from which we have two choices for a value (0 or 1). This gives $2 \times 4 = 8$ faces.

To count the number of edges, remember that for a three dimensional cube an edge is determined by specifying (and assigning to) all but one elements of our three vector. Thus selecting m and p to be 0 we have $(n, 0, 0)$ and $(n, 0, 0)$, where n takes on all values from $\{0, 1\}$ as vertices that specify one edge. To count the number of edges we can first specifying the *one* component that will change as we move along the given edge, and then specify a complete assignment of 0 and 1 to the remaining components. In four dimensions, we can pick the single component in four ways and specify the remaining components in $2^3 = 8$, ways giving $4 \cdot 8 = 32$ edges.

Problem 17 (the vector sum of the hours in a day)

Part (a): Since every vector can be paired with a vector pointing in the opposite direction the sum must be zero.

Part (b): We have

$$\sum_{i \neq 4} v_i = \left(\sum_{\text{all } i} v_i \right) - v_4 = 0 - v_4 = -v_4 \,,$$

with v_4 denoting the 4:00 vector.

Part (c): We have

$$\sum_{i \neq 1} v_i + \frac{1}{2} v_1 = \left(\sum_{\text{all } i} v_i \right) - v_1 + \frac{1}{2} v_1 = 0 - \frac{v_1}{2} = -\frac{v_1}{2} \,,$$

with v_1 denoting the 1:00 vector.

Problem 18 (more clock vector sums)

We have from Problem 17 that the vector sum of all the v_i's is zero,

$$\sum_{i \in \{1,2,\ldots 12\}} v_i = 0 \,.$$

Adding twelve copies of $(0, -1) = -\hat{\jmath}$ to each vector gives

$$\sum_{i \in \{1,2,\ldots 12\}} (v_i - \hat{\jmath}) = -12\hat{\jmath} \,.$$

But if in addition to the transformation above the vector 6:00 is set to zero and the vector 12:00 is doubled, we can incorporate those changes by writing out the above sum and making the terms summed equivalent to the specification in the book. For example we have

$$\begin{aligned}
\left(\sum_{i\neq\{6,12\}}(v_i-\hat{\jmath})\right)+(v_6-\hat{\jmath})+(v_{12}-\hat{\jmath}) &= -12\hat{\jmath} \\
\left(\sum_{i\neq\{6,12\}}(v_i-\hat{\jmath})\right)+(0-\hat{\jmath})+(2v_{12}-\hat{\jmath}) &= -v_6+v_{12}-12\hat{\jmath} \\
\left(\sum_{i\neq\{6,12\}}(v_i-\hat{\jmath})\right)+(0-\hat{\jmath})+(2v_{12}-\hat{\jmath}) &= -(0,1)+(0,-1)-12(0,1)=-10(0,1)\,.
\end{aligned}$$

The left hand side now gives the requested sum. In the last equation, we have written out the vectors in terms of their components to perform the summations.

Problem 26 (all vectors from a collection)

Not if the three vector are not degenerate, i.e. are not all constrained to a single line.

Problem 27 (points in common with two planes)

Since the plane spanned by u and v and the plane spanned by v and w intersect on the line v, all vectors cv will be in both planes.

Problem 28 (degenerate surfaces)

Part (a): Pick three vectors collinear, like

$$\begin{aligned}
u &= (1,1,1) \\
v &= (2,2,2) \\
w &= (3,3,3)
\end{aligned}$$

Part (b): Pick two vectors collinear with each other and the third vector not collinear with the first two. Something like

$$\begin{aligned}
u &= (1,1,1) \\
v &= (2,2,2) \\
w &= (1,0,0)
\end{aligned}$$

Problem 29 (combinations to produce a target)

Let c and d be scalars such that combine our given vectors in the correct way i.e.

$$c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \end{bmatrix}$$

which is equivalent to the system

$$\begin{aligned} c + 3d &= 14 \\ 2c + d &= 8 \end{aligned}$$

which solving for d using the second equation gives $d = 8 - 2c$ and inserting into the first equation gives $c + 3(8 - 2c) = 14$, which has a solution of $c = 2$. This with either of the equations above yields $d = -2$.

Section 1.2 (Lengths and Dot Products)

Problem 1 (simple dot product practice)

We have

$$\begin{aligned} u \cdot v &= -.6(3) + .8(4) = 1.4 \\ u \cdot w &= -.6(4) + .8(3) = 0 \\ v \cdot w &= 3(4) + 4(3) = 24 \\ w \cdot v &= 24\,. \end{aligned}$$

Chapter 2 (Solving Linear Equations)

Section 2.2 (The Idea of Elimination)

Problem 1

We should subtract 5 times the first equation. After this step we have

$$\begin{aligned} 2x + 3y &= 11 \\ -6y &= 6 \end{aligned}$$

or the system

$$\begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix}$$

The two pivots are 2 and -6.

Problem 2

the last equation gives $y = -1$, then the first equation gives $2x - 3 = 1$ or $x = 2$. Lets check the multiplication

$$\begin{bmatrix} 2 \\ 10 \end{bmatrix} (2) + \begin{bmatrix} 3 \\ 9 \end{bmatrix} (-1) = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \tag{1}$$

If the right hand changes to

$$\begin{bmatrix} 4 \\ 44 \end{bmatrix} \tag{2}$$

then -5 times the first component added to the second component gives $44 - 20 = 24$.

Chapter 3 (Vector Spaces and Subspaces)

Section 3.1

Problem 5

Part (a): Let M consist of all matrices that are multiples of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Part (b): Yes, since the element $1 \cdot A + (-1) \cdot B = I$ must be in the space.

Part (c): Let the subspace consist of all matrices defined by

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 6

We have $h(x) = 3(x^2) - 4(5x) = 3x^2 - 20x$.

Problem 7

Rule number eight is no longer true since $(c_1 + c_2)x$ is interpreted as $f((c_1 + c_2)x)$ and $c_1 x + c_2 x$ is interpreted as $f(c_1 x) + f(c_2 x)$, while in general for arbitrary functions these two are not equal i.e. $f((c_1 + c_2)x) \neq f(c_1 x) + f(c_2 x)$.

Problem 8

- The first rule $x + y = y + x$ is broken since $f(g(x)) \neq g(f(x))$ in general.
- The second rule is correct.
- The third rule is correct with the zero vector defined to be x.
- The fourth rule is correct if we define $-x$ to be the inverse of the function $f(\cdot)$, because then the rule $f(g(x)) = x$ states that $f(f^{-1}(x)) = x$, assuming an inverse of f exists.
- The seventh rule is not true in general since $c(x+y)$ is $cf(g(x))$ and $cx+cy$ is $cf(cg(x))$ which are not the same in general.
- The eighth rule is not true since the left hand side $(c_1 + c_2)x$ is interpreted as $(c_1 + c_2)f(x)$, while the right hand side $c_1x + c_2x$ is interpreted as $c_1f(c_2f(x))$ which are not equal in general.

Problem 9

Part (a): Let the vector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For $c \geq 0$ and $d \geq 0$. Then this set is the upper right corner in the first quadrant of the xy plane. Now note that the sum of any two vectors in this set will also be in this set but scalar multiples of a vector in this set may not be in this set. Consider

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix},$$

which is not be in the set.

Part (b): Let the set consist of the x and y axis (all the points on them). Then for any point x on the axis cx is also on the axis but the point $x + y$ will almost certainly not be.

Problem 10

Part (a): Yes

Part (b): No, since $c(b_1, b_2, b_3) = c(1, b_2, b_3)$ is not in the set if $c = \frac{1}{2}$.

Part (c): No, since if two vectors x and y are such that $x_1x_2x_3 = 0$ and $y_1y_2y_3 = 0$ there is no guarantee that $x + y$ will have that property. Consider

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Part (d): Yes, this is a subspace.

Part (e): Yes, this is a subspace.

Part (f): No this is not a subspace since if

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

has this property then cb should have this property but $cb_1 \leq cb_2 \leq cb_3$ might not be true. Consider

$$b = \begin{bmatrix} -100 \\ -10 \\ -1 \end{bmatrix} \quad \text{and} \quad c = -1\,.$$

Then $b_1 \leq b_2 \leq b_3$ but $cb_1 \leq cb_2 \leq cb_3$ is not true.

Problem 11

Part (a): All matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

for all $a, b \in \mathbb{R}$.

Part (b): All matrices of the form

$$\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$$

for all $a \in \mathbb{R}$.

Part (c): All matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

or diagonal matrices.

Problem 12

Let the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$, then $v_1 + v_2 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$ but $5 + 1 - 2(-2) = 10 \neq 4$ so the sum is not on the plane.

Problem 13

The plane parallel to the previous plane P is $x+y-2z=0$. Let the vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}$, which are both on P_0. Then $v_1 + v_2 = \begin{bmatrix} 2 \\ 1 \\ \frac{3}{2} \end{bmatrix}$. We then check that this point is on our plane by computing the required sum. We find that $2+1-2\left(\frac{3}{2}\right)=0$, and see that it is true.

Problem 14

Part (a): Lines, $\mathbb{R}^2$ itself, or $(0,0,0)$.

Part (b): $\mathbb{R}^4$ itself, hyperplanes of dimension four (one linear constraining equation among four variables) that goes through the origin like the following

$$ax_1 + bx_2 + cx_3 + dx_4 = 0\,.$$

Constraints involving two linear equation like toe above (going through the origin)

$$\begin{aligned} ax_1 + bx_2 + cx_3 + dx_4 &= 0 \\ Ax_1 + Bx_2 + Cx_3 + Dx_4 &= 0\,, \end{aligned}$$

which is effectively a two dimensional plane. In addition, constraints involving three equations like above and going through the origin (this is effectively a one dimensional line). Finally, the origin itself.

Problem 15

Part (a): A line.

Part (b): A point $(0,0,0)$.

Part (c): Let x and y be elements of $S \cap T$. Then $x+y \in S \cap T$ and $cx \in S \cap T$ since x and y are both in S and in T, which are both subspaces and therefore $x+y$ and cx are both in $S \cap T$.

Problem 16

A plane (if the line is in the plane to begin with) or all of $\mathbb{R}^3$.

Problem 17

Part (a): Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which are both invertible. Now $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which is not. Thus the set of invertible matrices is not a subspace.

Part (b): Let

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix},$$

which are both singular. Now $A+B = \begin{bmatrix} 7 & 6 \\ 4 & 6 \end{bmatrix}$, which is not singular, showing that the set of invertible matrices is not a subspace.

Problem 18

Part (a): True, since if A and B are symmetric then $(A+B)^T = A^T + B^T = A + B$ is symmetric. Also $(cA)^T = cA^T = cA$ is symmetric.

Part (b): True, since if A and B are skew symmetric then $(A+B)^T = A^T + B^T = -A-B = -(A+b)$ and $A+B$ is skew symmetric. Also if A is skew symmetric then cA is also since $(cA)^T = cA^T = -cA$.

Part (c): False since if $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ which is unsymmetric and $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, which is also unsymmetric then $A + B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ should be unsymmetric but its not. Thus the set of unsymmetric matrices is not closed under addition and therefore is not a subspace.

Problem 19

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, then the column space is given by

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 0 \\ 0 \end{bmatrix},$$

which is a *line* in the x-axis (i.e. all combinations of elements on the x-axis. If $B = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$ then the column space of B is $\begin{bmatrix} x_1 \\ 2x_2 \\ 0 \end{bmatrix}$ or the entire xy plane. If $C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}$ then Cx is given by $\begin{bmatrix} x_1 \\ 2x_2 \\ 0 \end{bmatrix}$ or a *line* in the xy plane.

Problem 20

Part (a): Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right]$$

Let E_{21} be given by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

Then we find that

$$E_{21} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b + 1 \end{array}\right],$$

so that $b_2 = 2b_1$ and $b_1 = -b_3$.

Part (b):

$$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let E_{21} and E_{31} be given by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

Then we see that

$$E_{31}E_{21} \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b + 1 \end{array}\right],$$

which requires that $b_1 + b_3 = 0$, or $b_1 = -b_3$.

Problem 21

A combination of the columns of B and C are also a combination of the columns of A. Those two matrices have the same column *span*.

Problem 22

For the first system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

we see that for any values of b the system will have a solution. For the second system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3 \end{array}\right]$$

we see that we must have $b_3 = 0$. For the third system

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

which is equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & b_3 - b_2 \end{array}\right],$$

so we must have $b_2 = b_3$.

Problem 23

Unless b is a combination of the previous columns of A. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ has a large column space. But if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $b = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ the column space does not change. Because b can be written as a linear combination of the columns of A and therefore adds no new information to the column space.

Problem 24

The column space of AB is contained in the and possibly equals the column space of A. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ which is of a smaller dimension than the original column space of A.

Problem 25

If $z = x + y$ is a solution to $Az = b + b^*$. If b and b^* are in the column space of A then so is $b + b^*$.

Problem 26

Any A that is a five by five invertible matrix has $\mathbb{R}^5$ as its column space. Since $Ax = b$ always has a solution then A is invertible.

Problem 27

Part (a): False. Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ then $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are each *not* in the column space but $x_1 + x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the column space. Thus the set of vectors not in the column space is not a subspace.

Part (b): True.

Part (c): True.

Part (d): False, the matrix I can add a full set of pivots (linearly independent rows). Let $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, with $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then A has a column space consisting of the zero vector and

$$A - I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

has all of $\mathbb{R}^2$ as its column space.

Problem 28

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Section 3.2

Problem 1

Fpr the matrix (a) i.e

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

let E_{21} be given by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that

$$E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

Now let E_{33} be given by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

So that

$$E_{33}E_{21}A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Which has pivot variables x_1 and x_3 and free variables x_2, x_4 and x_5. For the matrix (b)

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

let E_{32} be given by

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix},$$

so that

$$E_{32}A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix} = U.$$

Then the free variables are x_3 and the pivot variables are x_1 and x_2.

Problem 2

Since the ordinary echelon form for the matrix in (a) is

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we find a special solution that corresponds to each free vector by assigning ones to each free variable in turn and then solving for the pivot variables. For example, since the free variables are x_2, x_4, and x_5 we begin by letting $x_2 = 1$, $x_4 = 0$, and $x_5 = 0$. Then our system becomes

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \\ 0 \\ 0 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

which has a solution $x_3 = 0$ and $x_1 = -2$. So our special solution in this case is given by

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For the next special solution let $x_2 = 0$, $x_4 = 1$, and $x_5 = 0$. Then our special solution solves

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 1 \\ 0 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Which requires $x_3 = -2$ and $x_1 + 2(-2) = -4$ or $x_1 = 0$. Then our second special solution is given by

$$\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Our final special solution is obtained by setting $x_2 = 0$, $x_4 = 0$, and $x_5 = 1$. Then our system is

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \\ 1 \end{bmatrix} = 0$$

which reduces to solving

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$$

So that $x_3 = -3$ and $x_1 = -6 - 2(-3) = 0$ is given by

$$\begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Lets check our calculations. Create a matrix N with columns consisting of the three special solutions found above. We have

$$N = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & -3 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

And then the product of A times N should be zero. We see that

$$AN = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & -3 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

as it should. For the matrix in part (b) we have that

$$U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

then the pivot variables are x_1 and x_2 while the free variables are x_3. Setting $x_3 = 1$ we obtain the system

$$\begin{bmatrix} 2 & 4 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix},$$

so that $x_2 = -1$ and $x_1 = \frac{-2-(4)(-1)}{2} = 1$, which gives a special solution of

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Problem 3

From Problem 2 we have three special solutions

$$v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

then *any* solution to $Ax = 0$ can be expressed as a linear combination of these special solutions. The nullsapce of A contains the vector $x = 0$ only when there are no free variables or there exist n pivot variables.

Problem 4

The reduced echelon form R has ones in the pivot columns of U. For Problem 1 (a) we have

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then let $E_{13} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that

$$E_{13}U = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv R$$

The nullspace of R is equal to the nullspace of U since row opperations don't change the nullspace. For Problem 1 (b) our matrix U is given by

$$U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

so let $E_{12} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so that

$$E_{12}U = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now let $D = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then

$$DE_{12}U = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 5

For Part (a) we have that

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix},$$

then letting $E_{21} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ we get that

$$E_{21}A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then since $E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ we have that

$$A = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Where we can define the first matrix on the right hand side of the above to be L. For Part (b) we have that

$$A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix},$$

then letting E_{21} be the same as before we see that

$$E_{21}A = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}.$$

so that a decoposition of A is given by

$$A = E_{21}^{-1}U = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}.$$

Problem 6

For Part (a) since we have that $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ so we see that x_1 is a pivot variable and x_2 and x_3 are free variables. Then two special solutions can be computed by setting $x_2 = 1$, $x_3 = 0$ and $x_2 = 0$, $x_3 = 1$ and solving for x_1. In the first case we have $-x_1 + 3 = 0$ or $x_1 = 3$ giving a special vector of

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

In the second case we have $-x_1 + 5 = 0$ giving $x_1 = 5$, so that the second special vector is given by

$$v_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Thus all special solutions to $Ax = 0$ are contained in the set

$$c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} .$$

For Part (b) since we have that $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ so we see that x_1 and x_3 are pivot variable while x_2 is a free variables. To solve for the vector in the nullspace set $x_2 = 1$ and solve for x_1 and x_3. This gives

$$\begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ 1 \\ x_3 \end{bmatrix} = 0 ,$$

or the system

$$\begin{bmatrix} -1 & 5 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} .$$

This gives $x_3 = 0$ and $x_1 = 3$. So we have a special vector given by

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} .$$

For an mxn matrix the number of free variables plus the number of pivot variables equals n.

Problem 7

For Part (a) the nullspace of A are all points (x, y, z) such that

$$\begin{bmatrix} 3c_1 + 5c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ,$$

or the plane $x = 3y + 5z$. This is a plane in the xyz space. This space can also be described as all possible linear combinations of the two vectors

$$\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} .$$

For Part (b) the nullspace of A are all points that are multiples of the vector $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ which is a line in $\mathbb{R}^3$. Equating this vector to a point (x, y, z) we see that our line is given by $x = 3c$, $y = c$, and $z = 0$ or equivalently $x = 3y$ and $z = 0$.

Problem 8

For Part (a) since we have that $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}$. Let $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then we have that

$$DU = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix},$$

which is in reduced row echelon form. The identity matrix in this case is simply the scalar 1 giving

$$DU = \begin{bmatrix} \begin{bmatrix} 1 \end{bmatrix} & -3 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

where we have put a box around the "identity" in this case. For Part (b) since we have that $U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}$ so that defining $D = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$ we then have that

$$DU = \begin{bmatrix} 1 & -3 & -5 \\ 0 & 0 & 1 \end{bmatrix}.$$

The let $E_{13} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$ and we then get that

$$E_{13}DU = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for our reduced row echelon form. Our box around the identinty in the matrix R is around the pivot rows and pivot columns and is given by

$$\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{matrix} -3 \\ 0 \end{matrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

Problem 9

Part (a): False. This depends on what the reduced echelon matrix looks like. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then the reduced echelon matrix R is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, which has x_2 as a free variable.

Part (b): True. An invertible matrix is defined as one that has a complete set of pivots i.e. no free variables.

Part (c): True. Since the number of free variables plus the number of pivot variables equals n in the case of no free variables we have the maximal number of pivot variables n.

Part (d): True. If $m \geq n$, then by Part (c) the number of pivot variables must be less than n and this is equivalent to less than m. If $m < n$ then we have fewer equations than unknowns and when our linear system is reduced to echelon form we have a maximal set of pivot variables. We can have at most m, corresponding to the block identity in the reduced row echelon form in the mxm position. The remaining $n - m$ variables must be free.

Problem 10

Part (a): This is not possible since going from A to U involves zeroing elements below the diagonal only. Thus if an element is nonzero above the diagonal it will stay so for all elimination steps.

Part (b): The real requirement to find a matrix A is that A have three linearly independent columns/rows. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -3 \\ -1 & -2 & -2 \end{bmatrix}$, then with $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ we find that

$$EA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Continuing this process let $E' = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

$$E'EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I\,.$$

Part (c): This is not possible and the reason is as follows. R must have zeros above each of its pivot variables. What about the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$ which has no zero entries. Then

$$U = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which also equals R.

Part (d): If $A = U = 2R$, then $R = \frac{1}{2}A = \frac{1}{2}U$ so let

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{2}A = \frac{1}{2}U\,.$$

so take $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Problem 11

Part (a): Consider

$$\begin{bmatrix} 0 & 1 & x & x & x & x & x \\ 0 & 0 & 0 & 1 & 0 & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Part (b): Consider

$$\begin{bmatrix} 1 & x & 0 & x & x & 0 & 0 \\ 0 & 0 & 1 & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Part (c): Consider

$$\begin{bmatrix} 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Problem 12

Part (a): Consider

$$R = \begin{bmatrix} 0 & 1 & x & x & x & x & x & x \\ 0 & 0 & 0 & 1 & x & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 1 & x & x \end{bmatrix},$$

this is so that the pivot variables are x_2, x_4, x_5, and x_6. For the free variables to be x_2, x_4, x_5, and x_6 we we have

$$R = \begin{bmatrix} 1 & x & 0 & x & x & x & 0 & 0 \\ 0 & 0 & 1 & x & x & x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

Part (b): Consider

$$R = \begin{bmatrix} 0 & 1 & x & 0 & 0 & x & x & x \\ 0 & 0 & 0 & 1 & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 1 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 13

x_4 is certainly a free variable and the special solution is $x = (0, 0, 0, 1, 0)$.

Problem 14

Then x_5 is a free variable. The special solution is $x = (1, 0, 0, 0, -1)$.

Problem 15

If an mxn matrix has r pivots the number of special solutions is $n-r$. The nullspace contains only zero when $r = n$. The column space is $\mathbb{R}^m$ when $r = m$.

Problem 16

When the matrix has five pivots. The column space is $\mathbb{R}^5$ when the matrix has five pivots. Since $m = n$ then Problem 15 demonstrates that the rank must equal $m = n$.

Problem 17

If $A = \begin{bmatrix} 1 & -3 & -1 \end{bmatrix}$ and $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the free variables are y and z. Let $y = 1$ and $z = 0$ then $x = 3$ giving the first special solution of $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$. The second special solution is given by setting $y = 0$ and $z = 1$, then $x - 1 = 0$ or $x = 1$, so we have a second special solution of $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Problem 18

If $x - 3y - z = 12$, then expressing the vector (x, y, z) in iterms of y and z we find

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 19

For x in the nullspace of B means that $Bx = 0$ thus $ABx = A0 = 0$ and thus x is in the nullspace of AB. The nullspace of B is *contained* in the nullspace of AB. An obvious example when the nullspace of AB is larger than the nullspace of B is when

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

which has a nullspace given by the span of the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and has a nullspace given by the span of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which is larger than the nullspace of B.

Problem 20

If A is invertible then the nullspace of AB equals the nullspace of B. If v is an element of the nullspace of AB then $ABv = 0$ of $Bv = 0$ by multiplying both sides by A^{-1}. Thus v is an element of the nullspace of B.

Problem 21

We see that x_3 and x_4 are free variables. To determine the special solutions we consider the two assignments $(x_3, x_4) = (1, 0)$, and $(x_3, x_4) = (0, 1)$. Under the first we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

which give

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = 0\,.$$

In the same way under the second assignment we have

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = 0\,.$$

when we combine these two results we find that

$$\begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0\,,$$

so that A is given by

$$A = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

Problem 22

If $x_4 = 1$ and the other variables are solved for we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} (1)$$

or

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

so that A is given by

$$A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Problem 23

We have three equations with a rank of two which means that the nullity must be one. Let $A = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix}$ for some a, b, and c. Then if $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is to be in the nullity of A we must have

$$A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & a \\ 1 & 3 & b \\ 5 & 1 & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+2a \\ 1+3+2b \\ 5+1+2c \end{bmatrix} = 0\,.$$

Which can be made true if we take $a = \frac{1}{2}$, $b = -2$, and $c = -3$. Thus our matrix A in this case is

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}.$$

Problem 24

The number of equations equals three and the rank is two. We are requiring that the nullspace be of dimension two (i.e. spanned by two linearly independent vectors), thus $m = 3$ and $n = 4$. But the dimension of the vectors in the null space is three which is not equal to four. Thus it is not possible to find a matrix with such properties.

Problem 25

We ask will the matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$, work? Then if the column space contains $(1,1,1)$ then $m = 3$. If the nullspace is $(1,1,1,1)$ then $n = 4$. Reducing A we see that

$$A \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

So if $Av = 0$, then

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

Implying that $x - w = 0$, $y - w = 0$, and $z - w = 0$, thus our vector v is given by

$$v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = w \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and our matrix A does indeed work.

Problem 26

A key to solving this problem is to recognize that if the column space of A is *also* its nullspace then $AA = 0$. This is because AA represents A acting on each column of A and this produces zero since the column space is the nullspace. Thus we need a matrix A such that $A^2 = 0$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the requirement of $A^2 = 0$ means that

$$\begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives four equations for the unknowns a,b,c, and d. To find one solution let $a = 1$ then $d = -1$ by considering the $(1,2)$ element. Our matrix equation then becomes

$$\begin{bmatrix} 1 + bc & 0 \\ 0 & cb + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now let $1 + bc = 0$, which we can satisfy if we take $b = 1$ and $c = -1$. Thus with all of these unknowns specified we have that our A is given by

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

r	n-r=3-r
1	2
2	1
3	0

Table 1: All possible combinations of the dimension of the column space and the row space for a three by three matrix.

We can check this result. It is clear that A's row space is spanned by $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$ and its nullity is given by computing the R matrix

$$R = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right],$$

giving $n = \left[\begin{array}{c} 1 \\ -1 \end{array}\right]$.

Problem 27

In a three by three matrix we have $m = 3$ and $n = 3$. If we say that the column space has dimension r the nullity must then have dimension $n - r$. Now r can be either 1, 2, or 3. If we consider each possibility in tern we have Table 1, from which we see that we never have the column space equal to the row space.

Problem 28

If $AB = 0$ then the column space of B is contained in the nullity of A. For example the product AB can be written by recognizing this as the action of A on the columns of B. For example

$$AB = A\left[b^1|b^2|\cdots|b^n\right] = \left[Ab^1|Ab^2|\cdots|Ab^n\right] = 0\,,$$

which means that $Ab^i = 0$ for each i. Let $A = \left[\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right]$ which has nullity given by the span of $\left[\begin{array}{c} 1 \\ 1 \end{array}\right]$. Next consider $B = \left[\begin{array}{cc} 1 & 2 \\ 1 & 2 \end{array}\right]$. From which we see that $AB = 0$.

Problem 29

Almost sure to be the identity. With a random four by three matrix one is most likely to end with

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problem 30

Part (a): Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ then A has $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as its nullspace, but $A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as its nullspace.

Part (b): Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, then x_2 is a free variable. Now

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which has *no* free variables. A similar case happens with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

Then A has x_2 as a free variable and A^T has x_3 as a free variable.

Part (c): let A be given by

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which has x_1 and x_2 as pivot columns. While

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix},$$

has x_1 and x_3 as pivot columns.

Problem 31

If $A = [II]$, then the nullspace for A is $\begin{bmatrix} I \\ -I \end{bmatrix}$. If $B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix}$, then the nullspace for B is $\begin{bmatrix} I \\ -I \end{bmatrix}$. If $C = I$, then the nullspace for C is 0.

Problem 32

$x = (2, 1, 0, 1)$ is four dimensional so $n = 4$. The nullspace is a single vector so $n - r = 1$ or $4 - r = 1$ giving that $r = 3$ so we have three pivots appear.

Problem 33

We must have $RN = 0$. If $N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, then let $R = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$. The nullity has dimension of two and $n = 3$ therefore using $n - r = 2$, we see that $r = 1$. Thus we have only one nonzero in R. If $N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ the nullity is of dimension one and $n = 3$ so from $n - r = 1$ we conclude that $r = 2$. Therefore we have two nonzero rows in R.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

If $N = []$, we assume that this means that the nullity is the zero vector only. Thus the nullity is of dimension zero and $n = 3$ still so $n - r = 0$ means that $r = 3$ and have three nonzero rows in R

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 34

Part (a):

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Part (b):

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}.$$

They are all in reduced row echelon form.

Section 3.3 (The Rank and the Row Reduced Form)

Problem 1

Part (a): True

Part (b): False

Part (c): True

Part (d): False

Problem 5

If $R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then B is the rxr identity matrix, $C = D = 0$ and A is a r by $n - r$ matrix of zeros, since if it was not we would make pivot variables from them. The nullspace is given by $N = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

Problem 13

Using the expression proved in Problem 12 in this section we have that

$$\text{rank}(AB) \leq \text{rank}(A).$$

By replacing A with B^T, and B with A^T in the above we have that

$$\text{rank}(B^T A^T) \leq \text{rank}(A^T).$$

Now since transposes don't affect the value of the rank i.e. $\text{rank}(A^T) = \text{rank}(A)$, by the above we have that

$$\text{rank}(B^T A^T) = \text{rank}((AB)^T) = \text{rank}(AB) \leq \text{rank}(A^T) = \text{rank}(A)$$

proving the desired equivalence.

Problem 14

From problem 12 in this section we have that $\text{rank}(AB) \leq \text{rank}(A)$ but $AB = I$ so

$$\text{rank}(AB) = \text{I} = n$$

therefore we have that $n \leq \text{rank}(A)$, so equality must hold or $\text{rank}(A) = n$. A then is invertible and B must be its two sided inverse i.e. $BA = I$.

Problem 15

From problem 12 in this section we know that $\text{rank}(AB) \leq \text{rank}(A) \leq 2$, since A is 2x3. This means that BA cannot equal the identity matrix I, which has rank 3. An example of such matrices are

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then BA is

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I\,.$$

Problem 16

Part (a): Since R is the same for both A and B we have

$$\begin{aligned} A &= E_1^{-1} R \\ B &= E_2^{-1} R \end{aligned}$$

for two elementary elimination matrices E_1 and E_2. Now the nullspace of A is equivalent to the nullspace of R (they are related by an invertible matrix E_1), thus A and R have the same nullspace. This can be seen to be true by the following argument. If x is in the nullspace of A then

$$Ax = 0 = E_1^{-1} R x$$

so multiplying by E_1 on the left we have

$$Rx = E_1 0 = 0$$

proving that x is in the nullspace of R. In the same way if x is in the nullspace of R it must be in the nullspace of A. Therefore

$$\text{nullspace}(A) = \text{nullspace}(B)$$

The fact that $E_1 A = R$ and $E_2 A = R$ imply that A and B have the same row space. This is because E_1 and E_2 perform invertible row operations and as such don't affect the span of the rows. Since

$$E_1 A = R = E_2 B$$

each matrix A and B has the same row space.

Part (b): Since $E_1 A = R = E_2 B$ we have that $A = E_1^{-1} E_2 B$ and A equals an invertible matrix times B.

Problem 17

We first find the rank of the matrix A,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

from which we can see that A has rank 2. The elimination matrices used in this process were

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \quad E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

so

$$E_{33} D E_{21} A = R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then A can be reconstructed as

$$\begin{aligned} A &= E_{21}^{-1} D^{-1} E_{33}^{-1} R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} R \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 8 & 8 \end{bmatrix} R = E^{-1} R \end{aligned}$$

Then A can be written by taking the first $r = 2$ *columns* of E^{-1} and the first $r = 2$ *rows* of R giving

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Our results we can check as follows

$$
\begin{aligned}
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix}
\end{aligned}
$$

The above is the sum of two rank one matrices. Now for $B = [A\, A]$, concatenating the matrix A in this way does not change the rank. Thus the (COL)(($ROW)^T$ decomposition would take the first $r = 2$ columns of E^{-1} with the first $r = 2$ rows of R . When we concatenate matrices like this we find the reduced row echelon form for B to be that for A concatenated i.e.

$$R_B = [R\, R]\,,$$

and the elimination matrix is the *same*. Thus our two columns of E^{-1} are the same

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix}$$

and our two rows of R_B are the concatenation of the two rows in R or

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

As before one can verify that

$$[A\, A] = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Section 3.4 (The Complete Solution to $Ax = b$)

Problem 1

Let our augmented matrix A be,

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ 1 & -3 & 3 & 5 \end{array}\right]$$

then with

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

we have

$$E_{21}A = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right]$$

continuing by dividing by the appropriate pivots and eliminating the elements below and above each pivot we have

$$E_{21}A = \left[\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this expression we recognize the pivot variables of x_1 and x_3. The particular solution is given by $x_1 = -5$, $x_2 = 0$, and $x_3 = 1$. A homogeneous solution, is given by setting the free variable x_2, equal to one and solving for the pivot variables x_1, and x_3. When $x_2 = 1$ we have the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \end{bmatrix},$$

so $x_1 = -8$ and $x_3 = 0$. Thus our total solution is given by

$$x = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -8 \\ 1 \\ 0 \end{bmatrix}$$

Problem 2

Our system is given by

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Let our augmented system be

$$\begin{aligned} [A|b] &= \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 1 \\ 0 & 0 & 2 & 1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ &\Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 2 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

Which we see has rank 2. Thus since $n = 4$ the dimension of the null space is 2. The pivot variables are x_1 and x_3, and the free variables are x_2 and x_4. A particular solution can be obtained by setting $x_2 = x_4 = 0$ and solving for x_1 and x_3. Performing this we have the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

so our particular solution is given by

$$x_p = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}.$$

Now we have two special solutions to find for $Ax = 0$.

Problem 10

Part (a): False. The combination $c_1 x_p + c_2 x_n$ is not a solution unless $c_1 = 1$. E.g.

$$A(c_1 x_p + c_2 x_n) = c_1 A x_p + c_2 A x_n = c_1 b \neq b$$

Part (b): False. The system $Ax = b$ has an infinite number of particular solutions (if A is invertible then there is only one solution). For a general A this particular solution corresponds to a point on the space obtained by assigning values to the free variables. Normally, the zero vector is assigned to the free variables to obtain one particular solution. Any other arbitrary vector maybe assigned in its place.

Part (c): False. Let our solution be constrained to lie on the line passing through the points $(0, 1)$ and $(-1, 0)$, given by the equation $x - y = -1$. As such consider the system

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix},$$

this matrix has the row reduced echelon form of

$$R = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

thus x is a pivot variable and y is a free variable. Setting the value of $y = 0$ gives the particular solution $x = -1$, which has norm $||x_p|| = 1$. A point on this line exists that is closer to the origin, however, consider

$$||x_p||^2 = x^2 + y^2 = x^2 + (x + 1)^2$$

or the norm of all points on the given line. To minimize this take the derivative with respect to x and set this expression equal to zero,

$$\frac{||x_p||^2}{dx} = 2x + 2(x + 1) = 0\,.$$

Which has a solution given by $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. Computing the norm at this point we have

$$||x_p||^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} < 1\,,$$

which is smaller than what was calculated before. Thus showing that selecting the free variables set to zero does not necessary give a minimum norm solution.

Part (d): False. The point $x_n = 0$ is always in the nullspace. It happens that if A is invertible $x = 0$ is the only element of the nullspace.

Section 3.6 (Dimensions of the Four Subspaces)

Problem 3 (from ER find basis for the four subspaces)

Since we are given A in the decomposition ER we can begin by reading the rank of A from R which we see is two since R has two independent rows. We also see that the pivot variables are x_2 and x_4 while the free variables are x_1, x_3, and x_5. Thus a basis for the column space is given by taking two linearly independent column vectors from A. For example, we can take

$$\left[\begin{array}{c} 1 \\ 1 \\ 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} 3 \\ 4 \\ 1 \end{array}\right],$$

as a basis for the column space. A basis for the row space is given by two linearly independent rows. Two easy rows to take are the first and the second. Thus we can take

$$\left[\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{array}\right],$$

as a basis for the row space. A basis for the nullspace is given by finding the special solution when the free variables are sequentially assigned ones and then solving for the pivot variables. For example our first element of the nullspace is given by letting $(x_1, x_3, x_5) = (1, 0, 0)$, and solving for (x_2, x_4). We find $x_2 = 0$ and $x_4 = 0$ giving the first element in the nullspace of

$$\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right].$$

Our second element of the nullspace is given by letting $(x_1, x_3, x_5) = (0, 1, 0)$, and solving for (x_2, x_4). We find $x_2 = -2$ and $x_4 = 0$ giving the second element in the nullspace of

$$\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, our third element of the nullspace is given by letting $(x_1, x_3, x_5) = (0, 0, 1)$, and solving for (x_2, x_4). We find $x_2 = 0$ and $x_4 = -1$ giving the third element in the nullspace of

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

These three vectors taken together comprise a basis for the nullspace. A basis for the left nullspace can be obtained by the last $m = 3$ minus $r = 2$ (or one) rows of E^{-1}. Since $E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, we have that $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ from which we see that the last row of E^{-1} is given by

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

We can check that this element is indeed in the left nullspace of A by computing $v^T A$. We find that

$$\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

as it should.

Problem 4

Part (a): The matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

has the two given vectors as a column space and since the row space is $\mathbb{R}^2$ both $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

Part (b): The rank is one ($r = 1$) and the dimension of the nullspace is one. Since the rank plus the dimension of the nullspace must be n we see that $n = 1 + 1 = 2$. The number of components in both the column space vectors and the nullspace vector is three, which is not equal to two, we see that this is not possible.

Part (c): The dimension of the nullspace $n - r$ equals one plus the dimension of the left nullspace or $1 + (m - r)$ which must be held constant. We see that we need a matrix with a rank of one, $m = 1$, and $n = 2$. Lets try the matrix

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Which has $m = 1$, $r = 1$, and $n = 2$ as required. The dimenion of the nullity is $2 - 1 = 1$ and the dimension of the left nullspace is $1 - 1 = 0$ as required, thus everything is satisfied.

Part (d): Consider

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ a & \\ b & \end{bmatrix} = \begin{bmatrix} 3 + 3a & 1 + 3b \end{bmatrix} = 0.$$

Thus $a = -1$ and $b = -\frac{1}{3}$ so the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & -\frac{1}{3} \end{bmatrix}$ satisfies the required conditions.

Part (e): If the row space equals the column space then $m = n$. Then since the dimension of the nullspace is $n - r$ and the dimension of the left nullspace is also $n - r$ then these two spaces have equal dimension and don't contain linearly independent rows (equivalently columns).

Problem 5

Let $V = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. For B to have V as its nullspace we must have

$$B \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad B \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 0.$$

Which imposes two constraints on B. We can let $B = \begin{bmatrix} 1 & a & b \end{bmatrix}$ then the first condition requires that

$$B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 + a + b = 0,$$

and the second constraint requires that

$$B \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 + a = 0,$$

or $a = -2$, which when used in the first constraint gives that $b = -(1 + a) = 1$. Thus our matrix B is given by

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 6

Now A has rank two, $m = 3$, and $n = 4$. The dimension of its column space is two. The dimension of its row space is two, the dimension of its nullspace is $n - r = 2$. The dimension of its left nullspace is $m - r = 3 - 2 = 1$. To find basis for each of these spaces we simply need to find enough linearly independent vectors. For the column space we can take the vectors

$$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}.$$

For the row space pick

$$\begin{bmatrix} 0 \\ 3 \\ 3 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

For the left nullspace pick

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For B we have $r = 1$, $m = 3$, and $n = 1$. The dimension of its column space is one. The dimension of its row space is one, the dimension of its nullspace is $n - r = 0$. The dimension of its left nullspace is $m - r = 2$. For the column space we can take a basis given by the span of

$$\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

For the row space pick

$$\begin{bmatrix} 1 \end{bmatrix}.$$

For the left nullspace pick the empty set (or only the zero vector). For the left nullspace pick

$$\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 7

For A we have $m = n = r = 3$ then the dimension of the column space is three and has a basis given by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The dimension of the row space is also three and has the same basis. The dimension of the nullspace is zero and contains on the zero vector. The dimension of the left nullspace is zero and contains only the zero vector.

For b we have $m = 3$, $n = 6$, and $r = 3$ then the dimension of the column space is three and has the same basis as above. The dimension of the row space is still three and has a basis given by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The dimension of the nullspace is $6 - 3 = 3$ and a basis can be obtained from

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

The dimension of the left nullspace is $m - r = 3 - 3 = 0$ and contains only the zero vector.

Problem 8

For A we have $m = 3$, $n = 3 + 2 = 5$, and $r = 3$. Thus

$$\begin{aligned} \dim((C)(A)) &= 3 \\ \dim((C)(A^T)) &= 3 \\ \dim((N)(A)) &= n - r = 5 - 3 = 2 \\ \dim((N)(A^T)) &= m - r = 0\,. \end{aligned}$$

For B we have $m = 3 + 2 = 5$, $n = 3 + 3 = 6$, and $r = 3$. Thus

$$\begin{aligned} \dim((C)(A)) &= 3 \\ \dim((C)(A^T)) &= 3 \\ \dim((N)(A)) &= n - r = 3 \\ \dim((N)(A^T)) &= m - r = 5 - 3 = 2\,. \end{aligned}$$

For C we have $m = 3$, $n = 2$, and $r = 0$. Thus

$$\begin{aligned}
\dim((C)(A)) &= 0 \\
\dim((C)(A^T)) &= 0 \\
\dim((N)(A)) &= n - r = 2 \\
\dim((N)(A^T)) &= m - r = 3\,.
\end{aligned}$$

Problem 9

Part (a): First lets consider the equivalence of the ranks. The rank of A alone is equivalent to the rank of $B \equiv \begin{bmatrix} A \\ A \end{bmatrix}$ because we can simply subtract each row of A from the corresponding newly introduced row in the concatenated matrix B. Effectively, this is applying the elementary transformation matrix

$$E = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix},$$

to the concatenated matrix $\begin{bmatrix} A \\ A \end{bmatrix}$ to produced $\begin{bmatrix} A \\ 0 \end{bmatrix}$. Now for the matrix $C \equiv \begin{bmatrix} A & A \\ A & A \end{bmatrix}$ we can again multiply by E above obtaining

$$EC = \begin{bmatrix} I & 0 \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} A & A \\ 0 & 0 \end{bmatrix}.$$

Continuing to perform row operations on the top half of this matrix we can obtain $\begin{bmatrix} R & R \\ 0 & 0 \end{bmatrix}$ where R is the reduced row echelon matrix for A. Since this has the same rank as R the composite matrix has the same rank as the original. If A is m by n then B is $2m$ by n and A and B have the same row space and the same nullity.

Part (b): If A is m by n then B is $2m$ by n and C is $2m$ by $2n$. Then B and C have the same column space and left nullspace.

Problem 10

If a matrix with $m = 3$ and $n = 3$ with random entries it is likely that the matrix will be non-singular so its rank will be three and

$$\begin{aligned}
\dim((C)(A)) &= 3 \\
\dim((C)(A^T)) &= 3 \\
\dim((N)(A)) &= 0 \\
\dim((N)(A^T)) &= 0\,.
\end{aligned}$$

If A is three by five then $m = 3$ and $n = 5$ it is more likely that $\dim((C)(A)) = 3$ and $\dim((C)(A^T)) = 3$, while $\dim((N)(A)) = n - r = 2$, and $\dim((N)(A^T)) = m - r = 3 - 3 = 0$.

Problem 11

Part (a): If there exits a right hand side with no solution then when we perform elementary row operations on A we are left with a row of zeros in R (or U) that does not have the corresponding zero elements in Eb. Thus $r < m$ (since we must have a row of zeros). As always $r \leq m$.

Part (b): Because letting y be composed of r zeros stacked atop vectors with ones in each component i.e. in the case $r = 2$ and $m = 4$ consider the vectors

$$y_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad y_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

Then $y_1^T R = 0$ and $y_2^T R = 0$ so that $y^T(EA) = 0$ or equivalently $(E^T y)^T A = 0$. Therefore $E^T y$ is a nonzero vector in the left nullspace. Alternatively if the left nullspace is nonempty it must have a nonzero vector. Since the left nullspace dimension is given by $m - r$ which we know is greater than zero we have the existence of a non-zero element.

Problem 12

Consider the matrix A which I construct by matrix multiplication as

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix} .$$

If $(1, 0, 1)$ and $(1, 2, 0)$ are a basis for the row space then $\dim(A^T) = 2 = r$. To also be a basis for the nullspace means that $n - r = 2$ implying that $n = 4$. But these are vectors in $\mathbb{R}^3$ resulting in a contradiction.

Problem 13

Part (a): False. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} .$$

Then the row space is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the column space by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which are different.

Part (b): True. $-A$ is a trivial linear transformation of A and as such cannot alter the subspaces.

Part (c): If A and B share the same four spaces then $E_1 A = R$ and $E_2 B = R$ and we see that A and B are related by a linear transformation i.e. $A = E_1^{-1} E_2^{-1} B$. As an example pick

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then the subspaces are the same but A is not a multiple of B.

Problem 14

The rank of A is three and a basis for the column space is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 26 \end{bmatrix},$$

and lastly

$$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \cdots$$

Equivalently the three by three block composing the first three pivots of U is invertible so that an additional basis can be taken from the standard basis. A basis for the row space of dimension three is given by

$$(1, 2, 3, 4), \quad (0, 1, 2, 3), \quad (0, 0, 1, 2).$$

Problem 15

The row space and the left nullspace will not change. If $v = (1, 2, 3, 4)$ is in the column space of the original matrix the vector in the column space of the new matrix is $(2, 1, 3, 4)$.

Problem 16

If $v = (1, 2, 3)$ was a row of A then when we multiply by v this row would give the product of

$$\begin{bmatrix} 1 & 2 & 3 \\ x & x & x \\ \vdots & \vdots & \vdots \\ x & x & x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1^2 + 2^2 + 3^2 \\ x \\ \vdots \\ x \end{bmatrix} = \begin{bmatrix} 14 \\ x \\ \vdots \\ x \end{bmatrix},$$

which cannot equal zero.

Problem 17

For the matrix A given by $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, our matrix is rank two. The column span is all vectors $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$, the row space is all vectors $\begin{bmatrix} 0 \\ y \\ z \end{bmatrix}$, the nullspace is all vectors $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$, and finally, the left nullspace is all vectors $\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}$. For the matrix $I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. The rank is three and the row space is given by all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, the column space is all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and the left nullspace and the nullspace both contain only the zero vector.

Problem 18

We have

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{bmatrix},$$

using the elimination matrix $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}$. This matrix then reduces to

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & -3 & b_3 - 7b_1 - 2(b_2 - 4b_1) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & -3 & b_3 - 2b_2 + b_1 \end{bmatrix},$$

using the elimination matrix $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. The combination of the rows that produce the zero row is given by one times row one, minus two times the second row, one times the third row. Thus the vector

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is in the null space of A^T. A vector in the nullspace is given by setting $x_3 = 1$ and solving for x_1 and x_2. This gives the equation $x_1 + 2(-2) + 3(1) = 0$ or $x_1 = 4 - 3 = 1$. The vector then is

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

which is the same vector space as the left nullspace.

Problem 19

Part (a): Reducing our matrix to upper triangular form we have

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} &\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & -2 & b_3 - 4b_1 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - 4b_1 - b_2 + 3b_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \\ 0 & 0 & b_3 - b_2 - b_1 \end{bmatrix}.
\end{aligned}$$

Thus the vector $(-1, -1, 1)$ is in the left nullspace which has a dimension given by $m - r = 3 - 2 = 1$.

Part (b): Reducing our matrix to upper triangular form we have

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix} &\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 1 & b_4 - 2b_1 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & b_4 - 2b_1 + b_2 - 2b_1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & b_1 \\ 0 & -1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 2b_1 \\ 0 & 0 & b_4 + b_2 - 4b_1 \end{bmatrix}.
\end{aligned}$$

Thus the vectors in the left nullspace are given by

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix},$$

which has a dimension of $m - r = 4 - 2 = 2$.

Problem 20

Part (a): We must have $Ux = 0$ which has two pivot variables x_1 and x_3 and free variables x_2 and x_4. To find the nullspace we set $(x_2, x_4) = (1, 0)$ and solve for x_1 and x_3. Thus we get $4x_1 + 2 = 0$ or $x_1 = -\frac{1}{2}$ which gives a vector in the nullspace of

$$\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -3 \\ 0 \end{bmatrix}.$$

Now setting $(x_2, x_4) = (0, 1)$ and solving for x_1 and x_3 we need to solve $4x_1 + 2(0) + 0 + 1 = 0$ or $x_3 = -3$ which gives a vector in the nullspace of

$$\begin{bmatrix} \frac{1}{4} \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Part (b): The number of independent solutions of $A^T y$ are given by $m - r = 3 - 2 = 1$

Part (c): The column space is spanned by

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}.$$

Problem 21

Part (a): The vectors u and w.

Part (b): The vectors v and z.

Part (c): u and w are multiples of each other or are linearly dependent or v and z are multiples of each other or are linearly dependent.

Part (d): $u = z = (1, 0, 0)$ and $v = w = (0, 0, 1)$. Then

$$uv^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and

$$wz^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So that

$$A = uv^T + wz^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which has rank two.

Problem 22

Consider A decomposed as

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 2 \\ 4 & 1 & 1 \end{bmatrix} \end{aligned}$$

Problem 23

A basis for the row space is $(3,0,3)$ and $(1,1,2)$ which are independent. A basis for the column space is given by $(1,4,2)$ and $(2,5,7)$ which are also independent. A is not invertible because it is the product of two rank two matrices and therefore $\text{rank}(AB) \leq \text{rank}(B) = 2$. To be invertible we must have $\text{rank}(AB) = 3$ which it is not.

Problem 24

d is in the span of its rows. The solution is unique when the left nullspace contains only the zero vector.

Problem 25

Part (a): A and A^T have the same number of pivots. This is because they have the same rank they must have the same number of pivots.

Part (b): False. Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, then $y^T = \begin{bmatrix} -2 & 1 \end{bmatrix}$ is in the left nullspace of A but

$$y^T A^T = \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -4 \end{bmatrix} \neq 0\,,$$

and therefore is not in the left nullspace of A^T.

Part (c): False. Pick an invertible matrix say of size m by m then the row and column spaces are the entirety of $\mathbb{R}^m$. It is easy to imagine an invertible matrix such that $A \neq A^T$. For example let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Part (d): True, since if $A^T = -A$ then the rows of A are the negative columns of A and therefore have exactly the same span.

Problem 26

The rows of C are combinations of the rows of B. The rank of C cannot be greater than the rank of B, so the rows of C^T are the rows of A^T, so the rank of C^T (which equals the rank of C) cannot be larger than the rank of A^T (which equals the rank of A).

Problem 27

To be of rank one the two rows must be multiples of each other and the two columns must be multiples of each other. To make the rows multiples of each other assume row two is a multiple (say k) of row one i.e. $ka = c$ and $kb = d$. Thus we have $k = \frac{c}{a}$ and therefore $d = \frac{c}{a}b$. A basis for the row space is then given by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ and a basis for the nullspace is given by

$$\begin{bmatrix} -\frac{b}{a} \\ 1 \end{bmatrix} \propto \begin{bmatrix} -b \\ a \end{bmatrix}.$$

Problem 28

The rank of B is two and has a basis of the row space given by the first two rows in its representation, The reduced row echelon matrix looks like

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which is obtained by $EA = R$ where E is given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 4 (Orthogonality)

Section 4.1 (Orthogonality of the Four Subspaces)

Problem 1

For this problem A is 2x3 so $m = 2$ and $n = 3$ and $r = 1$. Then the row and column space has dimension 1. The nullspace of A has size $n - r = 3 - 1 = 2$. The left nullspace of A has size $m - r = 2 - 1 = 1$.

Problem 2

For this problem $m = 3$, $n = 2$, and $r = 2$. So the dimension of the nullspace of A is given by $2 - 2 = 0$, and the dimension of the left nullspace of A is given by $3 - 2 = 1$. The two components of x are x_r which is all of $\mathbb{R}^2$ and x_n which is the zero vector.

Problem 3

Part (a): From the given formulation we have that $m = 3$ and $n = 3$, obtained from the size (number of elements) of the column and nullspace vectors respectively. Then $n - r = 3 - r = 1$, we have a $r = 2$. This matrix seems possible and to obtain it, consider a matrix A as

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix},$$

which will have the requested properties.

Part (b): From the definition of the vectors in the row space we have $m = 3$, and $r = 2$ since there are only two vectors in the row space. Then the size of the nullspace imply that $n - r = n - 2 = 1$, so $n = 3$. Having the dimensions worked out we remember that for all matrices, the row space must be orthogonal to the nullspace. Checking for consistency in this example we compute these inner products. First we have

$$\begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

which holds true but the second requirement

$$\begin{bmatrix} 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4,$$

is not equal to zero, so the required matrix is not possible.

Part (c): To see if this might be possible let x be in the nullspace of A. Then to also be perpendicular to the column space requires $A^T x = 0$. So A and A^T must have the same nullspace. This will trivially be true if A is symmetric. Also we know that A cannot be invertible since the nullspace for A and A^T would then be trivial, consisting of only the zero vector. So we can try for a potential A the following

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Then $\mathbf{N}(A) = \mathbf{N}(A^T)$ is given by the span of the vector

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

which by construction is perpendicular to every column in the column space of A.

Part (d): This is not possible since from the statements given the vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ must be an element of the left nullspace of our matrix A and as such is orthogonal to every element of the column space of A. If the column space of A contains the vector $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ then checking orthogonality we see that

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 6$$

and the two vector are *not* orthogonal.

Part (e): The fact that the columns of add to the zero column means that the vector of all ones must be in the nullspace of our matrix. We can see if a two by two matrix of this form exists. We first investigate if we can construct a 2x2 example matrix that has the desired properties. The first condition given is that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

or in equations

$$\begin{aligned} a + b &= 0 \\ c + d &= 0 \end{aligned}$$

The second condition is that

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} a + c & b + d \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}. \tag{3}$$

So our total system of requirements on our unknown 2x2 system A is given by

$$\begin{aligned} a+b &= 0 \\ c+d &= 0 \\ a+c &= 1 \\ b+d &= 1 \end{aligned}$$

which in matrix form is given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Performing row reduction on the augmented matrix we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Since the last two equations contradict each other, one can conclude that this is *not* possible. Another way to see this same result is to notice that a row of all ones will be in the nullspace but also in the row space. Since the only vector in *both* these spaces must be the zero vector, we have a contradiction, showing that no such matrix exists.

Problem 4 (can the row space contain the nullspace)

It is not possible for the row space to contain the nullspace. To show this let $x \neq 0$ be a member of both, then from the second fundamental theorem of linear algebra (that the row space and the nullspace are orthogonal) we have $x^T x = 0$, which is not true unless $x = 0$.

Problem 5

Part (a): We have that y is perpendicular to b, since b is in the column space of A and y is in the left nullspace.

Part (b): If $Ax = b$ has no solution, then b is *not* in the column space of A and therefore $yb^T \neq 0$ and y is not perpendicular to b.

Problem 6

If $x = x_r + x_n$, then $Ax = Ax_r + Ax_n = Ax_r + 0 = Ax_r$. So x is in the column space of A because Ax_r is a linear combination of the columns of A.

Problem 7

For Ax to be in the nullspace of A^T, it must be in the left nullspace of A. But Ax is in the column space of A and these two spaces are orthogonal. Because Ax is in *both* spaces it must be the zero vector.

Problem 8

Part (a): For any matrix A, the column space of is perpendicular to its left nullspace. By the symmetry of A the left nullspace of A is the same as its nullspace.

Part (b): If $Ax = 0$ and $Ax = 5z$, then $z^T A^T = 5z^T$ or $z^T Ax = 5z^T x$. Since $Ax = 0$, we have that $5z^T x = 0$ or $z^T x = 0$. In terms of subspaces, x is in the nullspace and the left nullspace of A, while z is in the column space of A and therefore since the column space and the left nullspace are perpendicular we must have that x and z perpendicular.

Problem 9

The matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix},$$

has rank one. A row space given by the span of $[1,2]^T$, a column space given by the span of $[1,3]^T$, a nullspace given by $[-2,1]^T$, and a left nullspace given by the span of $[-3,1]^T$. With these vectors Figure 4.2 from the book would look like that seen in Figure XXX. We can verify the mapping properties of the matrix A by selecting a nonzero component along the two orthogonal spaces spanning the domain of A (its row space and its nullspace). For example, take $x_n = [1,2]^T$, and $x_r = [2,-1]^T$, two be vectors in the nullspace and row space of A respectively then define $x \equiv x_n + x_r = [3,1]^T$. We compute that

$$Ax = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

and as required

$$Ax_n = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_r = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}.$$

The matrix

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix},$$

has rank one. A row space given by the span of $[1,0]^T$, a column space given by the span of $[1,3]^T$, a nullspace given by $[0,1]^T$, and finally a left nullspace given by the span of $[-3,1]^T$. With these vectors Figure 4.2 from the book would look like that seen in Figure XXX. We can verify the mapping properties of the matrix B by selecting a nonzero component along the two orthogonal spaces spanning the domain of B (its row space and its nullspace). For example, take $x_n = [0, 2]^T$, and $x_r = [1, 0]^T$, be two vectors in the nullspace and row space of B respectively then define $x \equiv x_n + x_r = [1, 2]^T$. We compute that

$$Bx = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and as required (the component in the direction of the nullspace contributes nothing)

$$Bx_n = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Bx_r = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Problem 10 (row and nullspaces)

The matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

has rank two. A row space given by the span of $[1, -1]^T$, a column space given by the span of $[1, 0, 0]^T$, a nullspace given by $[1, 1]^T$, and a left nullspace given by the span of $[0, 1, 0]^T$ and $[0, 0, 1]^T$. With these vectors Figure 4.2 from the book would look like that seen in Figure XXX. We can verify the mapping properties of the matrix A by considering the vector x provided. Since x has components along the two orthogonal spaces spanning the domain of A (its row space and its nullspace) we have, since $x_n = [1, 1]^T$, and $x_r = [1, -1]^T$. We compute that

$$Ax = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

and as required

$$Ax_n = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_r = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 11

Let $y \in \mathcal{N}(A^T)$, then $A^T y = 0$, now $y^T A x = (y^T A x)^T$, since $y^T A x$ is a scalar and taking the transpose of a scalar does nothing. But we have that $(y^T A x)^T = x^T A^T y = x^T 0 = 0$, which proves that y is perpendicular to Ax.

Problem 12

The Fredholm alternative is the statement that exactly one of these two problems has a solution

- $Ax = b$
- $A^T y = 0$ such that $b^T y \neq 0$

In words this theorem can be stated that *either* b is in the column space of A or that there exists a vector in the left nullspace of A that is not orthogonal to b. To find an example where the second situation holds let

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then $Ax = b$ has no solution (since b is not in the column space of A). We can also show this by considering the augmented matrix $[A\, b]$ which is

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix},$$

since the last row is not all zeros, $Ax = b$ has no solution. For the second part of the Fredholm alternative, we desire to find a y such that $A^T y = 0$ and $b^T y \neq 0$. Now $A^T y$ is given by

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then we have that the vector y can be any multiple of the vector $[-2\, 1]^T$. Computing $b^T y$ we have $b^T y = 2(-2) + 1(1) = -3 \neq 0$, and therefore the vector $y = [-2, 1]^T$ is a solution to the second Fredholm's alternative.

Problem 13

If S is the subspace with only the zero vector then $S^\perp = \mathbb{R}^3$. If $S = \text{span}\{(1, 1, 1)\}$ then $S^\perp$ is all vectors y such that

$$y^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

or $y_1 + y_2 + y_3 = 0$. Equivalently the nullspace of the matrix A defined as

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

which has a nullspace given by the span of y_1 and y_2

$$y_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad y_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

If S is spanned by the two vectors $[2,0,0]^T$ and $[0,0,3]^T$, then $S^\perp$ consists of all vectors y such that

$$y^T \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 0 \quad \text{and} \quad y^T \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = 0$$

So $2y_1 = 0$ and $3y_3 = 0$ which imply that $y_1 = y_3 = 0$, giving $S^\perp = \text{span}\{[0,1,0]^T\}$.

Problem 14

$S^\perp$ is the nullspace of

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Therefore $S^\perp$ is a subspace of A even if S is not.

Problem 15

$L^\perp$ is the plane perpendicular to this line. Then $(L^\perp)^\perp$ is a line perpendicular to $L^\perp$, so $(L^\perp)^\perp$ is the same line as the original.

Problem 16

$V^\perp$ contains only the zero vector. Then $(V^\perp)^\perp$ contains all of $\mathbb{R}^4$, and $(V^\perp)^\perp$ is the same as V.

Problem 17

Suppose S is spanned by the vectors $[1,2,2,3]^T$ and $[1,3,3,2]^T$, then $S^\perp$ is spanned by the nullspace of the matrix A given by

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

Which has a nullspace given by selecting a basis for the free variables x_3 and x_4 and then solving for the pivot variables x_1 and x_2. Using the basis $[1,0]^T$ and $[0,1]^T$, if $x_3 = 1$, $x_4 = 0$, then $x_1 = 0$ and $x_2 = -1$, while if $x_3 = 0$ and $x_4 = 1$ then $x_1 = -5$ and $x_2 = 1$ and in vector form is spanned by

$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 18

If P is the plane given then $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ has this plane as its nullspace. Then $P^\perp$ are composed of the the elements of the left nullspace of A i.e. the nullspace of A^T. Since

$$A^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the nullspace of A^T equivalently $P^\perp$ is given by the span of the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Problem 19

We are asked to prove that if $S \subset V$ then $S^\perp \supset V^\perp$. To do this, let $y \in V^\perp$. Then for every element $x \in V$, we have $x^T y = 0$. But we can also say that for every element $x \in S$ it is also in V by the fact that S is a subspace of V and therefore $x^T y = 0$ so $y \in S^\perp$. Thus we have $V^\perp \subset S^\perp$.

Problem 20

The first column of A^{-1} is orthogonal to the span of the second through the last.

Problem 21 (mutually perpendicular column vectors)

$A^T A$ would be I.

Problem 22

$A^T A$ must be a diagonal matrix since it represents every column of A times every row of A. When the two columns are different the result is zero. When they are the same the norm (squared) of that column results.

Problem 23

The lines $3x+y=b_1$ and $6x+2y=b_2$ are parallel. They are the same line if $2b_1=b_2$. Then $[b_1,b_2]^T$ is perpendicular to the left nullspace of

$$A=\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$$

or $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Note we can check that this vector is an element of the left nullspace by computing

$$\begin{bmatrix} -2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = -2b_1+b_2 = -2b_1+2b_2 = 0$$

The nullspace of the matrix is the line $3x+y=0$. One vector in this nullspace is $[-1,3]^T$.

Problem 24

Part (a): As discussed in the book if two subspaces are orthogonal then they can only meet at the origin. But for the two planes given we have many intersections. To find them we solve the system given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0\,,$$

then the point (x,y,z) will be on both planes. Performing row reduction we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

so we see that $z=0$ and $x+y=0$, giving the fact that any vector that is a multiple of $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ is in both planes and these two spaces cannot be orthogonal.

Part (b): The two lines specified are described as the spans of the two vectors

$$\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

respectively. For their subspaces to be orthogonal, the subspace generating vectors must be orthogonal. In this case $\begin{bmatrix} 2 & 4 & 5 \end{bmatrix}^T \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = 2-12+10=0$ and they are orthogonal. We still need to show that they are not orthogonal components. To do so it suffices to find a vector orthogonal to one space that is not in the other space. Consider $\begin{bmatrix} 2 & 4 & 5 \end{bmatrix}$, which as a nullspace given by setting the free variables equal to a basis and solving for the pivot variables. Since the free variables are x_2 and x_3 we have a first vector in the nullspace given by setting $x_2 = 1$,$x_3 = 0$, which implies that $x_1 = -2$. Also setting $x_2 = 0$, $x_3 = 1$, we have that $x_1 = -\frac{5}{2}$, giving two vector of

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$$

Now consider the vector $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ it is orthogonal to $\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ and thus is in its orthogonal complement. This vector however is *not* in the span of $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. Thus the two spaces are not the orthogonal complement of each other.

Part (c): Consider the subspaces spanned by the vectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, respectively. They meet only at the origin but are *not* orthogonal.

Problem 25

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 7 \end{bmatrix},$$

then A has $[1\,,2\,,3]^T$ in both its row space and its nullspace. Let B be defined by

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 3 & -3 \end{bmatrix},$$

then B has $[1\,,2\,,3]^T$ in the column space of B and

$$B \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now v could not be both in the row space of A and in the nullspace of A. Also v could not both be in the column space of A and in the left nullspace of A. It could however be in the row space and the left nullspace or in the nullspace and the left nullspace.

Problem 26

A basis for the left nullspace of A.

Section 4.2 (Projections)

Problem 1 (simple projections)

Part (a): The coefficient of projection $\hat{x}$ is given by

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{1+2+2}{1+1+1} = \frac{5}{3}$$

so the projection is then

$$p = a\left(\frac{a^T b}{a^T a}\right) = \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the error e is given by

$$e = b - p = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

To check that e is perpendicular to a we compute $e^T a = \frac{1}{3}(-2+1+1) = 0$.

Part (b): The projection coefficient is given by

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{-1-9-1}{1+9+1} = -1\,.$$

so the projection p is then

$$p = \hat{x}a = -a = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

The error $e = b - p = 0$ is certainly orthogonal to a.

Problem 2 (drawing projections)

Part (a): Our projection is given by

$$p = \hat{x}a = \frac{a^T b}{a^T a}a = \cos(\theta)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ 0 \end{bmatrix}$$

Part (b): From Figure XXX of b onto a is zero. Algebraically we have

$$p = \hat{x}a = \frac{a^T b}{a^T a}a = \left(\frac{1-1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 3 (computing a projection matrix)

Part (a): The projection matrix P equals $P = \frac{aa^T}{a^Ta}$, which in this case is

$$P = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}{3} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

For this projection matrix note that

$$P^2 = \frac{1}{9} \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = P.$$

The requested product Pb is

$$Pb = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

Part (b): The projection matrix P equals $P = \frac{aa^T}{a^Ta}$, which in this case is

$$P = \frac{\begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \end{bmatrix}}{1+9+1} = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

For this projection matrix note that P^2 is given by

$$P^2 = \frac{1}{11^2} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} = \frac{1}{11^2} \begin{bmatrix} 11 & 33 & 11 \\ 33 & 99 & 33 \\ 11 & 33 & 11 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} = P.$$

The requested product Pb is then given by

$$Pb = \frac{1}{11} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 \\ 33 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Problem 4 (more calculations with projection matrices)

Part (a): Our first projection matrix is given by $P_1 = \frac{aa^T}{a^Ta}$ which in this case is

$$P_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Calculating P_1^2 we have that

$$P_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P_1 \,,$$

as required.

Part (b): Our second projection matrix is given by $P_2 = \frac{aa^T}{a^Ta}$ which in this case is

$$P_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Calculating P_2^2 we have that

$$P_2^2 = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = P_2 \,,$$

as required. In each case, P^2 should equal P because the action of the *second* application of our projection will not change the vector produced by the action of the *first* application of our projection matrix.

Problem 5 (more calculations with projection matrices)

We compute for the first project matrix P_1 that

$$P_1 = \frac{aa^T}{a^Ta} = \frac{1}{(1+4+4)}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} ,$$

and compute the second projection matrix P_2 by

$$P_2 = \frac{aa^T}{a^Ta} = \frac{1}{(4+4+1)}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & +1 \end{bmatrix} .$$

With these two we find that the product P_1P_2 is then given by

$$\begin{aligned} P_1P_2 &= \frac{1}{81}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & +1 \end{bmatrix} \\ &= \frac{1}{81}\begin{bmatrix} 4-8+4 & 4-8+4 & -2+4-2 \\ -8+16-8 & -8+16-8 & 4-8+4 \\ -8+16-8 & -8+16-8 & 4-8+4 \end{bmatrix} = 0 \,. \end{aligned}$$

An algebraic way to see this same result is to consider the multiplication of P_1 and P_2 in terms of the individual vectors i.e.

$$\begin{aligned} P_1P_2 &= \frac{a_1a_1^T}{a_1^Ta_1}\frac{a_2a_2^T}{a_2^Ta_2} \\ &= \frac{1}{a_1^Ta_1}\frac{1}{a_2^Ta_2}a_1a_1^Ta_2a_2^T \\ &= \frac{1}{a_1^Ta_1}\frac{1}{a_2^Ta_2}a_1(a_1^Ta_2)a_2^T = 0 \,, \end{aligned}$$

since for the vectors given we can easily compute that $a_1^T a_2 = 0$. Conceptually this result is expected since the vectors a_1 and a_2 are perpendicular and when we project a given vector onto a_1 we produce a vector that will still be *perpendicular* to a_2. Projecting this perpendicular vector onto a_2 will result in a zero vector.

Problem 6

From Problem 5 we have that P_1 given by

$$P_1 = \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \quad \text{so} \quad P_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and P_2 given by

$$P_2 = \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \quad \text{so} \quad P_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

and finally P_3 given by

$$\begin{aligned} P_3 &= \frac{a_3 a_3^T}{a_3^T a_3} = \frac{1}{4+1+4} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \\ &= \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} \quad \text{so} \quad P_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix}. \end{aligned}$$

Then we have that

$$p_1 + p_2 + p_3 = \frac{1}{9} \begin{bmatrix} 1+4+4 \\ -2+4-2 \\ -2-2+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We are projecting onto three orthogonal axis a_1, a_2, and a_3, since $a_3^T a_1 = -2-2+4 = 0$, $a_3^T a_2 = 4-2-2 = 0$, and $a_1^T a_2 = -2+4-2 = 0$.

Problem 7

From Problem 6 above we have that P_3 is given by

$$P_3 = \frac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$

So adding all three projection matrices we find that

$$P_1 + P_2 + P_3 = \frac{1}{9} \begin{bmatrix} 1+4+4 & -2+4-2 & -2-2+4 \\ -2+4-2 & 4+4+1 & 4-2-2 \\ -2-2+4 & 4-2-2 & 4+1+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as expected.

Problem 8

We have

$$\begin{aligned}
\hat{x}_1 &= \frac{a_1^T b}{a_1^T a_1} = 1 \quad \text{so} \quad p_1 = \hat{x}_1 a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\hat{x}_2 &= \frac{a_2^T b}{a_2^T a_2} = \frac{3}{5} \quad \text{so} \quad p_2 = \hat{x}_2 a_2 = \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{aligned}$$

This gives

$$p_1 + p_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Problem 9

The projection onto the plane a_1 and a_2 is the full $\mathbb{R}^2$ so the projection matrix is the identity I. Since A is a two by two matrix with linearly independent columns $A^T A$ is invertible. This product is given by

$$A^T A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix}$$

so that $(A^T A)^{-1}$ is given by

$$(A^T A)^{-1} = \frac{1}{4} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}.$$

The product $A(A^T A)^{-1} A^T$ can be computed. We have

$$\begin{aligned}
A(A^T A)^{-1} A^T &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 2 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = I,
\end{aligned}$$

as claimed.

Problem 10

When we project b onto a the coefficients are given by $\hat{x} = \frac{a^T b}{a^T a}$, so to project a_1 onto a_2 we would have coefficients and a projection given by

$$\begin{aligned}
\hat{x} &= \frac{a_2^T a_1}{a_2^T a_2} = \frac{1}{5} \\
p &= \hat{x} a_2 = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{aligned}$$

The projection matrix is given by $P_1 = \frac{a_2 a_2^T}{a_2^T a_2}$ and equals

$$P_1 = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

Then to project this vector back onto a_1 we obtain a coefficient and a projection given by

$$\begin{aligned} \hat{x} &= \frac{p^T a_1}{a_1^T a_1} = \frac{1}{5}\frac{1}{1} = \frac{1}{5} \\ \tilde{p} &= \hat{x} a_1 = \frac{1}{5}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

The projection matrix is given by $P_2 = \frac{a_1 a_1^T}{a_1^T a_1}$ and equals

$$P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

So that $P_2 P_1$ is given by

$$P_2 P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Which is not a projection matrix since it would have to be written proportional to a row which it can't be.

Problem 11

Remembering our projection theorems $A^T A\hat{x} = A^T b$ and $p = A\hat{x}$ we can evaluate the various parts of this problem.

Part (a): We find that $A^T A$ is given by

$$A^T A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix},$$

and $A^T b$ is given by

$$A^T b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

With this information the system for the coefficients $\hat{x}$ i.e. $A^T A\hat{x} = A^T b$ is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

which has a solution given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

so that $p = A\hat{x}$ is given by

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

With this projection vector we can compute its error. We find that $e = b - p$ is given by

$$e = b - p = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Part (b): We have for $A^T A$ the following

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}.$$

also we find that $A^T b$ is given by

$$A^T b = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}.$$

So that our system of normal equations $A^T A\hat{x} = A^T b$, becomes

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}.$$

This system has a solution given by

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 14 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

With these coefficients our projection vector p becomes

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

and our error vector $e = b - p$ is then given by

$$e = b - p = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = 0.$$

Problem 12

The projection matrix is given by $P_1 = A(A^T A)^{-1} A^T$. Computing P_1 we find that

$$\begin{aligned}
P_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

We can check that $P_1^2 = P_1$ as required by projection matrices. We have

$$P_1^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P_1 .$$

Now consider $P_1 b$ from which we have

$$P_1 b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

For the second part we again have $P_2 = A(A^T A)^{-1} A^T$, which is given by

$$\begin{aligned}
P_2 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\frac{1}{2} \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.
\end{aligned}$$

Then P_2^2 is given by

$$P_2^2 = \frac{1}{4} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = P_2 .$$

Now consider $P_1 b$ from which we have

$$P_2 b = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 8 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Problem 13

With $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, we will compute the projection matrix $A(A^TA)^{-1}A^T$. We begin by computing A^TA. We find that

$$A^TA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So P is four by four and we have that $Pb = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$.

Problem 14

Since b is in the span of the columns of A the projection will be b itself. Also $P \neq I$ since for vectors not in the column space of A their projection is not themselves. As an example let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix},$$

Then the projection matrix is given by $A(A^TA)^{-1}A^T$. Computing A^TA we find

$$A^TA = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}.$$

And then the inverse is given by

$$(A^TA)^{-1} = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}.$$

Which gives for the projection matrix the following

$$\begin{aligned} P &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} \left(\frac{1}{21} \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} \right) \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \\ &= \frac{1}{21} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 10 \\ 5 & 8 & -4 \end{bmatrix} \\ &= \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix} . \end{aligned}$$

So that $p = Pb$ is the given by

$$p = Pb = \frac{1}{21} \begin{bmatrix} 5 & 8 & -4 \\ 8 & 17 & 2 \\ -4 & 2 & 20 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 0 \\ 42 \\ 84 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = b \,.$$

Problem 15

The column space of $2A$ is the same as that of A, but $\hat{x}$ is not the same for A and $2A$ since $p_A = A\hat{x}$ and $p_{2A} = 2A\hat{x}$ while $p_A = p_{2A}$ since the column space of A and $2A$ are the same so the projections must be the same. Thus we have that

$$\hat{x}_A = 2\hat{x}_{2A} \,.$$

This can be seen by writing the equation for $\hat{x}_A$ and $\hat{x}_{2A}$ in terms of A. For example the equation for $\hat{x}_A$ is given by

$$A^T A \hat{x}_A = A^T b \,.$$

While that for $\hat{x}_{2A}$ is given by

$$4A^T A \hat{x}_{2A} = 2A^T b \,.$$

This latter equation is equivalent to $A^T A(2\hat{x}_{2A}) = A^T b$. Comparing this with the first equation we see that $\hat{x}_A = 2\hat{x}_{2A}$.

Problem 16

We desire to solve for $\hat{x}$ in $A^T A \hat{x} = A^T b$. With $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$ we have that

$$A^T A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} .$$

So that $\hat{x}$ is then given by

$$\begin{aligned} \hat{x} &= (A^T A)^{-1} A^T b \\ &= \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} . \end{aligned}$$

Problem 17 ($I - P$ is an idempotent matrix)

We have by expanding (and using the fact that $P^2 = P$) that

$$(I - P)^2 = (I - P)(I - P) = I - P - P + P^2 = I - 2P + P = I - P \,.$$

So when P projects onto the column space of A, $I - P$ projects onto the orthogonal complement of the column space of A. Or in other words $I - P$ projects onto the the left nullspace of A.

Problem 18 (developing an intuitive notion of projections)

Part (a): $I - P$ is the projection onto the vector spanned by $[-1, 1]^T$.

Part (b): $I - P$ is the projection onto the plane perpendicular to this line, i.e. $x + y + z = 0$. The projection matrix is derived from the column of

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

which has $x + y + z = 0$ as its left nullspace.

Problem 19 (computing the projection onto a given plane)

Consider the plane given by $x - y - 2z = 0$, by setting the free variables equal to a basis (i.e. $y = 1; z = 0$ and $y = 0; z = 1$) we derive the following two vectors in the nullspace

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} .$$

These are two vectors *in* the plane which we make into columns of A as

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with this definition we can compute $A^T A$ as

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} .$$

Then $(A^T A)^{-1}$ is given by

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} ,$$

and our projection matrix is then given by $P = A(A^T A)^{-1} A^T$ or

$$\begin{aligned} A(A^T A)^{-1} A^T &= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 1 & 2 \\ 5 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} . \end{aligned}$$

Problem 20 (computing the projection onto the same plane ... differently)

A vector perpendicular to the plane $x - y - 2z = 0$ is the vector

$$e = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

since then $e^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ for every x, y, and z in the plane. The projection onto this vector is given by

$$\begin{aligned} Q &= \frac{ee^T}{e^T e} \\ &= \frac{1}{1+1+4} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} . \end{aligned}$$

Using this result the projection onto the given plane is given by $I - Q$ or

$$\frac{1}{6} \begin{bmatrix} 6-1 & 1 & 2 \\ 1 & 6-1 & -2 \\ 2 & -2 & 6-4 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} ,$$

which is the same as computed earlier in Problem 19.

Problem 21 (projection matrices are idempotent)

If $P = A(A^T A)^{-1}A^T$ then

$$P^2 = (A(A^T A)^{-1}A^T)(A(A^T A)^{-1}A^T) = A(A^T A)^{-1}A^T = P\,.$$

Now Pb is in the column space of A and therefore its projection is *itself*.

Problem 22 (proving symmetry of the projection matrix)

Given the definition of the projection matrix $P = A(A^T A)^{-1}A^T$, we can compute its transpose directly as

$$P^T = (A(A^T A)^{-1}A^T)^T = A(A^T A)^{-T}A^T = A((A^T A)^T)^{-1}A^T = A(A^T A)^{-1}A^T\,.$$

which is the same definition as P proving that P is a symmetric matrix.

Problem 23

When A is invertible the span of its columns is equal to the entire space from which we are leaving i.e. $\mathbb{R}^n$, so the projection matrix should be the identity I. Therefore, since b is *in* $\mathbb{R}^n$ its projection into $\mathbb{R}^n$ must be itself. The error of this projection is then zero.

Problem 24

the nullspace of A^T is perpendicular to the column space $\mathcal{C}(A)$, by the second fundamental theorem of linear algebra. If $A^T b = 0$, the projection of b onto $\mathcal{C}(A)$ will be zero. From the expression for the projection matrix we can see that this is true because

$$Pb = A(A^T A)^{-1}A^T b = A(A^T A)^{-1}0 = 0\,.$$

Problem 25

The projection Pb fill the subspace S so S is the basis of P.

Problem 26

Since $A^2 = A$, we have that $A(A - I) = 0$. But since the rank of A is m, A is invertible we can therefore multiply both sides by A^{-1} to obtain $A - I = 0$ or $A = I$.

Problem 27

The vector Ax is in the nullspace of A^T. But Ax is always in the column space of A. To be in both spaces (since they are perpendicular) we must have $Ax = 0$.

Problem 28

From the information given Px is the second column of P. Then its length squared is given by $(Px)^T(Px) = x^T P^T Px = x^T P^2 x = x^T Px = p_{22}$, or the $(2, 2)$ element in P.

Section 4.3 (Least Squares Approximations)

Problem 1 (basic least squares concepts)

If our mathematical model of the relationship between b and t is a line given by $b = C + Dt$, then the four equations through the given points are given by

$$\begin{aligned} 0 &= C + D \cdot 0 \\ 8 &= C + D \cdot 1 \\ 8 &= C + D \cdot 3 \\ 20 &= C + D \cdot 4 \end{aligned}$$

If the measurements change to what is given in the text then we have

$$\begin{aligned} 1 &= C + D \cdot 0 \\ 5 &= C + D \cdot 1 \\ 13 &= C + D \cdot 3 \\ 17 &= C + D \cdot 4 \end{aligned}$$

Which has as an analytic solution given by $C = 1$ and $D = 4$.

Problem 2 (using the normal equations to solve a least squares problem)

For the b and the given points our matrix A is given by

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

The normal equations are given by $A^T A\hat{x} = A^T b$, or

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

or

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

which has as its solution $[C, D]^T = [1, 4]^T$. So the four heights with this $\hat{x}$ are given by

$$A\hat{x} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}.$$

With this solution by direct calculation the error vector $e = b - A\hat{x}$ is given by

$$e = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$$

The smallest possible value of $E = 1 + 9 + 25 + 9 = 44$.

Problem 3

From problem 2 we have $p = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix}$, so that $e = b - p$ is given by $e = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}$. Now consider $e^T A$ which is given by

$$e^T A = \begin{bmatrix} -1 & 3 & -5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

So the shortest distance is given by $||e|| = E = 44$.

Problem 4 (the calculus solution to the least squares problem)

We define $E = ||Ax - b||^2$ as

$$\begin{aligned} E &= (C + D \cdot 0 - 0)^2 + (C + D \cdot 1 - 8)^2 + (C + D \cdot 3 - 8)^2 \\ &+ (C + D \cdot 4 - 20)^2 \end{aligned}$$

so that taking derivatives of E we have

$$\begin{aligned}\frac{\partial E}{\partial C} &= 2(C + D \cdot 0 - 0) + 2(C + D \cdot 1 - 8) \\ &+ 2(C + D \cdot 3 - 8) + 2(C + D \cdot 4 - 20) \\ \frac{\partial E}{\partial D} &= 2(C + D \cdot 0 - 0) \cdot 0 + 2(C + D \cdot 1 - 8) \cdot 1 \\ &+ 2(C + D \cdot 3 - 8) \cdot 3 + 2(C + D \cdot 4 - 20) \cdot 4\,.\end{aligned}$$

where the strange notation used in taking the derivative above is to emphases the relationship between this procedure and the one obtained by using linear algebra. Setting each equation equal to zero and then dividing by two we have the following

$$\begin{aligned}&(C + D \cdot 0) + (C + D \cdot 1) + \\ &(C + D \cdot 3) + (C + D \cdot 4) = 0 + 8 + 8 + 20 = 36 \\ &(C + D \cdot 0) \cdot 0 + (C + D \cdot 1) \cdot 1 + \\ &(C + D \cdot 3) \cdot 3 + (C + D \cdot 4) \cdot 4 = 0 \cdot 0 + 8 \cdot 1 + 8 \cdot 3 + 20 \cdot 4 = 112\,.\end{aligned}$$

Grouping the unknowns C and D we have the following system

$$\begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}$$

Problem 5

The best horizontal line is given by the function $y = C$. By least squares the coefficient A is given by

$$A\hat{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} c = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

Which has normal equations given by $A^T Ax = A^T b$ or $4C = 16 + 20 = 36$, or $C = 9$. This gives an error of

$$e = b - A\hat{x} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} 9 = \begin{bmatrix} -9 \\ -1 \\ -1 \\ 11 \end{bmatrix}$$

Problem 6

We have $\hat{x} = \frac{a^T b}{a^T a} = \frac{8+8+20}{4} = 9$. Then

$$p = \hat{x}a = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$$

and

$$e = b - p = \begin{bmatrix} 0-9 \\ 8-9 \\ 8-9 \\ 20-9 \end{bmatrix}$$

so that $e^T a = \begin{bmatrix} -9 & -1 & -1 & +11 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$ as expected. Our error norm is given by $||e|| = ||b-p|| = \sqrt{81+1+1+121} = \sqrt{204}$.

Problem 7

For the case when $b = Dt$ our linear system is given by $A\hat{x} = b$ with $\hat{x} = [D]$ and

$$A = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 4 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.$$

With these definitions we have that $A^T A = [1+9+16] = [26]$, and $A^T b = [0+8+24+80] = [112]$, so that

$$\hat{x} = \frac{112}{26} = \frac{56}{13},$$

then Figure 1.9 (a) would look like

Problem 8

We have that

$$\hat{x} = \frac{a^T b}{a^T a} = \frac{0+8+24+80}{1+9+16} = \frac{56}{13}.$$

so that p is given by

$$p = \frac{56}{13} \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}.$$

In problems 1-4 the best line had coefficients $(C, D) = (1, 4)$, while in the combined problems 5-6 and 7-8 we found C and D given by $(C, D) = (9, \frac{56}{13})$. This is because $(1, 1, 1, 1)$ and $(0, 1, 3, 4)$ are *not* perpendicular.

Problem 9

Our matrix and right hand side in this case is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.$$

So the normal equations are given by

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix}.$$

and $A^T b$ is given by

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 9 & 16 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}.$$

In figure 4.9 (b) we are computing the best fit to the span of three vectors where "best" is measured in the least squared sense.

Problem 10

For the A given

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}.$$

The solution to the equation $Ax = b$ is given by performing Gaussian elimination on the augmented matrix $[A; b]$ as follows

$$\begin{aligned} [A;b] &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 27 & 8 \\ 1 & 4 & 16 & 64 & 20 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 3 & 9 & 27 & 8 \\ 0 & 4 & 16 & 64 & 20 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 12 & 60 & -12 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 8 \\ 0 & 0 & 6 & 24 & -16 \\ 0 & 0 & 0 & -84 & \text{XXX} \end{bmatrix}. \end{aligned}$$

Given

$$\hat{x} = \frac{1}{3} \begin{bmatrix} 0 \\ 47 \\ -28 \\ 5 \end{bmatrix},$$

then $p = b$ and $e = 0$.

Problem 11

Part (a): The best line is $1 + 4t$ so that $1 + 4\hat{t} = 1 + 4(2) = 9 = \hat{b}$

Part (b): The first normal equation is given by Equation 9 in the text and is given by

$$mC + \sum_i t_i \cdot D = \sum b_i \,,$$

by dividing by m gives the requested expression.

Problem 12

Part (a): For this problem we have $a^t a\hat{x} = a^t b$ given by

$$m\hat{x} = \sum_i b_i \,,$$

so $\hat{x}$ is then given by

$$\hat{x} = \frac{1}{m}\sum_i b_i \,,$$

or the mean of the b_i

Part (b): We have

$$e = b - \hat{x}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 - \hat{x} \\ b_2 - \hat{x} \\ \vdots \\ b_m - \hat{x} \end{bmatrix}$$

Then $||e|| = \sqrt{\sum_{i=1}^{m}(b_i - \hat{x})^2}$

Part (c): If $b = (1, 2, 6)^T$, then $\hat{x} = \frac{1}{3}(1 + 2 + 6) = 3$ and $p = (3, 3, 3)^T$, so the error e is given by

$$e = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} .$$

We can check $p^T e = 3(-2 - 1 + 3) = 0$ as it should. Computing our projection matrix P we have

$$P = \frac{aa^T}{a^T a} = \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} .$$

Problem 13

We will interpret this question as follows. For each instance the residual will be one of the values listed $(\pm 1, \pm 1, \pm 1)$. Considering $b - Ax = (\pm 1, \pm 1, \pm 1)$ we have by multiplying by $(A^T A)^{-1} A^T$ the following

$$(A^T A)^{-1} A^T (b - Ax) = (A^T A)^{-1} A^T b - (A^T A)^{-1} A^T A x = \hat{x} - x \,.$$

If the residual can equal any of the following vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

We first note that the average of all of these vectors is equal to zero. In the same way the action of $(A^T A)^{-1} A^T$ on each of these vectors would produce (each of the following should be multiplied by $1/3$)

$$3\,, -3\,, 1\,, 1\,, 1\,, -1\,, -1\,, -1\,,$$

which when summed gives zero.

Problem 14

Consider $(b - Ax)(b - Ax)^T$ and multiply by $(A^T A)^{-1} A^T$ on the left and $A(A^T A)^{-1}$ on the right, to obtain

$$(A^T A)^{-1} A^T (b - Ax)(b - Ax)^T A (A^T A)^{-1} \,.$$

Now since $B^T C = (C^T B)^T$ the above becomes remembering the definition of $\hat{x}$

$$\begin{aligned} (\hat{x} - x)(\left[A(A^T A)^{-1}\right]^T (b - Ax))^T &= (\hat{x} - x)((A^T A)^{-1} A^T (b - Ax))^T \\ &= (\hat{x} - x)(\hat{x} - x)^T \,. \end{aligned}$$

so that if the average of $(b - Ax)(b - Ax)^T$ is $\sigma^T I$ we have that the average of $(\hat{x} - x)(\hat{x} - x)^T$ is $(A^T A)^{-1} A^T (\sigma^2 I) A (A^T A)^{-1}$, to obtain $\sigma^2 (A^T A)^{-1} A^T A (A^T A)^{-1} = \sigma^2 (A^T A)^{-1}$.

Problem 15

The expected error $(\hat{x} - x)^2$ is $\sigma^2 (A^T A)^{-1} = \frac{\sigma^2}{m}$, so the variance drops significantly (as $O(1/m)$).

Problem 16

We have

$$\frac{1}{100} b_{100} + \frac{99}{100} \hat{x}_{99} = \frac{1}{100} \sum_i b_i \,.$$

Problem 17

Our equations are given by

$$\begin{aligned} 7 &= C + D(-1) \\ 7 &= C + D(1) \\ 21 &= C + D(2)\,. \end{aligned}$$

Which as a system of linear equations matrix are given by

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}.$$

The least squares solution is given by $A^T A x = A^T b$ which in this case simplify as follows

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} \quad \text{or} \\ \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} &= \begin{bmatrix} 35 \\ 42 \end{bmatrix}. \end{aligned}$$

Which gives for $[C, D]^T$ the following

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix},$$

so the linear line is $b = 9 + 4t$.

Problem 18

We have p given by

$$p = A\hat{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$$

that gives the values on the closest line. The error vector e is then given by

$$e = b - p = \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}.$$

Problem 19

Our matrix A is still given by $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, but now let $b = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}$, so that $\hat{x} = (A^T A)^{-1} A^T b = 0$. Each column of A is perpendicular to the error in the least squares solution and as such has $A^T b = 0$. Thus the projection is zero.

Problem 20

When $b = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix}$, we have

$$\begin{aligned} \hat{x} &= (A^T A)^{-1} A^T b \\ &= (A^T A)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} \\ &= (A^T A)^{-1} \begin{bmatrix} 35 \\ 42 \end{bmatrix} . \end{aligned}$$

Or inserting the value of $(A^T A)^{-1}$ we have

$$\hat{x} = \frac{1}{14} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix} .$$

Thus the closest line is given by $b = 9 + 4t$ and the error is given by

$$e = b - A\hat{x} = \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} - \begin{bmatrix} 5 \\ 13 \\ 17 \end{bmatrix} = 0\,.$$

Now $e = 0$ because this b is in the column space of A.

Problem 21 (the subspace containing the components of projections)

The error vector e must be perpendicular to the column space of A and therefore is in the left nullspace of A. The projection vector p must be in the column space of A, the projected basis $\hat{x}$ must be in the row space of A. The nullspace of A is the zero vector assuming that the columns of A are linearly independent which is generally true for least squares problems if $m > n$.

Problem 22

With A given by

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

we should form $A^T A\hat{x} = A^T b$ and solve for $\hat{x}$. Note that for this problem we have that $\sum t_i = 0$ and our line has coefficients given by

$$\begin{aligned} C &= \frac{1}{m}\sum_i b_i = \frac{1}{5}5 = 1 \\ D &= \frac{b_1 T_1 + \ldots + b_m T_m}{T_1^2 + T_2^2 + \ldots + T_m^2} = \frac{4(-2) + 2(-1) + -1(0) + 0(1) + 0(2)}{4+1+0+1+4} = \ldots . \end{aligned}$$

Then the least squares line is $C + Dt$.

Problem 23

With $P = (x, x, x)$ and $Q = (y, 3y, -1)$ then

$$||P - Q||^2 = (x-y)^2 + (x-3y)^2 + (x+1)^2 .$$

Then to find the minimum of this we set the x and y derivatives equal to zero

$$\begin{aligned} \frac{\partial ||P-Q||^2}{\partial x} &= 0 \\ \frac{\partial ||P-Q||^2}{\partial y} &= 0 , \end{aligned}$$

and solve for the unknowns x and y.

Problem 24

Now e is orthogonal to anything in the column space of A so that would be $p = A\hat{x}$, so $e^T p = 0$. We have for our error e the following

$$||e||^2 = (b-p)^T(b-p) = e^T(b-p) = e^T b = (b-p)^T b = b^T b - b^T p .$$

Problem 25

Since $||Ax - b||^2$ can be expressed as

$$\begin{aligned} ||Ax - b||^2 &= (Ax - b)^T(Ax - b) \\ &= (Ax)^T(Ax) - (Ax)^T b - b^T(Ax) + b^T b \\ &= ||Ax||^2 - 2b^T(Ax) + ||b||^2 . \end{aligned}$$

So the derivatives of $||Ax - b||^2$ will be zero when

$$2A^T Ax - 2A^T b = 0 ,$$

or

$$A^T Ax = A^T b .$$

These equations we recognized as the normal equations.

Section 4.4 (Orthogonal Bases and Gram-Schmidt)

Problem 1

Part (a): We check the dot product $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \neq 0$, and the second vector does not have norm equal to one so these vectors are only independent.

Part (b): We check the dot product $\begin{bmatrix} 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 0.4 \\ -0.3 \end{bmatrix} = 0.24 - 0.24 = 0$, so they are othogonal. The norm of each is given by

$$\begin{aligned} ||v_1|| &= \sqrt{0.36 + 0.64} = 1 \\ ||v_2|| &= \sqrt{0.16 + 0.09} = \sqrt{0.25} = 0.5\,. \end{aligned}$$

Part (c): Here we have that

$$v_1^T v_2 = -\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta) = 0\,,$$

and $||v_1|| = ||v_2|| = 1$ so the vectors are orthonormal.

Problem 2

We have

$$q_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad q_2 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

so that the matrix obtained by concatonating q_1 and q_2 as column is given by

$$Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$$

Then $Q^T Q$ is given by

$$Q^T Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the symmetric product QQ^T is given by

$$QQ^T = \frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

Problem 3

Part (a): Here $A^T A$ would be the three by three identity matrix times $4^2 = 16$.

Part (b): Here $A^T A$ would be

$$\begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Problem 4

Part (a): Let $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then QQ^T is given by

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Part (b): Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Part (c): Let the basis be composed of

$$\begin{bmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}$$

Problem 5

All vectors that lie in the plane must be in the nullspace of

$$A = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix},$$

which has a basis given by the span of v_1 and v_2 given by

$$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

These two vectors are not orthogonal. Now let w_1 be given by

$$w_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and $W_2 = v_2 - (v_2^T w_1)w_1$. Now as $v_2^T w_1 = \frac{1}{\sqrt{2}}2 = \sqrt{2}$ and $||w_1||^2 = 1$, we have the ratio above given by

$$\frac{(v_2^T w_1)}{||w_1||^2} w_1 = \sqrt{2}\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

So with this subcalculation we have W_2 given by

$$W_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore when we normalize we get w_2 equal to

$$w_2 = \frac{1}{||w_2||} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Problem 6

To show that a matrix Q is orthogonal we must show that $Q^T Q = I$. For the requested matrix $Q_1 Q_2$ consider the product $(Q_1 Q_2)^T (Q_1 Q_2)$. Since this is equal to $Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$, showing that $Q_1 Q_2$ is orthogonal.

Problem 7

The projection matrix P is given by $P = Q(Q^T Q)^{-1} Q^T = Q I^{-1} Q^T = Q Q^T$, so the projection onto b will be

$$p = Pb = QQ^T b = Q \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_m^T b \end{bmatrix} = (q_1^T b)q_1 + (q_2^T b)q_2 + \ldots + (q_m^T b)q_m$$

Problem 8

Part (a): For Q given by

$$Q = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \\ 0 & 0 \end{bmatrix}$$

we have

$$QQ^T = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then our projection matrix is given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that P^2 is then

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = P .$$

Part (b): Since $(QQ^T)(QQ^T) = QQ^TQQ^T = QQ^T$, we have that $P = QQ^T = (QQ^T)(QQ^T)$ so that P which equals QQ^T is the projection matrix onto the columns of the the matrix Q.

Problem 9 (orthonormal vectors are linearly independent)

Part (a): Assuming that $c_1q_1+c_2q_2+c_3q_3 = 0$ and taking the dot product of both sides with q_1 gives $c_1q_1^Tq_1 = 0$ implying that $c_1 = 0$. The same thing holds when we take the dot product with q_2 and q_3 showing that all c_i's must be zero and the q_i's are linearly independent.

Part (b): Defining $Q = [q_1\, q_2\, q_3]$, then to prove linearly dependence we are looking for an $x \neq 0$ such that $Qx = 0$. From $Qx = 0$ multiply on the left by Q^T to get $Q^TQx = 0$. Since $Q^TQ = I$ by the orthogonality of the q_i's we have that $x = 0$ showing that no nonzero x exists and the q_i's are linearly independent.

Problem 10

Part (a): To be in both planes we are looking for a variable $\begin{bmatrix} x \\ y \end{bmatrix}$ has

$$A = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}$$

Let $v_1 = \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \end{bmatrix}^T$ so that normalized we have

$$v_1 = \frac{1}{\sqrt{1+9+16+25+49}} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

then

$$\hat{v}_2 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \end{bmatrix} \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} \frac{1}{10^2} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

Normalizing we then have

$$v_2 = \frac{1}{\sqrt{49+9+16+25+1}} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}$$

Part (b): The vector closes to $[1\,,0\,,0\,,0\,,0]^T$ is given by $p = q_1(q_1^T b) + q_2(q_2^T b)$ or

$$\frac{1}{10} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} \frac{1}{10} + \frac{1}{10} \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix} \left(\frac{-7}{10} \right) = \frac{1}{50} \begin{bmatrix} 25 \\ -9 \\ -12 \\ 20 \\ 0 \end{bmatrix}.$$

Problem 11

This is $(q_1^T b)q_1 + (q_2^T b)q_2$.

Problem 12

Part (a): If the a_i's are orthogonal then $Ax = b$ is $[a_1\, a_2\, a_3]\, x = b$, and multiplying by A^T (which is the inverse of A) gives $A^T A x = A^T b$ or

$$x = \begin{bmatrix} a_1^T b \\ a_2^T b \\ a_3^T b \end{bmatrix}$$

Part (b): If the a's are orthogonal then

$$A^T A = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} \begin{bmatrix} a_1 a_2 a_3 \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & a_1^T a_3 \\ a_2^T a_1 & a_2^T a_2 & a_2^T a_3 \\ a_3^T a_1 & a_3^T a_2 & a_3^T a_3 \end{bmatrix} = \begin{bmatrix} a_1^T a_1 & 0 & 0 \\ 0 & a_2^T a_2 & 0 \\ 0 & 0 & a_3^T a_3 \end{bmatrix}$$

so from $A^T A x = A^T b = \begin{bmatrix} a_1^T b \\ a_2^T b \\ a_3^T b \end{bmatrix}$ we obtain

$$x = \begin{bmatrix} \frac{a_1^T b}{a_1^T a_1} \\ \frac{a_2^T b}{a_2^T a_2} \\ \frac{a_3^T b}{a_3^T a_3} \end{bmatrix}$$

Part (c): If the a's are independent then x_1 is the first row of A^{-1} times b.

Problem 13

We would let

$$\begin{aligned} A &= a \\ B &= b - \frac{a^T b}{a^T a} a = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \end{aligned}$$

We need to subtract two times a to make the result orthogonal to a.

Problem 14

We have

$$\begin{aligned} q_1 &= \frac{a}{||a||} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ q_2 &= \frac{B}{||B||} = \frac{1}{\sqrt{4+4}} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

Then we have

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & q_1^T b \\ 0 & 2\sqrt{2} \end{bmatrix}$$

with $q_1^T b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \frac{4}{\sqrt{2}}$, which implies that the above matrix decomposition is given by

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 4/\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix}.$$

We can check this result by multiplying the above matrices together. Performing the multiplication of the two matrices on the right together we have

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 4/\sqrt{2} \\ 0 & 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 4/2+2 \\ 1 & 4/2-2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix},$$

verifying the decomposition.

Problem 15

Part (a): With the matrix A given by

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$$

we will let $a = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$, so that $q_1 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$. Now let $b = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, then B is given by

$$\begin{aligned} B &= b - \frac{a^T b}{a^T a} a \\ &= \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} - \frac{(1-2-8)}{(1+4+4)} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Then q_2 is the normalized version of B and is given by

$$q_2 = \frac{B}{||B||} = \frac{1}{\sqrt{4+1+4}} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

Now to compute q_3 we pick a third vector say $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, that is linearly independent from a and b, we then have

$$\begin{aligned} C &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{c^T a}{a^T a} a - \frac{c^T b}{b^T b} b \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{18}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

Which gives for q_3 the following

$$q_3 = \frac{1}{\sqrt{26}} \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

Part (b): q_3 must be orthogonal to the columns and therefore is in the left nullspace.

Part (c): We have that p is given by

$$\begin{aligned} p &= \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 7 \end{bmatrix}^T \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \\ &= 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \end{aligned}$$

is one method, another would be by solving the normal equations $A^T A\hat{x} = A^T b$ which in this case turn out to be

$$A^T A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix} = 9 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} .$$

and $A^T b$ is given by

$$A^T b = \begin{bmatrix} 1+4-14 \\ 1-2+28 \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \end{bmatrix}$$

Then $\hat{x}$ is given by

$$\hat{x} = \frac{1}{9}\frac{1}{(2-1)} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 27 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 19 \\ 14 \end{bmatrix}$$

Problem 16

Find the projection of b onto a. We have that our coefficient $\hat{x}$ is given by

$$\hat{x} = \frac{b^T a}{a^T a} = \frac{4+10}{16+25+4+4} = \frac{2}{7} .$$

To find orthonormal vectors let

$$q_1 = \frac{1}{\sqrt{16+25+4+4}} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} .$$

and define B to be

$$B = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{b^T a}{a^T a} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{14}{49} \begin{bmatrix} 4 \\ 5 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} .$$

Normalizing this vector we then have

$$q_2 = \frac{1}{\sqrt{1+3(16)}} \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix} = \frac{1}{4\sqrt{3}} \begin{bmatrix} -1 \\ 4 \\ -4 \\ -4 \end{bmatrix}$$

Problem 17

We have

$$p = \frac{b^T a}{a^T a} a = \frac{1+3+5}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

with an error given by

$$e = b - p = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} .$$

Normalizing we have

$$q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad q_2 = \frac{1}{2} \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

Problem 18

If $A = QR$ then $A^T A = (R^T Q^T)(QR) = R^T R$ which we recognize as a lower triangular matrix times a upper triangular matrix. Therefore Gram-Schmidt on A corresponds to elimination on $A^T A$. If A is as given in this problem then

$$A^T A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix} ,$$

which reduces as

$$A^T A \Rightarrow \begin{bmatrix} 3 & 9 \\ 0 & 35 - 27 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 0 & 8 \end{bmatrix} .$$

Which has pivots equal to $||a||^2$ and $||e||^2$ respectively.

Problem 19

Part (a): True, since the inverse of an orthogonal matrix is its transpose.

Part (b): Yes, if Q has orthonormal columns then

$$||Qx||^2 = (Qx)^T (Qx) = x^T Q^T Q x = x^T x = ||x||^2 .$$

Problem 20

Let $q_1 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then B is given by

$$\begin{aligned} B &= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{(-2+1+3)}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}. \end{aligned}$$

so that $q_2 = \frac{B}{||B||}$ or

$$q_2 = \frac{1}{\sqrt{25+1+1+25}} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix} = \frac{1}{\sqrt{52}} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix}.$$

The projecting b onto the column space of A is equivalent to computing

$$\begin{aligned} p &= (q_1^T b) q_1 + (q_2^T b) q_2 \\ &= \frac{(-4-3+3)}{2} \left(\frac{1}{2} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{(20+3+3)}{\sqrt{52}} \left(\frac{1}{\sqrt{52}} \right) \begin{bmatrix} -5 \\ -1 \\ 1 \\ 5 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -7 \\ -3 \\ -1 \\ 3 \end{bmatrix}. \end{aligned}$$

So that the error vector $e = b - p$ is given by

$$e = b - p = \frac{1}{2} \begin{bmatrix} -8+7 \\ -6+3 \\ 6+1 \\ -3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 \\ -3 \\ 7 \\ -3 \end{bmatrix},$$

and then computing the inner product of e with each column of A we find (using Matlab notation that)

$$\begin{aligned} e^T A(:,1) &= \frac{1}{2}(-1-3+7-3) = 0 \quad \text{and} \\ e^T A(:,2) &= \frac{1}{2}(2+0+7-9) = 0\,, \end{aligned}$$

as required.

Problem 21

If $A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ so that $q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, we have B given by

$$\begin{aligned} B &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{A^T v}{A^T A} A \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{(1-1)}{A^T A} A \\ &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}. \end{aligned}$$

The next vector C is given by removing the projections along A and B. We find

$$\begin{aligned} C &= v - \frac{A^T v}{A^T A} A - \frac{B^T v}{B^T B} B \\ &= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \frac{9}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \end{aligned}$$

Problem 22

One could do this by performing elimination on $A^T A$ as in Problem 18 or just simply performing Gram-Schmidt on the columns of the matrix A. We have

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad q_1 = A\,.$$

With $v = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}^T$ we have that

$$B = v - \frac{v^T A}{A^T A} A = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},$$

so that $q_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. then in $v = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}^T$ we have a third orthogonal vector C as

$$\begin{aligned} C &= v - \frac{A^T v}{A^T A} A - \frac{B^T v}{B^T B} B \\ &= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} - \frac{4}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}. \end{aligned}$$

So that A is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

Problem 23

Part (a): We desire to compute a basis for the subspace for the plane given by

$$x_1 + x_2 + x_3 - x_4 = 0\,.$$

Consider the matrix A defined as $A = \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}$, then since we want to consider the nullspace of A we will assign ones to each free variables in succession and zeros to the other variables and then solve for the pivot variables. This will give us a basis for the nullspace. We find

$$\begin{aligned} x_2 = 1, x_3 = 0, x_4 = 0 &\Rightarrow x = \begin{bmatrix} -1 & 1 & 0 & 0 \end{bmatrix}^T \\ x_2 = 0, x_3 = 1, x_4 = 0 &\Rightarrow x = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}^T \\ x_2 = 0, x_3 = 0, x_4 = 1 &\Rightarrow x = \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

Part (b): The orthogonal complement to S are all vectors that are orthogonal to each component of the nullspace of A. This is the vector $\begin{bmatrix} 1 & 1 & 1-1 \end{bmatrix}^T$.

Part (c): If $b = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, then to decompose b into b_1 and b_2 consider the unit vector of the vector that spans the orthogonal complement i.e.

$$q_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix},$$

then b_2 given by

$$b_2 = (q_2^T b) q_2 = \frac{1}{2}(2)\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

Then

$$b_1 = b - b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

Problem 24

We would like to perform $A = QR$ when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We begin by computing q_1. We find

$$q_1 = \frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ c \end{bmatrix}.$$

and then B is given by

$$\begin{aligned} B &= \begin{bmatrix} b \\ d \end{bmatrix} - \left(\begin{bmatrix} b & d \end{bmatrix} \cdot \frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ c \end{bmatrix} \right) \frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a \\ c \end{bmatrix} \\ &= \begin{bmatrix} b \\ d \end{bmatrix} - \frac{ab+dc}{a^2+c^2}\begin{bmatrix} a \\ c \end{bmatrix} \\ &= \frac{ad-bc}{a^2+c^2}\begin{bmatrix} -c \\ a \end{bmatrix}. \end{aligned}$$

which is orthogonal to $\begin{bmatrix} a \\ c \end{bmatrix}$, and has a unit vector given by

$$\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} -c \\ a \end{bmatrix}.$$

So the matrix Q in the QR decomposition of A is given by

$$Q = \frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a & -c \\ c & a \end{bmatrix}.$$

Then R is given by (using Matlab notation)

$$R = \begin{bmatrix} q_1^T A(:,1) & q_1^T A(:,2) \\ 0 & q_2^T A(:,2) \end{bmatrix} = \frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & -cb+ad \end{bmatrix}.$$

To the decomposition of A is then given by

$$A = \left(\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a & -c \\ c & a \end{bmatrix} \right) \left(\frac{1}{\sqrt{a^2+c^2}}\begin{bmatrix} a^2+c^2 & ab+cd \\ 0 & -cb+ad \end{bmatrix} \right).$$

If $(a,b,c,d) = (2,1,1,1)$ then we obtain

$$A = \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \right) \left(\frac{1}{\sqrt{5}}\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} \right),$$

while if $(a,b,c,d) = (1,1,1,1)$ we obtain

$$A = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \right),$$

From which we see that the $(2,2)$ element of R in this case is zero.

Problem 25

Equation 8 is given by

$$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B$$

The first equation in 12 is given by

$$r_{kj} = \sum_{i=1}^{m} a_{ik} a_{ij} \,,$$

is the expression for the dot product between the kth column of Q and the jth column of A. Then $a_{ij} = a_{ij} - q_{ik} r_{kj}$ subtracts the projection onto the basis functions.

Problem 26

a and b may not be orthogonal so by subtracting projections along non-orthogonal vectors one would be double counting.

Problem 27

See the Matlab code `chap4_sect_4_4_prob_27.m`.

Problem 28

Equation 11 involves m multiplications from the summation and m divisions for the calculations of $q_{ik} = \frac{a_{ik}}{r_{kk}}$ giving a total of $O(2m)$ calculations. Each of these multiplications are performed multiple times. Thus we have

$$\begin{aligned}
\sum_{k=1}^{n} 2m + \sum_{j=k+1}^{n} 2m &= 2mn + \sum_{k=1}^{n} 2m(n - k - 1 + 1) \\
&= 2mn + 2m \sum_{k=1}^{n} (n - k) \\
&= 2mn + 2m \sum_{k=1}^{n-1} k \\
&= 2mn + 2m \left(\frac{n(n-1)}{2} \right) \\
&= mn^2 + mn \,,
\end{aligned}$$

which is the required number of flops.

Problem 29

Part (a): We desire to check that $Q^TQ = I$, when computing this product we have

$$
\begin{aligned}
Q^TQ &= c^2 \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \\
&= c^2 \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I\,,
\end{aligned}
$$

by picking $c = \frac{1}{2}$.

Part (b): We know that Q defined by

$$
Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix},
$$

which will be orthogonal if $c = \frac{1}{2}$ as in Part (a).

Problem 30

Projecting onto the first column of Q we have a coefficient given by $q_1^T b = \frac{1}{2}(-2) = -1$, so that we have a projection of

$$
p = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}.
$$

To project onto the first two columns of the matrix A we give

$$
\begin{aligned}
q_1^T b &= -1 \\
q_2^T b &= \frac{1}{2}(-2) = -1\,.
\end{aligned}
$$

So that p is now given by

$$
p = -\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.
$$

Problem 31

Now $Q = I - 2uu^T$ is a reflection matrix. If $u = [0, 1]^T$ then

$$uu^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so that Q is given by

$$Q = I - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If $r = \begin{bmatrix} x \\ y \end{bmatrix}$ then $Qr = \begin{bmatrix} x \\ -y \end{bmatrix}$. If $u = (0, 1/\sqrt{2}, 1/\sqrt{2})$ then

$$uu^T = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

so that Q is given by

$$Q = I - 2uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

If $r = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then $Qr = \begin{bmatrix} x \\ -z \\ -y \end{bmatrix}$.

Problem 32

Part (a): From the definition of Q we have

$$Qu = u - 2uu^T u = u - 2u = -u\,.$$

Part (b): If $u^T v = 0$ then we have

$$Qv = v - 2uu^T v = v\,.$$

Problem 33

What is special about the columns of W is that they are orthonormal. The inverse of W is then its transpose i.e.

$$W^{-1} = W^T = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}.$$

Chapter 5 (Determinants)

Section 5.1 (The Properties of Determinants)

Problem 1 (examples of properties of the determinant)

If $\det(A) = 2$, and A is 4 by 4 we then have

$$\begin{aligned}
\det(2A) &= 2^4 \det(A) = 2^4\, 2 = 32 \\
\det(-A) &= (-1)^4 \det(A) = 2 \\
\det(A^2) &= \det(A)^2 = 4 \\
\det(A^{-1}) &= \frac{1}{\det(A)} = \frac{1}{2}
\end{aligned}$$

Problem 2 (more examples with the determinant)

If $\det(A) = -3$, and A is 3 by 3 we then have

$$\begin{aligned}
\det(\frac{1}{2}A) &= \left(\frac{1}{2}\right)^3 \det(A) = -\frac{3}{8} \\
\det(-A) &= (-1)^3 \det(A) = -(-3) = 3 \\
\det(A^2) &= \det(A)^2 = 9 \\
\det(A^{-1}) &= \frac{1}{\det(A)} = -\frac{1}{3}
\end{aligned}$$

Problem 3 (true/false propositions with determinants)

Part (a): False. If we define A as

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

then $\det(A) = -2$ and we have $I + A$ given by

$$I + A = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix},$$

so $\det(I + A) = 10 - 6 = 4$, while $1 + \det(A) = 1 - 2 = -1$, which are not equal.

Part (b): True

Part (c): True

Part (d): False, let $A = I$ then

$$4A = 4I = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix},$$

so $\det(4A) = 16 \neq 4\det(A) = 4\det(I) = 4$.

Problem 4 (row exchanges of the identity)

If

$$J_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

then J_3 is obtained from I by exchanging rows one and three from the three by three identity matrix. If J_4 is given by

$$J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

then J_4 is obtained from the four by four identity matrix by exchanging the second and third rows and the first and fourth rows.

Problem 5 (more row exchanges of the identity)

We will propose an inductive argument to express the number of row exchanges needed to permute the reverse identity matrix J_n to the identity matrix I_n. From problem 4, we have the number of row exchanges needed when $n = 3$ and $n = 4$ is given by one and two respectively. For $n = 5$ the reverse identity matrix is given by

$$J_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and can be converted into the identity matrix with two exchanges; by exchanging rows one and five, and rows two and four. So we have that the determinant of J_5 is given by $(-1)^2 = 1$. For $n = 6$ the identity and the reverse identity are given by

$$I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

n	number of row exchanges
3	1
4	2
5	2
6	3
7	3

Table 2: The number of row exchanges needed to convert the identity matrix into the reverse identity matrix.

From which we can see that the reverse identity in this case has three row exchanges; row one and six, row two and five, row three and four. So we have that the determinant of J_6 is given by $(-1)^3 = -1$. For $n = 7$ we will have three row exchanges to obtain the reverse identity matrix, so the determinant of J_7 will be given by $(-1)^3 = -1$. A summary of our results thus far can be given in Table 2. From Table 2, the general rule seems to be that the number of exchanges required for transforming the n by n identity matrix to the n by n reverse identity matrix involves floor($\frac{n}{2}$) row exchanges. So to produce the J_{101} matrix we have floor($\frac{101}{2}$) $= 50$ row exchanges from the 101×101 identity matrix. From this the determinant of J_{101} is given by $(-1)^{50} = 1$.

Problem 6 (a row of all zeros gives a zero determinant)

If a matrix has a row of all zeros, we can replace that row with a row of non-zeros times a multiplier which is zero i.e. in the notation of the book take $t = 0$. Then part of rule number three, says that the determinant of this matrix is equal to t times the determinant of the matrix with the non-zero row. Since 0 times anything gives zero, the original determinant must be zero.

Problem 7 (determinants of orthogonal matrices)

An orthogonal matrix has the property that $Q^T Q = I$. Taking the determinant of both sides of this equation we obtain $|Q||Q^T| = 1$. Since $|Q| = |Q^T|$ we have that $|Q|^2 = 1$, or $|Q| = \pm 1$. Also from the above we have that for orthogonal matrices $Q^{-1} = Q^T$. By taking determinants of both sides we have that $|Q^{-1}| = |Q^T| = |Q|$.

Problem 8 (determinants of rotations and reflections)

If Q is a two-dimensional rotation, then

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then $|Q| = \cos(\theta)^2 + \sin(\theta)^2 = +1$. For a reflection Q is given by

$$Q = \left[\begin{array}{cc} 1 - 2\cos(\theta)^2 & -2\cos(\theta)\sin(\theta) \\ -2\cos(\theta)\sin(\theta) & 1 - 2\sin(\theta)^2 \end{array} \right]$$

so that

$$\begin{aligned} |Q| &= (1 - 2\cos(\theta)^2)(1 - 2\sin(\theta)^2) - 4\cos(\theta)^2\sin(\theta)^2 \\ &= 1 - 2(\cos(\theta)^2 + \sin(\theta)^2) = 1 - 2 = -1 \end{aligned}$$

Problem 9

If $A = QR$, then $A^T = R^T Q^T$ so the $|A^T| = |R^T||Q^T|$, and since R is upper triangular $|R^T| = |R|$ since both expressions are the product of the diagonal elements in each matrix. Also from the problem above we have that $Q^T = Q$ for an orthonormal matrix thus

$$|A^T| = |R^T||Q^T| = |R||Q| = |Q||R| = |QR| = |A|\,.$$

Problem 10

If the entries of every row of A add to zero, then from the determinant rule that $|A^T| = |A|$, and the fact that by subtracting a multiple of one row from another leaves the determinant unchanged we see that by subtracting a multiple of a column from another column leaves the determinant unchanged. Thus by repeatedly adding a multiple (one) of each column to each other (say accumulating the sum in the first column) we will obtain a column of zeros and therefore show that the determinant is zero.

If every row of A adds to *one* we can prove that $\det(A - I) = 0$ by recognizing that because of this fact every row of $A - I$ adds to zero and therefore the determinant must be zero by the previous part of this problem. This does not imply that $\det(A) = 1$ since if we let

$$A = \left[\begin{array}{cc} 2 & 0 \\ -1 & 1 \end{array} \right]$$

has every row adding to one but $\det(A) = 2 \neq 1$.

Problem 11

If $CD = -DC$, then the determinant of the left hand side is given by $|CD| = |C||D|$ and the determinant of the right hand side is given by $|-DC| = (-1)^n|DC| = (-1)^n|D||C|$. This shows that $(1 - (-1)^n)|D||C| = 0$, so $|D| = 0$, or $|C| = 0$, or $1 - (-1)^n = 0$, i.e. n is even.

Problem 12

The correct calculation is given by the following

$$\begin{aligned}
\det(A^{-1}) &= \det\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) \\
&= \frac{1}{(ad-bc)^2}\det\left(\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) \\
&= \frac{1}{(ad-bc)^2}(ad-cb) = \frac{1}{ad-bc}\,.
\end{aligned}$$

Problem 13

We have by applying row operations to the first example the following

$$\begin{aligned}
\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 5 \end{bmatrix} &= \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 0 & 5 \end{bmatrix} \\
&= \det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 1\cdot 2\cdot 3\cdot 4 = 24\,.
\end{aligned}$$

The second example is given by

$$\begin{aligned}
\det\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} &= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \\
&= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \\
&= \det\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = 2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4} = 4\,.
\end{aligned}$$

Problem 14

We have using row operations to simplify the determinant

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = \det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{bmatrix}.$$

Continuing in this fashion when we eliminate the element $b - a$ we obtain a $(3, 3)$ element of the above give by

$$(c^2 - a^2) - \frac{(c-a)}{(b-a)}(b^2 - a^2) = c^2 - cb - ca + ab = c(c-a) + b(a-c) = (c-b)(c-a)$$

so our determinant above becomes equal to

$$\det \begin{bmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & 0 & (c-b)(c-a) \end{bmatrix} = (b-a)(c-b)(c-a)\,,$$

as expected.

Problem 15

For the matrix A we know that its determinant must equal zero since it will be a three by three matrix but of rank one and therefore will not be invertible. Because it is not invertible its determinant must be zero. Another way to see this is to recognize that this matrix can be easily reduced (via elementary row operations) to a matrix with a row of zeros.

For the matrix K we see that $K^T = -K$, so that $|K^T| = |K|$ from Proposition 10 from this section of the book. We also know that $|-K| = (-1)^3|K|$ since K is a three by three matrix. Thus the determinant of K must satisfy $|K| = (-1)^3|K| = -|K|$, which when solved for for $|K|$, gives $|K| = 0$.

Problem 16

From the problem above we have shown that for a matrix K that is skew symmetric with m odd we have that $|K| = 0$. If m can be even giving a non zero determinant. For a four by four example consider the matrix K defined by

$$K = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

then we would have $|K|$ equal to (using elementary row operations)

$$(-1)\det\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} = (-1)\det\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

$$= (-1)\det\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$= (-1)^2\det\begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (-1)\cdot 1\cdot(-1) = 1\,.$$

Where the last equality is obtained by exchanging rows three and four.

Problem 17

The determinant of the first matrix (denoted A in this solution manual) the solution is formally,

$$\det(A) = \det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix},$$

which by subtracting the second row from the third gives

$$\det(A) = \det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 1 & 1 & 1 \end{bmatrix},$$

continuing we now subtract the first row from the second to obtain

$$\det(A) = \det\begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

from which since our matrix has two identical rows requires that its determinant must be zero.

For the second matrix (denoted by B in this solution manual) we have for the expression for the determinant the following

$$\det(A) = \det\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

Now by multiplying the first row by t and subtracting from the second and multiplying the first row by t^2 and subtracting from the third we have

$$\det(A) = \det \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix} .$$

Continuing using elementary row operations we have

$$\det(A) = \det \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^4-t(t-t^3) \end{bmatrix} .$$

The $(3,3)$ element of this matrix simplifies to $1-t^2$, which gives for the determinant of B the product of the diagonal elements or

$$1 \cdot (1-t^2) \cdot (1-t^2) .$$

This expression will vanish if $t = \pm 1$.

Problem 18

For the first U given by

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

from which we have $|U| = 1 \cdot 4 \cdot 6 = 24$. From this we have that $|U^{-1}| = \frac{1}{|U|} = \frac{1}{24}$, and $|U^2| = |U| \cdot |U| = |U|^2 = 24^2 = 416$.

For the second U given by

$$U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

we have $|U| = ad$, $|U^{-1}| = \frac{1}{|U|} = \frac{1}{ad}$ and $|U^2| = |U|^2 = a^2d^2$.

Problem 19 (multiple row operations in a single step)

One cannot do multiple row operations at one time and get the same value of the determinant. The correct manipulations are given by

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \det \begin{bmatrix} a & b \\ c-la & d-lb \end{bmatrix} \\ &= \det \begin{bmatrix} a-L(c-la) & b-L(d-lb) \\ c-la & d-lb \end{bmatrix} \\ &= \det \begin{bmatrix} a-Lc+Lla & b-Ld+Llb \\ c-la & d-lb \end{bmatrix} . \end{aligned}$$

The proposed matrix in the book is missing the terms Lla and Llb. Another way to show that the two determinants are not equal is to compute the second one directly. Which is given by

$$\begin{aligned}
(a - Lc)(d - lb) - (b - Ld)(c - la) &= ad - alb - Lcd + Llcb - (bc - lba - Ldc + Llad) \\
&= ad - bc + Llcb - Llad \\
&= ad - bc - Ll(ad - cb) \\
&= (ad - bc)(1 - Ll)
\end{aligned}$$

Problem 20

Following the instructions given and the matrix A we see that

$$\begin{aligned}
\det(A) &= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \det \begin{bmatrix} a & b \\ c+a & d+b \end{bmatrix} \\
&= \det \begin{bmatrix} -c & -d \\ c+a & d+b \end{bmatrix} \\
&= \det \begin{bmatrix} -c & -d \\ a & b \end{bmatrix} \\
&= (-1)\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = (-1)\det(B)
\end{aligned}$$

where in the transformations above we have used two rules. The first is that subtracting a multiple of one row from another row does not change the determinant and the second being that factoring a multiplier of a row out of the matrix multiples the determinant by an appropriate factor.

Problem 21

We have $|A| = 4 - 1 = 3$, $|A^{-1}| = \frac{1}{3^2}(4-1) = \frac{1}{3}$, and $|A - \lambda I| = (2-\lambda)^2 - 1$. Thus for $|A - \lambda I| = 0$ we must have

$$(2 - \lambda) = \pm 1$$

or $\lambda = 1$ or $\lambda = 3$. If $\lambda = 1$ then $A - \lambda I$ is given by

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If $\lambda = 3$ then $A - \lambda I$ is given by

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Problem 22

If A is given by

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} .$$

so we have that $|A| = 12 - 2 = 10$ and A^2 is given by

$$A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix} .$$

with a determinant given by $|A^2| = 100$, now A^{-1} is given by

$$A^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} .$$

so that $|A^{-1}| = \frac{1}{10}$. We now compute $A - \lambda I$ which gives

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} .$$

so that $|A - \lambda I| = (4-\lambda)(3-\lambda) - 2$. Now by setting this equal to zero and solving for λ we have that $|A - \lambda I| = 0$ is equivalent to $(\lambda - 2)(\lambda - 5) = 0$ giving that $\lambda = 2$ or $\lambda = 5$.

Problem 23

Since $|L| = 1$, we have that $|U| = 3(2)(-1) = -6$, so $|A| = |L| \cdot |U| = -6$. Then since $A = LU$ we have that $A^{-1} = U^{-1}L^{-1}$, so

$$|A^{-1}| = |U^{-1}||L^{-1}| = \frac{1}{|U|}\frac{1}{|L|} = -\frac{1}{6} .$$

Since $U^{-1}L^{-1}A = I$ we have the obvious identity that $|U^{-1}L^{-1}A| = 1$.

Problem 24

If $A_{ij} = i \cdot j$, then the A matrix is m by m and is given by the outer product

$$A = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ m \end{bmatrix} \begin{bmatrix} 1 & 2 & \ldots & m \end{bmatrix} .$$

Which is a rank one matrix and therefor has a determinant equal to zero, since it is not invertible. Multiple rows are multiples of a single row.

Problem 25

We are asked to prove that if $A_{ij} = i + j$ then $\det(A) = 0$. Lets consider the case when A is m by m and consider the first second and third rows of A. These rows are given by

$$\begin{array}{cccccc} 1+1 & 1+2 & 1+3 & 1+4 & \ldots & 1+m \\ 2+1 & 2+2 & 2+3 & 2+4 & \ldots & 2+m \\ 3+1 & 3+2 & 3+3 & 3+4 & \ldots & 3+m \end{array}$$

Now the determinant is unchanged if we subtract the second row from the first. Doing this gives for the first three rows the following

$$\begin{array}{cccccc} 1+1 & 1+2 & 1+3 & 1+4 & \ldots & 1+m \\ 2+1 & 2+2 & 2+3 & 2+4 & \ldots & 2+m \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{array}$$

Now subtracting the first row from the second row gives

$$\begin{array}{cccccc} 1+1 & 1+2 & 1+3 & 1+4 & \ldots & 1+m \\ 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{array}$$

Since this matrix has two repeated rows, the determinant must be zero.

Problem 26

For A we have

$$\det(A) = \det\begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} = (-1)\det\begin{bmatrix} c & 0 & 0 \\ 0 & 0 & b \\ 0 & a & 0 \end{bmatrix} = (-1)^2\det\begin{bmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} = abc\,.$$

For B we have

$$\begin{aligned} \det(B) &= \det\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} = (-1)\det\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & a & 0 & 0 \end{bmatrix} = (-1)^2\det\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & b & 0 \end{bmatrix} \\ &= (-1)^3\det\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} = -abcd\,. \end{aligned}$$

Finally for C we have

$$\begin{aligned}
\det(C) &= \det \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} = \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix} \\
&= \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-a-(b-a) \end{bmatrix} \\
&= \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = a(b-a)(c-b)\,.
\end{aligned}$$

Problem 27

Part (a): True. We know from a previous problem that $\text{rank}(AB) \leq \text{rank}(A)$ and since $\text{rank}(A) < m$, the product must have $\text{ran}(AB) \leq \text{rank}(A) < m$, and therefore AB cannot be invertible.

Part (b): True. Since elementary row operations change A into U and the determinant of U is the product of the pivots.

Part (c): False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\det(A - B) = 1$, but $\det(A) - \det(B) = 4 - 1 = 3$.

Part (d): True. If the product of A and B is defined in that way.

Problem 28

If $f(A) = \ln(\det(A))$, then for a two by two system our f is given by $f(A) = \ln(ad - bc)$. Defining $\Delta = ad - bc$, we have that

$$\begin{aligned}
\frac{\partial f}{\partial a} &= \frac{d}{\Delta} \\
\frac{\partial f}{\partial b} &= -\frac{c}{\Delta} \\
\frac{\partial f}{\partial c} &= \frac{b}{\Delta} \\
\frac{\partial f}{\partial d} &= \frac{a}{\Delta}
\end{aligned}$$

so that

$$\begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}\,.$$

Section 5.2 (Permutations and Cofactors)

Problem 1 (practice computing determinants)

For the matrix A using the formula $|A| = \sum \pm a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$, we have

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 3 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ &= 1(-1) - 2(-1) + 3(1) = -1 + 2 + 3 = 4 \neq 0 \end{aligned}$$

Since the determinant is not zero the columns *are* independent. For the matrix B we have

$$\begin{aligned} |B| &= 1 \begin{vmatrix} 4 & 4 \\ 6 & 7 \end{vmatrix} - 2 \begin{vmatrix} 4 & 4 \\ 5 & 7 \end{vmatrix} + 3 \begin{vmatrix} 4 & 4 \\ 5 & 6 \end{vmatrix} \\ &= 1(28 - 24) - 2(28 - 20) + 3(24 - 20) = 4 - 16 + 12 = 0\,. \end{aligned}$$

Since the determinant is zero the columns are *not* independent.

Problem 2 (more practice computing determinants)

For the matrix A using the formula $|A| = \sum \pm a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$, we have

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 0 \\ &= -1 - 1 = -2 \neq 0\,, \end{aligned}$$

Since the determinant is not zero the columns *are* independent. For the matrix B we have

$$\begin{aligned} |B| &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (45 - 48) - 2(36 - 42) + 3(32 - 35) = -3 + 12 - 9 = 0\,. \end{aligned}$$

Since the determinant is zero the columns are *not* independent.

Problem 3

We have that

$$|A| = x \begin{vmatrix} 0 & x \\ 0 & x \end{vmatrix} = 0\,,$$

since an entire column is zero. The rank of A is at most two, since the second column has no pivot.

Problem 4

Part (a): Since the rank of A is at most two, there can only be two linearly independent rows. As such this matrix must have a zero determinant.

Part (b): Formula 7 in the book is $\det(A) = \sum \det(P) a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$. In this expression every term will be zero because when we select columns we eventually have to select a zero in the three by three block in the lower left of the matrix A. These zeros in the multiplication is what makes every term zero.

Problem 5

For A we can expand the determinant about the first row giving

$$\begin{aligned} |A| &= 1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \\ &= 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 1 \left(-1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right) \\ &= -1(1) - 1(-1) = -1 + 1 = 0 \,. \end{aligned}$$

We can also compute $|A|$ by expanding about the last row of A given by

$$\begin{aligned} |A| &= -1 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} \\ &= -1(1) \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ &= -1(-1) - 1 = 1 - 1 = 0 \,. \end{aligned}$$

For the matrix B we can compute the determinant in the same way as with A. Expanding about the first row gives

$$|B| = 1 \begin{vmatrix} 3 & 4 & 5 \\ 4 & 0 & 3 \\ 0 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 3 & 4 \\ 5 & 4 & 0 \\ 2 & 0 & 0 \end{vmatrix} ,$$

followed by expanding each of the remaining determinants along the bottom row gives

$$|B| = 1 \begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} - 2(2) \begin{vmatrix} 3 & 4 \\ 4 & 0 \end{vmatrix} = -16 - 4(-16) = 48 \,.$$

Problem 6

By creating a matrix with *no* zeros we have certainly used the smallest number. One such matrix could be

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

then certainly $\det(A) = 0$. To create a matrix with as many zeros as possible and still maintain $\det(A) = 0$, consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix},$$

with a, b, c, d all nonzero. This matrix is certainly not singular but by setting any of a, b, c, or d equal to zero a singular matrix results.

Problem 7

Part (a): Our expression for the determinant is given by $|A| = \sum \pm a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$. Assuming our matrix has elements $a_{11} = a_{22} = a_{33} = 0$, we can reason which of the 3! terms in the determinant sum will be zero as follows. Obviously all permutations with a_{11} in them i.e. $(1, 2, 3)$, and $(1, 3, 2)$ will have a zero in them. Additionally, all permutations with a_{22} in them i.e. $(1, 2, 3)$, $(3, 2, 1)$ will be zero. The term $a_{33} = 0$ will cause the two permutations $(1, 2, 3)$ and $(2, 1, 3)$ to be zero. Since the permutation $(1, 2, 3)$ is counted three times in total we have four zero elements in the determinant sum.

Problem 8

To have $\det(P) = +1$ we must have an even number of row exchanges. Now the total number of five by five permutation matrices is given $5! = 120$. Half of this number are permutation matrices with an odd number of row exchanges and the other half have an even number of row exchanges so 60 have $\det(P) = -1$. Now

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

will require four exchanges to obtain the identity using row exchanges. Specifically, exchanging the first and the last row, then the second and the last row, and finally the third and

1	2	3	4	+	3	1	2	4	+
1	2	4	3	-	3	1	4	2	-
1	3	2	4	-	3	2	1	4	-
1	3	4	2	+	3	2	4	1	+
1	4	2	3	+	3	4	1	2	+
1	4	3	2	-	3	4	2	1	-
2	1	3	4	-	4	1	2	3	-
2	1	4	3	+	4	1	3	2	+
2	3	1	4	+	4	2	3	1	-
2	3	4	1	-	4	2	1	3	+
2	4	1	3	-	4	3	2	1	+
2	4	3	1	+	4	3	1	2	-

Table 3: An enumeration of the possible 4! permutations with + denoting a even permutation and $-$ denoting an odd permutation.

the last row we have that J transforms under these row operations as follows

$$
\begin{aligned}
J &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .
\end{aligned}
$$

Problem 9

Since we have that $\det(A) \neq 0$ then say $a_{1\alpha}a_{2\beta}\cdots a_{n\nu} \neq 0$, for some specification of the variables $(\alpha, \beta, \cdots, \nu)$. Construct the permutation that takes $(\alpha, \beta, \cdots, \nu) = (1, 2, 3, \cdots, n)$, i.e. the inverse permutation. Then in this case AP will have $a_{1\alpha}$ in position $(1, 1)$, a $a_{2\beta}$ in position $(2, 2)$, a $a_{3\cdot}$ in position $(3, 3)$, etc ending with $a_{n\nu}$ in position (n, n). This is because AP permutes the columns of A and will move $a_{1\alpha}$ to $(1, 1)$, etc.

Problem 10

Part (a): A systematic wave to do this problems would be to enumerate all of the possible permutations and separate them into positive and negative permutations. Consider the Table 3 for this enumeration.

Part (b): An odd permutation times an odd permutation is an even permutation.

Problem 11

For A given by

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix},$$

we have that $c_{11} = 6$, $c_{12} = -3$, $c_{21} = -1$, and $c_{22} = 2$ so our C is given by

$$C = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix}$$

For the matrix B given by

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}$$

we have $c_{11} = \begin{vmatrix} 5 & 6 \\ 0 & 0 \end{vmatrix} = 0$, $c_{12} = -\begin{vmatrix} 4 & 6 \\ 7 & 0 \end{vmatrix} = 42$, $c_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 0 \end{vmatrix} = -35$, $c_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix} = 0$, $c_{22} = \begin{vmatrix} 1 & 3 \\ 7 & 0 \end{vmatrix} = -21$, $c_{23} = -\begin{vmatrix} 1 & 2 \\ 7 & 0 \end{vmatrix} = 14$, $c_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$, $c_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6$, and finally $c_{33} = -\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 3$. Thus the cofactor matrix C is given by

$$C = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & 3 \end{bmatrix}.$$

The determinant of B is given by (expanding about the third row)

$$\det(B) = 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -21\,.$$

Problem 12 (the second derivative matrix)

For A given by Strang's "favorite" matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

we compute for the various cofactors the following: $c_{11} = +\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$, $c_{12} = -\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2$, $c_{13} = \begin{vmatrix} -1 & 2 \\ 0 & -1 \end{vmatrix} = 1$, $c_{21} = -\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2$, $c_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$, $c_{23} = -\begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} = 2$,

$c_{31} = \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} = 1$, $c_{32} = -\begin{vmatrix} 2 & 0 \\ -1 & -1 \end{vmatrix} = 2$, and finally $c_{33} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$. Thus the cofactor matrix C is given by

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{so} \quad C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

since C is symmetric. We then have for $C^T A$ the following

$$C^T A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

or four times the identity matrix. Note that

$$\det(A) = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 4\,.$$

so we see that $A^{-1} = \frac{1}{\det(A)} C^T$, as we know must be true.

Problem 13

As suggested in the text expanding $|B_4|$ using cofactors in the last row of B_4 we have

$$\begin{aligned} |B_4| &= 2\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix} \\ &= 2|B_3| + (-1)\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \\ &= 2|B_3| - |B_1|\,. \end{aligned}$$

Continuing our expansion we have that $|B_2| = 2 - 1 = 1$ and that

$$|B_3| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 1\,.$$

So we see that $|B_4| = 1$.

Problem 14

Part (a): We see that

$$\begin{aligned}
C_1 &= |0| = 0 \\
C_2 &= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \\
C_3 &= \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 \\
C_4 &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1)^2\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \,.
\end{aligned}$$

Part (b): We desire to compute the determinant of a matrix C_n of size $n \times n$ with all ones on the super and sub-diagonal as

$$|C_n| = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & & 0 & 1 & 0 \end{vmatrix} .$$

By expanding this determinant about the first row we have that

$$|C_n| = (-1)\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & & 0 & 1 & 0 \end{vmatrix} ,$$

which by further expanding about the first column gives

$$|C_n| = (-1)(1)\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & & 0 & 1 & 0 & 1 \\ 0 & 0 & \cdots & & 0 & 1 & 0 \end{vmatrix} = (-1)|C_{n-2}| ,$$

since we have removed *two* rows from the original C_n matrix. So since $|C_1| = 0$ we see from the above that $|C_3|, |C_5|, |C_7|, \cdots$ are all zero. Now $|C_2|$ will determine all even terms i.e. $|C_4|, |C_6|, |C_8|, \cdots$. We therefore have $|C_4| = 1, |C_8| = 1, |C_{12}| = 1, \cdots$ and $|C_6| = -1, |C_{10}| = -1, |C_{14}| = -1, \cdots$, so $|C_{10}| = -1$.

Problem 15

In Problem 14 (above) we have shown the desired relationships.

Problem 16

Part (a): We see that computing a few determinants that

$$
\begin{aligned}
|E_1| &= 1 \\
|E_2| &= 0 \\
|E_3| &= 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 0 - 1(1) = -1\,.
\end{aligned}
$$

To derive a recursive relationship consider define $|E_n|$ as

$$
|E_n| = \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix}\,.
$$

Now expand about the first row and we have that

$$
\begin{aligned}
|E_n| &= +1 \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\
&= |E_{n-1}| - \begin{vmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} = |E_{n-1}| - |E_{n-2}|\,,
\end{aligned}
$$

as we were reqested to show.

Part (b): With $E_1 = 1$ and $E_2 = 0$ we can iterate the above equation to find that

$$\begin{aligned} E_3 &= E_2 - E_1 = -1 \\ E_4 &= E_3 - E_2 = -1 - 0 = -1 \\ E_5 &= E_4 - E_3 = -1 - (-1) = 0 \\ E_6 &= E_5 - E_4 = 0 - (-1) = 1 \\ E_7 &= E_6 - E_5 = 1 - 0 = 1 \\ E_8 &= E_7 - E_6 = 1 - 1 = 0 \\ E_9 &= E_8 - E_7 = 0 - 1 = -1 \,. \end{aligned}$$

From these the pattern looks like

$$E_{2,5,8,\cdots} = 0 \quad \text{or} \quad E_{3n+2} = 0 \quad \text{for} \quad n = 0, 1, 2, \cdots$$

and

$$E_{3,4,9,10,15,16,\cdots} = -1 \,,$$

or $E_{3+6n} = -1$ and $E_{4+6n} = -1$ for $n = 0, 1, 2, \cdots$. Finally we hypothesis that

$$E_{6,7,12,13,18,19,\cdots} = 1 \,,$$

or $E_{6n} = 1$ and $E_{1+6n} = 1$ for $n = 0, 1, 2, \cdots$. Then E_{100} can be written as $E_{16\times 6+4}$ so looking at these patterns we see that $E_{6n+4} = -1$ so $E_{100} = -1$.

Problem 17

We define F_n to be

$$F_n = \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} .$$

so that expanding about the first row we find F_n to be

$$\begin{aligned} F_n &= 1 \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \\ &= F_{n-1} + F_{n-2} \end{aligned}$$

Problem 18

Thus linearity gives $|B_n| = |A_n| - |A_{n-1}| = (n+1) - (n-1+1) = 1$, where we have used the discussion in this section to evaluate $|A_n|$ and $|A_{n-1}|$.

Problem 19

The 4×4 Vandermonde determinant containings x^3 and not x^4 because a third degree polynomical requires four points to fit to. Thus a $n \times n$ Vandermonde determinant will have x^{n-1} in it. This determinant is zero if $x = a, b$ or c. The cofactor of x^3 is given by

$$
\begin{aligned}
\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} &= 1 \begin{vmatrix} b & b^2 \\ c & c^2 \end{vmatrix} - a \begin{vmatrix} 1 & b^2 \\ 1 & c^2 \end{vmatrix} + a^2 \begin{vmatrix} 1 & b \\ 1 & c \end{vmatrix} \\
&= bc^2 - cb^2 - a(c^2 - b^2) + a^2(c - b) \\
&= bc(c - b) - a(c - b)(c + b) + a^2(c - b) \\
&= (c - b)(c - a)(b - a)\,.
\end{aligned}
$$

Thus since V_4 is a polynomial with roots a,b, and c and the coefactor of x^3 represents the leading coefficient of the x^3 term in the total determinant. Thus

$$
V_4 = (c - b)(c - a)(b - a)(x - b)(x - a)(x - c)\,.
$$

Problem 20

We have that G_4 is defined by $G_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. Then $|G_4|$ is given by

$$
\begin{aligned}
|G_4| &= \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \\
&= (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix} \\
&= (-1) \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -3/2 \end{vmatrix} = (-1)(1)(1)(-2)(-3/2) = -3\,.
\end{aligned}
$$

We find

$$
\det(G_2) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1\,.
$$

and

$$
\det(G_3) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = (-1)(-1) + 1(1) = 2\,.
$$

So by the induction hypothesis we have that $\det(G_n) = (-1)^{n-1}(n - 1)$.

Problem 21

Part (a): The first statement is true since by applying elementary row opperations to the matrix $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ the pivots obtained will be determined from the matrices A and D only. Since the determinant is the product of the pivots it is equal to the products of the pivots from A and D.

Part (b): Let our large block matrix be

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & -1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}.$$

which has submatrices given by $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$. These have individual determinats given by $|A| = 1$, $|B| = 5$, $|C| = 1$, and $|D| = -5$. The determinant of the large block matrix is given by 15, while the product of $|A||D| = 1 \cdot (-5) = -5 \neq 15$. In addition, the expression

$$|A||D| - |C||B| = 1(-5) - 1(5) = -10\,,$$

which is not equal to the true determinant 15 either.

Part (c): Computing $AD - CB$ we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ -5 & -7 \end{bmatrix},$$

which has a determinant given by $42 - 15 = 27$, which is not equal to the true value either.

Problem 22

Part (a): Assuming that the index k refers to how many rows and columns the matrix L_k/U_k) subsumes i.e. L_1/U_1 are 1×1, L_2/U_2 are 2×2, etc. Then since the matrix L is lower triangular and constructed to have ones on its diagonal $|L_k| = 1$, for $k = 1, 2, 3$. The determinant of U_k wil then be $|U_1| = 2$, $|U_2| = 2 \cdot 3 = 6$, and $|U_3| = 2(3)(-1) = -6$. In the same way $|A_k| = |U_k|$.

Part (b): If A_1, A_2 and A_3 have determinants given by $2, 3$ and -1 the pivots are given by

$$p_1 = 2\,, p_2 = \frac{3}{2}\,, p_3 = \frac{-1}{2\left(\frac{3}{2}\right)} = -\frac{1}{3}\,.$$

Problem 23

Taking the determinant of the left hand side of the and using the determinant rule that row opperations don't change the the value of the determinant or the fact that the matrix $\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix}$ is lower triangular with ones on the diagonal we have that

$$\begin{aligned}
\text{LHS} &= \left| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right| = \left| \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right| = \left| \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} \right| \\
&= |A||D - CA^{-1}B|\,,
\end{aligned}$$

which is valid if A^{-1} exists. The above equals

$$|AD - ACA^{-1}B|\,,$$

by distributing $|A|$ into the determinant $|D - CA^{-1}B|$. If $AC = CA$ then this is equivalent to

$$|AD - CAA^{-1}B| = |AD - CB|\,.$$

Problem 24

Now

$$\det(M) = \det\left(\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \right) = \det\left(\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \right) \det\left(\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \right).$$

Noiw since $\det\left(\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} \right) = 1$ so that the above is give by

$$\det\left(\begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \right) = \det(AB)\,,$$

From Problem 21. If A is a single row and B is a single column then AB is a scalar and equals its own determinant. So we have that $\det(M) = AB$. For a 3×3 let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so that M is given by

$$M = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix}$$

Chapter 6 (Eigenvalues and Eigenvectors)

Section 6.1 (Introduction to Eigenvalues)

Problem 1

For the matrix A given

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

we have $\lambda_1 = 1$ and $x_1 = (0.6, 0.4)$ and $\lambda_2 = 1/2$ and $x_2 = (1, -1)$. For the square of A i.e. A^2 given by

$$A^2 = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix}$$

we have $\lambda_1 = 1$ and $x_1 = (0.6, 0.4)$ and $\lambda_2 = (1/2)^2$ and $x_2 = (1, -1)$. For A^∞ (given by)

$$A^\infty = \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

we have $\lambda_1 = 1$ and $x_1 = (0.6, 0.4)$ and $\lambda_2 = 0$ and $x_2 = (1, -1)$. To show why A^2 is halfway between A and A^∞ consider the common eigenvalues of all of them i.e.

$$x_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \quad \text{and} \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

These two vectors are linearly independent and thus span $\mathbb{R}^2$, that is they are a basis for $\mathbb{R}^2$. Consider the action of A^2 and $\frac{1}{2}(A + A^\infty)$ on this particular basis of $\mathbb{R}^2$. We have that

$$\begin{aligned} A^2 x_1 &= 1 x_1 = x_1 \\ \frac{1}{2}(A + A^\infty) x_1 &= \frac{1}{2}(1+1) x_1 = x_1 \end{aligned}$$

and

$$\begin{aligned} A^2 x_2 &= \frac{1}{4} x_2 \\ \frac{1}{2}(A + A^\infty) x_2 &= \frac{1}{2}(\frac{1}{2} + 0) x_2 = \frac{1}{4} x_2 \end{aligned}$$

Thus the action of A^2 and $\frac{1}{2}(A + A^\infty)$ is the *same* on a basis of $\mathbb{R}^2$ and therefore the two matrices must be identical.

Part (a): If we exchange two rows of A we obtain

$$\hat{A} = \begin{bmatrix} 0.2 & 0.7 \\ 0.8 & 0.3 \end{bmatrix},$$

which has eigenvalues given by

$$\begin{vmatrix} 0.2 - \lambda & 0.7 \\ 0.8 & 0.3 - \lambda \end{vmatrix} = 0$$

which when expanded can be factored into $(\lambda - 1)(2\lambda + 1) = 0$ and therefore has solutions given by $\lambda = 1$, and $\lambda = -1/2$. These are *not* the same as the eigenvalues of the original matrix A which were 1, and $1/2$.

Part (b): A zero eigenvalue means that A is not invertible. This property would *not* be changed by elimination.

Problem 2

For the matrix A given

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

we have eigenvalues given by the solutions λ of

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0\,,$$

which when expanded gives $(\lambda - 5)(\lambda + 1) = 0$, so the two eigenvalues are given by $\lambda = 5$ and $\lambda = -1$. The eigenvectors for A are given by the nullspace for (first for $\lambda = 5$)

$$\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In a similar way for $\lambda = -1$ we have

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The eigenvalues of $A + I$ are the eigenvalues of A plus 1 or $\lambda = 6$ and $\lambda = -1$. The eigenvectors of $A + I$ are the same as the eigenvectors of A.

Problem 3

For A defined by

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix},$$

the eigenvalues are given by solving

$$\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = 0\,,$$

which simplifies to $(\lambda - 4)(\lambda - 1) = 0$, so $\lambda = 4$ and $\lambda = -1$. The eigenvectors of A are given by the nullspaces of the following matrices (for $\lambda = -4$ first and then $\lambda = 1$)

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Rightarrow \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

The eigenvalues of A^{-1} are the inverses of the eigenvalues of A. When A has eigenvalues λ_1 and λ_2 its inverse has eigenvalues $1/\lambda_1$ and $1/\lambda_2$. The eigenvectors of A^{-1} are given by the nullspace of the following operators (for $\lambda = 1/4$ first and then for $\lambda = 1$)

$$\begin{bmatrix} -\frac{3}{4} - \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \quad \Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and

$$\begin{bmatrix} -\frac{3}{4} + 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \quad \Rightarrow \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

These eigenvectors are the *same* as the eigenvectors of A. That A and A^{-1} have the same eigenvectors can be seen from the simple expression $Ax = \lambda x$, which when we divide both sides by λ and multiply by A^{-1} gives

$$\frac{1}{\lambda} x = A^{-1} x,$$

showing that x is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

Problem 4

For A given by

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

we have eigenvalues given by the solutions to

$$\begin{vmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = 0$$

or $\lambda^2 + \lambda + 6 = 0$, which factors into $(\lambda + 3)(\lambda - 2) = 0$, giving the two values of $\lambda = -3$ or $\lambda = 2$. The eigenvectors are then given by the nullspaces of the following operators.

$$\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \quad \text{or} \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From these, the eigenvalues of A^2 are given by $(-3)^2 = 9$ and $2^2 = 4$, with the *same* eigenvectors as A. This is because when A has eigenvalues λ_i, A^2 will have eigenvalues λ_i^2.

Problem 5

For A we have eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1\,.$$

For B we have eigenvalues given by

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0 \Rightarrow \lambda = 1\,.$$

For the matrix $A+B$ we have eigenvalues given by

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 1\,,3\,.$$

So the eigenvalues of $A+B$ are *not equal to* the eigenvalues of A plus the eigenvalues of B. This would be true if A and B has the same eigenvectors which will happen if and only if A and B commute, i.e. $AB = BA$. Checking this fact for the matrices given here we have

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

while

$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

which are *not* equal so consequently A and B can't have the same eigenvectors.

Problem 6

From Problem 5 the eigenvalues of A and B are 1. The eigenvalues of the product AB are given by

$$|AB - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1 = 0\,,$$

which has roots given by

$$\lambda = \frac{3 \pm \sqrt{5}}{2}\,.$$

The eigenvalues of BA are given by

$$|BA - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 1 = 0\,,$$

which has the same roots as before and therefore BA has the same eigenvalues as AB. We note that the eigenvalues of AB/BA are *not* equal to the product of the eigenvalues of A and B. For this to be true A and B would need to have the same eigenvectors which they must not.

Problem 7

The eigenvalues of U are on its diagonal. They are also the pivots of A. The eigenvalues of L are on its diagonal, they are all ones. The eigenvalues of A are *not* the same as either the eigenvalues of U or L or the product of the eigenvalues of U and L (which would be the same as the product of the eigenvalues of U since the eigenvalues of L are all ones).

Problem 8

Part (a): If we know that x is an eigenvector one way to find λ is to multiply by A and "factor" out x.

Part (b): If we know that λ is an eigenvalue one way to find x is to determine the nullspace of $A - \lambda I$.

Problem 9

Part (a): Multiply $Ax = \lambda x$ by A on the left to obtain

$$A^2 x = \lambda A x = \lambda^2 x$$

Part (b): Multiply by $\frac{1}{\lambda}A^{-1}$ on both sides to get

$$\frac{1}{\lambda} x = A^{-1} x$$

Part (c): Add Ix on both sides of $Ax = \lambda x$ to get

$$(A + I)x = \lambda x + Ix = (\lambda + 1)x \,,$$

which shows that $\lambda + 1$ is an eigenvalue of $A + I$.

Problem 10

For A the eigenvalues are given by

$$|A - \lambda I| = \begin{vmatrix} 0.6 - \lambda & 0.2 \\ 0.4 & 0.8 - \lambda \end{vmatrix} = 0 \Rightarrow (0.6 - \lambda)(0.8 - \lambda) - 0.08 = 0 \,.$$

which gives $\lambda^2 - 1.4\lambda + 0.4 = 0$. To solve this we know that $\lambda = 1$ because A is a Markov matrices. The other root can be found by using the quadratic equation or factoring out the known root $\lambda = 1$ from the above quadratic. When that is done one finds that the second root is given by $\lambda = \frac{2}{5} = 0.4$. The eigenvectors for $\lambda = 1$ are given by considering the nullspace of the operator

$$A - I = \begin{bmatrix} -0.4 & 0.2 \\ 0.4 & -0.2 \end{bmatrix},$$

which has a nullspace given by the span of

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

For $\lambda = 0.4$ we have $A - \lambda I$ given by

$$\begin{bmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{bmatrix},$$

which has a nullspace given by the span of

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

For the matrix A^∞ our eigenvalues are given by $\lambda_1 = 1$ and $\lambda = \left(\frac{2}{5}\right)^\infty = 0$ and the same eigenvectors as A. Now A^∞ is obtained from the diagonalization of A i.e. $A = S\Lambda S^{-1}$. Which given the specific matrices involved is

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{5} \end{bmatrix} \left(\frac{1}{1+2}\right) \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}.$$

So that A^∞ is given by

$$\begin{aligned} A^\infty &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}. \end{aligned}$$

So A^{100} is then given by $A^{100} = S\Lambda^{100}S^{-1}$ or

$$\begin{aligned} A^{100} &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1^{100} & 0 \\ 0 & \left(\frac{2}{5}\right)^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{2}{5}\right)^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 & -\left(\frac{2}{5}\right)^{100} \\ 2 & \left(\frac{2}{5}\right)^{100} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 1 + 2\left(\frac{2}{5}\right)^{100} & 1 - \left(\frac{2}{5}\right)^{100} \\ 2 - 2\left(\frac{2}{5}\right)^{100} & 2 + \left(\frac{2}{5}\right)^{100} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix} + \left(\frac{2}{5}\right)^{100} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}, \end{aligned}$$

which we see is a slight perturbation of A^∞

Problem 11

Now P is a block diagonal matrix and as such has eigenvalues given by the eigenvalues of the block matrices on its diagonal. Since $\lambda = 1$ is the eigenvalue of the lower right block matrix and the upper right block is given by $\begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, which has eigenvalues given by solving for the roots of

$$\begin{vmatrix} 0.2-\lambda & 0.4 \\ 0.4 & 0.8-\lambda \end{vmatrix} = 0 \Rightarrow (0.2-\lambda)(0.8-\lambda) - 0.16 = 0$$

Multiplying this polynomial out we obtain $\lambda^2 - \lambda = 0$ or $\lambda = 0$ and $\lambda = 1$ as its roots. Now the eigenvectors for $\lambda = 1$ are given by computing an appropriate null space. We find

$$\begin{bmatrix} -0.8 & 0.4 & 0 \\ 0.4 & -0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -0.5 & 0 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so one eigenvector is given by $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and another is given by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. For the eigenvalue given by $\lambda = 0$ we have

$$\begin{bmatrix} 0.2 & 0.4 & 0 \\ 0.4 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the final eigenvector is given by $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. For P^{100} we have the same eigenvectors as for P and the eigenvalues given by $0^{100} = 0$ and $1^{100} = 1$. Thus everything for P^{100} is the same as for P. If two eigenvectors share the same λ then so do all linear combinations of the eigenvectors. Thus since $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ share the same eigenvalue of $\lambda = 1$ so will their sum $v_1 + v_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, which has no zero components. We can check this by computing

$$P \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 + 0.8 \\ 0.4 + 1.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Problem 12

The rank one projection matrix is given by $P = uu^T$, so P is given by

$$P = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix} = \frac{1}{36}\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{bmatrix}$$

Part (a): Now Pu is given by

$$\begin{aligned} Pu &= \frac{1}{36}\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix} \\ &= \frac{1}{6^3}\begin{bmatrix} 1+1+9+25 \\ 1+1+9+25 \\ 3+3+27+75 \\ 5+5+45+125 \end{bmatrix} = \frac{1}{6^3}\begin{bmatrix} 36 \\ 36 \\ 108 \\ 180 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 5 \end{bmatrix}. \end{aligned}$$

Thus u is an eigenvector with eigenvalue one.

Part (b): If v is perpendicular to u then $u^T v = v^T u = 0$ and $Pv = u(u^T v) = u \cdot 0 = 0$ so v is an eigenvector with eigenvalue $\lambda = 0$.

Part (c): To find three independent eigenvectors of P all with eigenvalues equal to zero we need to find three vectors perpendicular to u which means that each of these vectors must satisfy

$$\begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

or the three vectors that span the nullspace of A (where A is defined to be $A = \begin{bmatrix} 1 & 1 & 3 & 5 \end{bmatrix}$. Three vectors in the nullspace are given by "assigning a basis" to the variables x_2, x_3, and x_4 and computing x_1 from these. We find

$$\begin{aligned} x_2 &= 1\,, x_3 = 0\,, x_4 = 0 \Rightarrow x_1 = -1 \\ x_2 &= 0\,, x_3 = 1\,, x_4 = 0 \Rightarrow x_1 = -3 \\ x_2 &= 0\,, x_3 = 0\,, x_4 = 1 \Rightarrow x_1 = -5\,. \end{aligned}$$

Which gives the three vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 13

We find that

$$\det(Q - \lambda I) = \begin{vmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{vmatrix} = 0\,,$$

or when expanding the above determinant we find the characteristic equation for Q is given by

$$(\cos(\theta) - \lambda)^2 + \sin^2(\theta) = 0$$

or when expanding the quadratic we find that

$$\lambda^2 - 2\cos(\theta)\lambda + 1 = 0\,,$$

which using the quadratic equation gives for λ

$$\begin{aligned} \lambda &= \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} \\ &= \cos(\theta) \pm \sqrt{\cos^2(\theta) - 1} \\ &= \cos(\theta) \pm i\sin(\theta)\,. \end{aligned}$$

To find the eigenvectors we solve $(Q - \lambda I)x = 0$, which is given by

$$\begin{aligned} Q - \lambda I &= \begin{bmatrix} \cos(\theta) - (\cos(\theta) \pm i\sin(\theta)) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - (\cos(\theta) \pm i\sin(\theta)) \end{bmatrix} \\ &= \begin{bmatrix} \mp i\sin(\theta)) & -\sin(\theta) \\ \sin(\theta) & \mp i\sin(\theta)) \end{bmatrix} \\ &= \sin(\theta)\begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix}. \end{aligned}$$

which has eigenvectors given by $v_{1,2} = \begin{bmatrix} \pm i \\ 1 \end{bmatrix}$ i.e.

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Problem 14

The matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

will have eigenvalues given by the solution to

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0\,.$$

This simplifies to $-\lambda^3+1=0$ and has solutions given by $\lambda = e^{\frac{2\pi}{3}ik}$ for $k=0,1,2$. This gives

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\
\lambda_3 &= e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}$$

For the matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

will have eigenvalues given by the solution to

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0\,.$$

This simplifies to $-\lambda^3+\lambda^2+\lambda-1=0$, which has $\lambda=1$ as a root. Long division gives a factorization of $-(\lambda-1)^2(\lambda+1)=0$.

Problem 15

Consider the polynomial $\det(A-\lambda I)$ factored into its n factors as suggested in the text, ie.

$$\det(A-\lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda)\,.$$

Evaluating this polynomial at $\lambda=0$ we obtain

$$\det(A) = \prod_{i=1}^{n}\lambda_i\,.$$

Problem 16

If A has $\lambda_1=3$ and $\lambda_2=4$ then

$$\begin{aligned}
\det(A-\lambda I) &= (\lambda_1-\lambda)(\lambda_2-\lambda) \\
&= \lambda_1\lambda_2 - (\lambda_1+\lambda_2)\lambda + \lambda^2\,.
\end{aligned}$$

so let

$$\det(A-\lambda I) = 12 - 7\lambda + \lambda^2\,.$$

The quadratic formula gives then

$$\begin{aligned}
\lambda_1 &= \frac{a+d+\sqrt{(a+d)^2-4(ad-bc)}}{2} \\
\lambda_2 &= \frac{a+d-\sqrt{(a+d)^2-4(ad-bc)}}{2}\,.
\end{aligned}$$

Then $\lambda_1 + \lambda_2 = \frac{2(a+d)}{2} = a + d$, which is the linear term in the determinant equation i.e. $a + d = \lambda_1 + \lambda_2$.

Problem 17

We can always generate matrices with any specified eigenvalues by constructing them from

$$S \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} S^{-1},$$

with different choices for the eigenvector matrices S. For example pick eigenvectors given by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then our matrix A is given by

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{(1+2)} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 4 & -5 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 14 & -1 \\ -2 & 13 \end{bmatrix}. \end{aligned}$$

Note that other matrices can be generated in the same manner.

Problem 18

Part (a): the rank of A cannot be determined from the given information. For example, let A be given by

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

then A is diagonal and has eigenvalues as given and A has rank of two. Also consider A given by $A = S\Lambda S^{-1}$ as

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1 & 1 & 0 \\ -1/2 & 3/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{aligned}$$

This matrix has rank of three as can be seen by the following transformations

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 3 & -1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix},$$

which has rank three.

Part (b): We find $|B^T B| = |B^T||B| = |B|^2 = (0 \cdot 1 \cdot 2)^2 = 0$.

Part (c): The eigenvalues of $B^T B$ are given by 0^2, 1^1, and 2^2 or 0, 1, and 4.

Part (d): The eigenvalues of $B+I$ are the eigenvalues of B plus one, which gives 1, 2, and 3. The eigenvalues of $(B+I)^{-1}$ are the inverses of the eigenvalues of $B+I$ and are given by 1, $\frac{1}{2}$, and $\frac{1}{3}$.

Problem 19

Let our matrix A be given by

$$A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

then the trace of A must equal $0 + d = d = \lambda_1 + \lambda_2 = 4 + 7 = 11$, giving that $d = 11$. Also the determinant of A must equal $|A| = -c = \lambda_1 \lambda_2 = 28$, so $c = -28$. Thus we have determined A and it is

$$A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}.$$

Problem 20

Let A be given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}.$$

Then if the eigenvalues are -3, 0, and 3 we must have $\text{trace}(A) = 0+0+c = c = \lambda_1+\lambda_2+\lambda_3 = 0$ (or $c = 0$) and

$$\det(A) = -\begin{vmatrix} 0 & 1 \\ a & 0 \end{vmatrix} = a = \lambda_1 \lambda_2 \lambda_3 = 0\,.$$

Now from what we know about A we can now conclude that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & b & 0 \end{bmatrix}.$$

Now computing the characteristic equation for A we have that

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & b & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ b & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - b) = -\lambda^3 - b\lambda\,,$$

so we have that $b = 9$ and our matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 9 & 0 \end{bmatrix}.$$

Problem 21

We have that $\det(A - \lambda I) = \det(A^T - \lambda I)$, since $I^T = I$. Now let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, these are the examples from Problem 5 in this section. Then both A and B have $\lambda = 1$ with algebraic multiplicity of two. The eigenvectors of A can be computed by computing a basis for the nullspace of the operator $A - \lambda I$. We have that

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

or the span of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The eigenvectors of A^T are given by a basis for the nullspace of $A^T - \lambda I$. We find that

$$A^T - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

or the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Since these vectors are obviously not equivalent the eigenvectors of A and A^T are *different*.

Problem 22

We have

$$M = \begin{bmatrix} 0.6 & 0.8 & 0.1 \\ 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.5 \end{bmatrix}.$$

so that we find $M^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ given by

$$M^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.8 & 0.1 & 0.1 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So we know that M^T has an eigenvalue given by $\lambda = 1$, therefore M must have an eigenvalue $\lambda = 1$. Since a three by three singular Markov matrix must have two eigenvalues equal to zero and one and also must have trace$(M) = \frac{1}{2}$ we know that our third eigenvalue must satisfy

$$0 + 1 + \lambda = \frac{1}{2},$$

showing that $\lambda = -\frac{1}{2}$ as the third eigenvalue. To assemble M construct it from its eigenvalues by assigning random eigenvectors i.e. use the relationship $M = S\Lambda S^{-1}$. Now we can simplify things some by working with M^T which has the same eigenvalues and where we know that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector corresponding to $\lambda = 1$. Thus

$$M^T = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

To compute the inverse of S we augment M^T with the identity matrix and reduce the left hand side to the identity. We find

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & -\frac{1}{2} \end{array}\right].$$

Thus our inverse is given by

$$S^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

So that we find that

$$\begin{aligned} M^T &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 2 & -1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & \frac{3}{2} \\ 1 & 0 & 1 \\ \frac{3}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}, \end{aligned}$$

which is a valid Markov matrix.

Problem 23

Let $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $A_3 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a general matrix where we would like to determine a, b, c, and d. To do this, since $\lambda_1 = \lambda_2 = 0$ we have that from the trace and determinant identities that

$$\begin{aligned} 0 &= a + d \Rightarrow a = -d \\ 0 &= ad - cd \Rightarrow 0 = -d^2 - cb \Rightarrow d^2 = -cb\,. \end{aligned}$$

We can find a solution that satisfies this by letting $a = 1$, $d = -1$, so that $-cb = 1$ and we can take $c = -1$ and $b = 1$ obtaining

$$A_3 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then checking that the eigenvalues of A_3 are as they should be we find that setting $|A_3 - \lambda I| = 0$ that

$$\begin{aligned} |A_3 - \lambda I| &= \begin{vmatrix} 1-\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)(-1-\lambda) + 1 \\ &= -1 + \lambda^2 + 1 = \lambda^2 \end{aligned}$$

Now for each A_i we will check that $A_i^2 = 0$. For A_1 we have that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For A_2 we have that

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For A_3 we have that

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

In general when $a = -d$ and $d^2 = -cb$ then we have

$$\begin{bmatrix} -d & b \\ c & d \end{bmatrix} \begin{bmatrix} -d & b \\ c & d \end{bmatrix} = \begin{bmatrix} d^2 + bc & -db + bd \\ -cd + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem 24

We know since A is singular that at least one eigenvalue is zero. A corresponding eigenvector is given by any vector x such that

$$\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0\,.$$

Two such vectors are

$$\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A third eigenvector/eigenvalue combination in the rank one case (like we have here) is

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

This is because with this vector we have that

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} (2+2+2) = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

So $x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue six.

Problem 25

Note that $Ax = A(\sum_i c_i x_i) = \sum_i c_i A x_i = \sum_i c_i \lambda_i x_i$, and $Bx = \sum_i c_i \lambda_i x_i$ by the same logic. Since A and B have the same *action* on any vector x, they must represent the same linear transformation thus $A = B$.

Problem 26

Consider the expression $|A - \lambda I|$ we have

$$\left| \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} - \begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix} \right| = \left| \begin{bmatrix} B - \lambda I & C \\ 0 & D - \lambda I \end{bmatrix} \right| = |B - \lambda I| \, |D - \lambda I| \,,$$

since the lower left hand corner of $A - \lambda I$ is the zero matrix. We see that this expression vanishes whenever $|B - \lambda I| = 0$ or $|D - \lambda I| = 0$ which happen when $\lambda = 1, 2$ or $\lambda = 5, 7$ respectively. Thus the eigenvalues of A are given by $1, 2, 5$ and 7.

Problem 27

For our A since $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ we see that A is rank one with three eigenvalues given by zero (counted according to multiplicity) and one eigenvalue given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4 \,.$$

For rank one metrics we can easily compute the eigenvectors since they are given by the null vectors of the operator

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

these are given by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

each with eigenvalue zero and the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue four. For C we see that it has a rank of two and thus is not invertible and so one eigenvalue is zero. Since the sum of the rank plus the nullity of C must equal to four we know that the nullspace is of dimension two. Two vectors that span this space are given by

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

The other vectors with eigenvalues of two are given by

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 28

Since the eigenvalues of A were given by 0 with algebraic multiplicity 3 and 4 with algebraic multiplicity 1, the eigenvalues of $A-I$ are -1 with algebraic multiplicity 3 and 3 with algebraic multiplicity 1. If A is a 5x5 matrix of all ones, then A has eigenvalue 0 with multiplicity 4 and a single eigenvalue with value 5. $A-I$ will have 4 eigenvalues with value -1 and a single eigenvalue with value 4. The determinant of B is given by $(-1)^3 3 = -3$. The determinant of B with it is five by five is given by $(-1)^4 4 = 4$.

Problem 29

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ (an upper triangular matrix) the eigenvalues can be read off of the diagonal and are given by 1, 4, and 6. For B computing the characteristic equation we have

$$\begin{aligned} |B - \lambda I| &= \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 2-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 2-\lambda \\ 3 & 0 \end{vmatrix} \\ &= -\lambda(-\lambda(2-\lambda)) - 3(2-\lambda) \\ &= -\lambda^3 + 2\lambda^2 + 3\lambda - 6\,. \end{aligned}$$

From the expression for the determinant we see that $\lambda = 2$ must be a root of the above cubic equation. Factoring our $\lambda - 2$ from the above we see that the characteristic equation is equal to $(\lambda - 2)(-\lambda^2 + 3)$, so the other two roots are $\lambda = \pm\sqrt{3}$. For C we recognize it as a rank one matrix like

$$C = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

which has an eigenvalue/eigenvector combination given by

$$\lambda = 0 \quad \text{with} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\lambda = 6 \quad \text{with} \quad \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Problem 30

Consider $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = (a+b) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and we see that the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue $a+b$. Computing the characteristic equation of A i.e. $|A - \lambda I|$ we find that

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc)\,. \end{aligned}$$

Setting this to zero and solving using the quadratic equation we find that

$$
\begin{aligned}
\lambda &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2} \\
&= \frac{(a+d) \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4bd}}{2} \\
&= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bd}}{2} .
\end{aligned}
$$

From our one relationship among a, b, c, and d replace a with $a = c + d - b$ to obtain

$$
\lambda = \frac{c + 2d - b \pm \sqrt{(c+d-b)^2 - 2(c+d-b)d + d^2 + 4bc}}{2} .
$$

When we expand the terms in the under the radical in the above we find that they simplify to $(c+b)^2$, and our expression for λ then becomes

$$
\lambda = \frac{c + 2d \pm (c+b)}{2} = \begin{cases} \frac{2c+2d}{2} = c + d \\ \frac{2d-2b}{2} = d - b \end{cases}
$$

The first expression $c+d$ is what we found before. The second eigenvalue is given by $d-b$. A much easier way to calculate this value is to recognize that $\mathrm{tr}(A) = \lambda_1 + \lambda_2 = a+b+\lambda_2 = a+d$, so solving for λ_2 we find that $\lambda_2 = d - b$.

Problem 31

To exchange the first two rows and columns of A let $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Considering the nullspace of

$$
\begin{aligned}
A - 11I &= \begin{bmatrix} -10 & 2 & 1 \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{10} \\ 3 & -5 & 3 \\ 4 & 8 & -7 \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{10} \\ 0 & -\frac{22}{5} & \frac{33}{10} \\ 0 & \frac{44}{5} & -\frac{33}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{10} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 1 & -\frac{3}{4} \end{bmatrix} \\
&\Rightarrow \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{10} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 \end{bmatrix} ,
\end{aligned}
$$

which has a nullspace given by $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$. For the matrix PAP we have

$$
PAP - 11I = \begin{bmatrix} -5 & 3 & 3 \\ 2 & -10 & 1 \\ 8 & 4 & -7 \end{bmatrix} ,
$$

which would be worked in the same way as earlier.

Problem 32

Part (a): A basis for the nullspace is given by the span of u. A basis for the column space is is given by a span of $\{v, w\}$

Part (b): Let $x = \frac{1}{3}v + \frac{1}{5}w$, then

$$Ax = \frac{1}{3}Av + \frac{1}{5}Aw = \frac{3}{3}v + \frac{5}{5}w = v + w\,.$$

Then all solutions are given by

$$x = Cu + \frac{1}{3}v + \frac{1}{5}w\,.$$

Part (c): $Ax = u$ will have a solution if and only if u is in the same column space as A. This means that $u \in \text{Span}\{v, w\}$, or that

$$u = C_1 v + C_2 w\,.$$

This implies that u, v, and w are linearly independent in contradiction to the assumed independence of u, v, and w.

Section 6.2 (Diagonalizing a Matrix)

Problem 1

To factor $A = S\Lambda S^{-1}$ we first compute the eigenvalues and eigenvectors of A. The eigenvalues are given by finding the roots of the characteristic equation $|A - \lambda I| = 0$, which in this case becomes

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) = 0\,.$$

or $\lambda = 1$ or $\lambda = 3$. Then the eigenvectors associated with eigenvalue $\lambda = 1$ is given by the nullspace of $A - I$ or the matrix $\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, which is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The eigenvector associated with eigenvalue $\lambda = 3$ is given by the nullspace of the matrix $A - 3I$ or $\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus the matrix whos columns are given by the eigenvectors is given by

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

so that S^{-1} is given by

$$S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Thus A is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

This can easily be checked by multiplying the matrices above. For the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix},$$

Computing its eigenvalues we have to consider

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0\,.$$

Expanding the determinant of the above we have this equal to

$$\lambda(\lambda - 3) = 0\,,$$

so we see that $\lambda = 0$ or $\lambda = 3$. The eigenvalue associated with $\lambda = 0$ is given by the nullspace of A or the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ which is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The eigenvector associated with $\lambda = 3$ is given by the nullspace of $A - 3I$ i.e. the matrix $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}$. This matrix has a nullspace given by the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Thus the matrix S whos columns are the eigenvectors of A is given by

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{so} \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then we see that we can decompose A into the product $S\Lambda S^{-1}$ as

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$

which again can be checked by multiplying the matrices above together.

Problem 2

If $A = S\Lambda S^{-1}$ then

$$A^3 = (S\Lambda S^{-1})(S\Lambda S^{-1})(S\Lambda S^{-1}) = S\Lambda^2\Lambda S^{-1} = S\Lambda^3 S^{-1}\,,$$

and

$$A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}\,.$$

Problem 3

Then A can be assembled from its eigenvectors and eigenvalues by $A = S\Lambda S^{-1}$. We have $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ so $S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and then A is given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}.$$

Problem 4

If $A = S\Lambda S^{-1}$ the the eigenvalue matrix for A is Λ. The eigenvalue matrix for $A + 2I$ is given by $\Lambda + 2I$. The eigenvector matrix for $A + 2I$ is the same as that for A i.e. the matrix S. These are shown by the manipulations

$$S(\Lambda + 2I)S^{-1} = S\Lambda S^{-1} + 2SS^{-1} = A + 2I\,.$$

Problem 5

Part (a): False, A can still have an eigenvalue equal to zero.

Part (b): True, the matrix of eigenvectors S has an inverse.

Part (c): True, S has full rank and is therefore invertible.

Part (d): False, since S could have repeated eigenvalues and therefore possibly a non complete set of eigenvectors.

Problem 6

Then A is a diagonal matrix since $S = I = S^{-1}$ and $A = S\Lambda S^{-1} = \Lambda$. If the eigenvector matrix S is triangular then S^{-1} is also triangular. Forming the product $A = S\Lambda S^{-1}$ we see that left multiplying a triangular matrix S^{-1} onto Λ is multiplication of the the rows of S^{-1} by the diagonal elements of Λ the product $S^{-1}\Lambda$ is also triangular. Since S and ΛS^{-1} are both triangular their product is triangular and therefore A is triangular.

Problem 7

if $A = \left[\begin{array}{cc} 4 & 0 \\ 1 & 2 \end{array}\right]$ then A has eigenvectors given by

$$|A - \lambda I| = \left|\begin{array}{cc} 4-\lambda & 0 \\ 1 & 2-\lambda \end{array}\right| = (4-\lambda)(2-\lambda) = 0\,.$$

Which has solutions $\lambda = 2$ or $\lambda = 4$. The eigenvector associated with the eigenvalue $\lambda = 2$ is given by the nullspace of $A - 2I$ or the matrix $\left[\begin{array}{cc} 2 & 0 \\ 1 & 0 \end{array}\right]$ which is $\left[\begin{array}{c} 0 \\ 1 \end{array}\right]$. The eigenvector associated with $\lambda = 4$ is given by the nullspace of $A - 4I$ i.e. the matrix $\left[\begin{array}{cc} 0 & 0 \\ 1 & -2 \end{array}\right]$. Which has a nullspace given by the span of $\left[\begin{array}{c} 2 \\ 1 \end{array}\right]$. Thus all matrices that diagonalize A are given by

$$S = \begin{bmatrix} 0 & 2\beta \\ \alpha & \beta \end{bmatrix} \quad \text{so} \quad S^{-1} = \frac{1}{(-2\alpha\beta)} \begin{bmatrix} \beta & -2\beta \\ -\alpha & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\alpha} & \frac{1}{\alpha} \\ \frac{1}{2\beta} & 0 \end{bmatrix}.$$

The matrices that diagonalized A are the same ones that diagonalize A^{-1} so the S and S^{-1} above apply to the diagonalization of A^{-1} also.

Problem 8

We can assemble A from its eigenvectors using $S\Lambda S^{-1}$. We find

$$\begin{aligned} A &= S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \left(\frac{-1}{2}\right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{bmatrix} \left(\frac{-1}{2}\right) \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{bmatrix} \end{aligned}$$

Problem 9

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

In addition, A^3 is given by

$$A^3 = AA^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

and A^4 is given by

$$A^4 = AA^3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}.$$

Since $F_0 = 0, F_1 = 1, F_2 = 1, \cdots$ we have that if we define the vector u_n as

$$u_n = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix},$$

Then

$$u_{n+1} = \begin{bmatrix} F_{n+2} \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} + F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = Au_n.$$

With $u_0 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and iterating $u_{n+1} = Au_n$ we see that $u_n = A^n u_0$. If we want to compute F_{20} we extract the second component from u_{20}. Since $u_{20} = A^{20}u_0$, it will help to have u_0 written in terms of the eigenvectors of A. Doing this gives

$$u_0 = \frac{x_1 - x_2}{\lambda_1 - \lambda_2},$$

with $x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$, so that u_{20} becomes

$$u_{20} = \frac{\lambda_1^{20} x_1 - \lambda_2^{20} x_2}{\lambda_1 - \lambda_2}.$$

Now is since for the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ we have

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2},$$

the value of F_{20} is given by

$$\frac{\lambda_1^{20} - \lambda_2^{20}}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{20} - \left(\frac{1-\sqrt{5}}{2} \right)^{20} \right].$$

Problem 10

If $G_{k+2} = \frac{1}{2}(G_k + G_{k+1})$ then defining

$$u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix},$$

we have that

$$u_{k+1} = \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(G_k + G_{k+1}) \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$$

so that we have A given by

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues and eigenvectors of A are given by

$$|A - \lambda I| = \begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = -\lambda \left(\frac{1}{2} - \lambda \right) - \frac{1}{2} = 0$$

Thus we have solving for λ that $\lambda = -\frac{1}{2}$ and $\lambda = 1$. The eigenvectors are given by the nullspace of the operator $A - \lambda I$. For $\lambda = -\frac{1}{2}$ this is the matrix

$$\begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix},$$

which has a nullspace given by the span of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. For $\lambda = 1$ the matrix $A - \lambda I$ is

$$\begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix},$$

which has a nullspace given by the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Part (b): Powers of A can be obtained by $A^n = S\Lambda^n S^{-1}$, with

$$S = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

We then compute that A^n is given by

$$\begin{aligned} A^n &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \left(-\frac{1}{2}\right)^n & 0 \\ 0 & 1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\ \frac{1}{3}&\begin{bmatrix} \left(-\frac{1}{2}\right)^n & 1 \\ -2\left(-\frac{1}{2}\right)^n & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\ \frac{1}{3}&\begin{bmatrix} \left(-\frac{1}{2}\right)^n + 2 & -\left(-\frac{1}{2}\right)^n + 1 \\ -2\left(-\frac{1}{2}\right)^n + 2 & -\left(-\frac{1}{2}\right)^n + 1 \end{bmatrix}. \end{aligned}$$

From which we see that

$$A^\infty = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

Part (c): If $G_0 = 0$ and $G_1 = 1$ then $u_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so that $u_\infty = A^\infty u_0 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus $G_\infty = \frac{2}{3}$ the Gibonacci numbers approach $\frac{2}{3}$.

Problem 11

From the given pieces of the eigenvector decomposition $A = S\Lambda S^{-1}$ we recognize

$$S = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix},$$

so we have the decomposition of

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

Then powers of A are easy to compute. We find that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}.$$

From which we recognize that the requested multiplication is given by

$$\begin{aligned} S\Lambda^k S^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} &= S\Lambda^k \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= S\frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1^{k+1} - \lambda_2^{k+1} \\ \lambda_1^k - \lambda_2^k \end{bmatrix}. \end{aligned}$$

Which has a second component given by $F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$.

Problem 12

The original equation for the λ's is the characteristic equation given by

$$\lambda^2 - \lambda - 1 = 0\,,$$

Since solutions to the quadratic equation we see that multiplying by λ^k this equation can be written as

$$\lambda^{k+2} - \lambda^{k+1} - \lambda^k = 0\,,$$

or

$$\lambda^{k+2} = \lambda^{k+1} + \lambda^k\,.$$

Then the linear combination of λ_1^k and λ_2^k must satisfy this. Thus

$$F_k = \frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}\,,$$

So F_k will satisfy this recurrence relation and has values $F_0 = 0$ and $F_1 = \frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_2} = 1$.

Problem 13

Defining $u_0 = \left[\begin{array}{c} F_1 \\ F_0 \end{array}\right] = \left[\begin{array}{c} 1 \\ 2 \end{array}\right] = x_1 + x_2$, then

$$\begin{aligned} u_{20} &= A^{20}u_0 = A^{20}(x_1 + x_2) = \lambda_1^{20}x_1 + \lambda_2^{20}x_2 \\ &= \lambda_1^{20}\left[\begin{array}{c} \lambda_1 \\ 1 \end{array}\right] + \lambda_2^{20}\left[\begin{array}{c} \lambda_2 \\ 1 \end{array}\right] \end{aligned}$$

So the second component of this vector is given by $\lambda_1^{20} + \lambda_2^{20}$. Thus

$$F_{20} = \left(\frac{1+\sqrt{5}}{2}\right)^{20} + \left(\frac{1-\sqrt{5}}{2}\right)^{20}\,.$$

Problem 14

Given $F_{n+2} = F_n + F_{n+1}$ with initial conditions $F_0 = 0$ and $F_1 = 1$, we would like to prove that F_{3n} is an even number. One might be able to prove this by using the explicit representation of the Fibonacci numbers but it will probably be easier to prove by induction. Sine $F_3 = 2$ we have a starting condition of an induction proof to be true. Then assuming that F_{3k} is an even number for $k \leq n$ we desire to show that it is even for $F_{3(n+1)}$. Now consider $F_{3(n+1)}$ we have using the Fibonacci recurrence that

$$\begin{aligned} F_{3(n+1)} &= F_{3n+3} \\ &= F_{3n+2} + F_{3n+1} \\ &= F_{3n+1} + F_{3n} + F_{3n+1} \\ &= F_{3n} + 2F_{3n+1} \,. \end{aligned}$$

Thus since F_{3n} is even (by the induction hypothesis and $2F_{3n+1}$ is even we see that $F_{3(n+1)}$ is even. Thus our result is proven.

Problem 15

Part (a): True, $\lambda \neq 0$ and therefore A is invertible.

Part (b): This is possible but not definite. If the repeated eigenvalue has enough eigenvectors which is not in general true.

Part (c): It is possible if the $\lambda = 2$ eigenvalue does not have enough eigenvectors.

Problem 16

Part (a): False, the multiple eigenvector could correspond to a nonzero eigenvalue.

Part (b): This must be true of else if not we would have another distinct eigenvector.

Part (c): This is true. There are not enough eigenvectors to fill the eigenvector matrix S.

Problem 17

For the first matrix $A = \begin{bmatrix} 8 & b \\ c & 2 \end{bmatrix}$ since $\det(A) = \lambda_1 \lambda_2 = 25$ we have that

$$16 - bc = 25 \,,$$

or that $bc = -9$. Pick $b = 1$ and $c = -9$ giving $A = \begin{bmatrix} 8 & 1 \\ 9 & 2 \end{bmatrix}$. Then

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 8 - \lambda & 1 \\ -9 & 2 - \lambda \end{vmatrix} \\ &= (8 - \lambda)(2 - \lambda) + 9 = (\lambda - 5)^2 \,. \end{aligned}$$

An eigenvector for $\lambda = 5$ is given by the nullspace of the operator $A - 5I$ which is the matrix $\left[\begin{array}{cc} 3 & 1 \\ 9 & -3 \end{array}\right]$ or $\left[\begin{array}{c} 1 \\ -3 \end{array}\right]$. This matrix has only one eigenvector as requested. For the matrix $\left[\begin{array}{cc} 9 & 4 \\ c & 1 \end{array}\right]$ we must have $\mathrm{Tr}(A) = 10 = \lambda_1 + \lambda_2 = 10$ (which is true) and $\det(A) = 9 - 4c = 25$ or $c = -4$. Thus our matrix A is given by

$$A = \left[\begin{array}{cc} 9 & 4 \\ -4 & 1 \end{array}\right],$$

then the characteristic equation for A is given by

$$\begin{aligned} |A - \lambda I| &= \left|\begin{array}{cc} 9-\lambda & 4 \\ -4 & 1-\lambda \end{array}\right| \\ (9-\lambda)(1-\lambda) + 16 & \\ &= (\lambda - 5)^2 , \end{aligned}$$

as expected. We also have the eigenvectors for this matrix A given by the nullspace of $A-5I$, which in this case is the matrix $\left[\begin{array}{cc} 4 & 4 \\ -4 & -4 \end{array}\right]$ or the vector $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$. Finally, for the matrix $A = \left[\begin{array}{cc} 10 & 5 \\ -5 & d \end{array}\right]$ the determinant requirement gives

$$10d + 25 = 25 ,$$

or $d = 0$ so $A = \left[\begin{array}{cc} 10 & 5 \\ -5 & 0 \end{array}\right]$. Then the characteristic equation for A is given by

$$\begin{aligned} |A - \lambda I| &= \left|\begin{array}{cc} 10-\lambda & 5 \\ -5 & -\lambda \end{array}\right| \\ (\lambda^2 - 10\lambda + 25) = (\lambda - 5)^2 , & \end{aligned}$$

An the eigenvectors are given by the nullspace of $A - 5I$ or the matrix $\left[\begin{array}{cc} 5 & 5 \\ -5 & -5 \end{array}\right]$ or the vector $\left[\begin{array}{c} 1 \\ -1 \end{array}\right]$

Problem 18

The rank of $A - 3I$ is one and therefore since the rank plus the dimension of the nullspace must equal two we see that the nullspace has a dimension of $2 - 1 = 1$ and therefore there does not exist a complete set of eigenvectors for the $\lambda = 3$ eigenvalue. If we changed the $(1, 1)$ or the $(2, 2)$ element to 3.01 then the eigenvalues of A are given by 3 and 3.01 and since they are different we are guaranteed to have independent eigenvectors and A is diagonalizable.

Problem 19

If every λ has a magnitude less than one. Since A is a Markov matrix it has eigenvalues equal to one and therefore will not iterate to zero. For B it has eigenvalues given by solving $|B - \lambda I| = 0$ or

$$\begin{vmatrix} 0.6-\lambda & 0.9 \\ 0.1 & 0.6-\lambda \end{vmatrix} = (0.6-\lambda)^2 - 0.09 = 0\,,$$

or $\lambda = 0.3$ or $\lambda = 0.9$. Since $|\lambda_i| < 1$ we have $A^k \to 0$ as $k \to \infty$.

Problem 20

For A in Problem 19 we know since it is a Markov matrix that one eigenvalue is equal to one. Thus from the trace/determinant formulas its eigenvalues must satisfy

$$\lambda_1 + \lambda_2 = 1.2 \quad \text{and} \quad \lambda_1\lambda_2 = 0.36 - 0.16 = 0.2\,.$$

Thus we see that if $\lambda_1 = 1$ then $\lambda_2 = 0.2$. The eigenvector for $\lambda_1 = 1$ is given by the nullspace of $A - I = \begin{bmatrix} -0.4 & 0.4 \\ 0.4 & -0.4 \end{bmatrix}$ or the span of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda_2 = 0.2$ the eigenvector is given by the nullspace of the matrix $A - 0.2I = \begin{bmatrix} 0.4 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}$ or the span of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus our matrix of eigenvectors is given by

$$S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

with S^{-1} given by

$$S^{-1} = \frac{1}{-1-1}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix},$$

so that we have our eigenvalue decomposition given by $A = S\Lambda S^{-1}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}.$$

Thus since

$$\Lambda^k = \begin{bmatrix} 1^k & 0 \\ 0 & 0.2^k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2^k \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{as} k \to \infty\,,$$

the limit of A^k as $k \to \infty$ is given by

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which has the eigenvector corresponding to the $\lambda = 1$ eigenvalue in its columns.

Problem 21

The eigenvalues for B in Problem 19 are given by $\lambda_1 = 0.3$ and $\lambda_2 = 0.9$. For $\lambda = 0.3$ the eigenvectors are given by the nullspace of $\begin{bmatrix} 0.3 & 0.9 \\ 0.1 & 0.3 \end{bmatrix}$ or the span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. For $\lambda_2 = 0.9$ the eigenvectors are given by the nullspace of $\begin{bmatrix} -0.3 & 0.9 \\ 0.1 & -0.3 \end{bmatrix}$ or the span of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Thus to evaluate $B^{10}u_0$ we decompose u_0 in a basis provided by the eigenvectors of B. Doing this in matrix form we have

$$\begin{bmatrix} 3 & 3 & 6 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \end{bmatrix},$$

where I have concatenated the coefficient vectors used to expand each u_0. For example

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1^1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2^1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then this matrix of coefficients is given by

$$\begin{bmatrix} c_1^1 & c_1^2 & c_1^3 \\ c_2^1 & c_2^2 & c_2^3 \end{bmatrix} = \frac{1}{(-3-3)} \begin{bmatrix} 1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 6 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

or

$$\begin{aligned} \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= 1x_1 \\ \begin{bmatrix} 3 \\ -1 \end{bmatrix} &= -x_1 \\ \begin{bmatrix} 6 \\ 0 \end{bmatrix} &= -x_1 + x_2 \,. \end{aligned}$$

Which could have been obtained by inspection. Thus since $B^{10} = S\Lambda^{10}S^{-1}$, we have that since $S = \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix}$ and $S^{-1} = \frac{-1}{6} \begin{bmatrix} 1 & -3 \\ -1 & -3 \end{bmatrix}$ that

$$\begin{aligned} B^{10} &= \begin{bmatrix} -3 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.3^{10} & 0 \\ 0 & 0.9^{10} \end{bmatrix} \begin{bmatrix} -1/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -30.3^{10} & 3(0.9)^{10} \\ (0.3)^{10} & 0.9^{10} \end{bmatrix} \begin{bmatrix} -1/6 & 1/2 \\ 1/6 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}(0.3)^{10} + \frac{1}{2}0.9^{10} & -\frac{3}{2}(0.3)^{10} + \frac{3}{2}(0.9)^{10} \\ -\frac{1}{6}(0.3)^{10} + \frac{1}{6}0.9^{10} & \frac{1}{2}(0.3)^{10} + \frac{1}{2}(0.9)^{10} \end{bmatrix}. \end{aligned}$$

And more specifically we find that

$$B^{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = B^{10}x_2 = \lambda_2^{10}x_2 = (0.9)^{10}x_2 = (0.9)^{10} \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

$$B^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = B^{10}(-x_1) = -B^{10}x_1 = -\lambda_1^{10}x_1 = -(0.3)^{10} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = (0.3)^{10} \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

and finally that

$$
\begin{aligned}
B^{10}\begin{bmatrix} 6 \\ 0 \end{bmatrix} &= B^{10}(-x_1 + x_2) = -B^{10}x_1 + B^{10}x_2 \\
&= -\lambda_1^{10}x_1 + \lambda_2^{10}x_2 \\
&= -(0.3)^{10}\begin{bmatrix} -3 \\ 1 \end{bmatrix} + (0.9)^{10}\begin{bmatrix} 3 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Problem 22

A has eigenvalues given by the roots of

$$
\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0\,.
$$

Expanding the determinant above we find that the characteristic equation for A is given by

$$
(2-\lambda)^2 - 1 = 0\,,
$$

which has $\lambda = 1$, and $\lambda = 3$ as solutions. For the eigenvalue $\lambda_1 = 1$ the corresponding eigenvector is given by the nullspace of the matrix

$$
\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
$$

or the span of the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The eigenvalue $\lambda_2 = 3$ the corresponding eigenvector is given by the nullspace of the matrix

$$
\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},
$$

or the span of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus our matrix S and S^{-1} are given by

$$
S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.
$$

With these we see that A^k is given by

$$
\begin{aligned}
A^k &= S\Lambda^k S^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3^k \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) \\
&= \begin{bmatrix} 1 & 3^k \\ -1 & 3^k \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} 1+3^k & -1+3^k \\ -1+3^k & 1+3^k \end{bmatrix}
\end{aligned}
$$

Problem 23

Since B is upper triangular the eigenvalues of B are given by the elements on the diagonal and are therefore 3 and 2. The eigenvector for $\lambda = 2$ is given by the nullspace of

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The eigenvector for $\lambda = 3$ is given by the nullspace of

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus our matrix S and Λ are given by

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{so} \quad S^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus B^k is given by $S\Lambda^k S^{-1}$ which in this case is

$$\begin{aligned} B^k &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^k & 3^k \\ -2^k & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3^k & 3^k - 2^k \\ 0 & 2^k \end{bmatrix} \end{aligned}$$

Problem 24

If $A = S\Lambda S^{-1}$, then $|A| = |S\Lambda S^{-1}| = |S||\Lambda||S^{-1}| = |\Lambda|$. But since Λ is a diagonal matrix its determinant is the product of its diagonal elements. Thus we see that $|A| = \prod_{i=1}^{n} \lambda_i$. This quick proof works only when A is diagonalizable.

Problem 25

We have the product of A and B given by

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q & r \\ s & t \end{bmatrix} = \begin{bmatrix} aq + bs & ar + bt \\ cq + sd & cr + st \end{bmatrix},$$

so the trace of AB is given by $\text{Tr}(AB) = aq + bs + cr + dt$. The product in the other direction is given by

$$AB = \begin{bmatrix} q & r \\ s & t \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} qa + rc & qb + rd \\ sa + tc & sb + td \end{bmatrix},$$

Thus we have $\text{Tr}(BA) = aq + rc + sb + td$, which is the same as we had before.

Now choose A as S and B as ΛS^{-1}. Then the product $S(\Lambda S^{-1})$ has the same trace as the product in the reverse order i.e. $(\Lambda S^{-1})S = \Lambda$. The later matrix Λ, has its trace given by $\sum_{i=1}^{m} \lambda_i$. This argument again assumes that A is diagonalizable. For a general $m \times m$ matrix the product AB has elements given by $\sum_{k=1}^{m} a_{ik} b_{kj}$ and the product BA has terms given by $\sum_{k=1}^{m} b_{ik} a_{kj}$, so the trace of AB is given by summing the diagonal terms of AB or

$$\text{Tr}(AB) = \sum_{i=1}^{m} \left(\sum_{k=1}^{m} a_{ik} b_{ki} \right) .$$

while the trace of BA is given by summing the diagonal terms of BA or

$$\text{Tr}(BA) = \sum_{i=1}^{m} \left(\sum_{k=1}^{m} b_{ik} a_{ki} \right) .$$

We can see that these expressions are equal to each other, showing that the two traces are equal.

Problem 26

Now to have $AB - BA = I$ is impossible since the trace of the left hand side id given by

$$\text{Tr}(AB) - \text{Tr}(BA) = 0 \, ,$$

while the trace of the right hand side equals the trace of the $m \times m$ identity matrix or m. Let

$$A = E = \left[\begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right] ,$$

so that the products AB and BA are given by

$$AB = \left[\begin{array}{cc} 1 & -1 \\ -1 & 2 \end{array} \right] \quad \text{and} \quad BA = \left[\begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right] .$$

With these two matrices we see that the difference $AB - BA$ is given by $\left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right]$, which has a trace of zero as required.

Problem 27

If $A = S\Lambda S^{-1}$ and B in block form is given by $B = \left[\begin{array}{cc} A & 0 \\ 0 & 2A \end{array} \right]$ then we can decompose (factor) B as

$$B = \left[\begin{array}{cc} S\Lambda S^{-1} & 0 \\ 0 & S(2\Lambda)S^{-1} \end{array} \right] = \left[\begin{array}{cc} S & 0 \\ 0 & S \end{array} \right] \left[\begin{array}{cc} \Lambda & 0 \\ 0 & 2\Lambda \end{array} \right] \left[\begin{array}{cc} S^{-1} & 0 \\ 0 & S^{-1} \end{array} \right] .$$

We can easily check that this is indeed a factorization of B by explicitly multiplying the matrices on the right hand side together. We find multiplying the two right most matrices together that the above is equal to

$$\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} \Lambda S^{-1} & 0 \\ 0 & (2\Lambda)S^{-1} \end{bmatrix}.$$

Finally multiplying these two matrices together we have

$$\begin{bmatrix} S\Lambda S^{-1} & 0 \\ 0 & S(2\Lambda)S^{-1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix},$$

proving that we have found the decomposition for B. Thus the eigenvalue matrix for the block matrix $\begin{bmatrix} A & 0 \\ 0 & 2A \end{bmatrix}$ is given by $\begin{bmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{bmatrix}$ and the eigenvector matrices are given by

$$S = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} S^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix}.$$

Problem 28

Let our set $\mathcal{S}$ be defined as all four by four matrices such that

$$\mathcal{S} = \{\mathcal{A} | * = \mathcal{S}^{-\infty} \mathcal{A}\mathcal{S}\},$$

for a fixed given S. Then if A_1 and A_2 are in $\mathcal{S}$ we have that

$$A_1 + A_2 = S\Lambda_1 S^{-1} + S\Lambda S^{-1} = S(\Lambda_1 + \Lambda_2)S^{-1},$$

so we see that $A_1 + A_2$ is in $\mathcal{S}$. If $A_1 \in \mathcal{S}$ then $cA_1 = S(c\Lambda_1)S^{-1}$ so $cA_1 \in \mathcal{S}$. Thus $\mathcal{S}$ is a subspace. If $S = I$ then the only possible A's in S are the diagonal ones. This space has dimension four.

Problem 29

Suppose $A^2 = A$, then the column space of A must contain eigenvectors with $\lambda = 1$. In fact all *columns* of A are eigenvectors with eigenvalue equal to one. Thus all vectors in the column space are eigenvectors with eigenvalue $\lambda = 1$. The vectors with $\lambda = 0$ lie in the nullspace and from the first fundamental theorem of linear algebra the dimension of the column space plus the dimension of the nullspace equals n. Thus A will be diagonalizable since we are guaranteed to have enough (here n) eigenvectors.

Problem 30

When A has a nonempty nullspace we do indeed get $n-r$ linearly independent eigenvectors. If x is not in the nullspace of A there is no guarantee that $Ax = \lambda x$ for any constant λ. Thus the r vectors in the column space of A may have no basis (of the column space) such that $Ax = \lambda x$. In addition, the nullspace and the column space can overlap if for instance one of the nullspace vectors is in fact a *column* of the original A.

Problem 31

The eigenvectors of A for $\lambda = 1$ are given by the nullspace of

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

or the span of

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvectors of A for $\lambda = 9$ are given by the nullspace of

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}$$

or the span of

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ so that $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and therefore

$$R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Note that the product RR is given by

$$RR = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix},$$

which should be A, since if $R = S\sqrt{\Lambda}S^{-1}$ then

$$RR = S\sqrt{\Lambda}S^{-1}S\sqrt{\Lambda}S^{-1} = S\Lambda S^{-1}.$$

The square root of Λ would require the square roots of the numbers 9 and -1. The latter is imaginary and the product $R = S\sqrt{\Lambda}R^{-1}$ could *not* be real, since S and S^{-1} are both real but the matrix $\sqrt{\Lambda}$ is not. Therefore the product $S\sqrt{\Lambda}S^{-1}$ could not be real.

Problem 32

We have for $x^T x$ the following

$$\begin{aligned} x^T x &= x^T I x = x^T(AB - BA)x = x^T ABx - x^T BAx \\ &= (Ax)^T(Bx) + (Bx)^T(Ax) = 2(Ax)^T(Bx) \leq 2||Ax||||Bx||, \end{aligned}$$

where we have used the fact that $A^T = A$ and $B^T = -B$ to simplify the inner products

$$x^T ABx = (Ax)^T (Bx) \quad \text{and} \quad x^T BAx = -(Bx)^T (Ax) .$$

Thus $||x||^2 \leq 2||Ax||||Bx||$ so that

$$\frac{1}{2} \leq \frac{||Ax||}{||x||} \frac{||Bx||}{||x||} .$$

Problem 33

If A and B have the same independent eigenvectors and the same eigenvalues then $A = S\Lambda S^{-1}$ and $B = S\Lambda S^{-1}$ so we see that $A = B$.

Problem 34

If S is such that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$ then

$$AB = S\Lambda_1 S^{-1} \cdot S\Lambda_2 S^{-1} = S(\Lambda_1 \Lambda_2) S^{-1} = S(\Lambda_2 \Lambda_1) S^{-1} ,$$

since diagonal matrices commute and therefore

$$AB = S\Lambda_2 S^{-1} \cdot S\Lambda_1 S^{-1} = BA .$$

Problem 35

If A is diagonalizable then $A = S\Lambda S^{-1}$ and the product matrix

$$P \equiv (A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I) ,$$

can be simplified as

$$\begin{aligned} P &= (S\Lambda S^{-1} - \lambda_1 SS^{-1})(S\Lambda S^{-1} - \lambda_2 SS^{-1}) \cdots (S\Lambda S^{-1} - \lambda_n SS^{-1}) \\ &= S(\Lambda - \lambda_1 I)S^{-1} S(\Lambda - \lambda_2 I)S^{-1} S \cdots S(\Lambda - \lambda_n I)S^{-1} \\ &= S(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I) \cdots (\Lambda - \lambda_n I)S^{-1} . \end{aligned}$$

If we consider the product $(\Lambda - \lambda_1 I)(\Lambda - \lambda_2 I)\cdots(\Lambda - \lambda_n I)$, we recognize it as the product of diagonal matrices and we see that it is given by

$$
\left[\begin{array}{ccccc} 0 & & & & \\ & \lambda_2 - \lambda_1 & & & \\ & & \lambda_3 - \lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_n - \lambda_1 \end{array}\right] \times
$$
$$
\left[\begin{array}{ccccc} \lambda_1 - \lambda_2 & & & & \\ & 0 & & & \\ & & \lambda_3 - \lambda_2 & & \\ & & & \ddots & \\ & & & & \lambda_n - \lambda_2 \end{array}\right] \times \cdots \times
$$
$$
\left[\begin{array}{ccccc} \lambda_1 - \lambda_n & & & & \\ & \lambda_2 - \lambda_n & & & \\ & & \lambda_3 - \lambda_n & & \\ & & & \ddots & \\ & & & & 0 \end{array}\right] .
$$

This matrix product simplifies to a diagonal matrix Z who's diagonal elements are given by

$$
\begin{aligned}
d_{11} &= 0(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_n) = 0 \\
d_{22} &= (\lambda_2 - \lambda_1)0(\lambda_2 - \lambda_3)\cdots(\lambda_2 - \lambda_n) = 0 \\
d_{33} &= (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)0\cdots(\lambda_3 - \lambda_n) = 0 \\
&\vdots \\
d_{nn} &= (\lambda_n - \lambda_1)(\lambda_n - \lambda_2)\cdots(\lambda_n - \lambda_{n-1})0 = 0 \,.
\end{aligned}
$$

Since each diagonal element of a diagonal matrix is zero, the total product must also be zero i.e.

$$
(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I) = 0 \,.
$$

Problem 36

If $A = \left[\begin{array}{cc} -3 & 4 \\ -2 & 3 \end{array}\right]$ then the characteristic polynomial of A is given by

$$
|A - \lambda I| = \left|\begin{array}{cc} -3 - \lambda & 4 \\ -2 & 3 - \lambda \end{array}\right| = (-3 - \lambda)(3 - \lambda) + 8 = \lambda^2 - 1 \,.
$$

Now the matrix expression $A^2 - I$ which we compute equals

$$
\left[\begin{array}{cc} -3 & 4 \\ -2 & 3 \end{array}\right]\left[\begin{array}{cc} -3 & 4 \\ -2 & 3 \end{array}\right] - \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] = \left[\begin{array}{cc} 9 - 8 & -12 + 12 \\ 6 - 6 & -8 + 9 \end{array}\right] - \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] = 0 \,.
$$

Thus $A^2 = I$ and it looks like $A^{-1} = A$. To check this directly we can explicitly compute A^{-1} we find that

$$A^{-1} = \frac{1}{-9+8}\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -2 & 3 \end{bmatrix} = A\,,$$

as claimed.

Problem 37

Part (a): Always. A vector in the nullspace of A is automatically an eigenvector with eigenvalue zero.

Part (b): The eigenvectors with $\lambda \neq 0$ will span the column space if there are r independent vectors.

Section 6.3 (Applications to Differential Equations)

Problem 1

Let

$$A = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix},$$

to find the eigenvalues and eigenvectors. From the eigenvalue trace and determinant identity we have

$$\lambda_1 + \lambda_2 = 5 \quad \text{and} \quad \lambda_1\lambda_2 = 4$$

From which we can see that two eigenvalues are given by $\lambda = 1$ and $\lambda = 4$. For $\lambda = 1$ the eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix},$$

which has

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

as an eigenvector. For $\lambda = 4$, the eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix},$$

which has

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

as an eigenvector. Thus the two solutions to the given differential equation is given by

$$x_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t} \quad \text{and} \quad x_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t}$$

The general solution is then a linear combination of the above solutions. To have the general solution equal the given initial condition we have that

$$\begin{bmatrix} 5 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which gives $c_1 = 2$ and $c_2 = 3$. Thus the entire solution is given by

$$x(t) = 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{4t} .$$

Problem 2

Solving $\frac{dz}{dt} = z$ with $z(0) = -2$ gives $z(t) = -2e^t$. Then using this in the equation for y we have

$$\frac{dy}{dt} = 4y + 3z = 4y - 6e^t .$$

To solve this equation we solve the homogeneous part $\frac{dy}{dt} = 4y$ and then find a particular solution to the inhomogeneous part. The homogeneous solution is given by $y(t) = C_2 e^{4t}$ and a particular solution can be found by substituting a solution that looks like the inhomogeneous term. We try a solution of the form $y(t) = Ae^t$. When this is put into our inhomogeneous term we obtain

$$Ae^t - 4Ae^t = -6e^t ,$$

which gives $A = 2$. Thus we have a total solution for $y(t)$ given by

$$y(t) = C_2 e^{4t} + 2e^t .$$

To satisfy the initial condition of $y(0) = 5$ we have that C_2 must be given by the equation $C_2 + 2 = 5$ or $C_2 = 3$. Thus the solution to our full system is then

$$\begin{aligned} z(t) &= -2e^t \\ y(t) &= 3e^{4t} + 2e^t . \end{aligned}$$

Problem 3

If we define $v = y'$ we see that $y'' = 5v + 4y$ so our differential equation becomes the following system

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ 5y' + 4y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} .$$

In this case, our coefficient matrix A is given by $\begin{bmatrix} 0 & 1 \\ 4 & 5 \end{bmatrix}$. The two eigenvalues of this A must satisfy the trace determinant identities

$$\lambda_1 + \lambda_2 = 5 \quad \text{and} \quad \lambda_1 \lambda_2 = -4 .$$

From the first condition we see that $\lambda_1 = 5 - \lambda_2$ which when we put this into the second condition gives a quadratic for λ_2. Solving this gives

$$\lambda_2 = \frac{5 \pm \sqrt{41}}{2}.$$

We can verify these results by substituting $e^{\lambda t}$ directly into the differential equation $y'' = 5y' + 4y$ and solving for λ. When we do this we find that λ must satisfy

$$\lambda^2 - 5\lambda - 4 = 0,$$

the same characteristic equation we found earlier.

Problem 4

From the problems statement the functions $r(t)$ and $w(t)$ must satisfy

$$\begin{aligned} \frac{dr}{dt} &= 6r - 2w \\ \frac{dw}{dt} &= 2r + w. \end{aligned}$$

In matrix form our system is given by

$$\frac{d}{dt}\begin{bmatrix} r \\ w \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r \\ w \end{bmatrix}.$$

The coefficient matrix above has eigenvalues λ_1 and λ_2 that must satisfy

$$\lambda_1\lambda_2 = 10 \quad \text{and} \quad \lambda_1 + \lambda_2 = 7,$$

Thus by inspection $\lambda_1 = 2$ and $\lambda_2 = 5$ are the two eigenvalues. For $\lambda = 2$ the eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix},$$

which has

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

as an eigenvector. For $\lambda = 5$, the eigenvector is given by the nullspace of the following matrix

$$\begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

which has

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

as an eigenvector. Thus the total solutions to the given differential equation is given by a linear combination of the two solutions x_1 and x_2 given by

$$x_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} \quad \text{and} \quad x_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}.$$

That is $u(t)$ has the following form

$$u(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}.$$

The initial condition of $u(0)$ forces c_1 and c_2 to satisfy the following

$$\begin{bmatrix} 30 \\ 30 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Solving this linear system for c_1 and c_2 gives

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}.$$

Thus the entire solution is given by

$$u(t) = 10 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + 10 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t},$$

so the population of rabbits and wolves is given by

$$\begin{aligned} r(t) &= 10e^{2t} + 20e^{5t} \\ w(t) &= 20e^{2t} + 10e^{5t}. \end{aligned}$$

After a long time the ratio of rabbits to wolves is given by

$$\frac{r(t)}{w(t)} = \frac{10e^{2t} + 20e^{5t}}{20e^{2t} + 10e^{5t}} \to 2,$$

as $t \to \infty$.

Problem 5

Our differential equations become

$$\begin{aligned} \frac{dw}{dt} &= v - w \\ \frac{dv}{dt} &= w - v. \end{aligned}$$

Now consider the variable y defined as $y = v + w$. Taking the derivative of y we see that

$$\frac{dy}{dt} = \frac{dv}{dt} + \frac{dw}{dt} = w - v + v - w = 0.$$

So the function $y(t) = v(t) + w(t)$ is a constant for all time. This means that $y(t)$ is always equal to its initial condition $y(t) \equiv y(0)$. The constant value of y is easilty computed

$$y(0) = v(0) + w(0) = 30 + 10 = 40\,.$$

Defining the vector of unknowns u as $u = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$ then we have that u satisfies

$$\frac{du}{dt} = \begin{bmatrix} w - v \\ v - w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}.$$

In the above system of differential equations the coefficient matrix is given by $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, which has eigenvalues λ given by the solution of

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

Expanding this determinant we have $\lambda^2 + 2\lambda = 0$ or $\lambda = 0$ and $\lambda = -2$. The eigenvectors of A for $\lambda = -2$ are given by the nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, or the span of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The eigenvectors of A for $\lambda = 0$ are given by the nullspace of $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, or the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The total solutions to the given differential equation is given by

$$u(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Given the initial conditions of $v(0) = 30$ and $w(0) = 10$ to find c_1 and c_2 we regonize that they have to satisfy the initial condition requirement of u at 0. That is

$$\begin{bmatrix} 30 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which has a solution given by $c_1 = 10$ and $c_2 = 20$. In this case $u(t)$ is given by

$$u(t) = 10 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We can check that $v(t) + w(t) = 40$ for all time by adding the two functions found above. When we do this we find

$$10e^{-2t} + 20 - 10e^{-2t} + 20 = 40\,,$$

as required. When $t = 1$ we have that

$$u(1) = \begin{bmatrix} v(1) \\ u(1) \end{bmatrix} = \begin{bmatrix} 10e^{-2} + 20 \\ -10e^{-2} + 20 \end{bmatrix}.$$

Problem 6

Now our coefficient matrix is -1 times A means that the eigenvectors of $Ax = \lambda x$ becomes $-Ax = -\lambda x$. From which we see that the eigenvectors of $-A$ are the *same* as the eigenvectors of A, and the eigenvalues of $-A$ are the *negative* of the eigenvalues of A. Thus the two eigenvalues of $-A$ are given by $\lambda = 0$ and $\lambda = 2$, with eigenvectors given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so again the solution is given by

$$\begin{bmatrix} v(t) \\ w(t) \end{bmatrix} = 10 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus $v(t) = 10e^{2t} + 20 \to \infty$ as $t \to \infty$.

Problem 7

Let the vector u be defined as $u(t) = \begin{bmatrix} y \\ y' \end{bmatrix}$ then $\frac{du}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$, which has as its solution

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^{At} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

We can evaluate e^{At} using the definition in terms of a Taylor series, that is

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{6}A^3t^3 + \cdots$$

Now

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so that

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

From this we see that

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0) + y'(0)t \\ y'(0) \end{bmatrix},$$

The first component gives $y(t) = y(0) + y'(0)t$.

Problem 8

Substituting $y = e^{\lambda t}$ into our differential equation gives

$$\lambda^2 = 6\lambda - 9.$$

When we solve this for λ we find that $\lambda = 3$ is a double root. The matrix representation for $y'' = 6y' - 9y$ is given by

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

This coefficient matrix has eigenvalues given by the solution of $(\lambda - 3)^2 = 0$ as earlier. To look for the eigenvectors consider

$$\begin{bmatrix} -3 & 1 \\ -9 & 3 \end{bmatrix},$$

which has $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as the only eigenvector. To show that $y = te^{3t}$ is a second solution, evaluate the differential equation for this value of y. We compute

$$\begin{aligned} y &= te^{3t} \\ y' &= e^{3t} + 3te^{3t} \\ y'' &= 3e^{3t} + 3e^{3t} + 9te^{3t} = 6e^{3t} + 9te^{3t}. \end{aligned}$$

Then

$$6y' - 9y = 6e^{3t} + 18te^{3t} - 9te^{3t} = 6e^{3t} + 9te^{3t},$$

which is y^* showing how $y(t)$ satisfies the differential equation.

Problem 9

Part (a): We have

$$\begin{aligned} \frac{d}{dt}(u_1^2 + u_2^2 + u_3^2) &= 2u_1u_1' + 2u_2u_2' + 2u_3u_3 \\ &= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2) \\ &= 0. \end{aligned}$$

Since $u_1^2 + u_2^2 + u_3^2 = ||u||^2$, we see that $||u||$ must be a constant.

Part (b): $||e^{At}u(0)|| = ||u(0)||$ so e^{At} is an orthogonal matrix. When A is skew symmetric $Q = e^{At}$ is an orthogonal matrix.

Problem 10

Part (a): When $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we have two eigenvectors. The first $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with eigenvalue $\lambda = i$, and the second $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ with eigenvalue $\lambda = -i$. To superimpose these two vectors into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we have

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -i \end{bmatrix},$$

so our constants $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$.

Part (b): Thus the solution to

$$\frac{du}{dt} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \left[\begin{array}{c} u_1(t) \\ u_2(t) \end{array}\right],$$

is given by

$$u(t) = c_1 e^{it} \left[\begin{array}{c} 1 \\ i \end{array}\right] + c_2 e^{-it} \left[\begin{array}{c} 1 \\ -i \end{array}\right],$$

with $c_1 = c_2 = 1/2$ this becomes

$$u(t) = \frac{1}{2} e^{it} \left[\begin{array}{c} 1 \\ i \end{array}\right] + \frac{1}{2} e^{-it} \left[\begin{array}{c} 1 \\ -i \end{array}\right].$$

Using Euler's formula of

$$\begin{aligned} e^{it} &= \cos(t) + i\sin(t) \\ e^{-it} &= \cos(t) - i\sin(t). \end{aligned}$$

we have that $u(t)$ becomes

$$\begin{aligned} u(t) &= \frac{1}{2}(\cos(t) + i\sin(t)) \left[\begin{array}{c} 1 \\ i \end{array}\right] + \frac{1}{2}(\cos(t) - i\sin(t)) \left[\begin{array}{c} 1 \\ -i \end{array}\right] \\ &= \frac{1}{2} \left[\begin{array}{c} 2\cos(t) \\ -\sin(t) - \sin(t) \end{array}\right] = \left[\begin{array}{c} \cos(t) \\ \sin(t) \end{array}\right]. \end{aligned}$$

Problem 11

Part (a): The equation $\frac{d^2y}{dt^2} = -y$ is solved by $y(t) = A\cos(t) + B\sin(t)$. To have $y(0) = 1$ and $y'(0) = 0$ we must have $y(t) = \cos(t)$.

Part (b): We write the matrix form for the differential equation $y'' = -y$, by defining the vector u to be $u = \left[\begin{array}{c} y(t) \\ y'(t) \end{array}\right]$ so that

$$\frac{du}{dt} = \left[\begin{array}{c} y'(t) \\ y''(t) \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \left[\begin{array}{c} y(t) \\ y'(t) \end{array}\right].$$

From Part (a) we have that $y(t) = \cos(t)$, so $y'(t) = -\sin(t)$, then

$$u = \left[\begin{array}{c} \cos(t) \\ -\sin(t) \end{array}\right] \quad \text{and} \quad \frac{du}{dt} = \left[\begin{array}{c} -\sin(t) \\ -\cos(t) \end{array}\right],$$

which equals $\left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right] \left[\begin{array}{c} \cos(t) \\ -\sin(t) \end{array}\right] = \left[\begin{array}{c} -\sin(t) \\ -\cos(t) \end{array}\right]$ and $u(0) = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$, showing that this vector solution u solves the differential equation and has the correct initial conditions.

Problem 12

If A is invertible then a particular solution to

$$\frac{du}{dt} = Au - b\,,$$

will be u a constant if and only if $\frac{du}{dt} = 0$ or $0 = Au - b$ or $u = A^{-1}b$.

Part (a): For $\frac{du}{dt} = 2u - 8$. The particular solution is given by $2u = 8$ (or $u = 4$), and the homogeneous solution is given by $\frac{du}{dt} = 2u \Rightarrow u = Ce^{2t}$. Thus the complete solution is given by $u(t) = 4 + Ce^{2t}$.

Part (b): For $\frac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u - \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Then a particular solution is given by (again assuming u is a constant)

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

a particular solution is given by the solution to

$$\frac{du}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} u\,.$$

The coefficient matrix A is then given by $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, which has eigenvalues 2 and 3, with eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then the total solution is then

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}\,,$$

so that the total solution (particular plus the homogeneous) is given by

$$u = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}\,.$$

Problem 13

Assume that c is not an eigenvalue of A. Let $u = e^{ct}v$, where v is a constant vector. Then $\frac{du}{dt} = ce^{ct}v$ and

$$Au = Ae^{ct}v = e^{ct}Av\,,$$

so that the equation $\frac{du}{dt} = Au - e^{ct}b$ becomes

$$\begin{aligned} ce^{ct}e^{ct}Av - e^{ct}b & \\ cv &= Av - b \\ (A - cI)v &= b \\ v &= (A - cI)^{-1}b\,. \end{aligned}$$

Since c is not an eigenvector of A $A-cI$ is invertible, showing that $u = e^{ct}v = e^{ct}(A-cI)^{-1}b$ is a particular solution to the differential equation

$$\frac{du}{dt} = Au - e^{ct}b\,.$$

If c is an eigenvector of A, then $A-cI$ is not invertible and there exists a nonzero v such that $Av = cv$, so that when $e^{ct}v$ is substituted into our differential equation we have $cv = Av - b$ or $0 = -b$ a contradiction.

Problem 14

For a differential equation to be stable we require that $u \to 0$ as $t \to \infty$. For the differential equation $\frac{du}{dt} = Au$, when A is a matrix, this will happen when all the eigenvalues of A have negative real parts. For a two by two systems, this eigenvalue condition breaks down into conditions on the trace (T) and determinant (D) of A. The conditions are that $T \equiv a+d < 0$ and $D \equiv ad - bc > 0$. Since the eigenvalues of a two by two system $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are given by the characteristic equation or

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0\,.$$

This becomes

$$\begin{aligned} (a-\lambda)(d-\lambda) - bc &= 0 \\ \lambda^2 - (a+d)\lambda + ad - bc &= 0 \\ \lambda^2 - T\lambda + D &= 0\,, \end{aligned}$$

when expressed in terms of T and D. From which using the quadratic equation we find the roots given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}\,.$$

So the value of the expression $T^2 - 4D$ separates real from complex eigenvalues. Plotting $T^2 - 4D = 0$ on the determinant D v.s. trace axis T gives the following plot

Defining λ_1 and λ_2 as

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} \quad \text{and} \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}\,.$$

Part (a): For $\lambda_1 < 0$ and $\lambda_2 > 0$ let A be given by

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad A' = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Part (b): For $\lambda_1 > 0$ and $\lambda_2 > 0$ let A be given by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Part (c): For complex λ with real part we need $a > 0$. To find a matrix A that works we know that the components of A must satisfy

$$\begin{aligned} a + d &= \lambda_1 + \lambda_2 \\ ad - bc &= \lambda_1 \lambda_2 . \end{aligned}$$

From which we might try $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$. Then $\lambda_1 + \lambda_2 = 2$ and $\lambda_1 \lambda_2 = 2$. Now to obtain the required A we recall that $A = S \Lambda S^{-1}$ in this case would be given by

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} \left(\frac{1}{2} \right) \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \end{aligned}$$

which is not real and this experiment did not work. As another attempt consider A defined as $A = \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix}$ then $|A| = 2$ and $\text{Tr}(A) = 2$. Lets verify that indeed the eigenvalues are given by $1 \pm i$. The characteristic equation for this A is given by

$$\begin{vmatrix} 2-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0 ,$$

which has solutions given by

$$\lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = 1 \pm i ,$$

and thus this A works.

Problem 15

Consider the definition of the matrix exponential

$$e^{At} = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^3 + \frac{1}{24} A^4 t^4 + \frac{1}{5!} A^5 t^5 + \cdots$$

taking the time derivative of both sides of this expression we compute

$$\begin{aligned} \frac{d}{dt} e^{At} &= A + A^2 t + \frac{1}{2} A^3 t^2 + \frac{1}{6} A^4 t^3 + \frac{1}{4!} A^5 t^4 + \cdots \\ &= A(I + At + \frac{1}{2} A^2 t^2 + \frac{1}{6} A^3 t^3 + \frac{1}{4!} A^4 t^4 + \cdots) \\ &= A e^{At} . \end{aligned}$$

Problem 16

For the matrix $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, we see that the square of B is given by

$$B^2 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and thus all higher powers of B are also the zero matrix. Because of this property of the powers of B the matrix exponential is also simple to calculate

$$\begin{aligned} e^{Bt} &= I + Bt + \frac{1}{2}B^2t^2 + \frac{1}{6}B^3t^3 + \cdots \\ &= I + Bt = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Then

$$\frac{d}{dt}e^{Bt} = \frac{d}{dt}\begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

Problem 17

The solution at time $t+T$ can also be written as $e^{A(t+T)}u(0)$ and since we can view this as the solution at time T propagated for t more time we have

$$e^{At}e^{AT}u(0) = e^{A(t+T)}u(0),$$

so that we see

$$e^{At}e^{AT} = e^{A(t+T)}.$$

Problem 18

From the trace determinant identity for the eigenvalues for $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ we have that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1\lambda_2 = 0$. From which by trial and error we see that $\lambda_1 = 0$ and $\lambda_2 = 1$. The first eigenvector (for $\lambda_1 = 0$) is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and the second eigenvector (for $\lambda_2 = 1$) is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus $S = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ so that $S^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and the matrix of eigenvalues is $\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus A is given by

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then we have

$$
\begin{aligned}
e^{At} &= I + A + \frac{A^2t^2}{2} + \frac{A^3t^3}{3!} + \cdots \\
&= I + S\Lambda S^{-1} + S\Lambda^2 S^{-1}\frac{t^2}{2} + S\Lambda^3 S^{-1}\frac{t^3}{6} + \cdots \\
&= S\left[\Lambda + \Lambda^2\frac{t^2}{2} + \Lambda^3\frac{t^3}{6} + \cdots\right] S^{-1} \\
&= S\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots \end{array}\right] S^{-1} \\
&= S\left[\begin{array}{cc} 1 & 0 \\ 0 & e^t \end{array}\right] S^{-1} \\
&= \left[\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & e^t \end{array}\right]\left[\begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array}\right] \\
&= \left[\begin{array}{cc} e^t & -1 + e^t \\ 0 & 1 \end{array}\right]
\end{aligned}
$$

Note also that $e^{At} = Se^{\Lambda t}S^{-1}$ which may have been a quicker way of deriving the above.

Problem 19

For the general case if $A^2 = A$, then

$$
\begin{aligned}
e^{At} &= I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \cdots \\
&= I + At + \frac{At^2}{2} + \frac{At^3}{6} + \cdots \\
&= I + A(t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots) \\
&= I + A(e^t - 1)\,.
\end{aligned}
$$

For the specific case were $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]$ we see that indeed $A^2 = A$ as

$$
A^2 = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right] = A\,,
$$

so the above formula gives for e^{At}

$$
e^{At} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] + \left[\begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array}\right](e^t - 1) = \left[\begin{array}{cc} e^t & e^t - 1 \\ 0 & 1 \end{array}\right],
$$

the same as we had before.

Problem 20

For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, we have that $e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$ using Problem 18. For $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ we have $e^B = I + B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, since $B^2 = 0$ and all higher order terms in the Taylor expansion definition of e^B are zero. For the matrix $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we have

$$e^{A+B} = I + (e-1)(A+B)\,.$$

since $(A+B)^2 = A + B$. Thus e^{A+B} is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}.$$

Now consider the product of two matrices $e^A e^B$ which is given by

$$e^A e^B = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix} \neq e^{A+B} = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}.$$

And the product in the opposite order

$$e^B e^A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix} \neq e^A e^B\,.$$

Problem 21

For the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$, we have eigenvalues given by $\lambda = 1$ and $\lambda = 3$. The eigenvector for $\lambda = 1$ is given by the nullspace of $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, or the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The eigenvectors for $\lambda = 3$ are given by the nullspace of $\begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}$, or the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then $S = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ so that $S^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ with a matrix of eigenvalues given by $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. Thus we have that e^{At} is given by

$$\begin{aligned} e^{At} &= S e^{\Lambda t} S^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\right) \\ &= \frac{1}{2}\begin{bmatrix} 2e^t & -e^t + e^{3t} \\ 0 & 2e^{3t} \end{bmatrix} = \begin{bmatrix} e^t & -\frac{1}{2}e^t + \frac{1}{2}e^{3t} \\ 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

When $t = 0$ we have $e^{A \cdot 0} = e^0 = I$ and the right hand side of the above gives the same (the identity matrix).

Problem 22

If $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ then $A^2 = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = A$, so from Problem 19 we have that

$$\begin{aligned} e^{At} &= I + (e^t - 1)A \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^t - 1) \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e^t & 3(e^t - 1) \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Problem 23

Part (a): Since $(e^{At})^{-1} = e^{-At}$, them matrix e^{At} is *never* singular.

Section 6.4 (Symmetric Matrices)

Problem 1

$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N$, with $M^T = M$ and $N^T = -N$. For a square matrix

$$M = \frac{1}{2}(A + A^T) = \frac{1}{2} \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 6 \\ 4 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix}.$$

Then N must be given by

$$N = A - M = A - \frac{1}{2}(A + A^T) = \frac{1}{2}(A - A^T).$$

In this case we find that N is given by

$$N = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}.$$

Thus $A = M + N$ is decomposed as

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 3 & 3 & 3 \\ 6 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}.$$

Problem 2

If C is symmetric then A^TCA is also symmetric since

$$(A^TCA)^T = A^TC^TA = A^TCA\,.$$

When A is 6×3, A^T is 3×6 and C must be 6×6, so that finally A^TCA is 3×3.

Problem 3

The dot product of Ax with y equals

$$(Ax)^Ty = x^TA^Ty = x^TAy\,,$$

which is the dot product of x with Ay. If A is not symmetric then

$$(Ax)^Ty = x^TA^Ty\,.$$

Problem 4

Note that since A is symmetric so that it has real eigenvalues and orthogonal eigenvectors. The eigenvalues of A are given by

$$\begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda - 50 = 0\,,$$

This has solutions given by $\lambda = -5$ and $\lambda = 10$. The eigenvectors for $\lambda = -5$ are given by the nullspace of $\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$, or the span of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. The eigenvector for $\lambda = 10$ is given by the nullspace of $\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}$, or the span of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which is orthogonal to the previously computed eigenvector as it must be. To obtain an orthogonal matrix we need to normalize each vector giving

$$Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{so} \quad Q^{-1} = Q^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Thus

$$A = Q\Lambda Q^T = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -5 & 0 \\ 0 & 10 \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Problem 5

For $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$ since $A = A^T$ the eigenvalues must be real and the eigenvectors will be othogonol. To find the eigenvalues we find the roots of the characteristic polinomial

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = 0\,.$$

Expanding the determinant we find that it equals $\lambda(\lambda^2 - 9) = 0$ or $\lambda = 0$ and $\lambda = \pm 3$. For $\lambda_1 = -3$ the eigenvector is given by the nullspace of

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 1 & -1 & 3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which has a nullspace given by the span of $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. For $\lambda_2 = 0$ the eigenvector is given by the nullspace of

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which has a nullspace given by the span of $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. For $\lambda_3 = 3$ the eigenvector is given by the nullspace of

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 1 & -1 & -3/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & -1 & -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Which has a nullspace given by the span of $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$. Thus the matrix with columns of our eigenvectors is given by

$$\hat{Q} = \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}.$$

To make $\hat{Q}$ an orthogonal matrix we need to normalize each vector by its length. Thus we have that

$$Q = \frac{1}{\sqrt{4+4+1}} \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}$$

So that

$$Q^{-1} = Q^T = \frac{1}{3}\left[\begin{array}{ccc} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{array}\right]$$

and $\Lambda = \left[\begin{array}{ccc} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{array}\right]$, so that $A = Q\Lambda Q^T$ with the definitions of Q and Λ given above.

Problem 8

If $A^3 = 0$, then $\lambda = 0$ must be an eigenvalue of A. This is because we can recognize A^3 as A operating on the columns of A^2, which we are told results in the zero matrix. Thus each column of A^2 is an eigenvector of A with eigenvalue zeros. It is easy to find a 2×2 matrix that has $A^2 = 0$. One such matrix is $A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]$. I don't in general see why *all* of the eigenvalues of A must be zero. If $|A^3| = 0$, since $|A^3| = |A|^3$, we see that $|A^3| = 0$ is the same as $\left(\prod_i \lambda_i\right)^3 = 0$ so it seems that all is to be required is that we have *one* eigenvalue of A zero and the product will be zero. In the case when A is symmetric we know that it has an eigenvector decomposition with real eigenvalues and orthogonal eigenvectors. Thus $A = Q\Lambda Q^T$. In this case, from the third power of A we see that

$$A^3 = Q\Lambda^3 Q^T = 0 \Rightarrow \Lambda^3 = 0 \Rightarrow \Lambda = 0\,,$$

so that A must have all zero eigenvalues and in fact must be the zero matrix.

Problem 9

The characteristic equation of a 3×3 matrix A is a third order polynomial. As such, it can have at most two complex roots (which must be complex conjugates) and still be a real polynomial. Thus A must have at least one real eigenvalue. Another way to see this is to consider the trace of A. This must be real since it is a sum of the diagonal elements of A. By the trace, eigenvalue identity we have that $\text{Trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$, if all three of these λ's were complex then $\text{Tr}(A)$ would be complex. Thus at least one eigenvalue of A is real.

Problem 10

It is not stated the x must be real. For example consider the matrix $A = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right]$ then the characteristic equation is $\lambda^2 + 1 = 0$ or $\lambda = \pm i$. For $\lambda_1 = -i$, we have eigenvalues given by the nullspace of

$$\left[\begin{array}{cc} i & -1 \\ 1 & i \end{array}\right] \Rightarrow \left[\begin{array}{cc} i & 1 \\ 0 & 0 \end{array}\right],$$

or the span of $\begin{bmatrix} i \\ 1 \end{bmatrix}$. For the eigenvalue $\lambda_2 = +i$ the second eigenvector x_2 will be the complex conjugate of x_1, or $\begin{bmatrix} -i \\ 1 \end{bmatrix}$. Then the expression $\frac{x^T A x}{x^T x}$ will be complex (since the eigenvectors x are).

Problem 11

For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ the spectral theorem requires calculating $Q\Lambda Q^T$. We begin by computing the eigenvalues of A. We have

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)^2 - 1 = 0 \,.$$

The roots of this quadratic are given by $\lambda = 2$ and $\lambda = 4$. For $\lambda = 2$ the eigenvectors are given as the nullspace of

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} .$$

For $\lambda = 4$ we have the eigenvectors given by the vectors in the nullspace of

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{or} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

Then $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $Q^{-1} = Q^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Thus we have that

$$\begin{aligned} A &= Q\Lambda Q^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 4 & 4 \end{bmatrix} . \end{aligned}$$

From which we see that our spectral decomposition of A is given by

$$\begin{aligned} A &= \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 4 \end{bmatrix} \\ &= 2 \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -2 \end{bmatrix} \right) + 4 \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 4 & 4 \end{bmatrix} \right) . \end{aligned}$$

For the matrix B we perform the same manipulations as for A. First computing the eigenvalues we have

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = 0 \Rightarrow (9-\lambda)(16-\lambda) - 144 = 0 \,.$$

The roots of this quadratic are given by $\lambda = 0$ and $\lambda = 25$. From the spectral theorem for A we have the following decomposition

$$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots + \lambda_n x_n x_n^T \,.$$

This means that all eigenvalues with $\lambda = 0$ don't contribute to the decomposition above. Thus we only need to the calculate the eigenvector for $\lambda = 25$. This is given by the nullspace of

$$\begin{bmatrix} 9-25 & 12 \\ 12 & 16-25 \end{bmatrix} = \begin{bmatrix} -16 & 12 \\ 12 & -9 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}.$$

From which we see that the second eigenvector is given by $x_2 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Thus the spectral decomposition of B is given by

$$B = 25\left(\frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)\left(\frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}\right).$$

Problem 12

For the matrix $A = \begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}$, because $A^T = -A$, A must have imaginary eigenvalues. These are given by the characteristic equation or

$$\begin{vmatrix} -\lambda & 6 \\ -6 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 6^2 = 0 \Rightarrow \lambda = \pm 6i\,.$$

Consider the following 3×3 skew-symmetric matrix

$$B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix},$$

which has eigenvalues given by its characteristic equation

$$\begin{vmatrix} -\lambda & 1 & 2 \\ -1 & -\lambda & 3 \\ -2 & -3 & -\lambda \end{vmatrix} = 0\,.$$

Expanding the above determinant by cofactors we see that above is equivalent to

$$-\lambda\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ -3 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -\lambda & 3 \end{vmatrix} = 0\,.$$

or

$$-\lambda(\lambda^2 + 9) + 1(-\lambda + 6) - 2(3 + 2\lambda) = 0\,.$$

or simplifying

$$\lambda(\lambda^2 + 14) = 0\,.$$

So finally we see that $\lambda = 0$ or $\lambda = \pm i\sqrt{14}$.

Problem 15

For $Bx = \lambda x$ is given by
$$\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \lambda \begin{bmatrix} y \\ z \end{bmatrix}.$$
Which in components gives
$$\begin{aligned} Az &= \lambda y \\ A^T y &= \lambda z \,. \end{aligned}$$

Part (a): Multiplying the first equation n by A^T gives
$$A^T A z = \lambda A^T y = \lambda^2 z \,,$$
is an eigenvalue of $A^T A$.

Part (b): If $A = I$ then λ^2 is an eigenvalue I which are only ones. Thus $\lambda = \pm 1$, are the eigenvalues of B. Since B is of of size four by four we need four eigenvalues and they are $1, 1, -1, -1$. The eigenvectors of B can be obtained from the system of above. Thus z must be on eigenvalues of I and therefore is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$. In this same way $A^T y = \lambda z$ gives four systems for y (providing the four eigenvectors of B) the (since $A^T = I$ we can drop this obtaining).
$$\begin{aligned} y &= -\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \lambda = -1 \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y &= -\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \lambda = -1 \quad \text{and} \quad z = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ y &= 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \lambda = 1 \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y &= 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \lambda = 1 \quad \text{and} \quad z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$
Thus the eigenvector/eigenvalue system is given by
$$Q = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$
with $\text{Diag} = \text{diag}(-1, -1, 1, 1)$.

Problem 16

If $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then from $A^T A z = \lambda^2 z$ we have that $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} z = \lambda^2 z$, or

$$\begin{aligned} 2z &= \lambda^2 z \\ \lambda^2 &= 2 \\ \lambda &= \pm\sqrt{2}\,, \end{aligned}$$

If $z \neq 0$ any vector. Now 1 is 1×1 from the definition of B. Also $z = 0$ with *any* λ will work. To evaluate y consider $A^T y = -\sqrt{2}$ or

$$\begin{bmatrix} 1 & 1 \end{bmatrix} y = -\sqrt{2}\,.$$

so that

$$y = -\frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\,.$$

and consider $\begin{bmatrix} 1 & 1 \end{bmatrix} y = +\sqrt{2}$, for $\lambda = +\sqrt{2}$ so

$$y = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\,.$$

Finally consider if $z = 0$ and λ unknown to obtain

$$\begin{bmatrix} 1 & 1 \end{bmatrix} y = 0\,.$$

so that $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then the eigensystem for B is given by

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

with $Q^{-1} = Q^T$ as required and $\Lambda = \text{diag}(-\sqrt{2}, +\sqrt{2}, 0)$, where I have taken $\lambda_3 = 0$ since $B \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Problem 17

Every 2 by 2 symmetric system can be written as

$$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T = \lambda_1 P_1 + \lambda_2 P_2\,.$$

here P_1 and P_2 are projection matrices (when $||x_1|| = 1$ and $||x_2|| = 2$).

Part (a): Now we have

$$P_1 + P_2 = x_1 x_1^T + x_2 x_2^T = [x_1 x_2] \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} = QQ^T = I$$

since Q is an orthogonal matrix.

Part (b): Also we have

$$P_1 P_2 = x_1 x_1^T (x_2 x_2^T) = x_1 (x_1^T x_2) x_2^T = 0$$

since $x_1^T x_2 = 0$ as x_1 and x_2 can be made orthogonal (since A is symmetric).

Problem 18

Suppose $Ax = \lambda x$ and $Ay = 0y$ with $\lambda = 0$, here y is in the nullspace and x is in the column space.

$$\begin{aligned} x^T A &= \lambda x^T \\ x^T A y &= \lambda x^T y\,, \end{aligned}$$

since $Ay = 0$ then $\lambda x^T y = 0$ since $\lambda \neq 0$, then $x^T y = 0$. Also y in the nullspace and x in the column space but since $A = A^T$, x in the column space means x in the row space but the row space and the nullspace are orthogonal so $x^T y = 0$. If the second eigenvector is not zero say B, then we have $Ay = By$ and $Ax = \lambda x$ so we consider the matrix $B = A - \beta I$, so

$$Bx = (A - \beta I)x = Ax - \beta x = \lambda x - \beta x = (\lambda - \beta)x$$

so

$$By = (A - \beta I)y = Ay - \beta y = \lambda y - \beta y = 0\,.$$

So we see that x is an eigenvector of B with eigenvalues $\lambda - \beta$, and y is an eigenvector or B with eigenvalue 0), so x and y are orthogonal by the previous arguments.

Problem 19

For $B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix}$ which is not symmetric. It has eigenvalues given by

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & d-\lambda \end{vmatrix} = 0\,.$$

On expanding we have

$$(-1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & d-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1-\lambda \\ 0 & 0 \end{vmatrix} = -(1+\lambda)(1-\lambda)(d-\lambda) = 0\,.$$

So $\lambda = -1, d, +1$, which has eigenvectors given by (for $\lambda = -1$) the nullspace of the following matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & d+1 \end{bmatrix}.$$

Problem 27

See the Matlab file `prob_6_4_27.m`. There since A does not have linearly independent columns, the direct calculation of $A^T A$ will not be invertible. Since the projection matrix will project onto the columns of A we can take any set of linearly independent columns from A and construct the projection matrix using $A(A^T A)^{-1} A^T$ with A now understood to contain only linearly independent columns. When this is done Matlab gives computed eigenvectors with a dot product of exactly 1.0. Maybe there is an error somewhere?

Section 6.5 (Positive Definite Matrices)

Problem 15

Consider $x^T(A+B)x$ which by the distributive law equals $x^T Ax + x^T Bx$. Since both both A and B are positive definite we know that $x^T Ax > 0$ and $x^T Bx > 0$ for all $x \neq 0$. Since each term individually is positive, the sum $x^T Ax + x^T Bx$ must be positive for all $x \neq 0$. As this is the definition of positive definite, $A + B$ is positive definite.

Problem 19

If x is an eigenvector of A then

$$x^T Ax = x^T(\lambda x) = \lambda x^T x\,.$$

If A is positive definite then $x^T Ax > 0$. From the above we have that

$$\lambda x^T x > 0 \quad \text{or} \quad \lambda > 0$$

so the eigenvalues of a positive definite matrix must be positive.

Problem 20

Part (a): All the eigenvalues are positive so $\lambda = 0$ is not possible, therefore A is invertible

Part (b): To be positive definite a matrix must have positive (non-zero) diagonal elements. To achieve this for a permutation of the identity we must put all the ones on the diagonal giving the identity matrix.

Part (c): To be a positive definite projection matrix one must have

$$x^T P x > 0\,,$$

for every $x \neq 0$. If $P \neq I$, there exist non-zero x's that are in the orthogonal complement of the column space of P. These x's give $Px = 0$. Thus P will only be positive definite if it has a trivial column space orthogonal complement or $P = I$.

Part (d): A diagonal matrix as described gives

$$x^T D x > 0$$

for all $x \neq 0$ so D would be positive definite.

Part (e): Let A be give by

$$\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

Then $|A| = 2 - 1 = 1 > 0$, but $a = -1 < 0$ so A is not positive definite.

Section 6.6 (Similar Matrices)

Problem 1

If $B = M^{-1}AM$ and $C = N^{-1}BN$ we then have that

$$C = N^{-1}(M^{-1}AM)N = (MN)^{-1}A(MN)$$

So defining $T = MN$ we have $C = T^{-1}AT$. This states that if B is similar to A and C is similar to B then C is similar to A.

Problem 2

If $C = F^{-1}AF$ and also $C = G^{-1}BG$ then $F^{-1}AF = G^{-1}BG$ which gives

$$B = GF^{-1}AFG^{-1} = (FG^{-1})^{-1}A(FG^{-1})$$

Defining $M = FG^{-1}$ we see that $B = M^{-1}AM$, so if C is similar to A and C is similar to B then A is similar to B.

Problem 3

We are looking for a matrix M such that $A = M^{-1}BM$ or $MA = BM$. To find such a matrix let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

then $MA = BM$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or upon multiplying both sides we have

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix},$$

which to be satisfied imposes that $d = 0$ and $a = c$. If we let $a = 1$ and $b = 2$ the selected matrix becomes $M = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. For the next pair of A and B we have that $MA = BM$ or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or upon multiplying together the matrices on each side we have

$$\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -a+c & -b+d \end{bmatrix},$$

which after we set each component of the above equal gives the following system of equations

$$\begin{aligned} a &= -d \\ b &= -c \\ c &= -b \\ d &= -a \end{aligned}$$

Thus we have the restriction that $b = -c$ and $a = -d$. Picking $a = 1$ and $b = 2$ gives

$$M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

For the next pair of A and B we have that $MA = BM$ or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or upon multiplying together the matrices on each side we have

$$\begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 4a+3c & 4b+3d \\ 2a+c & 2b+d \end{bmatrix},$$

which after we set each component of the above equal gives the following system of equations

$$\begin{aligned} -3a + 3b - 3c &= 0 \\ 2a - 3d &= 0 \\ 2a - 3d &= 0 \\ 2b - 2c - 3d &= 0 \end{aligned}$$

This gives the following system for the coefficients a, b, c, and d

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Performing Gaussian elimination on our coefficient matrix produces

$$\begin{bmatrix} -3 & 3 & -3 & 0 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ -3 & 3 & -3 & 0 \\ 0 & 2 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 3 & -3 & -9/2 \\ 0 & 2 & -2 & -3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & -1 & -3/2 \\ 0 & 0 & 0 & -6 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Which implies that $d = 0$, $a = 0$, and $c = b$. If we take $b = 1$, our matrix M becomes

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Problem 4

If A has eigenvalues 0 and 1 it has two linearly independent eigenvectors and therefore can be factorized into $A = S\Lambda S^{-1}$, which says that A and Λ are similar. Now from Problem 2, since every matrix with eigenvalues 0 and 1 are similar to $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then they themselves are similar.

Problem 5

$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has $\lambda = 1$ only.

$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has $\lambda = -1$ and $\lambda = +1$.

$A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has $\lambda = 1$ and $\lambda = 0$.

$$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \text{ has } \lambda = 1 \text{ and } \lambda = 0.$$

$$A_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ has } \lambda = 1 \text{ and } \lambda = 0.$$

$$A_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ has } \lambda = 1 \text{ and } \lambda = 0. \text{ Thus } A_3,\ A_4,\ A_5, \text{ and } A_6 \text{ are similar.}$$

Problem 7

Part (a): If x is in the nullspace of A, then $Ax = 0$ so $M^{-1}x$ when multiplied on the left by $M^{-1}AM$ gives

$$M^{-1}AM(M^{-1}x) = M^{-1}Ax = M^{-1}0 = 0\,.$$

so $M^{-1}x$ is in the nullspace of $M^{-1}AM$.

Part (b): Since for every vector x in the nullspace of A there exists a vector $M^{-1}x$ in the nullspace of $M^{-1}AM$ and for every vector x in the nullspace of $M^{-1}AM$ there exists a vector Mx in the nullspace of A (since $M^{-1}AMx$ must then equal zero). Thus the nullspace of A and $M^{-1}AM$ have the same number of elements and therefore the dimension of the nullspace is the same.

Problem 8

No, the *order* or association of eigenvectors to eigenvalues could be different among the two matrices. If the association *is* the same I would think that $A = B$. With n independent eigenvectors again the answer is no to the question of $A = B$. The logic from the previous discussion still holds. If A has a double eigenvalue of 0 with a single eigenvector proportional to $(1, 0)$, then

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = M^{-1}AM$$

or

$$A = M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M^{-1}$$

with M a matrix the first column of which is the vector $[1, 0]^T$ and the second column of which must be linearly independent from the first column. This gives many possible A's. Consider two different M's

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$$

then the inverses are given by

$$M_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M_2^{-1} = \frac{1}{b} \begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix}$$

Thus $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and A_2 is given by

$$\begin{aligned} A_2 &= M_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M_2^{-1} \\ &= \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{1}{b} \begin{bmatrix} b & -a \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{b} \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \frac{1}{b} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

which does not equal A_1 unless $b = 1$. Thus in this case also there is the possibility of two different matrices with this property.

Chapter 8 (Applications)

Section 8.2 (Markov Matrices and Economic Models)

Problem 13

Since the rows/columns of B are linearly dependent we know that $\lambda = 0$ is an eigenvalue. The other eigenvalue can be obtained by the eigenvalue trace theorem or

$$-.2 - .3 = 0 + \lambda_2 \Rightarrow \lambda_2 = -0.5\,.$$

Since $\lambda_1 = 0$ when $e^{\lambda_1 t}$ multiplies x_1 we have only a multiplication by 1 to the eigenvector x_1. The factor $e^{\lambda_2 t}$ will decay to zero since $\lambda_2 < 0$ and therefore the steady state for this ODE is given by the eigenvector x_1 corresponding to $\lambda_1 = 0$, which in this case is give by

$$x = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}.$$

Therefore the solution, when decomposed in terms of its initial condition, will approach $c_1 x_1$.

Problem 14

The matrix $B = A - I$ has each column summing to 0. The steady state is the same as that of A, but with $\lambda_1 = 0$ and therefore $e^{\lambda_1 t} = 1$.

Problem 15

If each row of a matrix adds to a constant value (say C) this means that the vector $[1, 1, \ldots, 1]^T$ is an eigenvector of A, with the corresponding sum, C, the eigenvalue.

Problem 16

The required product is given by

$$\begin{aligned}(I - A)(I + A + A^2 + A^3 + \ldots) &= I + A + A^2 + A^3 + \ldots - A - A^2 - A^3 - A^4 - \ldots \\ &= I\end{aligned}$$

Problem 20

If A is a Markov matrix then $\lambda = 1$ is an eigenvalue of A and therefore $(I - A)^{-1}$ does not exist, so the given sum cannot sum to $(I - A)^{-1}$.

Chapter 9 (Numerical Linear Algebra)

Section 9.1 (Gaussian Elimination in Practice)

Problem 5

We wish to count the number of operations required to solve the following banded system $Ux = c$ with semiband width $w/2$ or

$$\begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,w/2} & \\ & u_{2,2} & u_{2,3} & \cdots & u_{2,w/2} & u_{2,w/2+1} \\ & & & & & \\ & & & & u_{n-1,n-1} & u_{n-1,n} \\ & & & & & u_{n,n} \end{bmatrix} x = c$$

so at row i we have non-zero elements in columns $j = i, i+1, i+2, \ldots i+w/2$, assuming that $i+w/2$ is less than n. Then a pseudo-code implementation of row-oriented back substitution would look something like the following

Counting the number of flops this requires, we have approximately two flops for every execution of the line $c(j) = c(j) - U(i,j) * c(j)$, giving the following expression for the number of flops

$$\sum_{i=n}^{1} \left[\left(\sum_{j=\min(n,i+w/2)}^{i+1} 2 \right) + 1 \right].$$

Now since

$$\sum_{j=\min(n,i+w/2)}^{i+1} 2 = O(2(w/2)) = O(w)$$

the above sum simplifies (using order notation) to

$$O(n + wn) = O(wn),$$

as requested.

Problem 6

If one knows L and U to solve $LUx = b$ requires one forward and one back solve. The back solve requires $O(1/2n^2)$ flops and the forward solve requires the same flop count $O(1/2n^2)$. Thus to solve $Ax = b$ when one has both L and U requires $O(n^2)$ operations. To solve for x when one has $A = QR$ one could first multiply by $Q^{-1} = Q^T$ to get $Rx = Q^T b$. The product of Q^T with b requires $O(n^2)$ flops, in addition to the back solve requires to "invert" R. Thus to solve $Ax = b$ when $A = QR$ requires $O((1 + 1/2)n^2) = O(3/2n^2)$ flops. Thus it is better to use the LU decomposition.

Problem 7

To invert an upper triangular matrix R we could repeatedly solve $Rx = e_i$ where e_i is the vector of all zeros with a 1 in the i-th component. When $i = 1$, $Rx = e_1$ requires only 1 flop, since x_2, x_3, through x_n are all zero. When $i = 2$, $Rx = e_2$ requires solving a 2x2 upper triangular matrix and as such requires $O(2^2/2) = O(2)$ operations. This is because in this case x_3, x_4, through x_n are all zero. Effectively the leading zeros in the back substitutions allow many of the unknown x_i's to be explicitly determined. In the same way solving $Rx = e_3$ requires $O(3^2/2)$ flops. So in general to solve $Rx = e_i$ requires $O(i^2/2)$ flops. Thus to compute the entire inverse of a triangular system R requires

$$\sum_{i=1}^{n} \frac{i^2}{2} = \frac{1}{2}\sum_{i=1}^{n} i^2 = \frac{1}{2}O(\frac{n^3}{3}) = O(\frac{n^3}{6}).$$

Problem 8

To solve $Ax = b$ for x with partial pivoting when,

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

we would first exchange the first two rows with a permutation matrix P to obtain

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 2 & 2 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}$$

where we have multiplied PA by E_{21} defined as

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}$$

so that we now have

$$E_{21}PA = \begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}.$$

Thus we have for our requested factorization of $PA = LU$ the following

$$PA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix} = LU.$$

For the second example where A is given by

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

we begin by exchanging the first two rows with a permutation P_1 to obtain

$$P_1A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

where the last transformation is obtained by multiplying the above matrix by the elementary elimination matrix E_{21} given by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

giving the following result for the matrix product thus far

$$E_{21}P_1A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}.$$

To continue our elimination with partial pivoting we next exchange rows 2 and 3 with a permutation matrix P_2 defined as

$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

then our chain of matrix products becomes

$$P_2E_{21}P_1A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Which can be obtained from $P_2E_{21}P_1A$ by multiplying on the left by the elementary elimination matrix E_{32} defined by

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}.$$

In total we then have $E_{32}P_2E_{21}P_1A = U$, which in matrix form is the following

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The next step is to pass the permutation matrices "through" the elementary elimination matrices so that we can get all elimination matrices on the left and all permutation matrices on the right. Something like $E_{32}\hat{E}_{21}P_2P_1A = U$. This can be performed by recognizing that the product of P_2 and E_{21} can be factored as

$$P_2E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1/2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \hat{E}_{21}P_2\,.$$

Thus the initial factorization of $E_{32}P_2E_{21}P_1A = U$, can be written as $E_{32}\hat{E}_{21}P_2P_1A = U$, and we then have that $P_2P_1A = \hat{E}_{21}^{-1}E_{32}^{-1}U$, which in matrix form is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} U\,.$$

which after we multiply all matrices in the above we can obtain our final $PA = LU$ decomposition as

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This can easily be checked for correctness by multiplying the matrices on both sides and showing that they are the same.

Problem 9

For the A given

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

we can compute specific elements of A^{-1} from the cofactor expansion formula, which is

$$A^{-1} = \frac{1}{\text{Det}(A)} C^T \quad \text{with} \quad C_{ij} = (-1)^{i+j} \text{Det}(M_{ij})$$

with M_{ij} the minor (matrix) of the (i,j)-th element. Then based on the A above we can investigate if the $(1,3)$, $(1,4)$, $(2,4)$, $(3,1)$, $(4,1)$, and $(4,2)$ elements of A^{-1} are zero. These are the elements of A which are zero and one might hope that a zero element in A would imply a zero element in A^{-1}. We can compute each element in tern. First $(A^{-1})_{1,3}$,

$$(A^{-1})_{1,3} = \frac{1}{\text{Det}(A)} C_{31} = \frac{1}{\text{Det}(A)} (-1)^{3+1} \text{Det}(M_{31})$$

Since every term in the inverse will depend on the value of $\text{Det}(A)$ we will compute it now. We find

$$\begin{aligned} \text{Det}(A) &= +1 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 1 \left[1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \right] - 1 \left[1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right] \\ &= - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \end{aligned}$$

Then we have that

$$\text{Det}(M_{31}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

so that $(A^{-1})_{1,3} = \frac{1}{-1}(1) = -1 \neq 0$.

Problem 10

We first find the LU factorization of the given A

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix}$$

obtained without partial pivoting. Note that in a realistic situation on would want to use partial pivoting since we assume that $\epsilon \ll 1$. Now our A can be reduced to

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix},$$

by multiplying A by the elementary elimination matrix E_{21} defined as

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{\epsilon} & 1 \end{bmatrix}.$$

Thus we have the direct LU factorization (without partial pivoting) given by

$$A = \begin{bmatrix} \epsilon & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix}.$$

Thus our system $Ax = b$ is given by

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix} \begin{bmatrix} \epsilon & 1 \\ 0 & 1 - \frac{1}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + \epsilon \\ 2 \end{bmatrix}.$$

Note that for this simple system we could solve $Ly = b$ and then solve $Ux = y$ exactly. Doing so would not emphasis the rounding errors that are present in this particular example. Thus we have chosen to solve this system by Gaussian elimination without pivoting using the teaching code `slu.m`. Please see the Matlab file `prob_9_1_10.m` for the requested computational experiments. There we see that without pivoting when ϵ is near 10^{-15} (near the unit round for double precision numbers) the error in the solution can be on the order of 10%. When one introduces pivoting (by switching the first two rows in this system) this error goes away and the solution is computed at an accuracy of $O(10^{-16})$.

Problem 14

To directly compute $Q_{ij}A$ would require two steps. First multiplying row i of A by $\cos(\theta)$ by row j of A by $-\sin(\theta)$ and adding these two rows. This step requires $2n$ multiplications and n additions. Second, multiply row i by $\sin(\theta)$ and adding to $\cos(\theta)$ multiplied by row j. Again requiring the same number of multiplications and additions as the first step. Thus in total we require $4n$ multiplications and $2n$ additions to compute $Q_{ij}A$.

Section 9.2 (Norms and Condition Numbers)

Problem 4

Since the condition number is defined as $\kappa(A) = ||A||||A^{-1}||$ from $||AB|| \leq ||A||||B||$ with $B = A^{-1}$ we have

$$||I|| \leq ||A||||A^{-1}|| = \kappa(A)\,,$$

but $||I|| = 1$ so $\kappa(A) \geq 1$ for every A.

Problem 5

To be symmetric implies the matrix is diagonalizable and $A = S\Lambda S^{-1}$ becomes $A = Q\Lambda Q^T$. Since every eigenvalue must be 1 we have $\Lambda = I$ and $A = QQ^T = I$, so A is actually the identity matrix.

Problem 6

If $A = QR$ then we have $||A|| \leq ||Q||||R|| = ||R||$. We also have $R = Q^T A$ so $||R|| \leq ||Q^T||||A|| = ||A||$. Thus $||A|| = ||R||$. To find an example of $A = LU$ such that $||A|| < ||L||||U||$. Let

$$L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

then we have

$$L^T L = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$

and

$$U^T U = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

Problem 7

Part (a): The triangle inequality gives $||(A+B)x|| \leq ||Ax|| + ||Bx||$

Part (b): It is easier to prove this with definition three from this section, that is

$$||A|| = \text{Max}_{x \neq 0} \frac{||Ax||}{||x||}\,.$$

Thus we have

$$\begin{aligned}
||A+B|| &= \text{Max}_{x\neq 0}\frac{||(A+B)x||}{||x||} \\
&\leq \text{Max}_{x\neq 0}\left(\frac{||Ax||+||Bx||}{||x||}\right) \\
&\leq \text{Max}_{x\neq 0}\frac{||Ax||}{||x||}+\text{Max}_{x\neq 0}\frac{||Bx||}{||x||} \\
&\leq ||A||+||B||
\end{aligned}$$

Problem 8

From $Ax = \lambda x$ we have that $||Ax|| = ||\lambda x|| = |\lambda| ||x||$, but since $||Ax|| \leq ||A|| ||x||$ we then have that $|\lambda| ||x|| \leq ||A|| ||x||$ or $|\lambda| \leq ||A||$ as requested.

Problem 9

Defining $\rho(A) = |\lambda_{\text{max}}|$ to find counter examples to the requested norm properties we will note that from previous discussions A and B cannot have the same eigenvectors or else $\lambda_A + \lambda_B = \lambda_{A+B}$. The requirement of not having the same eigenvalues can be simplified to the requirement that $AB \neq BA$. Thus diagonal matrices won't work for finding a counter example. Thus we look to the triangular matrices for counter examples. Consider A and B defined as

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}$$

Then since each matrix is triangular the eigenvalues are easy to calculate (they are the elements on the diagonal) and we have $\rho(A) = \rho(B) = 1$. Also note that

$$AB = \begin{bmatrix} 101 & 10 \\ 10 & 101 \end{bmatrix} \neq \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix} = BA$$

so A and B don't share the same eigenvectors and $\rho(A+B) \neq \rho(A) + \rho(B)$. Now the sum of A and B is given by

$$A + B = \begin{bmatrix} 1 & 10 \\ 10 & 1 \end{bmatrix}$$

which has eigenvalues given by the solution to $\lambda^2 - \text{Tr}(A+B)\lambda + \text{Det}(A+B) = 0$, which for this problem has $\lambda_1 = -9$ and $\lambda_2 = 11$ so $\rho(A+B) = 11$. Thus we see that

$$\rho(A+B) = 11 > \rho(A) + \rho(B) = 1 + 1 = 2$$

and we have a counterexample for the first condition (the triangle inequality for matrix norms). For the second condition we have the product AB given by

$$AB = \begin{bmatrix} 101 & 10 \\ 10 & 101 \end{bmatrix}$$

which has eigenvalues given by $\lambda_1 = 91$ and $\lambda_2 = 111$, thus we have

$$\rho(AB) = 111 > \rho(A)\rho(B) = 1\,,$$

providing a contradiction to the second triangle like inequality (this time for matrix multiplication). These eigenvalue calculations can be found in the Matlab file `prob_9_2_9.m`.

Problem 10

Part (a): The condition number of A is defined by $\kappa(A) = ||A||||A^{-1}||$, while the condition number of A^{-1} is defined by $\kappa(A^{-1}) = ||A^{-1}||||(A^{-1})^{-1}|| = ||A^{-1}||||A|| = \kappa(A)$

Part (b): The norm of A is given by $\lambda_{\max}(A^TA)^{1/2}$, and the norm of A^T is given by $\lambda_{\max}((A^T)^TA^T)^{1/2} = \lambda_{\max}(AA^T)^{1/2}$. From the SVD of A we have that $A^TA = V\Sigma^2V^T$ and $AA^T = U\Sigma^2U^T$, so both A^TA and AA^T have the *same* eigenvalues, i.e. the singular values of A and therefore $\lambda_{\max}(A^TA) = \lambda_{\max}(AA^T)$, showing that A and A^T have the *same* matrix norm.

Problem 11

From the definition of the condition number of a matrix $\kappa(A) = ||A||||A^{-1}||$, since A is symmetric $||A|| = \text{Max}(|\lambda(A)|)$ and A^{-1} will be symmetric so

$$||A^{-1}|| = \text{Max}(\lambda(A^{-1})) = \text{Max}\left|\frac{1}{\lambda(A)}\right| = \frac{1}{\text{Min}(|\lambda(A)|)}$$

From the A given we will have eigenvalues given by the solution of

$$\lambda^2 - \text{Tr}(A)\lambda + \text{Det}(A) = 0$$

which for this problem has solutions given by (these are computed in the Matlab file `prob_9_2_11.m`) $\lambda_1 = 0.00004999$, and $\lambda_2 = 2.00005$. Thus an estimate of the condition number is given by

$$\kappa(A) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} = \frac{2.00005}{0.00004999} = 40000\,.$$

Section 9.3 (Iterative Methods for Linear Algebra)

Problem 15 (eigenvalues and vectors for the 1,-2,1 matrix)

In general, for banded matrices, where the values on each band are constant, *explicit* formulas for the eigenvalues and eigenvectors can be obtained from the theory of finite differences. We will demonstrate this theory for the 1,-2,1 tridiagonal matrix considered here. Here we will change notation from the book and let the unknown vector, usually denoted by x be denoted by w. In addition, because we will use the symbol i for the imaginary unit ($\sqrt{-1}$), rather than the usual "i" subscript convention we will let our independent variable (ranging over components of the vector x or w) be denoted t. Thus notationally $x_i \equiv w(t)$. Converting our eigenvector equation $Aw = \lambda w$ into a *system* of equations we have that $w(t)$, must satisfy

$$w(t-1) - 2w(t) + w(t+1) = \lambda w(t) \quad \text{for} \quad t = 1, 2, \ldots, N\,,$$

with boundary conditions on $w(t)$ taken such that $w(0) = 0$ and $w(N+1) = 0$. Then the above equation can be written as

$$w(t-1) - (2+\lambda)w(t) + w(i+1) = 0\,.$$

Substituting $w(t) = m^t$ into the above we get

$$m^2 - (2+\lambda)m + 1 = 0\,.$$

Solving this quadratic equation for m gives

$$m = \frac{(2+\lambda) \pm \sqrt{(2+\lambda)^2 - 4}}{2}$$

From this expression if $|2+\lambda| \geq 2$ the expression under the square root is positive and the two roots are both *real*. With two real roots, the only solution that satisfies the boundary conditions is the trivial one ($w(t) = 0$). If $|2+\lambda| < 2$ then m is a complex number and the boundary conditions can be satisfied non-trivially. To further express this, define θ such that

$$2 + \lambda = 2\cos(\theta)$$

then the expression for m (in terms of θ) becomes

$$m = \frac{2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1}$$

or

$$m = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}$$

from the theory of finite differences the solution $w(t)$ is a linear combination of the two fundamental solutions or

$$w(t) = Ae^{i\theta t} + Be^{-i\theta t}\,. \tag{4}$$

Imposing the two homogeneous boundary condition we have the following system that must be solved for A and B

$$\begin{aligned} A + B &= 0 \\ Ae^{i\theta(N+1)} + Be^{-i\theta(N+1)} &= 0 \end{aligned}$$

Putting the first equation into the second gives

$$B(e^{i\theta(N+1)} - e^{-i\theta(N+1)}) = 0$$

Since B cannot be zero (else the eigenfunction $w(t)$ is identically zero) we must have θ satisfy

$$\sin(\theta(N+1)) = 0$$

Thus $\theta(N+1) = \pi n$ or

$$\theta = \frac{\pi n}{N+1} \quad \text{for} \quad n = 1, 2, \ldots, N$$

Tracing θ back to the definition of λ we have that

$$\lambda = -2 + 2\cos(\theta) = -2 + 2\cos(\frac{\pi n}{N+1})$$

Using the trigonometric identity

$$1 - \cos(\psi) = 2\sin(\frac{\psi}{2})^2$$

we get

$$\lambda_n = -4\sin(\frac{\pi n}{2(N+1)})^2 \quad \text{for} \quad n = 1, 2, 3, \ldots, N$$

For the eigenvalues of the $1, -2, 1$ discrete one dimensional discrete Laplacian. To evaluate the eigenvectors we go back to Eq. 4 using our new definition of θ. We get that

$$\begin{aligned} w(t) &\propto e^{i\theta t} - e^{-i\theta t} \\ &\propto \sin(\theta t) \\ &\propto \sin(\frac{\pi n}{N+1} t) \quad \text{for} \quad n = 1, 2, 3, \ldots, N \end{aligned}$$

Here the range of t is given by $t = 1, 2, \ldots, N$. These are the results given in the book when $n = 1$ i.e. we are considering only the first eigenvalue and eigenvector.

Problem 18 (an example of the QR method)

If A is given by

$$A = \left[\begin{array}{cc} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & 0 \end{array} \right] = QR$$

with a QR decomposition given by

$$QR = \left[\begin{array}{cc} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{array} \right] \left[\begin{array}{cc} 1 & x \\ 0 & y \end{array} \right]$$

Then expanding the matrix product above we must have for x and y the following equations to hold

$$\begin{aligned} x\cos(\theta) - y\sin(\theta) &= \sin(\theta) \\ x\sin(\theta) + y\cos(\theta) &= 0\,. \end{aligned}$$

Then solving the second equation for x we have $x = -\frac{y\cos(\theta)}{\sin(\theta)}$, which when put into the first equation gives

$$\left(\frac{-y\cos(\theta)}{\sin(\theta)} \right) \cos(\theta) - y\sin(\theta) = \sin(\theta)$$

which gives for y the solution of $y = -\sin(\theta)^2$. Thus we have for x that $x = \sin(\theta)\cos(\theta)$. With these two values our QR decomposition is given by

$$QR = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & \sin(\theta)\cos(\theta) \\ 0 & -\sin(\theta)^2 \end{bmatrix}$$

This gives for RQ product the following

$$\begin{aligned} RQ &= \begin{bmatrix} 1 & \sin(\theta)\cos(\theta) \\ 0 & -\sin(\theta)^2 \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) + \sin(\theta)^2\cos(\theta) & -\sin(\theta) + \sin(\theta)\cos(\theta)^2 \\ -\sin(\theta)^3 & -\cos(\theta)\sin(\theta)^2 \end{bmatrix} \end{aligned}$$

Showing that the $(2,1)$ entry is now $-\sin(\theta)^3$ as expected.

Problem 19

If A *is* an orthogonal matrix itself then the QR decomposition for A has $Q = A$ and $R = I$ so $RQ = IA = A$. Thus the QR method for computing the eigenvalues of A will fail.

Problem 20

If $A - cI = QR$, then let $A_1 = RQ + cI$, and by multiplying this equation by Q on the left we obtain

$$QA_1 = QRQ + cQ\,.$$

Next since $QR = A - cI$, the above QA_1 becomes

$$QA_1 = (A - cI)Q + cQ = AQ$$

Now multiplying by $Q^T = Q^{-1}$ on the left of the above we obtain $A_1 = Q^{-1}AQ$, so A_1 is a similarity transformation of A and therefore has the same eigenvalues as A.

Problem 21

From the given decomposition $Aq_j = b_{j-1}q_{j-1} + a_j q_j + b_j q_{j+1}$, since the q_j are orthogonal then $q_j^T q_i = \delta_{ij}$ so multiplying on the left by q_j^T gives

$$q_j^T A q_j = 0 + a_j q_j^T q_j + 0$$

so we have that $a_j = \frac{q_j^T A q_j}{q_j^T q_j}$. Our equation says that $AQ = QT$ where T is a tridiagonal matrix with main diagonal given by the a_j and b on the sub and super diagonal.

Problem 22

See the Matlab code `prob_9_3_21.m` and `lanczos.m`.

Problem 23

If A is symmetric, from the shifted QR method and Problem 20 we know that A_1 is related to A by $A_1 = Q^{-1}AQ$. Since $Q^{-1} = Q^T$ we have that $A_1 = Q^T AQ$, so the transpose of this expression gives

$$A_1^T = Q^T A^T Q = Q^T AQ = A_1$$

so A_1 is symmetric. Next let $A_1 = RAR^{-1}$ an show that A_1 is tridiagonal. Since R is upper triangular R^{-1} is upper triangular. Then A_1 is the product of an upper triangular matrix times a tridiagonal matrix times an upper triangular matrix. Now a tridiagonal matrix A, times an upper triangular matrix R^{-1} gives a matrix that is upper triangular with an additional nonzero subdiagonal. Such a matrix is called an upper Hessenberg matrix. Now an upper triangular matrix R times an upper Hessenberg matrix (AR^{-1}) will be upper Hessenberg, so the entire product RAR^{-1} is upper Hessenberg. From the first part of this problem A_1 is symmetric and therefore since $(A_1)_{ij} = 0$ for $i > j+1$ we must have $(A_1)_{ij} = 0$ for $j > i+1$ and A_1 is therefore triangular.

Problem 24

Following the hint in the book if $|x_i| \geq |x_j|$ for all j, then we have

$$|\sum_j a_{ij}x_j| = |x_i||\sum_j a_{ij}\frac{x_j}{x_i}| \leq |x_i|\sum_j |a_{ij}|\left|\frac{x_j}{x_i}\right| \leq |x_i|\sum_j |a_{ij}| < |x_i|\,.$$

Since the sum $\sum_j |a_{ij}| < 1$. Thus if x is an eigenvector with eigenvalue λ we have that the i-th component of $Ax = \lambda x$ is given by

$$\lambda x_i = \sum_j a_{ij}x_j$$

so taking the absolute value of both sides and using the above we obtain $|\lambda x_i| < |x_i|$ which by dividing by $|x_i|$ on both sides give $|\lambda| < 1$.

Problem 25

For the first A we have that (from the Gershgorin circle theorem) that

$$\begin{aligned} |\lambda - 0.3| &\leq 0.5 \\ |\lambda - 0.2| &\leq 0.7 \\ |\lambda - 0.1| &\leq 0.6 \end{aligned}$$

Since the sum of the absolute values of the elements along every row is less than 1, from problem 24 in this book we know that $|\lambda| < 1$, and therefore that $|\lambda|_{\max} < 1$. The three Gershgorin circles for the first A are given by the above. Thus incorporating the above we can derive that

$$
\begin{aligned}
-0.2 &\leq \lambda \leq 0.8 \\
-0.5 &\leq \lambda \leq 0.9 \\
-0.5 &\leq \lambda \leq 0.7
\end{aligned}
$$

Thus all eigenvalues must satisfy $-0.5 \leq \lambda 0.9$.

For the second matrix the rows don't add to something less than 1, so we can't conclude that $|\lambda| < 1$. But the Gershgorin circle theorem still holds and we can conclude that

$$
\begin{aligned}
|\lambda - 2| &\leq 1 \\
|\lambda - 2| &\leq 2 \\
|\lambda - 2| &\leq 1
\end{aligned}
$$

Thus the most restrictive condition holds and we have only that the eigenvalues of A can be bounded by $1 \leq \lambda \leq 3$.