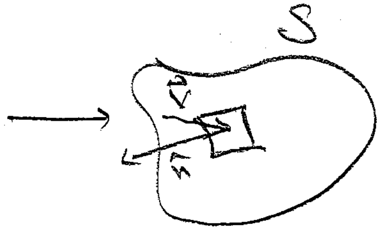
Pg 23 Acheson

① 1.1



Mass total = $\iiint_V \rho \, dV$

As $\vec{v}$ & $\hat{n}$ don't point in the same dir

$\frac{dM_T}{dt}$ = Amt flowing in.

$$\ominus \frac{1}{\Delta t} \iint_S \rho \, \vec{U} \circ \hat{n} \, \Delta t \, dS = \iint_S \rho \vec{U} \circ d\hat{S} = \iiint_V \nabla \circ (\rho \vec{U}) \, dV$$

∴ Setting two quantities equal gives

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV = - \iint \nabla \circ (\rho \vec{U}) \, dV$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \nabla \circ (\rho \vec{U}) = 0$$

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\bar{U}\cdot\nabla)\rho$$

& $\nabla\circ(\rho\bar{U}) = \nabla\rho\circ\bar{U} + \rho\nabla\circ\bar{U}$

$\therefore$ Above become $\left(\frac{\partial\rho}{\partial t} + \nabla\rho\circ\bar{U}\right) + \rho\nabla\circ\bar{U} = 0$

or $\frac{D\rho}{Dt} + \rho\nabla\circ\bar{U} = 0$

If $\nabla\circ\bar{U} = 0$ then $\frac{D\rho}{Dt} = 0$ or density does not change as you follow the fluid. & thus the fluid is uncompressible

Pg 24 Acheson

(1.2) fluid is in steady flow but ~~to~~ flow is not irrotational

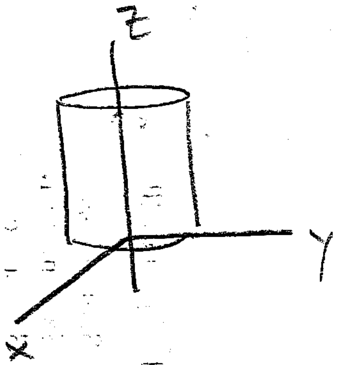
As $\nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -\Omega y & \Omega x & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + 2\Omega\hat{k} \neq 0$

thus $H = \frac{P}{\rho} + \frac{1}{2}\vec{u}^2 + gz$ is not constant for entire flow

Thus Bernoulli only gives that H is constant Along stream lines

or $(\vec{u}\cdot\nabla)H = 0$ The stream lines are given

by $\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dy}{dx} = \frac{v}{u} = \frac{x}{-y}$

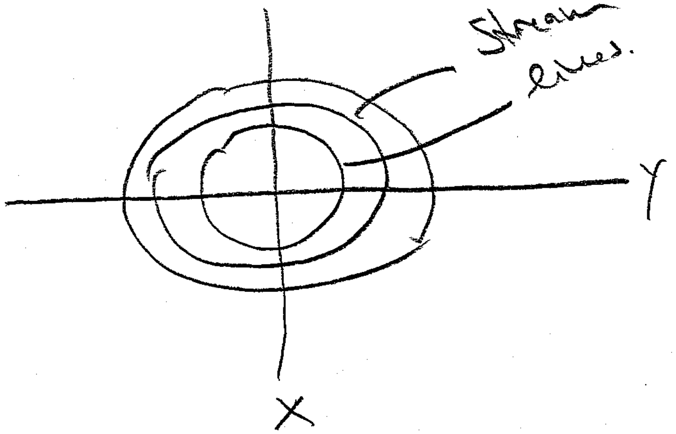


$$-\frac{y^2}{2} + C = \frac{x^2}{2}$$

$$y^2 = -x^2 + C'$$

$$\Rightarrow x^2 \; y^2 = C'$$

top down view:



Thus constant pressure exist only along these stream lines. + depends on Z.

Euler:

$$\frac{D\bar{u}}{Dt} = -\frac{1}{\rho}\nabla p + \vec{g} \qquad \nabla \circ \bar{u} = 0$$

$$\| \qquad\qquad\qquad \|$$

$$0 = 0 \quad \checkmark.$$

$$\cancelto{0}{\frac{\partial \bar{u}}{\partial t}} + (\bar{u}\circ\nabla)\bar{u} = -\frac{1}{\rho}\frac{\partial p}{\partial z}\hat{k} - g\hat{k}$$

$$\Downarrow$$

$$\left(-\Omega y\frac{\partial}{\partial x} + \Omega x\frac{\partial}{\partial y}\right)\left(-\Omega y\hat{i} + \Omega x\hat{j}\right)$$

$$-\Omega y(\Omega\hat{j}) + \Omega x(-\Omega)\hat{i} = -\frac{1}{\rho}\nabla p - g\hat{k}$$

ρ must depend on $\hat{i}$ & $\hat{j}$

Comp:

$$\therefore \quad \nabla p = -\rho g \hat{k} + \rho \Omega^2 x \hat{i} + \rho \Omega^2 y \hat{j}$$

$$\Rightarrow \quad p = -\rho g z + \rho \frac{\Omega^2 x^2}{2} + \rho \frac{\Omega^2 y^2}{2} + \text{const}$$

$$= -\rho g z + \frac{\rho \Omega^2}{2}(x^2 + y^2) + \text{cost}$$

At top surface

$$P(x,y) = \frac{\rho \Omega^2}{2}(x^2 + y^2) + \text{const}.$$

Pg 24 Acheson

(1.3) Rankine vortex

$$v_\theta = \begin{cases} \Omega r & r<a \\ \dfrac{\Omega a^2}{r} & r>a \end{cases} \qquad v_r = v_z = 0$$

Invisit. Euler: $$\frac{D\bar{v}}{Dt} = -\frac{1}{\rho}\nabla p + \bar{g}$$

Inside:

$$\cancel{\frac{\partial \bar{v}}{\partial t}}^{\,0} + (\bar{v}\cdot\nabla)\bar{v} = -\frac{1}{\rho}\nabla p + \bar{g}$$

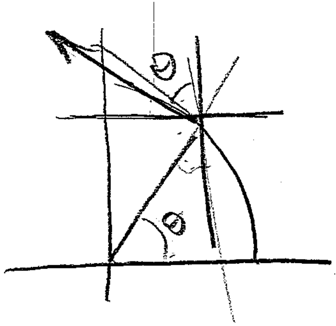
Also for ~~inside~~ outside the core it is important to note that $\nabla \times \bar{v} = 0$ ∴ we are in irrotational flow.

$\downarrow$ only in x, y, z.

See pg 4

$$u_x \frac{\partial \bar{v}}{\partial x} + u_y \frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}\hat{k}\right) + g\hat{k}$$

$$u_x \frac{\partial(v_\theta \hat{\theta})}{\partial x}$$



$$v_x = -v_\theta \cos\theta = -v_\theta \frac{x}{\sqrt{x^2+y^2}}$$

$$v_y = +v_\theta \sin\theta$$

$$= v_\theta \frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \quad v_x = -\Omega y \qquad v_y = +\Omega x$$

$$-\Omega y\left[\Omega \hat{j}\right] + \Omega x\left[-\Omega \hat{i}\right] = -\frac{1}{\rho}\nabla P + \bar{g}$$ sol'n Ex 1.2

$$\Rightarrow \quad p(x,y,z) = -\rho g z + \frac{\rho\Omega^2}{2} r^2 + \text{const.}$$

Outside

$$v_x = -\frac{\Omega a^2}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}}$$

$$v_y = \frac{\Omega a^2}{(x^2+y^2)} y$$

Then $(\bar{v} \cdot \nabla)\bar{v}$

$$\Rightarrow v_x \frac{\partial}{\partial x}\left[-\frac{\Omega a^2 x}{(x^2+y^2)}\hat{i} + \frac{\Omega a^2 y}{x^2+y^2}\hat{j}\right]$$

$$+ v_y \frac{\partial}{\partial y}\left[-\frac{\Omega a^2 x}{x^2+y^2}\hat{i} + \frac{\Omega a^2 y}{x^2+y^2}\hat{j}\right]$$

$$= v_x$$

$$v_\theta = \begin{cases} \Omega r & r<a \\ \frac{\Omega a^2}{r} & r>a \end{cases} \qquad v_r = v_z = 0$$

Euler w/ dissipation: Pg 353 gm

<u>$r<a$</u>

$$-\frac{\Omega^2 r^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \underline{\hat{r} \text{ eq}}$$

$$0 + \cancelto{0}{(\bar{v}\cdot\nabla)}\bar{v} + 0 = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \qquad \hat{\theta}$$

$$\frac{\partial p}{\partial z} = 0 \qquad \hat{z}$$

<u>$r>a$</u>

$$-\frac{\Omega^2 a^4}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \hat{r}$$

$$0 = -\frac{1}{r\rho}\frac{\partial p}{\partial \theta} \qquad \hat{\theta}$$

$$\frac{\partial p}{\partial z} = 0 \qquad \hat{z}$$

$\underline{r < a}$

$$p = f(r) \qquad + \qquad \frac{\partial p}{\partial r} = \rho \Omega^2 r$$

$$\Rightarrow p_{r<a} = \rho \frac{\Omega^2 r^2}{2} + C_1$$

$\underline{r > a}$

$$\frac{\partial p}{\partial r} = \frac{\Omega^2 a^4 \rho}{r^3} \quad \Rightarrow \quad p = -\frac{\Omega^2 a^4 \rho}{2 r^2} + C_2$$

$\underline{B.C.}$

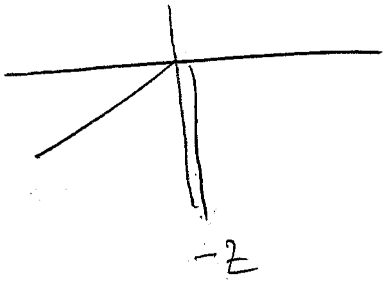
$$C_2 = p_\infty$$

$$p_{r<a}(r=a) = \rho \frac{\Omega^2 a^2}{2} + C_1 = p_{r>a}(r=a) = -\frac{\Omega^2 \rho a^2}{2} + p_\infty$$

$$\Rightarrow \quad C_1 = -\Omega^2 \rho a^2 + p_\infty$$

$$\therefore \quad P_{r<a} = \Omega^2 \rho \left(\frac{r^2}{2} - a^2 \right) + p_\infty$$

$$P_{r<a}(r=0) = -\Omega^2 \rho a^2 + p_\infty$$



If gravity is acting p's become

$$P_{r<a}(r,z) = \Omega^2 \rho \left(\frac{r^2}{2} - a^2\right) - \rho g z + p_\infty$$

$$P_{r>a}(r,z) = -\frac{\Omega^2 a^4 \rho}{2r^2} + p_\infty - \rho g z$$

Then $P_{r>a}(r=\infty, z=0) = p_0 = p_\infty$ = Atmospheric pressure

$\therefore$ must have $P_{r<a}(r=0, z=?) = -\Omega^2 \rho a^2 + p_0 - \rho g z = 0$

$$\Rightarrow \quad z = \frac{\Omega^2 \rho a^2 - p_0}{\rho g}$$

$$= \frac{\Omega^2 a^2}{g} - \frac{p_0}{\rho g}$$

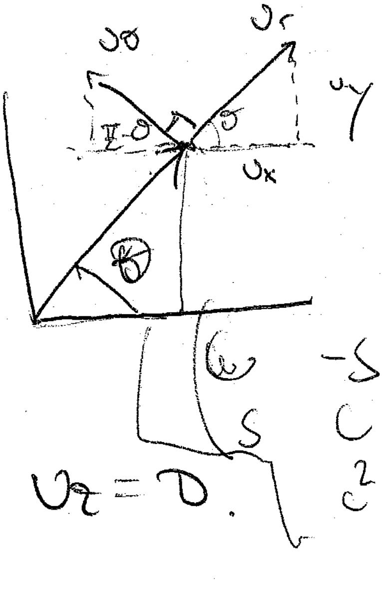
⇒ ~~$p/\rho + \frac{\Omega^2}{2}\rho(x^2+y^2) + \text{Const}$~~

$$p = \rho\frac{\Omega^2}{2}(x^2+y^2) - g\rho z + \text{Const} \qquad \text{on surface}$$

$$\Rightarrow \quad z = \text{Const} + \frac{\Omega^2}{2g}(x^2+y^2)$$

⑬ $v_\theta = \begin{cases} \Omega r & r<a \\ \frac{\Omega a^2}{r} & r>a \end{cases}$ Rankine vortex

$v_r = v_z = 0.$



$$v_x = +v_r\cos\theta - v_\theta\cos(\tfrac{\pi}{2}-\theta)$$
$$= v_r\cos\theta - v_\theta\sin\theta$$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} v_r \\ v_\theta \end{pmatrix}$$

$$v_y = v_r\sin\theta + v_\theta\cos\theta.$$

$v_z = 0.$ $c^2+s^2=1$

$$\therefore \quad v_x = v_r\left[\frac{x}{r}\right] - v_\theta\left[\frac{y}{r}\right]$$

$$v_y = v_r\frac{y}{r} + v_\theta\frac{x}{r}$$

? ~~$v_x = \Omega x$~~ As $v_r \equiv 0$.

$$v_x = -v_\theta \frac{y}{r} \qquad v_y = v_\theta \frac{x}{r} \qquad v_z \equiv 0.$$

$$\therefore\ v_x = -\Omega y \qquad v_y = \Omega x \qquad x^2+y^2 < a^2 \quad \text{I}$$

$$\& \ v_x = -\frac{\Omega a^2 y}{x^2+y^2} \qquad v_y = \frac{\Omega a^2 x}{x^2+y^2} \qquad x^2+y^2 > a^2. \quad \text{II}$$

Steady Euler eqs $(\bar{v}\cdot\nabla)\bar{v} = -\frac{1}{\rho}\nabla p + \bar{g}$. ; $\nabla\cdot\bar{v} = 0$

~~on surface~~ $\nabla\cdot\bar{v} = ?$ inside core

$= 0$ ✓.

$$\nabla\cdot\bar{v} = +\Omega a^2\left[\frac{+2xy}{(\quad)^2} + \frac{-2xy}{(\quad)^2}\right] = 0 \checkmark.$$ outside

components: w/o z.

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\delta z} \mp g$$

$$\Rightarrow \quad p = C(x,y) \mp g\rho z$$

inside: $$-\frac{1}{\rho}\frac{\partial p}{\delta x} = -\Omega y(0) + \Omega x(-\Omega)$$

$$-\frac{1}{\rho}\partial_y p = -\Omega^2 y$$

$$\Rightarrow \quad p = \frac{\rho\Omega^2}{2}(x^2+y^2) \mp g\rho z + C \qquad \text{inside.}$$

outside: get still $p = C(x,y) \mp g\rho z$

$$-\frac{1}{\rho}\partial_x p = \frac{-\Omega a^2 y}{x^2+y^2}\left(+\frac{\Omega a^2 y}{(x^2+y^2)^2}2x\right) + \frac{\Omega a^2 x}{x^2+y^2}\left(\frac{-\Omega a^2}{x^2+y^2} + \frac{\Omega a^2 y}{(x^2+y^2)^2}2y\right)$$

$$= \frac{\Omega^2 a^4}{(x^2+y^2)^3}\left[-2xy^2 - x(x^2+y^2) + 2xy^2\right]$$

$$-\frac{1}{\rho}\partial_x p = \frac{\Omega^2 a^4 x}{(x^2+y^2)^3}\left(-\cancel{2y^2} - x^2 \underbrace{- y^2 + \cancel{2y^2}}\right)$$

$$= \cancel{\frac{\Omega^2 a^4 x(-y^2 - x^2 - 2y)}{(x^2+y^2)^3}} \leftarrow y^2$$

$$= -\frac{\Omega^2 a^4 x}{(x^2+y^2)^2}$$

$$-\frac{1}{\rho}\partial_y p = -\frac{\Omega a^2 y}{x^2+y^2}\left[\frac{\Omega a^2}{(x^2+y^2)} - \frac{\Omega a^2 x}{(x^2+y^2)^2}2x\right]$$

$$+ \frac{\Omega a^2 x}{x^2+y^2}\left[\frac{-\Omega a^2 x}{(x^2+y^2)^2}2y\right]$$

$$= \frac{\Omega^2 a^4 y}{(x^2+y^2)^3}\left[-(x^2+y^2) + 2\cancel{x^2} - 2\cancel{x^2}\right]$$

$$= -\frac{\Omega^2 a^4 y}{(x^2+y^2)^2}$$

$$\therefore \quad \partial_x p = \rho\frac{\Omega^2 a^4 x}{(x^2+y^2)^2}$$

$$\partial_y p = \rho\frac{\Omega^2 a^4 y}{(x^2+y^2)^2}$$

$$\int \frac{y}{(x^2+y^2)^2}\,dy = ?\ \frac{1}{2}\int \frac{du}{(x^2+u)^2}$$

$u = y^2$

$du = 2y\,dy$

$$= \frac{1}{2}\frac{(-1)}{(x^2+u)^{+1}} + \text{Const}$$

$$\therefore\quad p = -\frac{\rho\Omega^2 a^4}{2(x^2+y^2)} + C(y,z)$$

$$\therefore\quad p = -\frac{\rho\Omega^2 a^4}{2(x^2+y^2)} - g\rho z + \text{const.}$$

~~Inside~~ outside

~~$p(r=0) = C$~~ Assuming pressure cont at $r=a$?? true Assumption ??

get $p = -\frac{\rho\Omega^2}{2}a^2 - g\rho z + C_1$ outside

$= \frac{\rho\Omega^2}{2}a^2 - g\rho z + C_2$ inside

$\Rightarrow\ C_1 - C_2 = \rho\Omega^2 a^2$

Then $p(r=0) = -g\rho z + C_2$

$p(r=\infty) = -g\rho z + C_1 \quad \therefore\ \Delta p = \rho\Omega^2 a^2$

$\underline{C_1}$

$\underline{C_2}$

$C_1 - C_2 > 0 \Rightarrow C_1 > C_2$

for free surface $p \equiv p_0$.

Then $p_0 = -g\rho z(\overset{r=0}{x=0, y=0}) + C_2$ inside

& $p_0 = -g\rho z(\overset{r=\infty}{\cancel{x=\infty},}) + C_1$ outside

Then $\left(z(r=\infty) - z(r=0) \right) = -C_1 + C_2 = -\rho\Omega^2 a^2$

$-g\rho$

$\Delta z = +\frac{\Omega^2 a^2}{g}$ Thus $z(\infty) > z(0)$

(1.4) $\nabla \circ \bar{u} = 0$ $\frac{\partial \bar{u}}{\partial t} + (\bar{u} \circ \nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nabla \chi$

$= -\frac{1}{\rho}\nabla p'$

$p' = p + \rho\chi$.

Pg 24 Acheson

(1.4)

$$\nabla \circ \vec{u} = 0$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\circ\nabla)\vec{u} = -\frac{\nabla}{\rho}(p+\rho\chi)$$ constant density

||

Dot · w/ $\vec{u}$

$$\vec{u}\circ\frac{\partial \vec{u}}{\partial t} + (\vec{u}\circ\nabla)\vec{u}\circ\vec{u} = -\frac{\nabla}{\rho}(p')\circ\vec{u}$$

||

$$\frac{\rho}{2}\frac{\partial \vec{u}^2}{\partial t} + \rho(\vec{u}\circ\nabla)(\vec{u}\circ\vec{u}) = -\nabla p'\circ\vec{u}$$

$$\int_V \frac{\rho}{2}\frac{\partial \vec{u}^2}{\partial t}\,dV + \int_V \rho(\vec{u}\circ\nabla)u^2 = -\int_V \nabla p'\circ\vec{u}\,dV$$

$$= \frac{d}{dt}\int_V \frac{\rho}{2}\bar{u}^2\, dV - \frac{\rho}{2}\int_V (\bar{u}\circ\nabla)u^2\, dV + \rho\int_V (\bar{u}\circ\nabla)u^2\, dV = -\int \nabla p'\circ u\, dV$$

$$\Rightarrow \quad \frac{d}{dt}\int_V \frac{\rho}{2} u^2\, dV = \int_V -\frac{\rho}{2}(\bar{u}\circ\nabla)u^2 - \nabla p'\circ u\, dV$$

$$= \int_V \left(-\frac{\rho}{2}(\bar{u}\circ\nabla)u^2 - \bar{u}\circ\nabla p'\right) dV$$

$$= \int_V (\bar{u}\circ\nabla)\left(-\frac{\rho u^2}{2} - p'\right) dV$$

$$= -\int_V \nabla\left(p' + \frac{\rho u^2}{2}\right)\circ\bar{u}\, dV$$

But $\nabla\circ\left(\left(p' + \frac{\rho U^2}{2}\right)\bar{U}\right) = \nabla\left(p' + \frac{\rho U^2}{2}\right)\circ\bar{U}$

$+\left(\cancelto{0}{\nabla\circ\bar{U}}\right)\left(p' + \frac{\rho U^2}{2}\right)$ w/ incompressibility

$$\therefore \quad \frac{d}{dt}\int_V \frac{\rho U^2}{2}\, dV = -\int_S \left(p' + \frac{\rho U^2}{2}\right)\bar{U}\circ\hat{n}\, dS \quad \checkmark$$

Pg 24 Acheson

(1.5) $$\frac{\partial \bar{u}}{\partial t} + \bar{\omega} \times \bar{u} + \nabla\left(\frac{1}{2}\bar{u}^2\right) = -\frac{1}{\rho}\nabla p - \nabla \chi$$

$$+ \quad \frac{D\rho}{Dt} + \rho \nabla \cdot \bar{u} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \bar{v} \cdot \nabla \rho + \rho \nabla \cdot \bar{v} = 0$$

$$\frac{D}{Dt}\left(\frac{\bar{\omega}}{\rho}\right) = \frac{\partial}{\partial t}\left(\frac{\bar{\omega}}{\rho}\right) + (\bar{v} \cdot \nabla)\left(\frac{\bar{\omega}}{\rho}\right)$$

$$= \frac{1}{\rho}\frac{\partial \bar{\omega}}{\partial t} - \bar{\omega}\,\frac{1}{\rho^2}\frac{\partial \rho}{\partial t} + v_x \frac{\partial}{\partial x}\left(\frac{\bar{\omega}}{\rho}\right) + v_y \frac{\partial}{\partial y}\left(\frac{\bar{\omega}}{\rho}\right) + v_z \frac{\partial}{\partial z}\left(\frac{\bar{\omega}}{\rho}\right)$$

$$= \frac{1}{\rho}\frac{\partial \bar{\omega}}{\partial t} + \frac{\bar{\omega}}{\rho^2}\left(\bar{v} \cdot \nabla \rho + \rho \nabla \cdot \bar{v}\right)$$

if $p = p(\rho)$

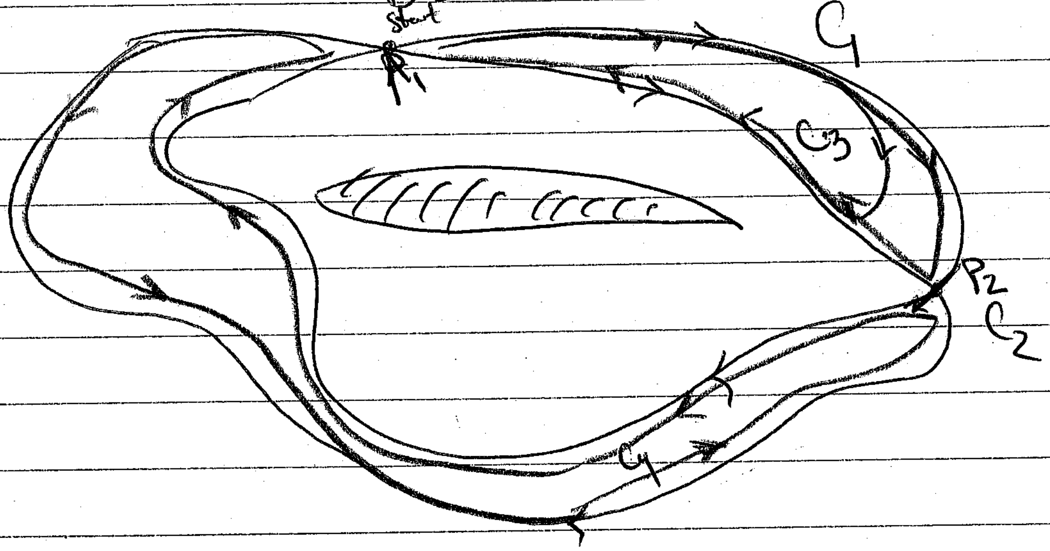
$$\nabla p = p'(\rho)\nabla\rho$$

$\Rightarrow$ $\cancel{\frac{D}{Dt}\left(\frac{\bar\omega}{\rho}\right)}$ $\therefore$ $\nabla(1/\rho) = -\frac{p'(\rho)}{\rho^2}\nabla\rho \times \nabla\rho = 0.$

$$\therefore \frac{D}{Dt}\left(\frac{\bar\omega}{\rho}\right) = \left(\left(\frac{\bar\omega}{\rho}\right)\cdot\nabla\right)\bar u$$

$\bar\rho$ Vorticity eq $\frac{D\bar\omega}{Dt} = (\bar\omega\cdot\nabla)\bar u.$

①.6



Show:

$$\int_{C_1}\bar u\cdot d\bar l = \int_{C_2}\bar u\cdot d\bar l$$

$$\int_{C_1}\bar u\cdot d\bar l = \int_{C_2}\bar u\cdot d\bar l + \int_{P_2\xrightarrow{C_2}P_1\xrightarrow{C_1}P_2}\bar u\cdot d\bar l + \int_{P_2\xrightarrow{C_1}P_1\xrightarrow{C_2}P_2}\bar u\cdot d\bar l$$

But on C_3: $P_2 \xrightarrow{C_2} P_1 \xrightarrow{C_1} P_2$

+ C_4: $P_2 \xrightarrow[C_1]{} P_1 \xrightarrow[C_2]{} P_2$

Stokes thm Apply

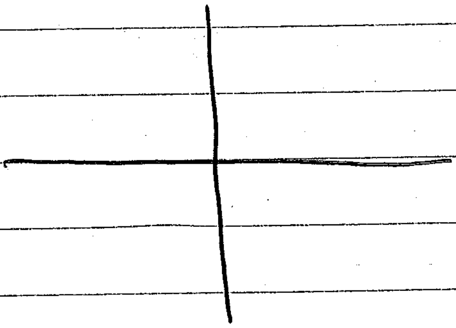
$$\int_{S_3} (\nabla \times \bar{u}) \cdot dA = 0$$ As flow is irrotational!! then

(1.7) $u = \alpha x \qquad v = -\alpha y \qquad w = 0$

$$\cancel{\frac{dx}{ds}} \qquad \frac{dx/ds}{u} = \frac{dy/ds}{v} =$$

$$\cancel{\frac{dy}{ds}} = \cancel{\frac{dx/ds}{u}} = \cancel{dy/ds}$$

$$\frac{dx/\cancel{ds}}{\cancel{\alpha} x} = \frac{dy/\cancel{ds}}{-\cancel{\alpha} y}$$



$$\frac{dy}{dx} = \frac{-y}{x}$$

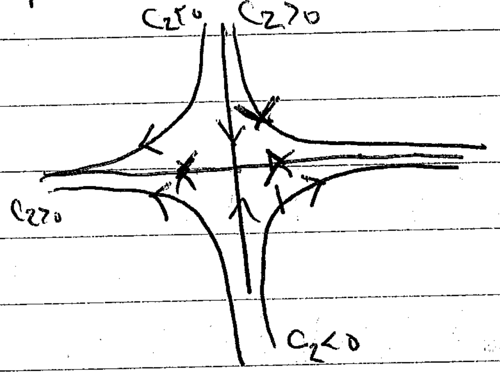
$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln y = -\ln x + C_1$$

$$y = \frac{C_2}{x} \qquad C_2 > 0$$

$x = 1 \quad y = C_2 \quad C_2 \rightarrow \infty$

$C_2 \rightarrow 0$



$U = \alpha x \quad \alpha > 0$

$V = -\alpha y$

Question asked

$$\frac{Dc(x,y,t)}{Dt} \overset{?}{=} 0 \qquad U_y \frac{\partial}{\partial y}$$

$$\Rightarrow \ -\alpha B x^2 y e^{-\alpha t} + (\bar{U} \cdot \nabla) B x^2 y e^{-\alpha t}$$

$$U_x (2Bxye^{-\alpha t}) + U_y B x^2 e^{-\alpha t}$$

$$\Rightarrow \ -\alpha B x^2 y e^{-\alpha t} + \alpha 2 B x^2 y e^{-\alpha t} + -\alpha y B x^2 e^{-\alpha t} = 0 \checkmark.$$

$x = \bar{X} e^{\alpha t} \qquad y = Y e^{-\alpha t} \qquad z = Z$

$$\left(\frac{\partial \bar{x}}{\partial t}\right)_{\bar{X}} = (\bar{X} \alpha e^{+\alpha t}, -\alpha Y e^{-\alpha t}, 0)$$

$$= (\alpha x, -\alpha y, 0) = \bar{U}.$$

$$\left(\frac{\partial \bar{U}}{\partial t}\right)_{\bar{X}} = (\bar{X} \alpha^2 e^{\alpha t}, \alpha^2 Y e^{-\alpha t}, 0)$$

$$= (\alpha^2 x, \alpha^2 y, 0)$$

$$\frac{D\bar{u}}{Dt} = 0, + (\bar{u}\circ\nabla)\bar{u} = \left(\alpha x \frac{\partial}{\partial x} \cancel{+} - \alpha y \frac{\partial}{\partial y}\right)(\alpha x, -\alpha y, 0)$$

$$= (\alpha^2 x, \alpha^2 y, 0)$$

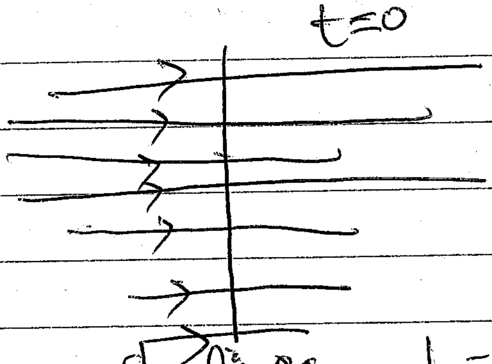
$$x = X e^{+\alpha t} \qquad y = Y e^{-\alpha t}$$

$$c = \beta X^2 \cancel{e^{+2\alpha t}} \cancel{e^{-\alpha t}} Y \cancel{e^{-\alpha t}} = \beta X^2 Y.$$ const, ind of t.

(1.8) $\frac{dx/ds}{u} = \frac{dy/ds}{v}$ At each time instant

$$\frac{dx/ds}{u_0} = \frac{dy/ds}{kt}$$

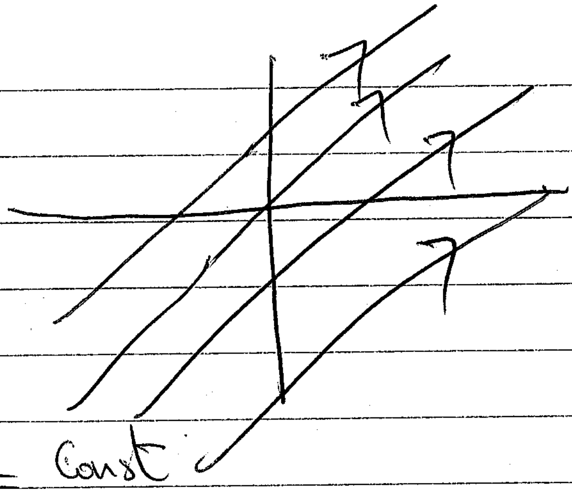
$$\frac{dy}{dx} = \frac{k}{u_0} t$$



$$y = \left(\frac{kt}{u_0}\right)x + y_0$$ St lines. $t_1 =$

particle paths $y(t) = \left(\frac{kt}{u_0}\right)x(t) + y_0$

$t > 0$



$$\frac{dx}{ds} = \text{Const}$$

v_0

$$\frac{dx}{ds} = v_0 C_1 \qquad \frac{dy}{dt} = C_1 k t$$

$$x = \cancel{v_0 C_1 t +}\ v_0 C_1 s + C_2 \qquad \Downarrow$$

$$v_0 C_1 t + C_2$$

$$y = \frac{C_1 k t^2}{2}$$

$$\frac{x - C_2}{v_0 C_1} = t$$

$$\Rightarrow \quad y = \frac{C_1 k}{2}\left(\frac{x - C_2}{v_0 C_1}\right)^2$$

$$\left.\frac{\partial x}{\partial t}\right|_{\bar{x}} = v_0 \qquad \left.\frac{\partial y}{\partial t}\right|_{\bar{X}} = kt \qquad \left.\frac{\partial z}{\partial t}\right|_{X} = 0$$

$$x = v_0 t + F_1(\bar{X}) \qquad y = \frac{kt^2}{2} + F_2(\bar{X}) \Rightarrow z = F_3(\bar{X})$$

$\bar{x} = \bar{X}$ when $t = 0$

& $F_1(\bar{X}) = X$, $F_2 = Y$, $F_3 = Z$

$$x = v_0 t + X$$

$$y = \frac{kt^2}{2} + Y$$

$$z = Z.$$

Pg 40 Acheson

Steady sol.: Changing to homo B.C.

$$\frac{\partial^2 u}{\partial y^2} = \frac{d^2 u}{dy^2} = 0 \qquad \begin{array}{l} u(0) = U \\ u(h) = 0 \end{array} \qquad u(y) = U(1 - y/h)$$

let $u = U(1 - y/h) + u_1$

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} \qquad u_1(y,0) = -U(1 - y/h)$$

$$u_1(0,t) = 0 \qquad u_1(h,t) = 0 \quad \Big\} \text{ homo b.c.}$$

Pg 39 Acheson

$p(y=h) = p_0$

$\Rightarrow \quad -\rho g h \cos\alpha + f(x) = p_0$

$\Rightarrow \quad f(x) = p_0 + \rho g h \cos\alpha.$

Pg 40

$$Q = \int_0^h u_x \, dy = gh$$

pg 43 Acheson

$\vec{v} = v_\theta(r,t)\hat{e}_\theta$ into

⇒ $$\frac{\partial \vec{v}}{\partial t} + (v\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 v$$

$$\frac{\partial v_\theta}{\partial t}\hat{e}_\theta - \frac{v_\theta^2}{r}\hat{e}_r = -\frac{1}{\rho}\left[\frac{\partial p}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{e}_\theta\right]$$

$$+ \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right)\hat{e}_\theta$$

$$-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

⇒

$$\frac{\partial v_\theta}{\partial t} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\frac{\partial p}{\partial \theta} = P(r,t) \Rightarrow p = P(r,t)\theta + f(r,t)$$

As $\frac{\partial p}{\partial z} = 0$

Pg 50 Acheson

㉑ (i) $R = \frac{UL}{\nu} = \frac{(150 m/s)(10 m)}{.15 \; cm^2/s} =$ Huge

$.15 \; cm^2 \left(\frac{1 m}{100 cm}\right)^2 \frac{1}{s}$

$\frac{\delta}{L} = O(R^{-1/2})$ . . .

(ii) $R = \frac{(2 cm)(5 cm/s)}{.01 \; cm^2/s} = 1000$

$\frac{\delta}{L} = O(R^{-1/2})$

(iii) . . .

(iv) $\frac{(10^{-3} cm)(10^{-2} cm/s)}{.01 \; cm^2/s} = 10^{-2}$. *

Low Reynolds # past spheres. ↑ Do!!

Pg 50 Acheson

(22)

$$(\bar{U}\circ\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{u} \qquad \nabla\circ\bar{u}=0$$

$$\cancel{y = X/a}$$

$$\cancel{\frac{\partial}{\partial y} = \frac{\partial}{\partial(X/a)} = \frac{1}{a}\frac{\partial}{\partial x} = a\frac{\partial}{\partial x}}$$

$\bar{u}=0$ on $x^2+y^2=a^2$

$\bar{u}\to(\bar{U},0,0)$ $x^2+y^2\to+\infty$

$$\bar{x}' = \bar{x}/a \qquad \bar{u}' = \bar{u}/U \qquad p' = \frac{p}{\rho U^2}$$

Then $\frac{\partial}{\partial x'} = a\frac{\partial}{\partial x}$ $\cancel{x/a}$ $\Rightarrow \frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial x'}$

eq

$$\cancel{\frac{U^2}{a}}\cancel{a}(\bar{u}'\circ\nabla')\bar{u}' = -\frac{1}{\cancel{\rho}}\cancel{\frac{a(\rho U^2)}{a}}\nabla'p' + \nu\frac{a^2U}{a^{\cancel{2}}}\nabla'^2\bar{u}'$$

$$\Rightarrow (\bar{u}'\circ\nabla')\bar{u}' = -\nabla'p' + \left(\frac{\cancel{a}\nu}{Ua}\right)\nabla'^2\bar{u}'$$

$$\Rightarrow (\bar{u}'\circ\nabla')\bar{u}' = -\nabla'p' + \frac{1}{R}\nabla'^2\bar{u}'$$

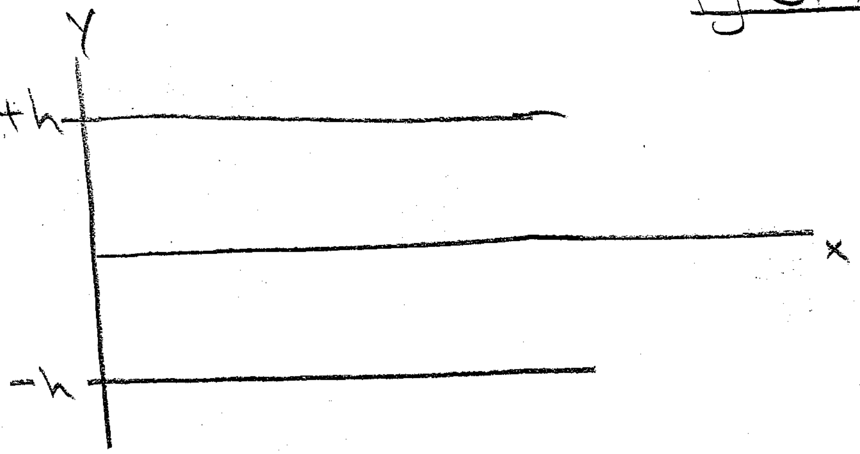
$$\nabla\circ\bar{u} = \cancel{\frac{U}{a}}\nabla'\circ\bar{u}' = 0$$

B.C $x'^2+y'^2=1$

$$\bar{u}\Rightarrow(1,0,0) \qquad x'^2+y'^2\to+\infty$$

Pg 51 Acheson

(2.3) (i)



$\nabla P = -\frac{dp}{dx}$

let $\bar{U} = (u(y),0,0)$ $\nabla\circ\bar{V}$ sat imedietly.

No other v or w velocity could exist for no slip condition at

$y = \pm h$

Eler becomes $\Rightarrow$ $\frac{\partial \bar{U}}{\partial t} + (\bar{U}\circ\nabla)\bar{U} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{U} + \bar{g}$

$$\Rightarrow \quad 0 + 0 = -\frac{1}{\rho}\nabla p + \nu\frac{d^2u}{dy^2}\hat{i} + g\hat{j}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2} \qquad 0 = \frac{\partial p}{\partial z}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g$$

2nd eq $\quad p(y) = \rho g y + A(x) \qquad$ As $\frac{\partial p}{\partial x} \neq 0$

B.C. $\quad u(y=-h) = 0$

$u(y=h) = 0$

$$u(y) = \frac{-1}{\rho \nu} \frac{\partial p}{\partial x} \frac{y^2}{2} + C_1 y + C_2$$

$\Rightarrow$ B.C. $\quad \frac{-1}{\rho \nu} \frac{\partial p}{\partial x} \frac{h^2}{2} - C_1 h + C_2 = 0$

$$\frac{-1}{\rho \nu} \frac{\partial p}{\partial x} \frac{h^2}{2} + C_1 h + C_2 = 0$$

$$2C_2 = +\frac{1}{\rho \nu} \frac{\partial p}{\partial x} h^2$$

$$C_2 = +\frac{1}{2\rho \nu} \frac{\partial p}{\partial x} h^2$$

& $C_1 = 0$

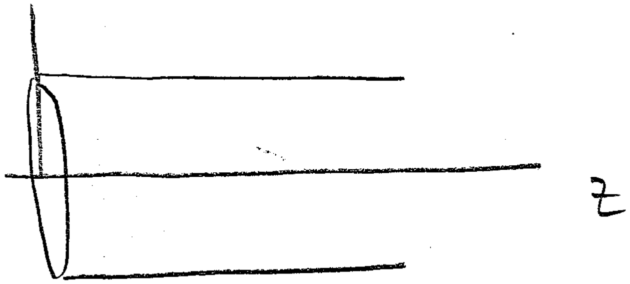
$$u(y) = \frac{-1}{2\rho\nu}\frac{\partial p}{\partial x}(y^2 - h^2)$$

$$\frac{\partial p}{\partial x} = 0 + A'(x) = \frac{\partial p}{\partial x}$$

$$A'(x) = \frac{\partial p}{\partial x}$$

$$\therefore\ p(x,y) = \rho g y + \frac{\partial p}{\partial x}x + C_1 \qquad C_1 = \text{const}$$

(ii)



Assum $\bar{u} = (0, 0, u(r))$ $\qquad \nabla\circ\bar{u} = 0$ ✓

Euler: $$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\circ\nabla)\bar{u} = \frac{-1}{\rho}\nabla p + \nu\nabla^2\bar{u} + \bar{g}$$

$$0 + \cancelto{0}{u_z(r)\frac{\partial u_z(r)}{\partial z}} = -\frac{1}{\rho}\nabla p + \nu\left[\nabla^2 u_z\right]\hat{k}$$

$$0 = -\frac{1}{\rho}\nabla p + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right]$$

<u>In comp:</u>

<u>Zeroing gravity:</u>

$$\Rightarrow \quad \frac{\partial p}{\partial r} = 0 \quad ; \quad \frac{\partial p}{\partial \phi} = 0$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[\quad '' \quad\right]$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{r}{\nu\rho}\frac{\partial p}{\partial z}$$

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\nu\rho}\frac{\partial p}{\partial z} + C_1$$

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\nu\rho}\frac{\partial p}{\partial z} + \frac{C_1}{r}$$

$$u_z(r) = \frac{r^2}{4\nu\rho}\frac{\partial p}{\partial z} + C_1 \ln r + C_2$$

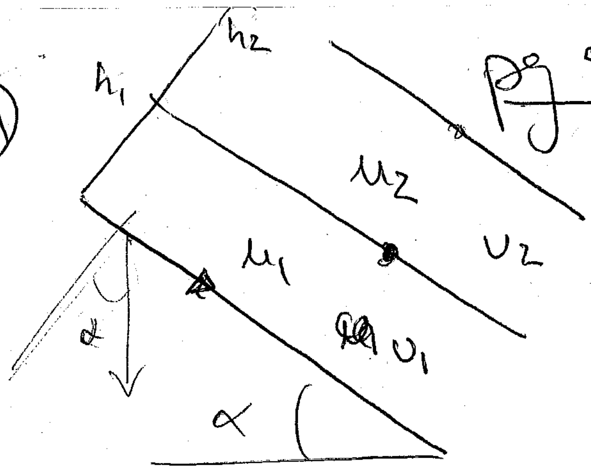
B.C. $u_z(r=0) < \infty \Rightarrow C_1 = 0$

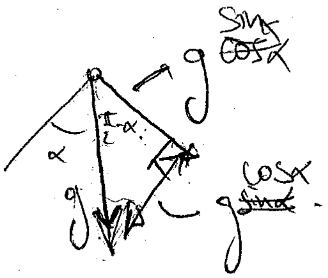
$u_z(r=R) = 0 \Rightarrow C_2 = -\frac{R^2}{4\nu\rho}\frac{\partial p}{\partial z}$

$$\therefore \quad u_z(r) = \frac{\partial p/\partial z}{4\nu\rho}\left(r^2 - R^2\right) = \frac{\overset{=P}{\left(-\partial p/\partial z\right)}}{4\boxed{\mu}}\left(R^2 - r^2\right)$$

$\rho\nu = \mu$

(2.4) pg 51 Acheson





$\nabla = \partial_x \hat{i} + \partial_y \hat{j} + \partial_z \hat{k}$

$\oint_C \nabla\phi\cdot dl = \phi|_C$

Steady flow

$$\cancel{\frac{\partial \bar{u}}{\partial t}}^{0} + \cancel{(\bar{u}\cdot\nabla)\bar{u}}^{0} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \bar{u} + \bar{g}$$

$$\bar{g} = \cancel{(g\cos\alpha, -g\sin\alpha, 0)}$$

$$= (g\sin\alpha, -g\cos\alpha, 0)$$

$$\bar{u} = (u(y), 0, 0)$$

$\hat{x}$ $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + g\sin\alpha$$

$\hat{y}$ $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - g\cos\alpha.$$

$\hat{z}$ $$0 = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

$P = -\rho g \cos\alpha \, y + C_1(x)$ Valid for Both regions.

$\frac{\partial P}{\partial x}$ ~~independent of~~ x.

$\Rightarrow$ ~~$C_1'(x)$~~ .

on $\underline{y = h_1}$

$P_1 = -\rho g \cos\alpha \, y + C_1(x)$ $0 < y < h_1$

$P_2 = -\rho g \cos\alpha \, y + C_2(x)$ $h_1 < y < h_2$

$P_1(h_1) = P_2(h_1)$

$P_2(h_2) = -\rho g \cos\alpha \, h_2 + \underline{C_2(x)} = \underline{P_0}$

$\underline{\text{No}}$ x dep.

$C_2(x) = P_0 + \rho g \cos\alpha \, h_2$

$\therefore$ $P_2 =$ ~~$\rho g \cos\alpha$~~

$P_0 - \rho g \cos\alpha (y - h_2)$ $h_1 < y < h_2$

& $P_2(h_1) = P_0 - \rho g \cos\alpha (\cancel{h_1} - h_2)$

$\overset{\text{set}}{=} P_1(h_1) = -\rho g \cos\alpha \, h_1 + C_1(x)$.

$$\Rightarrow \quad G(x) = p_0 + \rho g \cos\alpha \cdot h_2$$

$$\therefore \quad P_1 = p_0 + \rho g \cos\alpha\,(h_2 - y) \qquad 0 < y < h_1$$

Sheer on $y = h_2$ $\quad \mu_2 \frac{du_2}{dy} = 0$ No ~~force~~ shear

+ Sheer on $y = h_1$ $\quad \mu_1 \frac{du_1}{dy}\Big|_{h_1} = \mu_2 \frac{du_2}{dy}\Big|_{h_1}$

Then in Region II

$$\hat{x} \Rightarrow \quad 0 = \nu_2 \frac{d^2 u_2}{dy^2} + g\sin\alpha \quad \Rightarrow \frac{du_2}{dy} = -\frac{g}{\nu}\sin\alpha\, y + A$$

$$u_2 = -\frac{g}{\nu_2}\sin\alpha \frac{y^2}{2} + Ay + B$$

& Region I

$$\hat{x} \quad 0 = \nu_1 \frac{d^2 u_1}{dy^2} + g\sin\alpha.$$

$$u_1 = -\frac{g}{\nu_1}\sin\alpha \frac{y^2}{2} + A_1 y + B_1$$

In Region I $\quad u_1(y=0) = 0$

$$\Rightarrow B_1 = 0$$

Now B.C. are No slip at $y = h_1$ ✓

i.e. $u_1 = u_2 \quad y = h_1 \qquad u_1(y=0) = 0$ ✓

& stress at $y = h_2 \quad \mu_2 \left.\frac{du_2}{dy}\right|_{y=h_2} = 0.$

& $$\mu_1 \left.\frac{du_1}{dy}\right|_{y=h_1} = \mu_2 \left.\frac{du_2}{dy}\right|_{y=h_2}$$

Region II:

$$-\frac{g}{\nu_2}\sin\alpha = \frac{d^2 u_{II}}{dy^2}$$

$$u_{II}(y) = -\frac{g\sin\alpha}{\nu_2}\frac{y^2}{2} + A_2 y + B_2$$

$$\frac{d}{dy}u_{II}(y=h_2) = -\frac{g\sin\alpha}{\cancel{2}\nu_2}h_2 + A_2 = 0 \Rightarrow A_2 = \frac{g h_2 \sin\alpha}{\nu_2}$$

$$\therefore\ u_{II}(y) = -\frac{g\sin\alpha}{\nu_2}\frac{y^2}{2} + \frac{g h_2 \sin\alpha}{\nu_2}y + B_2$$

Region I w/ $u_1(y=0) = 0$

$$u_1(y) = -\frac{g\sin\alpha}{\nu_1}\frac{y^2}{2} + A_1 y$$

2 con

$u_1(h_1) = u_{11}(h_1)$

$$\Rightarrow \quad -\frac{g\sin\alpha}{\nu_1}\frac{h_1^2}{2} + A_1 h_1 = -\frac{g\sin\alpha}{\nu_2}\frac{h_1^2}{2} + \frac{g h_2 \sin\alpha}{\nu_2} h_1 + B_2$$

$$\mu_1 \frac{du_1}{dy}\Big|_{h_1} = \mu_2 \frac{du_1}{dy}\Big|_{h_1}$$

$$\Rightarrow \quad \mu_1\left[-\frac{g\sin\alpha}{\nu_1}h_1 + A_1\right] = \mu_2\left[-\frac{g\sin\alpha}{\nu_2}h_1 + \frac{g h_2 \sin\alpha}{\nu_2}\right]$$

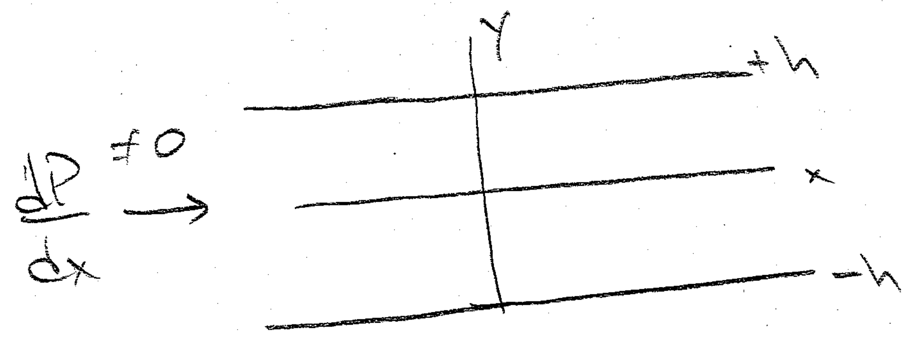
$$\Rightarrow \quad A_1 = \left(\frac{\mu_2}{\mu_1}\frac{g\sin\alpha}{\nu_2}\right)(h_2 - h_1) + \frac{g\sin\alpha}{\nu_1}h_1$$

Then

$$u_1(y) = -\frac{g\sin\alpha}{\nu_1}\left[\frac{y^2}{2}\right.$$

Pg 52 Acheson

㉕



Assume $\vec{u} = \hat{i}\,u(y,t)$ Then $\nabla \cdot \vec{u} = 0$ ✓

Euler:
$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\vec{u} + \overset{0}{\cancel{\text{gravity}}}$$

$$\underline{\hat{x}} \Rightarrow \quad \frac{\partial u}{\partial t} + 0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\underline{\hat{y}} \quad 0 + 0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} + 0 \Rightarrow \text{P ind of y}$$

$$\underline{\hat{z}} \Rightarrow \quad \text{P ind of z}$$

Let $-\frac{\partial P}{\partial x} = P$. get eq asked for

B.C $u(-h,t) = 0$ I.C. $u(y,0) = 0$ $\}$ $\not\Rightarrow$ sol $u=0$ As eq is inhomogenous

$u(h,t) = 0$

to solve let $u = u_0(y) + u_1(y,t)$

w/ $u_0(y)$ sol to steady state w/ B.C but not I.C

then $u_1(y,t)$ is chosen to satisfy homo B.V. + I.C +

$$u(y,0) = -u_0(y)$$

Steady state problem $\frac{d^2 u_0}{dy^2} = -\frac{P}{\nu\rho}$

$$\Rightarrow u_0(y) = -\frac{P}{\nu\rho}\frac{y^2}{2} + C_1 y + C_2$$

$$\left.\begin{matrix} u_0(-h) = 0 \\ + u_0(h) = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} C_1 = 0 \\ C_2 = \frac{h^2 P}{2\nu\rho} \end{matrix}$$

Now solve inhomo +

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2}$$

+ $u_1(y,0) = -u_0(y)$

$u_1(\pm h, t) = 0$

Heat eq: w/ Homo B.C.

Pg 227 Ashbaugh/Asmar.

(1) Sol

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-\lambda_n^2 t} \sin\left(\frac{n\pi}{L}\right)x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\lambda_n = \frac{cn\pi}{L}$$

$u_1 = u(y,t) - u_0(y)$

Pf →

$$\nu\frac{\partial^2 u_1}{\partial y^2} = \nu\frac{\partial^2 u}{\partial y^2} - \nu\frac{\partial^2 u_0}{\partial y^2}$$

$$\frac{\partial u_1}{\partial t} = \frac{\partial u}{\partial t}$$

$$\rightarrow \nu\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial u}{\partial t} \cancel{- \frac{P}{\rho}} - \nu\frac{\partial^2 u_0}{\partial y^2} = \frac{\partial u}{\partial t}$$

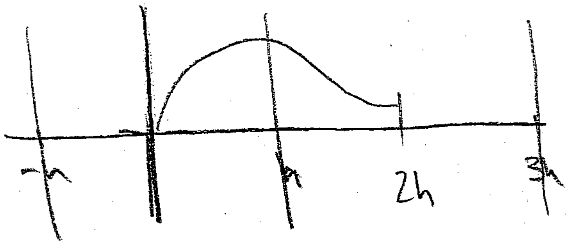
$$\parallel$$

$$\cancel{+\frac{P}{\nu\rho}}$$

I.C on $u_1(y,0) = u(y,0) - u_0(y)$

$= -u_0(y)$

B.C $u_1(\pm h, t) = 0 - 0 = 0$ ✓.



want B.C at 0 & 2h

Thus solve for u_2 & Add h to y to get u_1

$u_2(y,t) = u_1(y-h,t)$

$u_2(0,t) = u_1(-h,t) = 0$

$u_2(2h,t) = u_1(h,t) = 0$

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t}$$

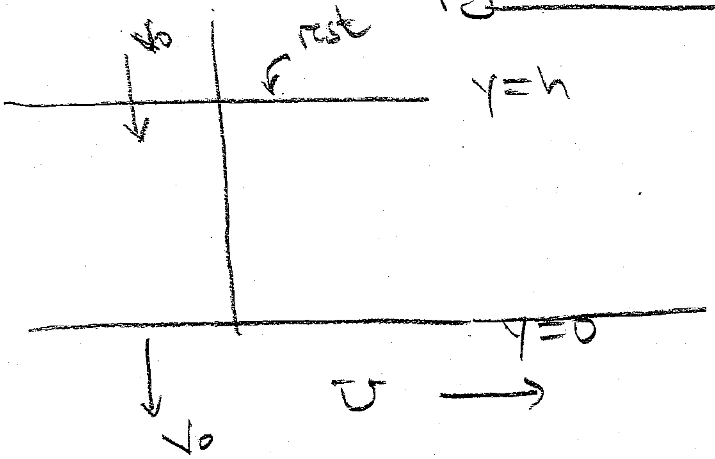
$$\frac{\partial^2 u_1}{\partial y^2} = \frac{\partial^2 u_2}{\partial y^2}$$

$$u_2(y,t) = \sum b_n$$

$$b_n = \frac{2}{2h}\int_0^{2h} \frac{P}{\nu\rho}(h^2-y^2)\sin\left(\frac{n\pi}{2h}y\right)dy$$

$$= \frac{P}{\nu\rho}\frac{1}{h}\Big[$$

Pg 52 Acheson



Assume

$$\bar{v} = u(x,y)\,\hat{i} - v_0\,\hat{j}$$

$$\nabla \cdot \bar{v} = \frac{\partial u}{\partial x} = 0 \Rightarrow u \text{ independent of } x$$

Euler

$$\cancel{\frac{\partial \bar{u}}{\partial t}}^{\,0} + (\bar{v}\cdot\nabla)\,\bar{v} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\bar{v}$$

$$\left(u\frac{\partial}{\partial x} - v_0\frac{\partial}{\partial y}\right)\bar{v} = -\frac{1}{\rho}\nabla P + \nu\frac{\partial^2 u}{\partial y^2}\,\hat{i}$$

$$\Rightarrow\; 0 - v_0\frac{\partial u}{\partial y}\,\hat{i} = -\frac{1}{\rho}\nabla P + \nu\frac{\partial^2 u}{\partial y^2}\,\hat{i}$$

$\underline{\hat{i}}$

$$-v_0\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$\hat{j} \Rightarrow \quad \frac{\partial P}{\partial y} = 0$ $\qquad \hat{k} \Rightarrow$ Pind of z

Assume $P \equiv 0$. Not told About any Press in x dir

$U'' + \frac{V_0}{\nu} U' = 0$ $\qquad$ B.C $\quad U(y=0) = U$

$U(y=h) = 0$

$\Rightarrow \quad U(y) =$

MA. gets same !!

Pg 52 Acheson

(2.7e) cont If $\frac{V_0 h}{\nu} \gg 1$

Then $e^{-V_0 h/\nu} \approx 0$

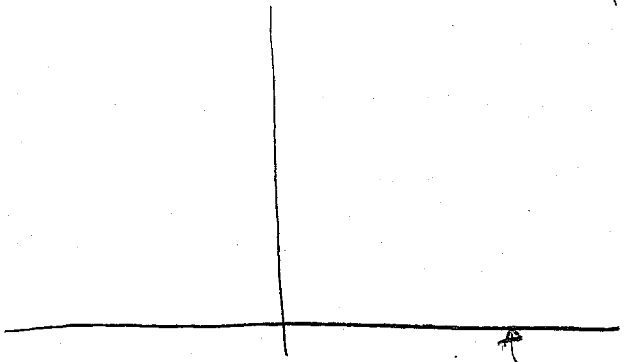
+ $u \approx U e^{-V_0 y/\nu}$

If $\frac{V_0 h}{\nu} \ll 1$ Then $\frac{V_0 y}{\nu}$ As $y<h$

$$\therefore \quad u \approx U \left(\frac{-\frac{V_0 y}{\nu} + \frac{V_0 h}{\nu}}{+\frac{V_0 h}{\nu}} \right) = \frac{U}{h}(h-y)$$

Pg 52 Acheson

(2.7)



$U\cos\omega t\,\hat{i} = \mathrm{Re}(U e^{i\omega t})$

Thus $\bar{u} = u(y,t)\hat{i}$ $\quad \nabla\cdot\bar{u} = 0$ ✓

Euler

$$\frac{\partial \bar{u}}{\partial t} + \cancelto{0}{(\bar{u}\cdot\nabla)\bar{u}} = -\cancelto{0}{\frac{1}{\rho}\nabla p} + \nu\nabla^2 u$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} \qquad u(0,t) = U e^{i\omega t}$$

& take real pt at end

Assume solution $u(y,t) = f(y)e^{i\omega t}$

$$i\omega \cancel{e^{i\omega t}} f(y) = \nu\frac{d^2 f}{dy^2}\cancel{e^{i\omega t}}$$

w/ B.C. $f(0) = U$

& $f(y\to\infty) = 0$

$$r^2 = \frac{\omega i}{\nu}$$

$$\Rightarrow r^2 - \frac{\omega i}{\nu} = 0$$

$$r = \pm\sqrt{\frac{\omega i}{\nu}}$$

$$= \pm\sqrt{\frac{\omega}{\nu}}\sqrt{+i}$$

took care of 2 roots w/ ±

$$i = 1e^{(i\pi/2 + \cancel{2\pi i})}$$

$$\sqrt{i} = e^{i\pi/4} = \Rightarrow +\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Ma $f(y) =$

$$\therefore f(y) = C_1 \exp\left\{\left(+\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) y\sqrt{\frac{\omega}{\nu}}\right\}$$

$$+ C_2 \exp\left\{\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) y\sqrt{\frac{\omega}{\nu}}\right\}$$

As $f(y) \to 0$ $y \to \infty$ $C_1 = 0$

$\therefore C_2 = U$

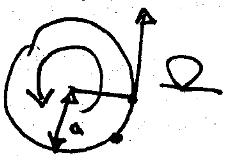
$$\therefore f(y) = U\exp\left\{\left(-\frac{1}{\sqrt{2}} y\sqrt{\boxed{\frac{\nu}{\omega}}}\right\} \exp\left\{-\frac{i}{\sqrt{2}} y\sqrt{\boxed{\frac{\nu}{\omega}}}\right\}$$

$$\Rightarrow U(y,t) = U\exp\left(-\frac{y}{\sqrt{2}}\sqrt{\boxed{\frac{\nu}{\omega}}}\right)\exp\left\{i\left(\frac{y}{\sqrt{2}}\sqrt{\boxed{\frac{\nu}{\omega}}} + \omega t\right)\right\}$$

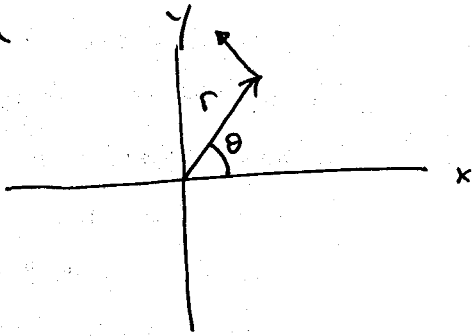
taking real pt

$$\Rightarrow U(y,t) = Ue^{-y\sqrt{\frac{\omega}{2\nu}}}\cos\left(\omega t - y\sqrt{\frac{\omega}{2\nu}}\right)$$

(2.8)



$u_\theta(a) = \Omega a$



Viscous Flow ⇒ B.C. no slip

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 u$$

$$\nabla\cdot u = 0$$

$$\vec{u} = u_\theta(r,t)\,\hat{e}_\theta$$

$$\nabla\cdot\bar{u} = 0$$

$$\Rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\Rightarrow \quad 0 + \frac{1}{r}\frac{\partial u_\theta(r,t)}{\partial \theta} + 0 = 0 \quad \Rightarrow \quad 0 = 0 \quad \checkmark$$

By eq 2.30 a steady state solution to the following

$$u_\theta = u_\theta(r) \equiv u(r)$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - u = 0$$ w/ ~~u(a) =~~

w/ B.C. $u(a) = \Omega a$ + $u(\infty) = 0$

Check $\vec{v} = \frac{\Omega a^2}{r}\hat{e}_\theta$ is an exact solution to the equations (steady state)

$\vec{v}(r=a) = \Omega a \hat{e}_\theta$ ✓ B.C.'s

$\vec{v}(r=\infty) = 0$ ✓

$$\frac{dv}{dr} = -\frac{\Omega a^2}{r^2} \quad ; \quad \frac{d^2v}{dr^2} = +2\frac{\Omega a^2}{r^3}$$

so $r^2\frac{d^2v}{dr^2} = \frac{2\Omega a^2}{r} \quad + \quad r\frac{dv}{dr} = -\frac{\Omega a^2}{r}$

$$r^2\frac{d^2v}{dr^2} + r\frac{dv}{dr} - v = 0$$

$$2\frac{\Omega a^2}{r} + -\frac{\Omega a^2}{r} - \frac{\Omega a^2}{r} = 0 \quad ✓$$

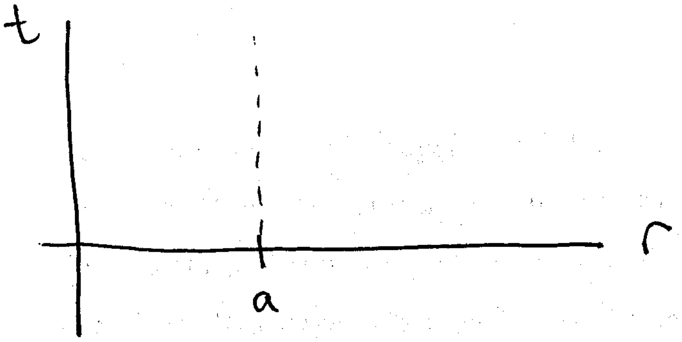
To solve the problem of what happens when the cylinder is started from rest consider the time dependent equation

$$* \quad \frac{\partial v_\theta}{\partial t} = \nu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right)$$

w/ $v_\theta(r,0) = 0$
$v_\theta(a,t) = \Omega a$
$v_\theta(\infty,t) = 0$

$\nu \equiv \frac{\mu}{\rho}$

ν = kinematic viscosity
μ = coefficient of viscosity



$u(r,0)$ needed ✓

$u(r_1,t)$ & $u(r_2,t)$ needed

Use seperation of variables

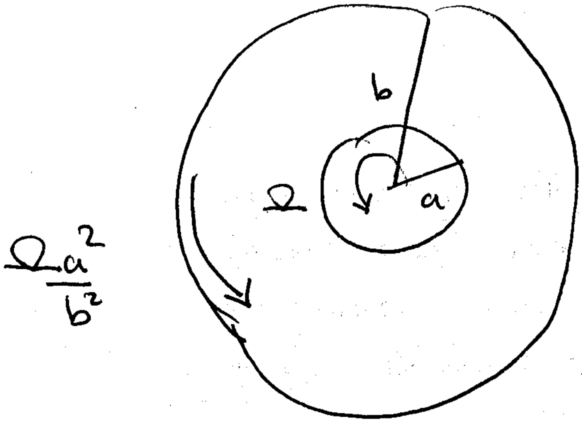
$u(r,t) = R(t)T(t)$ & put into eq *

$$R\cdot\frac{dT}{dt} = \nu\left[T\cdot\frac{d^2R}{dr^2} + \frac{T}{r}\frac{dR}{dr} - \frac{TR}{r^2}\right] \quad \div RT$$

$$\frac{1}{\nu T}\frac{dT}{dt} = \frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2}\right] = -\lambda$$

$$\Rightarrow \quad \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{R}{r^2} + \lambda R = 0$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - 1)R = 0$$



For the time independent solution

$$\nu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2} \right) = 0$$

w/ $v(a) = \Omega a$ $\qquad v(b) = \frac{\Omega a^2}{b^2} \cdot b = \frac{\Omega a^2}{b}$

$$\to \quad r^2 \frac{d^2 v}{dr^2} + r\frac{dv}{dr} - v = 0$$

$$v = r^{\alpha}$$

$$\alpha(a-1) + \alpha - 1 = 0$$

$$\alpha^2 - 1 = 0$$

$$\alpha = \pm 1$$

$$v(r) = A r^1 + B r^{-1}$$

$$v(a) = \Omega a = A a + \frac{B}{a}$$

$$v(b) = \frac{\Omega a^2}{b} = A b + \frac{B}{b}$$

$$\Omega ab = Aab + \frac{Bb}{a}$$

$$\frac{\Omega a^3}{b} = Aab + \frac{Ba}{b}$$

$$-$$

$$B\left(\frac{b}{a} - \frac{a}{b}\right) = \Omega\left(ab - \frac{a^3}{b}\right) = \Omega ab\left(1 - \frac{a^2}{b^2}\right)$$

$$B\left(\frac{b^2 - a^2}{ab}\right) = \Omega ab\left(\frac{b^2 - a^2}{b^2}\right)$$

$$B = \Omega\frac{a^2b^2}{b^2} = a^2\Omega$$

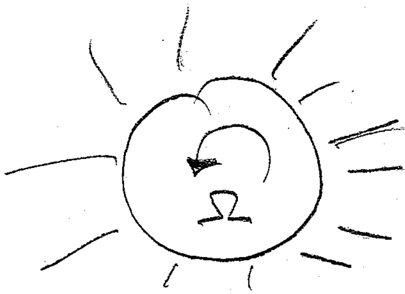
Then $\cancel{\Omega ab} = \cancel{Aa}b \qquad \cancel{\Omega}a = Aa + a\cancel{\Omega} \Rightarrow A = 0.$

$$v(r) = \frac{a^2\Omega}{r} \qquad \square$$

$v(a) = a\Omega$ ✓ $\qquad v(b) = \dfrac{a^2\Omega}{b}$ ✓

Pg 53 Acheson

(2.9)



$u_r(r=a) = -U$ B.C

Assume $\vec{U} = U_r(r)\hat{r} + u_\theta(r)\hat{\theta}$

$$\nabla \circ \vec{U} = \frac{1}{r}\frac{\partial}{\partial r}(rU_r) + \frac{1}{r}\cancel{\frac{\partial u_\theta}{\partial \theta}}^{\,0} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial r}(rU_r) = 0$$

$$u_r = \frac{f(\theta,z)}{r}$$

But there is No reason to Assume f would depend on θ, z. $U_r(r=a) = \frac{f}{a} = -U$

$f = -Ua$

$$\therefore \quad U_r(r) = -\frac{Ua}{r} \qquad r > a$$

Euler:

$$\frac{\partial U}{\partial t} + (\bar{U} \circ \nabla)\bar{U} = -\frac{1}{\rho}\nabla P + \nu\nabla^2 U$$

$\underline{\hat{r}}$

$$0 + (\bar{U}\cdot\nabla)\left(-\frac{Ua}{r}\right) - \frac{U_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial \rho} + \nu\left(\nabla^2\left(-\frac{Ua}{r}\right)\right)$$

$\underline{\hat{\theta}}$

$$(\bar{U}\cdot\nabla)(U_\theta) + \frac{U_r U_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 U_\theta - \frac{U_\theta}{r^2}\right)$$

$$\bar{U}\cdot\nabla = U_r\frac{\partial}{\partial r} \qquad /$$

Assume No pressure in θ direction $\frac{\partial p}{\partial \theta} = 0$

$$-\frac{Ua}{r}\frac{\partial}{\partial r}U_\theta + -\frac{Ua}{r}U_\theta = 0 + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right)U_\theta - \nu\frac{U_\theta}{r^2}$$

Mult by $\frac{-r}{Ua}$

$$\Rightarrow U_\theta' + U_\theta = +\frac{\nu}{Uar}U_\theta - \frac{\cancel{r}\nu}{Ua}\left[\frac{1}{\cancel{r}}\left(rU_\theta'' + U_\theta'\right)\right]$$

let $R = \frac{U a}{\nu}$

$$\Rightarrow \frac{r}{R} u_\theta'' + \frac{1}{R} u_\theta' + u_\theta' + u_\theta\left(1 - \frac{1}{Rr}\right) = 0$$

$$\Rightarrow r^2 u_\theta'' + r(R+1)u_\theta' + (R-1)u_\theta = 0$$

equidemention $u_\theta = x^\alpha$

$$\Rightarrow \alpha(\alpha-1) + (R+1)\alpha + (R-1) = 0$$

$$\Rightarrow \alpha^2 + R\alpha + R-1 = 0$$

$$\alpha = \frac{-R \pm \sqrt{R^2 - 4(R-1)}}{2} =$$

Then $\exists$ 2 sol. $u_\theta(r) = Ar^{\alpha_1} + Br^{\alpha_2}$

to be finite $r \to \infty$

$$R^2-4R+4 = (R-2)^2$$

$$\therefore \alpha = \frac{-R \pm (R-2)}{2} = -1, \; -R+1 \qquad R \neq 2.$$

$$\therefore v_\theta(r) = Ar^{-1} + Br^{1-R}$$

$R<2 \qquad 1-R > -1$

Thus $\Gamma =' 2\pi r v_\theta = 2\pi B r^{2-R}$ But $2-R > 0$

$$\therefore \Gamma \xrightarrow{r\to\infty} \infty$$

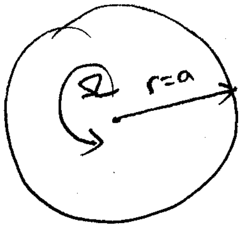
$R>2 \qquad 1-R < -1$

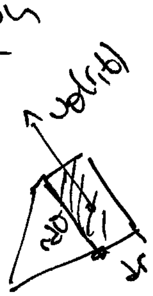
$$\& \; \Gamma = 2\pi B r^{2-R} \xrightarrow{r\to\infty} 0 \qquad \text{O.k}$$

As only one coeff can be eliminated it follows that $\exists \infty$ many solutions

(2.12)

Pg 54 Acheson





$dM = r d\theta\, dr \rho$

$\to 2\pi r\, dr \rho$

$$\bar{U} = U(r,t)\,\hat{e}_\theta \qquad \nabla \circ \bar{U} = \cancel{\frac{\partial u_r}{\partial r}}^{0} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta} u_\theta}^{0} + \cancel{\frac{\partial}{\partial z}}^{0}$$

$$\frac{\partial u_\theta}{dt} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)$$

$$= \frac{\nu}{r}\frac{\partial}{dr}\left(r\frac{\partial u_\theta}{dr}\right) - \frac{\nu u_\theta}{r^2}$$

if $\displaystyle E \equiv \frac{1}{2}\int_0^a (2\pi\rho) r u_\theta^2\, dr = 2\pi\rho\, E_{Book.}$

$$\cancel{\delta}\ \frac{dE}{dt} = \frac{2\pi\rho}{\cancel{2}}\cancel{2}\int_0^a r u_\theta \frac{\partial u_\theta}{\partial t}\, dr = 2\pi\rho\int_0^a r u_\theta \frac{\partial u_\theta}{\partial t}\, dr$$

Thu $$\frac{dE_\rho}{dt} + \frac{2\nu}{a^2}E = \cancel{\int_0^a} (2\pi\rho)\left[\int_0^a \left(r u_\theta \frac{\partial u_\theta}{\partial t} + \frac{\cancel{2}\nu}{a^2}\frac{r u_\theta^2}{\cancel{2}}\, dr\right)\right]$$

$$= 2\pi\rho\left[\int_0^a r u_\theta\left(\frac{\partial u_\theta}{\partial t} + \frac{\nu u_\theta}{a^2}\right) dr\right] \qquad r u_\theta\left(\frac{\partial u_\theta}{\partial t} + \frac{\nu u_\theta}{a^2}\right) dr$$

$$\leq 2\pi\rho\int_0^a \frac{\nu}{\cancel{r}}\frac{d}{dr}\left(r\frac{du_\theta}{dr}\right) u_\theta \cancel{r}\, dr = 2\pi\rho\nu\ \cancel{r}\frac{du_\theta}{dr}$$

$$= 2\pi\nu\rho\int_0^a \frac{d}{dr}\left(r\frac{du_\theta}{dr}\right) u_\theta\, dr$$

$$= 2\pi\nu\rho\left[\cancel{v_\theta r \frac{dv_\theta}{dr}}\Big|_0^a - \int_0^a \frac{dv_\theta}{dr} r \frac{dv_\theta}{dr}\, dr\right]$$

$v_\theta(a) = 0$; $r(0) = 0$

$$= -2\pi\nu\rho \int_0^a r\left(\frac{dv_\theta}{dr}\right)^2 dr \leq 0.$$

$\therefore \quad \frac{dE}{dt} + \frac{2\nu}{a^2} E \leq 0.$

$$\frac{dE}{dt} \leq \frac{-2\nu}{a^2} E$$

? $\frac{dE}{E} \leq \frac{-2\nu}{a^2} dt$

? $\ln E \leq \frac{-2\nu}{a^2} t + c$

$E \leq C e^{-2\nu/a^2 t}$ $\quad\therefore\quad$ $E \searrow 0$ As $t \to +\infty$

pg 54 Acheson   2-2-04

2.13

eq 2.28 $(u\cdot\nabla)u = -\frac{u_\theta^2}{r}\hat{e}_r$

& $\nu\nabla^2 u = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)\hat{e}_\theta$ 2.29

w/ $\vec{u} = u_\theta(r,t)\,\hat{e}_\theta$

eq (2.24) $(\vec{u}\cdot\nabla)\vec{u} = (\nabla\times\vec{u})\times\vec{u} + \nabla(\tfrac{1}{2}u^2)$

(2.25) $\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times(\nabla\times\vec{u})$

(2.26)
$$\nabla\times\vec{u} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix}$$

using $(\vec{u}\cdot\nabla)\vec{u} = (\nabla\times\vec{u})\times\vec{u} + \nabla(\tfrac{1}{2}u^2)$

$$\nabla\times\vec{u} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & ru_\theta & 0 \end{vmatrix} = \frac{1}{r}\Big[\hat{e}_r\Big(\cancelto{0}{-\frac{\partial(ru_\theta)}{\partial z}}\Big) - r\hat{e}_\theta\cdot 0 + \hat{e}_z\frac{\partial}{\partial r}(ru_\theta)\Big]$$

$$= \frac{1}{r}\Big[u_\theta + r\frac{\partial u_\theta}{\partial r}\Big]\hat{e}_z = \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\,\hat{e}_z$$

$$(\nabla\times\vec{u})\times\vec{u} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ 0 & 0 & \frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) \\ 0 & u_\theta & 0 \end{vmatrix}$$

$$= \hat{e}_r\left[-\frac{u_\theta}{r}\frac{\partial}{\partial r}(r u_\theta)\right] - \hat{e}_\theta[0] + \hat{e}_z[0]$$

$$= -\frac{u_\theta}{r}\frac{\partial}{\partial r}(r u_\theta)\,\hat{e}_r$$

Now $\nabla(\tfrac{1}{2}u^2) = \nabla(\tfrac{1}{2}u_\theta^2) =$

$$= \left(\hat{e}_r\frac{\partial}{\partial r} + \cancelto{0}{\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta}} + \cancelto{0}{\hat{e}_z\frac{\partial}{\partial z}}\right)(\tfrac{1}{2}u_\theta^2)$$

$$= \hat{e}_r\frac{\partial}{\partial r}(\tfrac{1}{2}u_\theta^2) = \hat{e}_r\, u_\theta \frac{\partial u_\theta}{\partial r}$$

Thus

$$(u\cdot\nabla)u = (\nabla\times\bar{u})\times\bar{u} + \nabla(\tfrac{1}{2}u^2)$$

$$= \left[-\frac{u_\theta}{r}\frac{\partial}{\partial r}(r u_\theta) + u_\theta\frac{\partial u_\theta}{\partial r}\right]\hat{e}_r$$

$$= \left[-\frac{u_\theta}{r}(u_\theta + r u_\theta') + u_\theta\cdot u_\theta'\right]\hat{e}_r$$

$$= \left(-\frac{u_\theta^2}{r} - \cancel{u_\theta u_\theta'} + \cancel{u_\theta u_\theta'}\right)\hat{e}_r$$

$$= -\frac{u_\theta^2}{r}\hat{e}_r$$ ✓ same

eq 2,29

$$\nabla^2 u = \nabla(\nabla \circ \vec{u}) - \nabla \times (\nabla \times \vec{u})$$

since $\nabla \times \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\hat{e}_z$

$$\nabla \times (\nabla \times \vec{u}) = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) \end{vmatrix}$$

$$= \frac{1}{r}\hat{e}_r\left[0-0\right] + \frac{1}{r} r\hat{e}_\theta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right] + 0$$

$$= \hat{e}_\theta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right]$$

$$\left\{\nabla \circ \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right. \quad so$$

$$\nabla \circ \vec{u} = \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right)\right] = 0$$

$$\nabla \circ u = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$$

$$\nabla(\nabla \circ \vec{u}) = \hat{e}_r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\left(\cancel{r}\frac{\partial u_\theta}{\partial r} + u_\theta\right)\right) \cdot \hat{e}_r$$

$$= \hat{e}_r\frac{\partial}{\partial r}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}u_\theta\right) = \left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + -\frac{1}{r^2}u_\theta\right)\hat{e}_r \checkmark$$

(2,14)   pg 54 Acheson

$$U = \alpha x F'(\eta)$$

$$V = -(\nu \alpha)^{1/2} F(\eta)$$

$$\eta = \left(\frac{\alpha}{\nu}\right)^{1/2} y$$

w/ B.C. $u(x,0) = 0$ + $v(x,0) = 0$

$\Rightarrow$ $F'(0) = 0$ + $F(0) = 0$

w/ B.C. $U \sim \alpha x$ as $y \to +\infty$ $\Rightarrow$ $F'(+\infty) = 1$

$V \sim -\alpha y$ as $y \to +\infty$ $\Rightarrow$ $F(+\infty) = \left(\frac{\alpha}{\nu}\right)^{1/2}$

Now the Navier-Stokes become w/

$U = (u(x,y), v(y), 0)$ N-S eqs

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + g$$

$$\nabla \cdot u = 0$$

x-component eq is: w/o pressure gradient + gravity.

$$U \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

y-component eq is:

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

$\nabla \cdot u = 0$ eq is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

w/ $u = \alpha x f'(\eta)$

$$\frac{\partial u}{\partial x} = \alpha f'(\eta) \; ; \; \frac{\partial^2 u}{\partial x^2} = 0 \; ; \; \frac{\partial u}{\partial y} = \alpha x f''(\eta)\left(\frac{\alpha}{\nu}\right)^{1/2} \; ; \; \frac{\partial^2 u}{\partial y^2} = \alpha x f'''(\eta)\left(\frac{\alpha}{\nu}\right)$$

w/ $v = -(\nu\alpha)^{1/2} f(\eta)$

$$\frac{\partial v}{\partial y} = -(\nu\alpha)^{1/2} f'(\eta)\left(\frac{\alpha}{\nu}\right)^{1/2} = -\alpha f'(\eta)$$

$$\frac{\partial^2 v}{\partial y^2} = -\alpha f''(\eta)\left(\frac{\alpha}{\nu}\right)^{1/2}$$

$$\frac{\partial v}{\partial x} = 0 \; ; \; \frac{\partial^2 v}{\partial x^2} = 0$$

∴ x-component eq becomes:

$$(\alpha x f'(\eta))(\alpha f'(\eta)) + (-(\nu\alpha)^{1/2} f(\eta))\left(\alpha x f''(\eta)\left(\frac{\alpha}{\nu}\right)^{1/2}\right)$$

$$= \nu\left[0 + \alpha x f'''(\eta)\left(\frac{\alpha}{\nu}\right)\right]$$

$$\Rightarrow \alpha^2 x (f'(\eta))^2 - \alpha^2 x f \cdot f'' = \alpha^2 x f''' \Rightarrow$$

$$f'(\eta)^2 - f f'' = f''' \quad *$$

∴ y-component eq becomes:

$$\alpha x f'(\eta) \cdot 0 + (-(\nu\alpha)^{1/2} f(\eta))(-\alpha f'(\eta))$$

$$= \nu\left[0 + -\alpha f''(\eta)\left(\frac{\alpha}{\nu}\right)^{1/2}\right]$$

$$\Rightarrow \quad (\nu x)^{1/2}\alpha\, f\cdot f' = -(\alpha\nu)^{1/2}\alpha\, f''$$

$$\Rightarrow \quad f\cdot f' = -f''$$

∴ Divergence eq becomes:

$\alpha f' + -\alpha f' = 0$ true!! ∀ η + f.

Q: I don't see how to get the +1 in this equation? It must be from the pressure term but I don't see how?

Assume DE is correct then I get

$$f''' + f f'' + 1 - f'^2 = 0 \Rightarrow f''' = f'^2 - 1 - f f''$$

$+ \quad f(0) = f'(0) = 0 \qquad f'(+\infty) = 1$

Solve this numerically.

let $f'(\eta) \cong \dfrac{f(\eta+h) - f(\eta)}{h}$; $f''(\eta) \cong$

let $$Y = \begin{pmatrix} f(\eta) \\ f'(\eta) \\ f''(\eta) \\ \end{pmatrix} \equiv \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Then $y_1' = y_2$

$y_2' = y_3$

$y_3' =$ [illegible] $= f'^2 - ff'' - 1 = y_2^2 - y_1 y_3 - 1$

[illegible]

So

$$\underline{y}' = \begin{bmatrix} y_2 \\ y_3 \\ y_2^2 - y_1 y_3 - 1 \end{bmatrix} = \underline{F}(\underline{y})$$

$$\underline{y}(0) = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} + \underline{y}(+\infty) = \begin{bmatrix} ? \\ +1 \\ ? \end{bmatrix}$$

It would then use a method like "euler" w/ a shooting method to produce a + a numerical solution.

Pg 54 Acheson

(2.14) $u = \alpha x f'(\eta)$

$v = -(\nu\alpha)^{1/2} f(\eta)$

	ν	cm^2/s
H_2O		.01
Air		.15

$$\eta = \left(\frac{\alpha}{\nu}\right)^{1/2} y$$

N.S.

$$\cancelto{0}{\frac{\partial \bar{u}}{\partial t}} + (\bar{u} \cdot \nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \bar{u} \qquad \nabla \cdot \bar{u} = 0$$

$\hat{x}$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\alpha x f' \cdot \alpha f'(\eta) + -(\cancel{\nu}\alpha)^{1/2} f \alpha x f''(\eta)\left(\frac{\alpha}{\cancel{\nu}}\right)^{1/2}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(0 + \alpha x \frac{\partial}{\partial y}\left(f''\left(\frac{\alpha}{\nu}\right)^{1/2}\right)\right)$$

$$\parallel$$

$$\alpha x\left(\frac{\alpha}{\cancel{\nu}}\right)^{1/2} f'''\left(\frac{\alpha}{\cancel{\nu}}\right)^{1/2}$$

$$\Rightarrow \alpha^2 x f'^2 - \alpha^2 x f f'' = \frac{1}{\alpha^2 x} \cdot \cancelto{0}{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \alpha^2 x f'''$$

← Don't discard

~~if no imposed pressure grad~~

$$f'^2 - f f'' = f''' - \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{1}{\alpha^2 x}$$

$$\Rightarrow f''' + f f'' - f'^2 - \frac{1}{\rho}\frac{\partial p}{\partial x}\frac{1}{\alpha^2 x} = 0 \qquad \star$$

$\hat{y}$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial}{\partial y}p + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\Rightarrow \alpha x f' \cancelto{0}{(-(\alpha\nu)^{1/2}} + -(\nu\alpha)^{1/2} f'(-(\nu\alpha))^{1/2} f'\left(\frac{\alpha}{\nu}\right)^{1/2}$$

$$= \frac{-1}{\rho}\frac{\partial p}{\partial y} + \cancel{\nu}\left(0 + -(\nu\alpha)^{1/2}\frac{\partial}{\partial y}\left(f'\left(\frac{\alpha}{\nu}\right)^{1/2}\right)\right)$$

$$\| \quad f''\left(\frac{\alpha}{\cancel{\nu}}\right)\Bigg)$$

$$\Rightarrow + \alpha\nu\cancel{\frac{\sqrt{\alpha}}{\sqrt{\nu}}}\, f f' = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \cancel{B}\,\alpha\cancel{(\nu\alpha)}^{1/2} f''$$

$$f'f = \frac{-1}{\rho\,\nu^{1/2}\alpha^{3/2}}\frac{\partial P}{\partial y} - f''$$ ** As η only fn of y.

~~$\Rightarrow \frac{\partial P}{\partial y}$ only fn of y. Also $\frac{\partial P}{\partial x} = f(y)\,\alpha x^2$~~

~~From * get $F_1(y) = f''' + ff'' - f'^2$~~

~~$\frac{1}{\rho}\frac{1}{\alpha^2 x}\frac{\partial P}{\partial x} = -F_1(y)$~~

~~$\frac{\partial P}{\partial x} = -\rho\alpha^2 x F_1(y)$~~

P ~~$= -\rho\frac{\alpha^2 x^2}{2}F_1(y) + F_2(y)$~~

look at y component:

$$f'f + f'' = \frac{d}{d\eta}\left(\frac{f^2}{2} + f'\right) = \frac{-1}{\rho\nu^{1/2}\alpha^{3/2}}\frac{\partial P}{\partial y}$$

$$= \frac{-1}{\rho\nu\alpha}\frac{\partial P}{\partial\left(\frac{\alpha^{1/2}}{\nu^{1/2}}y\right)} = \frac{-1}{\rho\nu\alpha}\frac{\partial P}{\partial\eta}$$

$$\Rightarrow \quad \frac{f^2}{2} + f' = \frac{-1}{\rho\nu\alpha}P + G(x)$$

$$\Rightarrow \quad P = \tilde{C}_1(x) - \rho\nu\alpha\left(\frac{f^2}{2} + f'\right)$$

$$\frac{\partial P}{\partial x} = \tilde{C}_1'(x) \qquad \frac{\partial P}{\partial x} \text{ fn of only } x.$$

$+$ ~~from eq *~~

$$\nabla\circ\bar{u} = 0 \Rightarrow \alpha f' + -(\nu\alpha)^{1/2} f' \left(\frac{\alpha}{\nu}\right)^{1/2} \equiv 0 \quad \checkmark$$

∴ ~~$\tilde{C}_1'(x) = -\rho\alpha^2 x\left(f''' + ff'' - f'^2\right)$~~

~~$\tilde{C}_1 = -\frac{\rho\alpha^2 x^2}{2}\left(\dots\dots\right)$~~

∴ If Assume $\frac{-1}{\rho}\frac{\partial P}{\partial x}\frac{1}{\alpha^2 x} \equiv 1$

$$\Rightarrow \quad P(x,y) = -\frac{1}{2}\rho\alpha^2 x^2 + C_1(y)$$

if $p = -\frac{1}{2}\rho\alpha^2 x^2 - \rho\nu\alpha\left(f' + \frac{f^2}{2}\right) + \text{const.}$

then. N.S $x \Rightarrow$

$$\frac{\partial P}{\delta x} = -\rho\alpha^2 x$$

$$\frac{\partial P}{\delta y} = -\rho\nu\alpha\left(f'' + ff'\right)\left(\frac{\alpha}{\nu}\right)^{1/2} = -\rho\nu^{1/2}\alpha^{3/2}\left(f'' + ff'\right)$$

<u>B.C.</u> at $y = 0$

$) \; u = 0 \Rightarrow f'(0) = 0$

$v = 0 \Rightarrow f(0) = 0$

~~f(y=+∞)~~

want ~~U(y=+∞)~~ = want $U \sim x\alpha$ $y \to +\infty$

$\Rightarrow f'(+\infty) = \underline{1}$

?? $= -V$

$$v \overset{y\to+\infty}{\sim} \overset{\nearrow 0 \text{ Assume}}{\cancel{f(+\infty)}} + f'(+\infty)(-\nu\alpha)\left(\frac{\alpha}{\nu}\right)^{1/2}$$

pg 75 Acheson

$$p_0 - p = T\frac{\partial^2 \eta}{\partial x^2} \quad \Rightarrow$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \frac{1}{2}u^2 + \overset{=gy}{\chi} = G(t)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \cancel{\frac{p_0}{\rho}} - \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} + \frac{1}{2}u^2 + g\eta = G(t) \quad \text{on} \quad y = \eta(x,t)$$

pick $G = \frac{p_0}{\rho}$ & linearize

$$\partial_t \phi + g\eta - \frac{T}{\rho}\frac{\partial^2 \eta}{\partial x^2} = 0 \quad \text{on} \quad y = 0$$

$$c = \frac{\omega}{k}$$

$$S = \frac{Tk^2}{\rho g}$$

Pg 87 Acheson

1) $\rho_0 \omega \hat{u}_1 = k\hat{p}_1$ 3) $\rho_0 i \omega \hat{v}_1 = \hat{p}_1' + \hat{\rho}_1 g$

2) $ik\hat{u}_1 + \hat{v}_1' = 0$ 4) $-i\omega\hat{\rho}_1 + \hat{v}_1 \frac{d\rho_0}{dy} = 0$

1) ⇒ 2) $ik\left(\frac{k\hat{p}_1}{\rho_0 \omega}\right) + \hat{v}_1' = 0$ *

$\frac{d}{dy} \Rightarrow \quad \frac{ik^2 \hat{p}_1'}{\rho_0 \omega} - \frac{ik^2 \hat{p}_1}{\rho_0^2 \omega}\rho_0' + \hat{v}_1'' = 0$ eq 3) + eq 1) ⇒ 2) *

$$\Rightarrow \frac{ik^2}{\rho_0\omega}\left(-\hat{\rho}_1 g + \rho_0 i\omega \hat{v}_1\right) - \frac{\cancel{ik^2}}{\rho_0^2 \cancel{\omega}}\rho_0'\left(-\frac{\hat{v}_1' \rho_0 \omega}{\cancel{ik^2}}\right) + \hat{v}_1'' = 0$$

eq 4)

$$\Rightarrow \frac{ik^2}{\rho_0\omega}\left(g\frac{i\rho_0'}{\omega}\hat{v}_1 + \rho_0 i\omega\hat{v}_1\right) + \frac{\rho_0'}{\rho_0}\hat{v}_1' + \hat{v}_1'' = 0$$

$$\Rightarrow \hat{v}_1'' + \frac{\rho_0'}{\rho_0}\hat{v}_1' + k^2\left[-\frac{g}{\omega^2}\frac{\rho_0'}{\rho_0} - \text{[illegible]}\right]\hat{v}_1 = 0$$

$$\hat{V}_1'' + \frac{\rho_0'}{\rho_0}\hat{V}_1' + k^2\left(\frac{N^2}{\omega^2} - 1\right)\hat{V}_1 = 0$$

des systems (i.e. surface

Assume wave on surface ?

find mean stream flow φ

$$N^2 = -\frac{g\rho_0'}{\rho_0}$$

let $\rho_0(y) = e^{-y/H}$ $\qquad \frac{\rho_0'}{\rho_0} = \frac{-\frac{1}{H}\cancel{e^{-y/H}}}{\cancel{e^{-y/H}}} = \frac{-1}{H}$

$$\Rightarrow \quad \hat{V}_1'' - \frac{1}{H}\hat{V}_1' + k^2\left(\frac{N^2}{\omega^2} - 1\right)\hat{V}_1 = 0$$

$$\lambda^2 - \frac{\lambda}{H} + k^2\left(\frac{N^2}{\omega^2} - 1\right) = 0 \qquad \hat{V}_1(y) = e^{\lambda y}$$

$$\lambda = \frac{1/H \pm \sqrt{1/H^2 - 4k^2\left(\frac{g}{H\omega^2} - 1\right)}}{2}$$

∴ Sol: $\hat{v}_1(y) = Ae^{\lambda_+ y} + Be^{\lambda_- y}$.

$\lambda_+ > 0 \qquad \lambda_- < 0$

∴ $B = 0$ else ~~$\equiv \ell$~~

∴ $$\hat{v}_1(y) = Ae^{\frac{y}{2H}} e^{+\sqrt{\frac{1}{4H^2} - k^2\left(\frac{N^2}{\omega^2} - 1\right)}\, y}$$

$$\equiv \ell$$

$$e^{i\sqrt{k^2\left(\frac{N^2}{\omega^2} - 1\right) - \frac{1}{4H^2}}\, y}$$

$$k^2 + \ell^2 + \frac{1}{4H^2} = \frac{k^2 N^2}{\omega^2}$$

$$\Rightarrow \quad \omega^2 = \frac{k^2 N^2}{k^2 + \ell^2 + \frac{1}{4H^2}}$$

PJ 91 Acheson

$$\partial_t u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{-1}{\rho}\frac{\partial p}{\partial x} \qquad .93$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g \qquad .94$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad .95$$

$$\frac{h_0}{L/(gh_0)^{1/2}} + (gh_0)^{1/2}\frac{h_0}{L} \sim \frac{h_0(gh_0)^{1/2}}{L} \quad ✓$$

$$\frac{(gh_0)^{1/2}}{L} + \frac{O(v)}{h_0} = 1 \qquad \cancel{O(v)} \sim \quad v \sim \frac{h_0(gh_0)^{1/2}}{L} \qquad u \sim (gh_0)^{1/2}$$

$$t \sim L/(gh_0)^{1/2}$$

$$.\underline{94} \qquad \frac{h_0(gh_0)^{1/2}}{L\left(L/(gh_0)^{1/2}\right)} + (gh_0)^{1/2}\frac{(gh_0)^{1/2}h_0}{L\cdot L} + \frac{h_0(gh_0)^{1/2}}{L}\frac{\cancel{h_0}(gh_0)^{1/2}}{L\,\cancel{h_0}}$$

$$\frac{gh_0^2}{L^2} \qquad \frac{gh_0^2}{L^2} \qquad \frac{gh_0^2}{L^2} \qquad g.$$

Pg 92 Acheson

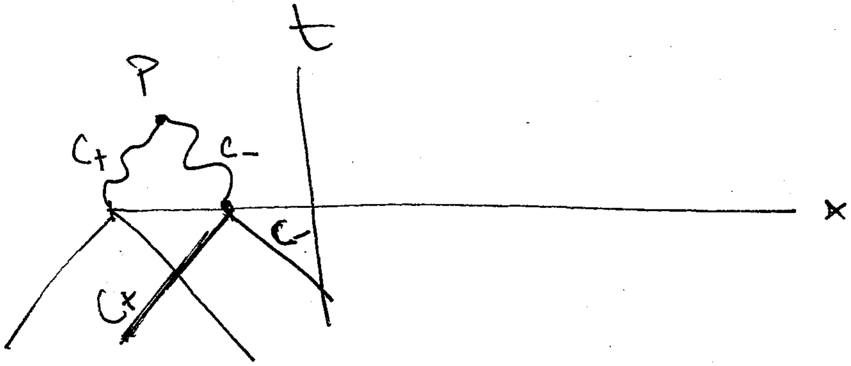
$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u+2c) = 0$$

$$\frac{dt}{ds} = 1 \qquad \underline{\frac{dx}{ds} = u+c} \qquad \frac{d(u+2c)}{ds} = 0.$$

$$t = s \qquad \frac{dx}{dt} = u+c \qquad \frac{d}{dt}(u+2c) = 0.$$

$$\frac{dx}{dt} = \pm c_0 \qquad \begin{matrix} x < 0 \\ t < 0 \end{matrix}$$

$$x = \pm c_0 t + \xi$$



$u + 2c = 2c_0$ Along C_+

$u - 2c = -2c_0$ " C_-

$\Rightarrow$ $u = 0$ $c = c_0$

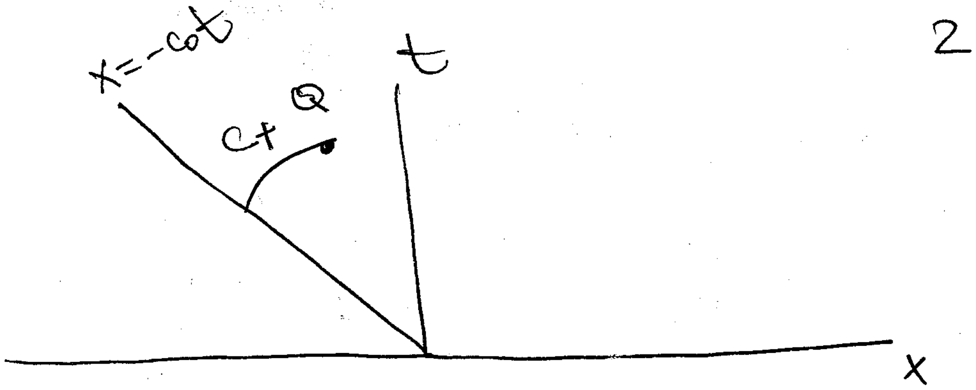
$\therefore$ Are st lines

w/ slope

$\frac{dx}{dt} = \pm c_0$

$$u + 2c = 2c_0$$

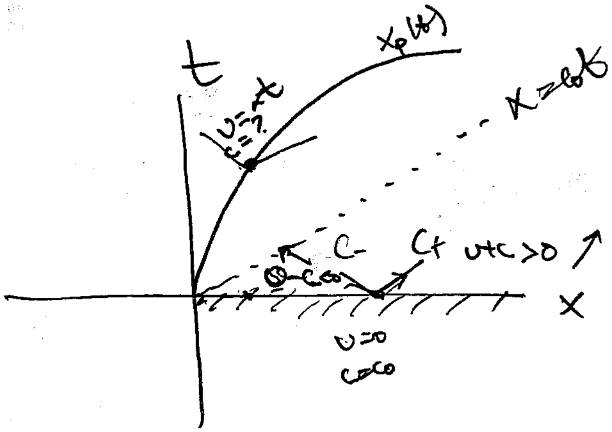
$$u - 2c = k$$



$$\frac{dx_p(t)}{dt} = u = \alpha t$$

$$x_p(t) = \frac{\alpha t^2}{2} + C.$$

$$x_p(t) = \frac{\alpha t^2}{2}$$



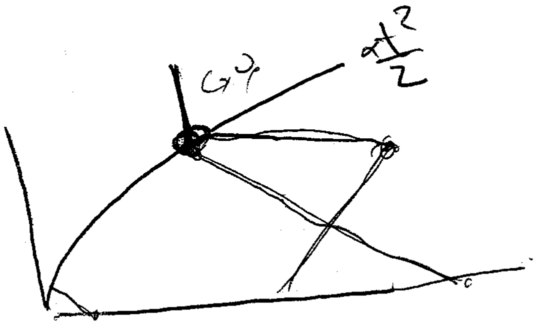
$$u - 2c = -2c_0 \qquad u = 2c - 2c_0$$

$$\left(\partial_t + (u+c)\partial_x\right)(u+2c) = 0$$

$$\to \left(\partial_t + (3c-2c_0)\partial_x\right)(\cancel{3}\cancel{4}c - \cancel{2}c_0) = 0$$

$$u - 2c = -2c_0 \quad \Rightarrow \quad c = \frac{1}{2}(u + 2c_0)$$

$$\partial_t(u+2c) + (u+c)\partial_x(u+2c) = 0.$$



$$\partial_t(2u) + \left(\frac{3}{2}u + c_0\right)\partial_x(2u) = 0$$

$$\partial_t u + \left(\frac{3}{2}u + c_0\right)\partial_x u = 0 \qquad *$$

$$\Rightarrow \quad u = F\left(x - \left(\frac{3}{2}u + c_0\right)t\right)$$ BC. at plate

$$\alpha t = F\left(\frac{1}{2}\alpha t^2 - \left(\frac{3}{2}\alpha t + c_0\right)t\right) =$$

Can Apply BC at $x = c_0 t$? i.e. $\underline{u = 0}$. No

$$0 = F\left(c_0 t - \frac{3}{2}ut - c_0 t\right) = F(0).$$

Both ~~B~~ u & c become ∞ at the Same time?

if $u - 2c = \text{const}$

& $u(x^*, t^*) = \infty \Rightarrow c(x^*, t^*) = \infty.$

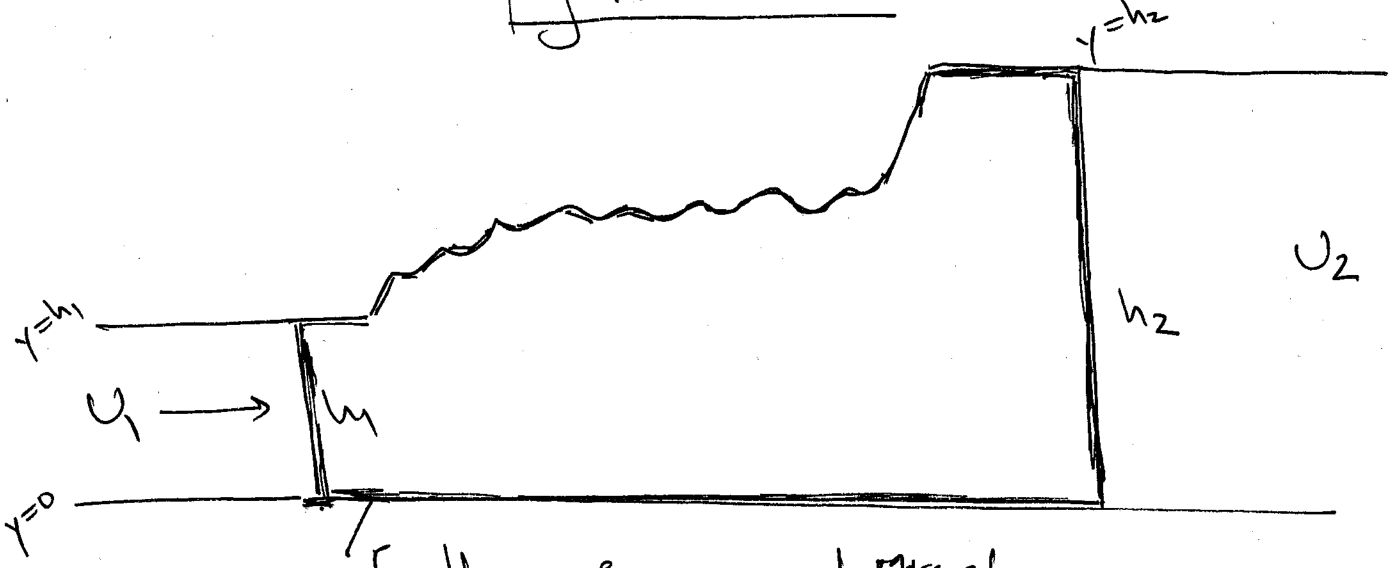
yes.

$\sqrt{gh_0} = c_0.$

$$\frac{\partial u}{\partial x} = 0 + \frac{1/2(-4\alpha)}{\sqrt{(c_0 - \frac{3}{2}\alpha t)^2 - 4\alpha(x - c_0 t)}} = \infty$$

same time As

Pg 100 Acheson



for this surface cons of Mass etc

Mass steady state in frame moving w/ bore

$$\cancel{\frac{d}{dt}\iint_{V_0}\rho\,dV}^{\,0} = -\iint_{S_1}\rho U_1\,dA + \iint_{S_2}\rho U_2\,dA$$

$$0 = -\cancel{\rho}U_1h_1 + \cancel{\rho}U_2h_2 \qquad U_1h_1 = U_2h_2$$

Mmnt:

$$-\int\rho u(\underset{\underset{-U}{\shortparallel}}{\vec{u}\cdot\hat{n}})\,d\xi \qquad \underline{P\hat{n}}$$

$$\cancel{\frac{d}{dt}\iint_S}^{\,0} = -\int\rho U_1^2\,d\xi + \int\rho U_2^2\,d\xi \mp \int_{S_1}P\,d\xi \mp \int_{S_2}P\,d\xi$$

$$= -\rho U_1^2h_1 + \rho U_2^2h_2 - \int_{S_1}P\,d\xi + \int_{S_2}P\,d\xi$$

$$P_1 = -g\rho(\xi - h_1) \qquad P_2 = -\rho g(\xi - h_2)$$

$$-\left(-g\rho \int_0^h (\xi - h)\, d\xi\right) = g\rho\left(\frac{\xi^2}{2} - h\xi\right)\Big|_0^h$$

$$= -\frac{g\rho}{2} h^2$$

Normal causes another sign flip.

$\therefore$ get $$\rho U_1^2 h_1 + g\rho\frac{h_1^2}{2} = \rho U_2^2 h_2 + g\rho\frac{h_2^2}{2}$$

$$U_1^2 h_1 + \frac{h_1^2}{2} = U_2^2 h_2 + \frac{g h_2^2}{2}.$$

$U_2 = \frac{U h_1}{h_2}$ Pg 101 Acheson

3,125 ⇒ $\frac{1}{2}gh_1^2 + h_1U_1^2 = \frac{1}{2}gh_2^2 + \frac{h_1^2U_1^2}{h_2}$

$F_1^2 = \frac{U_1^2}{gh_1}$ ∴ ÷ Above by gh_1

⇒

⇒ $$h_1\left(\frac{h_1}{h_2} - 1\right)U_1^2 = \frac{1}{2}g(h_1^2 - h_2^2)$$

$$U_1^2 = \frac{g\cancel{(h_1-h_2)}(h_1+h_2)}{2h_1\left(\frac{\cancel{h_1-h_2}}{h_2}\right)} = g\frac{h_2(h_1+h_2)}{2h_1}$$

÷ Above by gh_1

⇒ $$F_1^2 = \frac{h_2(h_1+h_2)}{2h_1^2}$$ & F_2 $(1 \leftrightarrow 2)$.

$h_2 > h_1$ $\frac{h_2}{h_1} \cdot \frac{(h_1+h_2)}{2h_1}$

pg 101 Acheson

$F_1^2 = \frac{(h_1+h_2)h_2}{2h_1^2}$ $F_2^2 = \frac{(h_1+h_2)h_1}{2h_2^2}$ $h_2 > h_1$

$<$ [illegible]

~~$h_1h_2 + h_2^2 > 2h_1^2$~~

~~$(h_1+h_2) > 2h_1\left(\frac{h_1}{h_2}\right)$~~

fix h_1. Min $F_1^2(h_2) = \frac{(h_1+h_2)h_2}{2h_1^2}$ on (h_1,∞)

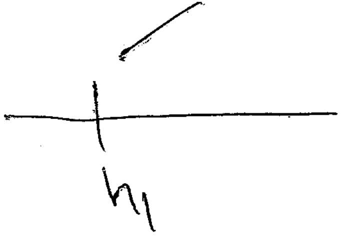
$\frac{dF_1^2}{dh_2} = \frac{1}{2h_1^2}\left[2h_2 + h_1\right] = 0$ $\frac{d^2F_1^2}{dh_2^2} = 2 > 0$

$\Rightarrow h_2 = -\frac{h_1}{2} < h_1$

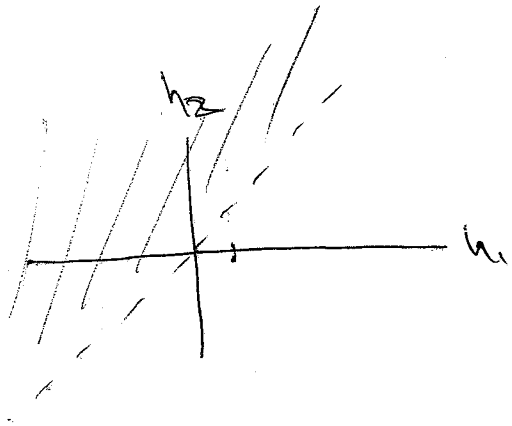
$\therefore h_2 = \frac{h_1}{2}$ is Min.

$\therefore$ ~~$F_1^2(h_2) \geq \frac{(3h_1/2)\,h_1/2}{2h_1^2} =$~~ [illegible] $\therefore$

∴ As $\frac{dF_1^2}{dh_2} > 0 \;\forall\; h_2 \geq h_1$



$F_1^2 \geq F_1^2(h_2 = h_1) = 1$.



& For $F_2^2 = \frac{(h_1+h_2)h_1}{2h_2^2}$ know $h_2 > h_1$

So

To Max $F_2^2(h_1,h_2)$ on upper ½ plane) put in $h_1 = h_2$
& Max 1-D prob.
2) Max over $(h_1,h_2) \in \mathbb{R}^2$ ½ plane

$F_2^2(h_2)$ & look for global/local
max min of h_2 on (h_1, ∞)

or " " "

Max Min of h_1 on $(0, h_2)$ * denom in h_2.

$$\frac{dF_2^2(h_1)}{dh_1} = \frac{1}{2h_2^2}\left[2h_1 + h_2\right] = 0. \Rightarrow h_1 = -\frac{h_2}{2}$$

out of range $(0, h_2)$

$\frac{dF_2^2}{dh_1} > 0 \;\forall\; h_1 \in (0, h_2)$

∴ min of F_2^2 at $h_1 = 0$ $F_2^2 > 0$

Max of F_2^2 at $h_1 = h_2$ $F_2^2 < 1$

Pg 103 Acheson

$$\left[\partial_t + (u+a)\partial_x\right]\left(u + \frac{2a}{r-1}\right) = 0 \qquad a = \left(\frac{rp}{\rho}\right)^{1/2}.$$

$$\left[\partial_t + (u-a)\partial_x\right]\left(u - \frac{2a}{r-1}\right) = 0 \quad (*).$$

$$(**)\quad u - \frac{2a}{r-1} = -\frac{2a_0}{r-1} \Rightarrow$$ ~~[illegible]~~ Neg Characteristic. ∴ put into 1st eq.

$\Rightarrow$ u or a + ** gives relation between 2

$$a = \frac{r-1}{2}u + a_0$$

$$\Rightarrow \left[\partial_t + \left(\frac{r+1}{2}u + a_0\right)\partial_x\right]\left(2u + \frac{2}{r-1}a_0\right) = 0$$

$$\left[\partial_t + \left(\frac{r+1}{2}u + a_0\right)\partial_x\right]u = 0.$$

$t_{Breaking} = ? \quad \Leftarrow$

$$u = \mathcal{F}\left(x - \left(\frac{\gamma+1}{2}u + a_0\right)t\right)$$

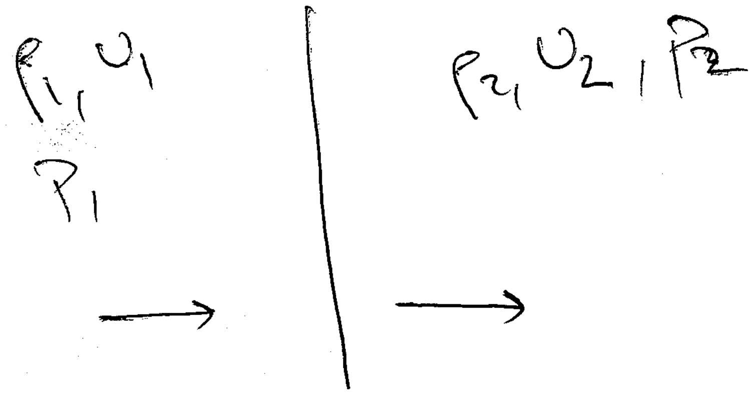
$$\frac{\partial u}{\partial x} = \mathcal{F}'(\xi)\left[1 - \frac{\gamma+1}{2}t_B \frac{\partial u}{\partial x}\right]$$

$$\Rightarrow \quad \partial_x u\left[1 + \mathcal{F}'(\xi)\,\frac{\gamma+1}{2}t_B\right] = \mathcal{F}'(\xi)$$

$$\partial_x u = \frac{\mathcal{F}'(\xi)}{1 + \mathcal{F}'(\xi)\frac{\gamma+1}{2}t_B = 0}$$

$$\Rightarrow \quad t_B = \min_{\xi} \frac{-1}{\left(\frac{2}{\gamma+1}\right)\mathcal{F}'(\xi)}$$

pg 103 Acheson



$$\rho_1 u_1 = \rho_2 u_2$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

$$\frac{1}{2}u_1^2 + \frac{a_1^2}{\gamma-1} = \frac{1}{2}u_2^2 + \frac{a_2^2}{\gamma-1}$$

Pg 111

(3.1) 1) invisid theory.
2) free surface $\frac{DF}{Dt} = 0$ on $y = \eta(x,t)$ free surface (1)

$F(x,y,t) = y - \eta(x,y,t)$

3) irrotational flow. use ϕ.
4) use unsteady Bernoulli for irrotational flow. on surface
5) linearize on free surface

get 2 free surface conditions

let $y = \eta(x,t)$
Then $y - \eta(x,t) \equiv F(x,y,t)$
Then particles of fluid on surface remain there
$\therefore \frac{DF}{Dt} = 0$

$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y=0$

$\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y=0$

try $\eta = A\cos(kx - \omega t)$
Assume $\phi = f(y)\sin(kx - \omega t)$
($\nabla^2\phi = 0$ irrotational)
then put into two free linearize surface conditions

gives $Ck = A\omega \quad -C\omega + gA = 0$

$$\Rightarrow \frac{\partial F}{\partial t} + (\vec{v}\cdot\nabla)F = 0$$

$$\Rightarrow -\frac{\partial \eta}{\partial t} + \text{[illegible]} + \frac{\partial \phi}{\partial x}\frac{\partial F}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow -\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial y} = 0 \qquad * \text{ on } y = \eta(x,t)$$

2nd surface condition comes from irrotational, time dependent

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho(\bar{u}\cdot\nabla)\bar{u} = -\nabla p - \nabla \chi.$$

$$\frac{\partial \bar{u}}{\partial t} + \underbrace{(\bar{u}\cdot\nabla)\bar{u}}_{\equiv} = -\frac{\nabla}{\rho}(p+\rho\chi)$$

let ~~$\bar{u} = \nabla\phi$~~

$$\overset{0 \text{ irrotational}}{\cancel{(\nabla\times\bar{u})}}\times\bar{u} + \nabla\left(\tfrac{1}{2}u^2\right)$$

$$\Rightarrow \quad \frac{\partial \bar{u}}{\partial t} = -\frac{\nabla}{\rho}\left(p + \rho\chi + \tfrac{1}{2}\rho u^2\right)$$

let $u = \nabla\phi$ & integrate

$$\frac{\partial \phi}{\partial t} = -\left(\frac{p}{\rho} + \chi + \tfrac{1}{2}u^2\right) + \text{const}(t)$$

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + \chi + \frac{1}{2}u^2 = C(t) \qquad \forall \text{ flow}$$

on surface $p \Rightarrow p_0 \quad \chi = gy$

picking $C(t)$ to be const & can get

$$\frac{\partial \phi}{\partial t} + g\eta + \frac{1}{2}u^2 = 0 \quad \text{on} \quad y = \eta(x,t) \quad **$$

lin on free surface

$\Rightarrow$ eq η is small

$$v(x,\eta,t) = v(x,0,t) + \eta \left.\frac{\partial v}{\partial y}\right|_{y=\eta} + O(\eta^2)$$

eq $*$ $\Rightarrow$ $\frac{\partial \eta}{\partial t} = v(x,0,t) = \frac{\partial \phi}{\partial y} \quad y=0$

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} \quad y = 0 \quad *'$$

for $**$ $\Rightarrow$ $\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y=0$ $**'$

~~Based~~ on Assum $\eta = A\cos(kx-\omega t)$ initial displacement of wave form.

$$+\omega A \sin(kx-\omega t) = \frac{\partial \phi}{\partial y}$$

Assum

$\Rightarrow \phi = f(y)\sin(kx-\omega t)$

Now $\phi \ni \nabla^2\phi = 0 \Rightarrow$

$$f''(y)\,\cancel{\sin(kx-\omega t)} - f(y)k^2\,\cancel{\sin(kx-\omega t)} = 0$$

$$f'' - k^2 f = 0 \Rightarrow f(y) = \cancel{Be}$$

$$= \cancel{C_1 \sin k}$$

$$= C_1 \sinh(ky) + C_2 \cosh(ky)$$

Now BC on $\phi(x,y) = \left(C_1 \sinh(ky) + C_2 \cosh(ky)\right) \sin(kx-\omega t)$

$\Rightarrow \phi\big|_{y=h} = 0.$ $\quad \cancel{u\big|_{y=h} = 0}$ $\quad = \cancel{\frac{\partial\phi}{\partial x}\Big|_{y=h} = 0}$ $\quad \boxed{v\big|_{y=h} = 0}$

$\cancel{C_1 \sinh(kh) = -C_2 \cosh(kh)}$

$\Rightarrow C_2 \cancel{= +C_1 \tanh(kh)}$ $\quad = \cancel{\frac{d}{dx}\sin(kx-\omega t)}\left(\Delta \text{ hyperbolics}\right|_{y=0} = 0.$

$\Rightarrow$ $\quad \cancel{\phi\big|_{y=h} = 0.}$

$$\phi(\cancel{x},y) = \cancel{C}\,C_1\left[\cancel{\sinh(ky) + \tanh(kh)\cosh(ky)}\right]\sin(kx-\omega t)$$

$$\frac{\partial\phi}{\partial y} = \left(C_1 k\cosh(ky) + C_2 k\sinh(ky)\right)\sin(kx-\omega t)$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=-h} \equiv 0 \Rightarrow$$

$$C_1\cosh(kh) = C_2\sinh(kh)$$

$$C_1 = C_2\tanh(kh)$$

$$\therefore \quad \phi = C_2\left[\tanh(kh)\sinh(ky) + \cosh(ky)\right]\sin(kx-\omega t)$$

Now ~~check~~ put η + ϕ into two surface conditions

$$\Rightarrow *' \Rightarrow \quad A\omega\sin(\cancel{kx}-\omega t) = C_2\sin(\cancel{kx}-\omega t)\left[\tanh(kh)\right]k$$

$$**' \Rightarrow \quad C_2(-\omega) + gA = 0$$

$$A = \frac{\omega C_2}{g} \quad \text{or} \quad C_2 = \frac{gA}{\omega}$$

$$\frac{\omega^2}{g} = k\tanh(kh)$$

$$\frac{\omega^2}{k^2} = \frac{g}{k}\tanh(kh) = c^2.$$

$$\therefore \quad \phi = \frac{g}{\omega}A\left(\tanh(kh)\sinh(ky) + \cosh(ky)\right)\sin(kx-\omega t)$$

Now: particle paths are easy

$$\frac{\partial\phi}{\partial x} = U = \frac{gk}{\omega}A\left(\tanh(kh)\sinh(ky) + \cosh(ky)\right)\cos(kx-\omega t)$$

$$v = \frac{gA}{\omega}k\left(\tanh(kh)\cosh(ky) + \sinh(ky)\right)\sin(kx-\omega t)$$

$$\frac{dx'}{dt} = \frac{gk}{\omega}A(\sim)\cos(k\bar{x}-\omega t)$$

$$x' = -\frac{gk}{\omega^2}A(\sim)\sin(k\bar{x}-\omega t)$$

$$\frac{dy'}{dt} = \frac{gAk}{\omega}(\sim)\sin(k\bar{x}-\omega t)$$

$$y' = \frac{gAk}{\omega^2}(\sim)\cos(k\bar{x}-\omega t)$$

$$\Rightarrow \quad \frac{x'^2}{\frac{g^2A^2k^2}{\omega^4}(\cdots\cdots)^2} + \frac{y'^2}{\frac{g^2A^2k^2}{\omega^4}(\cdots)^2} = 1$$

Not same!!

$$\tanh(kh)\left\{\begin{matrix}\sinh(ky)\\ \cosh(ky)\end{matrix}\right\} + \left\{\begin{matrix}\cosh(ky)\\ \sinh(ky)\end{matrix}\right\} \quad \text{v.s.}$$

eq 3.18 $\frac{\partial \eta}{\partial t} \sim O(A\omega)$

$$u \sim O\left(\frac{gkA}{\omega}\left(\tanh(kh)\sinh(ky) + \cosh(ky)\right)\right)$$

$$\frac{\partial \eta}{\partial x} \sim O(Ak)$$

$$v \sim O\left(\frac{gkA}{\omega}\left(\cancel{\tanh(kh)\cosh(ky) + \sinh(ky)}\right)\right)$$

if $u\frac{\partial \eta}{\partial x} \ll \frac{\partial \eta}{\partial t} + v$

$$\parallel$$

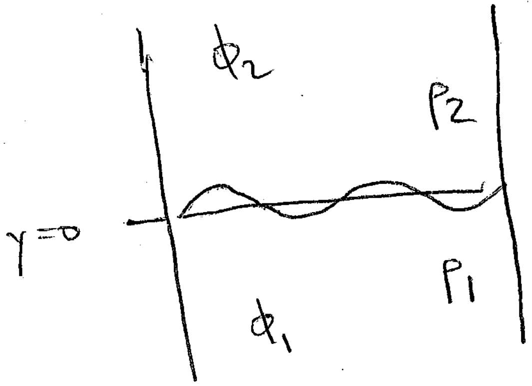
$$\frac{gk^2A^2}{\omega^2}\left(t\;(\,)\sinh(\,) + \cosh(\,)\right) \ll \cancel{A\omega}\;1$$

$$\frac{gkA}{\omega}$$

3,2

1) inviscid theory.

2) 2 interface conditions



on interface $y=\eta(x,t)$

Surface elts stay on surface

$\rho_1 > \rho_2$

1) $$\frac{DF}{Dt} = \frac{D}{Dt}\big(y-\eta(x,t)\big) = 0$$

$$\Rightarrow \quad -\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\left(-\frac{\partial\eta}{\partial x}\right) + \frac{\partial\phi}{\partial y} = 0$$

$$\Rightarrow \quad \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\phi}{\partial y} = 0. \quad \text{on } y=\eta(x,t)$$

2) Time dependent Bernoulli for irrotational flow

$$\Rightarrow \quad \nabla\frac{\partial\phi}{\partial t} + \overset{\nabla\times\bar{u}=0.}{\cancel{(\nabla\times\bar{u})}}\times\bar{u} + \nabla\left(\tfrac{1}{2}u^2\right) = -\frac{1}{\rho}\nabla p - \nabla\chi$$

$\Rightarrow$ integrate $\Rightarrow$

$$\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + \chi + \frac{1}{2}u^2 = b(t)$$ in both regions

∴ top region

$$\frac{\partial \phi_2}{\partial t} + \frac{P_2}{\rho_2} + g\eta + \frac{1}{2}u_2^2 = G_2(t)$$

bottom region

$$\frac{\partial \phi_2}{\partial t} + \frac{P_1}{\rho_1} + g\eta + \frac{1}{2}u_1^2 = G_1(t)$$

Now $P_1 = P_2$ (No surface tension)

?∴ $G_1(t) = G_2$ on surface $y = \eta(x,t)$

∴ ~~ϕ_2~~ $$\rho_2 \frac{\partial \phi_2}{\partial t} + P_2 + g\rho_2\eta + \frac{\rho_2}{2}u_2^2 = \rho_2 G_2$$

$$\rho_1 \frac{\partial \phi_1}{\partial t} + P_1 + g\rho_1\eta + \frac{\rho_1}{2}u_1^2 = \rho_1 G_1$$

Now $P_1 = P_2$ & ($\overset{?.}{\rho_2} \overset{?.}{G_2} = \overset{?}{\rho_1} G_1$ on interface ?

& linearize

$$\Rightarrow \quad \rho_2 \frac{\partial \phi_2}{\partial t} + g\rho_2\eta = \rho_1 \frac{\partial \phi_1}{\partial t} + g\rho_1\eta \quad \text{on } y = \eta(x,t)$$

Now: ~~in region 2 $\nabla^2\phi_2 = 0$~~

Assuming $\eta(x,t) = A\cos(kx - \omega t)$

Then looking at linearized particles staying on surface

$\Rightarrow$ [illegible] $= \frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial y}$ on $y=\eta(x,t)$

$\Rightarrow$ $\frac{\partial\eta}{\partial t} = \frac{\partial\phi_1}{\partial y}$ on $y=0$ *

$= \frac{\partial\phi_2}{\partial y}$

+ [illegible] w/

$\rho_2 \frac{\partial\phi_2}{\partial t} + g\rho_2\eta = \rho_1 \frac{\partial\phi_2}{\partial t} + g\rho_1\eta$ on $y=0$ **

~~Now solve~~ ~~$\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ in Region I, II~~

~~then~~ Assume $\eta = A\cos(kx-\omega t)$

Then solve $\nabla^2\phi_1 = \nabla^2\phi_2 = 0$ in Region I + II.

$\Rightarrow$ By * expect ϕ_1 + ϕ_2 $= \begin{cases} C_1 f_1(y) \sin(kx-\omega t) \\ (C_2 f_2(y)) \end{cases}$

w/ B.C for given Region

Then Region I *2 $\Rightarrow$ ~~$\omega A\sin(kx-\omega t)$~~

$=$ ~~$C_1 f(y)$~~ S

Solve $\nabla^2\phi_{1,2} = -k^2 f_{1,2} - f''_{1,2} = 0$

$\Rightarrow f_{1,2} = A_{1,2}e^{ky} + B_{1,2}e^{-ky}$

in Region I $\quad \phi \xrightarrow{y \to -\infty} 0$

$$f_1 = A_1 e^{ky} \qquad \phi_1 = A_1 e^{ky} \sin(kx - \omega t)$$

in Region II $\quad \phi \xrightarrow{y \to +\infty} 0$

$$f_2 = B_2 e^{-ky} \qquad \phi_2 = B_2 e^{-ky} \sin(kx - \omega t)$$

∴ eq * ⇒ ~~$A\omega = A_1 k$~~

$$A\omega \cancel{\sin(}\ \ ) = A_1 k e^{ky} \cancel{\sin(}\ \ ) \qquad y = 0$$

$$A\omega = A_1 k \qquad A_1 = A\frac{\omega}{k}$$

$$A\omega \cancel{\sin(}\ \ ) = -B_2 k e^{-ky} \sin(\cancel{kx} - \omega t) \qquad y = 0$$

$$A\omega = -B_2 k \Rightarrow B_2 = -\frac{A\omega}{k}$$

& eq **.

$$-\rho_2 B_2 \omega \cos(\cancel{kx} - \omega t) + g\rho_2 A \cancel{\cos(}\ \ )$$

$$= -\rho_1 A_1 \omega \cos(\cancel{\ \ }\ \ ) + g\rho_1 A \cancel{\cos(}\ \ )$$

$$\Rightarrow -\rho_2 B_2 \omega + g\rho_2 A = -\rho_1 A_1 \omega + g\rho_1 A$$

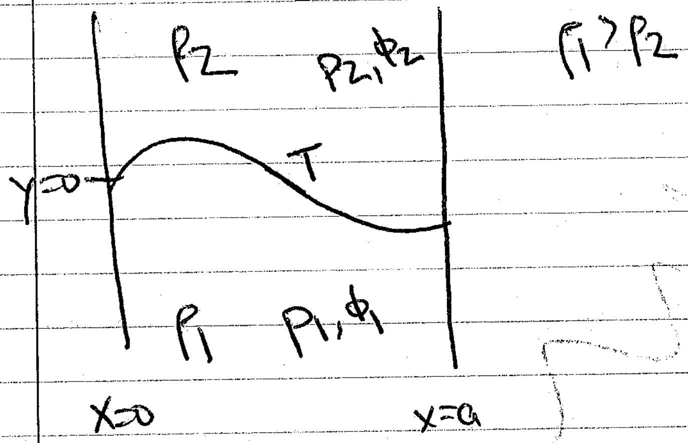
$$\frac{\rho_2 \omega^2}{k} + g\rho_2 = -\rho_1 \frac{\omega^2}{k} + g\rho_1$$

$$\frac{\omega^2}{k}(\rho_2 + \rho_1) = g(\rho_1 - \rho_2)$$

$$c^2 = \frac{g(\rho_1 - \rho_2)}{k(\rho_1 + \rho_2)}$$

Pg 111 Acheson

③③



Surface tension gives

$$-(p_1 - p_2) = T\frac{\partial^2\eta}{\partial x^2} \qquad y=0.$$

Unsteady Bernoulli gives $\frac{\partial\phi}{\partial t}+\frac{q}{2}$. for bottom

$$\frac{\partial\phi_1}{\partial t} + \cancelto{0\ \text{lin}}{\frac{1}{2}u_1^2} + \frac{p_1}{\rho_1} = C_1(t) \qquad \text{on } y=0$$

$$\frac{\partial\phi_2}{\partial t} + \cancelto{0\ \text{lin}}{\frac{1}{2}u_2^2} + \frac{p_2}{\rho_2} = C_2(t) \qquad \text{on } y=0.$$

set $\rho_1 C_1 = \rho_2 C_2$

$$\Rightarrow \rho_1\frac{\partial\phi_1}{\partial t} + \underline{P_1} + g\rho_1\eta = \rho_2\frac{\partial\phi_2}{\partial t} + P_2 + g\rho_2\eta$$

$$\rho_1\frac{\partial\phi_1}{\partial t} + g\rho_1\eta = \rho_2\frac{\partial\phi_2}{\partial t} + \rho_2 g\eta + T\frac{\partial^2\eta}{\partial x^2} \qquad y=0 \quad *$$

B.C. $\underset{x=0,a}{u_1 = 0}$. & $\frac{\partial\eta}{\partial t} = \frac{\partial\phi_{1,2}}{\partial y}$ on $y=0$ ** & ***

Input $\eta(x,t) = A\cos kx \cos(\omega t+\epsilon)$

try $\phi_{1,2} = f_{1,2}(y)\cos(kx)\sin(\omega t+\epsilon)$ could ωt !!

$\nabla^2\phi_{1,2} = 0$

$f''_{1,2}(y) - k^2 f_{1,2}(y) = 0$

$f_{1,2} = A_{1,2}e^{ky} + B_{1,2}e^{-ky}$

$\phi_1 \xrightarrow[y\to-\infty]{} 0 \Rightarrow A_1$ arb $B_1 = 0$

$\phi_2 \xrightarrow[y\to+\infty]{} 0 \Rightarrow A_2 = 0$ B_2 = arb.

$\phi_1 = A_1 e^{ky}\cos kx \sin(\omega t+\epsilon)$
$\phi_2 = B_2 e^{-ky}\cos kx \sin(\omega t+\epsilon)$

BC $x = 0, a \Rightarrow \frac{\partial\phi_{1,2}}{\delta x} = 0$

$\Rightarrow (\,''\,)\sin kx\big|_{y=0,a} = 0 \Rightarrow ka = n\pi$ $n = 1, 2, \ldots$

$k_n = \frac{n\pi}{a}$

$\phi_1 = A_n e^{\frac{n\pi}{a}y}\cos\frac{n\pi}{a}x\,\sin(\omega_n t+\epsilon_n)$

$\phi_2 = B_n e^{-\frac{n\pi}{a}y}\cos\left(\frac{n\pi}{a}x\right)\sin(\omega_n t+\epsilon_n)$

How know ω_n + ϵ_n + A_n + B_n
discrete now

?

Boubnowsy $\quad A_n \overset{?}{\sim} F(A)$

$\qquad\qquad\quad B_n \overset{?}{\sim} F(A)$

$\qquad\qquad\quad \omega_n \sim k_n$

Put in eq * * *

$\cancel{\rho_1}\ \frac{\partial\eta}{\partial t} = \frac{\partial\phi_2}{\partial y} \qquad y=0$

$-\omega A \cos k_n x \sin(\omega_n t+\epsilon_n) = -\frac{n\pi}{a} B_n \cancel{\cos}(\frac{n\pi}{a}x)\ \cancel{\sin(}\quad)$

$-\omega A = -\frac{n\pi}{a} B_n$

Put in * *

$\Rightarrow -\omega A \cos \cancel{k_n} x \sin \cancel{(\omega_n t+\epsilon_n)} = \frac{n\pi}{a} A_n \cos\cancel{(}\ \ |\ \ \cancel{s(}\quad)$

$-\omega A = \frac{n\pi}{a} A_n$

Put in *

$A_n\rho_1\omega_n \cos(\cancel{\ }x)\cos(\omega_n t+\epsilon_n) + \rho_1 g A \cos\cancel{(}\ x)\cos\cancel{(}\quad)$

$= \cancel{\rho_2}\cancel{\left(\frac{n\pi}{a}\right)} B_n\rho_2\omega_n \cos\cancel{(}\ x)\cos\cancel{(}\quad) + \rho_2 g A \cos\cancel{(}\ )\cancel{\cos}(\ )$

$A + T(\cancel{A\omega_n^2})(-k_n^2)\cos\cancel{(}\ )\cos\cancel{(}\quad)$

$\Rightarrow \underline{A_n}\rho_1\omega_n + \rho_1 g A = B_n\rho_2\omega_n + \rho_2 g A - T\left(\frac{\pi^2 n^2}{a^2}\right)A$

$\Rightarrow -\frac{a\omega_n^2}{n\pi}\rho_1\cancel{A} + \rho_1 g\cancel{A} = \frac{a\omega_n^2}{n\pi}\rho_2\cancel{A} + \rho_2 g\cancel{A} - T\frac{\pi^2 n^2}{a^2}\cancel{A}$

(over)

pg 113 Acheson

(3.6)

ρ_2 $U \longrightarrow$ ϕ_2 $\nabla^2\phi_2 = 0$ $u = U + \frac{\partial\phi_2}{\partial x}$

$v = \frac{\partial\phi_2}{\partial y}$

$y=0$ ———

ρ_1 ϕ_1 $\nabla^2\phi_1 = 0$ $u = \frac{\partial\phi_1}{\partial x}$

$v = \frac{\partial\phi_1}{\partial y}$

B.C. at $y=0$ $\frac{D(F)}{Dt} = \frac{D}{Dt}(y-\eta(x,t)) = 0$ $y=0$

$$\Rightarrow \frac{\partial}{\partial t}(y-\eta(x,t)) + u\frac{\partial}{\partial x}(y-\eta(x,t)) + v\frac{\partial}{\partial y}(y-\eta(x,t)) = 0$$

$$\Rightarrow -\frac{\partial\eta}{\partial t} + \overset{U}{\cancel{u}}\left(-\frac{\partial\eta}{\partial x}\right) + v = 0 \quad \text{At } y=0$$

$$\text{or} \quad -\frac{\partial\eta}{\partial t} + \frac{\partial\phi_{1,2}}{\partial x}\left(-\frac{\partial\eta}{\partial x}\right) + \frac{\partial\phi_{1,2}}{\partial y} = 0$$

$$\text{Lin BC} \Rightarrow \cancel{-\frac{\partial\eta}{\partial t} + \frac{\partial\phi_{1,2}}{\partial y}} = 0$$

$$\text{Bernoulli } \cancel{g\eta + \frac{\partial\phi_1}{\partial t} + \frac{p_1}{\rho_1} + \frac{1}{2}(\nabla\phi_1)^2 =} \text{ Const}$$

$$\cancel{p_1 = \text{Const} - \rho_1\frac{\partial\phi_1}{\partial t} - \frac{\rho_1}{2}(\nabla\phi_1)^2 -} gy$$

$$p_2 = \cancel{\text{''} - \rho_2 \cdots (\nabla\phi_2)^2 - gy}$$

~~$P_1 - P_2 = T \frac{\partial^2 \eta}{\partial x^2}$~~

~~lin B.C~~

$\Rightarrow$ ~~$\rho_1 \frac{\partial \phi_1}{\partial t} + \rho_1 g \eta = \rho_2 \frac{\partial \phi_2}{\partial t} + \rho_2 g \eta + T \frac{\partial^2 \eta}{\partial x^2}$~~

---

lin B.C for ϕ_2

$$u \; \frac{\partial \phi_2}{\partial x} = U + \frac{\partial \phi_2'}{\partial x}$$

$$-\frac{\partial \eta}{\partial t} - \left(U + \frac{\partial \phi_2}{\partial x}\right)\frac{\partial \eta}{\partial x} + \frac{\partial \phi_2}{\partial y} = 0$$

get $-\frac{\partial \eta}{\partial t} - U \frac{\partial \eta}{\partial x} + \frac{\partial \phi_2}{\partial y} = 0$ on $y =$ ~~$\eta(x,t)$~~ 0 **

for ϕ_1: $-\frac{\partial \eta}{\partial t} + \frac{\partial \phi_1}{\partial y} = 0$ on $y =$ ~~$\eta(x,t)$~~ 0 *

Bernolli: ϕ_1: $\frac{\partial \phi_1}{\partial t} + \frac{P_1}{\rho_1} + g\eta + \frac{1}{2}\cancel{(\nabla\phi)^2}^{\,0} = C_1(t)$

ϕ_2: $\frac{\partial \phi_2}{\partial t} + \frac{P_2}{\rho_2} + g\eta + \xrightarrow{} 0 = C_2(t)$

Not exactly see next pg

& $P_1 - P_2 = T \frac{\partial^2 \eta}{\partial x^2}$

get $\rho_1 \frac{\partial \phi_1}{\partial t} + g\rho_1 \eta = \rho_2 \frac{\partial \phi_2}{\partial t} + g\rho_2 \eta + T \frac{\partial^2 \eta}{\partial x^2}$

$$\cancel{\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} = U\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j}}$$

$$\cancel{\nabla\phi_2^{2} = U^2}$$

$$\nabla\phi_2 = (U, V) = \left(U + \frac{\partial\phi_2}{\partial x}\right)\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j}$$

$$\nabla\phi_2^{2} = \cancelto{0}{U^2} + U\frac{\partial\phi_2}{\partial x^2} + \cancelto{0}{\frac{\partial\phi_2}{\partial x}^2} + \cancelto{0}{\frac{\partial\phi_2}{\partial y}^2} \quad \text{Non linear}$$

const can absorb.

$\Rightarrow$ for $p_1 - p_2 = T\frac{\partial^2\eta}{\partial x^2}$

$$\rho_1\frac{\partial\phi}{\partial t} + g\rho_1\eta = \rho_2\frac{\partial\phi_2}{\partial t} + g\rho_2\eta + \rho_2 U\frac{\partial\phi_2}{\partial x} + T\frac{\partial^2\eta}{\partial x^2} \quad ***$$

---

if $\eta = Ae^{i(kx-\omega t)}$ then try sol

$$\phi_1 = f_1(y)e^{i(kx-\omega t)} \qquad \nabla^2\phi_1 = 0$$

$$\Rightarrow f_1'' - k^2 f_1 = 0$$

$$f_1(y) = C_1e^{ky} + C_2e^{-ky}$$

for $\phi_1 \xrightarrow{y\to-\infty} 0 \Rightarrow C_2 = 0$

$$f_1(y) = C_1e^{ky}$$

$$\phi_2 = f_2(y)e^{i(kx-\omega t)}$$

Sim reasoning gives $\nabla\cdot\bar{u} = 0$

$$\Rightarrow f_2(y) = C_3e^{ky} + C_4e^{-ky}$$

if $f_2(y) \xrightarrow{y\to+\infty} 0 \Rightarrow C_3 = 0.$

$$\phi_2 = C_3e^{-ky}e^{i(kx-\omega t)} \qquad \phi_1 = C_1e^{ky}e^{i(kx-\omega t)}$$

Then 3 eqs & 3 unknowns C_3, C_1 & $\omega(k)$

For eq * ~~$A(-\omega i)e$~~

$$-A(-\omega i)\cancel{e^{i(kx-\omega t)}} + C_1k\overset{1}{\cancel{e^{ky}}}\cancel{e^{i(kx-\omega t)}} = 0 \quad \text{on } y=0$$

$$A\omega i + C_1k = 0 \Rightarrow C_1 = -\frac{A\omega i}{k}$$

For eq **

$$-A(-\omega i)\cancel{e^{i(kx-\omega t)}} - UA(ik)\cancel{e^{i(kx-\omega t)}} + (-k)C_3\overset{y=0}{\cancel{e^{-ky}}}\cancel{e^{i(kx-\omega t)}} = 0 \quad y=0$$

$$A\omega i - AikU - kC_3 = 0$$

$$\Rightarrow C_3 = \frac{iA(\omega-kU)}{k}$$

for eq ***

$$\rho C_1 \cancel{e^{ky}}(-i\omega)\cancel{e^{i(kx-\omega t)}} + g\rho_1 A\cancel{e^{i(kx-\omega t)}} =$$

$$\rho_2 C_3 \cancel{e^{-ky}}(-i\omega)\cancel{e^{i(kx-\omega t)}} + g\rho_2 A\cancel{e^{i(kx-\omega t)}} + \rho_2 U C_3 \cancel{e^{-ky}}(ik)\cancel{e^{i(kx-\omega t)}}$$

$$+ TA(ik)^2\cancel{e^{i(kx-\omega t)}}$$

$$\rho e \Rightarrow \quad \rho_1\left(-\frac{\cancel{A}\omega i}{k}\right)(-i\omega) + g\rho_1\cancel{A} = \rho_2 \frac{i\cancel{A}(\omega-kU)}{k}(-i\omega)$$

$$+ g\rho_2\cancel{A} + \rho_2 U \frac{i\cancel{A}(\omega-kU)}{\cancel{k}}(i\cancel{k}) - T\cancel{A}k^2$$

$$\Rightarrow \quad -\rho_1\frac{\omega^2}{k} + g\rho_1 = \rho_2\frac{\omega}{k}(\omega-kU) + g\rho_2 - \rho_2 U(\omega-kU)$$

$$- Tk^2$$

mult k

$$\Rightarrow \quad \overset{\checkmark}{-\rho_1\omega^2} + g\rho_1 k = \rho_2\overset{\checkmark}{\omega}(\overset{\checkmark}{\omega-kU}) + g\rho_2 k \overset{\checkmark}{-\rho_2\omega Uk} + \overset{\checkmark}{k^2U^2\rho_2}$$

$$\overset{\checkmark}{-Tk^3}.$$

$$\Rightarrow \quad (\rho_1+\rho_2)\omega^2 - 2\rho_2 Uk\omega + \rho_2 U^2k^2 - k\left[k^2T + g(\rho_1-\rho_2)\right]$$

$$= 0$$

quadratic eq in w^2 for w to have 2 real roots alright but if it has one complex root have to have the complex conj + one are in trouble for w^2 to have only real root require the

$$b^2 - 4ac > 0$$

or $\cancel{4}\rho_2^2 U^2 k^2 > \cancel{4}(\rho_1+\rho_2)\Big(\cancel{|k|(Tk^2+(\rho_1-\rho_2)g)}$

$$\cdot\Big[\rho_2 U^2 k^2 - |k|\big[k^2 T + (\rho_1-\rho_2)g\big]\Big]\Big)$$

$$\rho_2^2 \cancel{U}^2 k^2 - (\rho_1+\cancel{\rho_2})\rho_2 U^2 k^2 \cancel{>} -(\rho_1+\rho_2)|k|\big[k^2T + (\rho_1-\rho_2)g\big]$$

$$\rho_1\rho_2 U^2 k^2 < (\rho_1+\rho_2)|k|\big(k^2 T + (\rho_1-\rho_2)g\big)$$

$$\frac{|k|}{k^2} = \frac{1}{|k|}$$

$$U^2 < \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\left[|k|T + (\rho_1-\rho_2)\frac{g}{|k|}\right]$$

Thus this must be true $\forall k$ or else $\exists$ a k $\ni$ this is not true + that mode will grow

Thus min $|k|T + \frac{(\rho_1-\rho_2)g}{|k|}$ over $k=(0,\infty)$

$$-T + \frac{(\rho_1-\rho_2)g}{|k|^2} = 0 \Rightarrow |k|$$

$$|k|^2 = \frac{(\rho_1-\rho_2)g}{T}$$

$$|k| = \frac{(\rho_1-\rho_2)^{1/2} g^{1/2}}{T^{1/2}}$$

plug in $v^2 \underset{\text{must be}}{<} (\rho_1^{-1}+\rho_2^{-1})\left[(\rho_1-\rho_2)^{1/2} g^{1/2} T^{1/2} + (\rho_1-\rho_2)^{1/2} g^{1/2} T^{1/2}\right]$

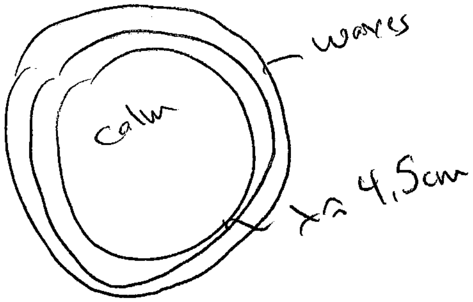
$$= 2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\left[(\rho_1-\rho_2)gT\right]^{1/2}$$

Thus system unstable if

$$v^2 > 2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\left((\rho_1-\rho_2)gT\right)^{1/2}$$

pg 113 Acheson

(37)



$$w^2 = gk + \frac{Tk^3}{\rho}$$

to see same wave length waves travel w/ speed c_g

Thus $t\,c_g(k)$ is outward growing radius

$$c_g(k) = \frac{g + 3Tk^2/\rho}{2(gk + Tk^3/\rho)^{1/2}}$$ group velocity has min at k^* which says

that All waves must travel faster than $c_g(k_{min})t$

$c_g(k)$ min at $k = \left(\frac{2}{\sqrt{3}} - 1\right)^{1/2}\left(\frac{\rho g}{T}\right)^{1/2}$

$$= \left(\frac{2}{\sqrt{3}} - 1\right)^{1/2}\left(\frac{10^3(9.81)}{.074}\right)^{1/2}$$

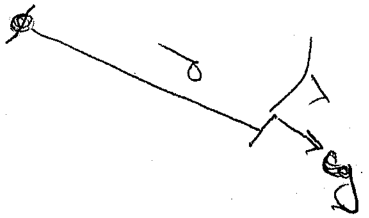
$$= 143.207 = \frac{2\pi}{\lambda}$$ $\rightarrow$ $\lambda = .04387$ m

$= 4.38$ cm!!

(3.8)

pg 113 Acheson

1



$w^2 = kg$

$\frac{dw}{dk} = \frac{\sqrt{g}}{2\sqrt{k}} = \frac{1}{2}\sqrt{\frac{g}{k}}$

$c_g t = d$

$k = \frac{w^2}{g}$

$\frac{t}{2}\frac{\sqrt{g}}{\omega}\sqrt{g} = d$

$t\frac{g}{2}\frac{T}{2\pi} = d$

waves w/ period T arrive at coast from d away in time

$t = \frac{4\pi d}{gT}$

$d = 5000$ km

know

$t = \frac{4\pi d}{gT}$

t_1 = time for waves of period T_1 to get to coast

$t_1 + 1$ day = time for waves of period T_2 to get to coast

$t_1 = \frac{4\pi d}{gT_1}$

$t_1 + 1\text{ day} = \frac{4\pi d}{gT_2}$

$\Rightarrow \frac{4\pi}{g} d\left(\frac{1}{T_2} - \frac{1}{T_1}\right) = 1\text{ day}$

$d = \frac{1\text{ day}\,(g)}{4\pi\left(\frac{1}{T_2} - \frac{1}{T_1}\right)}$

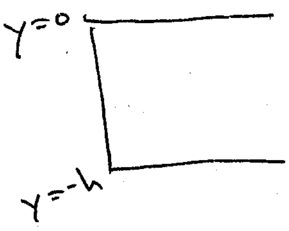
$d = 5.05 \times 10^6$ m ≈ 5000 km

pg 113 Acheson

③,⑨ Include affects of surface tension & finite depth on dispersion relation

ie Show $\omega^2 = Ck^2 + \omega_1 + \omega_2$

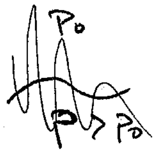
& if $h = \ldots$ $\omega_1 = -\omega_2$.



kinematic B.C $\frac{D(y-\eta(x,t))}{Dt} = 0$

$$\Rightarrow -\frac{\partial\eta}{\partial t} + \cancel{\frac{\partial\phi}{\partial x}\left(-\frac{\partial\eta}{\partial x}\right)}^{0} + \frac{\partial\phi}{\partial y}\cdot 1 = 0$$

$$\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \quad \text{on} \quad y = \eta(x,t) \quad *$$



0 nonlin

Bernoulli: $\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + g\eta + \cancel{\frac{1}{2}(\nabla\phi)^2}^{0} = C(t)$

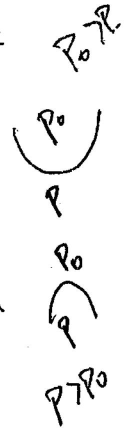
$p_0 - p = T\frac{\partial^2\eta}{\partial x^2}$

if $p_0 - p > 0$

η concave up

if $p_0 - p < 0$

η concave down



& $-p + p_0 = T\frac{\partial^2\eta}{\partial x^2}$ *

$$\Rightarrow \rho\frac{\partial\phi}{\partial t} + \rho g\eta - T\frac{\partial^2\eta}{\partial x^2} = 0 \quad **$$

let $\eta(x,t) = Ae^{i(kx-\omega t)}$

Try $\phi = f(y)e^{i(kx-\omega t)}$

$\Rightarrow f'' - k^2 f = 0$ for $\nabla^2\phi = 0$

$f(y) = C_1\cosh ky + C_2\sinh ky$ B.C. at $y=-h \Rightarrow \phi_y(y=-h) = 0$

$\Rightarrow f'(y=-h) = 0$

$$\Rightarrow f' = C_1 k \sinh ky + C_2 k \cosh ky$$

$$f'(-h) = 0 \Rightarrow \quad \cancel{C_2 = -\tanh}$$

$$-C_1 \cancel{k} \sinh kh + C_2 \cancel{k} \cosh kh = 0$$

$$C_2 = C_1 \tanh kh$$

$$\therefore \quad \phi =$$

$$= C_1(\cosh ky + \tanh kh \sinh ky)\, e^{i(kx-\omega t)}$$

Have 2 eqs * + ** + 2 unknown C_1 + $\omega(t)$

put in eq *

$$-A(i\omega)\cancel{e^{i(\ )}} = C_1(k\sinh ky + \tanh kh\, k \cosh ky)\cancel{e^{i(\ )}} \quad y=0$$

$$= C_1 k \tanh kh$$

$$C_1 = -\frac{i\omega A}{k \tanh kh}$$

put in eq **

$$\rho(-\omega i) f(y)\cancel{e^{i(\ )}} + \rho g A \cancel{e^{i(\ )}} - T(ik)^2 \cancel{e^{i(\ )}} A = 0$$

$$\cancel{-\omega i \rho C_1} \qquad \rho g A \quad + Tk^2 A = 0 \qquad \text{At } y=0$$

$$-\omega\rho\left(\frac{-i\omega \cancel{A}}{k\tanh kh}\right)\left(\cosh \hat{ky}^{\,1}\right) + \rho g \cancel{A} + Tk^2 \cancel{A} = 0$$

$$\frac{-\omega^2\rho}{k\tanh kh} + \rho g + Tk^2 = 0$$

$$\omega^2 = \frac{k}{\rho}(\tanh kh)\left(\rho g + Tk^2\right)$$

$$= \tanh(kh)\cdot\left[kg + \frac{Tk^3}{\rho}\right]$$

Q $C = \frac{\omega}{k} = \frac{\sqrt{\tanh(kh)}\left(kg + Tk^3/\rho\right)^{1/2}}{k}$

$$= \sqrt{\frac{\tanh(kh)}{k}}\left(g + \frac{Tk^2}{\rho}\right)^{1/2}$$

For long λ. k is small. +

~~$\tanh(kh) \approx kh +$~~

$C \approx C_0 + f(k)$

$=$

~~fer~~ $\tanh(0) = 0$

$\tanh' = \operatorname{sech}^2 \Big|_{x=0} = 1$

$\operatorname{sech}^{2\,\prime} = 2\operatorname{sech}^2\tanh \Big|\ = 0$ 2nd derivative

$\Big|$ 3rd derivative

$$\therefore \approx \sqrt{\frac{kh + \frac{2}{6}k^3h^3}{k}}\left(\sqrt{g}\right)\left(1 + \frac{Tk^2}{g\rho}\right)^{1/2} \quad 2$$

$$\approx \sqrt{h + \frac{k^2h^3}{3}}\,\sqrt{g}\left(1 + \frac{Tk^2}{2g\rho}\right)$$

$$\simeq \sqrt{h}\left(1+\frac{k^2h^2}{3}\right)^{1/2} \dots$$

$$\hat{=} \sqrt{hg}\left[1+\frac{k^2h^2}{6}\right]\left[1+\frac{Tk^2}{2g\rho}\right]$$

~~$= \sqrt{hg}\left[1+\frac{k}{2}\left[\frac{Tk}{g}+\frac{h}{3}\right]+\frac{Tk^3h}{4g}\right.$~~ $\left.\right]$

~~$C^2 = \frac{\omega^2}{k^2} = \tanh(kh)\left[\frac{g}{k}+Tk\right]$~~

~~$= \frac{g\tanh(kh)}{k} + Tk\tanh(kh)$~~

~~$\tanh(kh) = 0 + kh + 0 +$~~

$$\hat{=} \sqrt{hg}\left[1+\frac{k^2}{2}\left[\frac{h^2}{3}+\frac{T}{g\rho}\right]+\frac{Th^2k^4}{12g\rho}+\dots\right]$$

if minus instead of + + then $h^2 = \frac{3T}{\rho g}$

or $h = \left(\frac{3T}{\rho g}\right)^{1/2} = \left[\frac{3(.074\ \mathrm{N m^{-1}})}{(9.81\ \mathrm{m/s^2} \cdot 1000\ \mathrm{kg/m^3})}\right]^{1/2} = .00475\ \mathrm{m}$

$\hat{=}$ 5 mm.

PS 114 Acheson

(3.10) plane cap wave.

$$\omega(k)^2 = \frac{T}{\rho} k^3.$$

~~[illegible]~~

Then $\omega(k) = \sqrt{\frac{T}{\rho}}\, k^{3/2}$

$$c_g = \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{T}{\rho}}\, k^{1/2}$$

Then to see wave lengths of λ must be $x = c_g t$ from the source

$$\Rightarrow \quad c_g = \frac{x}{t} = \frac{3}{2}\sqrt{\frac{T}{\rho}}\, k^{1/2}$$

$$\frac{\rho}{T}\left(\frac{2}{3}\frac{x}{t}\right)^2 = k(x,t) \leftarrow \text{local } k.$$

Then $\omega(k) = \sqrt{\frac{T}{\rho}}\left(\frac{\rho}{T}\right)^{3/2}\left(\frac{2}{3}\frac{x}{t}\right)^3 = \left(\frac{\rho}{T}\right)^{-1/2}\left(\frac{\rho}{T}\right)^{3/2}\left(\quad\right)^3$

$$= \left(\frac{\rho}{T}\right)\left(\frac{2}{3}\frac{x}{t}\right)^3$$

get $\theta(x,t) =$ "$kx - \omega t$"

Then $\frac{\partial\theta}{\partial x} = k$ + $\frac{\partial\theta}{\partial t} = -\omega(x,t)$ locally.

Then $\frac{\partial \theta}{\delta x} = \frac{\rho}{T} \cdot \frac{4}{9} \cdot \frac{x^2}{t^2}$

$$\theta(x,t) = \frac{4\rho}{9T}\frac{x^3}{3t^2} + \theta_1(t)$$

$$\frac{\partial \theta}{\partial t} = -\frac{8}{27}\cancel{\frac{\rho}{T}\frac{x^3}{t^3}} + \theta_1' \overset{set}{=} -\omega(x,t) = -\cancel{\frac{\rho}{T}\cdot\frac{8}{27}\frac{x^3}{t^3}} .$$

$$\theta_1' = \text{const.}$$

$$\Rightarrow \quad \theta(x,t) = \frac{4}{27}\frac{\rho}{T}\frac{x^3}{t^2} + \epsilon . \qquad \frac{\lambda^3 x^3}{t^2} \Rightarrow \lambda \sim t^{2/3}$$

$$\therefore \quad \eta(x,t) = A(x,t)\cos\left\{\frac{4}{27}\frac{\rho}{T}\frac{x^3}{t^2} + t\right\}.$$

local $\lambda \sim \frac{T}{\rho}\frac{t^2}{x^2}$

Thus change in λ over distance δx is $\sim \frac{T}{\rho}\frac{t^2}{x^3}\delta x$.

$\therefore$ Change in λ over distan λ is

$\frac{T}{\rho}\frac{t^2}{x^3}\lambda$ $\quad\therefore$ fractional change is

if wave train is to be Approx sinusoidal

$$\frac{T}{\rho}\frac{t^2}{x^3} \ll 1$$

$$\Rightarrow \quad t^2 \ll \left(\frac{\rho}{T}\right)x^3$$

Pg 114 Acheson

3.11

$$\eta(x,0) = \int_{-\infty}^{\infty} a_0 e^{-\sigma(k-k_0)^2} e^{ikx}\, dk$$

$$= a_0 \int_{-\infty}^{\infty} e^{-\sigma(k^2 - 2kk_0 + k_0^2) + ikx}\, dk \qquad -ikx/\sigma$$

$$= a_0 \int_{-\infty}^{\infty} e^{-\sigma(k^2 + (-2k_0 - ix/\sigma)k + k_0^2}\, dk$$

$$= a_0 e^{-\sigma k_0^2} \int_{-\infty}^{\infty} e^{-\sigma(k^2 - (2k_0 + ix/\sigma)k)}\, dk$$

$$\left(\frac{2k_0 + ix/\sigma}{2}\right)^2 = \left(k_0 + \frac{ix}{2\sigma}\right)^2$$

$$= a_0 e^{-\sigma k_0^2} \int_{-\infty}^{\infty} e^{-\sigma\left[k^2 - (2k_0 + ix/\sigma)k + (k_0 + ix/2\sigma)^2\right] + \sigma(k_0 + ix/2\sigma)^2}\, dk$$

$$= a_0 \cancel{e^{-\sigma k_0^2}} \exp\left\{\sigma\left(\cancel{k_0^2} + \frac{ix}{\sigma}k_0 - \frac{x^2}{4\sigma^2}\right)\right\} \int_{-\infty}^{\infty} e^{-\sigma(k - (k_0 + ix/2\sigma))^2}\, dk$$

$$= a_0 e^{-x^2/4\sigma} e^{ixk_0} \int_{-\infty}^{\infty} e^{-\sigma k^2}\, dk$$

$$\rho^2 = \sigma k^2$$

$$d\rho = \sqrt{\sigma}\, dk$$

$$= a_0 e^{-x^2/4\sigma} e^{ixk_0} \int_{-\infty}^{\infty} e^{-\rho^2} \frac{d\rho}{\sqrt{\sigma}} =$$

$$= \frac{\cancel{2} a_0 e^{-x^2/4\sigma} e^{ixk_0}}{\sqrt{\sigma}} \underset{\overset{\|}{\sqrt{\pi}/\cancel{2}}}{\int_0^{\infty} e^{-\rho^2} d\rho}$$

$$\xi = \rho^2$$

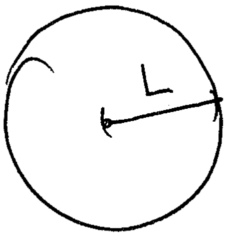
$$d\xi = 2\rho\, d\rho = 2\sqrt{\xi}\, d\rho$$

$$\eta(x,0) = a_0 \left(\frac{\pi}{\sigma}\right)^{1/2} \cos k_0 x\, e^{-x^2/4\sigma}$$

Show:
$$\int_{-\infty}^{\infty} e^{-\sigma(k-(k_0+ix/2\sigma))^2} dk = \int_{-\infty}^{\infty} e^{-\sigma k^2} dk.$$

pg 115 Acheson

3.13



Consider small amplitude sound waves

Mmt still holds.

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\cdot\nabla)\bar{u}\right) = -\nabla p$$

Mass:

$$\frac{\partial}{\partial t}(\rho) + \nabla\cdot(\rho\bar{u}) = 0$$

$S = \log(p\rho^{-\gamma})$

Assume const entropy of each particle $\frac{D}{Dt}(p\rho^{-\gamma}) = 0$

undisturb state & linearize

get $\frac{\partial^2 p_1}{dt^2} = a_0^2\nabla^2 p_1$

in spherical

$$\frac{\partial^2 p_1}{\partial t^2} = a_0^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p_1}{\partial r}\right)$$

So let $p_1(r,t) = f(r)\cos\omega t$.

$$-\omega^2 f(r) = a_0^2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)$$

$$\cancel{\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = -\left(\frac{\omega}{a_0}\right)^2 r^2 f}$$

$$\cancel{\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \left(\frac{\omega}{a_0}\right)^2 r^2 f = 0}$$

Now B.C. must be that $f(r)$ finite at $r=0$

+ v must vanish at $r=L$

3.61 ⇒ $\rho_0 \frac{\partial v_1}{\partial t} = -\nabla p_1$

$$-\omega^2 f = \frac{a_0^2}{r^2}\left[2r\frac{df}{dr} + r^2\frac{d^2f}{dr^2}\right]$$

$$= 2\frac{a_0^2}{r}\frac{df}{dr} + a_0^2\frac{d^2f}{dr^2}$$

$$\frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} + \left(\frac{\omega}{a_0}\right)^2 f = 0$$

mult by r^2

$$r^2\frac{d^2f}{dr^2} + 2r\frac{df}{dr} + \left(\frac{\omega}{a_0}\right)^2 r^2 f = 0 \quad *$$

"Bessels eq order 0" Not w/ 2.

let $r = \left(\frac{a_0}{\omega}\right)\xi$

$$\frac{d}{dr} = \left(\frac{\omega}{a_0}\right)\frac{d}{d\xi}$$

$$\frac{d^2}{dr^2} = \left(\frac{\omega}{a_0}\right)^2\frac{d^2}{d\xi^2}$$

~~$\frac{\xi}{\sqrt{2}}\frac{df}{d\left(\frac{\xi}{\sqrt{2}}\right)}$~~

~~$\sqrt{2}\,\xi\frac{df}{d(\xi/\sqrt{2})}$~~

||

~~let $t = \frac{1}{\sqrt{2}}$~~

$$\Rightarrow \xi^2\frac{d^2f}{d\xi^2} + 2\xi\frac{df}{d\xi} + \xi^2 f = 0$$

solve $f(\xi)$

Then $f(\omega/a_0\, r)$ is sol to *

let ~~$f = aF(bx)$ $aF(br)$~~

Then $\frac{df}{dr} = a\frac{dF}{d\xi}b.$

$$\frac{d^2f}{dr^2} = \frac{d}{dr}\left(a\frac{dF}{d\xi}b\right) = ab^2\frac{d^2F}{d\xi^2}$$

$$\Rightarrow \quad r^2ab^2\frac{d^2F}{d\xi^2} + 2(rb)\frac{dF}{d\xi} + \left(\frac{\omega}{a_0}\right)^2 r^2F = 0$$

$$\frac{d}{d\xi}\left(\xi^2\frac{df}{d\xi}\right) + \xi^2 f = 0$$

$f = \tan\xi$

$\frac{df}{d\xi} = \sec^2\xi$

$\xi = 0$ is ~~reg~~ ~~sing~~ ordinary pt to D.E..

$\Rightarrow \quad f = \sum_{\substack{\cancel{k=0} \\ k\in\mathbb{Z}}}^{\infty} a_k\xi^k$ pt in

$$\sum a_k(k)(k-1)\xi^k + 2\sum a_k k\xi^k + \sum a_k\xi^{k+2} = 0$$

$$\sum a_{k+2}\left[(k+2)(k+1) + 2(k+2)\right]\xi^{k+2} + \sum a_k\xi^{k+2} = 0$$

$$\Rightarrow \quad a_{k+2} = \frac{-a_k}{k(k+2)+3(k+2)} = \frac{-a_k}{(k+2)(k+3)} \quad k\in\mathbb{Z}$$

$a_0 \neq 0$

$k \to 2k$

$$a_{2(k+1)} = \frac{-a_{2k}}{(2(k+1))(2k+3)} = \frac{-a_{2k}}{2^2(k+1)(k+3/2)}$$

$$a_{2k} = \frac{(-1)^k \Gamma(1)\,\Gamma(3/2)\,a_0}{4^k\,\Gamma(k+1)\,\Gamma(k+3/2)}$$

$$\Gamma(z) = (z-1)\Gamma(z-1)$$

Check

$$a_{2k+2} = \frac{(-1)^{k+1}\,\Gamma(1)\,\Gamma(3/2)\,a_0}{4\,4^k\,\Gamma(k+1)(k+1)\,\Gamma(k+3/2)(k+3/2)}$$

$$= \frac{-1}{4(k+1)(k+3/2)}\left(a_{2k}\right) \quad ✓.$$

$k \to 2k-1$

$$a_{2k+1} = \frac{-a_{2k-1}}{(2k+1)(2k+2)} = -\frac{a_{2k-1}}{4(}$$

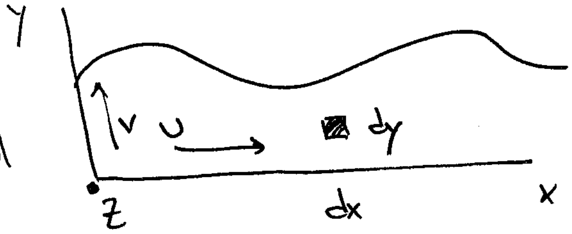
Pg 114 Acheson

(3.12) $\int dE$ = kenetic + potential. for 1 wave length.

$$= \int_{-\infty}^{\eta(x,t)} \int_{x_0}^{x_0+\lambda} \frac{1}{2}(u^2+v^2)\, \underbrace{\rho\, dx\, dy}_{mass}$$

$$+ \int_{\cancel{-\infty}\ 0}^{\eta(x,t)} \int_{x_0}^{x_0+\lambda} dx\, dy\, \rho g y$$

why 0 + not −∞? Normilization of potential energy thing?



$\cancel{mgy}$

$F = mg = -\frac{dU}{dy}$

$\boxed{U = -mgy}$

$dU = (dx\, dy\, \rho) g y$

Potential Energy:

$$\int_{\cancel{-\infty}\ 0}^{\eta(x,t)} \rho g y \lambda\, dy$$

if we are talking about deep H_2O $\omega^2 = gk$

$$\eta = A\cos(kx - \omega(k)t)$$

$$\phi = \frac{A\omega}{k} e^{ky} \sin(kx-\omega t)$$

Then $u = \frac{\partial\phi}{\partial x} = A\omega e^{+ky}\cos(kx-\omega t)$

$v = \frac{\partial\phi}{\partial y} = A\omega e^{ky}\sin(kx-\omega t)$

$$u^2+v^2 = A^2\omega^2 e^{2ky}$$

Must do y int 1st!!

Depends on x

$$\therefore\quad E(t) = \int_{-\infty}^{\boxed{\eta(x,t)}} \frac{\lambda}{2}\left(A^2\omega^2 e^{2ky}\right)\rho\, dy + \int_{\cancel{-\infty}\,0}^{\boxed{\eta(x,t)}} \rho g \lambda y\, dy$$

$$E = \int_{x_0}^{x_0+\lambda} \frac{\rho}{2}A^2\omega^2 \int_{-\infty}^{\eta(x,t)} e^{2ky}\, dy\, dx + \int_{x_0}^{x_0+\lambda} \rho g \int_{0}^{\eta(x,t)} y\, dy$$

$$= \frac{\rho}{2}A^2\omega^2 \int_{x_0}^{x_0+\lambda} \left.\frac{e^{2ky}}{2k}\right|_{-\infty}^{\eta(x,t)} dx + \frac{\rho g}{2}\int_{x_0}^{x_0+\lambda} \eta^2(x,t)\, dx$$

$$= \frac{\rho A^2\omega^2}{4k} \int_{x_0}^{x_0+\lambda} e^{2k\eta(x,t)}\, dx + \frac{\rho g A^2}{2}\int_{x_0}^{x_0+\lambda} \cos^2(kx-\omega(k)t)\, dx$$

$$\|$$

$$\frac{1}{2}\left(1+\cos\left(2(kx-\omega(k)t)\right)\right)$$

$$+\frac{\rho g A^2}{2}\lambda + \frac{\rho g A^2}{2}\cancelto{0}{\left.\frac{\sin\left(2(kx-\omega(k)t)\right)}{2k}\right|_{x_0}^{x_0+\lambda}}$$

$$\sin(2k(x_0+\lambda))$$
$$= \sin(2kx_0 + 2(2\pi)) = \sin(2kx_0)$$

$$= \frac{\rho A^2\omega^2}{4k}\int_{x_0}^{x_0+\lambda} e^{2kA\cos(kx-\omega t)}\,dx + \frac{\rho g A^2}{2}\lambda$$

$$\parallel$$

How do??

it taken to 0 As in hint

$$\approx \frac{\rho g A^2}{4\cancel{k}}\lambda\int_{8U_0}$$

$s^2 = 1 - c^2$

$U = 2k\cos(kx-\omega t)$

$dU = -2k^2\sin(kx-\omega t)\,dx$

$dU = -2k^2\left[1-\frac{U^2}{4k^2}\right]dx$

~~Must be 0 if $E = \frac{\rho g A^2}{2}$~~

Still missing factor??

$$= \frac{\rho g A^2}{4}\lambda + \frac{\rho g A^2}{\textcircled{2}}\lambda$$ — should be 4

$P_1 = -\rho\frac{\partial\phi}{\partial t}$ comes from Bernoulli As potential is constant at a point in space + $\frac{v^2}{2}$ ~ neglected.

$P_1 \int \cdot\, dy$ $\qquad \int_{-\infty}^{\eta(x,t)} P_1\,dy \cdot 1\, U$ $\qquad P_1 = -\rho\left[\frac{A(-\omega)}{k}e^{ky}\cos(kx-\omega t)\right]$

$$\int_{-\infty}^{\eta(x,t)} \rho\frac{\omega A}{k}\cos(kx-\omega t)\,e^{ky}\,U\,dy = \frac{\rho\omega A}{k}\cancel{\cos(kx-\omega t)\ \frac{e^{ky}}{k}}\Bigg|_{-\infty}^{\eta(x,t)}$$

$$= \frac{\rho\omega A}{k^2}\cancel{\cos(kx-\omega t)}\,e^{k\eta(x,t)}$$

Using hint

$\sim \cancel{\frac{\rho \omega A}{k^2} \cos(kx-\omega t)}$

+ ~~Average over time is~~ $\frac{2\pi}{T} = \omega$.

$$\cancel{\frac{1}{T}\int_{t_0}^{t_0+T} \cos(kx-\omega t)\,dt} =$$

---

putting in $U = A\omega$ . . .

using Hint

$$\int_{-\infty}^{\eta(x,t)} \frac{\rho\omega^2 A^2}{k} e^{2ky} \cos^2(kx-\omega t)\,dy = \frac{\rho\omega^2 A^2}{k}\cos^2(kx-\omega t)\ \frac{1}{2k}$$

Averaging over time $\frac{1}{T}\int_{t_0}^{t_0+T} \cos^2(kx-\omega t)\,dt$

$$\Rightarrow \quad \rho\frac{\omega^2 A^2}{2k^2}\cdot\frac{1}{2} = \rho\frac{\omega^2 A^2}{4k^2} = \frac{1}{4}\rho$$

$$\cancel{\frac{1}{T}\left(\frac{1}{2}\int\left(1+\cos(2(kx-\omega t))\right)\right)}dt$$

$\cancel{\frac{\omega^2}{k^2}}$ $\cancel{\frac{g}{k}}$ $\cancel{\omega^2 = gk}$

$$C_g = \frac{\omega}{k} \qquad \omega^2 = gk$$

$$\therefore \quad \bar{F} = \frac{1}{4}\rho g A^2 \boxed{\frac{1}{K}} \; ??$$

? Missing on ω

$$\frac{\bar{F}}{\bar{E}} = \frac{\frac{1}{4}\cancel{\rho g A^2} c}{\frac{1}{2}\cancel{\rho g A^2}} = \frac{1}{2}c$$

Pg 115 Acheson

3.13

$$\frac{\partial^2 p_1}{\partial t^2} = a_0^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p_1}{\partial r}\right)$$

let $p_1 = \frac{1}{r} h(r,t)$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 h}{\partial t^2} = a_0^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-\frac{1}{r^2}h + \frac{1}{r}\frac{\partial h}{\partial r}\right)\right)$$

$$= a_0^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(-h + r\frac{\partial h}{\partial r}\right))$$

= ~~$a_0^2 \frac{1}{r^2}\left(\frac{-\partial h}{\partial r}\right) + \frac{\partial h}{\partial r}$~~

$$\frac{1}{\cancel{r}}\frac{\partial^2 h}{\partial t^2} = \frac{a_0^2}{r^2}\left[-\frac{\partial h}{\partial r} + \frac{\partial h}{\partial r} + r\frac{\partial^2 h}{\partial r^2}\right] = \frac{a_0^2}{\cancel{r}}\frac{\partial^2 h}{\partial r^2}$$

$$\frac{\partial^2 h}{\partial t^2} = a_0^2 \frac{\partial^2 h}{\partial r^2}$$

B.C on gas must be that $u_r(r=L,t)=0$.

$h(r,t) =$ ~~[illegible]~~

$$= F_1(r - a_0 t) + F_2(r + a_0 t)$$

$$p_1(r,t) = \frac{1}{r}\left(F_1(r-a_0t) + F_2(r+a_0t)\right)$$

Pg Acheson

115

③ By eq 3.61

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\nabla p_1 = -\left[\frac{\partial}{\partial r} p_1 \hat{r} + \cancelto{0}{\frac{1}{r}\frac{\partial}{\partial \theta} p_1} \hat{\theta} + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial p_1}{\partial \phi}}\right]$$

$$= -\hat{r}\left[\frac{-1}{r^2}(F_1+F_2) + \frac{1}{r}(F_1'+F_2')\right]$$

$$\rho_0 \frac{\partial u_r}{\partial t}\hat{r} = \ldots$$

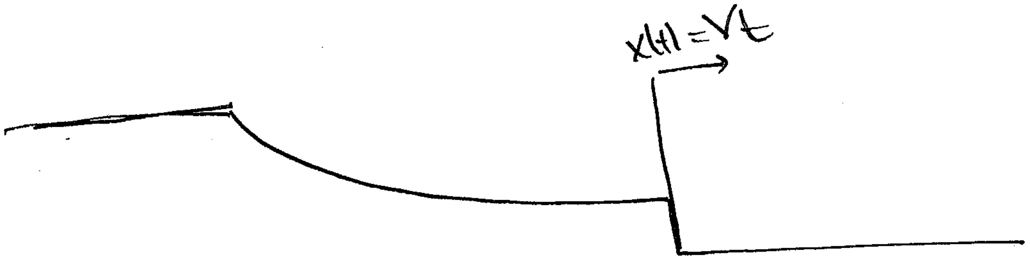
$$u = \frac{t}{\rho_0}\left[\frac{F_1+F_2}{r^2} - \frac{(F_1'+F_2')}{r}\right] + C$$

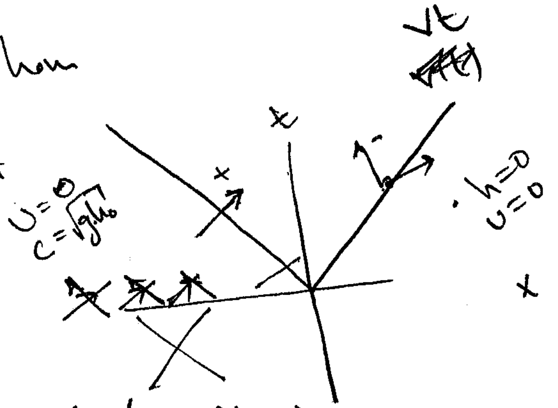
$$u(r=L) = 0$$

(3.18)

t=0 Pg 116 Acheson

$c=\sqrt{gh}$





Very far to the left characteristics hom

$h=h_0$ ⇒ Right R. invariant
$u=0$

$$u+2\sqrt{hg}=2\sqrt{h_0 g}$$

on plate must hom one entering characterist it is the left characterist + $u-2\sqrt{hg}=V-2\sqrt{h^* g}$ on $\frac{dx}{dt}=V-\sqrt{h^* g}$

where h^* is unknown.

for $t<0$ + $x<0$ H_2O is stationary + ∴ left.

Char are constant also $u\mp 2\sqrt{hg}=\mp 2\sqrt{h_0 g}$ on $\frac{dx}{dt}=\mp\sqrt{h_0 g}$

∴ get $u+2c=2\sqrt{gh_0}$

+ $u-2c=-2\sqrt{gh_0}$

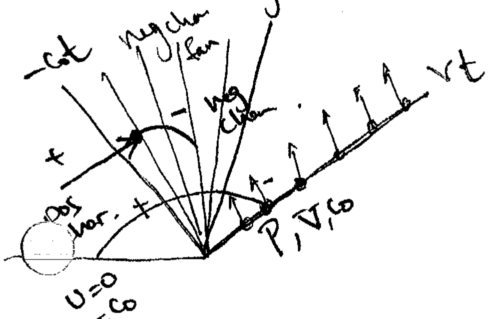
Add ⇒ $u=0$ sub ⇒ $c=\sqrt{gh_0}$

Boundary is $\frac{dx}{dt} = -c_0 = -\sqrt{gh_0}$

in Region to "Right" of $\frac{dx}{dt} = -c_0t$

get positive char. $U + 2\sqrt{gh} = 2\sqrt{gh_0}$

+ along some neg char $U - 2\sqrt{gh} = K$ K some constant



thus ⇒ U + $\sqrt{gh}$ are constants at this point + thus

$$\frac{dx}{dt} = U - \sqrt{gh} \text{ must be a constant}$$

at this point also. Preventing crossing of characteristics →

neg char must be st lines through the origin.

$$U - \sqrt{gh} = x/t.$$

Then solve $U - \sqrt{gh} = x/t$ $\quad 2U - \sqrt{gh} = 2x/t$ + Add

+ $U + 2\sqrt{gh} = 2\sqrt{gh_0}$

$$3U = 2\sqrt{gh_0} + 2(x/t)$$

$$U = \frac{2}{3}(\sqrt{gh_0} + x/t)$$

Then

$$\frac{2}{3}\left(\sqrt{gh_0}+x/t\right)-\sqrt{gh'} = x/t$$

$$-\sqrt{gh'} = \frac{1}{3}x/t - 2/3\sqrt{gh_0}$$

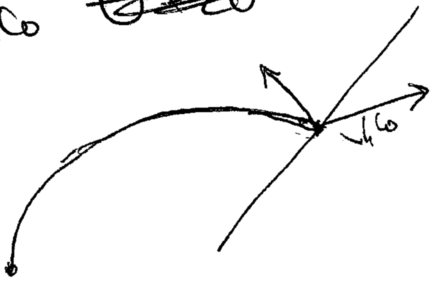
$$c = \frac{1}{3}\left(2c_0 - x/t\right)$$

Now Neg char come off of Vt w/ ~~slope~~ values At P.

$$u-2c = V-\underline{2c_0}$$

& if we Assume that positive char com from the undisturbed region

~~$U+2c$ $2c_0$~~ ~~$U+2c_0$~~ ~~$U+2c_0$~~ -



$u+2c = 2c_0$ then

Then $u-2c = V-2c_0$

$u+2c = 2c_0$

Add $u = \frac{V}{2}$

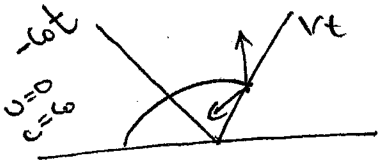
Sub $4c = -V+4c_0$

$c = c_0 - \frac{V}{4}$

Slope coming of for neg char. is $\frac{dx}{dt} = u-c = \frac{V}{2} - c_0 + \frac{V}{4}$

$= \frac{3V}{4} - c_0$

Starting over:



$u + 2c = 2c_0$

on plate $u = V$.

Then $V + 2c = 2c_0 \Rightarrow c = c_0 - \frac{V}{2}$

Thus $u + 2c = 2c_0$ along char coming of plate

$$\frac{dx}{dt} = u + c$$

& $u - 2c = V - 2c_0 + V = 2V - 2c_0$ along $\frac{dx}{dt} = u - c$

when these char intersect get

$$u - 2c = 2V - 2c_0$$
$$u + 2c = 2c_0$$

Add $u = V$ & sub $4c = -2V + 4c_0$

$$c = -\frac{V}{2} + c_0$$

where the intersect get st lines for characteristics

$$\frac{dx}{dt} = V \pm (c_0 - \tfrac{1}{2}V)$$

& $V + c_0 - \frac{1}{2}V = \frac{V}{2} + c_0$

$< 2c_0$

$V - c_0 + \frac{1}{2}V = \frac{3V}{2} - c_0$

1st set has $\frac{dx}{dt} > V$

2nd has $\frac{dx}{dt} < V$.

pg 117 Acheson

(3,18) Shallow thro theory. w/ $c = \sqrt{gh}$

$$\Rightarrow \left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u+2c) = 0$$

$$+ \quad \left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)(u-2c) = 0.$$

2 sets of characteristics.

Pg 117 Acheson

(3.19)

$$x = g(x_0)t + x_0$$

$$x = g(x_0 + \delta x_0)t + x_0 + \delta x_0$$

Subtr.

$$0 = (g(x_0 + \delta x_0) - g(x_0))t + \delta x_0$$

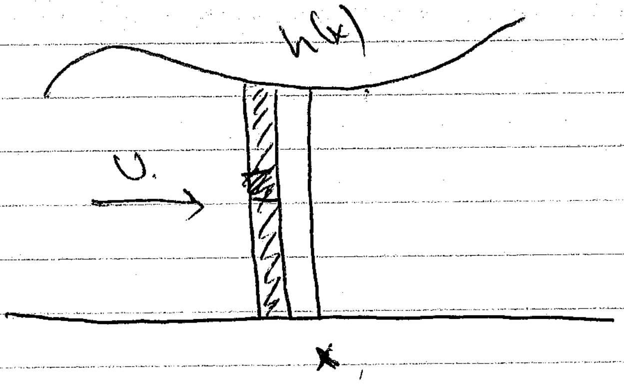
lim $\delta x_0 \to 0$

$$0 \simeq g'(x_0)\delta x_0 t + \delta x_0$$

$$= (g'(x_0)t + 1)\delta x_0$$

$$\Rightarrow \quad t_{BC} = \frac{-1}{g'(x_0)}$$

(3.20)



Flux any kind can write down ~~energ~~ energy of slab at

Then mult by u.

— ▨ dy

Potential and [illegible] = [illegible] [illegible] [illegible]
kinetic of moving slab.

$$\frac{\rho g U_1}{4h_2}(h_2-h_1)^3.$$

Pg 117 Acheson

(3.20) $$\Delta E = \rho g\left(U_1 h_1^2 - U_2 h_2^2\right) + \frac{1}{2}\rho\left(U_1^3 h_1 - U_2^3 h_2\right)$$

w/ $h_1 U_1 = h_2 U_2$

+ $\frac{1}{2} g h_1^2 + h_1 U_1^2 = \frac{1}{2} g h_2^2 + h_2 U_2^2$

$$U_2 = \frac{U_1 h_1}{h_2}$$

$$\Rightarrow \Delta E = \rho g\left(U_1 h_1^2 - \frac{U_1 h_1}{h_2} h_2^2\right) + \frac{1}{2}\rho\left(U_1^3 h_1 - \left(\frac{U_1 h_1}{\cancel{h_2}}\right)\left(U_2^2 \cancel{h_2}\right)\right)]$$

$$\Rightarrow \Delta E = \rho \frac{U_1}{h_2}\left[g\left(h_1 - h_1 h_2^2\right) + \frac{U_1^2}{2}\frac{h_1 h_2}{\cancel{h_2}} - \frac{h_1}{2} U_2^2 h_2\right]$$

$$= \frac{\rho U_1}{h_2}\left[g\left(h_1 - h_1 h_2^2\right) + \frac{1}{2}\left(U_1^2 h_1 h_2 - U_2^2 h_1 h_2\right)\right)$$

$$= \frac{\rho U_1}{h_2}\left[g\left(h_1 - h_1 h_2^2\right) + \frac{h_1 h_2}{2}\left(U_1^2 - U_2^2\right)\right]$$

$$U_1 = \frac{U_2 h_2}{h_1} \quad \& \quad U_1^2 h_1 = \frac{1}{2} g h_2^2 + h_2 U_2^2 - \frac{1}{2} g h_1^2$$

$$\delta E = \rho g \left[U_2 h_2 h_1 - U_2 h_2^2 \right]$$

$$+ \frac{\rho}{2} \left[\frac{U_2 h_2}{h_1} \left(\frac{1}{2} g h_2^2 + h_2 U_2^2 - \frac{1}{2} g h_1^2 \right) - U_2^3 h_2 \right]$$

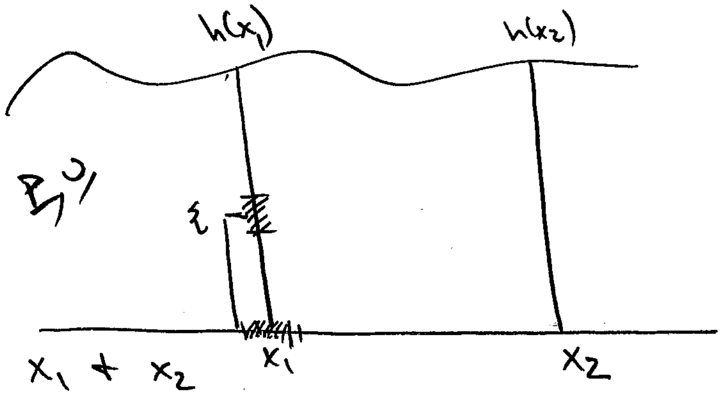
$$= \rho g U_2 h_2 (h_1 - h_2)$$

$$+ \frac{\rho}{2} U_2 h_2 \left[\frac{1}{2} g \left(\frac{h_2^2}{h_1} - h_1 \right) + \frac{h_2}{h_1} U_2^2 - U_2^2 \right]$$

$$= \rho U_2 h_2 \left[g(h_1 - h_2) + \frac{g}{4} \left(\frac{h_2^2}{h_1} - h_1 \right) + \left(\frac{h_2}{h_1} - 1 \right) U_2^2 \right]$$

Pg 117 Acheson

(3.20)



cons of energy between x_1 + x_2

$$\frac{d}{dt}\int_{x_1}^{x_2}\left(\frac{1}{2}\rho u^2 + \qquad\right) dx\cdot 1\cdot 1 = -\,u\Big(\underbrace{\qquad\qquad}_{\text{energy part}}\Big)\Big|_{x_1}^{x_2} +$$

kinetic energy + potential

1 dy

per slab $\displaystyle\int_{y=0}^{h(x)} 1\,dx\,dy\,\rho g\,\xi$ accurate

i.e. went only 1st order accurate expression

$$= dx\,\rho g\frac{\xi^2}{2} = dx\,\rho g\frac{h^2}{2}$$

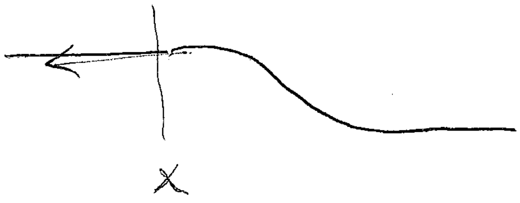
1st order in "hight"

$$dx\,\rho g\,h\,dh$$

$$+ \int_0^{h(x_1)} u p\,dy - \int_0^{h(x_2)} u p\,dy.$$

pg 118 Acheson

3.24



$$\frac{\partial u}{\partial t} + \left(\frac{1}{2}(r+1)u + a_0\right)\frac{\partial u}{\partial x} = \frac{2}{3}\nu\frac{\partial^2 u}{\partial x^2}$$

$u = f(\xi) \qquad \xi = x - Vt$

$\frac{\partial u}{\partial t} = f'(-V)$ $\qquad f(-\infty) = U_1$

$\qquad\qquad\qquad f(+\infty) = 0.$

$\frac{\partial u}{\partial x} = f'(\xi)$

$$\Rightarrow -Vf' + \left(\frac{1}{2}(r+1)f + a_0\right)f' = \frac{2}{3}\nu f''$$

$$\Rightarrow f'\left[\frac{1}{2}(r+1)f + a_0 - V\right] = \frac{2}{3}\nu f''$$

$$\Rightarrow f(a_0 - V) + \frac{1}{4}(r+1)\frac{f^2}{\cancel{2}} = \frac{2}{3}\nu f' + C$$

~~B.C. At $+\infty \Rightarrow f(+\infty) \approx 0 \Rightarrow C=0$~~

$\Rightarrow$

$$f\left[a_0 - V + \frac{1}{4}(r+1)\frac{f}{\cancel{2}}\right] = \frac{2}{3}\nu f' + C$$

$\Rightarrow$ let $\xi \to +\infty \qquad 0 = \frac{2}{3}\nu f' + C$

$$\Rightarrow f' \longrightarrow -\frac{3}{2\nu}C \quad \xi \to +\infty$$

Assume $f' \not\to 0$ As $\xi \to +\infty$

$f' \to C$ As $\xi \to +\infty$

$\Rightarrow f \to C\xi$ As $\xi \to +\infty$

But ~~$\Rightarrow$~~ fact that $f(+\infty) = 0$ unless $C = 0$

$\Rightarrow$

$$f\left[a_0 - V + \frac{1}{4}(r+1)\frac{f}{2}\right] = \frac{2}{3}\nu f'$$

let $\xi \to -\infty$

$$U_1\left[a_0 - V + \frac{1}{8}\frac{(r+1)}{4}U_1\right] = 0$$

$$\Rightarrow V = a_0 + \frac{(r+1)}{4}U_1$$

Then $$f\left[\left(\frac{r+1}{4}\right)(f - U_1)\right] = \frac{2}{3}\nu f'$$

$$\Rightarrow \frac{1}{4}(r+1)(f - U_1)f = \frac{2}{3}\nu f'$$

$$\frac{2}{3}\nu\frac{df}{(f-U_1)f} = \frac{1}{4}(r+1)\,d\xi$$

$$\frac{2}{3}\nu\left[\frac{1}{U_1}\left(\log\left|\frac{f-U_1}{f}\right|\right)\right] = \frac{\xi(r+1)}{4} + C_2$$

$$\log\left|\frac{f-v_1}{f}\right| = v_1\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi + C_2$$

$$-\left(\frac{f-v_1}{f}\right) = C_3 e^{+\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi v_1} \qquad f < v_1$$

$$-f + v_1 = C_3 e^{+\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi v_1} f$$

$$0 = 0 \checkmark \qquad \xi \to -\infty$$

$$v_1 \overset{?}{=} \infty \cdot 0 \checkmark. \quad \xi \to +\infty$$

$C_3 = v_1$ check ?

$$\lim_{\xi \to +\infty} v_1 e^{\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi v_1}$$

$$v_1 = f\left[C_3 e^{\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi v_1} + 1\right]$$

$$f(\xi) = \frac{v_1}{1 + C_3 e^{\frac{3}{8}\left(\frac{\gamma+1}{\nu}\right)\xi v_1}}$$ How get $C_3 = ?$ just take to be 1

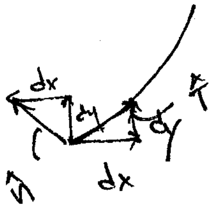
$$= \frac{v_1}{1 + C_3 e^{(x-vt)/\Delta}} \qquad \Delta = \frac{8}{3}\frac{\nu}{(\gamma+1)v_1}$$

$$f(0) = \frac{1}{2}v_1 = \qquad \Rightarrow C_3 = 1 \checkmark.$$

Pg 151 Acheson

(4.1) (ii)

$$\psi = \int_O^P u\,dy - v\,dx$$



$$\parallel$$

if

~~[illegible]~~ $(u,v)\circ(dy, -dx)$

$$= \int_O^P \bar{u}\circ\hat{n}\,dl$$

By Greens thm in plan

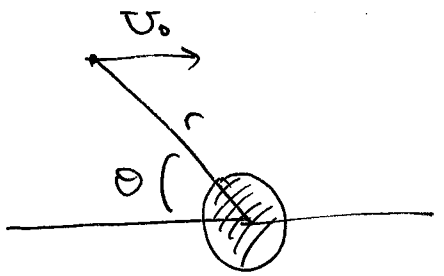
$$\int_O^P u\,dy + (-v)\,dx = \int_S \left(\frac{\partial u}{\partial x} - \frac{\partial(-v)}{\partial y}\right) dx\,dy = \int_S \cancel{\overset{=0}{\nabla\circ\bar{u}}}\,dx\,dy = 0$$

if O + P are same point

(4.2)

pg 223 Acheson

Spherical



let $\bar{U} = (U(r,\theta), V(r,\theta), 0)$

$\nabla \circ \bar{U} = 0$ in spherical

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cancel{\sin} U) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V\sin\theta) + \cancel{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(w)}^{0} = 0$$

Mult by $r^2\sin\theta$

$$\Rightarrow \frac{\partial}{\partial r}(r^2\sin\theta\, U) + \frac{\partial}{\partial\theta}(r\sin\theta\, V) = 0$$

let $f_\theta = r^2\sin\theta\, U \qquad f_r = -r\sin\theta\, V$

$\nabla \circ \bar{U} = 0$ ✓.

$$U = \frac{f_\theta}{r^2\sin\theta} \qquad V = \frac{-f_r}{r\sin\theta}.$$

Now: Stokes flow $\Rightarrow$

$$\nu\nabla^2\vec{U} = \frac{1}{\rho}\nabla p$$

$$\Rightarrow \nabla^2_{\text{spherical}}\bar{U} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}U\right) = \frac{1}{\rho\nu}$$

$$= \frac{1}{\mu}\frac{\partial p}{\partial r}.$$

$$\& \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = \frac{1}{\mu}\frac{1}{r}\frac{\partial p}{\partial\theta}.$$

put in stream fn.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\frac{f_\theta}{r^2\sin\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\frac{f_\theta}{r^2\sin\theta}\right)=\frac{1}{\mu}\frac{\partial p}{\partial r}.$$

$$\|$$

---

if $\nabla^2 u = \cancelto{0}{\nabla(\nabla\cdot\bar{u})} - \nabla\times(\nabla\times\bar{u})$.

$$\nabla\times\bar{u}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi\\ \partial_r & \partial_\theta & \partial_\phi\\ \frac{f_\theta}{r^2\sin\theta} & \frac{-f_r}{\sin\theta} & 0\end{vmatrix}$$

$$=\frac{1}{\cancel{r^2}\cancel{\sin\theta}}\Bigg[\hat{e}_r(0)+r\hat{e}_\theta(0)$$

$$+\cancel{r}\cancel{\sin\theta}\,\hat{e}_\phi\left(\frac{\partial}{\partial r}\left(\frac{-f_r}{\sin\theta}\right)-\frac{\partial}{\partial\theta}\left(\frac{f_\theta}{r^2\sin\theta}\right)\right)\Bigg]$$

$$=\frac{1}{r}\hat{e}_\phi\left[-\frac{f_{rr}}{\sin\theta}-\cancel{\frac{f_{\theta\theta}}{r^2\sin\theta}}\cancel{+\frac{f_\theta\cos\theta}{r^2\sin^2\theta}}\right]$$

$$=\cancel{\frac{-1}{r\sin\theta}\hat{e}_\phi\left[\frac{\partial^2 f}{\partial r^2}\right.}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\theta}\right)\Bigg]$$

$$=\frac{-1}{r\sin\theta}\hat{e}_\phi\underbrace{\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]f}_{E^2 f}.$$

get $\nabla p = -\mu\nabla\times(\nabla\times\bar{u})$

$$\nabla\times(\nabla\times\bar{u}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & -E^2\psi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\Big[\hat{e}_r\left(-\frac{\partial}{\partial\theta}E^2\psi\right) - r\hat{e}_\theta\left[\frac{\partial}{\partial r}\left(-E^2\psi\right)\right]$$

$$+ r\sin\theta\,\hat{e}_\phi(0)\Big]$$

$$\Rightarrow 1 \quad \frac{\partial p}{\partial r} = \frac{+\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}E^2\psi.$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{-\mu}{r^2\sin\theta}\,r\,\frac{\partial}{\partial r}\left(E^2\psi\right) = -\frac{\mu}{r\sin\theta}\frac{\partial}{\partial r}E^2\psi.$$

$$2 \quad \frac{\partial p}{\partial\theta} = \frac{-\mu}{\sin\theta}\frac{\partial}{\partial r}E^2\psi.$$

$$\frac{\partial}{\partial r}2 - \frac{\partial}{\partial\theta}1 \Rightarrow 0 = -\frac{\mu}{\sin\theta}\frac{\partial^2}{\partial r^2}E^2\psi - \frac{\mu}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}E^2\psi$$

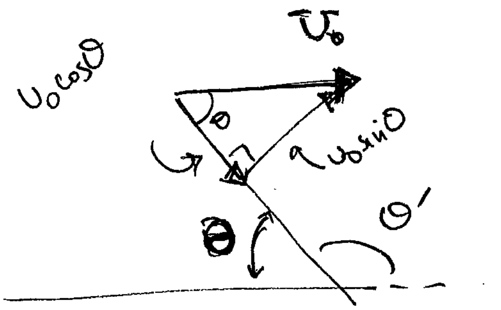
$$0 = -\frac{\mu}{\sin\theta}\left[\frac{\partial^2}{\partial r^2}E^2\psi + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial E^2\psi}{\partial\theta}\right)\right]$$

$$= \cancel{E}\left(\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)^2\psi = 0. \quad *$$

B.C. $u=0\Big|_{r=R}$ $v=0\Big|_{r=R}$

$\Rightarrow f_\theta(r=R)=0$ $u = U_0$

$+\ f_r(r=R)=0$



$\therefore$ $u(r\to+\infty) = -U_0\cos\theta =$

$v(r\to+\infty) = U_0\sin\theta =$

neg because now $\partial\theta$ is in dir to decrease $\angle\theta$

$\theta = \pi - \theta'$

$\cos(\pi-\theta')$

$= -\cos\theta'$. ✓.

$\sin(\pi-\theta')$

$= \overset{0}{\cancel{\sin\pi}}\cos\theta' - \sin\theta'\cos\pi$

$= \sin\theta'$

w/ chang of X.

$U_0\cos\theta = \dfrac{f_\theta(r\to+\infty)}{r^2\sin\theta}$

$+\ U_0\sin\theta = -\dfrac{f_r(r\to+\infty)}{r\sin\theta}$

$\Rightarrow f_\theta(r\to+\infty) = U_0 r^2\sin\theta\cos\theta$

$\Rightarrow f(r\to+\infty) = \dfrac{U_0 r^2}{2}\sin^2\theta.$

$f_r(+\infty) = +U_0 r\sin^2\theta$

$f = +U_0\dfrac{r^2}{2}\sin^2\theta..$

Try sol of form . . .

let $\psi = f(r) \sin^2\theta$. pt in eq $*$

$$\Rightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\underset{\sin^2\theta}{f''} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\cancel{f(r)}(2\cancel{\sin}\theta\cos\theta)}{\cancel{\sin\theta}}\right)\right] = 0$$

$$\Rightarrow \left[\partial_r^2 + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left\{f''\sin^2\theta \overset{-}{\cancel{+}} 2\frac{\sin^2\theta}{r^2}\right\}$$

$$\Rightarrow f^{IV}\sin^2\theta + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\cancel{\sin\theta}}f''2\cancel{\sin\theta}\cos\theta\right)$$

$$+ \quad - \frac{2\sin^2\theta}{r^3}(-2) + \frac{\sin\theta}{r^2}\partial_\theta\left(\frac{1}{\cancel{\sin\theta}}\left(-4\frac{\cancel{\sin\theta}\cos\theta}{r^2}\right)\right) = 0$$

Mistake some where.

$$\Rightarrow f^{IV}\sin^{\cancel{2}}\theta \overset{-}{\cancel{+}} 2\frac{\cancel{\sin}^{\cancel{2}}\theta}{r^2}f'' + 4\frac{\cancel{\sin}^{\cancel{2}}\theta}{r^3}$$

$$+ 4\frac{\cancel{\sin}^{\cancel{2}}\theta}{r^4} = 0$$

$$\Rightarrow f^{IV} - \frac{f^4}{r^2} + \frac{4}{r^3} + \frac{4}{r^4} = 0$$

<u>Check</u>: w/ pg 224 Acheson.

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2f}{dr^2} - \frac{2f}{r^2}\right) = 0$$

$$\Rightarrow \quad \frac{d^4f}{dr^2} - 2\frac{d^2}{dr^2}\left(\frac{f}{r^2}\right) - \frac{2}{r^2}\frac{d^2f}{dr^2} + \frac{4f}{r^4} = 0$$

$$\|$$

$$\frac{d}{dr}\left[\frac{f'}{r^2} - \frac{2f}{r^3}\right]$$

$$\|$$

$$\frac{f''}{r^2} - \frac{f'}{r^3} - \frac{2f'}{r^3} - \frac{2(-3)f}{r^4}$$

$$\Rightarrow \quad f^{IV} - \frac{2f''}{r^2} + \frac{2f'}{r^3} + \frac{4f'}{r^3} - \frac{12f}{r^4} - \frac{2}{r^2}f'' + \frac{4}{r^4}f = 0$$

$$\Rightarrow \quad f^{IV} - \frac{4f''}{r^2} + \frac{6f'}{r^3} - \frac{8}{r^4}f = 0 \qquad \times r^4$$

$$r^4 f^{IV} - 4r^2 f'' + 6r f' \cancel{+} - 8 \cancel{r^4} f = 0$$

let $f = r^\alpha$

$$\alpha(\alpha-1)(\alpha-2)(\alpha-3) - 4\alpha(\alpha-1) + 6\alpha - 8 = 0$$

Check $\alpha = 1$ $\quad -1(-2) - 2 = 0$ ✓.

This factors into

$$[(\alpha-2)(\alpha-3)-2][\alpha(\alpha-1)-2] = 0$$

& $\therefore$ $\alpha = -1, 1, 2, 4$

$$\therefore \; f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4$$

If we are to have $f(r \to +\infty) \sim \frac{U_0 r^2}{2}$

$$\Rightarrow \quad D = 0 \quad \& \quad C = \frac{U_0}{2}$$

Also $v_\theta(r=R) = 0$ & $v_r(r=R) = 0$ sphere is streamline of flow

$$\Rightarrow \quad \frac{A}{R} + BR + \frac{U_0 R^2}{2} = 0$$

$$\& \quad -\frac{A}{R^2} + B + U_0 R = 0$$

$$\Rightarrow \quad A = \frac{U_0 R^3}{4} \quad \& \quad B = -\frac{3aU_0}{4}$$

The Stokes solution is thus

$$\psi = \frac{1}{4}U_0\left(2r^2 + \frac{R^3}{r} - 3Rr\right)\sin^2\theta$$

w/ R = Radius of sphere

Then flow field is

$$U = \frac{\psi_\theta}{r^2\sin\theta} \qquad V = \frac{-\psi_r}{r\sin\theta}$$

$$\Rightarrow \psi_\theta = \frac{U_0}{2}\left(2r^2 + \frac{R^3}{r} - 3Rr\right)\sin\theta\cos\theta$$

$$\psi_r = \frac{U_0}{4}\left(4r - \frac{R^3}{r^2} - 3R\right)\sin^2\theta$$

$$\therefore U = \frac{U_0}{2}\left(2 + \frac{R^3}{r^3} - \frac{3R}{r}\right)\cos\theta = \frac{U_0}{2}\left(2 + \frac{R^3}{r^3} - \frac{3R}{r}\right)\frac{x}{r}$$

$$= \frac{U_0}{2}\left(\frac{2}{r} + \frac{R^3}{r^4} - \frac{3R}{r^2}\right)x$$

$$\& \; V = -\frac{U_0}{4}\left(4 - \frac{R^3}{r^3} - \frac{3R}{r}\right)\sin\theta = -\frac{U_0}{4}\left(4 - \frac{R^3}{r^3} - \frac{3R}{r}\right)\frac{y}{r}$$

$$= -\frac{U_0}{4}\left(\frac{4}{r} - \frac{R^3}{r^4} - \frac{3R}{r^2}\right)y$$

pg 268 Acheson

$$U\frac{\partial U}{\partial x} \sim \frac{U_0 U_0}{L}$$

$$V\frac{\partial U}{\partial y} \sim \frac{U_0\delta}{L}\cdot\frac{U_0}{\delta} = \frac{U_0^2}{L}.$$

keeping viscous term $\nu\frac{U_0}{\delta^2} \sim \frac{U_0^2}{L}$

$$\frac{\delta^2}{L} \sim \frac{\nu}{U_0}$$

$$\frac{\delta^2}{L^2} \sim \frac{\nu}{LU_0} \Rightarrow \left(\frac{\delta}{L}\right) \sim R^{-1/2}$$

---

solve $\epsilon u'' + u' = 1$ $\quad u(0)=0 \quad u(1)=2$ w/ B.L. theory

$p(x) = 1 > 0$ $\quad\therefore$ B.L at $x=0$.

outer sol: $\quad u' = 1$

$u = y + C$

$u(1) = 1 + C = 2 \Rightarrow C = 1$

$u_{outer}(y) = 1 + y.$

inner sol $\quad \epsilon u'' \cancel{+ u' = } 1$

Homo: $\quad \cancel{\epsilon u'' +} u' = 0$

$\Rightarrow \epsilon u' + u = \cancel{c_1}$

$u' + \cancel{\frac{1}{\epsilon}u} = \frac{c_1}{\epsilon}$

Do ~~scali~~ scaling + look at relative terms in terms of $\delta = y/\epsilon$

$\Rightarrow \frac{d}{dy}(\cancel{e^{y/\epsilon}}u) = \frac{c_1}{\epsilon}\cancel{e^{y/\epsilon}} \Rightarrow \cancel{e^{y/\epsilon}u(y)} = \frac{c_1}{\epsilon}\cancel{e^{y/\epsilon}} + c_2$

$$u(y) = \cancel{y + C_1 + C_2 e^{y/\epsilon}}$$

let ~~[illegible]~~ $\bar{Y} = y/\delta$

$$\frac{d}{dy} = \frac{1}{\delta}\frac{d}{dY}$$

$$\frac{d^2}{dy^2} = \frac{1}{\delta^2}\frac{d^2}{dY^2}$$

$$\frac{\epsilon}{\delta^2}\frac{d^2u}{dY^2} + \frac{1}{\delta}\frac{du}{dY} = 1$$

~~$\delta = \sqrt{\epsilon}$~~

let $\delta = \epsilon$

(**) $$\frac{1}{\epsilon}\frac{d^2u}{dY^2} + \frac{1}{\epsilon}\frac{du}{dY} = 1$$

$$\epsilon\delta^2 \frac{d^2u}{dy^2} + \delta\frac{du}{dy} = \delta^2$$

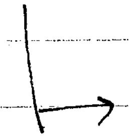
$\epsilon\delta^2 \sim \delta$ if $\frac{\epsilon}{\delta^2}\frac{d^2u}{dY^2} + \frac{du}{dY} =$

~~[illegible]~~ ~~[illegible]~~ idea that $\delta = \sqrt{\epsilon}$.

1st 2 ~~[illegible]~~ derivative terms must ~~[illegible]~~ balance

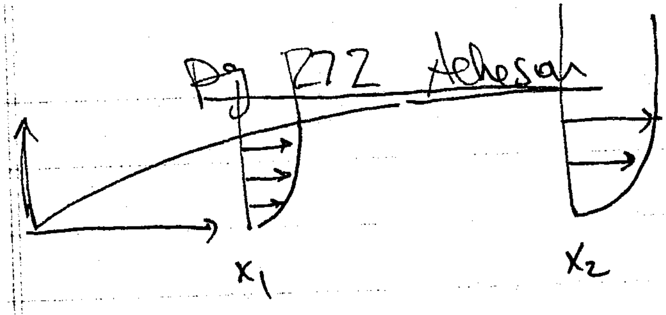
Solving **

$$u = A(1 - e^{-Y})$$



$$\lim_{Y\to+\infty} u = A = \lim_{y\to 0} u = 1$$

Pg 272 Acheson



$U = f(\eta)$ $\quad \eta = y/g(x)$ $\quad$ Assuming $g(x_1) < g(x_2)$

$U = \frac{\partial f}{\partial y}$ $\quad v = -\frac{\partial f}{\partial x}$.

let $u = U h(\eta) = \frac{\partial f}{\partial y}$

$f = U \int h(\eta)\, dy + k(x)$; $\eta = y/g(x)$

$ = U g(x) \int_0^\eta h(s)\, ds + k(x)$; $\quad dy = g(x)\, d\eta$

$f = 0$ At ~~x=0~~. $\eta = 0$ $\quad$ if plate is to be a streamline

$\Rightarrow k(x) = 0$.

$f = U g(x) f(\eta)$

$$u = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \eta} \cdot \frac{1}{g(x)} = U f'(\eta)$$

$$v = -\frac{\partial f}{\partial x} = -\left(U g'(x) f(\eta) + U g(x) f'(\eta) \left(\frac{-y}{g(x)^2} \right) g'(x) \right)$$

$$= \cancel{-Uf\left(g' f - y f g' \right.} \Big)$$

$u = f(\eta)$ $\eta = y/g(x)$ Assuming $g(x_1) < g(x_2)$

$u = \frac{\partial f}{\partial y}$ $v = -\frac{\partial f}{\partial x}$.

let $u = U h(\eta) = \frac{\partial f}{\partial y}$

$f = U \int h(\eta)\,dy + k(x)$; $\eta = y/g(x)$

$dy = g(x)\,d\eta$

$= U g(x) \int_0^{\eta} h(s)\,ds + k(x)$;

$f = 0$ At ~~zero~~. $\eta = 0$ if plate is to be a streamline

$\Rightarrow k(x) = 0$.

$f = U g(x) f(\eta)$

$u = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial \eta} \frac{1}{g(x)} = U f'(\eta)$

~~$v = -\frac{\partial f}{\partial x} = -\left(U g'(x) f(\eta) + U g(x) f'(\eta) \left(\frac{-y}{g(x)^2}\right) g'(x)\right)$~~

$= -U\left(g' f - \frac{y}{g} f' g'\right)$

$= U(\eta f' - f) g'$ put into 8.12.

4 a

$$U f'(\eta) \frac{\partial}{\partial x}\big(U f'(\eta)\big) + U(\eta f' - f) g' \frac{\partial}{\partial y}\big(U f'(\eta)\big)$$

$$= \nu \frac{\partial^2}{\partial y^2}\big(U f'(\eta)\big)$$

$$\Rightarrow U f'(\eta)\, U f''(\eta)\left(\frac{-y}{g^2} g'\right) + U(\eta f' - f) g' U f'' \frac{1}{g(x)}$$

$$= \nu \frac{d}{dy}\, U f''(\eta) \frac{1}{g}$$

$$= \nu \frac{U}{g^2} f'''(\eta)$$

$$\Rightarrow -U^{\cancel{2}} \frac{y}{g^{\cancel{2}}} g' f' f'' + U^{\cancel{2}}(\eta f' - f) g' \frac{f'}{\cancel{g}} = \nu \cancel{U} \frac{f'''}{g^{2}}$$

$$\cancel{-U \eta g' f' f''} + \cancel{U \eta f' f'' g'} - U f g' f'' = \frac{\nu f'''}{g}$$

$$f''' + \frac{U g g'}{\nu} f f'' = 0$$

$$g g' = \frac{\nu}{U} \quad \Rightarrow \quad \frac{g^2}{2} = \frac{\nu}{U} x + d$$

want g to become singular at $x = 0 \Rightarrow d = 0$.

$$^{1/2}$$

~~$= \nu \frac{\partial^2 (U f'(\eta))}{\partial y^2}$~~

~~$\Rightarrow U f'(\eta)\, U f''(\eta) \left(\frac{-y}{g^2} g'\right) + U(\eta f' - f) g'\, U f'' \frac{1}{g(x)}$~~

~~$= \nu \frac{d}{dy}\, U f''(\eta) \frac{1}{g}$~~

~~$= \nu \frac{U f'''(\eta)}{g^2}$~~

~~$\Rightarrow -U^2 \frac{y}{g^2} g' f' f'' + U^2 (\eta f' - f) g' \frac{f''}{g} = \nu \cancel{U} \frac{f'''}{g^2}$~~

~~$-U \eta g' f' f'' + U \eta f' f'' g' - U f g' f'' = \frac{\nu f'''}{g}$~~

~~$f''' + \frac{U g g'}{\nu} f f'' = 0$~~

$$\underline{g g' = \frac{\nu}{U} \quad \Rightarrow \quad \frac{g^2}{2} = \frac{\nu x}{U} + d}$$

want g to become singular at $x=0 \Rightarrow d=0$.

$$g = \left(\frac{2\nu x}{U}\right)^{1/2}$$

$f''' + f f'' = 0$. + B.C. @ $u(y=0)=0$
$\Rightarrow f'(0)=0$
$v(y=0) = 0 \Rightarrow f(0) = 0$.

5a ~~6~~ 5

∤ flow at $y \to +\infty \Rightarrow$ $u = U$ constant
$v = 0$.

$u = U \Rightarrow f'(+\infty) = 1. \Rightarrow$ ~~$f(+\infty) \to ?$~~
Also $f(+\infty) \sim \eta \quad \eta \to +\infty$

then $v = U(\eta f' - f)$
~~$\eta(f)$~~

$f''' = -ff'' \qquad f(0) = f'(0) = 0$
$f'(+\infty) = 1$

How solve num?
Solve $f''' = -ff''$
$f(0) = f'(0) = 0$
$f''(0) = t_k$

∤ want st $f(x, t_k) \ni \lim_{k \to +\infty} f(x, t_k) = f(+\infty) = 1$

BL thickness $\sim O\left(\left(\frac{\nu x}{U}\right)^{1/2}\right)$

$t_x = T_{\cancel{x}j} \overset{(0,1,0)}{n_j} = T_{x2} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Big|_{y=0}$

$v = U(\eta f' - f)g'$
$\frac{\partial v}{\partial x} = U(-y/g^2 g')f' + \eta f''(-y/g^2 g') - f'(-y/g^2 g'))g'$
$+ U(\eta f' - f)g''$

$U = U \Rightarrow f'(+\infty) = 1.$ ⇒ ~~$f(+\infty) \to ?$~~

Also $f(+\infty) \sim \eta$ $\eta \to +\infty$

~~then $v = U(\eta f' - f)$~~

~~$f''' = -ff''$~~ ~~$f(0) = f'(0) = 0$~~

~~$f'(+\infty) = 1$~~

~~How solve now?~~

~~Solve $f''' = -ff''$~~

~~$f(0) = f'(0) = 0$~~

~~$f''(0) = t_k$~~

~~+ want st $f(x,t_k) \Rightarrow \lim_{k\to+\infty} f(x,t_k) = f(+\infty) = 1$~~

~~BL thickness $\sim O\left(\left(\frac{\nu x}{U}\right)^{1/2}\right)$~~

$t_x = T_{xj}\overset{(0,1,0)}{n_j} = T_{x2} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\Big|_{y=0}$

$v = U(\eta f' - f)g'$

$\frac{\partial v}{\partial x} = U(-y/g^2 g')f' + \eta f''(-y/g^2 g') - f'(-y/g^2 g))g'$

$+ U(\eta f' - f)g''$

$\frac{\partial v}{\partial x}\Big|_{y=0} = U(0 + 0 + 0) + 0 = 0$

$\frac{\partial u}{\partial y}\Big|_{y=0} = U\frac{f''(\eta)}{g(x)}\Big|_{y=0} = Uf''(0)\left(\frac{U}{2\nu x}\right)^{1/2}$ decreases w/ x.

Pg 275 Acheson

$\bar{u} = u_r(r,\theta)\,\hat{e}_r$ polar

$$\nabla\circ\bar{u} = \text{[illegible]}\ \cancel{\frac{\partial u_r}{\partial r}} + \cancelto{0}{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}} + \cancelto{0}{\frac{\partial u_z}{\partial z}} = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = 0$$

$$\frac{\partial}{\partial r}(r u_r) = 0$$

$$\Rightarrow \quad r u_r = C(\theta, z)$$

$$u_r = \frac{C(\theta, z)}{r}$$ told $\bar{u} = u_r(r,\theta)\hat{e}_r$ w/o z depend

$$= \frac{C(\theta)}{r}$$

N-S

$$\cancelto{0}{\frac{\partial \bar{u}}{\partial t}} + (\bar{u}\circ\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \cancel{\nu\nabla^2\bar{u}}.$$

Not there
for BL theory
1st solve inviscid eqs, back
& then put viscosity
in

$$\|$$

$$u_r\frac{\partial}{\partial r}\bar{u} = -\frac{1}{\rho}\nabla p + \nu\cancelto{0}{\nabla^2} u_r$$

$\hat{r}$

$$u_r\frac{\partial u_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\cancelto{0}{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u_r} + \cancelto{0}{\frac{\partial^2}{\partial z^2}u_z}\right]$$

$$\left.\begin{aligned} 0 &= -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right] \quad p = p(r)$$

Assuming no pressure gradient i.e $p' = 0$.

$$\Rightarrow \quad \frac{C(\theta)}{\cancel{r}}\frac{\partial}{\partial r}\left(\frac{C(\theta)}{r}\right) = \frac{\nu}{\cancel{r}}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\frac{C(\theta)}{r}\right) + \frac{\nu}{r^2}\frac{\partial^2}{\partial\theta^2}\frac{C(\theta)}{r}$$

$$C^2\left(-\frac{1}{\cancel{r^2}}\right) = C\nu\frac{d}{dr}\left[r\underbrace{\left(\frac{-1}{r^2}\right)}_{-\frac{1}{r}}\right] + \frac{C''\nu}{r^2}$$

$$= C\nu\left(\frac{1}{\cancel{r^2}}\right) + \frac{C''\nu}{\cancel{r^2}}$$

$$C'' + C + \frac{C^2}{\nu} = 0$$

$$\frac{C^2(\theta)}{r}\left(\frac{-1}{r^2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

But from $\hat{\theta}$ eq $\frac{\partial p}{\partial \theta} = 0$.

$\Rightarrow$ p ind of θ. $\Rightarrow$ C cannot have θ dependence

$\Rightarrow$ C = const.

$$\frac{\partial p}{\partial r} = \frac{C^2\rho}{r^3} \qquad p = p_0 + \left(-\frac{1}{2}\right)\frac{C^2\rho}{r^2}$$

$$p = p_0 - \frac{C^2\rho}{2r^2} \qquad *$$

get then that $\bar{U} = -\frac{Q}{r}\hat{e}_r$.

in B.L use Bernoulli to get P.

$$p + \frac{1}{2}\rho U^2 = \text{const}$$

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx}$$

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}$$

At $y=0 \quad U = -\frac{Q}{x}$

$$\Rightarrow \quad -\frac{1}{\rho}\frac{dp}{dx} = -\frac{Q}{x}\,\frac{Q}{x^2} = -\frac{Q^2}{x^3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{Q^2}{x^3} + \nu\frac{\partial^2 u}{\partial y^2}$$

To satisfy $\nabla \cdot \bar{u} = 0$ in B.L introduce a stream fn

$$f \ni u = \frac{\partial f}{\partial y} \qquad v = -\frac{\partial f}{\partial x}$$

Then $\nabla \cdot \bar{u} = 0$ auto...

let $u = \left(\frac{-Q}{x}\right)h(\eta)$ main stream flow mult by fn that will modulate it.

w/ $\eta = O(y/g(x))$

Then $$f = \int_0^y -\frac{|Q|}{x} h(\eta)\, dy + k(x)$$

$$\eta = y/g(x) \qquad dy = g(x)\, d\eta$$

$$= \int_0^\eta -\frac{|Q|}{x} h(\eta) g(x)\, d\eta + k$$

$$= \cancel{\frac{|Q|}{x} g(x) \int_0^\eta h(s)\, d\eta} \quad -\frac{|Q|}{x} g(x) \int_0^\eta h(s)\, ds + k(x)$$

At $y=0$ $f=0$ $\Rightarrow$ $k(x)=0$.

Let $f = -\frac{|Q|}{x} g(x) f(\eta)$ $\qquad f(0)=0.$ $\quad y=0$ is streamline.

Then $U = -\frac{|Q|}{x} \frac{g(x) f'}{g} = -\frac{|Q|}{x} f'$ ✓ $\quad f'(0)=0.$ U vanishes on plate

To match w/ mainstream flow $\quad U \xrightarrow[\eta \to +\infty]{\cancel{y/g(x)} \to +\infty} -\frac{|Q|}{x}$

$$\Rightarrow f'(+\infty) = \begin{cases} 1 & \text{inflow} \\ -1 & \text{outflow.} \end{cases}$$

$$V = -\frac{\partial}{\partial x}\left(-\frac{|Q|}{x} g(x) f(\eta) \right) = -\frac{|Q|}{x^2} g(x) f(\eta) + \frac{|Q|}{x} g'(x) f(\eta)$$

$$+ \frac{|Q|}{x} g f' \frac{y}{g^2}(-g')$$

$$= -\frac{|Q|}{x^2} g f + \frac{|Q|}{x} g' f - \frac{|Q|}{x} \frac{\eta g'}{g} f'$$

~~Pt~~ ~~even~~ BC $v(y=0)=0$ Sat ✓.

Pt everything in $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{Q^2}{x^3}+\nu\frac{\partial^2 u}{\partial y^2}$

1st $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[-\frac{|Q|}{x}\frac{f''}{g}\right] = -\frac{|Q|}{xg^2}f'''$ ✓

$\Rightarrow$ + $\frac{\partial u}{\partial x} = +\frac{|Q|}{x^2}f' - \frac{|Q|}{x}f''\frac{y}{g^2}(-1)g' = \frac{|Q|}{x^2}f' + \frac{|Q|}{x}\frac{f''}{g^2}\eta g'$

$\frac{\partial u}{\partial y} = -\frac{|Q|}{x}\frac{f''}{g}$ put in

$$\Rightarrow -\frac{|Q|}{x}f'\left(\frac{|Q|}{x^2}f' + \frac{|Q|}{x}\frac{f''}{g^2}\eta g'\right) + -\frac{|Q|}{x}\frac{f''}{g}\left(-\frac{|Q|}{x^2}gf + \frac{|Q|}{x}g'f - \frac{|Q|}{x}\eta\frac{g'}{g}f'\right)$$

$$= -\frac{Q^2}{x^3} + -\nu\frac{|Q|}{xg^2}f'''$$

$$\Rightarrow -\frac{Q^2}{x^3}f'^2 - \cancel{\frac{Q^2}{x^2}\frac{f'f''}{g}\eta g'} + \frac{Q^2}{x^3}ff'' - \frac{Q^2}{x^2}\frac{ff''}{g}g' + \cancel{\frac{Q^2}{x^2}\frac{f''\eta g'f'}{g}}$$

$$= -\frac{Q^2}{x^3} - \frac{\nu|Q|}{xg^2}f'''$$

$$\Rightarrow -\frac{Q^2f'^2}{x^3} - \cancel{\frac{Q^2}{x^2}\frac{f''f'\eta g'}{g^2}} + \frac{Q^2}{x^3}f''f - \frac{Q^2f''fg'}{x^2g} + \cancel{\frac{Q^2f''f'\eta g'}{x^2g^2}}$$

$$= -\frac{Q^2}{x^3} - \frac{\nu|Q|f'''}{xg^2}$$

Mult by $\frac{xg^2}{\nu|Q|}$

$$\frac{Q^2}{|Q|} = \begin{cases} \frac{Q^2}{Q} & Q>0 \\ \frac{Q^2}{-Q} & Q<0 \end{cases} = \begin{cases} Q & Q>0 \\ -Q & Q<0 \end{cases} = \{|Q|.$$

$$\Rightarrow -f''' - \frac{Q^2 g^2 \cancel{\beta}}{|Q|\nu x^2} = -\frac{|Q|f'^2}{\nu x^2}g^2 + \frac{|Q|ff''}{x^2\nu}g^2$$

$$-\frac{|Q|ff''g'g}{x\nu}$$

$$\Rightarrow +f''' + \left(\frac{|Q|g^2}{\nu x^2}\right) = \cancel{\left(\frac{|Q|g^2}{\nu x^2}\right)f'^2 + \left(\frac{|Q|g'}{x\nu}\right)ff'' + \left(\frac{|Q|g^2}{x^2\nu}\right)ff''}$$

$$+\cancel{\left(\frac{|Q|g'}{x\nu}\right)\left[ff'' - \left(\frac{g}{x}\right)\right]}$$

$$\cancel{\left(\frac{|Q|g^2}{\nu x^2}\right)f'' + \left(\frac{|Q|g}{x\nu}\right)\left[1 - \left(\frac{g}{x}\right)\right]}ff''$$

$$= \left(\frac{|Q|g^2}{\nu x^2}\right)f'^2 + \left(\frac{|Q|g'g}{x\nu}\right)ff'' - \left(\frac{|Q|g^2}{\nu x^2}\right)ff''$$

3 terms like $\cancel{\frac{|Q|g'g}{x\nu}} = 1$ $\quad \frac{|Q|g^2}{\nu x^2} = 1$

$\cancel{gg' = \frac{x\nu}{|Q|}}$ $\quad \Rightarrow g = \left(\frac{\nu}{|Q|}\right)^{1/2}x$.

$\cancel{g^2 = \frac{x^2\nu}{2|Q|}} + C_1$

$\Rightarrow$ ~~want g to be singular at $x=0$~~ ∴ pick

~~$C_1 = 0$~~ $\quad g = x\cdot\left(\frac{\nu}{|Q|}\right)^{1/2}$.

Then $\frac{|\alpha| g' g}{x\nu} = \frac{\cancel{|\alpha|}(\cancel{\nu}/\cancel{|\alpha|})^{1/2} (\cancel{\nu}/\cancel{|\alpha|})^{1/2} \cancel{x}}{\cancel{x}\cancel{\nu}} \equiv 1$ also ✓.

$$\Rightarrow \quad f''' + 1 = f'^2 \qquad \text{BC} \quad \begin{array}{l} f(0) = 0 \\ f'(0) = 0 \\ f'(\infty) = \begin{cases} 1 & \text{inflow} \\ -1 & \text{outflow.} \end{cases} \end{array}$$

$$\Rightarrow \quad F'' + 1 - F^2 = 0.$$

$$\frac{F'^2}{2} + F - \frac{1}{3}F^3 = \text{const}$$

$$1 - \tfrac{1}{3} = \text{const} \qquad \text{At } y = +\infty$$

$$\Rightarrow \quad F'^2 = \frac{2}{3} - F + \frac{1}{3}F^3$$

Pg 277 Acheson

$$F'^2 = \frac{4}{3} - 2F + \frac{2}{3}F^3$$

Check $\frac{2}{3}(1-F)^2(2+F)$

$$= \frac{2}{3}(1-2F+F^2)(2+F)$$

$$= \frac{2}{3}(2+F-4F-\cancel{2F^2}+\cancel{2F^2}+F^3)$$

$$= \frac{4}{3}\cancel{4} - 2F + \frac{2}{3}F^3. \quad ✓.$$

$$1-F = 1-(G^2-2) = 3-G^2 \qquad F' = 2GG'$$

$$\frac{\cancel{2}}{3}(3-G^2)^2\cancel{G^2} = 2^{\cancel{2}}\cancel{G^2}G'^2$$

$$\frac{1}{6}(3-G^2)^2 = G'^2$$

$$\Rightarrow \quad G' = \pm\frac{1}{\sqrt{6}}(3-G^2)$$

$$G = \sqrt{3}\tanh H \qquad G' = \sqrt{3}$$

$$\Rightarrow \frac{\sqrt{3}H'}{\cosh^2 H} = \pm\frac{1}{\sqrt{6}}(3-3\tanh^2 H)$$

$$= \pm\frac{\cancel{3}}{\sqrt{6}}\frac{1}{\cosh^2 H}$$

$$1\frac{\cancel{\cosh H}}{\cancel{\cosh H}}H' - \frac{H'\sinh^2 H}{\cosh^2 H}.$$

$$= H'\left[1 - \frac{\sinh^2 H}{\cosh^2 H}\right]$$

$$= H'\frac{1}{\cosh^2 H}$$

$\Rightarrow$

[illegible]

$$H' = \pm \frac{\sqrt{3}}{\sqrt{6}} = \pm \frac{1}{\sqrt{2}}$$

$$H = \pm \frac{\eta}{\sqrt{2}} + c$$

$$G = \sqrt{3} \tanh\left(\pm \frac{\eta}{\sqrt{2}} + c \right)$$

$$F = \cdots \quad -2 + 3 \tanh^2\left(\pm \frac{\eta}{\sqrt{2}} + c \right) \xrightarrow{\eta \to \pm\infty} 1$$

Pg 307 Acheson

$$\rho = \rho(P,T)$$

$$= \rho(P,T_0) + \frac{\partial \rho}{\partial T}\Big|_P (T - T_0)$$

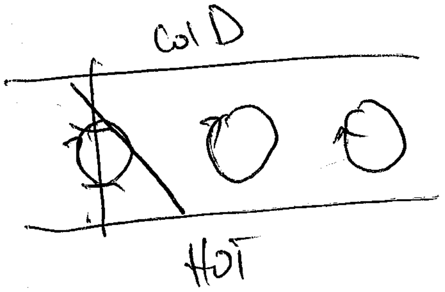
$$\cong \rho(P,T_0)\Big(1 + \underbrace{\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big|_P}_{-\alpha} \Delta T\Big)$$

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}\Big|_P.$$

0 if no time dependence

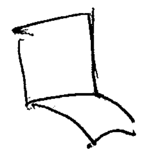
$$\cancel{\partial_t \rho}^{\,0} + \nabla\cdot(\rho\bar{u}) = 0$$

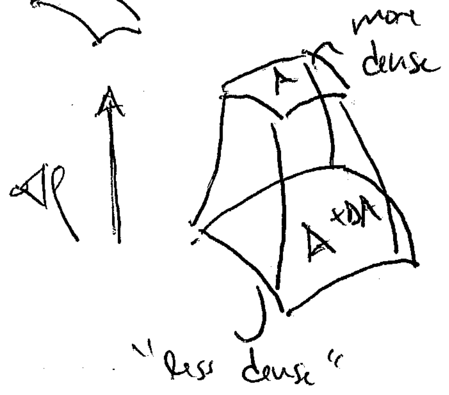
$$\nabla\rho\cdot\bar{u} + \rho\nabla\cdot\bar{u} = 0$$



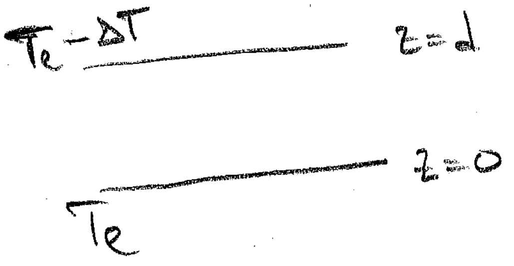
$\therefore$ $\nabla\rho$ = dir increase in ρ

Should be along $\hat{y}$.





~~T(z) =~~ $$0 = k\frac{d^2 T_0(z)}{dz^2}$$



$$T(z) = A + Bz$$

$$T(0) = A = T_e$$

$$T(d) = \cancel{T_e} + Bd = \cancel{T_e} - \Delta T$$

$$B = -\Delta T/d$$

$$T_0(z) = T_e - \frac{\Delta T}{d} z$$

Now $$\rho(z) = \bar{\rho}\left(1 - \alpha\left(T_0(z) - \bar{T}\right)\right)$$

$$= \bar{\rho}\left(1 - \alpha\left(T_e - \frac{\Delta T}{d} z - \bar{T}\right)\right)$$

$$= \bar{\rho}\left(1 - \alpha T_e + \alpha \bar{T} + \alpha \frac{\Delta T}{d} z\right)$$

$$= \bar{\rho}\left(1 + \alpha(\bar{T} - T_e) + \alpha \frac{\Delta T}{d} z\right)$$

Now presur ⇒ $$+\nabla p = -g\rho \hat{z} = -g\rho(z)\hat{z}$$

p ind of x & y

$$\frac{\partial p}{\partial z} = -g\bar{\rho}\left(1 + \alpha(\bar{T} - T_e) + \alpha \frac{\Delta T}{d} z\right)$$

$$P = P_0 - g\bar{\rho}\left(1 + \alpha(\bar{T} - T_e)\right)z - g\bar{\rho}\frac{\alpha}{d}\Delta T \frac{z^2}{2}$$

let $T = T_0(z) + T_1$ $\quad \bar{u} = \bar{u}_1$

$\rho = \rho_0(z) + \rho_1$

$p = p_0(z) + p_1$

get for $\nabla \circ \bar{u} = 0$

$\Rightarrow \nabla \circ \bar{u}_1 = 0$ cont.

<u>N-S</u>: $\rho\left(\frac{\partial \bar{u}}{\partial t} + (\bar{u}\circ\nabla)\bar{u}\right) = -\nabla p + \rho\nu\nabla^2\bar{u} + \rho\bar{g}$

$$\Rightarrow (\rho_0+\rho_1)\left(\frac{\partial \bar{u}_1}{\partial t} + \overset{O(\|u\|^2)}{\nearrow}\right) = -\nabla p_0 - \nabla p_1 + \rho_0\nu\nabla^2\bar{u}_1 + \rho_0\bar{g} + \rho_1\bar{g}$$

$$\Rightarrow \rho_0\frac{\partial \bar{u}_1}{\partial t} = \underline{\cancel{-\nabla p_0}} - \nabla p_1 + \rho_0\nu\nabla^2\bar{u}_1 + \cancel{\rho_0\bar{g}} + \rho_1\bar{g}$$

By order 1 eq

$$\Rightarrow \rho_0\frac{\partial \bar{u}_1}{\partial t} = -\nabla p_1 + \rho_0\nu\nabla^2\bar{u}_1 + \rho_1\bar{g}$$

Energy: $\frac{\partial T_1}{\partial t} + \underset{\|}{\bar{u}_1\circ\nabla T_0} = k\nabla^2 T_1$

$$\frac{\partial T_1}{\partial t} + w_1\frac{\partial T_0}{\partial z} = k\nabla^2 T_1$$

Boussinesq

$$\rho_0 + \rho_1 = \bar{\rho}\left(1 - \alpha(T_0 + T_1)\right)$$

$$\cancel{\rho_0} + \rho_1 = \bar{\rho}\cancel{(1 - \alpha T_0)} - \alpha\bar{\rho}T_1$$

$$\rho_1 = -\alpha\bar{\rho}T_1$$

Now in N-S eq replace $\rho_0(z) \longrightarrow \bar{\rho}$

Take $\nabla\times \longrightarrow$ 9.19

$$\cancel{\bar{\rho}}\frac{\partial}{\partial t}(\nabla\times\bar{v}_1) = \cancel{\bar{\rho}}\nu\nabla^2(\nabla\times\bar{v}_1) + \nabla\times(\rho_1\bar{g})$$

$$\Vert$$

$$\nabla\times(-\alpha\cancel{\bar{\rho}}T_1\bar{g})$$

$$\Rightarrow \left(\frac{\partial}{\partial t} - \nu\nabla^2\right)(\nabla\times\bar{v}_1) = -\alpha\nabla\times(T_1\bar{g})$$

$$\Vert$$

$$-\alpha\left(\nabla T_1\times\bar{g} + T_1\nabla\times\bar{g}\right)$$

$$\nabla\times\bar{g} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ & & \end{vmatrix} = 0$$ As $\bar{g}$ is constant vector

$$= -\alpha(\nabla T_1\times\bar{g})$$

$$\nabla\times(\nabla\times\bar{v}_1) = \nabla(\cancelto{0}{\nabla\circ\bar{v}_1}) - \nabla^2\bar{v}_1$$

But also

$$\nabla\times(\nabla T_1\times\bar{g}) = (\bar{g}\circ\nabla)\nabla T_1 - \cancelto{0}{(\nabla T_1\circ\nabla)}\bar{g} + \nabla T_1\cancelto{0}{(\nabla\circ\bar{g})}$$

$$-\bar{g}(\nabla\circ\nabla T_1)$$

$$=$$

$$\therefore\ \left(\frac{\partial}{\partial t}-\nu\nabla^2\right)(\nabla^2\bar{v}_1) = \alpha\left(\underset{\underset{-g\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}T_1\right)}{\parallel_{z\text{ comp}}}}{(\bar{g}\circ\nabla)\nabla T_1} - \underset{-\bar{g}\,\nabla^2 T_1}{\bar{g}(\nabla\circ\nabla T_1)}\right)$$

Take z comp

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2 w_1 = \alpha\left(-g\frac{\partial^2}{\partial z^2}T_1 + g\nabla^2 T_1\right)$$

$$= \alpha g\left(\frac{\partial^2}{\partial x^2}T_1 + \frac{\partial^2}{\partial y^2}T_1\right)$$

But ~~$\frac{\partial^2}{\partial x^2}T_1 + \frac{\partial^2}{\partial y^2}T_1$~~ = from $\frac{\partial T_1}{\partial t} - k\nabla^2 T_1 = -w_1\frac{dT_0}{dz}$

or $\left(\frac{\partial}{\partial t} - k\nabla^2\right)T_1 = -w_1\frac{dT_0}{dz}$ $\therefore$ hit Above w/ $(\partial_t - k\nabla^2)\rightarrow$

$$\Rightarrow \quad (\partial_t - \nu\nabla^2)(\partial_t - k\nabla^2)\nabla^2 w_1 = -\alpha g \frac{dT_0}{dz}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_1$$

Let $w_1 = W(z) f(x,y) e^{st}$ put in

Now $\nabla^2 w_1 = (fW'' + \nabla_\perp^2 f \cdot W)e^{st}$

$$\nabla_\perp^2 w_1 = W \nabla_\perp^2 f$$

Then $\nabla^2(\nabla^2 w_1) =$

$$(\partial_t - \nu\nabla_\perp^2 - \nu\partial_z^2)(\partial_t - k\nabla_\perp^2 - k\partial_z^2)(\nabla_\perp^2 + \partial_z^2) w_1$$

$$= -\alpha g \frac{dT_0}{dz} \nabla_\perp^2 w_1$$

$$\Rightarrow \quad (\partial_t - \nu\nabla_\perp^2 - \nu\partial_z^2)(\partial_t\nabla_\perp^2 + \partial_t\partial_z^2 - k\nabla_\perp^2\nabla_\perp^2$$

$$- k\nabla_\perp^2\partial_z^2 - k\partial_z^2\nabla_\perp^2 - k\partial_z^4) w_1 = -\alpha g \frac{dT_0}{dz}\nabla_\perp^2 w_1$$

$$\Rightarrow \quad \partial_t^2\nabla_\perp^2 + \partial_t^2\partial_z^2 - k\partial_t\nabla_\perp^2\nabla_\perp^2 - k\partial_t\nabla_\perp^2\partial_z^2 - k\partial_t\partial_z^2\nabla_\perp^2$$

$$- k\partial_t\partial_z^4 - \nu\partial_t\nabla_\perp^2\nabla_\perp^2 - \nu\partial_t\partial_z^2\nabla_\perp^2 + \nu k\nabla_\perp^2\nabla_\perp^2\nabla_\perp^2$$

$$(\partial_t - \nu\nabla_\perp^2 - \nu\partial_z^2)(\partial_t\nabla_\perp^2 + \partial_t\partial_z^2 - k\nabla_\perp^2\nabla_\perp^2 - 2k\nabla_\perp^2\partial_z^2 - k\partial_z^4)w_1 = -\alpha g \frac{dT_0}{dz}\nabla_\perp^2 w_1$$

$$\Rightarrow \Big(\partial_t^2\nabla_\perp^2 + \partial_t^2\partial_z^2 - k\partial_t\nabla_\perp^2\nabla_\perp^2 - 2k\partial_t\nabla_\perp^2\partial_z^2 - k\partial_t\partial_z^4$$

$$- \nu\partial_t\nabla_\perp^2\nabla_\perp^2 - \nu\partial_t\partial_z^2\nabla_\perp^2 + \nu k(\nabla_\perp^2)^3 + 2\nu k(\nabla_\perp^2)^2\partial_z^2$$

$$+ \nu k\nabla_\perp^2\partial_z^4 - \nu\partial_z^2\partial_t\nabla_\perp^2 - \nu\partial_t\partial_z^4 + \nu k\partial_z^2\nabla_\perp^2\nabla_\perp^2$$

$$+ 2\nu k\nabla_\perp^2\partial_z^4 + \nu k\partial_z^6\Big)w_1$$

$\Rightarrow$