pg 196 Taubes

$u(t,x) = e^{-rt} f(x-3t)$ solves $u_t = -3u_x - ru$

$u(t,0) = h(t)$

$u(t,0) = e^{-rt} f(-3t) = h(t)$ let $\tau = -3t$

$t = -\frac{\tau}{3}$

$\Rightarrow e^{+r\tau/3} f(\tau) = h(-\tau/3)$

$\Rightarrow f(\tau) = e^{-r\tau/3} h(-\tau/3)$ ✓

so $u(t,x) = e^{-rt}\left[e^{-\frac{r}{3}(x-3t)} h(-\frac{1}{3}(x-3t))\right]$

$= e^{-rt} e^{-\frac{rx}{3}} e^{rt} h(t - x/3)$

$= e^{-\frac{rx}{3}} h(t - x/3)$

(11) $u = \frac{a}{\sqrt{t}} e^{rt} e^{-x^2/4\mu t}$

$$u_t = -\frac{1}{2} a t^{-3/2} e^{rt} e^{-x^2/4\mu t} + a t^{-1/2} r e^{rt} e^{-x^2/4\mu t}$$

$$+ a t^{-1/2} e^{rt} e^{-x^2/4\mu t} \left(\frac{+x^2}{4\mu t^2} \right)$$

$$= \left(-\frac{1}{2} t^{-1} + r + \frac{x^2}{4\mu t^2} \right) a t^{-1/2} e^{rt} e^{-x^2/4\mu t}$$

$$u_x = a t^{-1/2} e^{rt} e^{-x^2/4\mu t} \left(\frac{-2x}{4\mu t} \right)$$

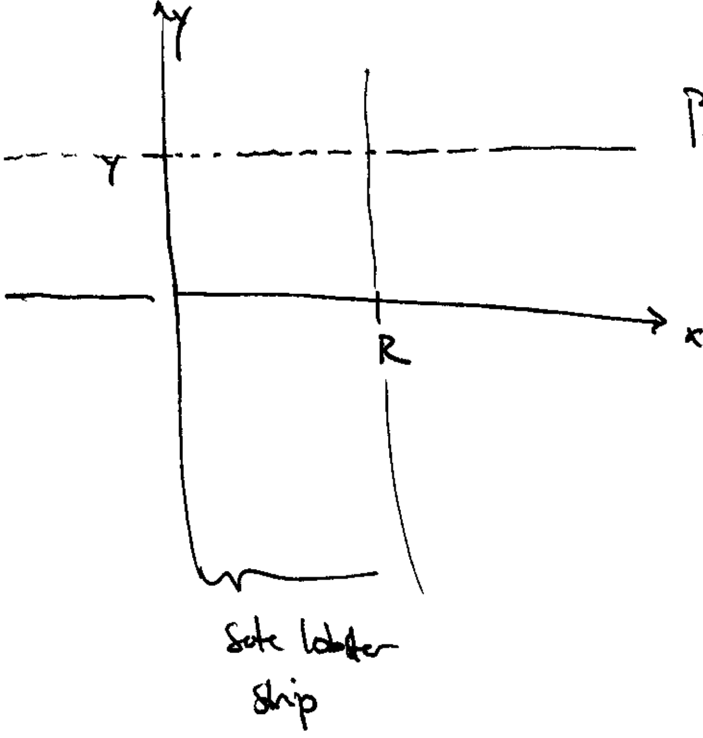
$$u_{xx} = a t^{-1/2} e^{rt} e^{-x^2/4\mu t} \left(\frac{-2x}{4\mu t} \right)^2 + a t^{-1/2} e^{rt} e^{-x^2/4\mu t} \left(-\frac{2}{4\mu t} \right)$$

$$= \left(\frac{x^2}{4\mu^2 t^2} - \frac{1}{2\mu t} \right) a t^{-1/2} e^{rt} e^{-x^2/4\mu t}$$

Form: $u_t = \mu u_{xx} + ru$

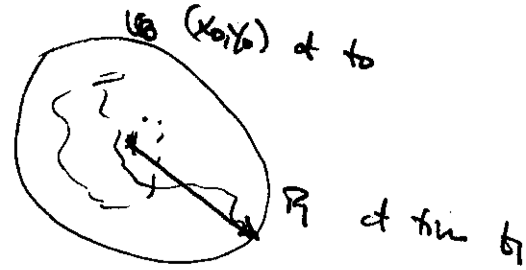
$$\left(-\frac{1}{2} t^{-1} + r + \frac{x^2}{4\mu t^2} \right) u = \left(\frac{x^2}{4\mu t^2} - \frac{1}{2t} \right) u + ru \qquad \checkmark$$

pg 247 Taubes



$u(t,x) = \frac{\#\text{ lobsters}}{\text{km}}$ at fixed $y = y_0$

$u_t = \mu u_{xx}$



~~$\mu \cong \frac{1}{R_1^2}$~~ How estimate μ?

$u_t = \mu u_{xx} + r u$.

$0 \le x \le R$ + BC $u(0,t) = 0$

$u(R,t) = 0$.

$R > \left(\frac{\mu \pi^2}{r}\right)^{1/2}$

$[r] = \frac{\text{growth rate } r \text{ per lobster}}{\text{per year}}$

$[\mu] = L^2/T$

$$\boxed{\mu \cong \frac{R_1^2}{t_1 - t_0}}$$

$u = AB$

Pg 320 Taubes

eq (6) $\frac{du}{dt} = 0$

$$-\frac{du}{dx} - \frac{1}{100}u(1-u) = 0$$

$$\frac{-du}{u(1-u)} = \frac{1}{100}dx \Rightarrow \frac{du}{u(1-u)} = \frac{-1}{100}dx$$

Partial fractions on left side

$$\frac{A}{u} + \frac{B}{1-u} = \frac{1}{u(1-u)}$$

$$A = B = 1$$

$$\therefore \quad \ln u + \ln(1-u)(-1) = -\frac{x}{100} + C_1$$

$$\ln\left[\frac{u}{1-u}\right] = -\frac{x}{2\cdot 50} + C_2$$

$$\frac{u}{1-u} = C_3 e^{-x/50} = C_3 e^{-x/100} \longrightarrow \text{evaluate at } x=0$$

$$\frac{1/2}{1/2} = C_3$$

$$u = C_3 e^{-x/100}(1-u)$$

$$(1 + C_3 e^{-x/100})u = C_3 e^{-x/100}$$

$$\Rightarrow u = \frac{e^{-x/100}}{1+e^{-x/100}} = \frac{1}{1+e^{x/100}} \quad ✓$$

Eg: 321 Tackes

$$\frac{d^2 u}{dx^2} + u - u^2 = 0 \qquad \text{eq sol} \quad + \; u(-\tfrac{\pi}{2}) = u(+\tfrac{\pi}{2}) = 3$$

$$\frac{d^2 u}{dx^2} = u^2 - u \qquad \text{Mult by } \frac{du}{dx}$$

$$\Rightarrow \frac{du}{dx} \cdot \frac{d^2 u}{dx^2} + (u - u^2)\frac{du}{dx} = 0$$

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{u^2}{2} - \frac{u^3}{3}\right] = 0$$

$$\Rightarrow \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{u^2}{2} - \frac{u^3}{3} = C_1$$

$$\left(\frac{du}{dx}\right)^2 + u^2 - \frac{2u^3}{3} = C_2$$

$$\frac{du}{dx} = \pm\left(C_2 - u^2 + \frac{2u^3}{3}\right)^{1/2}$$

(2) $g(0) = 0$ $g \neq 0$
$g(1) = 0$

(a) $\lambda g = \frac{d^2 g}{dx^2} - (x^2+1)g$

Then g must ≠ constant + hence max + mins
(if = constant const would have to be zero → ✗)

Assume Max $g > 0$, then $\frac{d^2g}{dx^2} \leq 0$ at max +

$-(x^2+1)g \leq 0$ so R.H.S. is negative.

$\Rightarrow \lambda < 0$

Assume Min $g < 0$, then $\frac{d^2g}{dx^2} \geq 0$ + $-(x^2+1)g > 0$ at

min $\Rightarrow \lambda < 0$

or both are true

∴ Proved stability.

(b) $\lambda g = \frac{d^2g}{dx^2} + (x^2+1)g$

Max $g > 0$ $\frac{d^2g}{dx^2} \leq 0$ + $(x^2+1)g > 0$ cant tell anything about the sign of

$\frac{d^2g}{dx^2} + (x^2+1)g$

(c) $\lambda g = \frac{d^2g}{dx^2} + (x^2-1)g$

Assume Max $g > 0$ $\quad \frac{d^2g}{dx^2} \le 0 \quad + \quad (x^2-1)g \le 0 \quad \forall x \in [0,1]$

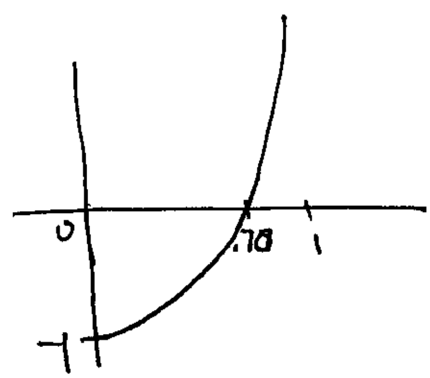
$\therefore$ LHS $\le 0 \Rightarrow \lambda < 0$

Assume Min $g < 0$ $\quad \frac{d^2g}{dx^2} \geq 0$

RHS ≥ 0

$\Rightarrow \lambda < 0 \quad \therefore$ stable

(d) $\lambda g = \frac{d^2g}{dx^2} + (2x^2-1)g$



Assume Max $g > 0$

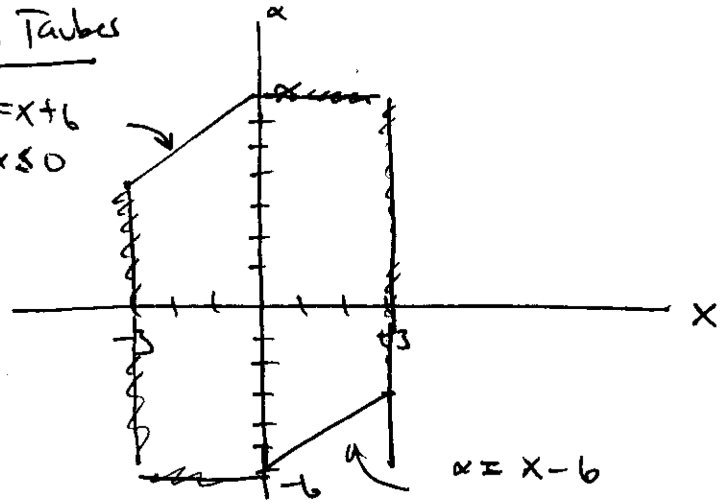
so $\frac{d^2g}{dx^2} < 0$

But $2x^2-1$ is Not Negative $\forall x \in [0,1]$ so No conclusion as to the sign of the RHS can be made

pg 398 Taubes

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha$$

$$\frac{d\alpha}{dt} = -\varepsilon x$$



on: $x = 3$

$$\frac{dx}{dt} = -\frac{27}{3} + 3 + \alpha = -6 + \alpha < 0 \quad \text{if} \quad \alpha < 6 \quad ✓$$

on $x = -3$

$$\frac{dx}{dt} = +9 - 3 + \alpha > 0 \quad \text{if} \quad \alpha > -6 \quad ✓$$

on $\alpha = +6$

$$\frac{d\alpha}{dt} = -\varepsilon x < 0 \quad \text{if} \quad x > 0 \quad ✓$$

on $\alpha = -6$

$$\frac{d\alpha}{dt} = -\varepsilon x > 0 \quad \text{if} \quad x < 0 \quad ✓$$

inside: $\alpha - x - 6 < 0$ outside: $\alpha - x - 6 > 0$

Thus show all trajectories on $\alpha = x + 6$ flow from outside to inside

i.e. $\frac{d}{dt}(\alpha - x - 6) < 0$ on $\alpha = x+6$ + $-3 \le x \le 0$

$$\|$$

$$\frac{d\alpha}{dt} - \frac{dx}{dt} < 0 \Rightarrow -\varepsilon x + \frac{x^3}{3} - x - \alpha \overset{?}{<} 0$$

$$\Rightarrow \quad -\varepsilon x + \frac{x^3}{3} - x - x - 6 \overset{?}{<} 0 \quad \text{when} \quad -3 \le x \le 0$$

$$\Rightarrow \quad -(\varepsilon+2)x + \frac{x^3}{3} - 6 < 0$$

let $F(x) \equiv -(\varepsilon+2)x + \frac{x^3}{3} - 6$ then Max $F'(x) = -(2+\varepsilon) + x^2 = 0$

$$x = \pm\sqrt{2+\varepsilon}$$

$$F''(x) = 2x$$

Thus $x_+ = +\sqrt{2+\varepsilon}$ is a min

$x_- = -\sqrt{2+\varepsilon}$ is a max only x_- is $\in [-3,0]$

Check $F(x_-) \overset{?}{<} 0$

$$+(\varepsilon+2)^{3/2} - \frac{1}{3}(\varepsilon+2)^{3/2} - 6 \overset{?}{<} 0$$

$$\frac{2}{3}(\varepsilon+2)^{3/2} - 6 < 0 \iff (\varepsilon+2)^{3/2} - 9 < 0 \quad \text{yes} ✓$$

iff $\quad (\varepsilon+2) < 9^{2/3} \iff \varepsilon < 9^{2/3} - 2$ ✓

~~[illegible]~~ For the wall at $\alpha = x - 6$

If $\quad x - 6 - \alpha < 0$ represents the "inside"

> 0 " " "outside"

~~Consider~~ To have ~~(0,0)~~ this boundary be a basin of attraction for the pt (0,0) we must have

$$\frac{d}{dt}(x-6-\alpha) < 0 \quad \text{on} \quad \alpha = x-6 \qquad 0 \le x \le +3$$

$$\Rightarrow -\frac{x^3}{3} + x + \alpha + \varepsilon x < 0$$

$$\Rightarrow -\frac{x^3}{3} + x + x - 6 + \varepsilon x < 0$$

$$\Rightarrow -\frac{x^3}{3} + (2+\varepsilon)x - 6 < 0$$

$$F(x) = -\frac{x^3}{3} + (2+\varepsilon)x - 6 \qquad F'(x) = -x^2 + (2+\varepsilon) = 0$$

$$\Rightarrow x = \pm\sqrt{2+\varepsilon}$$ only the positive root is valid

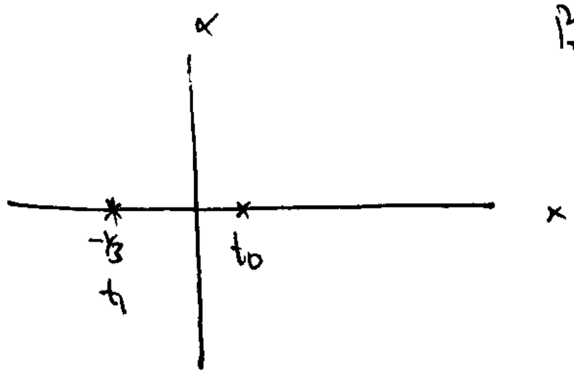
$$F''(x) = -2x < 0 \quad \therefore x = +\sqrt{2+\varepsilon} \text{ is a Max of } F(x)$$

$$F(+\sqrt{2+\varepsilon}) = -\frac{(2+\varepsilon)^{3/2}}{3} + (2+\varepsilon)^{3/2} - 6 = \frac{2}{3}(2+\varepsilon)^{3/2} - 6 \overset{?}{<} 0$$

$$\Leftrightarrow (2+\varepsilon)^{3/2} - 9 \overset{?}{<} 0$$

$$\Leftrightarrow \varepsilon < 9^{2/3} - 2$$ ✓

PY 399 Tables



$\Delta t = t_1 - t_0$

$$\frac{d\alpha}{dt} = -\varepsilon x$$

$$-\frac{d\alpha}{dt} = \varepsilon x \iff -\int_{t_0}^{t_1} \frac{d\alpha}{dt}\,dt = \varepsilon \int_{t_1}^{t_1} x\,dt \le 3\varepsilon \int_{t_1}^{t_1} dt$$

//

$$\alpha(t_0) - \alpha(t_1) \le 3\varepsilon \Delta t$$

||

$$0 - \tfrac{1}{3} \le 3\varepsilon\Delta t \Rightarrow \Delta t \ge \frac{1}{9\varepsilon}$$ for α to move from 0 to $\frac{1}{3}$

$\varepsilon \ll 1$ time required is very great.

How fast does x move? Say from $\frac{1}{2}$ to $-\frac{3}{2}$

$$\frac{dx}{dt} = -\frac{x^3}{3} + x + \alpha$$

$$\int_{t_0}^{t_1} \frac{dx}{dt}\,dt = -\int \frac{x^3}{3}\,dt + \int x\,dt + \int \alpha\,dt$$

||

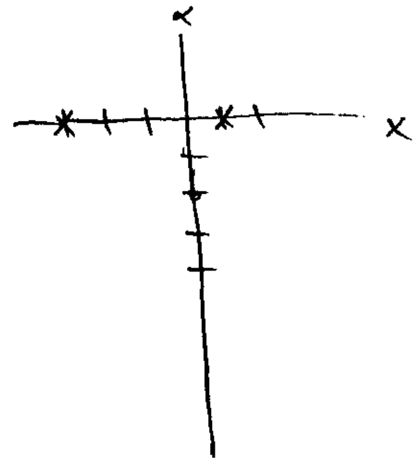
$$x(t_1) - x(t_0) = \ \ -\frac{1}{3}\int$$

$$-\frac{3}{2} - \frac{1}{2} = -2$$

$x(t_2) = 1/2 \quad x(t_3) = -3/2$

$t_3 > t_2$

$\alpha < -1/2 \qquad -\alpha > 1/2$



Estimate $\delta t = t_3 - t_2$

$$-\frac{dx}{dt} = \frac{x^3}{3} - x - \alpha$$

$$-\frac{dx}{dt} \geq \frac{1}{3}\left(x^3 - 3x + \frac{3}{2}\right) \quad \text{for} \quad -\frac{3}{2} \leq x \leq \frac{1}{2}$$

$F(x) \equiv \frac{1}{3}\left(x^3 - 3x + \frac{3}{2}\right)$ has a local ~~maximum~~ maximum. at $x = -1$

$$F'(x) = \frac{1}{3}(3x^2 - 3) = x^2 - 1 = 0 \rightarrow x = \pm 1 \quad \text{at} \quad x = -1$$

$$F''(x) = 2x$$

So $\frac{1}{3}(x^3 - 3x + 3/2)\Big|_{x=-\frac{1}{2}} = \frac{1}{3}\left(-\frac{1}{8} + \frac{3}{2} + \frac{3}{2}\right) = \frac{1}{3}\left(-\frac{1}{8} + 3\right)$

$$= \frac{1}{3}\left(-\frac{1}{8} + \frac{24}{8}\right) = \frac{23}{24} \ ?$$

$$\frac{1}{3}(x^3 - 3x + 3/2)\Big|_{x=-\frac{3}{2}} = \frac{1}{3}\left(-\frac{27}{8} + \frac{9}{2} + \frac{3}{2}\right) = \frac{1}{3}\left(\frac{9}{2} + \frac{3}{2}\right) = \frac{1}{3}(6) = 2$$

36
$\frac{27}{9}$

$$-\frac{dx}{dt} \geq \frac{1}{24}$$

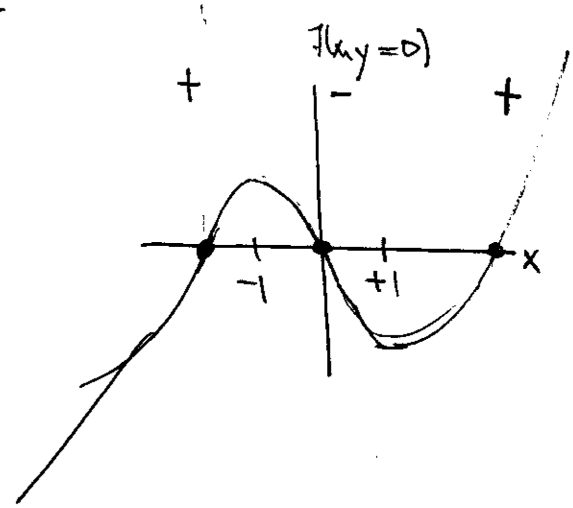
$$\underbrace{x(t_2) - x(t_3)}_{2} \gtrsim \frac{1}{24} \delta t$$

$$\Rightarrow \quad \delta t \leq 48 \qquad \text{v.s.} \qquad \Delta t \gtrsim \frac{1}{9\varepsilon}$$

Pg 411 Tasks

(2) (a) $F(x,y) = x^3 - 3x + y$

$$\frac{dx}{dt} = F(x,y)$$



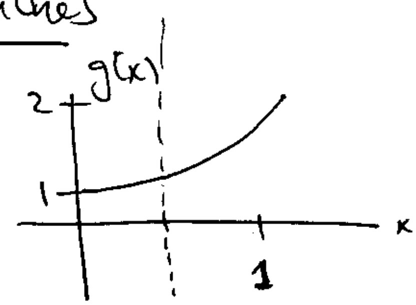
$$\frac{dF(x,y)}{dx} = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$\frac{d^2F}{dx^2} = 6x \Rightarrow x=-1 \text{ max } ; x = +1 \text{ min}$$

$$N_p = 2 \; ; \; N_n = 1 \Rightarrow N_p - N_n = 1$$

Pg 415 Tasks

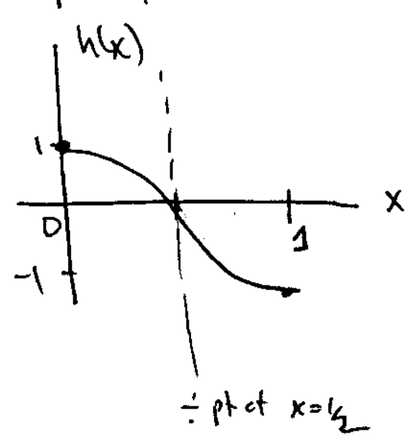
$$F(x) = \underbrace{1 + x^3}_{g(x)} + \underbrace{\cos(\pi x)}_{h(x)}$$



$$\underset{[0,1/2]}{\text{Max}}(g(x)) = 1 + \frac{1}{8} = \frac{9}{8}$$

$$\underset{[0,1/2]}{\text{Max}}(h(x)) = 1$$

$$\therefore \underset{[0,1/2]}{\text{Max}}(g+h) \le \frac{9}{8} + 1 = \frac{17}{8} \checkmark$$



$$\underset{[0,1/2]}{\text{Min}}(g(x)) = 1 \; ; \; \underset{[0,1/2]}{\text{Min}}(h(x)) = 0$$

[crossed out] $\min_{[0,1/2]} (h+g) \geq 1+0$ ✓

$\therefore \max(a+b) \leq \max(a)+\max(b)$
$\min(a+b) \geq \min(a)+\min(b)$

$\max_{[1/2,1]} (g(x)) = 2$; $\max_{[1/2,1]} (h(x)) = 0$

$\therefore \max_{[1/2,1]} (g+h) \leq 2$ ✓

$\min_{[1/2,1]} (g) = 1+\frac{1}{8} = \frac{9}{8}$; $\min_{[1/2,1]} (h) = 0$

So $\min_{[1/2,1]} (g+h) \geq \frac{9}{8}$

Then $\max_{[0,1]} (h+g) \leq \max\left(\underset{[0,1/2]}{h+g}, \underset{[1/2,1]}{h+g} \right) = \max\left(\frac{17}{8}, 2\right) = 2$

$\min_{[0,1]} (h+g) \geq \min\left(\underset{[0,1/2]}{h+g}, \underset{[1/2,1]}{h+g} \right) \geq \min\left(1, \frac{9}{8}\right) = 1$

M414 Tables

① $\frac{dx}{dt} = 3x + x^2 + 3$

Min + Max of $3x + x^2 + 3$ on $[0,1]$

$3 + 2x = 0 \Rightarrow x = -3/2 \notin [0,1]$

$3 \le \frac{dx}{dt} \le 3+1+3 = 7$

$\int_{t_0}^{t_1} dt \Rightarrow \quad 3\Delta t \le \underbrace{x(1) - x(0)}_{1} \le 7\Delta t$

$\therefore \quad \Delta t \le \frac{1}{3} \quad + \quad \Delta t \ge \frac{1}{7}$