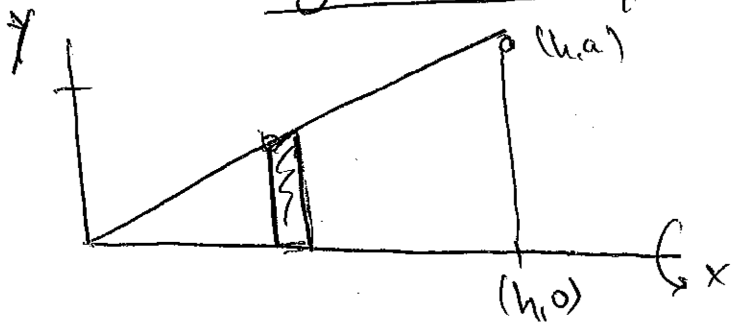
Pg 359 Boyce / DiPrima

(4)



$$\frac{y-a}{x-h} = \frac{a-0}{h-0}$$

$$y = a + \frac{a}{h}(x-h)$$

$$dV = 2\pi y(x)\, dx$$

$$V = 2\pi \int_0^h \left(a + \frac{a}{h}(x-h)\right) dx = 2\pi\left(ah + \frac{a}{h}\frac{(x-h)^2}{2}\right)\Bigg|_0^h$$

$$= 2\pi\left(ah + \frac{a}{2\cancel{h}}\left(0 - h^{\cancel{2}}\right)\right) = 2\pi a\left(h - \frac{h}{2}\right) = 2\pi \frac{ah}{2}$$

$$= \pi a h \overset{?}{=} \frac{1}{3}(\pi a^2) h$$

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if $a = 1/\sqrt{2}$ + $h = 1/\sqrt{2}$

$$V = \frac{\pi}{2}$$

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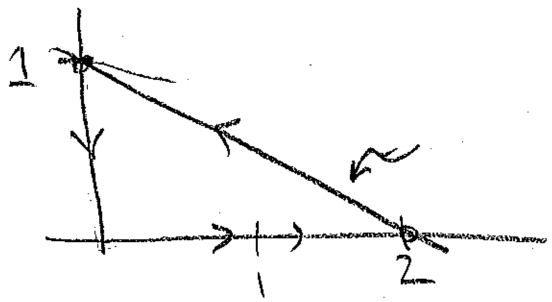

$$y - 0 = \left(\frac{\Delta y}{\Delta x}\right)(x-2)$$

$$y = \frac{-1}{2}(x-2) = -\frac{(x-2)}{2}$$

What is the connection between greens thm + complex analysis?

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(1) $\vec{F} = y\hat{i} + x^2\hat{j}$ $\quad$ C circle $x^2+y^2=a^2$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\Vert$$

on C let
$x(t) = a\cos t$
$y(t) = a\sin t$
$dx = -a\cancel{\cos t}\,dt \quad -a\sin t\,dt$
$dy = a\cos t\,dt$

$$\oint_C (a\sin t)(-a\,\cancel{\cos t\,dt}\ \sin t\,dt) + (a^2\cos^2 t)(a\cos t\,dt)$$

$$= \oint_{t=0}^{2\pi} \left(-a^2\sin^2 t + a^3\cos^3 t\right) dt$$

Finish

$$= -a^2 \underbrace{\frac{\sin^2 t}{2}\Big|_0^{2\pi}}_{0} + a^3\int_0^{2\pi}\cos^3 t\,dt$$

$$= a^3\int_0^{2\pi}(1-\sin^2 t)\cos t\,dt = -a^3\int_0^{2\pi}\sin^2 t\cos t\,dt$$

$v = \cancel{\cos t}\ \sin t$
$dv = +\cancel{\sin t\,dt}\ \cos t\,dt$

$$= -a^3\int_1^1 v^2\,dv = -a^3\,\frac{v^3}{3}\Big|_1^1 = 0.$$

RHS is:

go to polar:

$$\iint_{\Omega} (2x-1)\,dx\,dy$$

$x = r\cos\theta$ $\qquad$ $dx\,dy = r\,dr\,d\theta$

$y = r\sin\theta$

$$= \int_0^a \int_0^{2\pi} (2r\cos\theta - 1)\,r\,dr\,d\theta = \underbrace{\int_0^a \int_0^{2\pi} 2r^2\cos\theta\,dr\,d\theta}_{=0 \text{ as integral over } \cos\theta \text{ is}} - \underbrace{\int_0^a \int_0^{2\pi} r\,dr\,d\theta}$$

; $\left(-\frac{r^2}{2}\right|_0^a \int_0^{2\pi} d\theta$

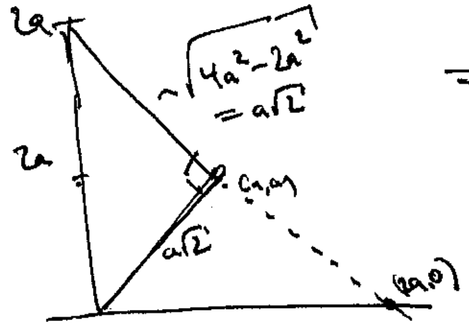
$= -\frac{a^2}{2}(2\pi)$

$= \underline{-a^2\pi}$ ✓

(5) $$\oint_C 2\,dx - 3\,dy \underset{\text{By Greens thm}}{=} \iint_\Omega \underbrace{\left(\frac{\partial(-3)}{\partial x} - \frac{\partial}{\partial y}(2)\right)}_{=0} dA$$

$$= 0.$$

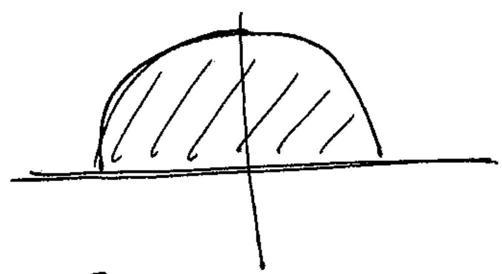
(6) $$\oint_C 2y\,dx - 3x\,dy \underset{\text{By Green}}{=} \iint_\Omega \left(\frac{\partial}{\partial x}(-3x) - \frac{\partial}{\partial y}(2y)\right) dA$$



$$= (-3-2)\iint_\Omega dA = -5$$

$$= -5\,\frac{(a\sqrt{2})(a\sqrt{2})}{2} = -5a^2.$$

(7) $$\oint xy\,dy \underset{\text{By Gr}}{=} \iint_\Omega \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(0)\right) dA$$



$$= \iint_\Omega y\,dA$$

$$= \{\text{y center of mass of this object}\}$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi} r\sin\theta\, r\,dr\,d\theta = \left.\frac{r^3}{3}\right|_0^a \int_0^\pi \sin\theta\,d\theta$$

$$= \frac{r^3}{3}\Big|_0^a \left(-\cos\theta\right)\Big|_0^\pi$$

$$= \frac{a^3}{3}\left(\cancel{\;}\, - (-1-1)\right) = \frac{2a^3}{3}$$

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(1) $\bar{v} = xy\sin y\,\hat{i} + x^2 z\,\hat{j} - y\cos z\,\hat{k}$

Find $\nabla\cdot\bar{v} = ?$ + $\nabla\times\bar{v} = ?$

$$\nabla\cdot\bar{v} = \frac{\partial}{\partial x}(xy\sin y) + \frac{\partial}{\partial y}(x^2 z) + \frac{\partial}{\partial z}(-y\cos z)$$

$$= y\sin y + 0 + y\sin z$$

$$= y(\sin y + \sin z)$$

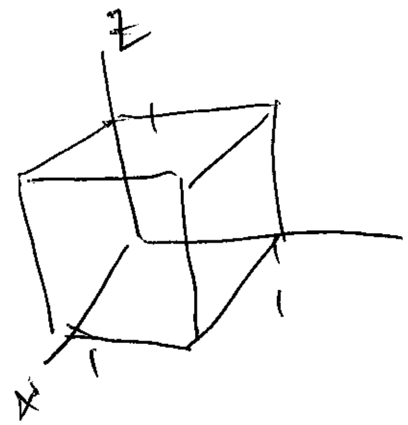
$$\nabla\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy\sin y & x^2 z & -y\cos z \end{vmatrix} =$$

$$\hat{i}\left(\frac{\partial}{\partial y}(-y\cos z) - \frac{\partial}{\partial z}(x^2 z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-y\cos z) - \frac{\partial}{\partial z}(xy\sin y)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(\cancel{+y\cos z}\ x^2 z) - \frac{\partial}{\partial y}(xy\sin y)\right)$$

$$= \hat{i}\left(-\cos z - x^2\right) + \hat{j}(0) + \hat{k}\left(2xz - x\sin y - xy\cos y\right)$$

① evaluate $\iint_S (2x\hat{i} - 3y\hat{j} + 4z\hat{k}) \cdot \hat{N}\, d\sigma$



$= \iint (\quad) \cdot \hat{i}\, d\sigma + \iint (\quad) \cdot (-\hat{i})\, d\sigma$

$+ \iint (\quad) \cdot \hat{j}\, d\sigma + \iint (\quad) \cdot (\hat{j})\, d\sigma$

$+ \iint (\quad) \cdot \hat{k}\, d\sigma + \iint (\quad) \cdot (-\hat{k})\, d\sigma$

$= \int_0^1\int_0^1 2x\, dy\, dz \Big|_{x=1} + \int_0^1\int_0^1 2x\, dy\, dz \Big|_{x=0} + \int_0^1\int_0^1 -3y\, dx\, dz \Big|_{y=1}$

$+ \int_0^1\int_0^1 3y\, dx\, dy \Big|_{y=0} + \iint 4z\, dx\, dy \Big|_{z=1} + \iint -4z\, dx\, dy \Big|_{z=0}$

$=$

$= 2 - 3 + 4 = 3.$

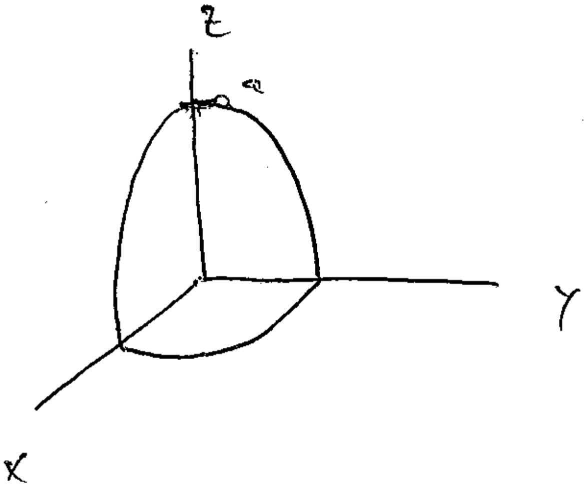
w/ Div. Thm: $\iint_S \vec{v} \circ \hat{N}\, d\sigma = \iiint_V \nabla \circ \vec{v}\, dV$

$\nabla \circ \vec{v} = 2 - 3 + 4 = 3$

$\therefore \iint_S \vec{v} \circ \hat{N}\, d\sigma = 3 \iiint dV = 3.$

evaluate:

(13) $\iint_S \bar{V} \cdot \hat{N} \, d\sigma \qquad \bar{V} = U z \hat{k}$



S is paraboloid

$z = a^2 - x^2 - y^2 \quad w/ \; z \geq 0$

$\hat{N}$ upper unit normal

$z = f(x,y)$

$$\hat{N}_{\pm} = \pm\left(\frac{-2x\hat{i} - 2y\hat{j} - \hat{k}}{\sqrt{1 + 4x^2 + 4y^2}}\right) \qquad \hat{N} = \pm\left(\frac{f_x\hat{i} + f_y\hat{j} - \hat{k}}{\sqrt{1 + f_x^2 + f_y^2}}\right)$$

take $-$ sign to get $\hat{k}$ to be positive

Div. Thm $\quad \iint_S \bar{V} \cdot \hat{N} \, d\sigma = \iiint_V (\nabla \cdot \bar{V}) \, dV$

$\nabla \cdot \bar{V} = U$

$$\therefore \iint_S \bar{V} \cdot \hat{N} \, d\sigma = U \iiint_V dV$$

$$= U \int_{\theta=0}^{2\pi} \int_{\rho=0}^{a} \int_{z=0}^{\overset{a^2-\rho^2}{a^2-x^2-y^2}} dz\, \rho\, d\rho\, d\theta$$

$$=$$

(18) Find total ~~force~~ vertical force on hemisphere

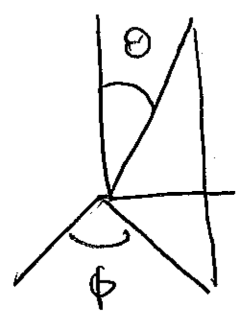
$x^2+y^2+z^2=a^2 \quad z \geq 0 \quad \bar{f}$ force per unit area

is $\bar{f} = \alpha z \hat{k}$

$$\bar{F} = \iint_S \bar{f}\, d\sigma$$

$$\bar{F}_{vertical} = \bar{F} \circ \hat{k} = \bar{F} \circ \hat{k}$$

$$\iint_S \bar{f} \circ \hat{k}\, d\sigma = \iint_S \alpha z\, d\sigma$$



As $z = a\cos\theta$ on sphere

$d\sigma = a^2 \sin\theta\, d\theta\, d\phi$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} a^2 a \cos\theta\ \alpha\ \sin\theta\, d\theta\, d\phi = \alpha a^3 (2\pi) \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta\, d\theta$$

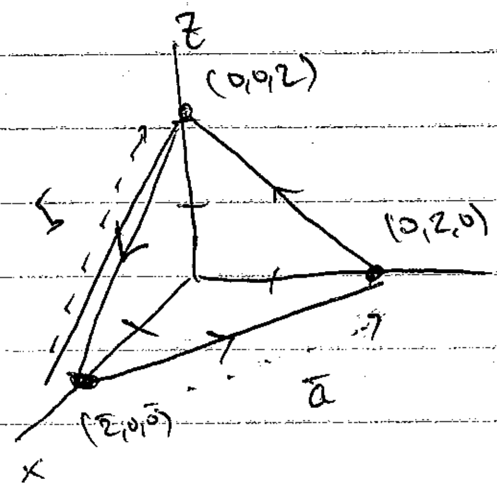
$$= \alpha a^3 (2\pi)\left(-\frac{\cos^2\theta}{2}\right)\Bigg|_0^{\pi/2} = \underline{\alpha a^3 \pi}$$

Pg 1051 Boyce / Diprima

(1) Use Stokes thm to evaluate the given vector around the given curve

$$\int_{\Gamma} \bar{v} \circ \hat{T} \, ds$$

w/ $\bar{v} = 2z\hat{i} - x\hat{j} + 3y\hat{k}$ Γ triangular path from $(2,0,0)$ to $(0,2,0)$ to $(0,0,3)$ to $(2,0,0)$



Stokes Thm:

$$\int_{\Gamma} \bar{v} \circ \hat{T} \, ds = \iint_{S} (\nabla \times \bar{v}) \circ \hat{N} \, d\sigma$$

$$\nabla \times \bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -x & 3y \end{vmatrix} = \hat{i}(3-0) + \hat{j}(2-0) + \hat{k}(-1-0)$$

$$= 3\hat{i} + 2\hat{j} - \hat{k}$$

$\hat{N} = ?$

$\bar{a} = (0-2)\hat{i} + (2-0)\hat{j} + (0-0)\hat{k}$

$\bar{b} = (0-2)\hat{i} + (0-0)\hat{j} + (2-0)\hat{k}$

$\rightarrow \bar{a} = -2\hat{i} + 2\hat{j} + \hat{k}$

$\bar{b} = -2\hat{i} + 0\hat{j} + 2\hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 2 & 1 \\ -2 & 0 & 2 \end{vmatrix} =$$

$$= -\hat{j}(-4+2) + 2(2\hat{i} + 2\hat{k})$$

$$= 4\hat{i} + 2\hat{j} + 4\hat{k} = \bar{N}$$

$$\hat{N} = \frac{\bar{N}}{\|\bar{N}\|} = \frac{\cancel{2}(2\hat{i} + \hat{j} + 2\hat{k})}{\cancel{2}\sqrt{4+1+4}} = \frac{(2\hat{i} + \hat{j} + 2\hat{k})}{3}$$

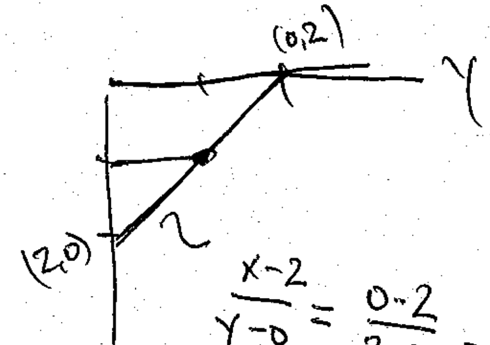
The $(\nabla \times \bar{v}) \cdot \hat{N} = \frac{1}{3}\left(6 + 2 - 2\right) = 2$

$$\therefore \int_{C} \bar{v} \cdot \hat{T}\, ds = 2 \iint_{S} d\sigma$$

project to xy plane from before
(hopefully)

$$d\sigma = \frac{dx\,dy}{|\bar{N} \cdot \hat{k}|} = \frac{dx\,dy}{(2/3)} = \frac{3}{2}dx\,dy$$

$$= 2\left(\frac{3}{2}\right)\iint_{\Omega_{xy}} dx\,dy$$



$$\frac{x-2}{y-0} = \frac{0-2}{2-0} = -1$$

$$\cancel{x} \rightarrow y = 2-x$$

$$= 3\int_{x=0}^{2}\int_{y=0}^{2-x} dy\,dx$$

$$= 3\int_{x=0}^{2}(2-x)\,dx = 3\left(2x - \frac{x^2}{2}\right\Big|_0^2$$

$$= 3(4-2) = 6 \,!!$$

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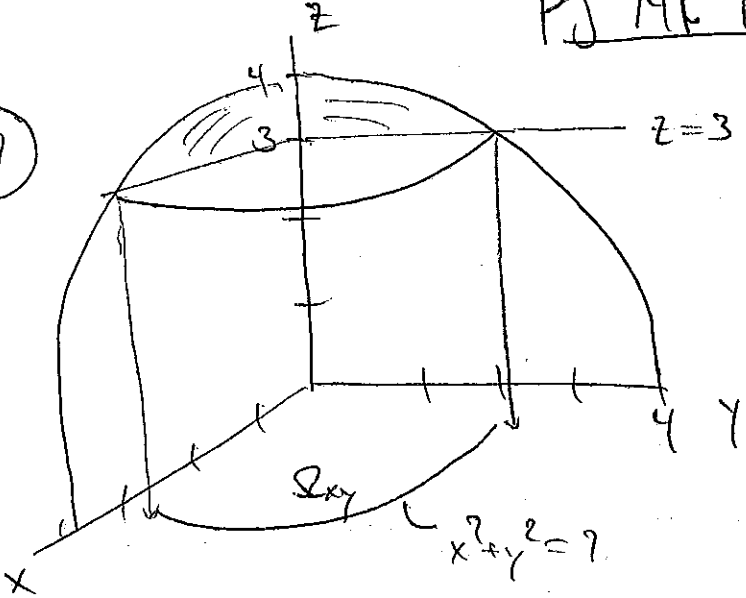
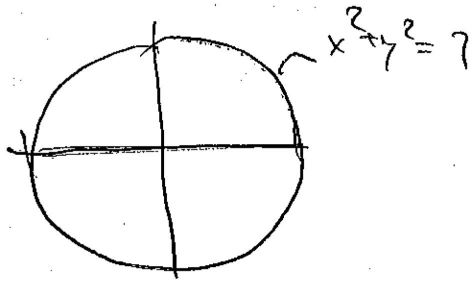
(7)



Ω_{xy} is when $z = 3$ in spheres eq

$$x^2+y^2+9=16$$

$$x^2+y^2=7$$



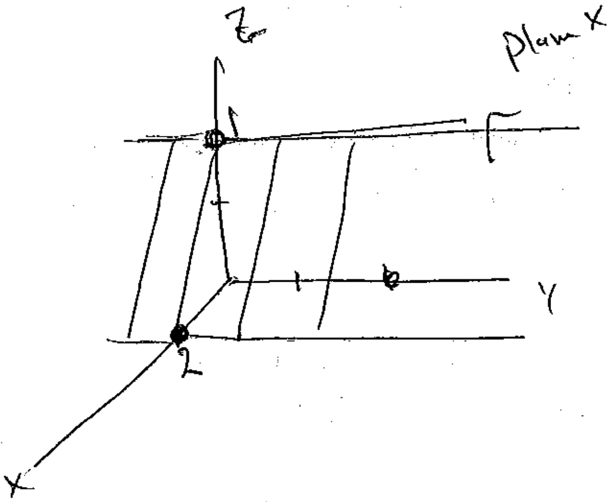
Inequalities:

$$3 \le z \le 16-x^2-y^2.$$

$$-\sqrt{7-x^2} \le y \le +\sqrt{7-x^2}$$

$$-7 \le x \le 7$$

(8)

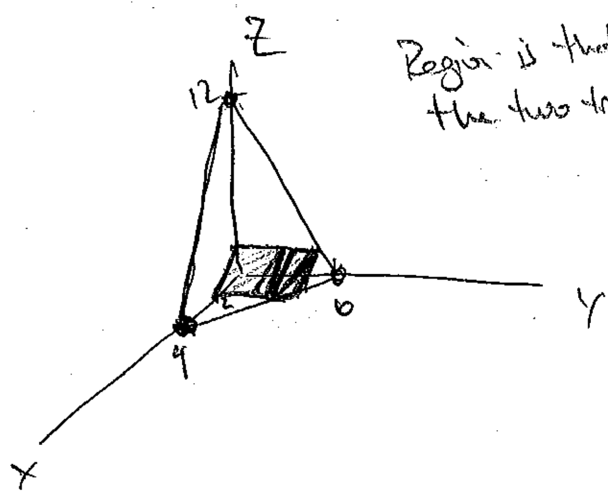


$x+2z=2$

has intercepts $(0,1)=(x,z)$

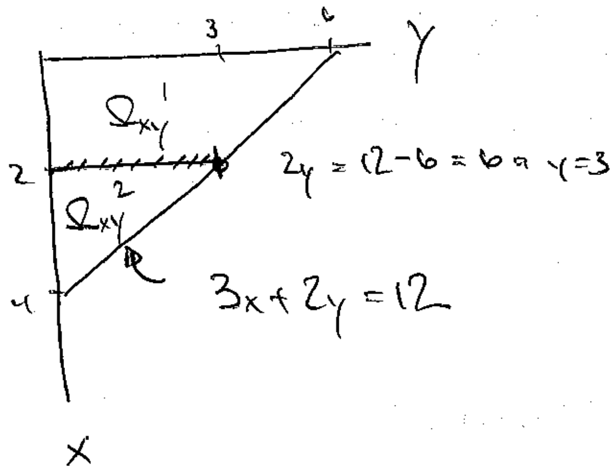
$+$ $(2,0)=(x,z)$

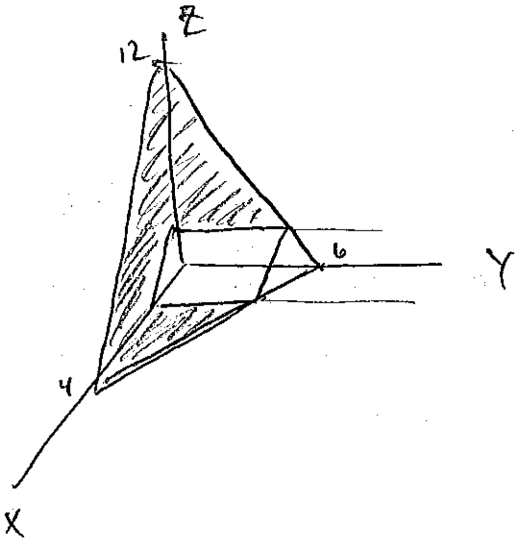
plane $3x+2y+z=12$



Region is that "inside" the two triangles

(a) Ω_{xy}





Then for Ω_{xy}^1:

$$\frac{2-x}{2} \le z \le 12 - 3x - 2y.$$

$$0 \le y \le \frac{12-3x}{2}$$

$$0 \le x \le 2$$

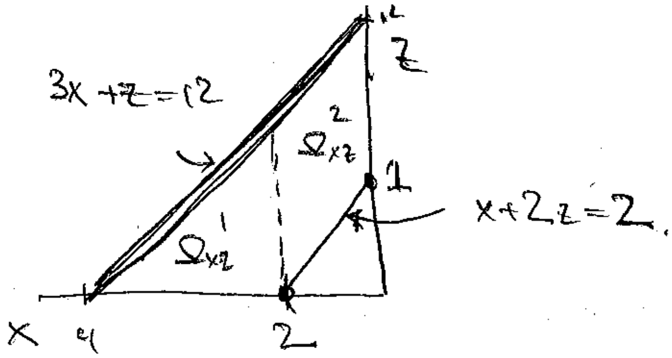
For Ω_{xy}^2:

$$0 \le z \le 12 - 3x - 2y$$

$$0 \le y \le \frac{12-3x}{2}$$

$$2 \le x \le 4$$

(b) Ω_{xz}



In Ω_{xz} , $0 \le y \le \dfrac{12-3x}{2}$

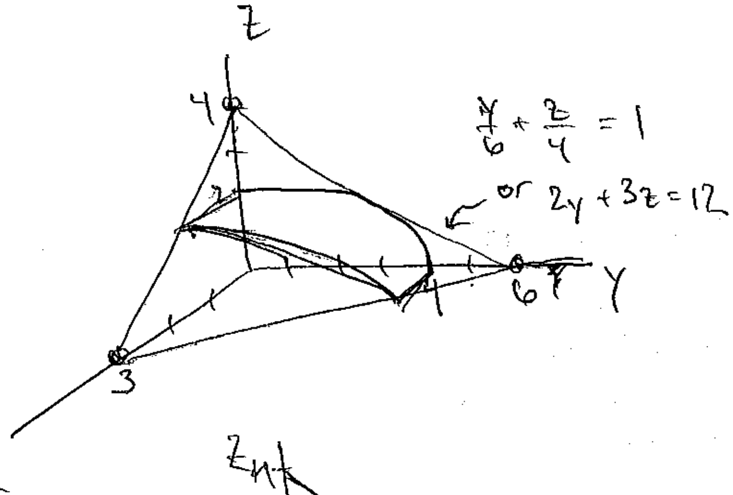
Ω^1_{xz} : $0 \le z \le 12-3x$

Ω^2_{xz} : $\dfrac{2-x}{2} \le z \le 12-3x$

& Ω^1_{xz} : $2 \le x \le 4$

Ω^2_{xz} : $1 \le x \le 2$

(9)



$4x+2y+3z=12$

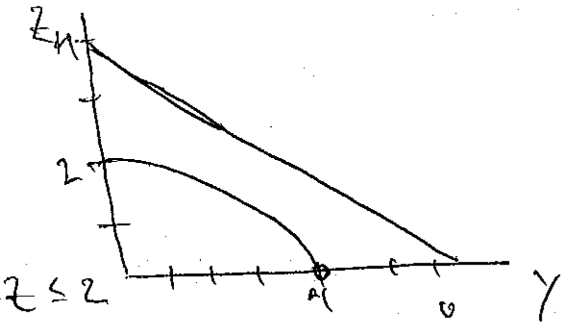
$\frac{x}{3}+\frac{y}{6}+\frac{z}{4}=1$

$\frac{dz}{dy}=-\frac{2}{3}$

? Do

$2y+3z=12$ +

$y=4-z^2$ int



$y=4-z^2$

$\dot{y}=-2z \qquad 0\le z\le 2$

$-2z \qquad 0\le 2z\le 4$

$-4\le -2z<0$

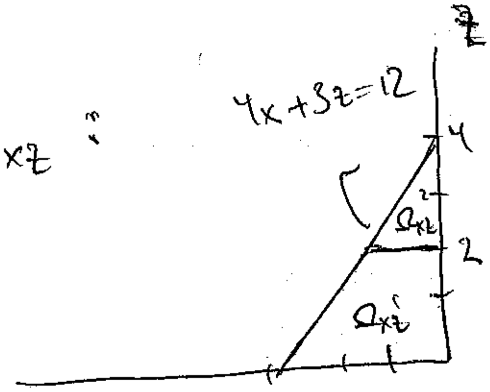
$8-2z^2+3z=12$

$2z^2-3z+4=0$

$z=\frac{+6\pm\sqrt{9-4(2)(4)}}{2(2)}$

$=6\pm\sqrt{<0}$ NO

(a) For Ω_{xz}:



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For Ω_{xz}^1: $\quad 4-z^2\le y\le \frac{12-3z}{2}$

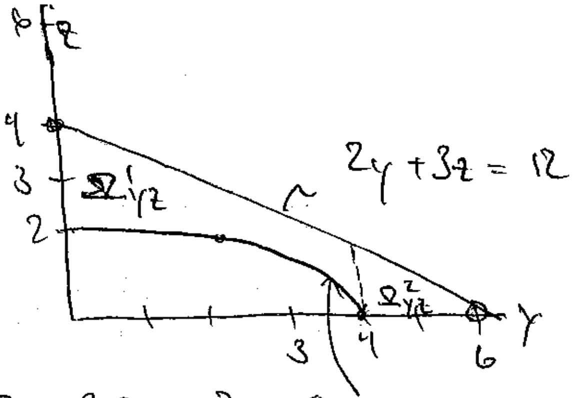
$0\le x\le \frac{12-3z}{4}$

$0\le z\le 2$

For Ω_{xz}^2: $\quad 0\le y\le \frac{12-4x-3z}{2}$ + $\quad 2\le z\le 4$

$0\le x\le \frac{12-3z}{4}$

(b) Ω_{yz}:



$$0 \le x \le \frac{12 - 3z - 2y}{4} \quad \left\{ z = \pm\sqrt{4-y} \right.$$

in Ω^1_{yz}:

$$\sqrt{4-y} \le z \le \frac{12-2y}{3}$$

$$0 \le y \le 4$$

in Ω^2_{yz}:

$$0 \le z \le \frac{12-2y}{3}$$

$$4 \le y \le 6$$

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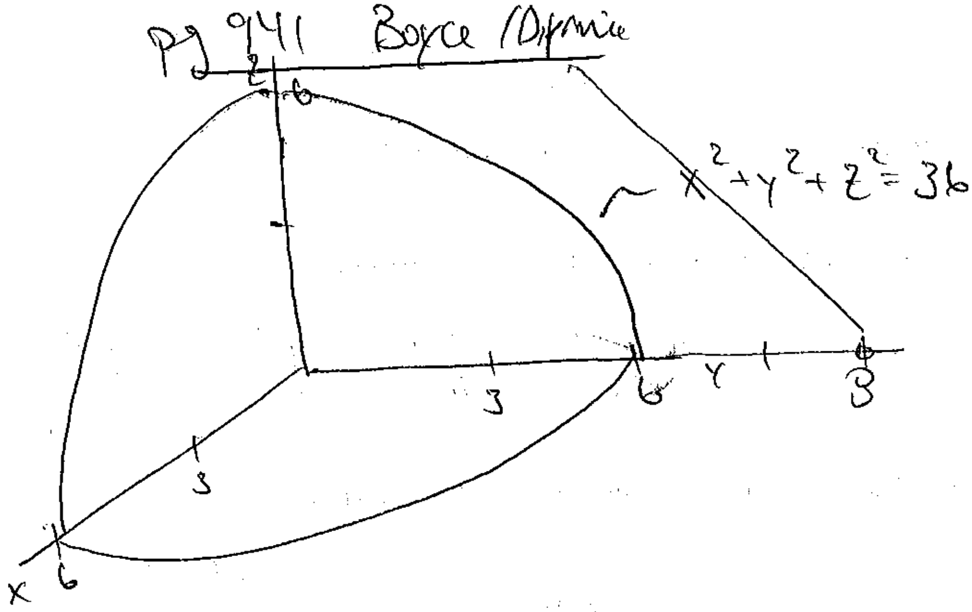
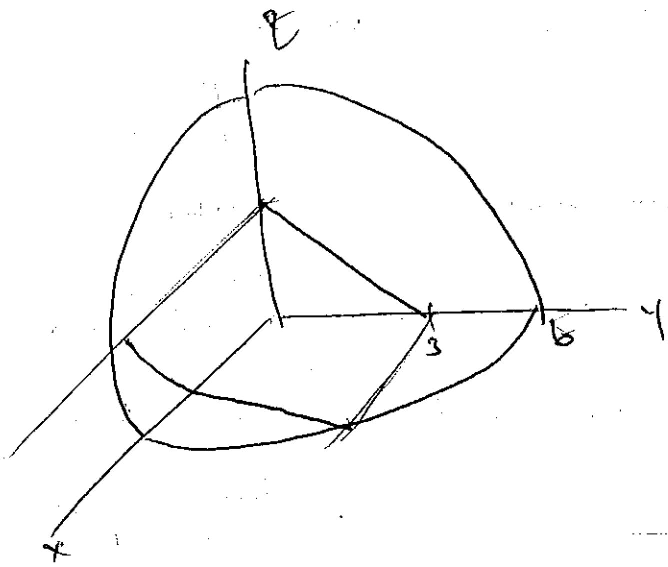


As stated the plan is above + not touching the sphere

consider $y+z=3$



Then projection curve onto $\mathcal{R}_{xy}$ is when

$y+z=3$

$+\ x^2+y^2+z^2=36$

$$\underbrace{3-y}_{\text{plane}} \le z \le \underbrace{36-(x^2+y^2)}_{\text{sphere}}$$

Ellimnate z to obtain an eq in only x + y That represents

The boundary of our region in xy plane (Ω_{xy})

$$z = 3 - y$$

$$x^2 + y^2 + (3-y)^2 = 36$$

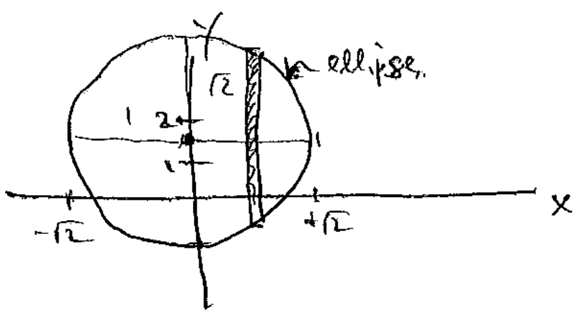
$$x^2 + y^2 + 9 - 6y + y^2 = 36$$

$$x^2 + 2y^2 - 6y = 36 - 9 = 30 - 3 = 27$$

$$\frac{x^2}{2} + y^2 - 3y = \frac{27}{2}$$

$$\frac{x^2}{2} + y^2 - 3y + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = \frac{27}{2}$$

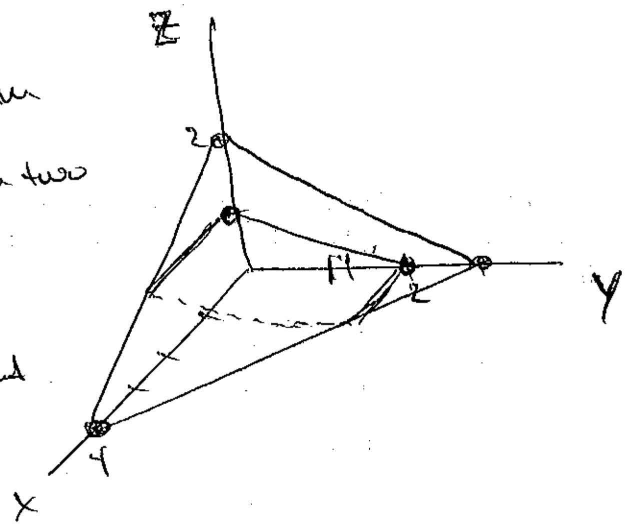
$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(y - \tfrac{3}{2}\right)^2 = \frac{27 \cdot 2}{4} + \frac{9}{4} = \frac{60 - 6 + 9}{4} = \frac{63}{4}.$$



$$\frac{3}{2} - \sqrt{\frac{63}{4} - \frac{x^2}{2}} \le y \le \frac{3}{2} + \sqrt{\frac{63}{4} - \frac{x^2}{2}}$$

$$-\sqrt{2} \le x \le \sqrt{2}.$$

(11) I 1st compute the region between the two planes.
Then I compute the region below the second plane



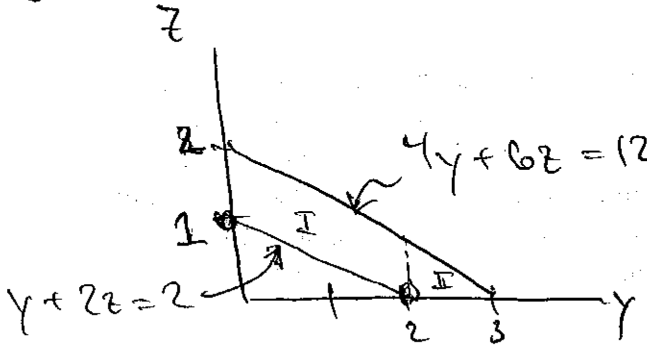
$y + 2z = 2$

$3x + 4y + 6z = 12$

$\frac{x}{4} + \frac{y}{3} + \frac{z}{2} = 1$

project to zy plane

$\Rightarrow$



$4y + 6z = 12 \Rightarrow 2y + 3z = 6$

$$0 \le x \le \frac{12 - 4y - 6z}{3}$$

I :
$$\frac{2-y}{2} \le z \le \frac{6-2y}{3}$$

$$0 \le y \le 2$$

II :
$$0 \le z \le \frac{6-2y}{3}$$

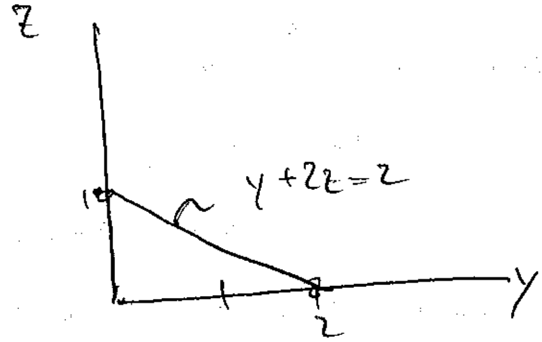
$$2 \le y \le 3$$

$$\therefore \quad M = \int_0^2 \int_{1-\frac{y}{2}}^{2-\frac{2}{3}y} \int_0^{4-\frac{4}{3}y-2z} (6-x-y)\, dx\, dz\, dy$$

$$+ \int_0^2 \int_0^{2-\frac{2}{3}y} \int_0^{4-\frac{4}{3}y-2z} (6-x-y)\, dx\, dz\, dy.$$

$$0 \le x \le 4 - \frac{4}{3}y - 2z$$

The Ω_{zy} is just

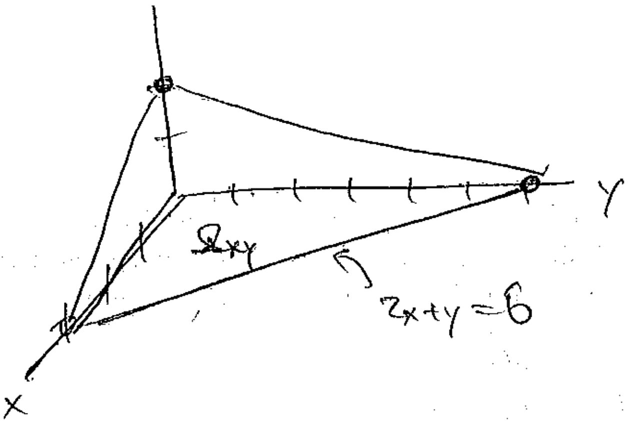


$$0 \le y \le 2 - 2z$$

& $\quad 0 \le z \le 1$

Pg 941 Boyce/DiPrima

(12)



$2x+y+3z=6$

project to Ω_{xy}:

$0 \le z \le \dfrac{6-2x-y}{3}$

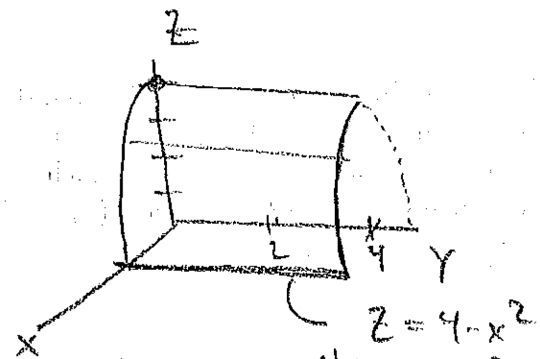
$0 \le x \le \dfrac{6-y}{2}$

$0 \le y \le 6$

$$I_z = \iiint \rho(x,y,z)(x^2+y^2)\,dV$$

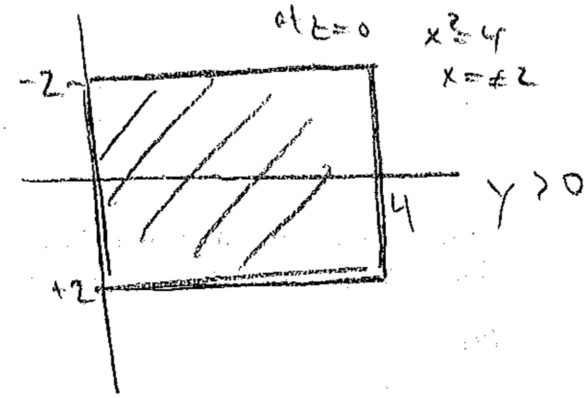
$$= \int_0^6 \int_0^{\frac{6-y}{2}} \int_0^{2-\frac{2}{3}x-\frac{y}{3}} k(x^2+y^2)\,dz\,dx\,dy$$

(14) $\bar{z} = \frac{1}{M}\iiint z\,\rho(x,y,z)\,dV$



$M = \iiint_V \rho(x,y,z)\,dV$

Projecting to the xy plane gives



$0 \le z \le 4 - x^2$

$0 \le y \le 4$

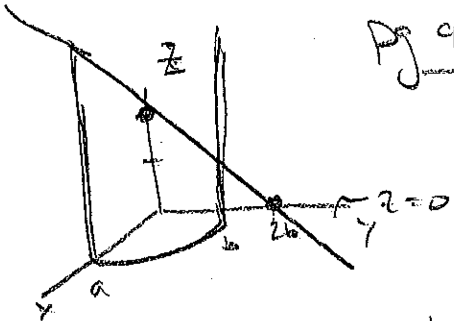
$-2 \le x \le 2$

$$M = \int_{-2}^{2}\int_{0}^{4}\int_{0}^{4-x^2} (6 - z)\,dz\,dy\,dx$$

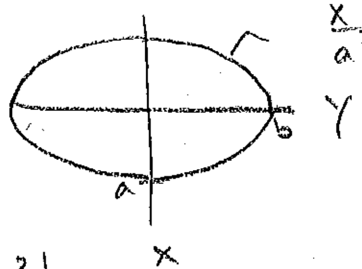
$$\bar{z} = \frac{1}{M}\int_{-2}^{2}\int_{0}^{4}\int_{0}^{4-x^2} (6 - z)\,z\,dz\,dy\,dx$$

Pg 942 Boyce/DiPrima

(5)



project to the xy plane



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \quad y = \pm b\sqrt{1-(x/a)^2}$

$\rho = k r^2 = k(x^2+y^2+z^2)$

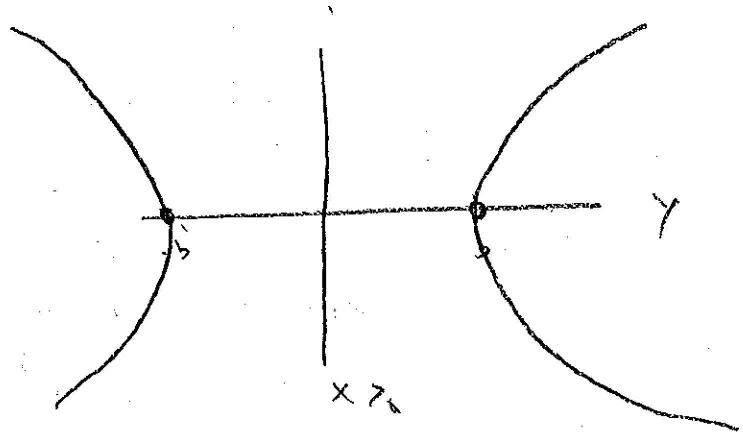
Then $\quad m = \iiint \rho \, dV = \int_{-a}^{a} \int_{-b\sqrt{1-(y/a)^2}}^{+b\sqrt{1-(x/a)^2}} \int_{0}^{2b-y} k(x^2+y^2+z^2)\, dz\, dy\, dx$

(16) $L_{yz} = \iiint_\Omega x\,\rho(x,y,z)\,dV$

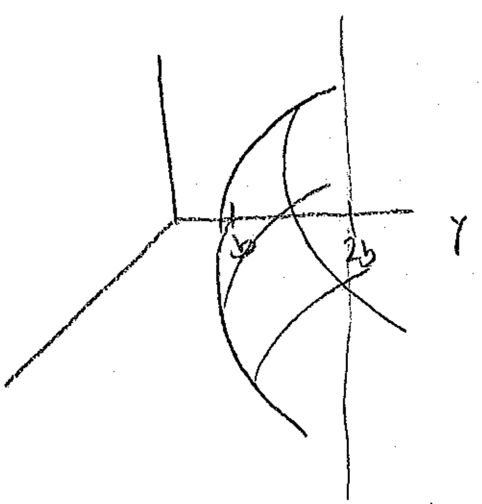
$L_{xz} = \iiint_\Omega y\,\rho(x,y,z)\,dV$

$-\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$ & plane $y = 2b$.

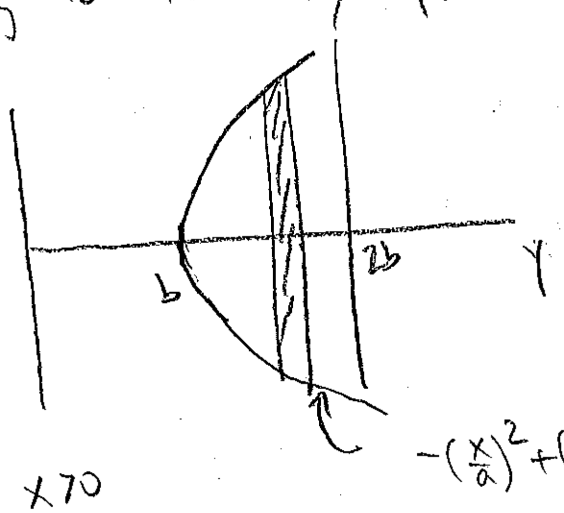
let $z = 0$ $\quad \left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$



Thus



Region looks like a convertable or a gondola "hood".
projecting to the xy plane we get the region



Then $-c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1} \le z \le c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}$

$$\left(\frac{z}{c}\right)^2 = -\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1$$

$$z = \pm c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}$$

$$x = \pm a\sqrt{\left(\frac{y}{b}\right)^2-1}$$

$$-a\sqrt{\left(\frac{y}{b}\right)^2-1} \le x \le a\sqrt{\left(\frac{y}{b}\right)^2-1}$$

$$b \le y \le 2b$$

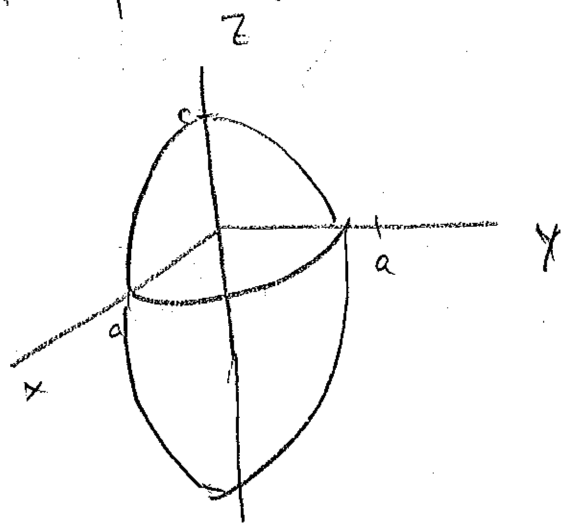
$$\therefore L_{xz} = \int_{b}^{2b}\int_{-a\sqrt{\left(\frac{y}{b}\right)^2-1}}^{a\sqrt{\left(\frac{y}{b}\right)^2-1}}\int_{-c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}}^{c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}} yK\,dz\,dx\,dy$$

(1) w/ $r^2 = x^2+y^2$ in cylindrical eq becomes

$$\frac{1}{a^2}(x^2+y^2) + \frac{z^2}{c^2} = 1$$

$$\frac{z^2}{c^2} + \frac{1}{a^2}r^2 = 1$$

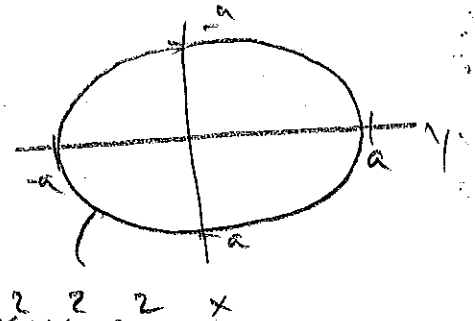
$$\Rightarrow \quad z = \pm c\sqrt{1 - r^2/a^2}$$



$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}}\int_{-c\sqrt{1-r^2/a^2}}^{c\sqrt{1-r^2/a^2}} dz\,dy\,dx$$

$$= 2c\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{1-r^2/a^2} \quad \ldots$$

projection in xy plane



$x^2+y^2 = a^2$

$-\sqrt{a^2-x^2} \le y \le +\sqrt{a^2-x^2}$

In polar

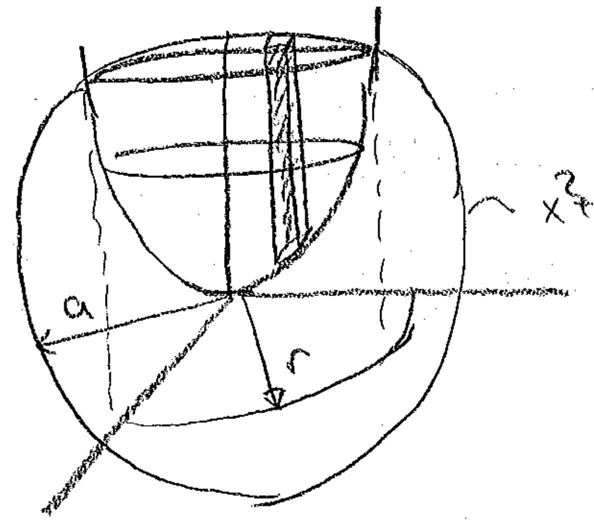
$$V = \int_0^{2\pi}\int_0^{a}\int_{-c\sqrt{1-r^2/a^2}}^{+c\sqrt{1-r^2/a^2}} dz\, r\,dr\,d\theta = 2\pi \cdot 2c\int_0^a \sqrt{1-r^2/a^2}\; r\,dr$$

let $v = r/a$ $\quad dv = dr/a$

$$= 4\pi c\int_0^1 \sqrt{1-v^2}\; a^2 v\,dv = 4\pi a^2 c\int_0^1 v\sqrt{1-v^2}\,dv = 4\pi a^2 c \left.\frac{(1-v^2)}{\frac{3}{2}(-2)}\right|_0^1$$

$$= \frac{4\pi a^2 c}{3}$$

(2)



$$az = \sqrt{12}\,(x^2+y^2) = \sqrt{12}\,r^2$$

$$x^2+y^2+z^2 = a^2$$

$$\Rightarrow r^2+z^2 = a^2.$$

Vol inside $\frac{\sqrt{12}}{a} r^2 \le z \le +\sqrt{a^2-r^2}$

$0 \le r \le ?$

$0 \le \theta \le 2\pi$

To find the upper limit of r elliminate z from $r^2+z^2 = a^2$

& $z = \frac{\sqrt{12}}{a} r^2$

$$\Rightarrow \frac{12}{a^2} r^4 + r^2 = a^2$$

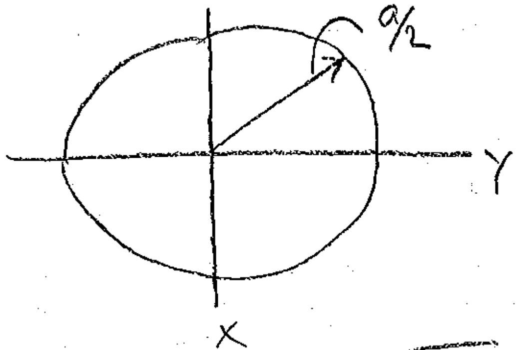
$$\Rightarrow r^4 + \frac{a^2}{12} r^2 - \frac{a^4}{12} = 0$$

$$r^2 = \frac{-\frac{a^2}{12} \pm \sqrt{\frac{a^4}{12^2} + 4\left(\frac{a^4}{12}\right)}}{2} = \frac{-\frac{a^2}{12} \pm \frac{a^2}{\sqrt{12}}\left(\frac{1}{12}+4\right)^{1/2}}{2}$$

$$r^2 = \frac{\frac{-a^2}{12} \pm \frac{a^2}{\sqrt{12}}\left(\frac{49}{12}\right)^{1/2}}{2} = \frac{a^2}{24}\left(-1 \pm 7\right) = \frac{a^2}{4}$$

take +

Then $r = a/2$. Thus projected into the xy plane we get



$$\therefore \quad V = \int_0^{2\pi}\int_0^{a/2}\int_{\frac{\sqrt{12}}{a}r^2}^{\sqrt{a^2-r^2}} dz\, r\, dr\, d\theta$$

$$= 2\pi \int_0^{a/2} \left(\sqrt{a^2-r^2}\, r - \frac{\sqrt{12}}{a} r^3 \right) dr$$

$$= 2\pi \left(\frac{(a^2-r^2)^{3/2}}{3/2(-2)} \Bigg|_0^{a/2} - \frac{\sqrt{12}}{a}\frac{r^4}{4} \Bigg|_0^{a/2} \right.$$

$$= 2\pi \left[\frac{-1}{3}\left(\left(\frac{3}{4}a^2\right)^{3/2} - a^3 \right) - \frac{\sqrt{12}}{4a}\left(\frac{a^4}{8}\right) \right]$$

$$= a^3 \frac{2\pi}{8}\left[-\frac{1}{3}\left(3^{3/2} - 8\right) - \frac{\sqrt{12}}{4}\right]$$

$$= \frac{\pi}{4} a^3 \left[\frac{8}{3} - \sqrt{3} - \frac{1}{2}\sqrt{3}\right]$$

$$= \frac{\pi}{4} a^3 \left[\frac{8}{3} - \frac{3}{2}\sqrt{3}\right] \triangleq \frac{\pi}{4} a^3 \left[2.\overline{6} - 1.5(1.7)\right]$$

$1.7 + .85$

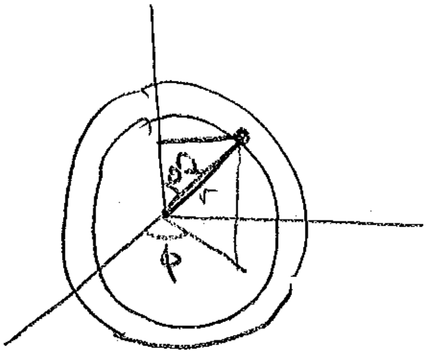
2.55

$\hat{=} .1$

$$= \frac{\pi}{4} a^3 (.068)$$

Pg 963 Boyce/DiPrima

(28)



$r_\perp = r\sin\theta$

$[I] = L^5 \frac{M}{L^3} = L^2 M$

$$I_z = \iiint r_\perp^2 \rho \, dV$$

$$= \iiint r^2 \sin^2\theta \, k \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b r^2 \sin^2\theta \, k \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2\pi k \left.\frac{r^5}{5}\right|_a^b \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{2\pi k}{5}(b^5 - a^5)\left[\int_0^{\pi} \sin\theta \, d\theta - \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta\right]$$

$$= \frac{2\pi k}{5}(b^5 - a^5)\left[-\cos\theta\Big|_0^{\pi} + \frac{\cos^3\theta}{3}\Big|_0^{\pi}\right]$$

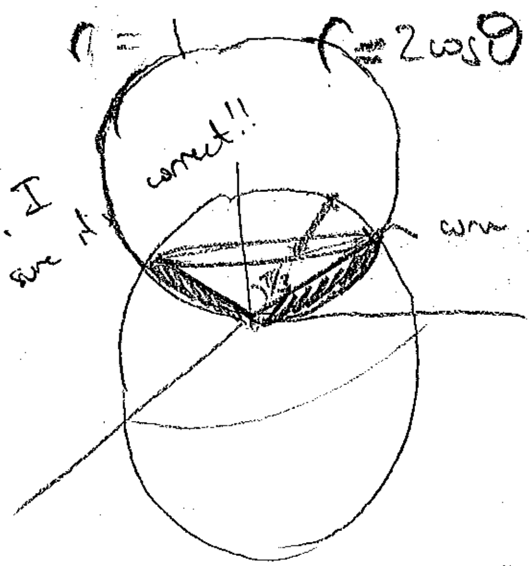
$$-(-1-1) + \frac{1}{3}(-1-1)$$

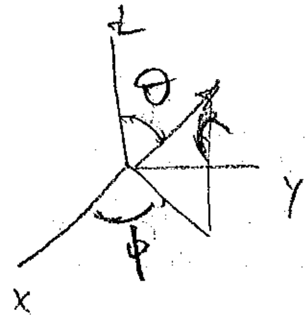
$$2 - \frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow I_z = \frac{8\pi k}{15}(b^5 - a^5)$$

(34)

Check this!! I am not sure it is correct!!





$\rho = 1$

$\& \quad 1 = 2\cos\phi$

$\Rightarrow \phi = \frac{\pi}{3}$

The $\quad V = \iiint_{\text{Top ice cream cone}} dV \qquad$ "Bottom Rynne"

$0 \le \theta \le \frac{\pi}{3} \qquad\qquad \frac{\pi}{3} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1 \qquad\qquad 0 \le r \le 2\cos\theta$

$0 \le \phi \le 2\pi \qquad\qquad 0 \le \phi \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 r^2 \sin\theta \, dr \, d\theta \, d\phi + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\pi/2}\int_0^{2\cos\theta} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\frac{V}{2\pi} = \frac{1}{3}\int_0^{\pi/3} \sin\theta \, d\theta + \frac{1}{3}\int_{\frac{\pi}{3}}^{\pi/2} 2^3 \cos^3\theta \, d\theta$$

$$\frac{3V}{2\pi} = -\left(\cos\frac{\pi}{3} - 1\right) + 8\int_{\frac{\pi}{3}}^{\pi/2} (\cos\theta - \sin^2\theta\cos\theta) \, d\theta$$

$$\frac{3V}{2\pi} = \underbrace{\left(1-\frac{1}{2}\right)}_{\frac{1}{2}} + 8\left[\left(\sin\theta\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} - \frac{\sin^3\theta}{3}\Big|_{\frac{\pi}{3}}^{\pi/2}\right]\right.$$

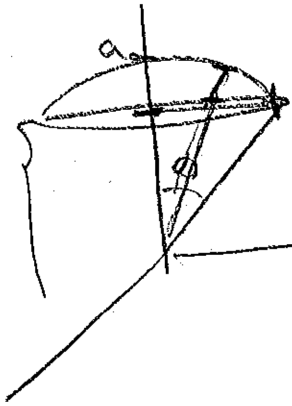
$$= \frac{1}{2} + 8\left[1-\frac{\sqrt{3}}{2}-\frac{1}{3}\left(1-\frac{3^{3/2}}{2^3}\right)\right]$$

$$\frac{1}{2} + 8\left[1-\frac{\sqrt{3}}{2}-\frac{1}{3}+\frac{\sqrt{3}}{8}\right]$$

$$\underbrace{\left(\frac{2}{3}-\frac{4\sqrt{3}}{8}+\frac{\sqrt{3}}{8}\right)}$$

$$8\left(\frac{2}{3}-\frac{3\sqrt{3}}{8}\right]$$

In ~~cylindrical~~ spherical vol of just "cap"



$a\cos\frac{\pi}{3} = \frac{a}{2}$

$z = \frac{a}{2} = r\cos\theta$ intersection of plane $z = \frac{a}{2}$ + sphere

$\Rightarrow r = \frac{a/2}{\cos\theta}$

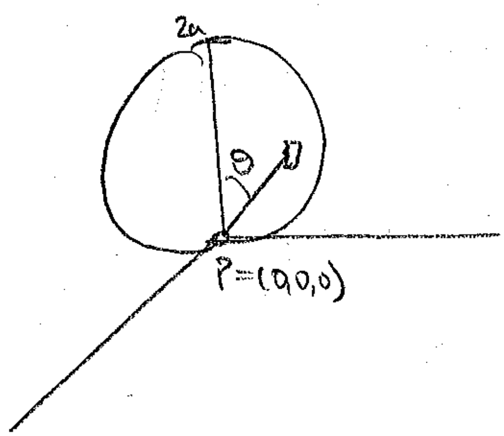
$0 \le \theta \le \frac{\pi}{3}$ $\qquad 0 \le \phi \le 2\pi$

$\frac{a/2}{\cos\theta} \le r \le a$

$$\therefore V_{cap} = \int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{a/2}{\cos\theta}}^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$\frac{V_{cap}}{2\pi} =$$

(35)



$\rho(x,y,z) = kr$

$$m = \iiint kr \, dV$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} kr \; r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2\pi k \int_0^{\pi/2} \sin\theta \underbrace{\int_{r=0}^{2a\cos\theta} r^3 dr}\; d\theta$$

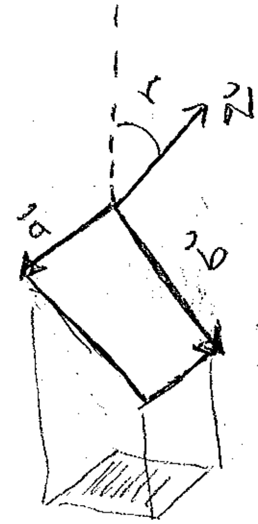
$$= 2\pi k \int_0^{\pi/2} \sin\theta \; \frac{r^4}{4}\Big|_0^{2a\cos\theta} d\theta$$

$$= \frac{\pi}{2} k \int_0^{\pi/2} \sin\theta \; 2^4 a^4 \cos^4\theta \, d\theta = 2^3 a^4 \pi k \left(-\frac{\cos^5\theta}{5}\right)\Big|_0^{\pi/2}$$

$$= \frac{8a^4\pi k}{5}(-1)(0-1) = \frac{8\pi a^4 k}{5}$$

Pg 1003 Boyce/Diprima

$$(\bar{a}\times\bar{b})\cdot\hat{k} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \Delta x & 0 & a_3\\ 0 & \Delta y & b_3\end{vmatrix}\cdot\hat{k}$$



$$= \hat{k}(\Delta x \Delta y)\cdot\hat{k} = \Delta x \Delta y$$

Pg 1004 Boyce Diprima

$$|\hat{N}(x_i^*, y_j^*)\cdot\hat{k}| = +\frac{1}{\left(f_x^2(x_i^*, y_j^*) + f_y^2(x_i^*, y_j^*) + 1\right)^{1/2}}$$

Then $$\Delta\sigma_{ij} = \frac{\Delta A_{ij}}{|\hat{N}(x_i^*, y_j^*)\cdot\hat{k}|} = \left(f_x^2(x_i^*, y_j^*) + f_y^2(x_i^*, y_j^*) + 1\right)^{1/2}\Delta A_{ij}$$

Pg 1007 Boyce/Diprima

$$\hat{N} = \frac{F_x\hat{i} + F_y\hat{j} + F_z\hat{k}}{\|\nabla F\|}$$

$$|\hat{N}\cdot\hat{k}| = \frac{|F_z|}{\|\nabla F\|}$$

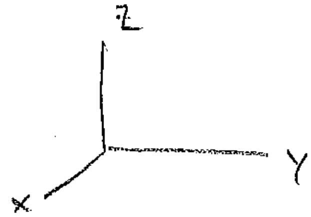
$$d\sigma = \frac{dA_{xy}}{|\hat{N}\cdot\hat{k}|} = \frac{(F_x^2 + F_y^2 + F_z^2)^{1/2}}{|F_z|}\, dA_{xy}$$

Pg 1009 Boyce / Diprima

$$\hat{N} = \frac{F_x\hat{i} + F_y\hat{j} + F_z\hat{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = \frac{x\hat{i} + z\hat{k}}{\sqrt{x^2+z^2}} = \frac{x}{a}\hat{i} + \frac{z}{a}\hat{k}$$

on surface

Pg 1010 Boyce / Diprima



onto yz planes: $x = f(y,z)$

$$A(S) = \iint_{\Omega_{yz}} \left(\frac{(F_x^2 + F_y^2 + F_z^2)^{1/2}}{|F_x|}\right) dA_{yz}$$

$F_x = -2x$

$F_y = 2y$

$F_z = 2z$

$$= \iint_{\Omega_{yz}} \frac{(x^2+y^2+z^2)^{1/2}}{|x|}\, dy\, dz$$

$$\boxed{A(S) = \iint_{\Omega} \frac{dA}{|\hat{N} \cdot \hat{n}|}}$$

normal to plane when taking the projection

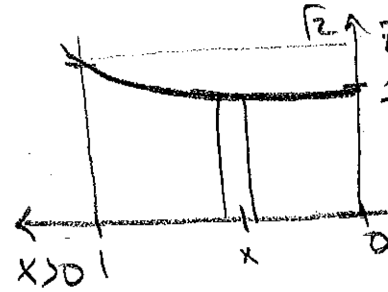
Must replace x w/ $x = x(y,z)$

$$= \iint_{\Omega_{yz}} \frac{(2(y^2+z^2)-1)^{1/2}}{(y^2+z^2-1)^{1/2}} = \int_{r=1}^{\sqrt{2}} \int_{\theta=0}^{\pi/2} \left(\frac{(2r^2-1)}{r^2-1}\right)^{1/2} r\, d\theta\, dr$$

V.S. projecting into xz plane

$$A(S) = \iint_{\Omega_{xz}} \frac{dA_{xz}}{|\hat{N} \cdot \hat{j}|} = \iint_{\Omega_{xz}} \frac{dA_{xz}(F_x^2 + F_y^2 + F_z^2)^{1/2}}{|F_y|}$$

$$= \iint_{\Omega_{xz}} dA_{xz} \, \frac{(x^2+y^2+z^2)^{1/2}}{|y|}$$



$y=0 \Rightarrow z^2 = 1+x^2$

$z^2 - x^2 = 1$

$$= \iint_{\Omega_{xz}} dA_{xz} \, \frac{(1+2x^2)^{1/2}}{\sqrt{1+x^2-z^2}}$$

for fixed x

$0 < z < \sqrt{1+x^2}$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \left(\frac{1+2x^2}{1+x^2-z^2}\right)^{1/2} dz \, dx$$

Note limits of x go from 0 to 1 not 1 to 0 for if the integrand was reversed we must be denoting a positive integral.

$$= \int_0^1 (1+2x^2)^{1/2} \int_0^{\sqrt{1+x^2}} \frac{dz}{(1+x^2-z^2)^{1/2}}$$

Remembering that

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

let $z = \sqrt{1+x^2}\, v$

$dz = \sqrt{}\, dv$

$$= \int_0^1 (\quad)^{1/2} \underbrace{\int_0^1 \frac{\sqrt{1+x^2}\, dv}{(1+x^2-(1+x^2)v^2)^{1/2}}}_{\displaystyle \int_0^1 \frac{dv}{\sqrt{1-v^2}} = \arcsin x \Big|_0^1 = \frac{\pi}{2}}$$

Pg 1011 Boyce / Diprima

let $\tan u = \sqrt{2}\, x$

$\sec^2 u \, du = \sqrt{2}\, dx$

$\tan u = \sqrt{2}$

$u = \tan^{-1}(\sqrt{2})$

$$A(S) = \frac{\pi}{2} \int_0^{\tan^{-1}(\sqrt{2})} \sec u \cdot \frac{1}{\sqrt{2}} \sec^2 u \, du$$

$(1+\tan^2 u)^{1/2} = ?$
$\sec u.$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{\tan^{-1}(\sqrt{2})} \sec^3 u \, du = \frac{\pi}{2\sqrt{2}} \int_0^{\tan^{-1}(\sqrt{2})} \sec u \, \sec^2 u \, du$$

$= \frac{\pi}{2\sqrt{2}}$

Pg 1012 Boyce/Diprima

$$M = \iint_{\Omega_{xy}} (a - z)\frac{a}{z}\, dA_{xy} \qquad \text{w/} \quad z = \pm\sqrt{a^2 - x^2} \quad \text{take } +.$$

$$= a\int_0^a\int_0^x \frac{\left(a - \sqrt{a^2 - x^2}\right)}{\sqrt{a^2 - x^2}}\, dy\, dx =$$

$$= a^2\int_0^a\int_0^x (a^2 - x^2)^{-1/2}\, dy\, dx - a\int_0^a\int_0^x dy\, dx$$

$$= a^2\int_0^a x(a^2 - x^2)^{-1/2}\, dy - a\int_0^a x\, dx$$

$$= a^2 \frac{(a^2 - x^2)^{1/2}}{(-\cancel{2})(1/\cancel{2})}\Bigg|_0^a - a\frac{x^2}{2}\Bigg|_0^a$$

$$= -a^2\left(0 - (a^2)^{1/2}\right) - \frac{a^3}{2} = a^3 - \frac{a^3}{2} = \frac{a^3}{2}.$$

Pg 1021 Boyce/Diprima

$$\frac{\partial \bar r}{\partial \theta} \times \frac{\partial \bar r}{\partial \phi} = \begin{vmatrix} \hat i & \hat j & \hat k \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \end{vmatrix}$$

$$= \hat i\left(-R^2\cos\theta\sin^2\phi\right) - \hat j\left(R^2\sin\theta\sin^2\phi\right) + \hat k\left(-R^2\sin^2\theta\sin\phi\cos\phi\right.$$

$$\left. - R^2\cos^2\theta\cos\phi\sin\phi\right)$$

$$= \hat i\left(-R^2\cos\theta\sin^2\phi\right) - R^2\sin\theta\sin^2\phi\,\hat j - \hat k\,R^2\sin\phi\cos\phi$$

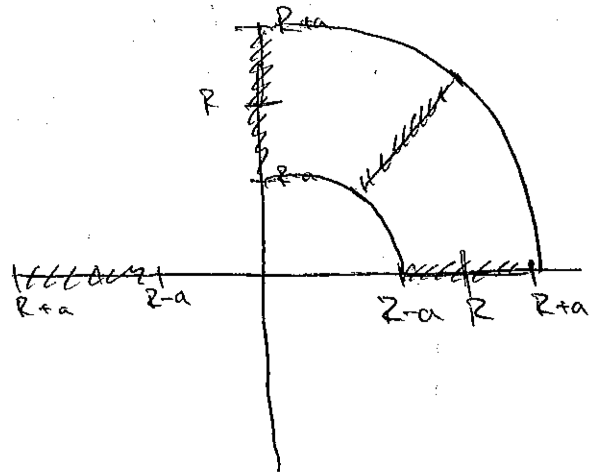
$$\left\|\frac{\partial \bar r}{\partial \theta} \times \frac{\partial \bar r}{\partial \phi}\right\| = R^2\left(\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi + \sin^2\phi\cos^2\phi\right)^{1/2}$$

$$= R^2\left(\sin^4\phi + \sin^2\phi\cos^2\phi\right)^{1/2}$$

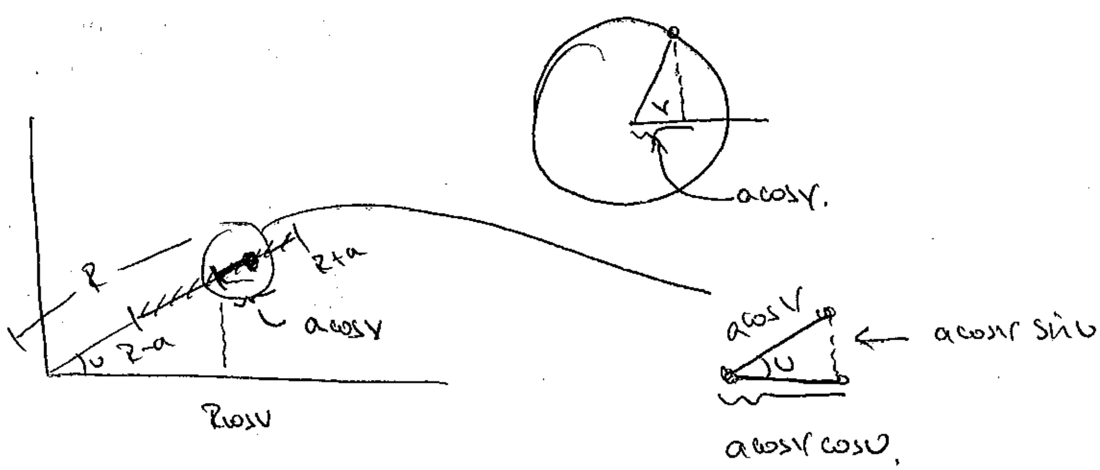
$$= R^2\sin\phi$$

pg 1022 Boyce / DiPrima

Top view:



Now for the circle at an angle u from the x axis & pointing at a point on that circle of radius a w/ $\measuredangle v$ The point would project onto the diameter of the circle



Thus the x component of the point is at

$$R\cos u + a\cos v \cos u = (R + a\cos v)\cos u.$$

The y component would be $R \sin u + (a \cos v) \sin u$

The z component is the easiest at $a \sin v$.

Pg 1023 Boyce / Diprima (1)

$$\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(R+a\cos v)\sin u & (R+a\cos v)\cos u & 0 \\ -a\sin v\cos u & -a\sin v\sin u & a\cos v \end{vmatrix}$$

$$= \hat{k}\left(a(R+a\cos v)\sin v\sin^2 u + a(R+a\cos v)\sin v\cos^2 u\right)$$

$$- a\cos v\left(\hat{i}\,(R+a\cos v)\cos u + \hat{j}(R+a\cos v)\sin u\right)$$

$$= a(R+a\cos v)\left\{\cos u\cos v\,\hat{i} + \sin u\cos v\,\hat{j} + \sin v\,\hat{k}\right\}$$

$$\left\|\frac{\partial \vec r}{\partial u} \times \frac{\partial \vec r}{\partial v}\right\|^2 = a^2(R+a\cos v)^2\left[\underbrace{(\cos^2 u + \sin^2 u)\cos^2 v + \sin^2 v}_{=1}\right]$$

PJ 1023 Boyce / Diprima

(1) $x = 2u - v \Rightarrow$ $v = 2u - x$ put into 1st & 2nd eq

$y = u + 2v$ $\qquad y = u + 4u - 2x = 5u - 2x$

$z = u - v$ $\qquad$ & $z = u - 2u + x = -u + x \Rightarrow u = -z + x$

put in above: $y = 5(-z+x) - 2x$

$y = -5z + 5x - 2x$

$y = -5z + 3x$ plane through origin entire surface

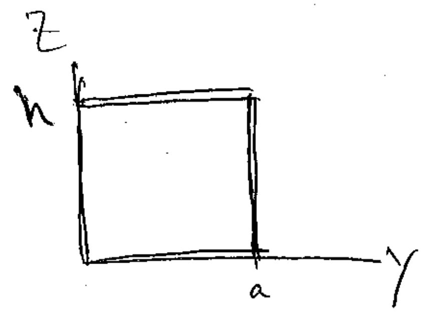
pg 1034 Boyce/DiPrima

$$F_z = -GM\rho \iint_S \frac{z\, d\sigma}{(x^2+y^2+z^2)^{3/2}} \qquad F(x,y,z) = x^2+y^2-a^2 = 0.$$

$$d\sigma = \frac{dA_{yz}}{|\hat{N}\cdot\hat{i}|} \qquad \hat{N} = \pm \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2+y^2}}$$

$$\hat{N}\Big|_{x^2+y^2=a^2} = \pm \frac{x\hat{i} + y\hat{j}}{a}$$

$$|\hat{N}\cdot\hat{i}| = \frac{x}{a}$$



$$\therefore\ F_z = -GM\rho(4) \iint_{\Omega_{yz}} \frac{z}{(a^2+z^2)^{3/2}} \frac{dA_{yz}}{(x/a)}$$

$$x = \pm\sqrt{a^2-y^2}$$

$$= -4GM\rho a \iint_{\Omega_{yz}} \frac{z}{(a^2+z^2)^{3/2}} \frac{dA_{yz}}{\sqrt{a^2-y^2}}$$

or

$$= -4GM\rho a \int_0^h \int_0^a \frac{z}{(a^2+z^2)^{3/2}} \frac{1}{(a^2-y^2)^{1/2}}\, dy\, dz$$

$$= -4GM\rho a \int_0^a \frac{(a^2+z^2)^{-1/2}}{(-1/2)(2)} \Big|_0^h \frac{1}{(a^2-y^2)^{1/2}}\, dy$$

$$= 4GM\rho a \int_0^a \left(\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{a}\right) \frac{1}{(a^2-y^2)^{1/2}}\, dy$$

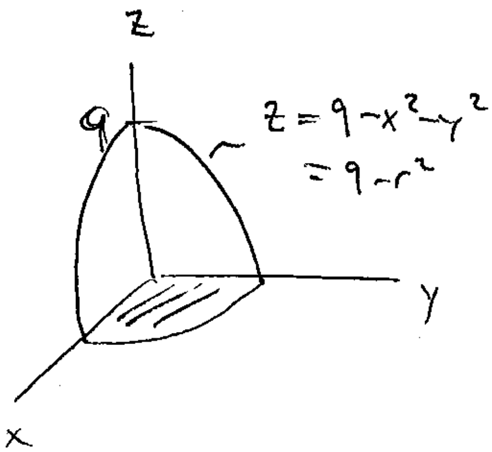
$$= -4GM\rho a \left(-(a^2+h^2)^{-1/2} + a^{-1}\right) \int_0^a \frac{dy}{\sqrt{a^2-y^2}}$$

$$= -4GM\rho a \left(\qquad \right) \underbrace{\arcsin\left(\frac{y}{a}\right)\Big|_0^a}_{\frac{\pi}{2}}$$

$$=$$

$$= -2\pi GM\rho\left(1 - a(a^2+h^2)^{1/2}\right)$$

8 $\vec{v} = (x^2+y^2)\hat{i} + 2xy\,\hat{j}$



$$\iint_S \vec{v}\circ\hat{N}\,d\sigma = \iiint_V (\nabla\circ\vec{v})\,dV$$

$$= \iiint (2x+2x)\,dV$$

in cylindrical $\displaystyle\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} 2(r\cos\theta)\,dz\,r\,dr\,d\theta$

$$= \cancel{\quad}\ 2\int_0^{2\pi}\int_0^{3}(9-r^2)\,r^2\cos\theta\,d\theta$$

$= 2\cdot 0\cdot \quad = 0$ from θ integral.

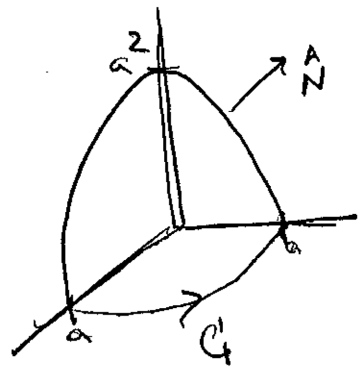
(21)

$$\frac{1}{3}\iint_S \vec{F}\circ\hat{n}\,d\sigma = \frac{1}{3}\iiint(\nabla\circ\vec{F})\,dV = \frac{3}{3}\iiint dV = V(D)$$

By divergence thm

(13) evaluate $\displaystyle\iint_S \nabla\times\vec{V}\circ\hat{N}\,d\sigma = ?$

$$\vec{V} = xyz\,\hat{i} + (x+z)\hat{j} + (x^2-y^2)\hat{k}.$$



$$\iint_S (\nabla\times\vec{V}\circ\hat{N})\,d\sigma = \int_C \vec{V}\circ\hat{\tau}\,ds$$

let in xy plane $\quad x = a\cos t \quad 0 \le t \le 2\pi$
$\quad y = a\sin t$

$$\hat{\tau} = \frac{\dot{x}\hat{i} + \dot{y}\hat{j}}{\sqrt{\dot{x}^2+\dot{y}^2}} \qquad ds = \sqrt{\dot{x}^2+\dot{y}^2}\,dt$$

$$\int_0^{2\pi} \Big((xyz)\,\dot{x} + (x+z)\,\dot{y}\Big)\,dt \Big|_{z=0} = \int_0^{2\pi} x\dot{y}\,dt = a^2\int_0^{2\pi} \cos t\cos t\,dt$$

$$= a^2\int_0^{2\pi} \frac{(1+\cos(2t))}{2}\,dt = \frac{a^2}{2}\left[2\pi + \frac{\sin(2t)}{2}\Big|_0^{2\pi}\right]$$

$$= \pi a^2$$