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$\ddot{x} + x + \epsilon(x^2-1)\dot{x} = 0$ let $x = x_0(\tau,T) + \epsilon x_1(\tau,T) + \epsilon^2 x_2(\tau,T) + O(\epsilon^3)$

w/ $T = \epsilon t$, $\tau = t$

$$\Rightarrow \partial_{\tau\tau}x_0 + \epsilon(\partial_{\tau\tau}x_1 + 2\partial_{\tau T}x_0) + O(\epsilon^2)$$

$$+ x_0 + \epsilon x_1 + \epsilon\left(x_0^2 - 1\right)\left(\partial_\tau x_0 + O(\epsilon)\right) = 0$$

$$\Rightarrow \partial_{\tau\tau}x_0 + \epsilon(\partial_{\tau\tau}x_1 + 2\partial_{\tau T}x_0) + x_0 + \epsilon x_1 + \epsilon(x_0^2-1)\partial_\tau x_0 + O(\epsilon^2) = 0$$

$\Rightarrow$ $O(1)$ $\quad \partial_{\tau\tau}x_0 + x_0 = 0$

$O(\epsilon)$ $\quad \partial_{\tau\tau}x_1 + x_1 = -2\partial_{\tau T}x_0 - (x_0^2-1)\partial_\tau x_0$

let $x_0 = r(T)\cos(\tau + \phi(T))$ put in $O(\epsilon)$ eq

$$\partial_{\tau\tau}x_1 + x_1 = -2\partial_T\left(-r(T)\sin(\tau+\phi(T))\right) - \left(r^2\cos^2(\tau+\phi) - 1\right)\left(-r\sin(\tau+\phi)\right)$$

$= \cancel{(-2(r}$

$$= 2\left[r'(T)\sin(\tau+\phi(T)) + r(T)\cos(\tau+\phi(T))\phi'(T)\right]$$

$$- r\sin(\tau+\phi)\left[r^2\cos^2(\tau+\phi) - 1\right]$$

$= -\cancel{1}\cancel{\left[-2\left(r'\sin(\tau+\phi(T)) + r(T)\cos(\tau+\phi(T))\phi'(T)\right)\right.} +$

using trig identity. get

$$\partial_{\tau\tau}x_1 + x_1 = 2\left[r'\sin(\tau+\phi) + r\cos(\tau+\phi)\phi'\right] + r\sin(\tau+\phi)$$

$$- r^3\sin(\tau+\phi)\cos^2(\tau+\phi)$$

$$\partial_{\tau\tau}x_1 + x_1 = 2\left[r'\sin(\tau+\phi) + r\cos(\tau+\phi)\,\phi'\right] + r\sin(\tau+\phi)$$
$$- \frac{r^3}{4}\left[\sin(\tau+\phi) + \sin 3(\tau+\phi)\right]$$

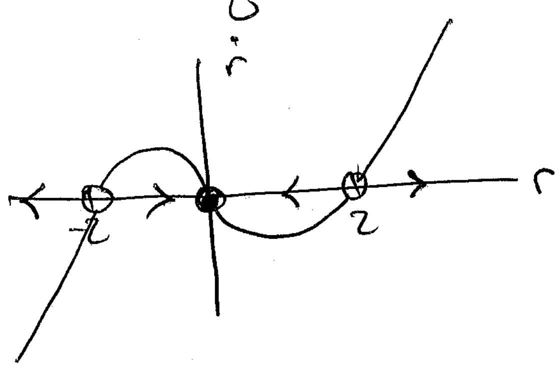
$$= \left(2r' + r - \frac{r^3}{4}\right)\sin(\tau+\phi)$$
$$+ \left(2r\phi'\right)\cos(\tau+\phi) + \underbrace{-\frac{r^3}{4}\sin(3(\tau+\phi))}_{\text{not secular}}$$

$$\Rightarrow \quad +2r' + r - \frac{r^3}{4} = 0$$

$$2r\phi' = 0$$

$$r' = \left(\frac{r^3}{4} - r\right)\frac{1}{2} = \frac{1}{2\cdot 4}r(r^2-4) = \frac{r}{8}(r^2-4)$$

$$= \frac{r}{8}(r-2)(r+2)$$



$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0$$

~~Let~~ Solve using 2 timing $\quad \tau = t$, $T = \epsilon t$

$$x = x_0(\tau, T) + \epsilon x_1(\tau, T) + \epsilon^2 x_2(\tau, T) + O(\epsilon^3)$$

$$\dot{x} = x_{0\tau} + \epsilon(x_{0T} + x_{1\tau}) + O(\epsilon^2)$$

$$\ddot{x} = x_{0\tau\tau} + \epsilon(x_{0T\tau} + x_{1\tau\tau}) + \epsilon x_{0\tau T} + O(\epsilon^2)$$

$$= x_{0\tau\tau} + \epsilon(x_{1\tau\tau} + 2x_{0T\tau}) + O(\epsilon^2)$$

put in (45)

$$x_{0\tau\tau} + \epsilon(x_{1\tau\tau} + 2x_{0T\tau}) + x_0 + \epsilon x_1 + O(\epsilon^2)$$

$$+\epsilon h(x_0 + \epsilon x_1 + \dots, x_{0\tau} + \epsilon(x_{0T} + x_{1\tau}) + \dots) = 0$$

$\Rightarrow$ $\cancel{x_{0\tau\tau} + \epsilon(x}$

$$x_{0\tau\tau} + x_0 + \epsilon(\underline{x_{1\tau\tau}} + 2x_{0T\tau} + \underline{x_1}) +$$

$$\epsilon h(x_0, x_{0\tau}) + O(\epsilon^2) = 0$$

$O(1)$ $\quad x_{0\tau\tau} + x_0 = 0$

$O(\epsilon^2)$ $\quad x_{1\tau\tau} + x_1 = -2x_{0\tau T} - h$

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7.3.8 Change (r,θ) to (x,y)

$$\dot{r} = r(1-r^2) + \mu r\cos\theta \qquad \dot{\theta} = 1$$

into x + y $\quad r^2 = x^2 + y^2$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$

$$\dot{r} = \frac{x}{r}\dot{x} + \frac{y}{r}\dot{y}$$

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$$

$$x\dot{y} - y\dot{x} = r^2 = x^2 + y^2$$

$\therefore$ $\cancel{\frac{x}{r}\dot{x} + y}$

$$x\dot{x} + y\dot{y} = r^2(1-r^2) + \mu r^2\cos\theta \quad + \quad x\dot{y} - y\dot{x} = x^2+y^2 = r^2$$

mult 1st eq by y

+ 2nd eq by x + Add

$$\underbrace{(y^2 + x^2)}_{r^2}\dot{y} = yr^2(1-r^2) + y\mu r^2\cos\theta + r^2x$$

$$\dot{y} = y(1-r^2) + y\mu\cos\theta + x$$

Mult 1st eq by $-x$

2nd eq by y + Add

$$\Rightarrow -x^2\dot{x} + -y^2\dot{x} = -xr^2(1-r^2) + \mu r^2\cos\theta(-x) + r^2 y$$

$$\Rightarrow \dot{x} = x(1-r^2) + \mu\cos\theta - y$$

$\therefore$ eqs become

$$\dot{y} = y(1-x^2-y^2) + \frac{\mu y x}{\sqrt{x^2+y^2}} + x$$

$$\text{+ } \dot{x} = x(1-x^2-y^2) + \frac{\mu x}{\sqrt{x^2+y^2}} - y$$

7.6.10

$$\sin\theta\cos^2\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2$$

Ma

$$= \frac{1}{4}\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) + \frac{1}{4}\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)$$

$$= \frac{1}{4}$$ ✓ ✓

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7.6.15

(a) $\ddot{x} + \sin x = 0 \quad \simeq \quad \ddot{x} + x - \frac{1}{6}x^3 = 0$

Try Poincaré method

$x(t) = x_0(T) + \epsilon x_1(T) + \cdots \qquad T = \omega t$

$= (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)t$

Then $\dot{x} = \omega \frac{dx}{dT}$

put in DE

$$\omega^2 \frac{d^2x}{dT^2} + x - \frac{1}{6}x^3 = 0$$

$$\Rightarrow (1 + 2\omega_1\epsilon + 2\omega_2\epsilon^2 + \omega_1^2\epsilon^2 + \cdots)\left[\frac{d^2x_0}{dT^2} + \epsilon\frac{d^2x_1}{dT^2} + \epsilon^2\frac{d^2x_2}{dT^2} + \cdots\right]$$

$+ \; x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots \quad - \frac{1}{6}\left(x_0^3 + 3x_0^2 x_1 \epsilon + 3x_0 x_1^2 \epsilon^2 + \cdots \right.$

1st small Amp let in $\ddot{x} + x - \frac{1}{6}x^3 = 0$

$x = \epsilon \tilde{x}$

Then $\epsilon \ddot{\tilde{x}} + \epsilon \tilde{x} - \frac{1}{6}\epsilon^3 \tilde{x}^3 = 0$

$\Rightarrow \ddot{\tilde{x}} + \tilde{x} - \frac{1}{6}\epsilon^2 \tilde{x}^3 = 0$ Then let $\epsilon' = \epsilon^2$ +

to get basic eq

$\ddot{x} + x - \frac{1}{6}\epsilon x^3 = 0$

Then Apply poincaré method As Above

$$\left(x_0 + \epsilon x_1 + \sum\right)^3$$

$$= \left(x_0 + \epsilon x_1 + \sum\right)\left(x_0 + \epsilon x_1 + \sum\right)\left(x_0 + \epsilon x_1 + \sum\right)$$

$$= \left(\left(x_0 + \epsilon x_1\right)^2 + 2\left(x_0 + \epsilon x_1\right)O(\epsilon^2) + O(\epsilon^4)\right)\left(x_0 + \epsilon x_1 + \sum\right)$$

$$= \left(x_0 + \epsilon x_1\right)^3 + \overset{\text{"}\ O(\epsilon^2)}{2\left(x_0 + \epsilon x_1\right)^2 O(\epsilon^2)} + O(\epsilon^4) + O(\epsilon^2) + O(\epsilon^4)$$

$$+ O(\epsilon^6)$$

~~Thus~~ $O(1)$:

$$\Rightarrow \quad \frac{d^2 x_0}{d\tau^2} + \epsilon\left(\frac{d^2 x_1}{d\tau^2} + 2\omega_1 \frac{d^2 x_0}{d\tau^2}\right) + \epsilon^2\left(\qquad\qquad\right) + \cdots$$

$$+ \quad x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$$

$$- \frac{\epsilon}{6}\left(x_0^3 + 3x_0^2 x_1 \epsilon + \cdots\right.$$

$O(1)$

$$\frac{d^2x_0}{dT^2} + x_0 = 0 \Rightarrow x_0(T) = Ae^{iT} + A^* e^{-iT}$$

$O(\epsilon)$

$$\frac{d^2x_1}{dT^2} + x_1 = -2\omega_1 \frac{d^2x_0}{dT^2} + \frac{1}{6}x_0^3$$

$$= +2\omega_1\left[Ae^{iT} + A^*e^{-iT}\right] + \frac{1}{6}\left[A^3e^{3iT} + 3A^2A^*e^{iT}\right.$$

$$\left. + 3AA^{*2}e^{-iT} + A^{*3}e^{-3iT}\right]$$

$$= \frac{1}{6}\left[A^3e^{3iT} + A^{*3}e^{-3iT}\right] + \left[2\omega_1A + \frac{A^2A^*}{2}\right]e^{iT}$$

$$+\left[2\omega_1A^* + \frac{A^{*2}A}{2}\right]e^{-iT}$$

pick $\omega_1 = -\frac{AA^*}{4} = \frac{-|A|^2}{4}$

Then sol

$$x_0(t) = Ae^{i\left(1-\frac{|A|^2}{4}\right)t} + A^* e^{-i\left(1-\frac{|A|^2}{4}\right)t}$$

7.6.18 $\ddot{x} + (a + \epsilon \cos t)x = 0 \quad a \hat{=} 1 \qquad T = \epsilon^2 t$

Should this be power series in ϵ^2? Don't think so.

$$x = X(t,T) = x_0(t,T) + \epsilon x_1(t,T) + \epsilon^2 x_2(t,T) + \cdots$$

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \epsilon^2 \frac{\partial \dot{x}}{\partial T} =$$

$$\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + 2\epsilon^2 \frac{\partial^2 x}{\partial t \partial T} + \epsilon^4 \frac{\partial^2 x}{\partial T^2} \qquad \text{put in above}$$

$$\Rightarrow \quad \cancel{\frac{\partial^2 x_0}{\partial t^2} + 2\epsilon^2 \frac{\partial^2 x_0}{\partial t \partial T} + \epsilon^4 \frac{\partial^2 x_0}{\partial T^2} + \epsilon\left(\frac{\partial^2 x_1}{\partial t^2} + 2\epsilon^2 \frac{\partial^2 x_1}{\partial t \partial T} + \epsilon^4 \frac{\partial^2 x_1}{\partial T^2}\right)} +$$

$$\frac{\partial^2 x_0}{\partial t^2} + \epsilon \frac{\partial^2 x_1}{\partial t^2} + \epsilon^2 \frac{\partial^2 x_2}{\partial t^2} + \epsilon^3 \frac{\partial^2 x_3}{\partial t^2} + \cdots$$

$$+ 2\epsilon^2 \left(\frac{\partial^2 x_0}{\partial t \partial T} + \epsilon \frac{\partial^2 x_1}{\partial t \partial T} + \epsilon^2 \frac{\partial^2 x_2}{\partial t \partial T} + \cdots \right)$$

$$+ \epsilon^4 \left(\frac{\partial^2 x_0}{\partial T^2} + \epsilon \frac{\partial^2 x_1}{\partial T^2} + \epsilon^2 \frac{\partial^2 x_i}{\partial T^i} + \cdots \right) \quad + a(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots)$$

$$+\ \epsilon\cos t\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\ \right) = 0$$

$$\Rightarrow\ \left(\frac{\partial^2 x_0}{\partial t^2} + a x_0\right) + \epsilon\left(\frac{\partial^2 x_1}{\partial t^2} + a x_1 + x_0\cos t\right) + \epsilon^2\left(\frac{\partial^2 x_2}{\partial t^2} + 2\frac{\partial^2 x_0}{\partial t\,\partial T} + x_1\cos t\right)$$

$$+\ \epsilon^3\left(\frac{\partial^2 x_3}{\partial t^2} + 2\frac{\partial^2 x_1}{\partial t\,\partial T} + a x_3 + \cos t\, x_2\right) + \cdots \qquad = 0$$

7.6.18 $\ddot{x} + (a + \epsilon \cos t)x = 0 \qquad a \approx 1$

two timing

$$x = x_0(t,T) + \epsilon x_1(t,T) + \cdots = \sum_{k=0}^{\infty} \epsilon^k x_k(t,T)$$

$$\dot{x} = \frac{\partial}{\partial t}x + \epsilon^2 \frac{\partial}{\partial T}x = \sum_{k=0} \epsilon^k \frac{\partial x_k}{\partial t} + \sum_{k=0} \epsilon^{k+2} \frac{\partial x_k}{\partial T}$$

$$\ddot{x} = \left(\frac{\partial^2}{\partial t^2} + 2\epsilon^2 \frac{\partial^2}{\partial t \partial T} + \epsilon^4 \frac{\partial^2}{\partial T^2}\right)x = \sum \epsilon^k \frac{\partial^2 x_k}{\partial t^2} + 2\sum \epsilon^{k+2} \frac{\partial^2 x_k}{\partial t \partial T} + \sum \epsilon^{k+4} \frac{\partial^2 x_k}{\partial T^2}$$

put in

$$\sum \epsilon^k \frac{\partial^2 x_k}{\partial t^2} + 2\sum \epsilon^{k+2} \frac{\partial^2 x_k}{\partial t \partial T} + \sum \epsilon^{k+4} \frac{\partial^2 x_k}{\partial T^2} + a\sum \epsilon^k x_k + \cos t \sum \epsilon^{k+1} x_k = 0$$

$\underline{O(1)}$

$$\frac{\partial^2 x_0}{\partial t^2} + a x_0 = 0$$

$$\Rightarrow \quad x_0(t,T) = A(T)\,e^{i\sqrt{a}\,t} + A^*(T)\,e^{-i\sqrt{a}\,t}$$

<u>$O(\varepsilon)$</u>

$$\frac{\partial^2 x_1}{\partial t^2} + \quad + a x_1 + \cos t\, x_0 = 0$$

$$\frac{\partial^2 x_1}{\partial t^2} + a x_1 = -\Big[$$

7.6.19 $\quad \ddot{x} + x + \epsilon x^3 = 0 \qquad \begin{array}{l} x(0) = a \\ \dot{x}(0) = 0 \end{array}$

(a) $\quad \tau = \omega t$

$$\frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}$$

$$\Rightarrow \quad \omega^2 \frac{d^2 x}{d\tau^2} + x + \epsilon x^3 = 0.$$

(b) $\quad x(\tau, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k x_k(\tau)$

$$\Rightarrow \quad (1 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \dots)^2 \sum_{k=0}^{\infty} \epsilon^k x_k''(\tau) + \sum \epsilon^k x_k(\tau) + \epsilon\left(\sum \epsilon^k x_k\right)^3 = 0$$

$$\Rightarrow \quad (1 + 2\omega_1 \epsilon + 2\omega_2 \epsilon^2 + \omega_1^2 \epsilon^2 + \dots) \sum \epsilon^k x_k'' + \sum + \epsilon\left(x_0^3 + 3x_0^2 x_1 \epsilon + 3x_0 x_1^2 \epsilon^2 + \dots\right)$$

$\underline{O(1)}$

$$x_0'' + x_0 = 0$$

$\underline{O(\epsilon)}$

$$x_1'' + 2\omega_1 x_0'' + x_1 + x_0^3 = 0 \quad ✓$$

(c) I.C. $\quad x_0(0) = a$

$\quad x_k(0) = 0 \quad k \geq 1$

$$\mathring{x}(0) = \omega \frac{dx(0)}{d\tau} = (1 + \epsilon\omega_1 + \cdots)(x_0' + \epsilon x_1' + \cdots)$$

$$= x_0' + \epsilon(\omega_1 x_0' + x_1') + \cdots$$

$x_0'(0) = 0 \quad \omega_1 \cancel{x_0'(0)}^{0} + x_1'(0) = 0 \quad \Rightarrow \quad x_1'(0) = 0. \quad \Rightarrow \quad x_k'(0) = 0.$

(d) I.C $x_0(\tau) = a\cos\tau$

(e) $O(\epsilon)$

$$x_1'' + x_1 = +2\omega_1 a\cos\tau - \frac{a^3}{8}\left(e^{i\tau} + e^{-i\tau}\right)^3$$

$$= -\frac{a^3}{8}\left(e^{3i\tau} + 3e^{i\tau} + 3e^{-i\tau} + e^{-3i\tau}\right)$$

$$-\frac{a^3}{4}\left(\cos 3\tau + 3\cos\tau\right)$$

$$= \left(2\omega_1 a - \frac{3}{4}a^3\right)\cos\tau - \frac{a^3}{4}\cos 3\tau$$

Pick $\omega_1 = \frac{3a^2}{8}$

$$\therefore\ x_{1\,part} = -\frac{a^3}{4}\left[\frac{e^{3i\tau}}{-8} + \frac{e^{-3i\tau}}{-8}\right] = \frac{a^3}{16}\left[\cos 3\tau\right]$$

$$x_1(\pi) = B + \frac{a^3}{16} = 0 \Rightarrow B =$$

$$= \frac{a^3}{16}\cos 3\tau - \frac{a^3}{16}\cos\tau$$

$$\therefore \quad x(t) = a\cos\left(1+\frac{3a^2}{8}\epsilon\right)t + \frac{\epsilon a^3}{16}\left(\cos 3\left(1+\frac{3a^2}{8}\epsilon\right)t + \cos\left(1+\frac{3a^2}{8}\epsilon\right)t\right)$$

7.6.20

$$\ddot{x} + x + \epsilon x^3 = 0 \qquad x(0) = a, \ \dot{x}(0) = 0.$$

$$x(t) = \sum_{k=0} \epsilon^k x_k(t).$$

$O(1)$

$$\ddot{x}_0 + x_0 = 0$$

$$x_0(t) = a\cos t$$

$O(\epsilon)$

$$\ddot{x}_1 + x_1 + x_0^3 = 0$$

$$\ddot{x}_1 + x_1 = -\frac{a^3}{8}\left[e^{3it} + 3e^{it} + 3e^{-it} + e^{-3it}\right]$$

$$x_{1part} = -\frac{a^3}{4}\left[-\frac{1}{8}(2\cos 3t)\right] - \frac{3a^3}{4}\left[\left[\frac{1}{D^2+1}(e^{it} + e^{-it})\right]\right]$$

$$= \frac{1(2)}{32}\cos 3t - \frac{3}{4}\left[\frac{1}{(D-i)(D+i)}e^{it} + \frac{1}{(D+i)(D-i)}e^{-it}\right]$$

$$= \frac{1}{16}\cos 3t - \frac{3}{4}\left[\frac{1}{2i}\,\frac{1}{D-i}e^{it}(1) + \frac{-1}{2i}\,\frac{1}{D+i}e^{-it}\right]$$

Mult All by a^3

$$= \frac{\cos 3t}{16} - \frac{3}{2i\cdot 4}\left[e^{it}t - e^{-it}t\right]$$

$$= \quad '' \quad - \frac{3}{4}t\sin t$$

B.C + I.C $x_1(0)=0$ $\dot{x}_1(0)=0$

$$\Rightarrow \quad x_1(t) = \frac{-1}{16}\cos t + \frac{\cos 3t}{16} - \frac{3}{4}t\sin t$$

$$\therefore \quad x(t) = a\cos t + \epsilon a^3\left[\frac{1}{16}(\cos 3t - \cos t) - \frac{3}{4}t\sin t\right]$$

7.6.21

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x = 0$$

let $\tau = \omega t = (\cancel{1} + \omega_1\epsilon + \omega_2\epsilon^2 + \cdots)t$

$$\frac{dx}{dt} = \frac{d\tau}{dt}\frac{dx}{d\tau} = (1+\omega_1\epsilon + \omega_2\epsilon^2 + \ldots)\frac{dx}{d\tau}$$

$$\frac{d^2x}{dt^2} = (1+\epsilon\omega_1 + \epsilon^2\omega_2 + \cdots)^2\frac{d^2x}{d\tau^2}$$

I.C. $x(0) = a$ if $x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \ldots$

$\Rightarrow$ $x_0(0) = a$ $x_n(0) = 0$ $\forall\ n \geq 1$

$\dot{x}(0) = b$ $\frac{dx}{dt} = (1+\omega_1\epsilon + \omega_2\epsilon^2 + \ldots)\frac{dx}{d\tau} = (1+\omega_1\epsilon + \omega_2\epsilon^2 + \ldots)\left(\frac{dx_0}{d\tau} + \epsilon\frac{dx_1}{d\tau} + \epsilon^2\frac{dx_2}{d\tau^2} + \ldots\right)$

$$= \frac{dx_0}{d\tau} + \epsilon\left(\omega_1\frac{dx_0}{d\tau} + \frac{dx_1}{d\tau}\right) + \epsilon^2\left(\omega_2\frac{dx_0}{d\tau} + \omega_1\frac{dx_1}{d\tau} + \frac{dx_2}{d\tau}\right) + \cdots$$

$\Rightarrow$ $\left.\frac{dx_0}{d\tau}\right|_{\tau=0} = b$ $\left.\omega_1\frac{dx_0}{d\tau} + \frac{dx_1}{d\tau}\right|_0 = 0 \Rightarrow \frac{dx_1(0)}{d\tau} = -\omega_1 b$

$$\frac{dx_2(0)}{d\tau} = -\omega_2 b - \omega_1(-\omega_1 b) = (-\omega_2 + \omega_1^2)b \quad \cdots$$

put in DE

$$\left(1+\epsilon\omega_1+\epsilon^2\omega_2+\cdots\right)^2\left(\frac{d^2x_0}{d\tau^2}+\epsilon\frac{d^2x_1}{d\tau^2}+\epsilon^2\frac{d^2x_2}{d\tau^2}+\epsilon^3\frac{d^2x_3}{d\tau^2}+\cdots\right)$$

$$+\,\epsilon\left[\left(x_0+\epsilon x_1+\epsilon^2x_2+\epsilon^3x_3+\cdots\right)^2-1\right]\left(1+\omega_1\epsilon+\omega_2\epsilon^2+\omega_3\epsilon^3+\cdots\right)\left(\frac{dx_0}{d\tau}+\epsilon\frac{dx_1}{d\tau}+\epsilon^2\frac{dx_2}{d\tau}+\cdots\right)$$

$$+\;x_0+\epsilon x_1+\epsilon^2x_2+\epsilon^3x_3+\cdots = 0$$

$$=\left(1+\epsilon(2\omega_1)+\epsilon^2(2\omega_2+\omega_1^2)+\epsilon^3(2\omega_3+2\omega_1\omega_2)+\cdots\right)\left(x_0''+\epsilon x_1''+\epsilon^2x_2''+\epsilon^3x_3''+\cdots\right)$$

$$+\,\epsilon\left[x_0^2+\epsilon(2x_1x_0)+\epsilon^2(2x_2x_0+x_1^2)+\epsilon^3(2x_3x_0+2x_1x_2)+\cdots-1\right]\cdot$$

$$\left[x_0'+\epsilon(x_1'+\omega_1x_0')+\epsilon^2(\omega_2x_0'+\omega_1x_1'+x_2')+\epsilon^3(\cdots)\right]\cdot$$

$$+\,x_0+\epsilon x_1+\epsilon^2x_2+\cdots = 0$$

$$x_0'' + \epsilon\left(x_1'' + 2\omega_1 x_0''\right) + \epsilon^2\left((2\omega_2+\omega_1^2)x_0'' + 2\omega_1 x_1'' + x_2''\right) + O(\epsilon^3)$$

$$+ \epsilon\left[\left(x_0^2 - 1\right)x_0' + \epsilon\left(2x_1x_0x_0' + x_0^2x_1' + \omega_1 x_0^2 x_0'\right) + O(\epsilon^2)\right]$$

$$+ \sum \epsilon^t x_t = 0.$$

<u>$O(1)$</u>

$$x_0'' + x_0 = 0 \qquad x_0(\tau) = A\cos(\tau) + B\sin(\tau)$$

$A = a$. $\quad B = +b$. $\quad x_0(\tau) = a\cos\tau + b\sin\tau$. put in.

<u>$O(\epsilon)$</u>

$$x_1'' + 2\omega_1\left(-a\cos\tau - b\sin\tau\right) + \left(a\cos\tau + b\sin\tau\right)^2\left(-a\sin\tau + b\cos\tau\right) - 1$$

$$+ x_1 = 0.$$

$$x_1'' + x_1 = 2\omega_1 a\cos\tau + 2\omega_1 b\sin\tau + 1$$

$$+ (a\sin\tau - b\cos\tau)(a^2\cos^2\tau + 2ab\cos\tau\sin\tau + b^2\sin^2\tau)$$

$$= 2\omega_1 a\cos\tau + 2\omega_1 b\sin\tau + 1$$

$$+ a^3\sin\tau\cos^2\tau + 2a^2b\cos\tau\sin^2\tau + ab^2\sin^3\tau - a^2b\cos^3\tau$$

$$- 2ab^2\cos^2\tau\sin\tau - b^3\cos\tau\sin^2\tau.$$

$$\ddot{x} + \epsilon(x^2-1)\dot{x} + x = 0$$

Poincaré

let $x(t) = \sum_{k=0} \epsilon^k x_k(T)$ $T = \omega t = (1+\omega_1\epsilon + \omega_2\epsilon^2 + \cdots)t$

$$\partial_t = \omega \partial_T$$

$$\frac{\partial T}{\partial t}\partial_T = \omega \partial_T$$

Then $\ddot{x} = \omega^2 \partial_T^2$

$\therefore$ get

$$(1+\epsilon\omega_1 + \cdots)^2 \sum \epsilon^k \partial_T^2 x_k(T) + \epsilon((x_0^2 + 2x_0x_1\epsilon + \cdots) - 1)\cdot$$

$$(1+\epsilon\omega_1 + \cdots)\sum \epsilon^k \partial_T x_k + \sum \epsilon^k x_k = 0$$

Then

$$\Rightarrow \quad \left(1 + 2\epsilon\omega_1 + \epsilon^2\omega_1^2 + \cdots\right)\sum \epsilon^k \partial_T^2 x_k + \epsilon\left(x_0^2 + 2x_0x_1\epsilon + \cdots - 1\right)\cdot$$

$$\left(1 + \epsilon\omega_1 + \cdots\right)\sum \epsilon^k \partial_T x_k + \sum \epsilon^k x_k = 0$$

$\underline{O(1)}$

$$\partial_T^2 x_0 + x_0 = 0 \quad \Rightarrow \quad x_0 = Ae^{iT} + A^* e^{-iT}$$

$\underline{O(\epsilon)}$

$$\partial_T^2 x_1 + 2\omega_1 \partial_T^2 x_0 + x_0^3 + x_1 = 0$$

$$\partial_T^2 x_1 + x_1 = -2\omega_1\left[-x_0\right] - x_0^3$$

$$= 2\omega_1\left(Ae^{iT} + A^* e^{-iT}\right) - \left(\qquad \right)$$

just need secular terms to compute ω_1

$$\therefore \quad 2\omega_1 A e^{iT} \quad O(e^{3iT}) + 3A^2A^* e^{iT} + \text{c.c.}$$

$$\Rightarrow \quad 2\omega_1 \cancel{A} = -3\cancel{A^2}A^*$$

$$\Rightarrow \quad \omega_1 = -\frac{3}{2}|A|^2.$$

7.6.22

$$\ddot{x} + x + \epsilon x^2 = 0 \qquad x(0) = a \quad \dot{x}(0) = 0.$$

Poincaré $\quad x(t) = \sum \epsilon^k x_k(T) \qquad T = \omega t$

$$= (1 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \cdots)t$$

Then $\quad \dot{x}(t) = \omega \frac{d}{dT} x(T) = (1 + \omega_1 \epsilon + \omega_2 \epsilon^2 + \cdots) \sum \epsilon^k x_k'(T)$

$$= x_0'(T) + \epsilon(\omega_1 x_0'(T) + x_1'(T)) + \epsilon^2(\omega_2 x_0'(T) + \omega_1 x_1'(T) + x_2'(T)) + \cdots \Big|_{T=0}$$

$$x_0'(0) = 0$$

$$x_1'(0) = 0 = \cdots \qquad x_k'(0) = 0.$$

put in

$$(1+2\omega_1\epsilon+\cdots)\sum \epsilon^k x_k''(T) + \sum \epsilon^k x_k$$

$$+\ \epsilon\left(x_0^2 + 2\epsilon x_0 x_1 + \cdots \right) = 0$$

$\underline{O(1)}$:

$$x_0'' + x_0 = 0 \quad \Rightarrow \quad w/\ I.C. \quad x_0(T) = a\cos T.$$

$\underline{O(\epsilon)}$

$$2\omega_1 x_0'' + x_1'' + x_1 + x_0^2 = 0$$

$$x_1'' + x_1 = +2\omega_1(x_0) - x_0^2$$
$$= 2\omega_1 a\cos t - a^2\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2T\right)$$

$$\Rightarrow \quad \omega_1 = 0$$

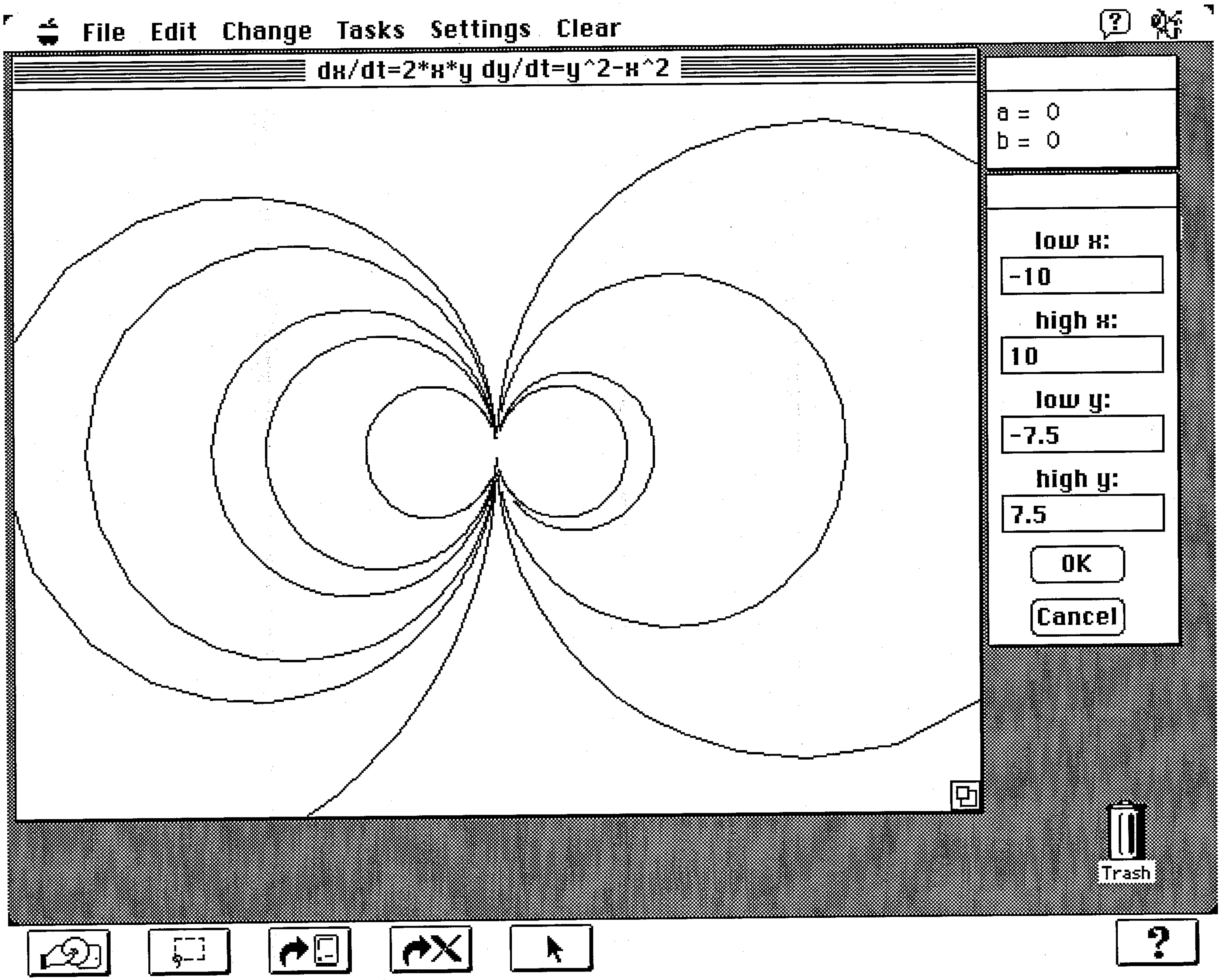
Then $\quad x_{1\,part}(t) = -\frac{a^2}{2} - \frac{a^2}{2} \frac{\cos 2T}{(-4+1)}$

$$= -\frac{a^2}{2}\left[1 + \frac{-1}{3}\cos 2T\right]$$

$$x_1(t) = B + -\frac{a^2}{\cancel{2}}\left(\frac{\cancel{2}}{3}\right) = 0 \quad \Rightarrow \quad B = \frac{a^2}{3}$$

$$x_1(\cancel{t}) = \frac{a^2}{3}\cos T - \frac{a^2}{2} + \frac{a^2}{6}\cos 2T$$

Then $O(\epsilon^2)$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-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
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prob6.1.11.ps

Tue Oct 15 23:03:12 1996

corfu / HP LaserJet 4Si

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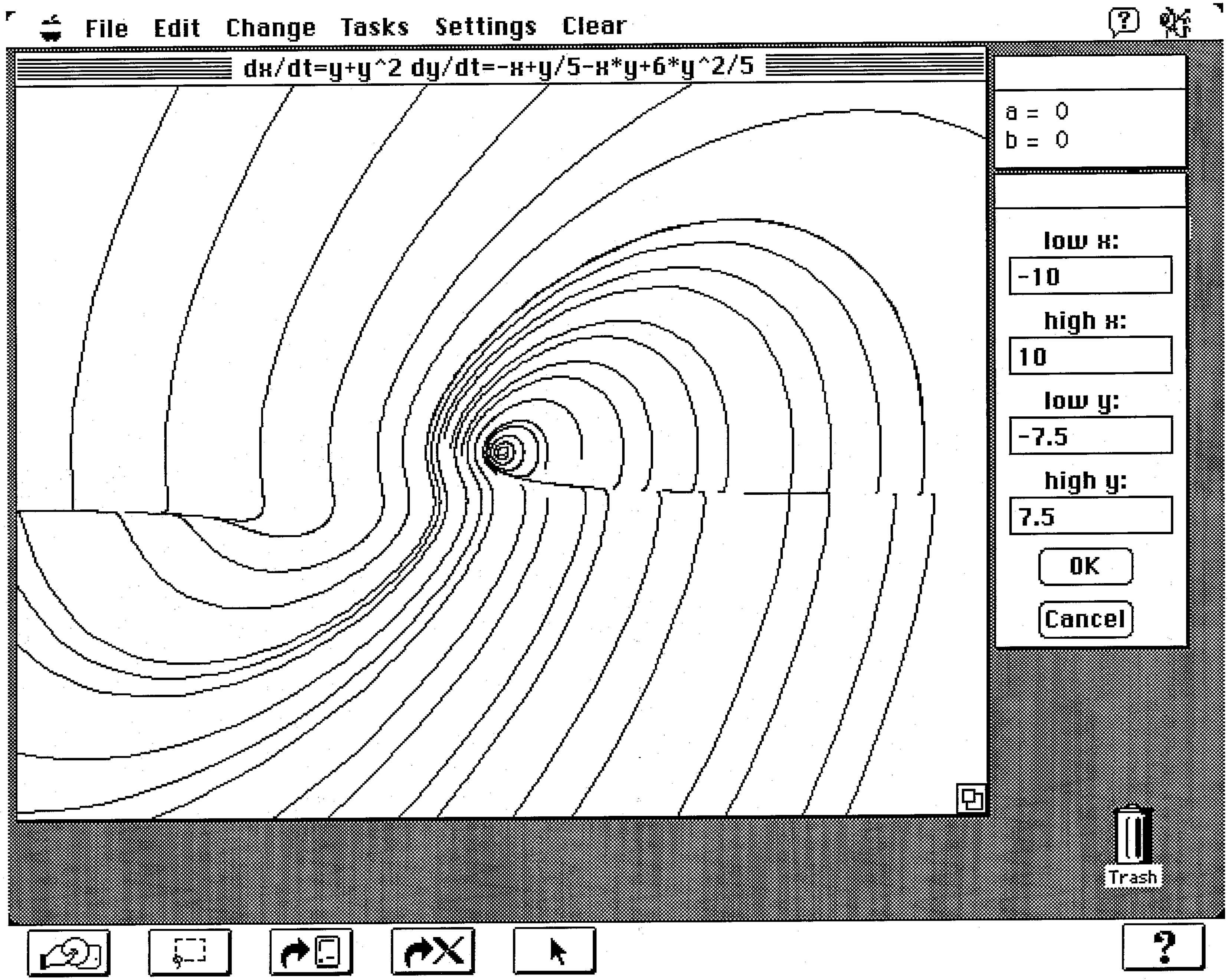
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b = 0
low x:
-10
high x:
10
low y:
-7.5
high y:
7.5
OK
Cancel
Trash
?

pg 243 Strogatz

$$abx^2 - x + ab = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4a^2b^2}}{2ab}$$

$$\dot{x} = -ax + y$$

$$\dot{y} = \frac{x^2}{1+x^2} - by$$

$$\dot{x} = 0 \Rightarrow y = ax$$

$$\dot{y} = 0 \Rightarrow y = \frac{1}{b}\frac{x^2}{1+x^2}$$

$$A = \begin{pmatrix} -a & 1 \\ \frac{2x}{1+x^2} - \frac{x^2(2x)}{(1+x^2)^2} & -b \end{pmatrix} = \begin{pmatrix} -a & 1 \\ \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} & -b \end{pmatrix}$$

$$= \begin{pmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{pmatrix}$$

$$\tau^2 - 4\Delta = a^2 + 2ab + b^2 - 4ab = 0 \Rightarrow (a-b)^2 = 0$$

$$\Delta = ab - \frac{2x^*}{(1+x^{*2})^2} = ab\left[1 - \frac{2}{(1+x^{*2})}\right] = ab\left[\frac{1+x^{*2} - 2}{1+x^{*2}}\right]$$

$$= ab\left[\frac{x^{*2} - 1}{1+x^{*2}}\right]$$

As $\Delta = ab - \dfrac{2x^*}{(1+(x^*)^2)^2} < ab$ if $x^* > 1$

Thus $\tau^2 - 4\Delta > (a+b)^2 - 4ab = (a-b)^2 > 0$

$\tau = \mu \qquad \Delta = -(\mu+2)$

If $\Delta < 0$ then saddle $\Rightarrow$ $-(\mu+2) < 0$

$\mu + 2 > 0$

$\mu > -2$

If $\mu < -2$ $\Delta > 0$ Then $\tau < 0$

$\tau^2 - 4\Delta = \mu^2 + 4(\mu+2) = \mu^2 + 4\mu + 4 + 4 =$

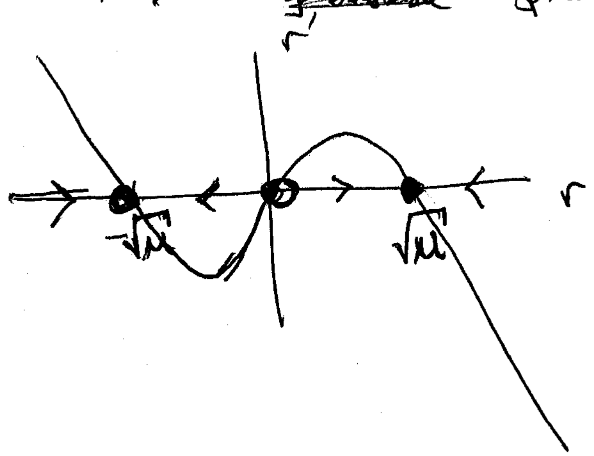
$(\mu+2)^2 + 4 > 0$ then stable node

Pg 250 Strogatz

$r' = \mu r - r^3 = r(\mu - r^2)$ if $\mu < 0$ $\mu - r^2 < 0$ $\forall r$.

then $r = 0$ is globally stable.

For $\mu > 0$ ~~spiral at the origin~~ $\mu - r^2 = (\sqrt{\mu} - r)(\sqrt{\mu} + r)$



$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$= (\mu r - r^3)\cos\theta - r(\omega + br^2)\sin\theta$$
$$= (\mu - r^2)r\cos\theta - (\omega + br^2)r\sin\theta$$
$$= (\mu - r^2)x - (\omega + br^2)y$$
$$= (\mu - x^2 - y^2)x - (\omega + bx^2 + by^2)y$$
$$= \mu x - \omega y +$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$= (\mu - r^2)r\sin\theta + (\omega + br^2)r\cos\theta$$

$$\Rightarrow \dot{y} = (\mu - r^2)y + (\omega + br^2)x$$

$$= (\mu - (x^2+y^2))y + (\omega + b(x^2+y^2))x$$

$$= \mu y + \omega x + \text{cubic terms}$$

$$A = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}$$

$$|A - \lambda I| = \lambda^2 - 2\mu\lambda + (\mu^2 + \omega^2) = 0$$

$$\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4\mu^2 - 4\omega^2}}{2} = \mu \pm \frac{1}{2}\sqrt{-4\omega^2}$$

$$= \mu \pm i\omega$$

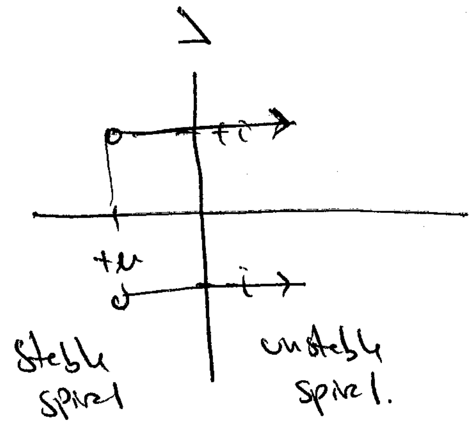
Pg 253 Strogatz

Char eq is:

$$\lambda^2 - 2\mu\lambda + (\mu^2+1) = 0$$

$$\lambda = \frac{2\mu \pm \sqrt{4\mu^2 - 4(\mu^2+1)}}{2} = \frac{2\mu \pm \sqrt{-4}}{2}$$

$$= \mu \pm i$$



let $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$r^2 = x^2 + y^2$$

$$2r\dot{r} = 2x\dot{x} + 2y\dot{y} \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$$

$$= x(\mu x - y + xy^2)$$

$$+ y(x + \mu y + y^3)$$

$$\Rightarrow r\dot{r} = \mu x^2 - \cancel{xy} + x^2y^2 + \cancel{xy} + \mu y^2 + y^4$$

$$= \mu(x^2+y^2) + y^2(x^2+y^2)$$

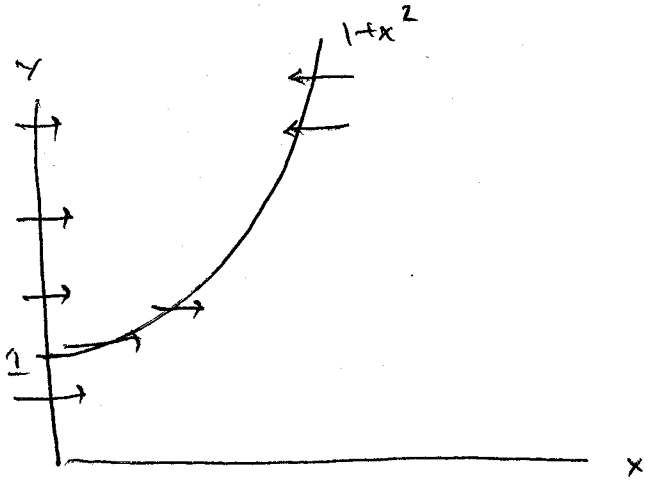
$$= \mu r^2 + r^2y^2 \Rightarrow \dot{r} = \mu r + ry$$

Pg 251 Strogatz

$\dot{x} = 0$

$\Rightarrow \quad x - a = \dfrac{-4xy}{1+x^2} \quad \Rightarrow \quad \dfrac{(1+x^2)}{4x}(a-x) = y$

$\dot{y} = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad y = 1+x^2$



on $x = 0$ $\quad \dot{x} > 0$

on $y = 1+x^2$

$\dot{x} = a - x - 4x = a - 5x$

x large $\Rightarrow \dot{x} < 0$

x small $\Rightarrow \dot{x} > 0$

pg 259 Strogatz

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$

$$J = \begin{pmatrix} -1 - \frac{4y}{1+x^2} + \frac{8x^2y}{(1+x^2)^2} & -\frac{4x}{1+x^2} \\ b\left(1 - \frac{y}{1+x^2}\right) + bx\left(+\frac{2yx}{(1+x^2)^2}\right) & bx\left(\frac{-1}{1+x^2}\right) \end{pmatrix}$$

$$= \frac{1}{1+x^2}\begin{pmatrix} -1-x^2-4y + \frac{8x^2y}{(1+x^2)} & -4x \\ b(1+x^2-y) + \frac{2bx^2y}{1+x^2} & -bx \end{pmatrix}$$

eqs for eq pts are

$$\frac{4x^*y^*}{1+x^{*2}} = a - x^* \qquad y^* = 1 + x^{*2}$$

$$J\Big|_{x^*,y^*} = \frac{1}{1+x^{*2}}\begin{pmatrix} -5y^* + 8x^{*2} & -4x^* \\ 2bx^{*2} & -bx^* \end{pmatrix}$$

$$= \frac{1}{1+x^{*2}} \begin{pmatrix} -5+3x^{*2} & -4x^{*} \\ 2bx^{*2} & -bx^{*} \end{pmatrix}$$

$$\Delta = \frac{5bx^{*} - 3bx^{*3} + 8bx^{*3}}{(1+x^{*2})^2} = \frac{5bx^{*} + 5bx^{*3}}{(1+x^{*2})^2}$$

$$= \frac{5bx^{*}}{1+x^{*2}}.$$

$$\tau = \frac{-5 - bx^{*} + 3x^{*2}}{1+x^{*2}}$$

For $\tau > 0$

$$\Rightarrow \quad -5 - bx^{*} + 3x^{*2} > 0$$

$$\Rightarrow \quad -5 - b\frac{a}{5} + \frac{3a^2}{25} > 0$$

$$\Rightarrow \quad \frac{ab}{5} < \frac{3a^2}{25} - 5$$

$$b < \frac{3a}{5} - \frac{25}{a}$$

<u>Pg 260 Strogatz</u>

$$\Delta = \frac{5\left(\frac{3a}{5} - \frac{25}{a}\right)\left(\frac{a}{5}\right)}{1+\left(\frac{a}{5}\right)^2} = \frac{a\left(3a^2 - 5\cdot 25\right)}{5a\left(1+\left(\frac{a}{5}\right)^2\right)}$$

$$\frac{\delta(3a^2-\delta\cdot\delta^2)}{\delta^2+a^2} = \frac{15a^2-\delta^4}{a^2+\delta^2}$$

Pg 261 Strogatz

$\dot{r} = \mu r + r^3 - r^5$ If $\mu < 0$

$= r(\mu - r^2 - r^4)$

Zeros of $r^4 + r^2 - \mu = 0$

$$r^2 = \frac{-1 \pm \sqrt{1+4\mu}}{2}$$

Has 2 real solutions if

$1 + 4\mu > 0$

$\Rightarrow 4\mu > -1$

~~[illegible]~~ $\mu > -1/4$

thus if $\mu < -1/4$ ~~[illegible]~~ has no real solution.

can't take taylor series about $\mu \approx -\frac{1}{4}$ As not analytic there

$-\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\mu}$ let $\mu \approx -\frac{1}{4}+$ $\underline{\mu > -1/4}$

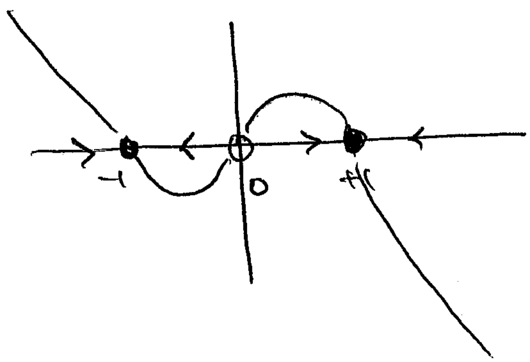
$-\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4(\mu+1/4-1/4)}$

$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{4(\mu+1/4)} = -\frac{1}{2} \pm \sqrt{\mu+1/4}$

$r \; -\frac{1}{2} \pm$

Pg 262 Strogatz

$\dot{r} = r(1-r^2) = r(1-r)(1+r)$



$\theta = \frac{\pi}{2} \qquad \sin\theta = 1$

Pg 264 Strogatz

$\ddot{x} + x + \epsilon \dot{x} x^2 - \epsilon \dot{x} = 0$

let $u^2 = \epsilon x^2 \Rightarrow u = \sqrt{\epsilon}\, x \Rightarrow x = \frac{u}{\sqrt{\epsilon}}$

$\sqrt{\epsilon} \quad \frac{\ddot{u}}{\cancel{\sqrt{\epsilon}}} + \frac{u}{\cancel{\sqrt{\epsilon}}} + u^2 \frac{\dot{u}}{\cancel{\sqrt{\epsilon}}} - \frac{\epsilon \dot{u}}{\sqrt{\epsilon}} = 0$

Pg 2

Pg 266 Strogatz Pendulum scaling.

eq for damped pendulum driven by constant torque is:

$$mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta = \Gamma$$

$mL^2\ddot{\theta}$: $I\alpha$ — $b\dot{\theta}$: damping — $mgL\sin\theta$: torque due to gravity — Γ: constant torque.

Non dimensionalize let $\tilde{t} = Tt$

$$\Rightarrow \frac{mL^2}{T^2}\theta'' + \frac{b}{T}\theta' + mgL\sin\theta = \Gamma$$

$$\Rightarrow \theta'' + \frac{Tb}{mL^2}\theta' + \underbrace{\frac{\cancel{m}g\cancel{L}T^2}{\cancel{m}L^{\cancel{2}}}}_{=1}\sin\theta = \frac{\Gamma T^2}{mL^2}$$

to maximize non linearity

$$\Rightarrow T = \sqrt{\frac{L}{g}}$$

$$\& \quad \alpha = \frac{Tb}{mL^2} = \frac{L^{1/2}}{g^{1/2}}\frac{b}{mL^2} = \frac{b}{g^{1/2}mL^{3/2}}$$

$$\& \quad \Gamma_0 = \frac{\Gamma \cancel{L}}{gmL^{\cancel{2}}} = \frac{\Gamma}{gmL}$$

then eq becomes $\theta'' + \alpha\theta' + \sin\theta = \Gamma_0$.

Pg 266 Strogatz

$$\tilde{t} = \left(\frac{2eI_c}{\hbar C}\right)^{1/2} t$$

$$I = I_B/I_c \qquad \alpha = \frac{\hbar}{2eI_c}$$

$$\frac{\cancel{\hbar C}}{2e}\phi'' \frac{1}{\left(\frac{2eI_c}{\cancel{\hbar C}}\right)^{-1}} + \frac{\hbar}{2eR}\phi'\left(\frac{2eI_c}{\hbar C}\right)^{1/2} + I_c\sin\phi$$

$$= I_B$$

$$\Rightarrow I_c\phi'' + \frac{\hbar}{2eR}\frac{1}{\left(\frac{\hbar C}{2eI_c}\right)^{1/2}}\phi' + I_c\sin\phi = I_B$$

$$\Rightarrow I_c\phi'' + \frac{\hbar}{2eR}\frac{(2eI_c)^{1/2}}{(\hbar C)^{1/2}}\phi' + I_c\sin\phi = I_B$$

$$\Rightarrow I_c\phi'' + \frac{I_c^{1/2}\sqrt{\hbar}}{\sqrt{2}\sqrt{e}R\sqrt{C}}\phi' + I_c\sin\phi = I_B$$

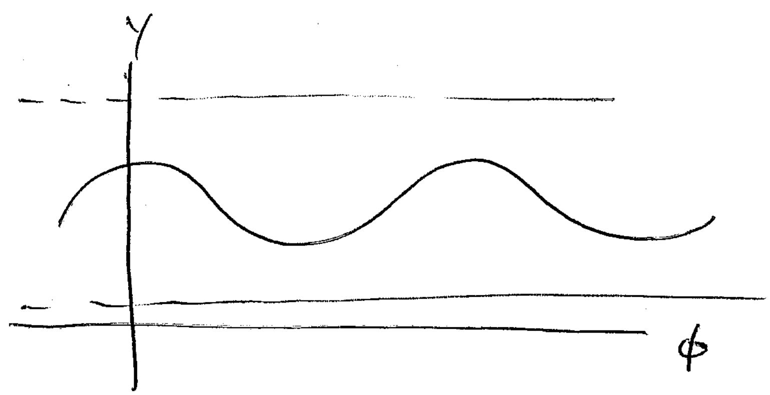
$$\phi'' + \underbrace{\frac{\sqrt{\hbar}}{\sqrt{2}\,I_c^{1/2}\sqrt{e}\,R\sqrt{C}}}_{\alpha}\phi' + \sin\phi = I$$

$$J = \begin{pmatrix} 0 & 1 \\ -\cos\phi^* & -\alpha \end{pmatrix}$$

$\tau = -\alpha$, $\Delta = \cos\phi^* = \pm\sqrt{1-I^2}$

$\Delta > 0$

$\tau^2 - 4\Delta = \alpha^2 - 4\sqrt{1-I^2} = \frac{\hbar}{2eI_c R^2 C} - 4\sqrt{1 - I_B^2/I_c^2}$



If $y > \alpha^{-1}(I - \sin\phi)$

$\Rightarrow I - \sin\phi - \alpha y < 0$ flow down and

If $y < \alpha^{-1}(I - \sin\phi) \Rightarrow$ " > 0

$y_1 \cancel{< \alpha^{-1}(I - \sin\alpha) < \alpha^{-1}(I-1)}$

$y_2 \cancel{> \alpha^{-1}(I - \sin\alpha)}$

Pg 269 Strogatz

Is (5) the energy for (3) or (4) ?

Is E conserved along trajectory

$$\dot{E} = y\dot{y} + \sin\phi\,\dot{\phi} = y(I - \sin\phi - \alpha y) + \sin\phi\, y$$

$$= Iy - \alpha y^2 \neq 0$$

Pg 277 Strogatz

$$\omega^* = \frac{\omega_2(k_1 + \cancel{k_2}) + k_2(\omega_1 - \cancel{\omega_2})}{k_1 + k_2} = \frac{k_1\omega_2 + \omega_1 k_2}{k_1 + k_2}$$

$$\Delta\omega_2 = \omega_2 - \omega^* = \frac{\omega_2(\cancel{k_1} + k_2) - k_1\cancel{\omega_2} - k_2\omega_1}{k_1 + k_2} = \frac{k_2(\omega_2 - \omega_1)}{k_1 + k_2}$$

$$\Delta\omega_1 = \omega_1 - \omega^* = \frac{\omega_1 k_1 + \cancel{\omega_1 k_2} - k_1\omega_2 - \cancel{k_2\omega_1}}{k_1 + k_2}$$

$$= \frac{k_1(\omega_1 - \omega_2)}{k_1 + k_2}$$

Then $\left|\frac{\Delta\omega_1}{\Delta\omega_2}\right| = \left|\frac{k_1}{k_2}\right|$

Pg 283 Strogatz

$$\phi_i(t) = \phi^*(t) + \eta_i(t)$$

$$\dot\phi_i = \Omega + a\sin\phi_i + \frac{1}{N}\sum_{j=1}^{N}\sin\phi_j$$

ith eq is:

$$\dot\phi^*(t) + \dot\eta_i(t) = \Omega + a\sin(\phi^* + \eta_i) + \frac{1}{N}\sum_{j=1}^{N}\sin(\phi^* + \eta_j)$$

$$\Rightarrow \dot\phi^*(t) + \dot\eta_i = \Omega + a\sin\phi^* + a\cos\phi^*\,\eta_i + O(\eta_i^2)$$

$$+ \frac{1}{N}\sum_{j=1}^{N}\left(\sin\phi^* + \cos\phi^*\,\eta_j + O(\eta_j^2)\right)$$

$$\Rightarrow \cancel{\dot\phi^*} + \dot\eta_i = \cancel{\Omega} + \cancel{a\sin\phi^*} + (a + \cancel{1})\sin\phi^*$$

$$+ [a\cos\phi^*]\,\eta_i + \frac{\cos\phi^*}{N}\sum_{j=1}\eta_j$$

$$\Rightarrow \dot\eta_i = (a\cos\phi^*)\,\eta_i + \cos\phi^*\,\frac{1}{N}\sum_{j=1}\eta_j$$

let $\mu = \frac{1}{N}\sum_{j=1}^{N}\eta_j$

$$\xi_i = \eta_{i+1} - \eta_i$$

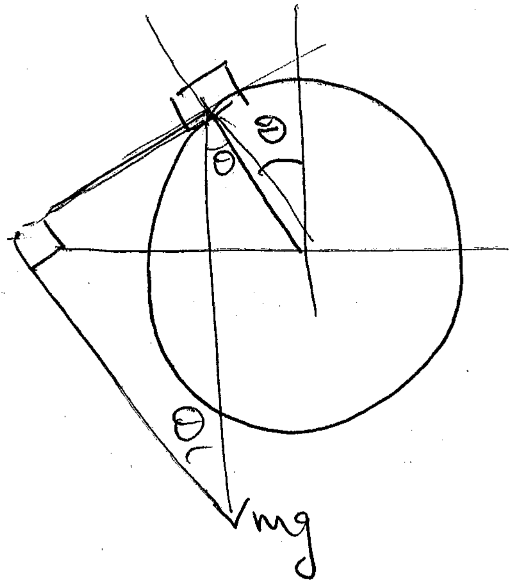
$$\mathring{\xi}_i = \mathring{\eta}_{i+1} - \mathring{\eta}_i = (a\cos\phi^*)\,\xi_i$$

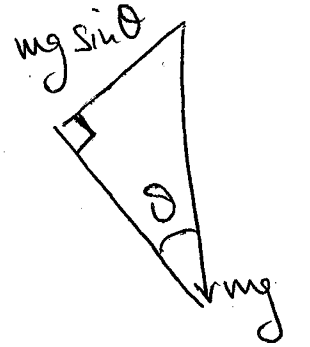
$$\frac{d\xi_i}{\xi_i} = \frac{a\cos\phi^*}{\Omega + (a+1)\sin\phi^*}\,d\phi^*$$

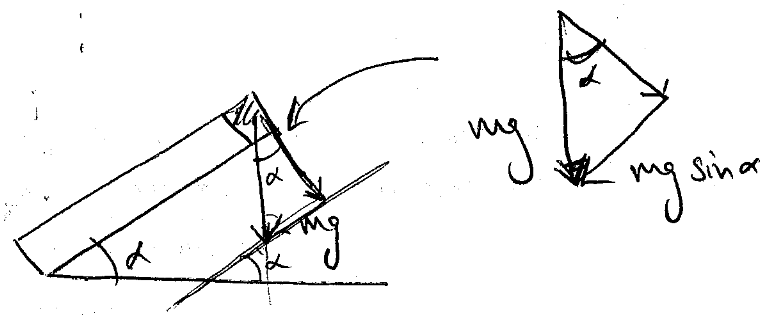
$$\oint \frac{d\xi_i}{\xi_i} = \int_0^{2\pi}$$

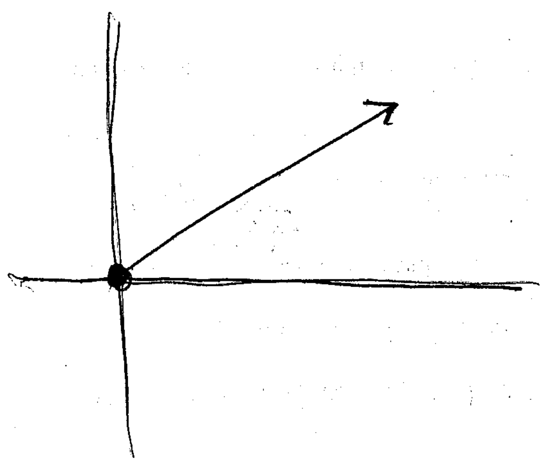
$$\ln \frac{\xi_i(T)}{\xi_i(0)} =$$

pg 307 Strogetz









eq (2) $\frac{\partial m}{\partial t} = Q - km - \omega \frac{\partial m}{\partial \theta}$

$$\Rightarrow \sum_{n=0}^{\infty} \dot{a}_n(t) \sin(n\theta) + \dot{b}_n \cos(n\theta) = -\omega \sum_{n=0}^{\infty} a_n n \cos n\theta - b_n n \sin n\theta$$

$$+ \sum_{n=0}^{\infty} q_n \cos n\theta - k \sum$$

$\Rightarrow$ ~~$\dot{a}_n = +wb_n n - kb_n$~~

$$\dot{a}_n = + w b_n n - k a_n$$

$$\dot{b}_n = - w a_n n + q_n - k b_n$$

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pg 309 Strogatz

$$a_1 = \frac{wb_1}{k}$$

$$wa_1 = -kb_1 + q_1$$

$$a_1 = \frac{\nu w}{\pi g r}$$

$$\Rightarrow \quad w^2 \frac{b_1}{k} = -kb_1 + q_1$$

$$w^2 b_1 = -k^2 b_1 + k q_1$$

$$(w^2 + k^2) b_1 = k q_1$$

$$b_1 = \frac{k q_1}{w^2 + k^2}$$

$$w\left(\frac{b_1}{k} - \frac{\nu}{\pi g r}\right) = 0$$

$$b_1 = \frac{k \nu}{\pi g r}$$

pg 312 Strogatz

$(x, y) \to (-x, -y)$

$-\dot{x} = \sigma(-y + x) \Rightarrow \dot{x} = \sigma(y - x)$

$-\dot{y} = -rx + y + xz \Rightarrow \dot{y} = rx - y - xz$

$\dot{z} = xy - bz$

$\nabla \cdot f = \frac{\partial}{\partial x}(\sigma(y-x)) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(xy - bz)$

$= -\sigma + -1 - b$

$\dot{V} = (-\sigma - 1 - b)V$

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pg 314 Strogatz

$\dot{x} = 0 \Rightarrow y = x$ put into 2 eqs below

$\dot{y} = 0 \Rightarrow rx - y - xz = 0$

$\dot{z} = 0 \Rightarrow xy = bz$

$rx - x - xz = 0 \Rightarrow x(r - 1 - z) = 0 \Rightarrow x = 0$ or $z = r - 1$

$x^2 = bz$

If $x = 0$ $z = 0 \Rightarrow (0,0,0)$ vs. $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Delta = \sigma - r\sigma = \sigma(1-r)$$

$$\tau^2 - 4\Delta = (\sigma+1)^2 - 4\sigma(1-r) = \sigma^2 + \underline{2\sigma} + 1 - \underline{4\sigma} + 4\sigma r$$

$$= \sigma - 2\sigma + 1 + 4\sigma r = (\sigma-1)^2 + 4\sigma r > 0$$

$$\frac{1}{2}\dot{V} = \cancel{-(x^2 - (r+1)xy + y^2)}$$

$$= -(x^2 - (r+1)xy) - y^2 - bz^2$$

$$= -\left(x^2 - (r+1)xy + \left(\frac{r+1}{2}\right)^2 y^2\right) + \left(\frac{r+1}{2}\right)^2 y^2 - y^2 - bz^2$$

$$= -\left(x - \left(\frac{r+1}{2}\right)y\right)^2 - \left[1 - \left(\frac{r+1}{2}\right)^2\right] y^2 - bz^2$$

If $0 < r < 1$

$1 < r+1 < 2$

$\frac{1}{2} < \frac{r+1}{2} < 1$

Then $\frac{1}{4} < \left(\frac{r+1}{2}\right)^2 < 1$

$-1 < -\left(\frac{r+1}{2}\right)^2 < -\frac{1}{4}$

$0 < 1 - \left(\frac{r+1}{2}\right)^2 < 3/4$

Pg 340 Strogatz

$\dot{u} = \sigma(v - u)$
$\dot{v} = ru - v - 20uw$

$\dot{w} =$

$\vec{e} = \vec{d} - \vec{r} = (u - u_r, v - v_r, w - w_r)$

$\dot{e}_1 = \quad - \sigma(v - u) -$

$= \sigma(e_2 - e_1)$

$\dot{e}_2 = r\cancel{e_1}^{0} - e_2 - 20uw + 20u(t)w_r$

$= r\cancel{e_1}^{0} - e_2 - 20u(e_3)$

$\dot{e}_3 = \delta uv - \delta uv_r - be_3 \quad = \quad \delta u(e_2) - be_3$

Pg 355 Strogatz

If $r_n = r^n$

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \frac{r^{n-1}(r-1)}{r^n(r-1)} = \frac{r^{n-1}}{r^n} = \frac{1}{r}$$

$$\therefore \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta$$

or dropping the limit

$$r_n - r_{n-1} = \delta(r_{n+1} - r_n)$$

$$0 = \delta r_{n+1} - (1+\delta) r_n + r_{n-1}$$ constant coeff difference eq.

$$0 = \delta \lambda^2 - (1+\delta)\lambda + 1$$

$$\lambda = \frac{(1-\delta) \pm \sqrt{(1+\delta)^2 - 4\delta}}{2\delta} = \frac{1-\delta \pm \sqrt{1+2\delta+\delta^2-4\delta}}{2\delta}$$

$$= \frac{1-\delta \pm \sqrt{1-2\delta+\delta^2}}{2\delta} = \frac{1-\delta \pm (1-\delta)}{2\delta}$$

$$= 0,\ \frac{2-2\delta}{2\delta} = \frac{1}{\delta} - 1$$

pg 358 Strogatz

$x^* = 1 - \frac{1}{r}$ is stable if $f'(x^*) = \cancel{rx^*(1-x^*)}$

$r(1-2x^*)$

$= \cancel{r\left(1-\frac{1}{r}\right)\left(\frac{1}{r}\right)} = \cancel{\frac{1}{r}}$

$= r\left(1 - 2 + \frac{2}{r}\right)$

$= r\left(-1 + \frac{2}{r}\right)$

$= -r + 2$

$\cancel{|f'(x^*)| < 1}$

$\cancel{\Rightarrow \left|1 - \frac{1}{r}\right| < 1}$

$\cancel{-1 < 1 - \frac{1}{r} < 1}$

$\cancel{-2 < -\frac{1}{r} < 0}$

$\cancel{0 < \frac{1}{r} < 2}$

$\cancel{\frac{1}{2} < r < \infty}$

x^* is stable if

$|f'(x^*)| < 1$

$\Rightarrow -1 < 2 - r < 1$

$-3 < -r < -1$

$1 < r < 3$

pg 359 Strogatz

& $r = 3$

$$p, q = \frac{r+1}{2r}\Big|_{r=3} = \cancel{\frac{1}{2} + \frac{1}{2r}}\Big|_{r=3} = \cancel{\frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{\ }}$$

$$= \frac{4}{2 \cdot 3} = \frac{2}{3}$$

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pg 361 Strogatz

$$f(x) = rx(1-x)$$

& $f'(x) = r(1-x) + rx(-1) = r - rx - rx = r(1-2x)$

Then

$$\lambda = f'(p) f'(q) = r^2(1-2p)(1-2q)$$

$$= r^2\left(1 - 2(p+q) + 4pq\right)$$

But from pg 359 $\quad p, q = \dfrac{r+1 \pm \sqrt{(r-3)(r+1)}}{2r}$

Thus $\quad p + q = \dfrac{r+1}{2r} + \dfrac{r+1}{2r} = \dfrac{r+1}{r}$

$$p \cdot q = \frac{1}{4r^2}\left((r+1)^2 - (r-3)(r+1)\right)$$

$$= \frac{1}{4r^2}\left(\cancel{r^2} + 2r + 1 - \cancel{r^2} - r + 3r + 3\right)$$

$$p\cdot q = \frac{1}{4r^2}(4r+4) = \frac{r+1}{r^2}$$

Then $\lambda = r^2\left[1 - 2\left(\frac{r+1}{r}\right) + 4\left(\frac{r+1}{r^2}\right)\right]$

$$= r^2 - 2r(r+1) + 4(r+1)$$

$$= r^2 - 2r^2 - 2r + 4r + 4$$

$$= 4 + 2r - r^2$$

Thus 2 cycle stable iff $|\lambda| < 1$

$$\Rightarrow |4+2r-r^2| < 1$$

$$-1 < 4+2r-r^2 < 1$$

$$0 < 5+2r-r^2$$

$$r = \frac{-2\pm\sqrt{4-4(-1)(5)}}{2(-1)}$$

$$= \frac{-2\pm\sqrt{4+20}}{-2}$$

$$= \frac{2\mp\sqrt{24}}{2} = 1\mp\sqrt{6}$$

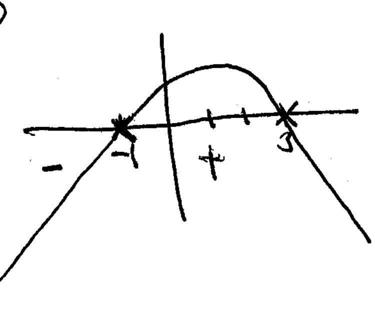
$$0 < (r-(1-\sqrt{6}))(r+(1-\sqrt{6}))$$

$$3+2r-r^2 < 0$$

$$(3-r)(1+r) < 0$$

$r = 3 \quad r = -1$

$(+)(-)$



$\Rightarrow$ ~~$-1 < r < 3$~~

Thus this side is true for

$r \in (-\infty,-1) \cup (3,\infty)$

Pg 361 Strogatz

$$|4+2r-r^2|<1$$

$$-1<4+2r-r^2<1$$

Solve 1st $\quad -1<4+2r-r^2$

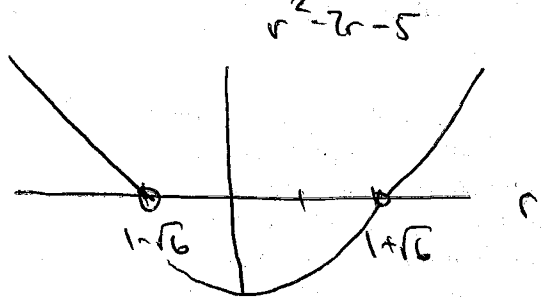
$\Leftrightarrow \quad r^2-2r-5<0$

Zeros:

$$r = \frac{2\pm\sqrt{4-4(-5)}}{2} = \frac{2\pm 2\sqrt{1+5}}{2} = 1\pm\sqrt{6}$$

$\Leftrightarrow \quad (r-(1+\sqrt{6}))(r-(1-\sqrt{6}))<0$

$\Leftrightarrow$



Thus $\Rightarrow$

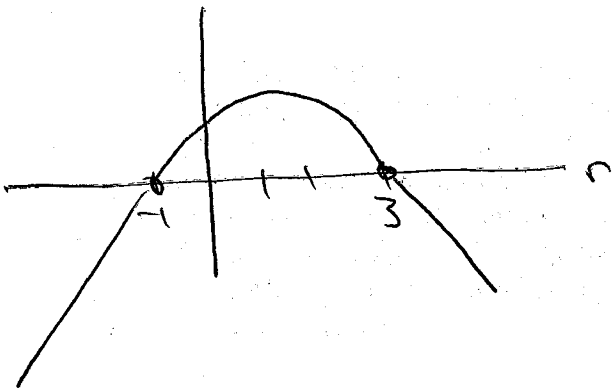
$$1-\sqrt{6}<r<1+\sqrt{6}\simeq 3.449$$

Solve next:

$$4+2r-r^2<1$$

$\Leftrightarrow \quad 3+2r-r^2<0$

$\Leftrightarrow \quad (3-r)(1+r)<0$



Thus $\Rightarrow$ $r < -1$ or $r > 3$

combining these ranges w/ $r > 0$ $\Rightarrow$

$$3 < r < 3.449 = 1 + \sqrt{6}.$$

pg 370 Strogatz

$f(x) = r\sin\pi x$

$f'(x) = \pi r\cos\pi x = 0$

$\pi x = \frac{\pi}{2} \Rightarrow x = \frac{1}{2}$

Then $f(1/2) = r$

$f(x) = rx(1-x)$

$f'(x) = r(1-2x) = 0$

$\Rightarrow x = 1/2$

$f(1/2) = r\frac{1}{2}(\frac{1}{2}) = \frac{r}{4}$

Thus $r\sin\pi x$'s maxima is 4xs ~~B~~ as large as the maximum of $rx(1-x)$'s the ~~logistic~~ logistic's.

Why does the x-axis ~~B~~ of $r\sin\pi x$ scale by 1/4 of that of the logistic's map?

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pg 380 Strogatz

$\lambda = (4)pq = 0 \qquad p = 0$ or $q = 0$

$f^2(x, R_1) = (\cancel{R_1 - x^2}) [R_1 - x^2](R_1 - x^2) = p$ or q

$= R_1 - (R_1 - x^2)^2 \Big|_{x = p \text{ or } q}$

i.e. p & q are both fixed pts for the ~~2nd~~ map $f^2(x)$

pt the one that is 0.

$0 = R_1 - R_1^2 \Rightarrow R_1 = 0, R_1 = 1$ If $R_1 = 0$

Then $q = f(p, R_1) = 0$ & q is also 0 (if p is)

Thus we must pick $R_1 = 1$.

Note that for $f = r - x^2$

$f' = -2x = 0 \Rightarrow x = 0$ & $f_{max} = r$.

$f^2(x, R_1) =$

$\cancel{f(x, R_0) \hat{=} \alpha^2 f}$

$f^2(\frac{x}{\alpha}, R_1) \hat{=} \alpha f^4(\frac{x}{\alpha^2}, R_2)$

& $f(x, R_0) \hat{=} \alpha f^2(\frac{x}{\alpha}, R_1)$

$\Rightarrow f(x, R_0) \hat{=} \alpha^2 f^4(\frac{x}{\alpha^2}, R_2)$

From $g_i(x) = \lim_{n\to\infty} \alpha^n f^{(2^n)}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$

$g_\infty(x) = \lim_{n\to\infty}$

$X = 0$

$g(0) - \alpha g^2(0)$

$\parallel$

$1 = \alpha g(1) \Rightarrow \alpha =$

$x \to X/\alpha$ let $V = X/\alpha \Rightarrow X = \alpha V$ $\quad V = \alpha X$

$\alpha V_{n+1} = -(1+\mu)\alpha V_n + a\alpha^2 V_n^2 + O(V_n^3)$

$V_{n+1} = -(1+\mu)V_n + a\alpha V_n^2 + O(V_n^3)$ pick $\alpha = 1/a$

$p = -(1+\mu)q + q^2$

$q = -(1+\mu)p + p^2$

8b $\quad (p-q) = -(1+\mu)(q-p) + q^2 - p^2$

$1 = +(1+\mu) + -(q+p)$

$q + p = \mu$

* Note we are just using the result that the sum of two roots equals the value of c in a quadratic & the product equals the value of b in a quadratic.

Multiply.

$$pq = +(1+\mu)^2 pq - qp^2(1+\mu) - pq^2(1+\mu) + (pq)^2$$

$$0 = 2\mu pq + \mu^2 pq - pq(1+\mu)(\underbrace{p+q}_{\mu}) + p^2q^2$$

$$0 = \underbrace{\mu(\mu+2)pq - pq(1+\mu)(\mu)} + p^2q^2$$

$$0 = \mu pq + (pq)^2$$

$$\Rightarrow pq = -\mu$$

$$p = -\frac{\mu}{q}$$

$$p + -\frac{\mu}{p} = \mu$$

$$p^2 - \mu p - \mu = 0 \qquad p = \frac{\mu \pm \sqrt{\mu^2 - 4(-\mu)}}{2}$$

$$= \frac{\mu \pm \sqrt{\mu^2 + 4\mu}}{2} =$$

Thus $p = \dfrac{\mu + \sqrt{\mu^2 + 4\mu}}{2}$ $\qquad q = \dfrac{\mu - \sqrt{}}{2}$

Pg 389 Strogatz

Ex 10.2.2

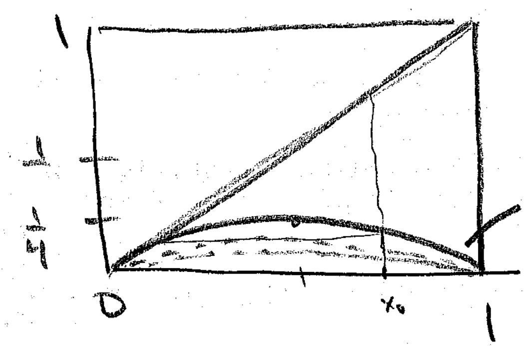
Logistic map:

$x_{n+1} = r x_n(1-x_n)$ then if $0 \le r \le 1$

$x^* = 0$ is globally stable

Max of $r x_n(1-x_n)$ for $x_0 \in [0,1]$ is at $x = 1/2$ w/

Max value of $r/4$



family of parabolas w/ $0 \le r \le 1$

Claim: x & $rx(1-x)$ intersect at only $x=0$

$x_1 = 0$ is an solution & so is $1 = r(1-x_2)$

$$-\left(\frac{1}{r} - 1\right) = x_2$$

$\Rightarrow$ $x_2 = 1 - \frac{1}{r}$

But $1 - \frac{1}{r}$ is $-\infty < 1 - \frac{1}{r} < 0$.

Check.

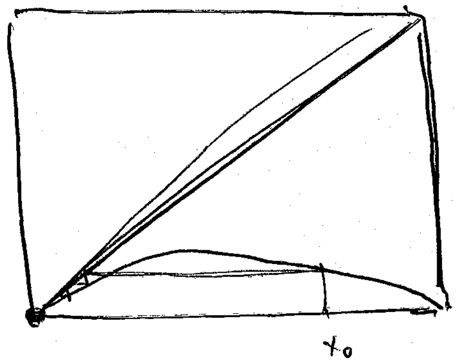
$$0 \le r \le 1$$

$$1 < \frac{1}{r} < \infty$$

$$-\infty < -\frac{1}{r} < -1$$

$$-\infty < 1 - \frac{1}{r} < 0. \quad ✓.$$

Thus the plot blown up looks like



& All trajectories head to the fixed pt $x^* = 0$.

pg 396 strogatz

i

(10.7.3) $g(x)$ +

If ~~$g(x)$~~ $\alpha g^2\left(\frac{x}{\alpha}\right) = g(x)$

$\mu g(x/\mu)$ is also a soln

~~$\mu g\left(\frac{g(x/\alpha)}{\mu}\right)$~~ Now $\mu g(\xi/\mu) \circ \mu g(\xi/\mu)$ compose

$$= \mu g\left(\frac{1}{\mu} \mu g(\xi/\mu)\right) = \mu g^2(\xi/\mu) = g(\xi)$$

from * $\therefore$ $\mu g(x/\mu)$ is also a soln

g

Pg 430 Strogatz

(x, y)

T': $x' = x$, $y' = 1 + y - ax^2$

T'': $x'' = bx'$, $y'' = y'$

$\Rightarrow$ $x'' = bx$, $y'' = 1 + y - ax^2$

T''': $x''' = y''$, $y''' = x''$

$\Rightarrow$ $x''' = 1 + y - ax^2$, $y''' = bx$

Pg 431 Strogatz

$x_{n+1} = y_n + 1 - ax_n^2$ $\qquad$ $y_{n+1} = bx_n$

$x_{n+1} - 1 + ax_n^2 = y_n$ $\qquad$ $x_n = \dfrac{y_{n+1}}{b}$

$y_n = x_{n+1} - 1 + \dfrac{a}{b^2} y_{n+1}^2$

— — — —

Pg 432 Strogatz

$x_{n+1} = 1 - ax_n^2 + y_n$ $\qquad$ $y_{n+1} = bx_n$

$$\frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} = \begin{pmatrix} -2x_n a & 1 \\ b & 0 \end{pmatrix} \quad \therefore \quad \underbrace{\left| \frac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} \right|}_{} = -b$$

det of $\dfrac{\partial(x_{n+1}, y_{n+1})}{\partial(x_n, y_n)} = -b$