Solutions to the Problems in Calculus by William E. Boyce and Richard C. DiPrima

John L. Weatherwax*

Introduction

Here you'll find some notes that I wrote up as I worked through this excellent book. I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

All comments (no matter how small) are much appreciated. In fact, if you find these notes useful I would appreciate a contribution in the form of a solution to a problem that I did not work, a mathematical derivation of a statement or comment made in the book that was unclear, or a correction to a typo (spelling, grammar, etc) about these notes. Sort of a "take a penny, leave a penny" type of approach. Remember: pay it forward.

*wax@alum.mit.edu

Chapter 1 (Functions)

Section 1.4 (Functions)

Section 1.5 (Examples of Functions)

Problem 22

WWX: working here.

Section 1.6 (Trigonometric Functions)

Problem 4

Using the cosine sum formula we find

$$\cos\left(\frac{7\pi}{6}+\frac{7\pi}{4}\right)=\cos\left(\frac{7\pi}{6}\right)\cos\left(\frac{7\pi}{4}\right)-\sin\left(\frac{7\pi}{6}\right)\sin\left(\frac{7\pi}{4}\right)=-\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}=-\frac{(\sqrt{3}+1)}{2\sqrt{2}}\,.$$

Problem 8

Using the half-angle formula for $\sin(\cdot)$ we have

$$\sin\left(\frac{1}{2}\left(\frac{\pi}{6}\right)\right)=+\sqrt{\frac{1-\cos(\pi/6)}{2}}=\sqrt{\frac{1-(\sqrt{3}/2)}{2}}\,.$$

Chapter 6 (The Integral)

Section 6.5 (integration by substitution)

additional examples at evaluating integrals

Integrate:

$$\int \frac{\cos(x)dx}{\sqrt{1+\sin(x)}} \, .$$

Let $u = \sin(x)$ then $du = \cos(x)dx$ and we get

$$\int \frac{du}{\sqrt{1+u}} = 2(1+u)^{1/2} + C = 2(1+\sin(x))^{1/2} + C \, .$$

Integrate:

$$\int \frac{\sin^{-1}(x)dx}{\sqrt{1+x^2}} \, .$$

Let $u = \sin^{-1}(x)$ then $du = \frac{dx}{\sqrt{1-x^2}}$ and we get

$$\int u du = \frac{u^2}{2} + C = \frac{1}{2}(\sin^{-1}(x))^2 + C \, .$$

Integrate:

$$\int \frac{\tan(x)dx}{\cos^2(x)} \, .$$

First write this integral as

$$-\int \left(-\frac{\sin(x)}{\cos^3(x)}\right) dx$$

Let $u = \cos(x)$ then $du = -\sin(x)dx$ and we get

$$-\int \frac{du}{u^3} = -\frac{u^{-2}}{(-2)} + C = \frac{1}{2u^2} + C = \frac{1}{2\cos^2(x)} + C \, .$$

Integrate:

$$I = \int \frac{dx}{(1-\sin(x))} \, .$$

First write this integral as

$$
\begin{aligned}
I &= \int \frac{(1+\sin(x))dx}{1-\sin^2(x)}dx \\
&= \int \frac{(1+\sin(x))dx}{\cos^2(x)}dx \\
&= \int \frac{dx}{\cos^2(x)} + \int \frac{\sin(x)}{\cos^2(x)}dx \\
&= \int \sec^2(x)dx - \int \left(-\frac{\sin(x)}{\cos^2(x)}\right) dx\,.
\end{aligned}
$$

In the second integral let $u = \cos(x)$ then $du = -\sin(x)dx$ and we get

$$
\begin{aligned}
I &= \tan(x) - \int \frac{du}{u^2} \\
&= \tan(x) - \frac{u^{-1}}{(-1)} + C \\
&= \tan(x) + \frac{1}{\cos(x)} + C\,.
\end{aligned}
$$

Integrate:

$$
I = \int e^{\ln\left(\sqrt{x}\right)}dx = \int \sqrt{x}dx = \frac{2}{3}x^{3/2} + C\,.
$$

Integrate:

$$
I = \int \frac{\cos(\sqrt{x})}{2\sqrt{x}}dx\,.
$$

Let $u = \sqrt{x}$ so that $du = \frac{dx}{2\sqrt{x}}$ and we get

$$
\begin{aligned}
I &= 2\int \cos(u)du = -2\sin(u) + C \\
&= -2\sin\left(\sqrt{x}\right) + C\,.
\end{aligned}
$$

Integrate:

$$
I = \int \frac{dx}{\sqrt{x^2+2x+2}}
$$

We have

$$
\begin{aligned}
I &= \int \frac{dx}{\sqrt{x^2+2x+1+1}} \\
&= \int \frac{dx}{\sqrt{(x+1)^2+1}}\,.
\end{aligned}
$$

Let $u = x+1$ then $du = dx$ to get

$$
I = \int \frac{du}{\sqrt{u^2+1}} = \arcsin(u) + C = \arcsin(x+1) + C\,.
$$

Integrate:

$$I = \int \frac{(3x-7)dx}{(x-1)(x-2)(x-3)}\,.$$

We will do this with partial fractions. We need to find A, B, and C such that

$$3x - 7 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)\,.$$

Let $x = 2$ to get

$$6 - 7 = 0 + B(1)(-1) + 0 \quad \text{so} \quad B = 1\,.$$

Let $x = 3$ to get

$$9 - 7 = 0 + 0 + C(2)(1) \quad \text{so} \quad C = 1\,.$$

Let $x = 1$ to get

$$3 - 7 = A(-1)(-2) + 0 + 0 \quad \text{so} \quad A = -2\,.$$

Thus we have written our integral as

$$\begin{aligned} I &= \int \left(-\frac{2}{x-1} + \frac{1}{x-2} + \frac{1}{x-3} \right) dx \\ &= -2\ln|x-1| + \ln|x-2| + \ln|x-3| + C\,. \end{aligned}$$

Integrate:

$$I = \int x^2 e^x dx\,.$$

We will do this with integration by parts. Let $u = x^2$ so that $du = 2xdx$ and $v = e^x$ so that $dv = e^x dx$ and the above becomes

$$I = x^2 e^x - 2\int xe^x dx\,.$$

To evaluate this second integral again use integration by parts by letting $u = x$, so that $du = dx$ and $v = e^x$ so that $dv = e^x dx$ and we get

$$\begin{aligned} I &= x^2 e^x - 2\left[xe^x - \int e^x dx \right] \\ &= x^2 e^x - 2xe^x + 2e^x + C\,. \end{aligned}$$

Integrate:

$$I = \int \sqrt{x^2+1}\,dx\,.$$

Let $x = \tan(\theta)$ so that $dx = \sec^2(\theta)d\theta$ and we get

$$\begin{aligned} I &= \int \sec(\theta)\sec^2(\theta)d\theta = \int \sec^3(\theta)d\theta \\ &= \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\int \sec(\theta)d\theta \\ &= \frac{1}{2}\sec(\theta)\tan(\theta) + \frac{1}{2}\ln|\sec(\theta) + \tan(\theta)| + C \\ &= \frac{x}{2}\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1} + x\right| + C\,. \end{aligned}$$

Integrate:

$$I = \int \frac{e^t}{1+e^{2t}} dt\,.$$

Let $u = e^t$ so that $du = e^t dt$ then

$$I = \int \frac{du}{1+u^2} = \arctan(u) + C = \arctan\big(e^t\big) + C\,.$$

Integrate:

$$I = \int \frac{1}{e^x + e^{-x}} dx$$

We multiply by $\frac{e^x}{e^x}$ to get

$$I = \int \frac{e^x}{e^{2x}+1} dx = \arctan(e^x) + C\,.$$

Integrate:

$$I = \int \frac{1}{1+\sqrt{x}} dx\,.$$

Multiply by $\frac{\sqrt{x}}{\sqrt{x}}$ to get

$$\int \frac{\sqrt{x}dx}{\sqrt{x}(1+\sqrt{x})}\,.$$

Integrate by parts where $u = \sqrt{x}$ so that $du = \frac{dx}{2\sqrt{x}}$ and $v = 2\ln|1+\sqrt{x}|$ so that $dv = \frac{dx}{\sqrt{x}(1+\sqrt{x})}$ and we get

$$I = 2\sqrt{x}\ln|1+\sqrt{x}| - 2\int \frac{\ln|1+\sqrt{x}|}{2\sqrt{x}} dx\,.$$

Let $u = 1+\sqrt{x}$ so that $dw = \frac{dx}{2\sqrt{x}}$ in this second integral to get

$$I = 2\sqrt{x}\ln|1+\sqrt{x}| - 2\int \ln|w| dw\,.$$

Use integration by parts on this second integral with $u = \ln|w|$ so $du = \frac{dw}{w}$ with $v = w$ with $dv = dw$ and we get

$$\begin{aligned}
I &= 2\sqrt{x}\ln|1+\sqrt{x}| - 2\left[w\ln|w| - \int dw\right] \\
&= 2\sqrt{x}\ln|1+\sqrt{x}| - 2\left[w\ln|w| - w\right] \\
&= 2\sqrt{x}\ln|1+\sqrt{x}| - 2(1+\sqrt{x})\ln|1+\sqrt{x}| - 2(1+\sqrt{x})\,.
\end{aligned}$$

Integrate:

$$I = \int \frac{1}{\sqrt{1+\sqrt{x}}} dx\,.$$

Let $u = 1 + \sqrt{x}$ so that $\sqrt{x} = u - 1$ and

$$du = \frac{dx}{2\sqrt{x}} \Rightarrow dx = 2du\sqrt{x} = 2(u-1)du\,,$$

and we get for I

$$\begin{aligned} I &= \int \frac{2(u-1)}{\sqrt{u}} du = 2\int (\sqrt{u} - u^{-1/2}) du \\ &= 2\frac{u^{3/2}}{3/2} - 2\frac{u^{1/2}}{1/2} + C = \frac{4}{3}u^{3/2} - 4u^{1/2} + C \\ &= \frac{4}{3}(1+\sqrt{x})^{3/2} - 4(1+\sqrt{x})^{1/2} + C\,. \end{aligned}$$

Integrate:

$$I = \int t^{2/3}(t^{5/3}+1)^{2/3} dt\,.$$

Let $u = t^{5/3} + 1$ so $du = \frac{5}{3}t^{2/3}dt$ and we get

$$\begin{aligned} I &= \frac{3}{5}\int \frac{5}{3}t^{2/3}(t^{5/3}+1)^{2/3} dt \\ &= \frac{3}{5}\frac{(t^{5/3}+1)^{5/3}}{(5/3)} + C = \frac{9}{25}(t^{5/3}+1)^{5/3} + C\,. \end{aligned}$$

Integrate:

$$I = \int \frac{\cot(x)dx}{\ln(\sin(x))}\,.$$

Let $u = \ln(\sin(x))$ so that $du = \frac{\cos(x)}{\sin(x)}dx = \cot(x)dx$ and we get

$$\begin{aligned} I &= \int \frac{du}{u} = \ln|u| + C \\ &= \ln|\ln(\sin(x))| + C = \ln(\ln(\sin(x))) + C\,. \end{aligned}$$

For the following integrals we will only give hits as to how they maybe evaluated

Integrate:

$$I = \frac{dt}{\sqrt{1-e^{-t}}}$$

Hint let $u = (1-e^{-t})^{-1/2}$.

Integrate:

$$I = \int \frac{dx}{e^x - 1}\,.$$

Hint multiply by $\frac{e^{-x}}{e^{-x}}$.

Integrate:

$$I = \int \frac{du}{(e^u - e^{-u})^2}\,.$$

Hint multiply by $\frac{e^{2u}}{e^{2u}}$, and use the integration substitution $v = e^{2u}$.

Integrate:

$$I = \int e^x \cos(2x) dx$$

Hint let $u = e^v$ with $dv = \cos(2x)$ and use integration by parts twice.

Integrate:

$$I = \int \frac{dx}{x(1 + x^{1/3})}$$

Hint let $u = 1 + x^{1/3}$

Integrate:

$$I = \int \frac{z^5 dz}{\sqrt{1 + z^2}}\,.$$

Hint use integration by parts with $u = z^4$ and $dv = z(1 + z^2)^{-1/2}$ three times.

Integrate:

$$I = \int (\sin^{-1}(x))^2 dx\,.$$

Hint use integration by parts with $u = (\sin^{-1}(x))^2$ and $dv = dx$.

Integrate:

$$I = \int \frac{x^3}{(x^2 + 1)^2}\,.$$

Hint use integration by parts with $u = x^2$ and $dv = \frac{xdx}{(x^2+1)^2}$.

Integrate:

$$I = \int x\sqrt{2x + 1} dx$$

Hint use integration by parts with $u = x$ and $dv = (2x + 1)^{1/2} dx$.

Integrate:

$$I = \int e^{-x} \tan^{-1}(e^x) dx$$

Hint $u = \tan^{-1}(e^x)$ and $dv = e^{-x}$ and use integration by parts.

Show:

$$I = \int \ln\big(x + \sqrt{x}\big) = \ln\big(\sqrt{x}\big) + \ln\big(1 + \sqrt{x}\big)$$

Integrate:

$$I = \int \frac{x}{x^2 + 4x + 3}\,.$$

Hint use partial fractions.

Integrate:

$$I = \int \frac{\sqrt{x^2 - a^2}}{x} dx$$

Hint divide by x and let $\frac{a}{x} = \sin(\theta)$.

Integrate:

$$I = \int \frac{dx}{x(3\sqrt{x} + 1)}\,.$$

Let $u = 3\sqrt{x} + 1$.

Integrate:

$$I = \int \frac{\cot(\theta) d\theta}{1 + \sin^2(\theta)}\,.$$

Hint let $u = \sin(\theta)$.

Integrate:

$$I = \int \frac{e^{4t} dt}{1 + e^{2t})^{2/3}}\,.$$

Hint $u = e^{2t}$.

Integrate:

$$I = \int \frac{x dx}{\sqrt{1 - x}}$$

Hint let $u = 1 - x$.

Integrate:

$$I = \int \ln\Big(x + \sqrt{x^2 - 1}\Big) dx$$

Hint use integration by parts with $u = \ln\big(x + \sqrt{x^2 - 1}\big)$ and $dv = dx$.

Integrate:

$$I = \int \sin^{-1}(\sqrt{x}) dx$$

Hint use integration by parts with $u = \sin^{-1}(\sqrt{x})$ and $dv = dx$.

Chapter 7 (Applications of the Integral)

Section 7.2 (Volume)

Problem 4

To evaluate this volume, construct a line from $(0, 0)$ to (h, a). Such a line has a form $y(x) = a + \frac{a}{h}(x - h)$. Rotating this line about the x-axis we would have a differential of volume given by $dV = 2\pi y(x) dx$. The total volume is then the integral of this differential or

$$V = 2\pi \int_0^h \left(a + \frac{a}{h}(x - h) \right) dx = 2\pi \left(ah + \frac{a}{h} \left. \frac{(x - h)^2}{2} \right|_0^h \right) = \pi a h \, .$$

If $a = \frac{1}{\sqrt{2}}$ and $h = \frac{1}{\sqrt{2}}$ we get $V = \frac{\pi}{2}$.

Chapter 8 (Elementary Transcendental Functions)

Section 8.7 (The Hyperbolic and Inverse Hyperbolic Functions)

Problem 37

Part (a): A definition of the arctanh function is the value of y (given x) such that $x = \tanh(y)$. From the domain and range of the tanh function we see that the domain of the function $y = \text{arctanh}(x)$ is $-1 < x < +1$ and the range is $-\infty < y < \infty$.

We desire to take the derivative of $y = \text{arctanh}(x)$. We can do this by first solving for x and then taking the derivative implicitly. Solving for x gives

$$x = \tanh(y)\,.$$

Taking the derivative with respect to x gives

$$1 = \text{sech}^2(y)\frac{dy}{dx}\,.$$

But we can take the identity $\cosh^2(x) - \sinh^2(x) = 1$, and divide by $\cosh^2(x)$ to get an equivalent identity of $\text{sech}^2(x) = 1 - \tanh^2(x)$, so that our derivative becomes

$$\frac{dy}{dx} = \frac{1}{1 - \tanh^2(y)} = \frac{1}{1 - x^2}\,,$$

with a domain of $|x| < 1$.

Review Problems (Elementary Transcendental Functions)

Problem 33

We want to integrate $\int \frac{dx}{\sqrt{x}(1+x)}$. Let $u = \sqrt{x}$, so that $x = u^2$, and $du = \frac{dx}{2\sqrt{x}} = \frac{dx}{2u}$. With these our integral becomes

$$\int \frac{2du}{1+u^2} = 2\tan^{-1}(u) + C = 2\tan^{-1}(\sqrt{x}) + C\,.$$

Chapter 9 (Methods of Integration)

Section 9.2 (Integration by Parts)

Problem 10 (the integral of $\ln(x)$)

When we let $u = \ln(x)$ and $dv = dx$ using the integration by parts formula $\int u dv = uv - \int v du$ we have

$$\int \ln(x)dx = x\ln(x) - \int x\left(\frac{1}{x}\right)dx = x\ln(x) - x + c\,. \tag{1}$$

We can check that this is correct by taking the derivative of the given expression where we find

$$\ln(x) + 1 - 1 = \ln(x)\,,$$

as it should.

Problem 12 (the integral of $x\ln(x)$)

Let $u = x$ and $dv = \ln(x)dx$ and then use the result from Problem 10 in Equation 1 to get

$$\begin{aligned}\int x\ln(x)dx &= x(x\ln(x) - x) - \int 1(x\ln(x) - x)dx \\ &= x^2\ln(x) - x^2 - \int x\ln(x)dx + \frac{x^2}{2} + c\,.\end{aligned}$$

Solving for $\int x\ln(x)dx$ in the previous expression we find

$$\int x\ln(x)dx = \frac{1}{2}\left(x^2\ln(x) - \frac{x^2}{2}\right) + c = \frac{x^2}{4}(2\ln(x) - 1) + c\,. \tag{2}$$

Problem 14 (the integral of $\ln(x)^2$)

Let $u = \ln(x)$ and $dv = \ln(x)dx$ and then use the result from Problem 10 in Equation 1 to get

$$\begin{aligned}\int \ln(x)^2 dx &= \ln(x)(x\ln(x) - x) - \int \frac{1}{x}(x\ln(x) - x)dx \\ &= x\ln(x)^2 - x\ln(x) - \int \ln(x)dx + x + c \\ &= x\ln(x)^2 - x\ln(x) - x\ln(x) + x + x + c \\ &= x(\ln(x)^2 - 2\ln(x) + 2) + c\,. \end{aligned} \tag{3}$$

Problem 16 (the integral of $x\ln(x)^2$)

Let $u = x$ and $dv = \ln(x)^2$ with the solution to Problem 14 in Equation 3 we have

$$\int x\ln(x)^2 dx = x(x\ln(x)^2 - 2x\ln(x) + 2x) - \int (x\ln(x)^2 - 2x\ln(x) + 2x)dx$$

Then solving for $\int x\ln(x)^2 dx$ we get

$$2\int x\ln(x)^2 dx = x^2\ln(x)^2 - 2x^2\ln(x) + 2x^2 + 2\int x\ln(x)dx - \frac{2}{2}x^2 + c\,.$$

Using the results from Problem 12 in Equation 2 this becomes

$$\begin{aligned}\int x\ln(x)^2 dx &= \frac{1}{2}x^2\ln(x)^2 - x^2\ln(x) + \frac{x^2}{2} + \frac{x^2}{4}(2\ln(x) - 1) + c\\ &= \frac{1}{2}x^2\ln(x)^2 - \frac{1}{2}x^2\ln(x) + \frac{x^2}{4} + c\,.\end{aligned}$$

We can check this result using the result from Problem 28 with $p = 1$ where we get

$$\int x\ln(x)^2 dx = \frac{x^2}{2}\ln(x)^2 - \frac{2}{4}x^2\ln(x) + \frac{2}{8}x^2 + c\,,$$

the same as before.

Problem 18 (the integral of $x^2\ln(x)$)

Since $\int x\ln(x)dx = \frac{\ln(x)^2}{2} + c$ when we write $x^2\ln(x)$ as $x(x\ln(x))$ and use integration by parts with $u = x$ and $dv = x\ln(x)$ to get

$$\begin{aligned}\int x^2\ln(x)dx &= \frac{x}{2}\ln(x)^2 - \int x\ln(x)dx\\ &= \frac{x}{2}\ln(x)^2 - \frac{\ln(x)^2}{2} + c\,.\end{aligned}$$

Problem 20 (the integral of $x(x+10)^{50}$)

$$\begin{aligned}\int x(x+10)^{50}dx &= x\frac{(x+10)^{51}}{51} - \int \frac{(x+10)^{51}}{51}dx\\ &= x\frac{(x+10)^{51}}{51} - \frac{(x+10)^{52}}{52(51)} + c\,.\end{aligned}$$

Problem 27 (the integral of $x^p \ln(x)$)

Let $u = x^p$ and $dv = \ln(x)dx$ (and thus that $\int x\ln(x)dx = x\ln(x) - x$) with integration by parts we have

$$\begin{aligned}\int x^p \ln(x)dx &= x^p(x\ln(x) - x) - \int px^{p-1}(x\ln(x) - x)dx \\ &= x^{p+1}\ln(x) - x^{p+1} - p\int x^p\ln(x)dx + p\int x^p dx\,.\end{aligned}$$

Solving for $\int x^p \ln(x)dx$ we first get

$$(1+p)\int x^p\ln(x)dx = x^{p+1}\ln(x) - x^{p+1} + \frac{p}{p+1}x^{p+1} + c\,,$$

or

$$\int x^p\ln(x)dx = \frac{x^{p+1}}{p+1}\ln(x) - \frac{p}{(p+1)^2}x^{p+1} + c \quad \text{when} \quad p \neq 1\,. \tag{4}$$

Problem 28 (the integral of $x^p \ln(x)^2$)

Let $u = x^p$ and $dv = \ln(x)^2$ and use the results from Problem 14 namely Equation 3 to get

$$\begin{aligned}\int x^p\ln(x)^2dx &= x^p x(\ln(x)^2 - 2\ln(x) + 2) - \int px^{p-1}x(\ln(x)^2 - 2\ln(x) + 2)dx \\ &= x^{p+1}(\ln(x)^2 - 2\ln(x) + 2) - p\int x^p\ln(x)^2dx + 2p\int x^p\ln(x)dx - 2p\int x^p dx\,.\end{aligned}$$

Solving for $\int x^p\ln(x)^2dx$ and using the result from Problem 27 namely Equation 4 we have

$$(1+p)\int x^p\ln(x)^2dx = x^{p+1}(\ln(x)^2 - 2\ln(x) + 2) - 2p\left[\frac{x^{p+1}}{p+1}\ln(x) - \frac{x^{p+1}}{(p+1)^2}\right] - \frac{2p}{p+1}x^{p+1} + c\,.$$

Thus when $p \neq 1$ we have

$$\begin{aligned}\int x^p\ln(x)^2dx &= \frac{x^{p+1}}{p+1}\ln(x)^2 + \frac{1}{p+1}\left[-2 + \frac{2p}{p+1}\right]x^{p+1}\ln(x) \\ &\quad + \frac{1}{p+1}\left[2 - \frac{2p}{(p+1)^2} - \frac{2p}{p+1}\right]x^{p+1} + c \\ &= \frac{x^{p+1}}{p+1}\ln(x)^2 - \frac{2}{(p+1)^2}x^{p+1}\ln(x) + \frac{2}{(p+1)^3}x^{p+1} + c\end{aligned} \tag{5}$$

Problem 33 (the integral of $\ln(x)^n$)

Let $u = \ln(x)^n$ and $dv = dx$ so using integration by parts $\int udv = uv - \int vdu$ we get

$$\begin{aligned} \int \ln(x)^n dx &= x\ln(x)^n - \int xn\ln(x)^{n-1}\frac{1}{x}dx \\ &= x\ln(x)^n - n\int \ln(x)^{n-1}dx\,. \end{aligned}$$

Section 9.6 (Partial Fractions)

Problem 1

From the formulas given in the book

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}\,,$$

or

$$1 = A(x+1) + Bx\,.$$

Let $x = 0$ and $x = -1$ to get $1 = A$ and $1 = -B$. Thus

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}\,,$$

so the integral is given by

$$\int \frac{1}{x(x+1)} = \int \frac{dx}{x} - \int \frac{1}{x+1} = \ln|x| - \ln|x+1| + c = \ln\left|\frac{x}{x+1}\right| + c\,.$$

Problem 2

We write

$$\frac{5x-13}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}\,,$$

or

$$5x - 13 = A(x-3) + B(x-2)\,.$$

Let $x = 3$ and $x = 2$ to get

$$2 = B \quad \text{and} \quad -3 = -A\,.$$

Thus $A = 3$ and $B = 2$ and our integral is given by

$$\int \frac{5x-13}{(x-2)(x-3)}dx = \int \frac{3}{x-2}dx + \int \frac{2dx}{x-3} = 3\ln|x-2| + 2\ln|x-3| + c\,.$$

Problem 3

We write

$$\frac{-4}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}\,,$$

or

$$-4 = A(x+2) + B(x-2)\,.$$

Let $x = 2$ to get $-4 = 4A$ or $A = -1$. Let $x = -2$ to get $-4 = -4B$ so $B = 1$. Thus the integral we want is

$$\begin{aligned}\int \frac{-4}{x^2-4}dx &= -\int \frac{1}{x-2}dx + \int \frac{1}{x+2}dx \\ &= -\ln|x-2| + \ln|x+2| + c = \ln\left|\frac{x+2}{x-2}\right| + c\,.\end{aligned}$$

Problem 4

We write

$$\frac{5x+1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}\,,$$

or

$$5x+1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2\,.$$

Let $x = 1$ to get $6 = 3B$ or $B = 2$. Let $x = -2$ to get $-9 = C(9)$ or $C = -1$. Equating the coefficients of the $O(x^2)$ terms gives

$$0 = A + C \Rightarrow A = -C = 1\,.$$

Thus the integral we seek can be written as

$$\begin{aligned}\int \frac{5x+1}{(x-1)^2(x+2)}dx &= \int \frac{1}{x-1}dx + \int \frac{2}{(x-1)^2}dx - \int \frac{1}{x+2}dx \\ &= \ln|x+1| - 2(x-1)^{-1} - \ln|x+2| + c \\ &= \ln\left|\frac{x+1}{x+2}\right| - 2(x-1)^{-1} + c\,.\end{aligned}$$

Problem 5

We write

$$\frac{x^2+4x+5}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}\,,$$

or

$$x^2+4x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)\,.$$

Let $x = -2$ to get $4 - 8 + 5 = 0 + B(-1)(1)$ or $B = -1$. Let $x = -3$ to get $9 - 12 + 5 = 0 + 0 + C(-2)(-1)$ to get $C = 1$. Let $x = -1$ to get $1 - 4 + 5 = A(1)(2)$ or $A = 1$. Then we have shown that

$$\begin{aligned}\int \frac{x^2+4x+5}{(x+1)(x+2)(x+3)}dx &= \int \frac{1}{x+1}dx - \int \frac{1}{x+2}dx + \int \frac{1}{x+3} \\ &= \ln|x+1| - \ln|x+2| + \ln|x+3| + c \\ &= \ln\left|\frac{(x+1)(x+3)}{x+2}\right| + c\,.\end{aligned}$$

Problem 6

Note that

$$\frac{x^3+2x}{x^2-x-2}\,,$$

is not a proper rational fraction so we need to use long division to reduce it to a proper rational fraction. This is done with the following steps

$$\begin{array}{r|l} & \qquad\qquad\qquad x+1 \\ \hline x^2-x-2 & x^3 \qquad\quad +2x \\ & -x^3+x^2+2x \\ \hline & \qquad x^2+4x \\ & \quad -x^2 \;\; +x+2 \\ \hline & \qquad\qquad 5x+2 \end{array}$$

From this expression we have shown that

$$\frac{x^3+2x}{x^2-x-2} = x + 1 + \frac{5x+2}{x^2-x-2}\,.$$

Note that $x^2 - x - 2 = (x-2)(x+1)$ so we can write

$$\frac{5x+2}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}\,,$$

or

$$5x + 2 = A(x+1) + B(x-2)\,.$$

Let $x = -1$ to get $-3 = B(-3)$ or $B = 1$. Let $x = 2$ to get $12 = 3A$ or $A = 4$. Thus we have shown that

$$\frac{5x+2}{x^2-x-2} = \frac{4}{x-2} + \frac{1}{x+1}\,,$$

and our integral is given by

$$\begin{aligned}\int \frac{x^3+2x}{x^2-x-2}dx &= \int (x+1)dx + \int \frac{4}{x-2}dx + \int \frac{dx}{x+1} \\ &= \frac{x^2}{2} + x + 4\ln|x-2| + \ln|x+1| + c\,.\end{aligned}$$

Problem 7

Note that by the rules for partial fractions the integrand can be written as

$$\frac{x}{(x+1)^2(x-1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2},$$

or

$$x = A(x+1)(x-1)^2 + B(x-1)^2 + C(x-1)(x+1)^2 + D(x+1)^2 .$$

Let $x = 1$ so that $1 = 4D$ or $D = \frac{1}{4}$. Let $x = -1$ so that $-1 = B(2)^2$ or $B = -\frac{1}{4}$. Expanding the polynomial on the right-hand-side of the above expression we get

$$x = A(x^3 - x^2 - x + 1) + B(x^2 - 2x + 1) + C(x^3 + x^2 - x + 1) + D(x^2 + 2x + 1) .$$

Thus equating the coefficients of $O(x^3)$ terms we get $0 = A + C$ and equating the coefficients of the $O(x^2)$ terms

$$0 = -A + B + C + D = -A - \frac{1}{4} + C + \frac{1}{4} .$$

These last two equations taken together imply that $A = C = 0$ and our fraction is then written as

$$\frac{x}{(x+1)^2(x-1)^2} = -\frac{1}{4(x+1)^2} + \frac{1}{4(x-1)^2} .$$

With this expression our integral becomes

$$\int \frac{x}{(x+1)^2(x-1)^2} dx = +\frac{1}{4(x+1)} - \frac{1}{4(x-1)} + c .$$

Problem 8

Note that by the rules of partial fractions our integrand can be written as

$$\frac{3x+2}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4},$$

or

$$3x + 2 = A(x^2+4) + (Bx+C)(x+2)$$

Let $x = -2$ so that $-4 = 8A$ or $A = -\frac{1}{2}$. Let $x = \pm 2i$ say $x = 2i$ then we get

$$6i + 2 = (2iB + C)(2i + 2) = -4B + 4iB + 2iC + 2C = (4B + 2C)i + (2C - 4B) .$$

Equating the real and imaginary parts of the above we must have

$$\begin{aligned} 6 &= 4B + 2C \\ 2 &= 2C - 4B . \end{aligned}$$

When we solve this system for B and C we get $B = \frac{1}{2}$ and $C = 2$. Thus we have shown that

$$\frac{3x+2}{(x+2)(x^2+4)} = -\frac{1}{2}\frac{1}{x+2} + \frac{\frac{1}{2}x + 2}{x^2+4} .$$

To integrate this we have

$$\begin{aligned}\int \frac{3x+2}{(x+2)(x^2+4)}dx &= -\frac{1}{2}\ln|x+2| + \frac{1}{2}\int \frac{x}{x^2+4}dx + 2\int \frac{dx}{x^2+4} \\ &= -\frac{1}{2}\ln|x+2| + \frac{1}{4}\ln|x^2+4| + 2\int \frac{dx}{x^2+4} \\ &= -\frac{1}{2}\ln|x+2| + \frac{1}{4}\ln|x^2+4| + \frac{2}{2}\arctan\left(\frac{x}{2}\right) + c \\ &= -\frac{1}{2}\ln|x+2| + \frac{1}{4}\ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + c\,.\end{aligned}$$

Problem 9

The given fraction is not proper and thus we need to do long division on it.

$$\begin{array}{r|l} & 1 \\ \hline x^3+x & x^3+3x^2-x+3 \\ & -x^3 \qquad\quad -x \\ \hline & \qquad 3x^2-2x+3 \end{array}$$

Thus we have

$$\frac{x^3+3x^2-x+3}{x^3+x} = 1 + \frac{3x^2-2x+3}{x(x^2+1)}\,.$$

Now the remaining fraction can be written as

$$\frac{3x^2-2x+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}\,,$$

or

$$3x^2-2x+3 = A(x^2+1) + (Bx+C)x\,.$$

Expanding the right-hand-side of the above gives

$$Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A\,.$$

Thus

$$\begin{aligned} A+B &= 3 \\ C &= -2 \\ A &= 3\,. \end{aligned}$$

Thus $B = 0$ and we have

$$\frac{3x^2-2x+3}{x(x^2+1)} = \frac{3}{x} - \frac{2}{x^2+1}\,.$$

So to integrate we have

$$\int \frac{x^3+3x^2-x+3}{x^3+x}dx = x + 3\int \frac{dx}{x} - 2\int \frac{dx}{x^2+1} = x + 3\ln|x| - 2\arctan(x) + c\,.$$

Problem 10

The given fraction can be written as

$$\frac{x^2+5x+6}{(x^2+4)(x^2+9)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+9}\,,$$

or

$$\begin{aligned} x^2+5x+6 &= (Ax+B)(x^2+9) + (Cx+D)(x^2+4) \\ &= Ax^3 + 9Ax + Bx^2 + 9B + Cx^3 + 4Cx + Dx^2 + 4D \\ &= (A+C)x^3 + (B+D)x^2 + (9A+4C)x + (9B+4D)\,. \end{aligned}$$

Thus we have

$$\begin{aligned} A+C &= 0 \\ B+D &= 1 \\ 9A+4C &= 5 \\ 9B+4D &= 6\,. \end{aligned}$$

Solving this system of equations we have $A=1$, $B=\frac{2}{5}$, $C=-1$, and $D=-\frac{3}{5}$. Thus we get

$$\frac{x^2+5x+6}{(x^2+4)(x^2+9)} = \frac{x+\frac{2}{5}}{x^2+4} - \frac{x+\frac{3}{5}}{x^2+9}\,,$$

Then to integrate we have

$$\begin{aligned} \int \frac{x^2+5x+6}{(x^2+4)(x^2+9)}dx &= \int \frac{x}{x^2+4}dx + \frac{2}{5}\int \frac{1}{x^2+4}dx - \int \frac{x}{x^2+9}dx - \frac{3}{5}\int \frac{1}{x^2+9}dx \\ &= \frac{1}{2}\ln\big(x^2+4\big) + \frac{1}{5}\arctan\left(\frac{x}{2}\right) - \frac{1}{2}\ln\big(x^2+9\big) - \frac{1}{5}\arctan\left(\frac{x}{3}\right) + c\,. \end{aligned}$$

Problem 11

The quadratic in the denominator has roots given by

$$\frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2\pm\sqrt{-4}}{2}\,.$$

Thus this is an irreducible quadratic. The formula for partial fractions in this case is given by

$$\frac{2x^2+4}{x(x^2+2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}\,,$$

or

$$2x^2+4 = A(x^2+2x+2) + (Bx+C)x\,.$$

Expanding the right-hand-side we get

$$Ax^2 + 2Ax + 2A + Bx^2 + Cx = (A+B)x^2 + (2A+C)x + 2A\,.$$

Thus for this to match the left-hand-side requires

$$
\begin{aligned}
2 &= A + B \\
0 &= 2A + C \\
4 &= 2A\,.
\end{aligned}
$$

Thus $A = 2$, $B = 0$, and $C = -4$ and we have shown that

$$\frac{2x^2+4}{x(x^2+2x+2)} = \frac{2}{x} - \frac{4}{x^2+2x+2} = \frac{2}{x} - \frac{4}{(x+1)^2+1}\,.$$

Thus we can evaluate the integral given as

$$\int \frac{2x^2+4}{x(x^2+2x+2)}dx = 2\ln|x| - 4\arctan(x+1) + c\,.$$

Problem 12

Using the rules of partial fractions we can write

$$\frac{3}{x(x^2-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}\,,$$

or

$$3 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)\,.$$

Let $x = 1$ to get $3 = 2C$ or $C = \frac{3}{2}$. Let $x = -1$ to get $3 = -B(-2)$ or $B = \frac{3}{2}$. Let $x = 0$ to get $3 = A(-1)$ or $A = -3$ and we have shown

$$\frac{3}{x(x^2-1)} = -\frac{3}{x} + \frac{3}{2(x+1)} + \frac{3}{2(x-1)}\,.$$

With this expression the integral we seek is given by

$$\int \frac{3}{x(x^2-1)}dx = -3\ln(x) + \frac{3}{2}\ln|x+1| + \frac{3}{2}\ln|x-1| + c\,.$$

Problem 13

Using the rules of partial fractions we can write

$$\frac{1}{x^4-1} = \frac{1}{(x^2+1)(x-1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1}\,,$$

or

$$1 = (Ax+B)(x-1)(x+1) + C(x^2+1)(x+1) + D(x^2+1)(x-1)\,.$$

Let $x = 1$ to get $1 = C(2)(2)$ or $C = \frac{1}{4}$. Let $x = -1$ to get $1 = D(2)(-2)$ or $D = -\frac{1}{4}$. Let $x = i = \sqrt{-1}$ to get

$$1 = (Ai + B)(-1 - 1) \quad \text{so} \quad A = 0 \quad \text{and} \quad B = -\frac{1}{2}\,.$$

Thus the integral we seek can be evaluated as

$$\begin{aligned}
\int \frac{1}{x^4 - 1} dx &= -\frac{1}{2} \int \frac{dx}{x^2 + 1} + \frac{1}{4} \int \frac{dx}{x - 1} + \frac{1}{4} \int \frac{dx}{x + 1} \\
&= -\frac{1}{2} \arctan(x) + \frac{1}{4} \ln|x - 1| + \frac{1}{4} \ln|x + 1| + c \\
&= -\frac{1}{2} \arctan(x) + \frac{1}{4} \ln|(x - 1)(x + 1)| + c\,.
\end{aligned}$$

Problem 14

To evaluate this integral we will need to reduce the fraction to proper form. Thus we need to perform polynomial long division. We find

$$\begin{array}{r|rrrr}
 & & & 3x & +2\,, \\ \hline
x^2+4 & 3x^3 & +2x^2 & +2x & +6 \\
 & -3x^3 & & -12x & \\ \hline
 & & 2x^2 & -10x & +6 \\
 & & -2x^2 & & -8 \\ \hline
 & & & -10x & -2
\end{array}$$

so that

$$\frac{3x^3 + 2x^2 + 2x + 6}{x^2 + 4} = 3x + 2 - \frac{10x + 2}{x^2 + 4}\,.$$

From this we see that our integral can be given by

$$\begin{aligned}
\int \frac{3x^3 + 2x^2 + 2x + 6}{x^2 + 4} &= \frac{3x^2}{2} + 2x - 10 \int \frac{x}{x^2 + 4} dx - 2 \int \frac{dx}{x^2 + 4} \\
&= \frac{3x^2}{2} + 2x - \frac{10}{2} \ln(x^2 + 4) - \arctan\left(\frac{x}{2}\right) + c\,.
\end{aligned}$$

Problem 15

Using the rules of partial fractions we can write

$$\frac{1}{(x - 1)^2 (x + 1)^2} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{x + 2} + \frac{D}{(x + 2)^2}\,,$$

or

$$1 = A(x - 1)(x + 2)^2 + B(x + 2)^2 + C(x + 2)(x - 1)^2 + D(x - 1)^2\,. \tag{6}$$

Let $x = 1$ to get $1 = 9B$ or $B = \frac{1}{9}$. Let $x = -2$ to get $1 = 9D$ or $D = \frac{1}{9}$. Taking the x derivative of Equation 6 we get

$$0 = A(x+2)^2 + 2A(x-1)(x+2) + 2B(x+2) + C(x-1)^2 + 2C(x+2)(x-1) + 2D(x-1)\,.$$

Let $x = 1$ in this derivative to get

$$0 = 9A + 6B = 9A + \frac{6}{9}\,.$$

Thus $A = -\frac{2}{27}$. Let $x = -2$ in the derivative to get

$$0 = C(-3)^2 + 2D(-3) = 9C - 6D = 9C - \frac{2}{3}\,.$$

Thus $C = \frac{2}{27}$. Using these we get that we can expand our integrand as

$$\frac{1}{(x-1)^2(x+2)^2} = -\frac{2}{27}\frac{1}{x-1} + \frac{1}{9}\frac{1}{(x-1)^2} + \frac{2}{27}\frac{1}{x+2} + \frac{1}{9}\frac{1}{(x+2)^2}$$

Using this our integral can be computed as

$$\begin{aligned}\int \frac{dx}{(x-1)^2(x+2)^2} &= -\frac{2}{27}\ln|x-1| - \frac{1}{9(x-1)} + \frac{2}{27}\ln|x+2| - \frac{1}{9(x+2)} + c \\ &= \frac{2}{27}\ln\left|\frac{x+2}{x-1}\right| - \frac{1}{9(x-1)} - \frac{1}{9(x+2)} + c\,.\end{aligned}$$

Problem 16

The roots of the quadratic polynomial in the denominator is given by $\frac{-2\pm\sqrt{4-20}}{2}$ which are two imaginary numbers, thus the quadratic is irreducible. The rules of partial fractions give

$$\frac{3x-4}{(x^2+2x+5)(x^2+2)} = \frac{Ax+B}{x^2+2x+5} + \frac{Cx+D}{x^2+2}\,,$$

or

$$3x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+2x+5)\,.$$

Expanding the right-hand-side of the above gives

$$(A+C)x^3 + (B+2C+D)x^2 + (2A+5C+2D)x + (2B+5D)\,.$$

For this expression to equal the original left-hand-side (or $3x-4$) for all x, we must have

$$\begin{aligned} A + C &= 0 \\ B + 2C + D &= 0 \\ 2A + 5C + 2D &= 3 \\ 2B + 5D &= -4\,. \end{aligned}$$

When we solve for A, B, C, and D in the above we get $A=-1$, $B=-2$, $C=1$, and $D=0$. Thus the integrand for this problem can be written as

$$\frac{3x-4}{(x^2+2x+5)(x^2+2)} = -\frac{x}{x^2+2x+5} + \frac{x}{x^2+2}\,.$$

Thus the integral can now be evaluated as

$$\begin{aligned}
\int \frac{3x-4}{(x^2+2x+5)(x^2+2)}dx &= -\int \frac{x}{(x+1)^2+4}dx + \int \frac{x}{x^2+2}dx \\
&= -\int \frac{x+1}{(x+1)^2+4}dx + \int \frac{1}{(x+1)^2+4}dx + \int \frac{x}{x^2+2}dx \\
&= -\frac{1}{2}\ln\left((x+1)^2+4\right) + \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + \frac{1}{2}\ln\left(x^2+2\right) + c\,.
\end{aligned}$$

Problem 17

The rules of partial fractions state that we can write our fraction as

$$\frac{1}{(x+b)(x+d)} = \frac{A}{x+b} + \frac{B}{x+d}\,,$$

for some A and B. The above is the same as

$$1 = A(x+d) + B(x+b)\,.$$

Let $x=-d$ to get $1=B(b-d)$ or $B=\frac{1}{b-d}$ assuming that $b\neq d$. Let $x=-b$ to get $1=A(d-b)$ so $A=\frac{1}{d-b}$. Thus our integral can be written as

$$\begin{aligned}
\int \frac{dx}{(x+b)(x+d)} &= \frac{1}{d-b}\int \frac{dx}{x+b} + \frac{1}{b-d}\int \frac{dx}{x+d} \\
&= \frac{1}{d-b}\ln|x+b| + \frac{1}{b-d}\ln|x+d| + c \\
&= \frac{1}{d-b}\ln\left|\frac{x+b}{x+d}\right| + c\,.
\end{aligned}$$

Problem 18

Note that our fraction can be written as

$$\frac{1}{(ax+b)(cx+d)} = \frac{1}{ac\left(x+\frac{b}{a}\right)\left(x+\frac{d}{c}\right)}\,.$$

The rules of partial fractions state that we can write

$$\frac{1}{\left(x+\frac{b}{a}\right)\left(x+\frac{d}{c}\right)} = \frac{A}{x+\frac{b}{a}} + \frac{B}{x+\frac{d}{c}}\,,$$

for some A and B. The above is the same as

$$1 = A\left(x + \frac{d}{c}\right) + B\left(x + \frac{b}{a}\right).$$

Let $x = -\frac{d}{c}$ in the above to get $1 = B\left(\frac{b}{a} - \frac{d}{c}\right) = B\left(\frac{bc-ad}{ac}\right)$. Thus

$$B = \frac{ac}{bc - ad},$$

assuming that $bc - ad \neq 0$. Let $x = -\frac{b}{a}$ in the above to get $1 = A\left(-\frac{b}{a} + \frac{d}{c}\right) = A\left(\frac{ad-cb}{ac}\right)$. Thus

$$A = \frac{ac}{ad - cb},$$

again assuming that $bc - ad \neq 0$. Thus we have shown that we can write our integrand as

$$\begin{aligned}
\frac{1}{(ax+b)(cx+d)} &= \frac{1}{ac}\left[\frac{ac}{ad-cb}\frac{1}{\left(x+\frac{b}{a}\right)} - \frac{ac}{ad-cb}\frac{1}{\left(x+\frac{d}{c}\right)}\right] \\
&= \frac{1}{ad-cb}\left(\frac{1}{x+\frac{b}{a}} - \frac{1}{x+\frac{d}{c}}\right).
\end{aligned}$$

Thus the integral we seek is given by

$$\begin{aligned}
\int \frac{dx}{(ax+b)(cx+d)} &= \frac{1}{ad-cb}\left(\ln\left|x+\frac{b}{a}\right| - \ln\left|x+\frac{d}{c}\right|\right) + c_1 \\
&= \frac{1}{ad-cb}\ln\left|\frac{x+\frac{b}{a}}{x+\frac{d}{c}}\right| + c_1 = \frac{1}{ad-cb}\ln\left|\frac{\frac{1}{a}(ax+b)}{\frac{1}{c}(cx+d)}\right| + c_1 \\
&= \frac{1}{ad-cb}\ln\left|\frac{ax+b}{cx+d}\right| + \frac{1}{ad-cb}\ln\left|\frac{c}{a}\right| + c_1 \\
&= \frac{1}{ad-cb}\ln\left|\frac{ax+b}{cx+d}\right| + c_2.
\end{aligned}$$

Here c_1 and c_2 are constants.

Problem 19

The rules of partial fractions state that we can write

$$\frac{1}{a^2 - x^2} = -\frac{1}{x^2 - a^2} = -\left(\frac{A}{x-a} + \frac{B}{x+a}\right),$$

for some A and B. The above is the same as

$$-1 = -A(x+a) - B(x-a).$$

Let $x = a$ to get $-1 = -B(-2a)$ or $B = -\frac{1}{2a}$. Let $x = +a$ to get $-1 = -2aA$ or $A = \frac{1}{2a}$. Thus we have shown that

$$\frac{1}{a^2 - x^2} = -\frac{1}{2a(x-a)} + \frac{1}{2a(x+a)}.$$

Using this expression we have that our integral given by

$$\int \frac{dx}{a^2 - x^2} = -\frac{1}{2a}\ln|x-a| + \frac{1}{2a}\ln|x+a| + c = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c.$$

Problem 20

The rules of partial fractions state that we can write

$$\frac{1}{(a^2-x^2)^2} = \frac{1}{(a-x)^2(a+x)^2} = \frac{A}{a-x} + \frac{B}{(a-x)^2} + \frac{C}{a+x} + \frac{D}{(a+x)^2}\,.$$

for some A, B, C, and D. The above is the same as

$$1 = A(a-x)(a+x)^2 + B(x+a)^2 + C(a+x)(a-x) + D(a-x)^2\,.$$

Let $x = a$ to get $1 = B(4a^2)$ or $B = \frac{1}{4a^2}$ Let $x = -a$ to get $1 = D(2a)^2$ or $D = \frac{1}{4a^2}$. To determine the coefficients A and C we take the derivative of the equation above. We find

$$0 = A(-1)(a+x)^2 + 2B(x+a) + C(a-x)^2 + 2C(a+x)(a-x)(-1) + 2D(a-x)(-1)\,.$$

Let $x = -a$ in the above to get

$$0 = C(2a)^2 + 2D(-1)(2a)\,.$$

Let $x = a$ in the above to get

$$0 = A(-1)(2a)^2 + 2B(2a)\,.$$

Solving these two equations for A and C we find $A = \frac{1}{4a^3}$ and $C = \frac{1}{4a^3}$. Thus we have shown that

$$\frac{1}{(a^2-x^2)^2} = \frac{1}{4a^3}\frac{1}{a-x} + \frac{1}{4a^2}\frac{1}{(a-x)^2} + \frac{1}{4a^3}\frac{1}{a+x} + \frac{1}{4a^2}\frac{1}{(a+x)^2}\,.$$

Using this expression we can now perform the desired integration. We find

$$\begin{aligned}
\int \frac{dx}{(a^2-x^2)^2} &= -\frac{1}{4a^3}\ln|a-x| + \frac{1}{4a^2}\frac{1}{a-x} + \frac{1}{4a^3}\ln|a+x| - \frac{1}{4a^2}\frac{1}{a+x} + c \\
&= \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right| + \frac{1}{4a^2}\frac{1}{a-x} - \frac{1}{4a^2}\frac{1}{a+x} + c \\
&= \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right| + \frac{1}{4a^2}\left(\frac{a+x-a+x}{a^2-x^2}\right) + c \\
&= \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right| + \frac{1}{2a^2}\left(\frac{x}{a^2-x^2}\right) + c\,.
\end{aligned}$$

Problem 21

The rules of partial fractions state that we can write our integrand as

$$\frac{x}{a^2-x^2} = \frac{A}{a-x} + \frac{B}{a+x}\,.$$

for some A and B. The above is the same as

$$1 = A(a+x) + B(a-x)\,.$$

Let $x=-a$ to get $-a=B(2a)$ so $B=-\frac{1}{2}$. Let $x=a$ to get $a=A(2a)$ so $A=\frac{1}{2}$. Thus we have shown that

$$\frac{x}{a^2-x^2}=\frac{1}{2(a-x)}-\frac{1}{2(a+x)}\,.$$

Thus we can integrate as

$$\int \frac{x}{a^2-x^2}=-\frac{1}{2}\ln|a-x|-\frac{1}{2}\ln|a+x|+c=-\frac{1}{2}\ln|(a-x)(a+x)|+c=-\frac{1}{2}\ln|a^2-x^2|+c\,.$$

Chapter 9 Review Problems

Problem 44

In the integral

$$\int \frac{\cos(\arcsin(x))}{\sqrt{1-x^2}}dx\,,$$

let $v=\arcsin(x)$ so that $dv=\frac{dx}{\sqrt{1-x^2}}$ and the integral is given by

$$\int \cos(v)dv=\sin(v)+c=\sin(\arcsin(x))+c\,.$$

Chapter 12 (Infinite Series)

Review Problems (Infinite Series)

Problem 42

This statement is false. As a counter example let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n}$ then both individual series diverge but the product series $a_n b_n = \frac{1}{n^2}$ converges.

Chapter 13 (Taylor's Approximation and Power Series)

Section 13.6 (The Binomial and Some Other Series)

Problem 1

To derive the Taylor series of the given $f(x)$ we have

$$
\begin{aligned}
f(x) &= (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right)x^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{-2k+3}{2}\right)x^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(2\cdot 1-3)(2\cdot 2-3)(2\cdot 3-3)\cdots(2k-3)}{2^k k!}x^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k(-1)(1)(3)(5)\cdots(2k-3)}{2^k k!}x^k \\
&= 1+\frac{x}{2}+\sum_{k=2}^{\infty} \frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-3)x^k}{2^k k!}\,.
\end{aligned}
$$

Applying the ratio test to the above series we have

$$
\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \limsup_{k\to\infty} \frac{\frac{1\cdot 3\cdot 5\cdots(2k-3)(2k-1)|x|^{k+1}}{2^{k+1}(k+1)!}}{\frac{1\cdot 3\cdot 5\cdots(2k-3)|x|^k}{2^k k!}} = \limsup_{k\to\infty} |x|\frac{2k-1}{2(k+1)} = |x|\,.
$$

Thus to have convergence we require this limit to be less than one or $|x|<1$. This implies that $\rho=+1$.

Problem 3

To derive the Taylor series for the given $f(x)$ about x_0 we have

$$
f(x) = (4+x)^{-1/2} = 4^{-1/2}\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(1+\frac{x}{4}\right)^{-1/2}\,.
$$

From which we see that

$$\begin{aligned}
f(x) &= \frac{1}{2}\sum_{k=0}^{\infty}\left(\begin{array}{c}-1/2\\k\end{array}\right)\left(\frac{x}{4}\right)^k \\
&= \frac{1}{2}\left(1+\sum_{k=1}^{\infty}\left(\begin{array}{c}-1/2\\k\end{array}\right)\left(\frac{x}{4}\right)^k\right) \\
&= \frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-k+1\right)\left(\frac{x}{4}\right)^k\right) \\
&= \frac{1}{2}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k(1)(3)(5)\cdots(2k-1)}{2^k k!}\left(\frac{x}{4}\right)^k\right).
\end{aligned}$$

Applying the ratio test to the above series we have

$$\limsup_{k\to\infty}\frac{|a_{k+1}|}{|a_k|} = \limsup_{k\to\infty}\frac{(2k+1)}{2(k+1)}\left(\frac{|x|}{4}\right) = \frac{|x|}{4}.$$

To have convergence we require $\frac{|x|}{4} < 1$ which implies that $\rho = +4$.

Problem 5

We desire the Taylor series of $f(x) = \ln\left(\sqrt{1+x^2}+x\right)$ at the point $x_0 = 0$. Taking the derivative of f we find that

$$\begin{aligned}
f'(x) &= \frac{\left(\frac{1}{2}(1+x^2)^{-1/2}(2x)+1\right)}{\sqrt{1+x^2}+x} = \frac{\frac{x}{\sqrt{1+x^2}}+1}{\sqrt{1+x^2}+x} \\
&= 1\left(\frac{1}{\sqrt{1+x^2}}\right) = \sum_{k=0}^{\infty}\left(\begin{array}{c}-1/2\\k\end{array}\right)x^{2k}.
\end{aligned}$$

Now we can expand the binomial coefficient in the above as

$$\begin{aligned}
\left(\begin{array}{c}-1/2\\k\end{array}\right) &= \frac{1}{k!}\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-k+1\right) \\
&= \frac{1}{2^k k!}(-1)(-1-2)\cdots(-1-2k+2) = \frac{(-1)^k(1\cdot 3\cdot 5\cdots(2k-1))}{2^k k!}.
\end{aligned}$$

Thus we find that $f'(x)$ is given as a Taylor series by

$$f'(x) = \sum_{k=0}^{\infty}\frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!}x^{2k}.$$

Integrating this expression term by term we find that our desired function $f(x)$ is given by

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k 1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!(2k+1)}x^{2k+1} + C,$$

where C is an integration constant that needs to be determined. Setting $x = 0$ in the above series gives $f(0) = C$, while setting $x = 0$ in the original expression for $f(x)$ gives $f(0) = \ln(1) = 0$. From which we see that we should take $C = 0$.

Problem 14

We find that our desired $f(x)$ can be written as

$$\begin{aligned} f(x) &= (8-x)^{1/3} = 2\left(1-\frac{x}{8}\right)^{1/3} \\ &= 2\sum_{k=0}^{\infty}\left(\begin{array}{c} 1/3 \\ k \end{array}\right)\left(-\frac{x}{8}\right)^k \\ &= 2\left(1+\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\cdots\left(\frac{1}{3}-k+1\right)\left(-\frac{x}{8}\right)^k\right). \end{aligned}$$

Computing the first few terms in this series for $k=1$ and $k=2$ we have

$$f(x) = 2\left(1-\left(\frac{1}{3}\right)\left(\frac{x}{8}\right)+\frac{1}{2}\left(-\frac{2}{3}\right)\left(\frac{x^2}{64}\right)+\cdots\right) = 2\left(1-\frac{x}{24}-\frac{x^2}{192}+\cdots\right).$$

Problem 16

Part (a): We have, by taking the term by term derivative of the given expression that

$$f'(x) = \sum_{k=1}^{\infty}\left(\begin{array}{c} \alpha \\ k \end{array}\right)kx^{k-1} = \sum_{k=0}^{\infty}\left(\begin{array}{c} \alpha \\ k+1 \end{array}\right)(k+1)x^k.$$

For $|x|<1$. Note that since the $k=0$ term is a constant its derivative is zero.

Part (b): After multiplying $f'(x)$ by $1+x$ we have

$$(1+x)f'(x) = \sum_{k=0}^{\infty}\left[\left(\begin{array}{c} \alpha \\ k+1 \end{array}\right)(k+1)x^k + \left(\begin{array}{c} \alpha \\ k+1 \end{array}\right)(k+1)x^{k+1}\right].$$

Considering the second summation above we see that it is equal to $\sum_{k=1}^{\infty}\left(\begin{array}{c} \alpha \\ k \end{array}\right)kx^k$, by shifting the index on k. Since we can include the $k=0$ term in this summation (the multiplication of k makes this term zero) we can write it as $\sum_{k=0}^{\infty}\left(\begin{array}{c} \alpha \\ k \end{array}\right)kx^k$. Which when put into the first summation gives us the result that $(1+x)f'(x)$ is equal to

$$\sum_{k=0}^{\infty}\left[\left(\begin{array}{c} \alpha \\ k+1 \end{array}\right)(k+1)+\left(\begin{array}{c} \alpha \\ k \end{array}\right)k\right]x^k,$$

as requested.

Part (c): We have that the expression on the right hand side is equal to

$$
\begin{aligned}
\binom{\alpha}{k+1}(k+1) + \binom{\alpha}{k} k &= \left(\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(k+1)+1)}{(k+1)!}\right)(k+1) \\
&+ \left(\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}\right) k \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!} \\
&+ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)k}{k!} \\
&= \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}(\alpha-k+k) \\
&= \alpha \binom{\alpha}{k}.
\end{aligned}
$$

Part (d): Using the identity in Part (c) and the expression found in Part (b) we see that

$$(1+x)f'(x) = \alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \alpha f(x)$$

so that a differential equation satisfied by $f(x)$ is given by

$$\frac{f'}{f} = \frac{\alpha}{1+x}.$$

When we integrate both sides of this expression we obtain

$$\ln(f(x)) = \alpha \ln(1+x) + C_1,$$

with C_1 an integration constant. To evaluate this constant take $x = 0$. Since $f(0) = 1$ we see that $C_1 = 0$ and so we can finally conclude that

$$f(x) = (1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Vectors and Three-Dimensional Analytic Geometry

Vectors in Two Dimensions

Problem 1

We have

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= (-2,1)+(3,-2)=(1,-1)\\
\mathbf{a}-\mathbf{b} &= (-2,1)-(3,-2)=(-5,3)\\
2\mathbf{a}-3\mathbf{b} &= 2(-2,1)-3(3,-2)=(-4,2)-(9,-6)=(-13,8)\,.
\end{aligned}$$

Problem 2

We have

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= (1,0)+(-2,1)=(-1,1)\\
\mathbf{a}-\mathbf{b} &= (1,0)-(-2,1)=(3,-1)\\
2\mathbf{a}-3\mathbf{b} &= 2(2,0)-3(-2,1)=(2,0)-(6,3)=(-4,-3)\,.
\end{aligned}$$

Problem 3

We have

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= (1,1)+(1,-1)=(2,0)\\
\mathbf{a}-\mathbf{b} &= (1,1)-(1,-1)=(0,2)\\
2\mathbf{a}-3\mathbf{b} &= 2(1,1)-3(1,-1)=(2,2)-(3,-3)=(-1,5)\,.
\end{aligned}$$

Problem 4

We have

$$\begin{aligned}
\mathbf{a}+\mathbf{b} &= (2,-5)+(1,4)=(3,-1)\\
\mathbf{a}-\mathbf{b} &= (2,-5)-(1,4)=(1,-9)\\
2\mathbf{a}-3\mathbf{b} &= 2(2,-5)-3(1,4)=(4,-10)-(3,12)=(1,-22)\,.
\end{aligned}$$

Problem 5

We have

$$\overrightarrow{\mathbf{PQ}} = (4, -2) - (2, 1) = (2, -3)\,.$$

Problem 6

We have

$$\overrightarrow{\mathbf{PQ}} = (2, -2) - (-1, -4) = (3, 2)\,.$$

Problem 7

We have

$$\overrightarrow{\mathbf{PQ}} = (-1, 0) - (2, 4) = (-3, -4)\,.$$

Problem 8

We have

$$\overrightarrow{\mathbf{PQ}} = (1, 4) - (5, -2) = (-4, 6)\,.$$

Problem 9

We have

$$\begin{aligned}\overrightarrow{\mathbf{PQ}} &= \mathbf{b} - \mathbf{a} = (3, -2) - (2, 1) = (1, -3) \\ \overrightarrow{\mathbf{QP}} &= \mathbf{a} - \mathbf{b} = (2, 1) - (3, -2) = (-1, 3)\,.\end{aligned}$$

Problem 10

We know the value of the vector $\overrightarrow{\mathbf{PQ}}$ and it can be computed as

$$\overrightarrow{\mathbf{PQ}} = \mathbf{b} - \mathbf{a} = (-3, 2)\,.$$

Since we know $\mathbf{a}$ this is

$$\mathbf{b} - (2, -1) = (-3, 2)\,,$$

or

$$\mathbf{b} = (2, -1) + (-3, 2) = (-1, 1)\,.$$

Problem 11

If we have $\mathbf{a} = \mathbf{b}$ then

$$(x + y, x - y) = (2, 3)\,,$$

or

$$\begin{aligned} x + y &= 2 \\ x - y &= 3\,. \end{aligned}$$

If we add these two equations we get $x = \frac{5}{2}$. If we subtract these two equations we get $y = -\frac{1}{2}$.

Problem 12

If we have $\mathbf{a} = 2\mathbf{b}$ then

$$(x + y, 2) = 2(3, x - 2y)\,,$$

or

$$(x + y, 2) = (6, 2x - 4y)\,.$$

This is equivalent to the system

$$\begin{aligned} x + y &= 6 \\ 2x - 4y &= 2\,. \end{aligned}$$

From the last equation we have $x = 1 + 2y$. If we put this into the first equation we can solve for y to find $y = \frac{5}{3}$. This means that $x = \frac{13}{3}$.

Problem 13

We have $||\mathbf{a}|| = \sqrt{2^2 + 1^2} = \sqrt{5}$.

Problem 14

We have $||\mathbf{a}|| = \sqrt{1 + 9} = \sqrt{10}$.

Problem 15

We have $||\mathbf{a}|| = \sqrt{9 + 16} = 5$.

Problem 16

We have $||\mathbf{a}|| = \sqrt{x^2 + x^2} = 2|x|$.

Problem 17

We have

$$\begin{aligned} ||\mathbf{a}|| &= \sqrt{(x+y)^2 + (x-y)^2} = \sqrt{x^2 + 2xy + y^2 + x^2 - 2xy + y^2} \\ &= \sqrt{2x^2 + 2y^2} = \sqrt{2}\sqrt{x^2 + y^2}\,. \end{aligned}$$

Problem 18

We have

$$||\mathbf{a}|| = \sqrt{9x^2 + 16x^2} = \sqrt{25x^2} = 5|x|\,.$$

Problem 19

Let the vector $\mathbf{b} = (x, y)$ then we want $\mathbf{b}^T\mathbf{a} = 0$ which is $x + 2y = 0$. Thus $x = -2y$. This means that

$$||\mathbf{b}|| = \sqrt{x^2 + y^2} = \sqrt{4y^2 + y^2} = \sqrt{5}|y|\,.$$

If we want $||\mathbf{b}|| = 5$ then we need to have $|y| = \sqrt{5}$ so $y = \pm\sqrt{5}$. This means that $x = -2y = \mp 2\sqrt{5}$ and our vector $\mathbf{b}$ is

$$\mathbf{b} = (\mp 2\sqrt{5}, \pm\sqrt{5}) = \pm\sqrt{5}(-2, 1)\,.$$

Lets check that this value for $\mathbf{b}$ has the needed requirements. We have

$$\mathbf{b}^T\mathbf{a} = \pm\sqrt{5}(-2 + 2) = 0\,,$$

and

$$||\mathbf{b}|| = \sqrt{4(5) + 5} = \sqrt{25} = 5\,.$$

Problem 20

We have

$$\overrightarrow{\mathbf{PQ}} = (-1, 4) - (2, 1) = (-3, 3)\,.$$

This means that

$$||\overrightarrow{\mathbf{PQ}}|| = \sqrt{3^2 + 3^2} = 3\sqrt{2}\,.$$

Thus

$$\widehat{\mathbf{PQ}} = \frac{\overrightarrow{\mathbf{PQ}}}{||\overrightarrow{\mathbf{PQ}}||} = \frac{1}{\sqrt{2}}(-1,1)\,.$$

Then with this we have

$$\mathbf{b} = 8\widehat{\mathbf{PQ}} = \frac{8}{\sqrt{2}}(-1,1)\,.$$

Problem 21

We have

$$||(2,-3)|| = \sqrt{4+9} = \sqrt{13}$$
$$||(\lambda,1)|| = \sqrt{\lambda^2+1}\,.$$

Setting these two equal gives

$$\lambda^2 + 1 = 13 \quad \text{or} \quad \lambda^2 = 12 \quad \text{or} \quad \lambda = \pm 2\sqrt{3}\,.$$

Problem 22

Let

$$\mathbf{a} = (a_x, a_y)$$
$$\mathbf{b} = (b_x, b_y)$$
$$\mathbf{c} = (c_x, c_y)\,.$$

Then we have

$$(\mathbf{a}+\mathbf{b})+\mathbf{c} = (a_x+b_x, a_y+b_y) + (c_x, c_y) = (a_x+b_x+c_x, a_y+b_y+c_y)\,,$$

and

$$\mathbf{a}+(\mathbf{b}+\mathbf{c}) = (a_x, a_y) + (b_x+c_x, b_y+c_y) = (a_x+b_x+c_x, a_y+b_y+c_y)\,,$$

which are equal.

Problem 23

Let $\mathbf{a} = (a_x, a_y)$ then

$$(\lambda\mu)\mathbf{a} = (\lambda\mu a_x, \lambda\mu a_y)$$
$$\lambda(\mu\mathbf{a}) = \lambda(\mu a_x, \mu a_y) = (\lambda\mu a_x, \lambda\mu a_y)\,,$$

which are equal.

Problem 24

Let

$$\begin{aligned}\mathbf{a} &= (a_x, a_y) \\ \mathbf{b} &= (b_x, b_y)\,.\end{aligned}$$

Then we have

$$\lambda(\mathbf{a}+\mathbf{b}) = \lambda(a_x + b_x, a_y + b_y) = (\lambda a_x + \lambda b_x, \lambda a_y + \lambda b_y)\,,$$

and

$$\lambda\mathbf{a} + \lambda\mathbf{b} = (\lambda a_x, \lambda a_y) + (\lambda b_x, \lambda b_y) = (\lambda a_x + \lambda b_x, \lambda a_y + \lambda b_y)\,,$$

which are the same.

Problem 25

The balance of forces in the vertical direction gives

$$+T\sin(\theta) + T\sin(\theta) - W = 0 \quad \text{so} \quad T = \frac{W}{2\sin(\theta)}\,.$$

The Dot Product

Problem 1

We have

$$\mathbf{a}\cdot\mathbf{b} = 2 - 3 = -1\,,$$

and

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{-1}{\sqrt{1+1}\sqrt{4+9}} = -\frac{1}{\sqrt{26}}\,.$$

Problem 2

We have

$$\mathbf{a}\cdot\mathbf{b} = 2 + 2 = 4\,,$$

and

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{4}{\sqrt{1+4}\sqrt{4+1}} = \frac{4}{5}\,.$$

Problem 3

We have

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2 = 0\,,$$

so that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = 0 \quad \text{so} \quad \theta = \frac{\pi}{2}\,.$$

Problem 4

We have

$$\mathbf{a} \cdot \mathbf{b} = -2 + 6 = 4\,,$$

so that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{4}{\sqrt{4+9}\sqrt{1+4}} = \frac{4}{\sqrt{65}}\,.$$

Problem 5

We have

$$\mathbf{a} \cdot \mathbf{b} = 4 - 1 = 3\,.$$

so that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{3}{\sqrt{2}\sqrt{17}} = \frac{3}{\sqrt{34}}\,.$$

Problem 6

We have

$$\mathbf{a} \cdot \mathbf{b} = 6 - 10 = -4\,,$$

so that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{-4}{\sqrt{9+4}\sqrt{4+25}} = -\frac{4}{\sqrt{13}\sqrt{29}} = -\frac{4}{13\sqrt{2}}\,.$$

Problem 7

We have

$$\mathbf{a} \cdot \mathbf{b} = 2 + 6 = 8\,,$$

so that

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{8}{\sqrt{4+4}\sqrt{1+9}} = \frac{2}{\sqrt{5}}\,.$$

Problem 8

We have

$$\mathbf{a}\cdot\mathbf{b} = 3 - 2 = 1\,,$$

so that

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{1}{\sqrt{1+4}\sqrt{9+1}} = \frac{1}{5\sqrt{2}}\,.$$

Problem 9

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(-3,-4)}{\sqrt{9+16}} = \frac{1}{5}(-3,-4)\,.$$

Problem 10

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(2,-5)}{\sqrt{4+25}} = \frac{(2,-5)}{\sqrt{29}}\,.$$

Problem 11

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(3x,-4x)}{\sqrt{9x^2+16x^2}} = \frac{(3x,-4x)}{5|x|} = \left(\frac{3x}{5|x|}, -\frac{4x}{5|x|}\right)\,.$$

Problem 12

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(x,x)}{\sqrt{x^2+x^2}} = \frac{(x,x)}{|x|} = \left(\frac{x}{|x|}, \frac{x}{|x|}\right)\,.$$

Problem 13

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(-5,12)}{\sqrt{25+144}} = \frac{(-5,12)}{\sqrt{169}} = \frac{(-5,12)}{13}\,.$$

Problem 14

We could take

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||} = \frac{(x, y)}{\sqrt{x^2 + y^2}}.$$

Problem 15

As $-\mathbf{a} = (-1, -3)$ we could take

$$\mathbf{u} = \frac{-\mathbf{a}}{||\mathbf{a}||} = \frac{(-1, -3)}{\sqrt{1 + 9}} = \frac{(-1, -3)}{\sqrt{10}}.$$

Problem 16

As $-\mathbf{a} = -4\mathbf{i} + 3\mathbf{j}$ we could take

$$\mathbf{u} = \frac{-\mathbf{a}}{||\mathbf{a}||} = \frac{(-4, 3)}{\sqrt{16 + 9}} = \left(-\frac{4}{5}, \frac{3}{5}\right).$$

Problem 17

A vector perpendicular to $\mathbf{a}$ is proportional to $\pm(-(-2), 1) = \pm(2, 1)$ and thus has a unit vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{\pm(2, 1)}{\sqrt{5}}.$$

Problem 18

A vector perpendicular to $\mathbf{a}$ is proportional to $\pm(-5, 12)$ and thus has a unit vector $\mathbf{u}$ given by

$$\mathbf{u} = \frac{\pm(-5, 12)}{\sqrt{25 + 144}} = \frac{\pm(-5, 12)}{\sqrt{169}} = \frac{\pm(-5, 12)}{13}.$$

Problem 19

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u}_{\mathbf{a}} = \frac{(1, 1)}{\sqrt{2}}.$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{2+1}{\sqrt{2}} = \frac{3}{\sqrt{2}} .$$

Problem 20

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u_a} = \frac{(2,1)}{\sqrt{5}} .$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{2+1}{\sqrt{5}} = \frac{3}{\sqrt{5}} .$$

Problem 21

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u_a} = \frac{(2,1)}{\sqrt{5}} .$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{2-3}{\sqrt{5}} = -\frac{1}{\sqrt{5}} .$$

Problem 22

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u_a} = \frac{(3,2)}{\sqrt{9+4}} = \frac{(3,2)}{\sqrt{13}} .$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{6-6}{\sqrt{13}} = 0 .$$

Problem 23

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u_a} = \frac{(1,-2)}{\sqrt{5}} .$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{x - 2x}{\sqrt{5}} = -\frac{x}{\sqrt{5}} \,.$$

Problem 24

Note that the unit vector in the direction of $\mathbf{a}$ is given by

$$\mathbf{u_a} = \frac{(x, x)}{\sqrt{x^2 + x^2}} = \frac{(x, x)}{|x|\sqrt{2}} \,.$$

Then the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $\mathbf{b} \cdot \mathbf{u_a}$ which in this case is

$$\frac{x - 2x}{|x|\sqrt{2}} = -\frac{x}{|x|\sqrt{2}} \,.$$

Problem 25

Recall that the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{u_a})\mathbf{u_a} \,.$$

To compute this note that

$$\mathbf{u_a} = \frac{(2, -1)}{\sqrt{4 + 1}} = \frac{(2, -1)}{\sqrt{5}} \,,$$

and

$$\mathbf{b} \cdot \mathbf{u_a} = \frac{2 + 3}{\sqrt{5}} = \sqrt{5} \,.$$

Thus

$$\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{u_a})\mathbf{u_a} = \sqrt{5}\frac{(2, -1)}{\sqrt{5}} = (2, -1) \,.$$

Problem 26

Recall that the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{u_a})\mathbf{u_a} \,.$$

To compute this note that

$$\mathbf{u_a} = \frac{(-3, 4)}{5} \,,$$

and

$$\mathbf{b} \cdot \mathbf{u_a} = \frac{-6 - 4}{5} = -2 \,.$$

Thus

$$\text{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b} \cdot \mathbf{u_a})\mathbf{u_a} = -2\mathbf{u_a} = \frac{(6, -8)}{5} \,.$$

Problem 27

Recall that the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a}\,.$$

To compute this note that

$$\mathbf{u_a} = \frac{(1,1)}{\sqrt{2}}\,,$$

and

$$\mathbf{b}\cdot\mathbf{u_a} = \frac{2+3}{\sqrt{2}} = \frac{5}{\sqrt{2}}\,.$$

Thus

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a} = \frac{5}{\sqrt{2}}\left(\frac{(1,1)}{\sqrt{2}}\right) = \frac{5}{2}(1,1)\,.$$

Problem 28

Recall that the projection of $\mathbf{b}$ onto $\mathbf{a}$ is given by

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a}\,.$$

To compute this note that

$$\mathbf{u_a} = \frac{(2,-3)}{\sqrt{4+9}} = \frac{(2,-3)}{\sqrt{13}}\,,$$

and

$$\mathbf{b}\cdot\mathbf{u_a} = \frac{4x-3x}{\sqrt{13}} = \frac{x}{\sqrt{13}}\,.$$

Thus

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = (\mathbf{b}\cdot\mathbf{u_a})\mathbf{u_a} = \frac{x}{\sqrt{13}}\left(\frac{(2,-3)}{\sqrt{13}}\right) = \left(\frac{2x}{13}, \frac{-3x}{13}\right)\,.$$

Problem 29

Let $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ then

$$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y\,,$$

while

$$\mathbf{b}\cdot\mathbf{a} = b_x a_x + b_y a_y\,,$$

which are the same.

Problem 30

Let $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ then

$$\lambda(\mathbf{a} \cdot \mathbf{b}) = \lambda(a_x b_x + a_y b_y) = \lambda a_x b_x + \lambda a_y b_y \,,$$

while

$$(\lambda \mathbf{a}) \cdot \mathbf{b} = (\lambda a_x, \lambda a_y) \cdot (b_x, b_y) = \lambda a_x b_x + \lambda a_y b_y \,,$$

which are the same.

Problem 31

We have that

$$\begin{aligned} ||a+b|| &= ||(4,1)|| = \sqrt{16+1} = \sqrt{17} = 4.123106 \\ ||a|| + ||b|| &= \sqrt{1+4} + \sqrt{9+1} = \sqrt{5} + \sqrt{10} = 5.398346 \,, \end{aligned}$$

showing that

$$||a+b|| \leq ||a|| + ||b|| \,,$$

is true in this case.

Problem 32

We have that

$$\begin{aligned} ||a+b|| &= ||(3,2)|| = \sqrt{9+4} = \sqrt{13} \\ ||a|| + ||b|| &= \sqrt{1+1} + \sqrt{4+9} = \sqrt{2} + \sqrt{13} \,, \end{aligned}$$

showing that

$$||a+b|| \leq ||a|| + ||b|| \,,$$

is true in this case.

Problem 33

We have that

$$\begin{aligned} ||a+b|| &= ||(5,3)|| = \sqrt{25+9} = \sqrt{34} = 5.830952 \\ ||a|| + ||b|| &= \sqrt{1+4} + \sqrt{9+16} = \sqrt{5} + 5 = 7.236068 \,, \end{aligned}$$

showing that

$$||a+b|| \leq ||a|| + ||b|| \,,$$

is true in this case.

Problem 34

We have that

$$||a+b|| = ||(2,-2)|| = \sqrt{4+4} = 2\sqrt{2}$$
$$||a|| + ||b|| = \sqrt{2} + \sqrt{1+9} = \sqrt{2} + \sqrt{10}\,,$$

showing that

$$||a+b|| \leq ||a|| + ||b||\,,$$

is true in this case.

Problem 35

Let $\mathbf{x} = c_1\mathbf{a}$ and $\mathbf{y} = c_2\mathbf{b}$. The triangle inequality with $\mathbf{x}$ and $\mathbf{y}$ is

$$||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||\,,$$

or

$$||c_1\mathbf{a} + c_2\mathbf{b}|| \leq ||c_1\mathbf{a}|| + ||c_2\mathbf{b}||\,.$$

Now replacing the above with

$$||c_1\mathbf{a}|| = |c_1|||\mathbf{a}||$$
$$||c_2\mathbf{b}|| = |c_2|||\mathbf{b}||\,,$$

we get the desired expression.

Problem 36

Part (a): Starting from the origin, if we draw two vectors $\mathbf{a}$ and $\mathbf{b}$ in the x-y plane then the vector $\mathbf{a}-\mathbf{b}$ is the one that if we "add to" the vector $\mathbf{b}$ we get $\mathbf{a}$. This is just the statement that

$$(\mathbf{a}-\mathbf{b}) + \mathbf{b} = \mathbf{a}\,,$$

which is also true algebraically. Geometrically this means that "walking" along $\mathbf{b}$ from tail to tip and then along $\mathbf{a}-\mathbf{b}$ from tail to tip we get to the tip of $\mathbf{a}$.

Part (b): Consider the triangle inequality $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}||+||\mathbf{y}||$ with $\mathbf{x} = \mathbf{b}$ and $\mathbf{y} = \mathbf{a}-\mathbf{b}$. Then this is

$$||\mathbf{a}|| \leq ||\mathbf{b}|| + ||\mathbf{a}-\mathbf{b}||\,,$$

or

$$||\mathbf{a}-\mathbf{b}|| \geq ||\mathbf{a}|| - ||\mathbf{b}||\,. \tag{7}$$

Part (c): Consider the triangle inequality $||\mathbf{x}+\mathbf{y}|| \le ||\mathbf{x}|| + ||\mathbf{y}||$ with $\mathbf{x} = \mathbf{a}$ and $\mathbf{y} = \mathbf{b} - \mathbf{a}$. Then this is

$$||\mathbf{b}|| \le ||\mathbf{a}|| + ||\mathbf{b} - \mathbf{a}|| \,,$$

or

$$||\mathbf{b} - \mathbf{a}|| \ge ||\mathbf{b}|| - ||\mathbf{a}|| \,. \tag{8}$$

Part (d): As $||\mathbf{a} - \mathbf{b}|| = ||\mathbf{b} - \mathbf{a}||$ from Equation 7 we have

$$||\mathbf{a} - \mathbf{b}|| \ge ||\mathbf{a}|| - ||\mathbf{b}|| \,,$$

and from Equation 8 we have

$$||\mathbf{a} - \mathbf{b}|| \ge ||\mathbf{b}|| - ||\mathbf{a}|| \,.$$

Taken together this means that

$$||\mathbf{a} - \mathbf{b}|| \ge |||\mathbf{a}|| - ||\mathbf{b}||| \,.$$

Problem 37

Part (a): Notice that $\mathbf{a} - \mathbf{r}$ is proportional to $\mathbf{a} - \mathbf{b}$ thus there is a z such that

$$\mathbf{a} - \mathbf{r} = z(\mathbf{a} - \mathbf{b}) \,.$$

Solving for $\mathbf{r}$ gives

$$\mathbf{r} = \mathbf{a} - z(\mathbf{a} - \mathbf{b}) \,.$$

If we let $t = -z$ we have the desired expression.

Part (b): These are the points beyond P or Q but still on the segment joining P and Q.

Problem 38

We compute

$$||\mathbf{r}|| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1 \,.$$

Drawing this vector in the x-y plane we see that the vector $\mathbf{r}$ makes an angle θ with the x-axis.

Vectors in Three Dimensions

Problem 1

We have

$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (3, -2, 4) - (2, 1, -1) = (1, -3, 5) \,.$$

Problem 2

We have
$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (1, -3, 2) - (3, 2, -4) = (-2, -5, 2)\,.$$

Problem 3

Using
$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}\,,$$
we have
$$\overrightarrow{\mathbf{Q}} = \overrightarrow{\mathbf{PQ}} + \overrightarrow{\mathbf{P}}\,.$$
Thus in this case we have
$$\overrightarrow{\mathbf{Q}} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + 2\mathbf{i} - \mathbf{j} + 2\mathbf{k} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\,.$$

Problem 4

Using
$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}\,,$$
we have
$$\overrightarrow{\mathbf{P}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{PQ}}\,.$$
Thus in this case we have
$$\overrightarrow{\mathbf{P}} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k} - (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = \mathbf{i} + 5\mathbf{j} - 4\mathbf{k}\,.$$

Problem 5

We have
$$3\mathbf{a} - 2\mathbf{b} = 3(2, -1, 3) - 2(-3, 2, 1) = (6, -3, 9) - (-6, 4, 2) = (12, -7, 7)\,.$$

Problem 6

We have
$$3\mathbf{a} - 2\mathbf{b} = (12, 0, 3) - (4, -4, 6) = (8, 4, -3)\,.$$

Problem 7

We have

$$3\mathbf{a} - 2\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} - (-2\mathbf{i} + 6\mathbf{k}) = 8\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}\,.$$

Problem 8

We have

$$3\mathbf{a} - 2\mathbf{b} = -3\mathbf{i} + 6\mathbf{j} - 9\mathbf{k} - (4\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}) = -7\mathbf{i} - 4\mathbf{j} - \mathbf{k}\,.$$

Problem 9

We have

$$||\mathbf{a}|| = \sqrt{9 + 4 + 1} = \sqrt{14}\,.$$

Problem 10

We have

$$||\mathbf{a}|| = \sqrt{4 + 4 + 9} = \sqrt{17}\,.$$

Problem 11

We have

$$||\mathbf{a}|| = \sqrt{4 + 1 + 16} = \sqrt{21}\,.$$

Problem 12

We have

$$||\mathbf{a}|| = \sqrt{9 + 25 + 1} = \sqrt{35}\,.$$

Problem 13

We have

$$||\overrightarrow{\mathbf{PQ}}|| = ||\overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}|| = ||(3, -2, 4) - (2, 1, -1)|| = \sqrt{1 + 9 + 25} = \sqrt{35}\,.$$

Problem 14

We have

$$||\overrightarrow{\mathbf{PQ}}|| = ||\overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}}|| = ||(1,-3,2)-(3,2,-4)|| = ||(-2,-5,6)|| = \sqrt{4+25+36} = \sqrt{65}\,.$$

Problem 15

We have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 6-4+2=4 \\ \cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{4}{\sqrt{9+1+4}\sqrt{4+16+1}} = \frac{4}{\sqrt{14}\sqrt{21}} = \frac{4}{7\sqrt{6}}\,. \end{aligned}$$

Problem 16

We have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 4+0+1=5 \\ \cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{5}{\sqrt{16+1}\sqrt{1+1+1}} = \frac{5}{\sqrt{17}\sqrt{3}} = \frac{5}{\sqrt{51}}\,. \end{aligned}$$

Problem 17

We have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 1+2-3=0 \\ \cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = 0\,. \end{aligned}$$

Thus $\theta = \frac{\pi}{2}$.

Problem 18

We have

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= -2+0-3=-5 \\ \cos(\theta) &= \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||} = \frac{-5}{\sqrt{4+1+1}\sqrt{1+9}} = \frac{-5}{\sqrt{6}\sqrt{10}} = -\frac{5}{2\sqrt{15}}\,. \end{aligned}$$

Problem 19

The component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is given by

$$||\mathbf{b}||\cos(\theta) = ||\mathbf{b}||\left(\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}\right) = \frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||}\,. \tag{9}$$

For the vectors given here we have

$$\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{2+1-6}{\sqrt{4+1+9}} = -\frac{3}{\sqrt{14}}\,.$$

Problem 20

The component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is given by Equation 9. For the vectors given here we have

$$\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{4+0+2}{\sqrt{1+1+4}} = \frac{6}{\sqrt{6}} = \sqrt{6}\,.$$

Problem 21

The component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is given by Equation 9. For the vectors given here we have

$$\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{-2+3+4}{\sqrt{4+9+16}} = \frac{5}{\sqrt{29}}\,.$$

Problem 22

The component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is given by Equation 9. For the vectors given here we have

$$\frac{\mathbf{a}\cdot\mathbf{b}}{||\mathbf{a}||} = \frac{-2+3+4}{\sqrt{1+1+1}} = \frac{5}{\sqrt{3}}\,.$$

Problem 23

To be perpendicular we need to have $\mathbf{a}\cdot\mathbf{b} = 0$. Now

$$\mathbf{a}\cdot\mathbf{b} = 6\lambda + \lambda - 3 = 7\lambda - 3\,.$$

To have this equal zero we need $\lambda = \frac{3}{7}$.

Problem 24

To be perpendicular we need to have $\mathbf{a} \cdot \mathbf{b} = 0$. Now

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2\lambda^2 + 12\,.$$

To have this equal zero we need to have $\lambda = \pm\sqrt{7}$.

Problem 25

We want

$$\mathbf{u} = \frac{\mathbf{a}}{||\mathbf{a}||}\,.$$

Now $||\mathbf{a}|| = \sqrt{9+1+4} = \sqrt{14}$. Thus

$$\mathbf{u} = \frac{1}{\sqrt{14}}(3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\,.$$

Problem 26

We want

$$\mathbf{u} = -\frac{\mathbf{a}}{||\mathbf{a}||}\,.$$

Now $||\mathbf{a}|| = \sqrt{4+9+16} = \sqrt{29}$. Thus

$$\mathbf{u} = -\frac{1}{\sqrt{29}}(2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})\,.$$

Problem 27

The vector from $\mathbf{P}$ to $\mathbf{Q}$ is

$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (1,-1,1) - (-2,0,4) = (3,-1,-3)\,.$$

Thus

$$||\overrightarrow{\mathbf{PQ}}|| = \sqrt{9+1+9} = \sqrt{19}\,.$$

Thus

$$\mathbf{u} = \frac{1}{\sqrt{19}}(3,-1,-3)\,.$$

Problem 28

Let $\mathbf{u} = (u_x, u_y, u_z)$ then we need to have $\mathbf{u}$ such that

$$\mathbf{a} \cdot \mathbf{u} = 0 = \mathbf{b} \cdot \mathbf{u} \,.$$

Note that

$$\mathbf{a} \cdot \mathbf{u} = 2u_x - 3u_y + u_z = 0 \tag{10}$$
$$\mathbf{b} \cdot \mathbf{u} = u_x + u_y + u_z = 0 \,. \tag{11}$$

From Equation 11 we have

$$u_z = -u_x - u_y \,, \tag{12}$$

which we put into Equation 10 to get

$$2u_x - 3u_y + (-u_x - u_y) = 0 \,,$$

which simplifies to give $u_x = 2u_y$. Using this in Equation 12 we get that

$$u_z = -2u_y - u_y = -3u_y \,.$$

Thus $(u_x, u_y, u_z) = (2u_y, u_y, -3u_y)$. To be a unit vector means that $||\mathbf{u}|| = 1$ or

$$\sqrt{4u_y^2 + u_y^2 + 9u_y^2} = |u_y|\sqrt{14} \,.$$

Lets take $u_y = \pm\frac{1}{\sqrt{14}}$ and then get

$$\mathbf{u} = (u_x, u_y, u_z) = (2u_y, u_y, -3u_y) = \pm\frac{1}{\sqrt{14}}(2, 1, -3) \,.$$

Problem 29

If $\mathbf{a}$ and $\mathbf{b}$ are sides of a parallelogram then the diagonal are given by the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. If these two diagonals are perpendicular that means that their dot product must be zero or

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0 \,,$$

or expanding the left-hand-side gives

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0 \,,$$

or

$$||\mathbf{a}||^2 - ||\mathbf{b}||^2 = 0 \quad \text{or} \quad ||\mathbf{a}|| = ||\mathbf{b}|| \,,$$

showing that the sides must all be of the same length.

Problem 30

Using results from the previous problem we are asked to evaluate

$$\begin{aligned}||\mathbf{a}+\mathbf{b}||^2 + ||\mathbf{a}-\mathbf{b}||^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} \\ &= 2\mathbf{a}\cdot\mathbf{a} + 2\mathbf{b}\cdot\mathbf{b} = 2||\mathbf{a}||^2 + 2||\mathbf{b}||^2\,,\end{aligned}$$

which is the sum of squares of all four sides.

Problem 31

Let the cube be constructed in the positive octant of the x-y-z Cartesian coordinate system (assume a side of length one). Then the diagonal is represented by the vector

$$\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}\,.$$

An example edge is given by the vector $\mathbf{i}$. The cosign of the angle between these two is then given by

$$\cos(\theta) = \frac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{i}}{\sqrt{1+1+1}\sqrt{1}} = \frac{1}{\sqrt{3}}\,.$$

This means that

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right) = 0.9553166 = 54.73561^\circ\,.$$

Problem 32

If $\mathbf{a}$ and $\mathbf{b}$ are sides of a parallelogram then the diagonal are given by the vectors $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ thus in this case we have

$$\begin{aligned}\mathbf{a}+\mathbf{b} &= 0\mathbf{i} - \mathbf{j} + 4\mathbf{k} = -\mathbf{j} + 4\mathbf{k} \\ \mathbf{a}-\mathbf{b} &= 2\mathbf{i} - 3\mathbf{j} + 0\mathbf{k} = 2\mathbf{i} - 3\mathbf{j}\,.\end{aligned}$$

The angle between these two vectors is given by

$$\begin{aligned}\cos(\theta) &= \frac{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})}{||\mathbf{a}+\mathbf{b}||\,||\mathbf{a}-\mathbf{b}||} \\ &= \frac{0+3+0}{\sqrt{1+16}\sqrt{4+9}} = \frac{3}{\sqrt{221}} = 0.201802\,.\end{aligned}$$

Thus $\theta = 1.367599 = 78.35765^\circ$.

Problem 33

Let the cube be constructed in the positive octant of the x-y-z Cartesian coordinate system (assume a side of length one). Then the diagonal is represented by the vector

$$\mathbf{d} = \mathbf{i} + \mathbf{j} + \mathbf{k}\,.$$

A diagonal to one of its faces is given by the vector $\mathbf{i} + \mathbf{j}$. The cosign of the angle between these two is then given by

$$\cos(\theta) = \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} + \mathbf{j})}{\sqrt{1 + 1 + 1}\sqrt{1 + 1}} = \sqrt{\frac{2}{3}} = 0.8164966\,.$$

This means that

$$\theta = 0.6154797 = 35.26439^{\circ}\,.$$

Problem 34

Now each of the given vectors can be written as the difference between the position vectors pointing to the endpoints. Thus the expression we are given can be written as

$$(\overrightarrow{\mathbf{B}} - \overrightarrow{\mathbf{A}}) + (\overrightarrow{\mathbf{C}} - \overrightarrow{\mathbf{B}}) + (\overrightarrow{\mathbf{D}} - \overrightarrow{\mathbf{C}}) + (\overrightarrow{\mathbf{E}} - \overrightarrow{\mathbf{D}}) = 0\,,$$

or simplifying we get

$$-\overrightarrow{\mathbf{A}} + \overrightarrow{\mathbf{E}} = 0\,,$$

showing that the point A must correspond with the point E.

The Cross Product

Problem 1

We find

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 2 & 4 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & -1 \\ 2 & 4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \\ &= \mathbf{i}(-4 + 2) - \mathbf{j}(8 + 1) + \mathbf{k}(4 + 1) = -2\mathbf{i} - 9\mathbf{j} + 5\mathbf{k}\,. \end{aligned}$$

Problem 2

We find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= \mathbf{i}(-2-1) - \mathbf{j}(1-3) + \mathbf{k}(-1-6) = -3\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}\,.$$

Problem 3

We find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -2 & -3 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}$$
$$= \mathbf{i}(1+3) - \mathbf{j}(1+2) + \mathbf{k}(-3+2) = 4\mathbf{i} - 3\mathbf{j} - \mathbf{k}\,.$$

Problem 4

We find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 1 & 1 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -3 \\ 1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}$$
$$= \mathbf{i}(3) - \mathbf{j}(-2+3) + \mathbf{k}(2) = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,.$$

Problem 5

We find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & 0 \\ b_2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & 0 \\ b_1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$
$$= (a_1 b_2 - a_2 b_1)\mathbf{k}\,.$$

Problem 6

We find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ 0 & 0 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & 0 \\ 0 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & 0 \\ 0 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}$$
$$= a_2b_3\mathbf{i} - a_1b_3\mathbf{j}\,.$$

Problem 7

Note that $\mathbf{b}+\mathbf{c} = -\mathbf{i}+5\mathbf{j}+4\mathbf{k}$. Thus

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ -1 & 5 & 4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -1 \\ 5 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix}$$
$$= \mathbf{i}(-4+5) - \mathbf{j}(8-1) + \mathbf{k}(10-1) = \mathbf{i} - 7\mathbf{j} + 9\mathbf{k}\,.$$

Problem 8

Note that $\mathbf{a}+\mathbf{b} = 3\mathbf{i}+\mathbf{j}+2\mathbf{k}$. Thus

$$(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ -2 & 3 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix}$$
$$= \mathbf{i}(1-6) - \mathbf{j}(3+4) + \mathbf{k}(9+2) = -5\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}\,.$$

Problem 9

In one way to evaluate this we first compute

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}$$
$$= \mathbf{i}(2-9) - \mathbf{j}(1+6) + \mathbf{k}(3+4) = -7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}\,.$$

Then we compute

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -14+7-7 = -14\,.$$

We can compute this "directly" if we note that it is a scalar triple product and can be evaluated using determinants as

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 2 & -1 & -1 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = 2(2-9) - (-1)(1+6) + (-1)(3+4) = -14+7-7 = -14\,,$$

the same as before.

Problem 10

Note that $(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b} = \mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})$ which is a scalar triple product. Thus

$$\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & -1 \\ -2 & 3 & 1 \end{vmatrix} = 1(-1+3) - 2(2-2) + 3(6-2) = 14\,.$$

Note that another way to evaluate this is as follows

$$(\mathbf{a}\times\mathbf{c})\cdot\mathbf{b} = (\mathbf{c}\times\mathbf{b})\cdot\mathbf{a} = -(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = -\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = -(-14) = 14\,,$$

using the result from Problem 9.

Problem 11

Note that $\mathbf{b}\times\mathbf{c} = -7\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}$ from Problem 9. Thus

$$\begin{aligned}\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ -7 & -7 & 7 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -1 \\ -7 & 7 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ -7 & 7 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ -7 & -7 \end{vmatrix} \\ &= \mathbf{i}(-7-7) - \mathbf{j}(14-7) + \mathbf{k}(-14-7) = -14\mathbf{i} - 7\mathbf{j} - 21\mathbf{k}\,.\end{aligned}$$

Problem 12

Note that

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -1 \\ 2 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} \\ &= \mathbf{i}(-3+2) - \mathbf{j}(6+1) + \mathbf{k}(6+1) = -\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}\,.\end{aligned}$$

Thus

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -7 & 7 \\ -2 & 3 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -7 & 7 \\ 3 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 7 \\ -2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -7 \\ -2 & 3 \end{vmatrix} \\ &= \mathbf{i}(-7-21) - \mathbf{j}(-1+14) + \mathbf{k}(-3-14) = -28\mathbf{i} - 13\mathbf{j} - 17\mathbf{k}\,.\end{aligned}$$

Problem 13

This would be $\mathbf{u} = \pm\frac{\mathbf{a}\times\mathbf{b}}{||\mathbf{a}\times\mathbf{b}||}$. Now

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & 1 \\ 1 & -4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}$$
$$= \mathbf{i}(8-1) - \mathbf{j}(-4-3) + \mathbf{k}(1+6) = 7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}\,.$$

Using this result we have that

$$||\mathbf{a}\times\mathbf{b}|| = \sqrt{3(7^2)} = 7\sqrt{3}\,.$$

Thus

$$\mathbf{u} = \pm\frac{7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}}{7\sqrt{3}} = \pm\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}\,.$$

Problem 14

This would be $\mathbf{u} = \pm\frac{\mathbf{a}\times\mathbf{b}}{||\mathbf{a}\times\mathbf{b}||}$. Now

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & -1 \\ 3 & 0 & 4 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 3 \\ 3 & 0 \end{vmatrix}$$
$$= \mathbf{i}(12) - \mathbf{j}(-8+3) + \mathbf{k}(0-9) = 12\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}\,.$$

Using this result we have that

$$||\mathbf{a}\times\mathbf{b}|| = \sqrt{12^2 + 5^2 + 9^2} = \sqrt{250} = 5\sqrt{10}\,.$$

Thus

$$\mathbf{u} = \pm\frac{12\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}}{5\sqrt{10}}\,.$$

Problem 15

We compute

$$\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (0,3,0) - (0,0,0) = (0,3,0)$$
$$\overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}} = (2,5,8) - (0,0,0) = (2,5,8)\,.$$

Thus we have

$$\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 0 \\ 2 & 5 & 8 \end{vmatrix} = 0 + 3\begin{vmatrix} \mathbf{i} & \mathbf{k} \\ 2 & 8 \end{vmatrix} + 0$$
$$= 3(8\mathbf{i} - 2\mathbf{k}) = 24\mathbf{i} - 6\mathbf{k}\,.$$

Now the area we seek is given by $\frac{1}{2}||\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}||$ or

$$\frac{1}{2}\sqrt{24^2 + 6^2} = 3\sqrt{17}\,.$$

Problem 16

We compute

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} &= \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (2,1,3) - (-1,-2,1) = (3,3,2) \\
\overrightarrow{\mathbf{PR}} &= \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}} = (1,4,0) - (-1,-2,1) = (2,6,-1)\,.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & 2 \\ 2 & 6 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 3 & 2 \\ 6 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 3 \\ 2 & 6 \end{vmatrix} \\
&= \mathbf{i}(-3-12) - \mathbf{j}(-3-4) + \mathbf{k}(18-6) \\
&= -15\mathbf{i} + 7\mathbf{j} + 12\mathbf{k}\,.
\end{aligned}$$

Now the area we seek is given by $\frac{1}{2}||\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}||$ or

$$\frac{1}{2}\sqrt{15^2 + 7^2 + 12^2} = \frac{1}{2}\sqrt{418}\,.$$

Problem 17

From the problem we compute

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} &= (1,1,0) \\
\overrightarrow{\mathbf{PR}} &= (-1,3,0) \\
\overrightarrow{\mathbf{PS}} &= (1,0,4)\,.
\end{aligned}$$

Then the volume of the parallelepiped is given by the absolute value of the scalar triple product defined as

$$\overrightarrow{\mathbf{PQ}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PS}})\,. \tag{13}$$

To evaluate this we find

$$\overrightarrow{\mathbf{PQ}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PS}}) = \begin{vmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4(3+1) = 16\,,$$

As this value is not zero these three points are not coplanar.

Problem 18

From the problem we compute

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} &= \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (2,1,4) - (-1,2,2) = (3,-1,2) \\
\overrightarrow{\mathbf{PR}} &= \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}} = (-1,4,1) - (-1,2,2) = (0,2,-1) \\
\overrightarrow{\mathbf{PS}} &= \overrightarrow{\mathbf{S}} - \overrightarrow{\mathbf{P}} = (3,0,5) - (-1,2,2) = (4,-2,3)\,.
\end{aligned}$$

Then the volume of the parallelepiped is given by the absolute value of the scalar triple product defined in Equation 13. To evaluate this we find

$$\overrightarrow{\mathbf{PQ}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PS}}) = \left| \begin{array}{ccc} 3 & -1 & 2 \\ 0 & 2 & -1 \\ 4 & -2 & 3 \end{array} \right| = 3 \left| \begin{array}{cc} 2 & -1 \\ -2 & 3 \end{array} \right| + 4 \left| \begin{array}{cc} -1 & 2 \\ 2 & -1 \end{array} \right| = 12 - 12 = 0\,.$$

As this value is zero these three points are coplanar.

Problem 19

From the problem we compute

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} &= \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (0,\lambda,1) - (2,4,-1) = (-2,\lambda-4,2) \\
\overrightarrow{\mathbf{PR}} &= \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}} = (-2,1,2) - (-2,4,-1) = (-4,-3,3) \\
\overrightarrow{\mathbf{PS}} &= \overrightarrow{\mathbf{S}} - \overrightarrow{\mathbf{P}} = (1,1,0) - (2,4,-1) = (-1,-3,1)\,.
\end{aligned}$$

To be coplanar the volume of the parallelepiped must be zero. The volume of the parallelepiped is given by the absolute value of the scalar triple product defined in Equation 13. To evaluate this we find

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PS}}) &= \left| \begin{array}{ccc} -2 & \lambda-4 & 2 \\ -4 & -3 & 3 \\ -1 & -3 & 1 \end{array} \right| = -2 \left| \begin{array}{cc} -3 & 3 \\ -3 & 1 \end{array} \right| + 4 \left| \begin{array}{cc} \lambda-4 & 2 \\ -3 & 1 \end{array} \right| - \left| \begin{array}{cc} \lambda-4 & 2 \\ -3 & 3 \end{array} \right| \\
&= -2(-3+9) + 4(\lambda-4+6) - (3\lambda-12+6) = \lambda+2\,,
\end{aligned}$$

when we simplify. This value is zero when $\lambda = -2$.

Problem 20

From the problem we compute

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} &= \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{P}} = (1,1,1) - (0,2,-3) = (1,-1,4) \\
\overrightarrow{\mathbf{PR}} &= \overrightarrow{\mathbf{R}} - \overrightarrow{\mathbf{P}} = (2,0,-1) - (0,2,-3) = (2,-2,2) \\
\overrightarrow{\mathbf{PS}} &= \overrightarrow{\mathbf{S}} - \overrightarrow{\mathbf{P}} = (\lambda,2\lambda,-\lambda) - (0,2,-3) = (\lambda,2\lambda-2,-\lambda+3)\,.
\end{aligned}$$

To be coplanar the volume of the parallelepiped must be zero. The volume of the parallelepiped is given by the absolute value of the scalar triple product defined in Equation 13. To evaluate this we find

$$\begin{aligned}
\overrightarrow{\mathbf{PQ}} \cdot (\overrightarrow{\mathbf{PR}} \times \overrightarrow{\mathbf{PS}}) &= \begin{vmatrix} 1 & -1 & 4 \\ 2 & -2 & 2 \\ \lambda & 2\lambda - 2 & -\lambda + 3 \end{vmatrix} \\
&= \lambda \begin{vmatrix} -1 & 4 \\ -2 & 2 \end{vmatrix} - (2\lambda - 2) \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + (-\lambda + 3) \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} \\
&= 18\lambda - 12 \,,
\end{aligned}$$

when we simplify. This value is zero when $\lambda = \frac{2}{3}$.

Problem 21

From the problem we compute

$$\begin{aligned}
\overrightarrow{\mathbf{P_1P_2}} &= (x_2 - x_1, y_2 - y_1) \\
\overrightarrow{\mathbf{P_1P_3}} &= (x_3 - x_1, y_3 - y_1) \,.
\end{aligned}$$

From the text, the area of this triangle is given by $\frac{1}{2}||\overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}}||$. We find

$$\overrightarrow{\mathbf{P_1P_2}} \times \overrightarrow{\mathbf{P_1P_3}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \,.$$

The norm of the above cross product vector is the absolute value of the 2×2 determinant on the right-hand-side of the above. We can write the above 2×2 determinant as a 3×3 determinant (called D) as

$$D \equiv \begin{vmatrix} 0 & 0 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \,.$$

We will now use operations which leave the value of the determinant unchanged to convert D into the form given in this problem thereby showing the desired equivalence. To do that we add the first row to the second row and then the third row to get

$$D = \begin{vmatrix} 0 & 0 & 1 \\ x_2 - x_1 & y_2 - y_1 & 1 \\ x_3 - x_1 & y_3 - y_1 & 1 \end{vmatrix} \,.$$

Next we multiply the third column by x_1 and add that column to the first column to get

$$D = \begin{vmatrix} x_1 & 0 & 1 \\ x_2 & y_2 - y_1 & 1 \\ x_3 & y_3 - y_1 & 1 \end{vmatrix} \,.$$

Next we multiply the third column by y_1 and add that column to the second column to get

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} .$$

Now we exchange columns twice to get

$$D = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} .$$

Finally we take the transpose of this to get

$$D = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} .$$

Taking the absolute value and multiplying by $\frac{1}{2}$ gives the area.

Problem 22

Part (a): We have

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x + c_x & b_y + c_y & b_z + c_z \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} a_y & a_z \\ b_y + c_y & b_z + c_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_x & a_z \\ b_x + c_x & b_z + c_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_x & a_y \\ b_x + c_x & b_y + c_y \end{vmatrix} \\
&= \mathbf{i}(a_y(b_z + c_z) - a_z(b_y + c_y)) - \mathbf{j}(a_x(b_z + c_z) - a_z(b_x + c_x)) + \mathbf{k}(a_x(b_y + c_y) - a_y(b_x + c_x)) \\
&= \mathbf{i}(a_y b_z - a_z b_y) - \mathbf{j}(a_x b_z - a_z b_x) + \mathbf{k}(a_x b_y - a_y b_x) \\
&+ \mathbf{i}(a_y c_z - a_z c_y) - \mathbf{j}(a_x c_z - a_z c_x) + \mathbf{k}(a_x c_y - a_y c_x) \\
&= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} .
\end{aligned}$$

Part (b): This would be done in a similar way to Part (a) above.

Problem 23

Part (a): We have

$$\begin{aligned}
(\alpha \mathbf{a}) \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \alpha a_x & \alpha a_y & \alpha a_z \\ b_x & b_y & b_z \end{vmatrix} = \mathbf{i} \begin{vmatrix} \alpha a_y & \alpha a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} \alpha a_x & \alpha a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} \alpha a_x & \alpha a_y \\ b_x & b_y \end{vmatrix} \\
&= \alpha \mathbf{i}(a_y b_z - a_z b_y) - \alpha \mathbf{j}(a_x b_z - a_z b_x) + \alpha \mathbf{k}(a_x b_y - a_y b_x) = \alpha(\mathbf{a} \times \mathbf{b}) .
\end{aligned}$$

Part (b): This would be done in a similar way to Part (a) above.

Problem 24

We want to show that

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = 2(\mathbf{b}\times\mathbf{a})\,.$$

To show this we consider

$$
\begin{aligned}
(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x+b_x & a_y+b_y & a_z+b_z \\ a_x-b_x & a_y-b_y & a_z-b_z \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} a_y+b_y & a_z+b_z \\ a_y-b_y & a_z-b_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_x+b_x & a_z+b_z \\ a_x-b_x & a_z-b_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_x+b_x & a_y+b_y \\ a_x-b_x & a_y-b_y \end{vmatrix} \\
&= \mathbf{i}((a_y+b_y)(a_z-b_z)-(a_y-b_y)(a_z+b_z)) \\
&- \mathbf{j}((a_x+b_x)(a_z-b_z)-(a_x-b_x)(a_z+b_z)) \\
&+ \mathbf{k}((a_x+b_x)(a_y-b_y)-(a_x-b_x)(a_y+b_y)) \\
&= \mathbf{i}(a_ya_z - a_yb_z + a_zb_y - b_yb_z - a_ya_z - a_yb_z + a_zb_y + b_yb_z) \\
&- \mathbf{j}(a_xa_z - a_xb_z + a_zb_x - b_xb_z - a_xa_z - a_xb_z + a_zb_x + b_xb_z) \\
&+ \mathbf{k}(a_xa_y - a_xb_y + a_yb_x - b_xb_y - a_xa_y - a_xb_y + a_yb_x + b_xb_y) \\
&= 2\mathbf{i}(a_zb_y - a_yb_z) - 2\mathbf{j}(a_zb_x - a_xb_z) + 2\mathbf{k}(-a_xb_y + a_yb_x) \\
&= 2\mathbf{i}\begin{vmatrix} b_y & b_z \\ a_y & a_z \end{vmatrix} - 2\mathbf{j}\begin{vmatrix} b_x & b_z \\ a_x & a_z \end{vmatrix} + 2\mathbf{k}\begin{vmatrix} b_x & b_y \\ a_x & a_y \end{vmatrix} \\
&= 2(\mathbf{b}\times\mathbf{a})\,.
\end{aligned}
$$

Problem 25

Part (a): We want to evaluate $\mathbf{i}\times(\mathbf{b}\times\mathbf{c})$. To start this problem we notice that

$$
\begin{aligned}
\mathbf{b}\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_y & b_z \\ c_y & c_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_x & b_z \\ c_x & c_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_x & b_y \\ c_x & c_y \end{vmatrix} \\
&= \mathbf{i}(b_yc_z - b_zc_y) - \mathbf{j}(b_xc_z - b_zc_x) + \mathbf{k}(b_xc_y - b_yc_x)\,.
\end{aligned}
$$

This means that

$$
\begin{aligned}
\mathbf{i}\times(\mathbf{b}\times\mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ b_yc_z - b_zc_y & -b_xc_z + b_zc_x & b_xc_y - b_yc_x \end{vmatrix} \\
&= -1\begin{vmatrix} \mathbf{j} & \mathbf{k} \\ -b_xc_z + b_zc_x & b_xc_y - b_yc_x \end{vmatrix} \\
&= -1(\mathbf{j}(b_xc_y - b_yc_x) + \mathbf{k}(b_xc_z - b_zc_x)) = \mathbf{j}(b_yc_x - b_xc_y) + \mathbf{k}(b_zc_x - b_xc_z) \\
&= \mathbf{j}b_yc_x - \mathbf{k}b_zc_x - \mathbf{j}b_xc_y - \mathbf{k}b_xc_z = c_x(\mathbf{b} - b_x\mathbf{i}) - b_x(\mathbf{c} - c_x\mathbf{i}) \\
&= c_x\mathbf{b} - b_x\mathbf{c}\,.
\end{aligned}
$$

Part (b): Next we want to evaluate $\mathbf{j} \times (\mathbf{b} \times \mathbf{c})$. We find

$$
\begin{aligned}
\mathbf{j} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ b_y c_z - b_z c_y & -b_x c_z + b_z c_x & b_x c_y - b_y c_x \end{vmatrix} \\
&= 1 \begin{vmatrix} \mathbf{i} & \mathbf{k} \\ b_y c_z - b_z c_y & b_x c_y - b_y c_x \end{vmatrix} \\
&= \mathbf{i}(b_x c_y - b_y c_x) - \mathbf{k}(b_y c_z - b_z c_y)) = \mathbf{i} b_x c_y + \mathbf{k} b_z c_y - \mathbf{i} c_x b_y - \mathbf{k} c_z b_y \\
&= c_y(\mathbf{i} b_x + \mathbf{k} b_z) - b_y(\mathbf{i} c_x - \mathbf{k} c_z) = c_y(\mathbf{b} - b_y \mathbf{j}) - b_y(\mathbf{c} - c_y \mathbf{j}) \\
&= c_y \mathbf{b} - b_y \mathbf{c}\,.
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
\mathbf{k} \times (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ b_y c_z - b_z c_y & -b_x c_z + b_z c_x & b_x c_y - b_y c_x \end{vmatrix} \\
&= -1 \begin{vmatrix} \mathbf{i} & \mathbf{j} \\ b_y c_z - b_z c_y & -b_x c_z + b_z c_x \end{vmatrix} \\
&= \mathbf{i}(b_x c_z - b_z c_x) - \mathbf{j}(b_z c_y - b_y c_z)) = \mathbf{i} b_x c_z + \mathbf{j} b_y c_z - \mathbf{i} c_x b_z - \mathbf{j} c_y b_z \\
&= c_z(\mathbf{b} - b_z \mathbf{k}) - b_z(\mathbf{c} - c_z \mathbf{k}) \\
&= c_z \mathbf{b} - b_z \mathbf{c}\,.
\end{aligned}
$$

Part (c): We can compute what we seek by breaking $\mathbf{a}$ up into its components and using what we computed above as

$$
\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= a_x \mathbf{i} \times (\mathbf{b} \times \mathbf{c}) + a_y \mathbf{j} \times (\mathbf{b} \times \mathbf{c}) + a_z \mathbf{k} \times (\mathbf{b} \times \mathbf{c}) \\
&= (a_x c_x \mathbf{b} - a_x b_x \mathbf{c}) + (a_y c_y \mathbf{b} - a_y b_y \mathbf{c}) + (a_z c_z \mathbf{b} - a_z b_z \mathbf{c}) \\
&= (a_x c_x + a_y c_y + a_z c_z)\mathbf{b} - (a_x b_x + a_y b_y + a_z b_z)\mathbf{c} \\
&= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,,
\end{aligned}
$$

as we were to show.

Problem 26

Part (a): From the previous problem we have

$$
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}\,.
$$

Using the fact that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ in the above we get

$$
(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}\,.
$$

Replace the "letters" in the above to get the same thing but with different "letters"

$$
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}\,.
$$

Part (b): We now seek to determine when $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. Using the above this is equivalent to

$$(\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \,,$$

or

$$(\mathbf{b} \cdot \mathbf{a})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \,.$$

This means that the vector $\mathbf{c}$ must be parallel to the vector $\mathbf{a}$.

Problem 27

Part (a): The relationships $\mathbf{d} \cdot \mathbf{a} = 0$ and $\mathbf{d} \cdot \mathbf{b} = 0$ give

$$d_1 a_1 + d_2 a_2 + d_3 a_3 = 0 \tag{14}$$

$$d_1 b_1 + d_2 b_2 + d_3 b_3 = 0 \,. \tag{15}$$

We will solve for d_3 in both. If we solve for d_3 in Equation 14 we get

$$d_3 = -\frac{d_1 a_1 + d_2 a_2}{a_3} \,,$$

while Equation 15 gives

$$d_3 = -\frac{d_1 b_1 + d_2 b_2}{b_3} \,.$$

If we set these two equal we get

$$-\frac{d_1 a_1 + d_2 a_2}{a_3} = -\frac{d_1 b_1 + d_2 b_2}{b_3} \,,$$

or

$$d_1 a_1 b_3 + d_2 a_2 b_3 = d_1 b_1 a_3 + d_2 b_2 a_3 \,,$$

or

$$d_1(a_1 b_3 - a_3 b_1) = d_2(a_3 b_2 - a_2 b_3) \,, \tag{16}$$

which is one of the desired equations.

Next we will solve for d_2 in both. If we solve for d_3 in Equation 14 we get

$$d_2 = -\frac{d_3 a_3 + d_1 a_1}{a_2} \,,$$

while Equation 15 gives

$$d_2 = -\frac{d_3 b_3 + d_1 b_1}{b_2} \,.$$

If we set these two equal and simplify (as before) we get

$$d_3(a_2 b_3 - a_3 b_2) = d_1(a_1 b_2 - a_2 b_1) \,, \tag{17}$$

which is the second of the desired equations.

Part (b): Using the given expression we find

$$
\begin{aligned}
||\mathbf{d}||^2 &= ||\mathbf{a}||^2||\mathbf{b}||^2 \sin^2(\theta) \\
&= ||\mathbf{a}||^2||\mathbf{b}||^2(1-\cos^2(\theta)) \\
&= ||\mathbf{a}||^2||\mathbf{b}||^2 - |\mathbf{a}\cdot\mathbf{b}|^2 \\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2 \\
&= a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2 + a_3^2b_3^2 \\
&- (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_1a_2b_1b_2 + 2a_1a_3b_1b_3 + 2a_2a_3b_2b_3) \\
&= a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 + a_1^2b_3^2 - 2a_1a_3b_1b_3 + a_3^2b_1^2 + a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 \\
&= (a_1b_2-a_2b_1)^2 + (a_1b_3-a_3b_1)^2 + (a_2b_3-a_3b_2)^2 \,,
\end{aligned}
$$

the desired expression.

Part (c): If we multiply the expression for $||\mathbf{d}||^2$ above by d_1^2 we get

$$
\begin{aligned}
d_1^2||\mathbf{d}||^2 &= (d_1(a_1b_2-a_2b_1))^2 + (d_1(a_1b_3-a_3b_1))^2 + (d_1(a_2b_3-a_3b_2))^2 \\
&= ((a_2b_3-a_3b_2)d_3)^2 + ((a_2b_3-a_3b_2)d_2)^2 + (d_1(a_2b_3-a_3b_2))^2 \\
&= (a_2b_3-a_3b_2)^2||\mathbf{d}||^2 \,.
\end{aligned}
$$

Assuming that $||\mathbf{d}||^2 \neq 0$ we have

$$
d_1^2 = (a_2b_3-a_3b_2)^2 \quad \text{or} \quad d_1 = \pm(a_2b_3-a_3b_2) \,.
$$

Using this in Equation 16 we get

$$
d_2 = \pm(a_3b_1-a_1b_3) \,,
$$

while using this in Equation 17 we get

$$
d_3 = \pm(a_1b_2-a_2b_1) \,.
$$

Part (d): From the geometric definition of $\mathbf{a}\times\mathbf{b}$ to have $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a}\times\mathbf{b}$ form a right handed coordinate system means that when $\mathbf{a}=\mathbf{i}$ and $\mathbf{b}=\mathbf{j}$ we must have $\mathbf{a}\times\mathbf{b}$ in the direction of $\mathbf{k}$ and with a magnitude of

$$
||\mathbf{i}||\mathbf{j}|| \sin(90^\circ) = 1 \,.
$$

Thus $\mathbf{i}\times\mathbf{j}=\mathbf{k}$. Using the formulas we have derived above for d_i in this case we see that

$$
\begin{aligned}
d_1 &= \pm(0-0) = 0 \\
d_2 &= \pm(0-0) = 0 \\
d_3 &= \pm(1-0) = \pm 1 \,.
\end{aligned}
$$

Thus to agree with the argument above we must take the plus sign in the above expression.

Planes and Lines

Problem 1

Following the book, the vector equation of a plane is

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0\,. \tag{18}$$

Here

$$\mathbf{r} - \mathbf{r}_0 = (x-2)\mathbf{i} + (y+1)\mathbf{j} + (z-3)\mathbf{k}\,.$$

Thus the equation of the plane is

$$-(x-2) + 4(y+1) + 5(z-3) = 0\,,$$

or expanding we get

$$-x + 4y + 5z = -2 - 4 + 15 = 9\,.$$

Problem 2

Following the book the vector equation of a plane is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. Here

$$\mathbf{r} - \mathbf{r}_0 = (x-3)\mathbf{i} + y\mathbf{j} + (z+4)\mathbf{k}\,.$$

Thus the equation of the plane is

$$y + 2(z+4) = 0\,,$$

or expanding we get

$$y + 2z = -8\,.$$

Problem 3

A normal to this plane is given by

$$\begin{aligned}
\mathbf{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2-1 & 0+2 & 1-3 \\ 0-1 & 5 & -1 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 2 & -2 \\ -1 & 5 & -1 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 2 & -2 \\ 5 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -3 & -2 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & 2 \\ -1 & 5 \end{vmatrix} \\
&= \mathbf{i}(-2+10) - \mathbf{j}(3-2) + \mathbf{k}(-15+2) = 8\mathbf{i} - \mathbf{j} - 13\mathbf{k}\,.
\end{aligned}$$

A point on this plane is $\mathbf{r}_0 = (1, -2, 3)$. Then the equation of this plane is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ which in this case is

$$8(x-1) - (y+2) - 13(z-3) = 0\,,$$

or expanding we get

$$8x - y - 13z = -29\,.$$

Problem 4

A normal to this plane is given by

$$\begin{aligned}
\mathbf{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4-2 & 0-1 & 2+1 \\ -3 & -3 & 1+1 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -3 & -3 & 2 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -1 & 3 \\ -3 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & -1 \\ -3 & -3 \end{vmatrix} \\
&= \mathbf{i}(-2+9) - \mathbf{j}(3-9) + \mathbf{k}(-6-3) = 7\mathbf{i} + 5\mathbf{j} - 9\mathbf{k}\,.
\end{aligned}$$

A point on this plane is $\mathbf{r}_0 = (2, 1, -1)$. Then the equation of this plane is $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ which in this case is

$$7(x-2) + 5(y-1) - 9(z+1) = 0\,,$$

or expanding we get

$$7x + 5y - 9z = 28\,.$$

Problem 5

From its "coefficients" a normal to this plane is given by

$$\mathbf{n} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}\,.$$

A plane thought the point is $\mathbf{r}_0 = (3, 1, -2)$ and with normal $\mathbf{n}$ is given by the equation $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. In this case is

$$2(x-3) - (y-1) + 5(z+2) = 0\,,$$

or expanding

$$2x - y + 5z = -5\,.$$

Problem 6

From the book, we know that the vector in the "direction" of the line is formed from the coefficients of the numbers in the denominator of the fractions on the left-hand-side of

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{-1} = t\,,$$

so

$$\mathbf{n} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}\,.$$

A plane thought the point is $\mathbf{r}_0 = (4, -1, 3)$ and with normal $\mathbf{n}$ is given by the equation $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$. In this case is

$$3(x-4) - 2(y+1) - (z-3) = 0\,,$$

or expanding

$$3x - 2y - z = 11\,.$$

Problem 7

From the book, we know that the vector in the "direction" of the line is formed from the coefficients of the numbers in the denominator of the fractions. Thus one line is in the "direction"

$$\mathbf{l}_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}\,,$$

and the other line is in the "direction"

$$\mathbf{l}_2 = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}\,.$$

A vector normal to this plane is given by $\mathbf{n} = \mathbf{l}_1 \times \mathbf{l}_2$. We find

$$\begin{aligned}\mathbf{n} = \mathbf{l}_1 \times \mathbf{l}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ -1 & 4 & 2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & -1 \\ 4 & 2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} \\ &= \mathbf{i}(-4+4) - \mathbf{j}(6-1) + \mathbf{k}(12-2) = -5\mathbf{j} + 10\mathbf{k}\,.\end{aligned}$$

We now need a point on this plane. Notice that the point $\mathbf{r}_0 = (2, -1, 0)$ is on each line. The equation of the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ in this case is given by

$$-5(y+1) + 10(z-0) = 0\,,$$

or expanding we get

$$-5y + 10z = 5\,,$$

or dividing by -5 this gives

$$y - 2z = -1\,.$$

Problem 8

For our plane P_0 to be perpendicular to another plane P_1 means that the normal vector of P_0 is perpendicular to the normal vectors of P_1. For plane P_0 to be perpendicular to both P_1 and P_2 means that the normal vector of P_0 is perpendicular to *both* normal vectors of P_1 and P_2.

Thus a vector normal to this plane is given by $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2$. We find

$$\begin{aligned}\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ -2 & 1 & 3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \\ &= \mathbf{i}(-6-1) - \mathbf{j}(3+2) + \mathbf{k}(1+4) = -7\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}\,.\end{aligned}$$

With $\mathbf{r}_0 = (3, 2, -1)$ the equation of the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ in this case is given by

$$-7(x-3) - 5(y-2) + 5(z+1) = 0\,,$$

or expanding we get

$$-7x - 5y + 5z = -36\,.$$

Problem 9

Lets find two points on this line. As each fraction equals t if we set $t = 0$ we note that $P = (-2, 3, -1)$ is one point on this line. If we take $t = 1$ we get another point $Q = (-3, 5, 2)$ on the plane. With these two points and the desired point $R = (2, 2, -3)$ we now have three points on our plane. Using these three points a normal to this plane is given by

$$\begin{aligned}
\mathbf{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3+2 & 5-3 & 2+1 \\ 4 & -1 & -2 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ 4 & -1 & -2 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 2 \\ 4 & -1 \end{vmatrix} \\
&= \mathbf{i}(-4+3) - \mathbf{j}(2-12) + \mathbf{k}(1-8) = -\mathbf{i} + 10\mathbf{j} - 7\mathbf{k}\,.
\end{aligned}$$

The equation of the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ in this case with $\mathbf{r}_0 = (2, 2, -3)$ is given by

$$-(x-2) + 10(y-2) - 7(z+3) = 0\,,$$

or expanding we get

$$x + 10y - 7z = -39\,.$$

Problem 10

The normal to this plane must be parallel to

$$\mathbf{n} = (3, 0, -2) - (1, -3, 2) = (3-1)\mathbf{i} + (0+3)\mathbf{j} + (-2-2)\mathbf{k} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}\,.$$

The equation of the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$ in this case with $\mathbf{r}_0 = (1, -3, 2)$ is given by

$$2(x-1) + 3(y+3) - 4(z-2) = 0\,,$$

or expanding we get

$$2x + 3y - 4z = 1\,.$$

Problem 11

Setting this expression equal to t and solving for x, y, and z we get

$$\begin{aligned}
x &= 2 + 3t \\
y &= -1 + 2t \\
z &= 4 - t\,.
\end{aligned}$$

Problem 12

Solving each of these expressions for t I find

$$\frac{x+1}{6} = \frac{y-2}{-3} = \frac{z-5}{4} = t\,.$$

Problem 13

The equation of this line would be given by

$$\mathbf{r} - \mathbf{r}_0 = t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,,$$

or

$$(x-2)\mathbf{i} + (y+1)\mathbf{j} + (z-3)\mathbf{k} = t(\mathbf{i} + 2\mathbf{j} - \mathbf{k})\,.$$

This means that

$$\begin{aligned} x &= 2 + t \\ y &= -1 + 2t \\ z &= 3 - t\,. \end{aligned}$$

Problem 14

The equation of this line would be given by

$$\mathbf{r} - \mathbf{r}_0 = t(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\,,$$

or

$$(x+3)\mathbf{i} + (y-2)\mathbf{j} + (z+5)\mathbf{k} = t(2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})\,.$$

This means that

$$\begin{aligned} x &= -3 + 2t \\ y &= 2 + 3t \\ z &= -5 - 5t\,. \end{aligned}$$

Solving for t in each of these gives

$$t = \frac{x+3}{2} = \frac{y-2}{3} = \frac{z+5}{-5}\,.$$

Problem 15

A vector $\mathbf{v}$ "parallel" to this line is given by

$$\mathbf{v} = (2-4)\mathbf{i} + (-1-2)\mathbf{j} + (6-3)\mathbf{k} = -2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}\,.$$

Then the equation of the line or $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ is given by

$$(x-4)\mathbf{i} + (y-2)\mathbf{j} + (z-3)\mathbf{k} = -2t\mathbf{i} - 3t\mathbf{j} + 3t\mathbf{k}\,,$$

or

$$\begin{aligned} x - 4 &= -2t \\ y - 2 &= -3t \\ z - 3 &= 3t\,. \end{aligned}$$

Solving these for t we get the symmetric equations

$$t = \frac{x-4}{-2} = \frac{y-2}{-3} = \frac{z-3}{3}\,.$$

Problem 16

The vector $\mathbf{v}$ that this line is "parallel" to is given by the denominators of the given fraction or

$$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}\,.$$

Then the equation of the line or $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ with $\mathbf{r}_0 = (-1, 0, 3)$ is given by

$$(x+1)\mathbf{i} + y\mathbf{j} + (z-3)\mathbf{k} = 2t\mathbf{i} - 3t\mathbf{j} - t\mathbf{k}\,,$$

or

$$\begin{aligned} x &= -1 + 2t \\ y &= -3t \\ z &= 3 - t\,, \end{aligned}$$

for the scalar parametric equations.

Problem 17

A normal to the plane is given by $\mathbf{n} = \mathbf{j} + 2\mathbf{k}$. The equation of the line is then given by $\mathbf{r} - \mathbf{r}_0 = t\mathbf{n}$ with $\mathbf{r}_0 = (-2, 6, 1)$ or

$$(x+2)\mathbf{i} + (y-6)\mathbf{j} + (z-1)\mathbf{k} = 0\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}\,,$$

or

$$
\begin{aligned}
x &= -2 \\
y &= 6 + t \\
z &= 1 + 2t \,,
\end{aligned}
$$

for the scalar parametric equations. Solving these for t we get

$$
t = y - 6 = \frac{z-1}{2} \,,
$$

for the symmetric equations.

Problem 18

The normal to this plane is given by

$$
\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \,.
$$

The equation of the line is then given by $\mathbf{r} - \mathbf{r}_0 = t\mathbf{n}$ with $\mathbf{r}_0 = (3, -1, 4)$ or

$$
(x-3)\mathbf{i} + (y+1)\mathbf{j} + (z-4)\mathbf{k} = 2t\mathbf{i} + 3t\mathbf{j} - t\mathbf{k} \,,
$$

or

$$
\begin{aligned}
x &= 3 + 2t \\
y &= -1 + 3t \\
z &= 4 - t \,,
\end{aligned}
$$

for the scalar parametric equations.

Problem 19

Our line must run parallel to a vector that is perpendicular to the two normals of the two planes. These two normals are

$$
\begin{aligned}
\mathbf{n}_1 &= \mathbf{i} - 2\mathbf{k} \\
\mathbf{n}_2 &= 3\mathbf{j} + \mathbf{k} \,.
\end{aligned}
$$

A vector that is perpendicular to these two vectors is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2$. We find

$$
\begin{aligned}
\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \\
&= \mathbf{i}(0+6) - \mathbf{j}(1-0) + \mathbf{k}(3) = 6\mathbf{i} - \mathbf{j} + 3\mathbf{k} \,.
\end{aligned}
$$

We now need a point on this line. If we take $y = -3$ then the second plane gives $z = 0$. Using this in the first plane and we find $x = 4$ to give the point $\mathbf{r}_0 = (4, -3, 0)$. The equation of the line is then given by $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ or

$$(x-4)\mathbf{i} + (y+3)\mathbf{j} + z\mathbf{k} = 6t\mathbf{i} - t\mathbf{j} + 3t\mathbf{k}\,,$$

or

$$\begin{aligned} x &= 4 + 6t \\ y &= -3 - t \\ z &= 3t\,, \end{aligned}$$

for the scalar parametric equations.

Problem 20

Following the procedure used in the previous problem we have

$$\begin{aligned} \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 3 & -5 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 1 \\ 3 & -5 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} \\ &= \mathbf{i}(5-3) - \mathbf{j}(-5-2) + \mathbf{k}(3+2) = 2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}\,. \end{aligned}$$

We now need a point on this line. That is we are looking for a solution to the system

$$\begin{aligned} x - y + z &= 7 \\ 2x + 3y - 5z &= 9\,. \end{aligned}$$

If we solve the first equation for x we get $x = 7 - z + y$. If we put that into the second equation and solve for y (in terms of z) we get

$$y = \frac{7z}{5} - 1\,.$$

From this functional form we can get nice integer solutions if we take $z = 5$ so that $y = 6$ and then $x = 7 - 5 + 6 = 8$. Thus a point on the line is $\mathbf{r}_0 = (8, 6, 5)$. The equation of the line is then given by $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$ or

$$(x-8)\mathbf{i} + (y-6)\mathbf{j} + (z-5)\mathbf{k} = t(2\mathbf{i} + 7\mathbf{j} + 5\mathbf{k})\,,$$

or

$$\begin{aligned} x &= 8 + 2t \\ y &= 6 + 7t \\ z &= 5 + 5t\,, \end{aligned}$$

for the scalar parametric equations. Solving these for t the symmetric equations for this line are given by

$$t = \frac{x-8}{2} = \frac{y-6}{7} = \frac{z-5}{5}\,.$$

Problem 21

A normal to this plane is given by $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})\,.$$

We now need to find a point Q on this plane. If we take $x = y = 0$ then $z = 7$ so the point $(x, y, z) = (0, 0, 7)$ is on the plane. Using this value for Q we find

$$\overrightarrow{\mathbf{QP}} = 2\mathbf{i} + \mathbf{j} - 12\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{\sqrt{3}}|2 + 1 - 12| = \frac{9}{\sqrt{3}} = 3\sqrt{3}\,.$$

Problem 22

A normal to this plane is given by $\mathbf{N} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})\,.$$

We now need to find a point Q on this plane. If we take $x = y = 0$ then $z = -12$ so the point $(x, y, z) = (0, 0, -12)$ is on the plane. Using this value for Q we find

$$\overrightarrow{\mathbf{QP}} = -\mathbf{i} - 3\mathbf{j} + 14\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{\sqrt{14}}| - 2 - 9 - 14| = \frac{25}{\sqrt{14}}\,.$$

Problem 23

A normal to this plane is given by $\mathbf{N} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. From the parametric equations given we can derive the symmetric equations for this line as

$$t = \frac{x - 3}{-3} = \frac{y}{2} = \frac{z + 1}{4}\,.$$

From these a vector parallel to this line is given by $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. For our line to be parallel to the plane means that the vector $\mathbf{a}$ must be perpendicular to $\mathbf{N}$. To check this we compute

$$\mathbf{N} \cdot \mathbf{a} = -3(2) - 2 + 8 = 0\,.$$

Thus $\mathbf{a}$ is perpendicular to $\mathbf{N}$ and our line is parallel to the plane.

Problem 24

A normal to this plane is given by $\mathbf{N} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. From the symmetric equations given for this line a vector parallel to this line is given by $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. For our line to be parallel to the plane means that the vector $\mathbf{a}$ must be perpendicular to $\mathbf{N}$. To check this we compute

$$\mathbf{N} \cdot \mathbf{a} = -4 + 9 + 1 = 6 \neq 0\,.$$

Thus $\mathbf{a}$ is not perpendicular to $\mathbf{N}$ and our line is not parallel to the plane.

Problem 25

The parametric equations for this line are

$$\begin{aligned} x &= 2 - 2t \\ y &= -1 + 3t \\ z &= 2 - t\,. \end{aligned}$$

If we put these into the equation of the plane we get

$$4 - 4t - 3 + 9t - 2 + t = 11 \quad \text{or} \quad t = 2\,.$$

Using this value in the parametric equations above give the point $(x, y, z) = (-2, 5, 0)$.

Problem 26

Note that the vectors $\mathbf{p} - \mathbf{r}$ and $\mathbf{q} - \mathbf{r}$ are in the plane. A normal to this plane is then given by the cross product of these two vectors or

$$\mathbf{n} = (\mathbf{p} - \mathbf{r}) \times (\mathbf{q} - \mathbf{r})\,.$$

Using the associative property of the cross product we can write this as

$$\mathbf{n} = (\mathbf{p} \times \mathbf{q}) - (\mathbf{p} \times \mathbf{r}) - (\mathbf{r} \times \mathbf{q}) + (\mathbf{r} \times \mathbf{r})\,.$$

Using the properties of the cross product $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{a} = 0$ the above is equivalent to

$$\mathbf{n} = (\mathbf{p} \times \mathbf{q}) + (\mathbf{r} \times \mathbf{p}) + (\mathbf{q} \times \mathbf{r})\,,$$

as we were to show.

Problem 27

A normal to this plane is given by $\mathbf{N} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{3}(2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\,.$$

A vector parallel to this line is given by $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Now note that it the line is not parallel to the plane the distance between them will be zero as the line will eventually intersect the plane. We find that $\mathbf{N} \cdot \mathbf{a} = -6 - 2 + 8 = 0$ so our line is parallel to our plane.

A point on the plane is $Q = (0, 0, 10)$. A point on the line is $P = (2, 0, -1)$. Thus

$$\overrightarrow{\mathbf{QP}} = 2\mathbf{i} - 11\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{3}|4 - 22| = 6\,.$$

Problem 28

A normal to this plane is given by $\mathbf{N} = 5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$. A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{45}}(5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})\,.$$

The symmetric equations for this line are given by

$$t = \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{-3}\,.$$

From these a vector parallel to this line is given by $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Now note that it the line is not parallel to the plane the distance between them will be zero as the line will eventually intersect the plane. We find that

$$\mathbf{N} \cdot \mathbf{a} = 5(2) - 4(1) + 2(-3) = 10 - 4 - 6 = 0\,.$$

so our line is parallel to our plane.

A point on the plane is $Q = (0, 0, 6)$. A point on the line is $P = (1, -1, 2)$. Thus

$$\overrightarrow{\mathbf{QP}} = \mathbf{i} - \mathbf{j} - 4\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{\sqrt{45}}|5 + 4 - 8| = \frac{1}{\sqrt{45}}\,.$$

Problem 29

From the symmetric form of the lines we find vectors parallel to each line given by

$$\begin{aligned} \mathbf{a} &= \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \\ \mathbf{b} &= 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}\,. \end{aligned}$$

A vector perpendicular to **a** and **b** is given by

$$\mathbf{N} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ 4 & -1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix}$$
$$= \mathbf{i}(6-2) - \mathbf{j}(2+8) + \mathbf{k}(-1-12) = 4\mathbf{i} - 10\mathbf{j} - 13\mathbf{k}\,.$$

A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{285}}(4\mathbf{i} - 10\mathbf{j} - 13\mathbf{k})\,.$$

A point on the first line is $P = (2, -1, 1)$. A point on the second line is $Q = (-1, 2, -3)$. Thus

$$\overrightarrow{\mathbf{QP}} = 3\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{\sqrt{285}}|12 + 30 - 52| = \frac{10}{\sqrt{285}}\,.$$

Problem 30

The symmetric equations for these two lines are

$$t = \frac{x}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$$
$$t = \frac{x-2}{-1} = \frac{y+1}{3} = \frac{z}{1}\,.$$

Vectors parallel to each linear are then given by

$$\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{b} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}\,.$$

A vector perpendicular to **a** and **b** is given by

$$\mathbf{N} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ -1 & 3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}$$
$$= \mathbf{i}(-1-3) - \mathbf{j}(2+1) + \mathbf{k}(6-1) = -4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}\,.$$

A unit normal is then

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{5\sqrt{2}}(-4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})\,.$$

A point on the first line is $P = (0, -1, 3)$. A point on the second line is $Q = (2, -1, 0)$. Thus

$$\overrightarrow{\mathbf{QP}} = 2\mathbf{i} + 3\mathbf{k}\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{QP}} \cdot \mathbf{n}|$ or

$$\frac{1}{5\sqrt{2}}|-8 + 0 + 15| = \frac{7}{5\sqrt{2}}\,.$$

Problem 31

This problem presents one way to compute the needed normal vector $\mathbf{n}$. The following problem demonstrates another way.

From the given symmetric equations a vector parallel to the line is given by

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}\,.$$

A point on the line is given by $A = (1, -1, 0)$. Thus

$$\overrightarrow{\mathbf{AP}} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) - (\mathbf{i} - \mathbf{j}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}\,.$$

Now to find a vector $\mathbf{N}$ that is perpendicular to the line and through the point P thus it would take the form

$$\mathbf{N} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) - ((1 + 2t)\mathbf{i} + (-1 + 3t)\mathbf{j} - t\mathbf{k}) = (1 - 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (1 + t)\mathbf{k}\,,$$

for $t \in \mathbb{R}$. For $\mathbf{N}$ to be perpendicular to $\mathbf{a}$ we need $\mathbf{N} \cdot \mathbf{a} = 0$ which in this case is

$$2(1 - 2t) + 3(2 - 3t) - (1 + t) = 0 \quad \text{so} \quad t = \frac{1}{2}\,.$$

This means that

$$\mathbf{N} = \frac{1}{2}\mathbf{j} + \frac{3}{2}\mathbf{k}\,.$$

Making this a unit vector then gives

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{10}}(\mathbf{j} + 3\mathbf{k})\,.$$

The distance we seek is given by $|\overrightarrow{\mathbf{AP}} \cdot \mathbf{n}|$ or

$$\frac{1}{\sqrt{10}}|0 + 2 + 3| = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{10}}{2}\,.$$

Problem 32

From the given parametric equations we can compute the symmetric equations. We find

$$t = \frac{x + 2}{1} = \frac{y - 4}{-1} = \frac{z - 1}{3}\,.$$

From the symmetric equations a vector parallel to the line is given by

$$\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}\,.$$

A point on the line is given by $A = (-2, 4, 1)$. Thus

$$\overrightarrow{\mathbf{AP}} = (\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - (-2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) = 3\mathbf{i} - \mathbf{j} - 5\mathbf{k}\,.$$

Note that the vector $\mathbf{b} = \mathbf{a} \times \overrightarrow{\mathbf{AP}}$ will be perpendicular to the plane containing the line and the point. Computing this I find

$$\begin{aligned}
\mathbf{b} = \mathbf{a} \times \overrightarrow{\mathbf{AP}} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 3 & -1 & -5 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ -1 & -5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 3 & -5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \\
&= \mathbf{i}(5+3) - \mathbf{j}(-5-9) + \mathbf{k}(-1+3) = 8\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}\,.
\end{aligned}$$

Now the vector $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ will be perpendicular to both the line (i.e. the vector $\mathbf{a}$) and the normal to the plane (i.e. the vector $\mathbf{b}$) and will thus be the vector we seek. I find

$$\begin{aligned}
\mathbf{N} = \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ 8 & 14 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 3 \\ 14 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 8 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 8 & 14 \end{vmatrix} \\
&= \mathbf{i}(-2-42) - \mathbf{j}(2-24) + \mathbf{k}(14+8) = -44\mathbf{i} + 22\mathbf{j} + 22\mathbf{k} = 22(-2\mathbf{i} + \mathbf{j} + \mathbf{k})\,.
\end{aligned}$$

Making this a unit vector then gives

$$\mathbf{n} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k})\,.$$

As *another* way (similar to what was done in the previous problem) to find a vector $\mathbf{N}'$ that is perpendicular to the line and through the point P is to recall that it would take the form

$$\mathbf{N}' = (\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) - ((-2+t)\mathbf{i} + (4-t)\mathbf{j} + (1+3t)\mathbf{k}) = (3-t)\mathbf{i} + (-1+t)\mathbf{j} + (-5-3t)\mathbf{k}\,,$$

for $t \in \mathbb{R}$. For $\mathbf{N}'$ to be perpendicular to $\mathbf{a}$ we need $\mathbf{N}' \cdot \mathbf{a} = 0$ which in this case is

$$(3-t) - (-1+t) + 3(-5-3t) = 0 \quad \text{so} \quad t = -1\,.$$

This means that

$$\mathbf{N}' = 4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\,.$$

Making this a unit vector then gives

$$\mathbf{n}' = \frac{\mathbf{N}'}{||\mathbf{N}'||} = \frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} - \mathbf{k})\,.$$

Note that this is the negative of the vector $\mathbf{n}$ found above. As we are taking the absolute values below both vectors will give the equivalent distance.

The distance we seek is given by $|\overrightarrow{\mathbf{AP}} \cdot \mathbf{n}| = |\overrightarrow{\mathbf{AP}} \cdot \mathbf{n}'|$ or

$$\frac{1}{\sqrt{6}}|6+1+5| = \frac{\sqrt{11}}{\sqrt{6}}\,.$$

Problem 33

For this problem we will take $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Part (a): In this case $\mathbf{a}\cdot\mathbf{r} = 0$ is

$$a_x x + a_y y + a_z z = 0\,,$$

or the equation of a plane thought the origin with the normal vector $\mathbf{a}$.

Part (b): In terms of components this expression is

$$|a_x x + a_y y + a_z z| = c\sqrt{a_x^2 + a_y^2 + a_z^2}\,.$$

If $\mathbf{a}\cdot\mathbf{r} > 0$ then this is

$$a_x x + a_y y + a_z z = +c\sqrt{a_x^2 + a_y^2 + a_z^2}\,,$$

which is a plane with normal $\mathbf{a}$ not through the origin. If $\mathbf{a}\cdot\mathbf{r} < 0$ then this above is

$$a_x x + a_y y + a_z z = -c\sqrt{a_x^2 + a_y^2 + a_z^2}\,,$$

which is another plane with normal $\mathbf{a}$ and not though the origin.

If we divide both sides of the original expression by $||\mathbf{a}||$ we get

$$\left|\frac{\mathbf{a}}{||\mathbf{a}||}\cdot\mathbf{r}\right| = c\,,$$

which states that the distance from the origin to the plane is c.

Part (c): This is equivalent to $\mathbf{r}\cdot\mathbf{r} - \mathbf{a}\cdot\mathbf{r} = 0$ or

$$x^2 + y^2 + z^2 - a_x x - a_y y - a_z z = 0\,.$$

If we "complete the square" in the above we get

$$\left(x - \frac{a_x}{2}\right)^2 + \left(y - \frac{a_y}{2}\right)^2 + \left(z - \frac{a_z}{2}\right)^2 = \frac{a_x^2}{4} + \frac{a_y^2}{4} + \frac{a_z^2}{4}\,.$$

which is the equation of a sphere with a center at $\frac{1}{2}\mathbf{a}$ and a radius R such that $R^2 = \frac{1}{4}||\mathbf{a}||^2$ so $R = \frac{||\mathbf{a}||}{2}$.

Problem 35

These two lines are parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$. Then a normal to each line is given by $\mathbf{N} = \mathbf{a}\times\mathbf{b}$. Two points on the lines are $\mathbf{r}_1$ and $\mathbf{r}_2$ so a vector connecting these two point is $\mathbf{r}_1 - \mathbf{r}_2$. These two lines will intersect if and only if the distance between them is zero or from the discussion in the text if

$$\left|\left(\frac{\mathbf{a}\times\mathbf{b}}{||\mathbf{a}\times\mathbf{b}||}\right)\cdot(\mathbf{r}_1 - \mathbf{r}_2)\right| = 0\,.$$

This is equivalent to the given expression.

Vector Functions of One Variable

Problem 1

For this we have
$$\lim_{t\to 0}(e^{-t}\mathbf{i} + 2\cos(t)\mathbf{j} + (t^2-1)\mathbf{k}) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}\,.$$

Problem 2

For this we have
$$\lim_{t\to \pi}(e^{2t}\mathbf{i} - \sin(t)\mathbf{j} + t^3\mathbf{k}) = e^{2\pi}\mathbf{i} - 0\mathbf{j} + \pi^3\mathbf{k} = e^{2\pi}\mathbf{i} + \pi^3\mathbf{k}\,.$$

Problem 3

Because of the denominator in the expression $\frac{2t}{t-1}$ the limit does not exist.

Problem 4

For this we have
$$\lim_{t\to 1}\left(\frac{1}{t}\mathbf{i} + \frac{t-2}{t+1}\mathbf{j} + \frac{t-1}{2t}\mathbf{k}\right) = \mathbf{i} - \frac{1}{2}\mathbf{j}\,.$$

Problem 5

For this we have
$$\lim_{t\to 0}\frac{1}{t}\left(\sin(2t)\mathbf{i} + 3t\mathbf{j} + \tan(t)\mathbf{k}\right) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}\,.$$
In the above we have used L'Hospital's rule to evaluate the component limits.

Problem 6

For this we have
$$\lim_{t\to 2}\left(\frac{t^2-4}{t-2}\mathbf{i} + \frac{\sqrt{t}-\sqrt{2}}{t-2}\mathbf{j}\right) = \lim_{t\to 2}\left((t+2)\mathbf{i} + \frac{\sqrt{t}-\sqrt{2}}{(\sqrt{t}-\sqrt{2})(\sqrt{t}+\sqrt{2})}\mathbf{j}\right) = 4\mathbf{i} + \frac{1}{2\sqrt{2}}\mathbf{j}\,.$$

Problem 7-9

These can be proven by converting the left and right hand-side into its **i**, **j**, and **k** components and showing that they are equal.

Problem 10

For this we have

$$\begin{aligned}\lim_{t\to 0}(3\mathbf{u}(t) - 2\mathbf{v}(t)) &= 3\lim_{t\to 0}\mathbf{u}(t) - 2\lim_{t\to 0}\mathbf{v}(t) \\ &= 3(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - 2(\mathbf{i} - \mathbf{j}) = \mathbf{i} + 7\mathbf{j} - 3\mathbf{k}\,.\end{aligned}$$

Problem 11

For this we have

$$\begin{aligned}\lim_{t\to 0}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \lim_{t\to 0}\mathbf{u}(t)\cdot\lim_{t\to 0}\mathbf{v}(t) \\ &= (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\cdot(\mathbf{i} - \mathbf{j}) = 1 - 2 = -1\,.\end{aligned}$$

Problem 12

For this we have

$$\begin{aligned}\lim_{t\to 0}(\mathbf{u}(t)\times\mathbf{v}(t)) = \lim_{t\to 0}\mathbf{u}(t)\times\lim_{t\to 0}\mathbf{v}(t) = (\mathbf{i} + 2\mathbf{j} - \mathbf{k})\times(\mathbf{i} - \mathbf{j}) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & -1 & 0\end{vmatrix} \\ = \mathbf{k}\begin{vmatrix}1 & 2 \\ 1 & -1\end{vmatrix} + 1\begin{vmatrix}\mathbf{i} & \mathbf{j} \\ 1 & -1\end{vmatrix} &= \mathbf{k}(-1-2) + (-\mathbf{i} - \mathbf{j}) = -\mathbf{i} - \mathbf{j} - 3\mathbf{k}\,.\end{aligned}$$

Problem 13

For this we have

$$\mathbf{F}'(t) = 2t\mathbf{i} - \sin(t)\mathbf{j} + 2\cos(t)\mathbf{k}\,.$$

Problem 14

For this we have

$$\mathbf{F}'(t) = -e^{-t}\mathbf{i} + (e^t + te^t)\mathbf{j} + 4\mathbf{k}\,.$$

Problem 15

For this we have

$$\mathbf{F}'(t) = (2\sin(2t) + 4t\cos(2t))\mathbf{i} + 3\sin(t)\mathbf{j} + 12(2t-1)\mathbf{k}\,.$$

Problem 16

For this we have

$$\mathbf{F}'(t) = (\cos(t) - t\sin(t))\mathbf{i} + (\sin(t) + t\cos(t))\mathbf{j} - 6\mathbf{k}\,.$$

Problem 17

For this we have

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}(t) + \mathbf{v}(t)) &= \mathbf{u}'(t) + \mathbf{v}'(t) \\
&= -e^{-t}\mathbf{i} - 2\sin(t)\mathbf{j} + 2t\mathbf{k} + 2e^{2t}\mathbf{i} - \cos(t)\mathbf{j} + 3t^2\mathbf{k} \\
&= (2e^{2t} - e^{-t})\mathbf{i} - (\cos(t) + 2\sin(t))\mathbf{j} + (2t + 3t^2)\mathbf{k}\,.
\end{aligned}$$

Problem 18

For this we have

$$\begin{aligned}
\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) \\
&= (-e^{-t}\mathbf{i} - 2\sin(t)\mathbf{j} + 2t\mathbf{k})\cdot(e^{2t}\mathbf{i} - \sin(t)\mathbf{j} + t^3\mathbf{k}) \\
&+ (e^{-t}\mathbf{i} + 2\cos(t)\mathbf{j} + (t^2-1)\mathbf{k})\cdot(2e^{2t}\mathbf{i} - \cos(t)\mathbf{j} + 3t^2\mathbf{k}) \\
&= (-e^{t} + 2\sin^2(t) + 2t^3) + (2e^{t} - 2\cos^2(t) + 3t^2(t^2-1)) \\
&= e^{t} + 2(\sin^2(t) - \cos^2(t)) + 5t^4 - 3t^2\,.
\end{aligned}$$

Problem 19

For this we have

$$\begin{aligned}
\frac{d}{dt}(3\mathbf{u}(t) - 2\mathbf{v}(t)) &= 3\mathbf{u}'(t) - 2\mathbf{v}'(t) \\
&= 3(-e^{-t}\mathbf{i} - 2\sin(t)\mathbf{j} + 2t\mathbf{k}) - 2(2e^{2t}\mathbf{i} - \cos(t)\mathbf{j} + 3t^2\mathbf{k}) \\
&= (-3e^{-t} - 4e^{2t})\mathbf{i} + (-6\sin(t) + 2\cos(t))\mathbf{j} + (6t - 6t^2)\mathbf{k}\,.
\end{aligned}$$

Problem 20

For this we have

$$\begin{aligned}\frac{d}{dt}(4\mathbf{u}(t) - 3\mathbf{v}(t)) &= 4\mathbf{u}'(t) - 3\mathbf{v}'(t)\\ &= 4(2\mathbf{i} - 2t\mathbf{j} + 4t^3\mathbf{k}) - 3(2t\mathbf{i} + 6\mathbf{j} + 5\mathbf{k})\\ &= (8 - 6t)\mathbf{i} - (4t + 18)\mathbf{j} + (16t^3 - 15)\mathbf{k}\,.\end{aligned}$$

Problem 21

For this we have

$$\begin{aligned}\frac{d}{dt}(\mathbf{u}(t)\cdot\mathbf{v}(t)) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)\\ &= (2\mathbf{i} - 2t\mathbf{j} + 4t^3\mathbf{k})\cdot(t^2\mathbf{i} + 6t\mathbf{j} + 5t\mathbf{k}) + (2t\mathbf{i} - t^2\mathbf{j} + t^4\mathbf{k})\cdot(2t\mathbf{i} + 6\mathbf{j} + 5\mathbf{k})\\ &= 2t^2 - 12t^2 + 20t^4 + 4t^2 - 6t^2 + 5t^4 = -12t^2 + 25t^4\,.\end{aligned}$$

Problem 22

It might be faster to work this problem by first taking the cross product and then taking the derivative of that expression. For the cross product of these two vectors I find

$$\begin{aligned}\mathbf{u}(t)\times\mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2t & -t^2 & t^4\\ t^2 & 6t & 5t \end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix} -t^2 & t^4\\ 6t & 5t \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2t & t^4\\ t^2 & 5t \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2t & -t^2\\ t^2 & 6t \end{vmatrix}\\ &= \mathbf{i}(-5t^3 - 6t^5) - \mathbf{j}(10t^2 - t^6) + \mathbf{k}(12t^2 + t^4)\,.\end{aligned}$$

This then means that

$$\frac{d}{dt}(\mathbf{u}(t)\times\mathbf{v}(t)) = \mathbf{i}(-15t^2 - 30t^4) - \mathbf{j}(20t - 6t^5) + \mathbf{k}(24t + 4t^3)\,.$$

Problem 23

As $\mathbf{v}(t)\cdot\mathbf{u}(t) = \mathbf{u}(t)\cdot\mathbf{v}(t)$ the answer to this problem is the same as the answer to Problem 21.

Problem 24

As $\mathbf{v}(t)\times\mathbf{u}(t) = -\mathbf{u}(t)\times\mathbf{v}(t)$ the answer to this problem is the same as the negative of the answer to Problem 22.

Problem 25

Part (a): For this vector we have $x = 2t - 3$ so $t = \frac{x+3}{2}$. If we put that into $y = -4t^2$ we get

$$y = -4\left(\frac{x+3}{2}\right)^2 = -(x+3)^2\,,$$

which is a parabola with its vertex at $(-3, 0)$ and "opening downwards". Now as t goes from $-\infty$ to ∞ the particle moves on this parabola from left to right.

Part (b): For this expression for $\mathbf{r}(t)$ we find

$$\mathbf{r}'(t) = 2\mathbf{i} - 8t\mathbf{j}\,,$$

thus $\mathbf{r}'(-1) = 2\mathbf{i} + 8\mathbf{j}$. The desired unit vector is then

$$\frac{1}{\sqrt{68}}(2\mathbf{i} + 8\mathbf{j}) = \frac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})\,.$$

Part (c): When $t = -1$ we find $\mathbf{r}(-1) = -5\mathbf{i} - 4\mathbf{j}$. The tangent to the curve goes thought the point $\mathbf{r}(-1)$ and in the direction of $\mathbf{r}'(-1)$. This means that we have

$$\begin{aligned} x(t) &= -5 + t \\ y(t) &= -4 + 4t \\ z(t) &= 0\,, \end{aligned}$$

for $-\infty < t < +\infty$.

Part (d): A normal vector for this plane is given by $\mathbf{N} = \mathbf{i} + 4\mathbf{j}$. Next recall that the equation of a plane is given by Equation 18. In this case this becomes

$$(x + 5) + 4(y + 4) = 0\,.$$

Problem 26

Part (a): Let $x(t) = \cosh(t)$ and $z(t) = 2\sinh(t)$ then using $\cosh^2(\theta) - \sinh^2(\theta) = 1$ we find

$$x^2 - \frac{z^2}{4} = 1\,.$$

This is a hyperbola with two branches

$$x = \pm\sqrt{1 + \frac{z^2}{4}}\,.$$

When $t = 0$ we find $(x(0), z(0)) = (1, 0)$. As $t \to -\infty$ we have $(x(t), z(t)) \to (+\infty, -\infty)$ and as $t \to +\infty$ we have $(x(t), z(t)) \to (+\infty, +\infty)$. This means that our "particle" is in the plane $y = 0$ and on the right-most curve starting at $(x, z) = (+\infty, -\infty)$ and moving towards the point $(x, z) = (1, 0)$ before moving away from it towards $(x, z) = (+\infty, +\infty)$.

Part (b): We find $\mathbf{r}(0) = 1\mathbf{i}$ which is the point $(x, y, z) = (1, 0, 0)$. We also find

$$\mathbf{r}'(t) = \sinh(t)\mathbf{i} + 2\cosh(t)\mathbf{k}\,.$$

This means that $\mathbf{r}'(0) = 2\mathbf{k}$. A unit tangent vector is then $\mathbf{k}$.

Part (c): The parametric equations of the line through the point $(1, 0, 0)$ and parallel to the vector $\mathbf{k}$ are given by

$$\begin{aligned} x - 1 &= 0 \\ y - 0 &= 0 \\ z - 0 &= \tau\,. \end{aligned}$$

for $-\infty < \tau < \infty$.

Part (d): A normal vector to this plane is given by $\mathbf{N} = 2\mathbf{k}$. The equation of a plane is given by Equation 18 which in this case is given by

$$0(x - 1) + 0(y - 0) + 2(z - 0) = 0 \quad \text{or} \quad z = 0\,.$$

Problem 27

Part (a): From this parametric representation of our vector function we find

$$\begin{aligned} \mathbf{r}(0) &= 2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 2\mathbf{i} \\ \mathbf{r}'(t) &= -2\sin(t)\mathbf{i} + 3\cos(t)\mathbf{j} + 4\mathbf{k} \quad \text{and thus} \\ \mathbf{r}'(0) &= 0\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} = 3\mathbf{j} + 4\mathbf{k}\,. \end{aligned}$$

A unit tangent vector to the curve at the point $t = 0$ is then given by

$$\frac{\mathbf{r}'(0)}{||\mathbf{r}'(0)||} = \frac{3\mathbf{j} + 4\mathbf{k}}{\sqrt{9 + 16}} = \frac{1}{5}(3\mathbf{j} + 4\mathbf{k})\,.$$

Part (b): A parametric equation for the line tangent to the curve is then given by

$$\begin{aligned} x(\tau) &= 2 + 0\tau = 2 \\ y(\tau) &= 0 + 3\tau = 3\tau \\ z(\tau) &= 0 + 4\tau = 4\tau\,. \end{aligned}$$

Part (c): A vector normal to the desired plane is given by $\mathbf{N} = \mathbf{r}'(0) = 3\mathbf{j} + 4\mathbf{k}$. Using this in Equation 18 gives

$$0(x - 2) + 3(y - 0) + 4(z - 0) = 0\,, \text{or} \quad 3y + 4z = 0\,.$$

Problem 28

Part (a): The given vector curve will pass though the point $(4, -3, 2)$ for a t value given by the solution to the equations

$$
\begin{aligned}
4 &= t^2 + 3 \\
-3 &= -3t \\
2 &= 2t^2 \,.
\end{aligned}
$$

From the second of these we see that $t = 1$. Also for this vector function we compute

$$
\begin{aligned}
\mathbf{r}'(t) &= 2t\mathbf{i} - 3\mathbf{j} + 4t\mathbf{k} \quad \text{thus} \\
\mathbf{r}'(1) &= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \,.
\end{aligned}
$$

A unit tangent vector to the curve at $t = 1$ is then given by

$$
\frac{\mathbf{r}'(1)}{||\mathbf{r}'(1)||} = \frac{2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}}{\sqrt{4+9+16}} = \frac{2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}} \,.
$$

Part (b): A parametric equation for the line tangent to the curve at the given point is

$$
\begin{aligned}
x(\tau) &= 4 + 2\tau \\
y(\tau) &= -3 - 3\tau \\
3(\tau) &= 2 + 4\tau \,,
\end{aligned}
$$

for $-\infty < \tau < +\infty$.

Part (c): The plane normal to the curve has a normal vector given by $\mathbf{r}'(1)$. Using that and Equation 18 the equation of the plane through this point takes the form

$$
2(x-4) + 3(y+3) + 4(z-2) = 0 \,.
$$

Problem 29

Lets call these two curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(\tau)$ respectively. Then from what we are given we find

$$
\begin{aligned}
\mathbf{r}_1'(t) &= 2t\mathbf{i} - 3\mathbf{j} + 4t\mathbf{k} \\
\mathbf{r}_2'(\tau) &= 2\mathbf{i} + \mathbf{j} + \tau\mathbf{k} \,.
\end{aligned}
$$

We first note that for $\mathbf{r}_1(t)$ will go though the point $(4, -3, 2)$ when $t = 1$ and $\mathbf{r}_2(\tau)$ to go through the point $(4, -3, 2)$ when $\tau = 2$. Next from these two expressions we can compute that

$$
\begin{aligned}
\mathbf{r}_1'(1) &= 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \\
\mathbf{r}_2'(2) &= 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \,.
\end{aligned}
$$

Using these we find that

$$
\cos(\theta) = \frac{\mathbf{r}_1'(1) \cdot \mathbf{r}_2'(2)}{||\mathbf{r}_1'(1)||||\mathbf{r}_2'(2)||} = \frac{4-3+8}{\sqrt{4+9+16}\sqrt{4+1+4}} = \frac{3}{\sqrt{29}} = 0.557086 \,.
$$

This gives $\theta = 56.14549°$.

Problem 30

Lets call these two curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(\tau)$ respectively. We first note that for $\mathbf{r}_1(t)$ will go though the point $(1, 0, 1)$ when $t = 2$ and $\mathbf{r}_2(\tau)$ to go through the point $(1, 0, 1)$ when $\tau = 0$. Next from these two expressions we find

$$
\begin{aligned}
\mathbf{r}_1'(t) &= -\pi \sin(\pi t)\mathbf{i} - 2\pi \cos(\pi t)\mathbf{j} + \mathbf{k} \\
\mathbf{r}_2'(\tau) &= 2\mathbf{i} + 8\tau\mathbf{j} - \pi \sin(\pi\tau)\mathbf{k}\,.
\end{aligned}
$$

From these we compute that

$$
\begin{aligned}
\mathbf{r}_1'(2) &= 0\mathbf{i} - 2\pi\mathbf{j} + \mathbf{k} \\
\mathbf{r}_2'(0) &= 2\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}\,.
\end{aligned}
$$

Using these we find that

$$
\cos(\theta) = \frac{\mathbf{r}_1'(2) \cdot \mathbf{r}_2'(0)}{||\mathbf{r}_1'(2)||||\mathbf{r}_2'(0)||} = 0\,.
$$

This gives $\theta = \frac{\pi}{2} = 90°$.

Problem 31

Part (a): For this vector function we have

$$
\begin{aligned}
x(t) &= a\cos(\omega t) \\
y(t) &= b\sin(\omega t)\,,
\end{aligned}
$$

and thus

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,
$$

which is the equation of an ellipse with semi-major and semi-minor axis of lengths a and b.

Part (b): The vector expression for the velocity of the particle is given by

$$
\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}\,.
$$

Part (c): The direction towards the origin at any time t is given by $-\mathbf{r}(t)$. A unit vector in that direction is then given by

$$
\mathbf{u} = -\frac{\mathbf{r}(t)}{||\mathbf{r}(t)||} = \frac{-a\cos(\omega t)\mathbf{i} - b\sin(\omega t)\mathbf{j}}{\sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}}\,.
$$

The desired component in the direction of $\mathbf{r}'(t)$ is then $\mathbf{u} \cdot \mathbf{r}'(t)$. We compute this to be

$$
\frac{a^2\omega\sin(\omega t)\cos(\omega t) - b^2\omega\sin(\omega t)\cos(\omega t)}{\sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}} = \frac{(a^2 - b^2)\omega\sin(\omega t)\cos(\omega t)}{\sqrt{a^2\cos^2(\omega t) + b^2\sin^2(\omega t)}}\,.
$$

Problem 32

We are told that $||\mathbf{u}(t)|| = c$ or $||\mathbf{u}||^2 = c^2$ where c is a constant. In terms of the components of $\mathbf{u}$ this is equivalent to

$$u_1^2(t) + u_2^2(t) + u_3^2(t) = c^2 \,.$$

Taking the derivative of this with respect to t gives

$$2u_1(t)u_1'(t) + 2u_2(t)u_2'(t) + 2u_3(t)u_3'(t) = 0 \,,$$

which can also be written in vector form as

$$\mathbf{u}(t) \cdot \mathbf{u}'(t) = 0 \,.$$

This is the statement that $\mathbf{u}(t)$ and $\mathbf{u}'(t)$ are perpendicular.

Problem 33

From the definition of the dot product we have

$$\mathbf{u}(t) \cdot \mathbf{v}(t) = u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t) \,.$$

Then using the product rule in each scalar product we have

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}(t) \cdot \mathbf{v}(t) &= \frac{du_1(t)}{dt}v_1(t) + \frac{du_2(t)}{dt}v_2(t) + \frac{du_3(t)}{dt}v_3(t) + u_1(t)\frac{dv_1(t)}{dt} + u_2(t)\frac{dv_2(t)}{dt} + u_3(t)\frac{dv_3(t)}{dt} \\
&= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \,.
\end{aligned}$$

Problem 34

Recall that the cross product of two vectors is given by

$$\begin{aligned}
\mathbf{u}(t) \times \mathbf{v}(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1(t) & u_2(t) & u_3(t) \\ v_1(t) & v_2(t) & v_3(t) \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2(t) & u_3(t) \\ v_2(t) & v_3(t) \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1(t) & u_3(t) \\ v_1(t) & v_3(t) \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{vmatrix} \\
&= \mathbf{i}(u_2(t)v_3(t) - u_3(t)v_2(t)) - \mathbf{j}(u_1(t)v_3(t) - u_3(t)v_1(t)) + \mathbf{k}(u_1(t)v_2(t) - u_2(t)v_1(t)) \,.
\end{aligned}$$

Then using the product rule in each scalar product we have

$$\begin{aligned}
\frac{d}{dt}\mathbf{u}(t) \times \mathbf{v}(t) &= \mathbf{i}(u_2'(t)v_3(t) - u_3'(t)v_2(t)) - \mathbf{j}(u_1'(t)v_3(t) - u_3'(t)v_1(t)) + \mathbf{k}(u_1'(t)v_2(t) - u_2'(t)v_1(t)) \\
&+ \mathbf{i}(u_2(t)v_3'(t) - u_3(t)v_2'(t)) - \mathbf{j}(u_1(t)v_3'(t) - u_3(t)v_1'(t)) + \mathbf{k}(u_1(t)v_2'(t) - u_2(t)v_1'(t)) \\
&= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \,.
\end{aligned}$$

Arc Length and Curvature

WWX: Working here.

Dynamics of Particles

Kepler's Laws

Cylinders, Surfaces of Revolution, and Quadric Surfaces

Chapter 16 (Differentiation of Functions of Several Variables)

Section 16.4 (the Gradient and the Directional Derivative)

Problem 13 (increasing most rapidly)

For the $f(x,y)$ given we have $\nabla f = (x+y)\hat{\imath}+(2x+2y)\hat{\jmath}$, so that $||\nabla f|| = \sqrt{(x+y)^2+4(x+y)^2} = \sqrt{5}|x+y|$, which is the rate of change where f is increasing most most rapidly.

Problem 14 (increasing most rapidly)

For the $f(x,y,z)$ given we have $\nabla f = (y^2+6zx)\hat{\imath}+(2xy+2z^2)\hat{\jmath}+(4yz+3x^2)\hat{k}$, so that $||\nabla f|| = \sqrt{(x+y)^2+4(x+y)^2} = \sqrt{5}|x+y|$, which is the rate of change where f is increasing most most rapidly.

Problem 42 (a heat seeking particle)

The maximum heat flow would be in the direction of the gradient. In this problem this is

$$\begin{aligned} \nabla T &= T_0 e^{-(x^2+3y^2)/5}\left(-\frac{2x}{5}\hat{\imath}-\frac{6y}{5}\hat{\jmath}\right) \\ &= -\frac{T_0}{5}e^{-(x^2+3y^2)/5}\left(2x\hat{\imath}+6y\hat{\jmath}\right) \end{aligned}$$

Therefore at (a,b) the insect follows, the path in (x,y) that has slope given by

$$\frac{dy}{dx} = \frac{(\nabla T)_y}{(\nabla T)_x}\,.$$

In this problem we obtain that this expression is

$$\frac{dy}{dx} = \frac{6y}{2x} = \frac{3y}{x}\,,$$

or solving this differential equation

$$\begin{aligned} \frac{dy}{y} &= 3\frac{dx}{x} \\ \ln(|y|) &= 3\ln(|x|)+C \\ |y| &= C|x|^3\,. \end{aligned}$$

Evaluating this expression at the point (a,b) to determine the constant C we find that $C = b/a^3$ and thus the path is given by

$$y = \frac{b}{a^3}x^3\,.$$

Chapter 17 (Multiple Integrals)

Section 17.5 (Triple Integrals)

Problem 7 (the cap of a sphere)

See the scanned notes for a sketch of the given projection. The projection on to the xy-plane happens when $z = 3$. Thus projecting the $3D$ region onto the xy-plane requires $x^2 + y^2 + 9 = 16$, or $x^2 + y^2 = 7$. The inequalities describing this volume then become

$$\begin{aligned} 3 \le & \quad z \quad \le 16 - x^2 - y^2 \\ -\sqrt{7 - x^2} \le & \quad y \quad \le +\sqrt{7 - x^2} \\ -7 \le & \quad x \quad \le +7 \,. \end{aligned}$$

Problem 8 (a region bounded by two planes)

See the scanned notes for a sketch of the given region. From that figure one sees that the plane $x + 2z = 2$ has xz intercepts given by $(x, z) = (0, 1)$ and $(x, z) = (2, 0)$ with y arbitrary. The plane $3x + 2y + z = 12$ has intercepts on the three coordinate axis given by $(9, 0, 0), (0, 6, 0)$, and $(0, 0, 12)$. Thus the volume sought is the region above the plane $x + 2z = 2$ and below the plane $3x + 2y + z = 12$.

Part (a): When we project our volume onto the xy-plane we have to break the region into two regions Ω^1_{xy} and Ω^2_{xy}, since the smaller plane $x + 2z = 2$ intersects the xy axis when $z = 0$ at the point $x = 2$. Letting Ω^1_{xy} be the upper trapezoid of Ω_{xy} we have for Ω^1_{xy} the following mathematical description

$$\begin{aligned} \frac{2 - x}{2} \le & \quad z \quad \le 12 - 3x - 2y \\ 0 \le & \quad y \quad \le \frac{12 - 3x}{2} \\ 0 \le & \quad x \quad \le 2 \,. \end{aligned}$$

Letting Ω^2_{xy} denote the lower trapezoid of Ω_{xy} we have the following mathematical description

$$\begin{aligned} 0 \le & \quad z \quad \le 12 - 3x - 2y \\ 0 \le & \quad y \quad \le \frac{12 - 3x}{2} \\ 2 \le & \quad x \quad \le 4 \,. \end{aligned}$$

Part (b): Projecting into the xz plane we need to break into two regions. For all points in the region Ω_{xz} for y we have

$$0 \le y \le \frac{12 - 3x}{2} \,.$$

In Ω^1_{xz}, the first of the two regions we break Ω_{xz} up into we have

$$\begin{aligned} 0 \le \;& z \le 12 - 3x \\ 2 \le \;& x \le 4\,, \end{aligned}$$

while in Ω^2_{xz}, the second of the two regions we break Ω_{xz} up into we have

$$\begin{aligned} \frac{2-x}{2} \le \;& z \le 12 - 3x \\ 1 \le \;& x \le 2\,. \end{aligned}$$

Problem 9 (between a plane and a cylinder)

Part (a): See the scanned notes for a diagram of the integration region. For Ω_{xz} we have to break the integration region into two parts depending on whether or not we are above/below the z location where the cylinder intersects the xz-plane. This intersection occurs when $y = 0$ or $z = +2$ (taking the positive root of $z^2 = 4$). Letting Ω^1_{xz} denote the projection onto the xz-plane below the line $z = 2$ we have defining equations for Ω^1_{xz} given by

$$\begin{aligned} 4 - z^2 \le \;& y \le \frac{12-3z}{2} \\ 0 \le \;& x \le \frac{12-3z}{4} \\ 0 \le \;& z \le 2\,. \end{aligned}$$

Letting Ω^2_{xz} denote the projection onto the xz-plane above the line $z = 2$ we have defining equations given by

$$\begin{aligned} 0 \le \;& y \le \frac{12-4x-3z}{2} \\ 0 \le \;& x \le \frac{12-3z}{4} \\ 2 \le \;& z \le 4\,. \end{aligned}$$

Part (b): See the scanned notes for a diagram of this integration region. For Ω_{yz} we again have to break our integration region up into two components depending on whether we are above/below the cylinders intersection on the y-axis. The cylinder intersects the y-axis when $x = 0$ and $z = 0$ of $y = 4$. If we denote Ω^1_{yz} as the region of the yz-plane below the intersection $y = 4$ we have

$$\begin{aligned} 0 \le \;& x \le \frac{12-3z-2y}{4} \\ \sqrt{4-y} \le \;& z \le \frac{12-2y}{3} \\ 0 \le \;& y \le 4\,. \end{aligned}$$

If we denote by Ω_{yz}^2 the region above the intersection $y = 4$ we have

$$\begin{array}{rcl} 0 \leq & x & \leq \dfrac{12 - 3z - 2y}{4} \\ 0 \leq & z & \leq \dfrac{12 - 2y}{3} \\ 4 \leq & y & \leq 6\,. \end{array}$$

Problem 10 (between a plane and a sphere)

See the scanned notes for a diagram of the desired integration region. To be above the plane and inside the sphere requires that our z variable satisfy

$$3 - y \leq z \leq 36 - x^2 - y^2\,.$$

Next consider the plane $y + z = 3$. The curve resulting from the intersection of this plane and the sphere $x^2 + y^2 + z^2 = 36$ is given by the equation $x^2 + y^2 + (3 - y)^2 = 36$. We can expand the given quadratic $(3 - y)^2$, combine terms, and complete the square in the variable y to obtain the following for the expression for the curve projected into the xy-plane

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{63}{4}\,.$$

One can find the algebra to derive this on the scanned notes. This is the expression for an ellipse and we can integrate over it by letting y range as

$$\frac{3}{2} - \sqrt{\frac{63}{4} - \frac{x^2}{2}} \leq y \leq \frac{3}{2} + \sqrt{\frac{63}{4} - \frac{x^2}{2}}\,,$$

while x ranges over

$$-\sqrt{2} \leq x \leq \sqrt{2}\,.$$

These three inequalities complete the specification of our integration region.

Problem 12

To evaluate this we project onto xy-plane where

$$\begin{array}{rcl} 0 \leq & z & \leq \dfrac{6 - 2x - y}{3} \\ 0 \leq & x & \leq \dfrac{6 - y}{2} \\ 0 \leq & y & \leq 6\,. \end{array}$$

Thus we have

$$I_z = \iiint \rho(x, y, z)(x^2 + y^2)dV = \int_0^6 \int_0^{\frac{6-y}{2}} \int_0^{2-\frac{2}{3}x-\frac{y}{3}} k(x^2 + y^2)dzdxdy\,.$$

Problem 14

Recall the definition of the z-coordinate of the center of mass is given by

$$\bar{z} = \frac{1}{M} \iiint z\rho(x, y, z)dV\,,$$

with M given by

$$M = \iiint_V \rho(x, y, z)dV\,.$$

Projecting the integration region into the xy-plane gives the following limits

$$\begin{array}{rcl} 0 \leq & z & \leq 4 - z^2 \\ 0 \leq & y & \leq 4 \\ -2 \leq & z & \leq 2\,. \end{array}$$

With these M is given by

$$M = \int_{-2}^{2} \int_0^4 \int_0^{4-x^2} (6 - z)dzdydx\,.$$

Once we have computed this we compute $\bar{z}$ as

$$\bar{z} = \frac{1}{M} \int_{-2}^{2} \int_0^4 \int_0^{4-x^2} (6 - z)zdzdydx\,.$$

Problem 15

To evaluate this integral we project onto the xy-plane which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

or an ellipse. Solving for y in the above expression gives

$$y = \pm b\sqrt{1 - \left(\frac{x}{a}\right)^2}\,.$$

If our density is given by $\rho = kr^2 = k(x^2 + y^2 + z^2)$, then M is given by

$$M = \iint \rho dV = \int_{-a}^{+a} \int_{-b\sqrt{1-(y/a)^2}}^{b\sqrt{1-(y/a)^2}} \int_0^{2b-y} k(x^2 + y^2 + z^2)dzdydx\,.$$

Problem 16

To begin recall the definition of the first moment with respect to the xz-plane is given by

$$L_{xz} = \iiint_{\Omega} y\rho(x,y,z)dV\,.$$

We want to consider the region Ω bounded by one sheet of the hyperboloid

$$-\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1\,. \tag{19}$$

and the plane $y = 2b$. Let $z = 0$ and we get $\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$. The region looks like a convertible or a "gondola hood". Projecting onto the xy-plane since when we solve for z in Equation 19 we get

$$z = \pm c\sqrt{\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 - 1}\,.$$

The region that we want to integrate over has z bounded by

$$-c\sqrt{\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 - 1} \le z \le +c\sqrt{\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 - 1}\,.$$

When $z = 0$ solving for x in $\left(\frac{y}{b}\right)^2 - \left(\frac{x}{a}\right)^2 = 1$ we get

$$x = \pm a\sqrt{\left(\frac{y}{b}\right)^2 - 1}\,,$$

thus the range for x is given by

$$-a\sqrt{\left(\frac{y}{b}\right)^2 - 1} \le x \le +a\sqrt{\left(\frac{y}{b}\right)^2 - 1}\,,$$

and the range for y is given by $b \le y \le 2b$. Thus for L_{xz} we get

$$L_{xz} = \int_b^{2b} \int_{-a\sqrt{\left(\frac{y}{b}\right)^2-1}}^{a\sqrt{\left(\frac{y}{b}\right)^2-1}} \int_{-c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}}^{+c\sqrt{\left(\frac{y}{b}\right)^2-\left(\frac{x}{a}\right)^2-1}} ykdzdxdy\,.$$

Section 17.6 (Integration using Cylindrical Coordinates)

Problem 1 (the volume of an ellipsoid)

To introduce the substitution $r^2 = x^2 + y^2$ needed for cylindrical coordinates we write our equation as

$$\frac{1}{a^2}(x^2 + y^2) + \frac{z^2}{c^2} = 1\,,$$

or in terms of r

$$\frac{z^2}{c^2} + \frac{1}{a^2}r^2 = 1\,,$$

or solving for z we have our ellipsoid given by

$$z = \pm c\sqrt{1 - \left(\frac{r}{a}\right)^2}\,.$$

Then the equation for our volume V becomes

$$V = \int_{-a}^{+a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-c\sqrt{1-r^2/a^2}}^{c\sqrt{1-r^2/a^2}} dzdydx = 2c\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\sqrt{1-r^2/a^2}dydx\,.$$

Expressing the xy-integration in polar rather than Cartesian we can express the integral for V as

$$V = \int_0^{2\pi}\int_0^a\int_{-c\sqrt{1-r^2/a^2}}^{c\sqrt{1-r^2/a^2}} dzrdrd\theta = (2\pi)2c\int_0^a\sqrt{1-r^2/a^2}rdr\,.$$

To integrate this last integral we let $v = r/a$ so that $dv = dr/a$ and the above becomes

$$V = 4\pi c\int_0^1\sqrt{1-v^2}a^2vdv = 4\pi a^2c\int_0^1 v\sqrt{1-v^2}dv = 4\pi a^2c\left(\frac{1-v^2}{(3/2)(-2)}\right|_0^1 = \frac{4\pi a^2c}{3}\,.$$

Problem 2 (the volume of Ω)

First note that the given paraboloid can be written in polar as

$$az = \sqrt{2}(x^2+y^2) = \sqrt{2}r^2\,.$$

Next the bounds of the region we want to integrate is given by

$$\begin{aligned}
\frac{\sqrt{12}}{a}r^2 \leq & \quad z \quad \leq \sqrt{a^2-r^2} \\
0 \leq & \quad r \quad \leq R \\
0 \leq & \quad \theta \quad \leq 2\pi\,,
\end{aligned}$$

Where we don't know the value of R the upper limit of the variable r. To find the upper limit on r we eliminate z from $r^2 + z^2 = a^2$ and $z = \frac{\sqrt{12}}{a}r^2$ to give

$$\frac{12}{a^2}r^4 + r^2 = a^2\,,$$

or

$$r^4 + \frac{a^2}{12}r^2 - \frac{a^4}{12} = 0\,.$$

Solving this quadratic equation for r^2 we have

$$\begin{aligned}
r^2 &= \frac{-\frac{a^2}{12} \pm \sqrt{\frac{a^4}{12^2} + 4\left(\frac{-a^4}{12}\right)}}{2} = \frac{-\frac{a^2}{12} \pm \frac{a^2}{\sqrt{12}}\left(\frac{1}{12}+4\right)^{1/2}}{2} \\
&= \frac{-\frac{a^2}{12} \pm \frac{a^2}{\sqrt{12}}\left(\frac{49}{12}\right)^{1/2}}{2} = \frac{a^2}{24}(-1 \pm 7) = \frac{a^2}{4}\,,
\end{aligned}$$

where we have taken the positive sign since we must have $r^2 > 0$. Thus $r = \frac{a}{2}$ and our volume V when we project into the xy-plane becomes

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^{a/2}\int_{\frac{\sqrt{12}}{a}r^2}^{\sqrt{a^2-r^2}} dzrdrd\theta \\
&= 2\pi \int_0^{a/2}\left(\sqrt{a^2-r^2}r - \frac{\sqrt{12}}{a}r^3\right)dr \\
&= 2\pi\left(\left(\frac{(a^2-r^2)^{3/2}}{(3/2)(-2)}\Big|_0^{a/2} - \left(\frac{\sqrt{12}}{a}\left(\frac{r^4}{4}\right)\Big|_0^{a/2}\right)\right)\right. \\
&= 2\pi\left[-\frac{1}{3}\left(\left(\frac{3}{4}a^2\right)^{3/4} - a^3\right) - \frac{\sqrt{e12}}{4a}\left(\frac{a^4}{8}\right)\right],
\end{aligned}$$

which could probably be simplified.

Section 17.7 (Integration using Spherical Coordinates)

Problem 28 (a moment of inertia problem)

We have $r_\perp = r\sin(\theta)$ and the moment of inertia is defined as $I_z = \iiint r_\perp^2 \rho dV$. To compute this using spherical coordinates we have

$$\begin{aligned}
I_z &= \iiint r^2\sin^2(\theta)kdV \\
&= \int_0^{2\pi}\int_0^{\pi}\int_a^b r^2\sin(\theta)^2 kr^2\sin(\theta)drd\theta d\phi \\
&= 2\pi k\left(\frac{r^5}{5}\Big|_a^b\right)\left(\int_0^\pi \sin^3(\theta)d\theta\right) \\
&= \frac{2\pi k}{5}(b^5-a^5)\left(\int_0^\pi \sin(\theta)d\theta - \int_0^\pi \cos^2(\theta)\sin(\theta)d\theta\right) \\
&= \frac{2\pi k}{5}(b^5-a^5)\left(-\cos(\theta)|_0^\pi + \frac{\cos(\theta)^3}{3}\Big|_0^\pi\right) \\
&= \frac{8\pi k}{15}(b^5-a^5).
\end{aligned}$$

Problem 34 (the volume of a region within two spheres)

The volume of any region is given by the triple integral $V = \iiint dV$, which we evaluate by splitting it into two pieces. We have the top "ice cream cone" shape which is defined by the limits

$$
\begin{aligned}
0 \leq \ & \theta & \leq \frac{\pi}{3} \\
0 \leq \ & r & \leq 1 \\
0 \leq \ & \phi & 2\pi \,,
\end{aligned}
$$

and the “bottom rind“ is defined by the limits

$$
\begin{aligned}
\frac{\pi}{3} \leq \ & \theta & \leq \frac{\pi}{2} \\
0 \leq \ & r & \leq 2\cos(\theta) \\
0 \leq \ & \phi & 2\pi
\end{aligned}
$$

Thus the volume V is given by the sum of two integrals

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 r^2 \sin(\theta) dr d\theta d\phi + \int_0^{2\pi} \int_{\pi/3}^{\pi/2} \int_0^{2\cos(\theta)} r^2 \sin(\theta) dr d\theta d\phi \,.$$

So that to evaluate this we have

$$
\begin{aligned}
\frac{V}{2\pi} &= \frac{1}{3} \int_0^{\pi/3} \sin(\theta) d\theta + \frac{1}{3} \int_{\pi/3}^{\pi/2} 2^3 \cos(\theta)^3 d\theta \quad \text{or} \\
\frac{3V}{2\pi} &= -(\cos\left(\frac{\pi}{3}\right) - 1) + 8 \int_{\pi/3}^{\pi/2} (\cos(\theta) - \sin(\theta)^2 \cos(\theta)) d\theta \quad \text{or} \\
\frac{3V}{2\pi} &= \left(1 - \frac{1}{2}\right) + 8 \left[\sin(\theta)\Big|_{\pi/3}^{\pi/2} - \frac{\sin^3(\theta)}{3}\Bigg|_{\pi/3}^{\pi/2} \right] \quad \text{or} \\
\frac{3V}{2\pi} &= 8 \left(\frac{2}{3} - \frac{3\sqrt{3}}{8} \right) \,,
\end{aligned}
$$

when we simplify. We might try to evaluate this expression in cylindrical coordinates where the limits are given by

$$
\begin{aligned}
0 \leq \ & \theta & \leq \frac{\pi}{3} \\
0 \leq \ & \phi & \leq 2\pi \\
\frac{a/2}{\cos(\theta)} \leq \ & r & \leq a \,,
\end{aligned}
$$

and so the volume is given by

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{a/2}{\cos(\theta)}}^a r^2 \sin(\theta) dr d\theta d\phi \,.$$

Problem 35 (the mass of a sphere)

We are told that that our density is given via $\rho(x, y, z) = kr$ and we want to compute the total mass M given by $M = \iiint krdV$. Thus we get

$$M = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos(\theta)} krr^2 \sin(\theta) dr d\theta d\phi \,.$$

This can be evaluated using

$$\begin{aligned}
M &= 2\pi k \int_0^{\pi/2} \sin(\theta) \int_{r=0}^{2a\cos(\theta)} r^3 dr d\theta \\
&= 2\pi k \int_0^{\pi/2} \sin(\theta) \left. \frac{r^4}{4} \right|_0^{2a\cos(\theta)} d\theta \\
&= \frac{\pi}{2} k \int_0^{\pi/2} \sin(\theta) 2^4 a^4 \cos^4(\theta) d\theta \\
&= 2^3 a^4 \pi k \left(- \left. \frac{\cos^5(\theta)}{5} \right|_0^{\pi/2} \right. \\
&= \frac{8a^4\pi k}{5}(-1)(0-1) = \frac{8\pi a^4 k}{5} .
\end{aligned}$$

Chapter 18 (Line and Surface Integrals)

Section 18.3 (Green's Theorem)

Problem 1

We desire to verify Green's theorem

$$\oint_C Pdx + Qdy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,, \tag{20}$$

with $F = y\mathbf{i} + x^2\mathbf{j}$ and the curve C given by the circle $x^2 + y^2 = a^2$. To parameterize this curve C, let $x(t) = a\cos(t)$ and $y(t) = a\sin(t)$, so that $dx = -a\sin(t)dt$ and $dy = a\cos(t)dt$ so that the contour integral (the left hand side of the Green's theorem) becomes

$$I \equiv \oint_C Pdx + Qdy = \oint_C (a\sin(t))(-a\sin(t)dt) + (a^2\cos^2(t))(a\cos(t)dt)\,.$$

With this we see that I becomes

$$\begin{aligned}
I &= \int_{t=0}^{2\pi} (-a^2\sin^2(t) + a^3\cos^3(t))dt \\
&= -a^2 \int_0^{2\pi} \left(\frac{1-\cos(2t)}{2} \right) dt + a^3 \int_0^{2\pi} (1-\sin^2(t))\cos(t)dt \\
&= -\frac{a^2}{2}(2\pi) + \frac{a^2}{2}\int_0^{2\pi} \cos(2t)dt - a^3 \int_0^{2\pi} \sin^2(t)\cos(t)dt \\
&= -a^2\pi + \frac{a^2}{2}\left(\frac{\sin(2t)}{2} \Big|_0^{2\pi} \right) - a^3 \int_0^{2\pi} \sin^2(t)\cos(t)dt \\
&= -a^2\pi - a^3 \int_0^{2\pi} \sin^2(t)\cos(t)dt\,.
\end{aligned}$$

To evaluate the remaining integral let $v = \sin(t)$ so that $dv = \cos(t)dt$ and the integral above becomes

$$-a^3 \int_0^0 v^2 dv = -a^3 \left. \frac{v^3}{3} \right|_0^0 = 0\,.$$

While the right hand side of Green's theorem is given by

$$\iint_\Omega (2x-1)dxdy\,.$$

Converting this integral to polar by using $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $dxdy = rdrd\theta$ our right-hand-side integral becomes

$$\begin{aligned}
\int_0^a \int_0^{2\pi} (2r\cos(\theta) - 1)rdrd\theta &= \int_0^a \int_0^{2\pi} 2r^2\cos(\theta)drd\theta - \int_0^a \int_0^{2\pi} rdrd\theta \\
&= 0 - \left(\left. \frac{r^2}{2} \right|_0^a \right) \int_0^{2\pi} d\theta = -\frac{a^2}{2}(2\pi) = -a^2\pi\,,
\end{aligned}$$

the same expression, proving the equivalence.

Problem 5

By Green's theorem we have

$$\oint_C 2dx - 3dy = \iint_\Omega \left(\frac{\partial}{\partial x}(-3) - \frac{\partial}{\partial y}(2) \right) dA = 0\,.$$

Problem 6

By Green's theorem we have

$$\begin{aligned}\oint_C 2ydx - 3xdy &= \iint_\Omega \left(\frac{\partial}{\partial x}(-3x) - \frac{\partial}{\partial y}(2y) \right) dA \\ &= (-3-2) \iint_\Omega dA = -5(a\sqrt{2})\frac{a\sqrt{2}}{2} = -5a^2\,,\end{aligned}$$

where we have evaluated $\iint_\Omega dA$ (the area of the region Ω) using elementary geometry.

Problem 7

By Green's theorem we have

$$\oint xydy = \iint_\Omega \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(0) \right) dA = \iint_\Omega ydA\,,$$

which we recognize as the is the y coordinate of the center of mass of the object Ω. To evaluate this expression we will convert the integral from Cartesian coordinates to polar coordinates. We find

$$\begin{aligned}\int_{r=0}^{a} \int_{\theta=0}^{\pi} r\sin(\theta) r dr d\theta &= \left. \frac{r^3}{3} \right|_0^a \int_0^\pi \sin(\theta) d\theta \\ &= \left(\frac{a^3}{3} \right) (-\cos(\theta)|_0^\pi = \left(\frac{a^3}{3} \right)(1+1) = \frac{2a^3}{3}\,.\end{aligned}$$

Section 18.4 (Surface Area and Surface Integrals)

Notes on the expression $N = \frac{a \times b}{||a \times b||}$

We compute the cross product of $a \times b$ first and then dot with $\mathbf{k}$ to find

$$(\bar{a} \times \bar{b}) \cdot \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & a_3 \\ 0 & \Delta y & b_3 \end{vmatrix} = (\mathbf{k}\,\Delta x\,\Delta y) \cdot \mathbf{k} = \Delta x \Delta y\,,$$

as given in the book for $\Delta\sigma(N \cdot k)$. If our surface is given via $z = f(x, y)$ then from the expression for $N(x_i^*, y_i^*)$ given in the book, the required denominator of $\Delta\sigma_{ij}$ is given by

$$|\hat{N}(x_i^*, y_i^*) \cdot \hat{k}| = \frac{+1}{\left[f_x^2(x_i^*, y_i^*) + f_y^2(x^*, y^*) + 1\right]^{1/2}}.$$

Then our differential of surface area $\Delta\sigma_{ij}$ is given by

$$\Delta\sigma_{ij} = \frac{\Delta A_{ij}}{|\hat{N}(x_i^*, y_i^*) \cdot \hat{k}|} = \left[f_x^2(x_i^*, y_i^*) + f_y^2(x^*, y^*) + 1\right]^{1/2} \Delta A_{ij},$$

the same as given in the book. If our surface is given by an equation of the form $F(x, y, z) = 0$ then the normal vector is

$$\vec{N} = \frac{F_x\hat{i} + F_y\hat{j} + F_z\hat{k}}{||\nabla F||},$$

so that $|N \cdot \hat{k}| = \frac{|F_z|}{||\nabla F||}$ and our differential of surface area under this parametrization of the surface is given by

$$d\sigma = \frac{dA_{xy}}{|\hat{N} \cdot \hat{k}|} = \frac{(F_x^2 + F_y^2 + F_z^2)^{1/2}}{|F_z|} dA_{xy}.$$

Notes on example 3

From the expression for N given in this section and the function $F(x, y, z)$ given in this example we have

$$\hat{N} = \frac{F_x\hat{i} + F_y\hat{j} + F_z\hat{k}}{\sqrt{F_x^2 + F_y^2 + F_z^2}} = \frac{x\hat{i} + z\hat{j}}{\sqrt{x^2 + z^2}} = \frac{x}{a}\hat{i} + \frac{z}{a}\hat{k}.$$

Notes on example 4

In general the expression for the surface area $A(S)$ is given by

$$A(S) = \iint_\Omega \frac{dA}{|\hat{N} \cdot \hat{n}|},$$

where $\hat{n}$ is the normal to the plane where by we are taking the projection. If we project into the yz plane where the surface is given via $x = f(y, z)$ the surface area is given by the general expression

$$A(S) = \iint_{\Omega_{yz}} \left(\frac{F_x^2 + F_y^2 + F_z^2}{|F_x|}\right) dA_{yz}.$$

Since for this example we have $F_x = -2x$, $F_y = 2y$, and $F_z = 2z$, the above becomes

$$A(S) = \iint_{\Omega_{yz}} \frac{(x^2 + y^2 + z^2)^{1/2}}{|x|} dydz.$$

We must replace x with its functional expression in terms of y and z or $x = x(y, z)$ and get

$$A(S) = \iint_{\Omega_{yz}} \frac{(2(y^2+z^2)-1)^{1/2}}{(y^2+z^2-1)^{1/2}} = \int_{r=1}^{\sqrt{2}} \int_{\theta=0}^{\pi/2} \left(\frac{2r^2-1}{r^2-1} \right)^{1/2} r d\theta dr \,,$$

where we have converted the integral we obtained into polar coordinates. We can compare this result with what we get if we evaluate this integral in the xz-plane, where in general we get

$$A(S) = \int_{\Omega_{xz}} \frac{dA_{xz}}{|\hat{N} \cdot \hat{j}|} = \iint_{\Omega_{xz}} \frac{dA_{xz}(F_x^2 + F_y^2 + F_z^2)^{1/2}}{|F_y|} \,.$$

For this problem in particular we have when we use $y = \sqrt{1+x^2-z^2}$ we find

$$A(S) = \iint_{\Omega_{xz}} dA_{xz} \frac{(x^2+y^2+z^2)^{1/2}}{|y|} = \iint_{\Omega_{xz}} dA_{xz} \frac{(1+2x^2)^{1/2}}{\sqrt{1+x^2-z^2}} \,.$$

The range of limits can be obtained by letting $y = 0$ (to project into the xz-plane) where we have $z^2 = 1 + x^2$. For a fixed x then we find $0 < z < \sqrt{1+x^2}$ and the limits of x go from 0 to 1 *not* 1 to 0 for if the integrand was removed (replaced with a 1) we must have a result that denotes a positive integral. Thus we get

$$A(S) = \int_0^1 \int_0^{\sqrt{1+x^2}} \left(\frac{1+2x^2}{1+x^2-z^2} \right)^{1/2} dz dx \,.$$

The above becomes

$$\int_0^1 (1+2x^2)^{1/2} \int_0^{\sqrt{1+x^2}} \frac{dz}{(1+x^2-z^2)^{1/2}} dx \,.$$

Let $z = \sqrt{1+x^2}v$ so that $dz = \sqrt{1+x^2}dv$, and remembering that $\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$ we get

$$\begin{aligned}
A(S) &= \int_0^1 (1+2x^2)^{1/2} \int_0^1 \frac{\sqrt{1+x^2}dv}{(1+x^2-(1+x^2)v^2)^{1/2}} dx \\
&= \int_0^1 (1+2x^2)^{1/2} \int_0^1 \frac{dv}{\sqrt{1-v^2}} dx \\
&= \int_0^1 (1+2x^2)^{1/2} \left(\arcsin(x)|_0^1\right. dx = \frac{\pi}{2} \int_0^1 (1+2x^2)^{1/2} dx \,.
\end{aligned}$$

To finish this integral we let $\tan(u) = \sqrt{2}x$ so that $\sec^2(u)du = \sqrt{2}dx$ to get for $A(S)$

$$\begin{aligned}
A(S) &= \frac{\pi}{2} \int_0^{\tan^{-1}(\sqrt{2})} \sec(u) \left(\frac{1}{\sqrt{2}} \sec^2(u) \right) du \\
&= \frac{\pi}{2\sqrt{2}} \int_0^{\tan^{-1}(\sqrt{2})} \sec^3(u) du = \frac{\pi}{2\sqrt{2}} \int_0^{\tan^{-1}(\sqrt{2})} \sec(u)\sec^2(u) du = \frac{\pi}{2\sqrt{2}} \,.
\end{aligned}$$

Notes on example 5

Following the books derivation we obtain and then evaluate the given expression for M. We have

$$
\begin{aligned}
M &= \iint_{\Omega_{xy}} (a-z)\frac{a}{z} dA_{xy} \quad \text{with} \quad z = \pm\sqrt{a^2-x^2} \\
&= a\int_0^a \int_0^x \frac{(a-\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} dy dx \\
&= a^2 \int_0^a \int_0^x (a^2-x^2)^{-1/2} dy dx - a\int_0^a \int_0^x dy dx \\
&= a^2 \int_0^a x(a^2-x^2)^{-1/2} dy - a\int_0^a x dx \\
&= a^2 \left(\left. \frac{(a^2-x^2)^{1/2}}{(-2)(1/2)} \right|_0^a - a \left(\left. \frac{x^2}{2} \right|_0^a \right.\right. \\
&= -a^2(0-(a^2)^{1/2}) - \frac{a^3}{2} = \frac{a^3}{2}.
\end{aligned}
$$

Section 18.5 (Parametric Equations of Surfaces)

Notes on Surface Area in Parametric Coordinates

Following the derivation given in the book we need to compute

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} &= \left| \begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ -R\sin(\theta)\sin(\phi) & R\cos(\theta)\sin(\phi) & 0 \\ R\cos(\theta)\cos(\phi) & R\sin(\theta)\cos(\phi) & -R\sin(\phi) \end{array} \right| \\
&= \hat{i}(-R^2\cos(\theta)\sin(\phi)^2) - \hat{j}(R^2\sin(\theta)\sin^2(\phi)) \\
&+ \hat{k}(-R^2\sin^2(\theta)\sin(\phi)\cos(\phi) - R^2\cos^2(\theta)\cos(\phi)\sin(\phi)) \\
&= \hat{i}(-R^2\cos(\theta)\sin^2(\phi)) - R^2\sin(\theta)\sin^2(\phi)\hat{j} - \hat{k}R^2\sin(\phi)\cos(\phi).
\end{aligned}
$$

So that we have that the norm of this vector is given by

$$
\begin{aligned}
\left|\left|\frac{\partial \bar{r}}{\partial \theta} \times \frac{\partial \bar{r}}{\partial \phi}\right|\right| &= R^2(\cos^2(\theta)\sin^4(\phi) + \sin^2(\theta)\sin^4(\phi) + \sin^2(\phi)\cos^2(\phi))^{1/2} \\
&= R^2(\sin^4(\phi) + \sin^2(\phi)\cos^2(\phi))^{1/2} = R^2\sin(\phi).
\end{aligned}
$$

As claimed in the book.

Notes on Example 5

Given the parametric representation of the points on the torus we compute $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ and then we need to evaluate the cross product of these two vectors to compute the surface area.

We thus find

$$
\begin{aligned}
\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(R + a\cos(v))\sin(u) & (R + a\cos(v))\cos(u) & 0 \\ -a\sin(v)\cos(u) & -a\sin(v)\cos(u) & a\sin(v) \end{vmatrix} \\
&= \hat{k}(a(R + a\cos(v))\sin(v)\sin^2(u) + a(R + a\cos(v))\sin(v)\cos^2(u)) \\
&+ a\cos(v)(\hat{i}(R + a\cos(v))\cos(u) + \hat{j}(R + a\cos(v))\sin(u)) \\
&= a(R + a\cos(v))\left(\cos(u)\cos(v)\hat{i} + \sin(u)\cos(v)\hat{j} + \sin(v)\hat{k}\right) .
\end{aligned}
$$

Therefore with this the norm becomes

$$
\left|\left|\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right|\right|^2 = a^2(R + a\cos(v))^2((\cos^2(u) + \sin^2(u))\cos^2(v) + \sin^2(v)) = a^2(R + a\cos(v))^2 ,
$$

the same as computed in the book.

Problem 1 (a surface in parametric form)

We are told that our surface has the following form

$$
\begin{aligned}
x &= 2u - v \\
y &= u + 2v \\
z &= u - v ,
\end{aligned}
$$

Solving the first equation for v give $v = 2u - x$ putting this into the second and third equation give

$$
\begin{aligned}
y &= u + 4v - 2x = 5u - 2x \quad \text{and} \\
z &= u - 2u + x = -u + x \Rightarrow u = -z + x .
\end{aligned}
$$

Putting this last equation for u into the one for y gives

$$
y = 5(-z + x) - 2x = -5z + 3x ,
$$

which is a plane thought the origin.

Section 18.6 (The Divergence and the Curl)

Problem 1 (practice with the divergence and the curl)

With $\mathbf{v} = xy\sin(y)\mathbf{i} + x^2z\mathbf{j} - y\cos(z)\mathbf{k}$, compute $\boldsymbol{\nabla} \cdot \mathbf{v}$ and $\boldsymbol{\nabla} \times \mathbf{v}$.

The expression $\boldsymbol{\nabla} \cdot \mathbf{v}$ is given by

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{v} &= \frac{\partial}{\partial x}(xy\sin(y)) + \frac{\partial}{\partial y}(x^2z) + \frac{\partial}{\partial z}(-y\cos(z)) \\
&= y\sin(y) + 0 + y\sin(z) = y(\sin(y) + \sin(z)) .
\end{aligned}
$$

The expression $\boldsymbol{\nabla} \times \mathbf{v}$ is given by

$$
\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy\sin(y) & x^2 z & -y\cos(z) \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(-y\cos(z)) - \frac{\partial}{\partial z}(x^2 z)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(-y\cos(z)) - \frac{\partial}{\partial z}(xy\sin(z))\right) \\
&+ \mathbf{k}\left(\frac{\partial}{\partial x}(x^2 z) - \frac{\partial}{\partial y}(xy\sin(y))\right) \\
&= \mathbf{i}(-\cos(z) - x^2) + \mathbf{j}(0) + \mathbf{k}(2xz - x\sin(y) - xy\cos(y)) \,.
\end{aligned}
$$

Section 18.7 (The Divergence Theorem)

Notes on Example 1

From the text we have that F_3 is given by

$$
F_3 = -GM\rho \iint_S \frac{z d\sigma}{(x^2 + y^2 + z^2)^{3/2}} \,.
$$

With $F(x, y, z) = x^2 + y^2 - a^2$ our differential of surface area is $d\sigma = \frac{dA_{yz}}{|\hat{N}\cdot\hat{i}|}$ where

$$
\hat{N} = \pm \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} \,.
$$

When we evaluate this unit vector on $x^2 + y^2 = a^2$ we get that $\hat{N}$

$$
\hat{N} = \pm \frac{x\hat{i} + y\hat{j}}{a} \,.
$$

This gives that $|\hat{N} \cdot \hat{i}| = \frac{x}{a}$. Using this we find F_3 becomes

$$
\begin{aligned}
F_3 &= -GM\rho(4) \iint_{\Omega_{yz}} \frac{z}{(a^2 + z^2)^{3/2}} \frac{dA_{yz}}{(x/a)} \\
&= -4GM\rho a \iint_{\Omega_{yz}} \frac{z}{(a^2 + z^2)^{3/2}} \frac{dA_{yz}}{\sqrt{a^2 - y^2}} \\
&= -4GM\rho a \int_0^h \int_0^a \frac{z}{(a^2 + z^2)^{3/2}} \frac{1}{\sqrt{a^2 - y^2}} dz dy \\
&= -4GM\rho a \left(\left. \frac{(a^2 + z^2)^{-1/2}}{(-1/2)(2)} \right|_0^h \int_0^a \frac{1}{(a^2 - y^2)^{1/2}} dy \right. \\
&= 4GM\rho a \left(\frac{1}{(a^2 + h^2)^{1/2}} - \frac{1}{a} \right) \arcsin\left(\frac{y}{a}\right)\Big|_0^a \\
&= -4GM\rho a \left(\frac{1}{a} - \frac{1}{(a^2 + h^2)^{1/2}} \right) \left(\frac{\pi}{2}\right) \\
&= -2\pi GM\rho(1 - a(a^2 + h^2)^{1/2}) \,.
\end{aligned}
$$

Problem 1 (practice with the divergence theorem)

For this problem we desire to evaluate the integral

$$I \equiv \iint_S (2x\mathbf{i} - 3y\mathbf{j} + 4z\mathbf{k}) \cdot \mathbf{N} d\sigma \,,$$

over the unit cube both directly and by using the divergence theorem. We begin by performing this integration directly. By breaking the total integration into pieces over the individual six faces of the cube the above integral above becomes (here we use the notation () as a shorthand to denote the integrand)

$$\begin{aligned} I &= \iint () \cdot \mathbf{i} d\sigma + \iint () \cdot (-\mathbf{i}) d\sigma + \iint () \cdot \mathbf{j} d\sigma + \iint () \cdot (-\mathbf{j}) d\sigma \\ &+ \iint () \cdot \mathbf{k} d\sigma + \iint () \cdot (-\mathbf{k}) d\sigma \\ &= \int_0^1 \int_0^1 2x dy dz + \int_0^1 \int_0^1 2x dy dz + \int_0^1 \int_0^1 (-3y) dy dz + \int_0^1 \int_0^1 3y dx dy \\ &+ \int_0^1 \int_0^1 4z dx dy + \int_0^1 \int_0^1 -4z dx dy = 2 - 3 + 4 = 3 \,. \end{aligned}$$

In the above the six integrals are the integrals over the faces where one coordinate is held constant over the entire integration region. For each integral we hold $x = 1$, $x = 0$, $y = 1$, $y = 0$, $z = 1$, and $z = 0$ constant.

As a second method to evaluate this integral we will use the divergence theorem, which states that the desired integral is equal to the following

$$\iint_S \mathbf{v} \cdot \mathbf{N} d\sigma = \iiint \boldsymbol{\nabla} \cdot \mathbf{v} dv \,.$$

The divergence of $\mathbf{v}$ is given by $\boldsymbol{\nabla} \cdot \mathbf{v} = 2 - 3 + 4 = 3$. Therefore

$$\iint_S \mathbf{v} \cdot \mathbf{N} d\sigma = 3 \iiint dV = 3 \,,$$

the same result as before.

Problem 8 (some practice with the divergence theorem)

Our vector field for this problem is given by

$$\vec{v} = (x^2 + y^2)\hat{i} + 2xy\hat{j} \,.$$

Thus the divergence theorem gives

$$\iint_S \vec{v} \cdot \hat{N} d\sigma = \iiint_V (\nabla \cdot \vec{v}) dV = \iiint (2x + 2x) dV \,.$$

To evaluate this integral we convert to cylindrical where we get

$$\int_0^{2\pi}\int_0^3\int_0^{9-r^2} 2(r\cos(\theta))dzrdrd\theta = 2\int_0^{2\pi}\int_0^3 (9-r^2)r^2\cos(\theta)d\theta = 2\cdot 0 = 0\,,$$

when we evaluate the θ integral.

Problem 13 (practice with the divergence theorem)

For this problem we want to evaluate $\iint_S \mathbf{v}\cdot\mathbf{N}d\sigma$ when $\mathbf{v} = Uz\mathbf{k}$ and S is the portion of the paraboloid $z = a^2 - x^2 - y^2$ with $z \geq 0$ and $\mathbf{N}$ the upper unit normal. Recall that an explicit representation of the two inner/outer normals when we have a surface expressed in the form $z = f(x, y)$ is given by

$$\mathbf{N}_\pm = \pm\left(\frac{f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}\right) = \pm\left(\frac{-2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}}\right)\,.$$

In the above, to get the upper unit normal we take the minus sign so that the coefficient $\mathbf{k}$ is positive. To evaluate this flux, $\iint_s \mathbf{v}\cdot\mathbf{N}d\sigma$, we will use the divergence theorem given by

$$\iint_S \mathbf{v}\cdot\mathbf{N}d\sigma = \iiint_V (\boldsymbol{\nabla}\cdot\mathbf{v})dV\,.$$

Now since $\boldsymbol{\nabla}\cdot\mathbf{v} = U$, the above becomes

$$\begin{aligned}
\iint_S \mathbf{v}\cdot\mathbf{N}d\sigma &= U\iiint_V dV = U\int_{\theta=0}^{2\pi}\int_{\rho=0}^{a}\int_{z=0}^{a^2-x^2-y^2} dz\rho d\rho d\theta \\
&= U(2\pi)\int_{\rho=0}^{a}(a^2 - x^2 - y^2)\rho d\rho = 2\pi\int_{\rho=0}^{a}(a^2-\rho^2)\rho d\rho \\
&= 2\pi\left(\frac{a^2\rho^2}{2} - \frac{\rho^4}{4}\right|_0^a = \pi\left(a^2a^2 - \frac{a^4}{2}\right) = \frac{\pi a^4}{2}\,.
\end{aligned}$$

Problem 17 (practice with the divergence theorem)

We desire to find the total vertical force on the hemisphere $x^2 + y^2 + z^2 = a^2$ for $z \geq 0$, where the force per unit area is $\bar{f} = \alpha z\mathbf{k}$. We have for a total force $\bar{F}$ the following

$$\bar{F} = \iint_S \bar{f}d\sigma\,.$$

So the the vertical force is $F_{\text{vertical}} = \bar{F}\cdot\hat{z} = \bar{F}\cdot\mathbf{j}$, and gives

$$\iint_S \bar{f}\cdot\mathbf{j}d\sigma = \iint_S \alpha z d\sigma\,.$$

As $z = a\cos(\theta)$ on the sphere in spherical coordinates and the differential of area $d\sigma$ (again in spherical coordinates) is given by $d\sigma = a^2 \sin(\theta) d\theta d\phi$, we see that the integral above becomes

$$\begin{aligned}\int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{2\pi} a^2 a\cos(\theta)\alpha\sin(\theta)d\theta d\phi &= \alpha a^3 (2\pi) \int_{\theta=0}^{2\pi} \cos(\theta)\sin(\theta)d\theta \\ &= \alpha a^3(2\pi)\left(\frac{-\cos^2(\theta)}{2}\right|_0^{\pi/2} = \alpha a^3 \pi\,.\end{aligned}$$

Problem 21 (an expression for the volume)

From the given expression and the divergence theorem we find

$$\frac{1}{3}\iint_S \vec{F}\cdot\hat{n}d\sigma = \frac{1}{3}\iiint(\nabla\cdot\vec{F})dV = \frac{3}{3}\iiint dV = V(D)\,,$$

as we were to show.

Section 18.8 (Stokes' Theorem)

Problem 1 (practice with Stokes' theorem)

We desire to use Stokes theorem to evaluate $\int_\Gamma \mathbf{v}\cdot\mathbf{T}ds$ for the vector $\mathbf{v} = 2z\mathbf{i} - x\mathbf{j} + 3y\mathbf{k}$ and the curve Γ the triangular path from $(2,0,0)$ to $(0,2,0)$ to $(0,0,3)$ and back to $(2,0,0)$. We begin by recalling Stokes' theorem

$$\int_\Gamma \mathbf{v}\cdot\mathbf{T}ds = \iint(\boldsymbol{\nabla}\times\mathbf{v})\cdot\mathbf{N}d\sigma\,.$$

Thus to compute this we first need to compute $\boldsymbol{\nabla}\times\mathbf{v}$. We find

$$\boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z & -x & 3y\end{vmatrix} = \mathbf{i}(3-0) + \mathbf{j}(2-0) + \mathbf{k}(-1-0) = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}\,.$$

We next compute $\mathbf{N}$, the normal to the given surface. We begin by computing two vectors in the plane, $\mathbf{a}$ and $\mathbf{b}$ as

$$\begin{aligned}\mathbf{a} &= (0-2)\mathbf{i} + (2-0)\mathbf{j} + (0-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= (0-2)\mathbf{i} + (0-0)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 0\mathbf{j} + 2\mathbf{k}\,.\end{aligned}$$

These represent vectors spanning the edges of the given planer triangle. Then the normal vector $\mathbf{N}$ using these is given by the cross product, which we find

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ -2 & 0 & 2\end{vmatrix} = -\mathbf{j}(-4+2) + 2(2\mathbf{i} + 2\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} = \mathbf{N}\,.$$

We now need to normalize this vector to compute $\hat{\mathbf{N}}$. We find

$$\hat{\mathbf{N}} = \frac{\mathbf{N}}{||\mathbf{N}||} = \frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})\,.$$

Thus the required product in Stokes' theorem is $(\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{N} = \frac{1}{3}(6+2-2) = 2$. Using Stokes' theorem we now have

$$\int_\Gamma \mathbf{v} \cdot \mathbf{T} ds = 2 \iint_S d\sigma\,.$$

To evaluate this remaining integral we project our integration region onto the x-y plane. From the discussion in the book we have a differential of surface area for this projection given by

$$d\sigma = \frac{dxdy}{|\mathbf{N}\cdot\mathbf{k}|} = \frac{dxdy}{(2/3)} = \frac{3}{2}dxdy\,.$$

Thus our integral above becomes

$$\begin{aligned}
2\left(\frac{3}{2}\right)\iint_{\Omega_{xy}} dxdy &= 3\int_{x=0}^{2}\int_{y=0}^{2-x} dydx \\
&= 3\int_{x=0}^{2}(2-x)dx = 3\left(2x - \frac{x^2}{2}\right|_0^2 = 3(4-2) = 6\,,
\end{aligned}$$

when we specify the limits of the integration in the x-y plane.

Problem 13 (more practice with Stokes' theorem)

We want to evaluate $\iint_S \nabla \times \vec{v} \cdot \hat{N}$ where our vector field is given by

$$\vec{v} = xyz\hat{i} + (x+z)\hat{j} + (x^2 - y^2)\hat{k}\,.$$

To do that we will use Stokes' theorem which is

$$\iint_S \nabla \times \vec{v} \cdot \hat{N} d\sigma = \int_C \vec{v} \cdot \hat{\tau} dS\,.$$

Consider the curve C in the xy plane parametrized by $x(t) = a\cos(t)$ and $y(t) = a\sin(t)$ for $0 \le t \le 2\pi$. Then we have

$$\hat{\tau} = \frac{\dot{x}\hat{i} + \dot{y}\hat{j}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\,,$$

and

$$dS = \sqrt{\dot{x}^2 + \dot{y}^2}dt\,.$$

Using these we get that the right-hand-side of Stokes' theorem is given by

$$\begin{aligned}
\int_C \vec{v} \cdot \hat{\tau} dS &= \int_0^{2\pi} \left.((xyz)\dot{x} + (x+z)\dot{y})\right|_{z=0} dt \\
&= \int_0^{2\pi} x\dot{y}dt = a^2\int_0^{2\pi}\cos(t)\cos(t)dt \\
&= a^2\int_0^{2\pi}\frac{1+\cos(2t)}{2}dt = \frac{a^2}{2}\left(2\pi + \frac{1}{2}\sin(2t)|_0^{2\pi}\right) = \pi a^2\,.
\end{aligned}$$