$$\frac{-v_{i-1,j} + 2v_{ij} - v_{i+1,j}}{h_x^2} + \frac{-v_{i,j-1} + 2v_{ij} - v_{i,j+1}}{h_y^2} = f_{ij}$$

w/ B.C. $v_{ij} = 0$ if $i=0$ or $i=M$ or $j=0$ or $j=N$

$\Rightarrow$ Grouping all actions on a column of x's (fixed i)

$$-\frac{v_{i,j-1}}{h_y^2} + \left(\frac{2}{h_x^2} + \frac{2}{h_y^2}\right) v_{ij} - \frac{v_{i,j+1}}{h_y^2} - \frac{v_{i-1,j}}{h_x^2} - \frac{v_{i+1,j}}{h_x^2} = f_{ij}$$

fixing i

$$\underbrace{\begin{pmatrix} 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & -\frac{1}{h_y^2} & 0 \cdots \\ -\frac{1}{h_y^2} & 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & -\frac{1}{h_y^2} \\ & -\frac{1}{h_y^2} & 2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) \end{pmatrix}}\begin{pmatrix} v_{i,1} \\ v_{i,2} \\ v_{i,3} \\ \vdots \\ v_{i,N-1} \end{pmatrix} + \underbrace{\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} v_{i-1,1} \\ v_{i-1,2} \\ \vdots \\ v_{i-1,N-1} \end{pmatrix}} + \begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} v_{i+1,1} \\ v_{i+1,2} \\ \vdots \\ v_{i+1,N-1} \end{pmatrix}$$

Tridiagonal Matrix of size $(N-1)\times(N-1)$

w/ entries of $-\frac{1}{h_y^2}$ on sub + superdiagonal

+ $2\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)$ on diagonal

See next page for these terms

$$+ \begin{pmatrix} -\frac{1}{h_x^2} & 0 & & \\ 0 & -\frac{1}{h_x^2} & 0 & \\ & & \ddots & \end{pmatrix} \begin{pmatrix} V_{i-1,1} \\ V_{i-1,2} \\ \vdots \\ V_{i-1,N-1} \end{pmatrix} + \underbrace{\begin{pmatrix} -\frac{1}{h_x^2} & 0 & & \\ 0 & -\frac{1}{h_x^2} & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}}_{I} \begin{pmatrix} V_{i+1,1} \\ V_{i+1,2} \\ \vdots \\ V_{i+1,N-1} \end{pmatrix}$$

$$= \begin{pmatrix} f_{i,1} \\ f_{i,2} \\ \vdots \\ f_{i,N-1} \end{pmatrix}$$

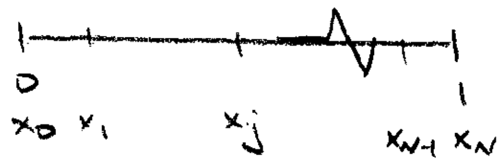
In Blok form

$$\begin{pmatrix} A & I & & & \\ I & A & I & & \\ & I & A & I & \\ & & & \ddots & \\ & & & I & A \end{pmatrix} \begin{pmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vdots \\ \vec{V}_{M-1} \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ \vdots \\ \bar{f}_{M-1} \end{pmatrix}$$

w/ Block tridiagonal matrix of size $(M-1)(M-1)$

(1.1) $-u''(x) + \sigma u(x) = f(x) \qquad 0 < x < 1$

$u'(0) = u'(1) = 0$



2nd order discretization of eq

$$-\frac{(v_{i-1} - 2v_i + v_{i+1})}{h_x^2} + \sigma v_i = f_i \qquad \forall i \text{ in "bulk"} \quad i = 1, \cdots, N-1$$

Using 1st order $\approx \dfrac{u(x_1) - u(x_{-1})}{2h_x} = 0 \rightarrow v_1 = v_{-1}$
derivative approximation

+ For Node N

$$u_x\Big|_{x_N=1} \approx \frac{u(x_{N+1}) - u(x_{N-1})}{2h_x} = 0 \Rightarrow \quad v_{N+1} = v_{N-1}$$

Then For $i=0$ No eq is specified
$i = N$ No eq is specified.

$$\Rightarrow \quad -\frac{1}{h^2} v_{i-1} + \left(\frac{2}{h^2} + \sigma\right) v_i + -\frac{1}{h^2} v_{i+1} = f_i$$

$$\begin{pmatrix} \\ \\ \\ \end{pmatrix} \begin{pmatrix} v_1 \\ \\ \\ \end{pmatrix}$$

I see 2 ways out of the problem of how to incorporate The eqs $V_1 = V_{-1}$ & $V_{N+1} = V_{N-1}$ into the solution & hence a well posed problem.

1) Assume periodic Boundary conditions $\Rightarrow$ $V_{-1} = V_{N-1}$
$V_{N+1} = V_1$

Then $V_1 = V_{-1} = V_{N-1}$
& $V_{N+1} = V_{N-1} \Rightarrow V_1 = V_{N-1}$ } then equations duplicate each other

& an obvious 3rd condition for periodicity would be $V_0 = V_N$

$$V_0 - V_N = 0 \qquad \text{1st eq at Node } i=0$$

$$-\frac{1}{h^2} V_{i-1} + \left(\frac{2}{h^2} + \sigma\right) V_i + -\frac{1}{h^2} V_{i+1} = \Phi_i \qquad i = 1, 2, \cdots, N-1$$

$$V_1 - V_{N-1} = 0$$

$$\begin{pmatrix} 1 & & & & -1 \\ -\frac{1}{h^2} & \frac{2}{h^2}+\sigma & \frac{1}{h^2} & & \\ & -\frac{1}{h^2} & \frac{2}{h^2}+\sigma & -\frac{1}{h^2} & \\ & & & & \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ \vdots \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_N \end{pmatrix} \qquad ?$$

2nd way is apply eq

$$-\frac{1}{h^2}V_{i-1} + \left(\frac{2}{h^2}+\sigma\right)V_i - \frac{1}{h^2}V_{i+1} = f_i \quad \text{at} \quad i=0$$

$$\& \quad i=N$$

$$\Rightarrow \quad -\frac{1}{h^2}\underbrace{V_{-1}}_{\| \atop V_1} + \left(\frac{2}{h^2}+\sigma\right)V_0 - \frac{1}{h^2}V_1 = f_0$$

$$\Rightarrow \quad \left(\frac{2}{h^2}+\sigma\right)V_0 - \frac{2}{h^2}V_1 = f_0$$

one eq !!

& at $i=N$

$$-\frac{1}{h^2}V_{N-1} + \left(\frac{2}{h^2}+\sigma\right)V_N - \frac{1}{h^2}\underbrace{V_{N+1}}_{\| \atop V_{N-1}} = f_N$$

$$\rightarrow \quad -\frac{2}{h^2}V_{N-1} + \left(\frac{2}{h^2}+\sigma\right)V_N = f_N$$

Thus eq used $\forall$ node is

$$\left(\frac{2}{h^2}+6\right)V_0 - \frac{2}{h^2}V_1 = f_0$$

$$-\frac{1}{h^2}V_{i-1} + \left(\frac{2}{h^2}+6\right)V_i - \frac{1}{h^2}V_{i+1} = f_i$$

$$-\frac{2}{h^2}V_{N-1} \quad \left(\frac{2}{h^2}+6\right)V_N = f_N$$

$$\Rightarrow \begin{pmatrix} \frac{2}{h^2}+6 & -\frac{2}{h^2} & 0 & \cdots & & \\ -\frac{1}{h^2} & \frac{2}{h^2}+6 & -\frac{1}{h^2} & & & \\ & -\frac{1}{h^2} & \frac{2}{h^2}+6 & -\frac{1}{h^2} & & \\ & & & & & \\ & & & & -\frac{2}{h^2} & \frac{2}{h^2}+6 \end{pmatrix} \begin{pmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_N \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ \vdots \\ f_N \end{pmatrix}$$

Seems to me there are $N+1$ unknowns

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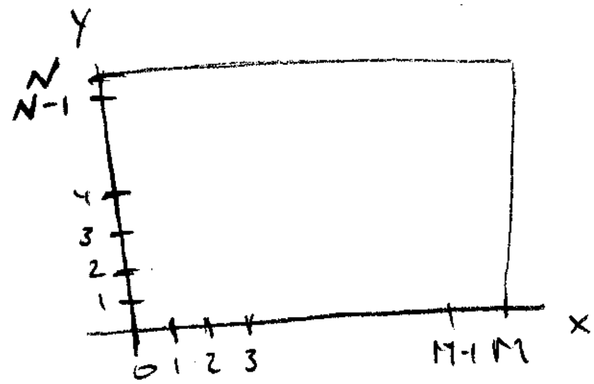
05-10-01 · 1.

(1,2)

$$-\epsilon(U_{xx}+U_{yy}) + aU_x = f(x)$$



$$-\epsilon\left[\frac{V_{i-1,j}-2V_{i,j}+V_{i+1,j}}{h_x^2} + \frac{V_{i,j-1}-2V_{i,j}+V_{i,j+1}}{h_y^2}\right]$$

$$+\ \underbrace{a\frac{(V_{i+1,j}-V_{i-1,j})}{2h_x}}\ = f(x_i)$$

w/ B.C. ∋

$V_{ij} = 0$ if $i=0$, $i=M$ or $j=0$ or $j=N$

$1 \leq i \leq M-1$ + $1 \leq j \leq N-1$

$$\Rightarrow \underbrace{\left(-\frac{\epsilon}{h_x^2}-\frac{a}{2h_x}\right)}_{\text{diff column}}V_{i-1,j} + \underbrace{\left(\frac{2\epsilon}{h_x^2}+\frac{2\epsilon}{h_y^2}\right)}_{\text{same column}}V_{ij} + \underbrace{\left(-\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right)}_{\text{diff column}}V_{i+1,j}$$

$$+\ \underbrace{-\frac{\epsilon}{h_y^2}V_{i,j-1}}_{\text{same column}} \underbrace{-\frac{\epsilon}{h_y^2}V_{i,j+1}}_{\text{same column}} = f(x_i) \qquad 1 \leq j \leq N-1 \;+\; 1 \leq i \leq M-1$$

Now let $\vec{V} = \begin{pmatrix}\vec{V}_1\\ \vec{V}_2\\ \vec{V}_3\\ \vdots\\ \vec{V}_{M-1}\end{pmatrix}$ Then w/ $\vec{V}_1 = \begin{pmatrix}V_{11}\\ V_{12}\\ V_{13}\\ \vdots\\ V_{1N-1}\end{pmatrix}$ $\vec{V}_2 = \begin{pmatrix}V_{2,1}\\ V_{2,2}\\ \vdots\\ V_{2N-1}\end{pmatrix}$

$$\begin{pmatrix}2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix}V_{i1}\\ V_{i2}\\ \vdots\\ V_{iN-1}\end{pmatrix}$$

See next page for larger version of this

$$
\underbrace{\begin{pmatrix}
2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & -\frac{\epsilon}{h_y^2} & & & \\
-\frac{\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & -\frac{\epsilon}{h_y^2} & & \\
& -\frac{\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right) & -\frac{\epsilon}{h_y^2} & \\
& & & \ddots & \\
& & & -\frac{\epsilon}{h_y^2} & 2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)
\end{pmatrix}}_{A}
\begin{pmatrix} V_{i1} \\ V_{i2} \\ \vdots \\ \vdots \\ V_{iN-1} \end{pmatrix}
$$

A tridiagonal matrix of size $(N-1)(N-1)$

w/ diagonals filled w/ $2\epsilon\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)$ +

\+ subdiagonals filled w/ $-\frac{\epsilon}{h_y^2}$ + $-\frac{\epsilon}{h_y^2}$.

Then

$$
\begin{pmatrix}
A & I & & & \\
I & A & I & & \\
& & \ddots & & \\
& & I & A & I \\
& & & I & A
\end{pmatrix}
\begin{pmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vdots \\ \vec{V}_{M-1} \end{pmatrix}
=
\begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vdots \\ \vec{f}_{M-1} \end{pmatrix}
$$

$\vec{f}_i = \begin{pmatrix} f(x_i) \\ f(x_i) \\ \vdots \\ f(x_i) \end{pmatrix}$ constant vector matrix.

w/ $I = -\left(\frac{\epsilon}{h_x^2}+\frac{a}{2h_x}\right)$ Identity.

Then # of Blocks in Block diagonal matrix is $\underbrace{(M-1)\cdot(M-1)}_{\text{size of Block diagonal matrix}}$

each Block of size $(N-1)\cdot(N-1)$

total matrix is the of size $(M-1)(N-1)\cdot(M-1)(N-1)$

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$$v_j^{(1)} = (1-\omega) v_j^{(0)} + \omega v_j^* = (1-\omega) v_j^{(0)} + \omega \left[\frac{1}{2} \left(v_{j-1}^{(0)} + v_{j+1}^{(0)} + h^2 f_j \right) \right]$$

$$\Rightarrow \quad \vec{V}^{(1)} = \left[I(1-\omega) + \omega \underbrace{D^{-1}(L+U)}_{P_J} \right] \vec{V}^{(0)} + \omega D^{-1} \vec{f}$$

$$\bar{v}^{(1)} = \left[(1-\omega)I + \omega D^{-1}(L+U)\right]\bar{v}^{(0)} + \omega D^{-1}f$$

$$= \bar{v}^{(0)} - \omega\left[I - D^{-1}(L+U)\right]\bar{v}^{(0)}$$

$$+ \omega D^{-1} f$$

$r = f - Av$

$\Rightarrow \quad Av = f - r$

$\Rightarrow (D-L-U)v = f - r$

$\Rightarrow \left[I - D^{-1}(L+U)\right]v = D^{-1}(f-r$

$$= \bar{v}^{(0)} - \omega D^{-1}(\cancel{f} - \bar{r}^{(0)})$$

$$+ \omega D^{-1}\cancel{f}$$

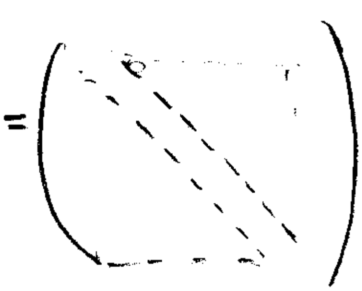
$$= \bar{v}^{(0)} + \omega D^{-1}\bar{r}^{(0)}$$

$U - V = A^{-1}r$

$U = V + A^{-1}r \Rightarrow$ stationary linear iteration from

Picking B close to A^{-1}

$$v^{(1)} = v^{(0)} + Br^{(0)}$$

$A =$ 

$$(D-L-U)u = f$$

$$(D-L)u = Uu + f$$

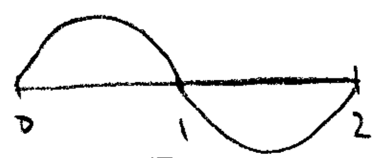
$$u = (D-L)^{-1}Uu + (D-L)^{-1}f$$

For u_i product only incorporates values of u from $i+1$ to $N-1$

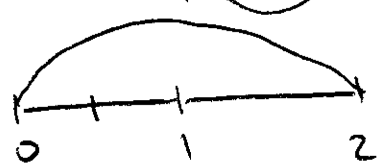
at i $(D-L)^{-1}f$ gives the solution to u using only f from 1 to $i-1$.

$$V_j = \sin\left(\frac{jk\pi}{N}\right) \qquad 0 \le j \le N, \quad 1 \le k \le N-1$$

$\sin\left(\frac{\pi x}{N}\right)$ $N=1$



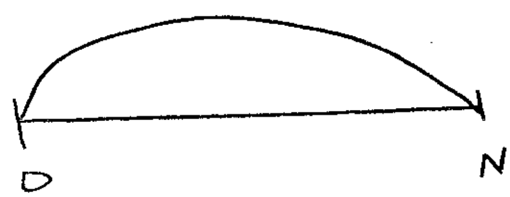
$\sin\left(\frac{\pi x}{N}\right)$ $N=2$



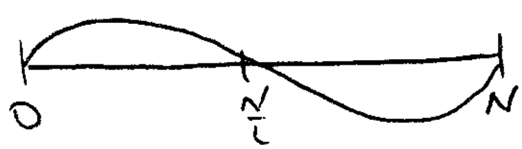
wave modes get longer.

Fix N.

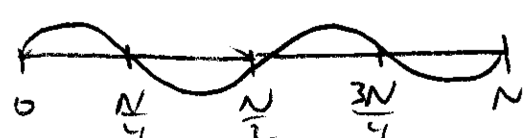
$\sin\left(\frac{k\pi x}{N}\right)$ $k=1$



$\sin\left(\frac{k\pi x}{N}\right)$ $k=2$



$\sin\left(\frac{k\pi x}{N}\right)$ $k=3$



$$\frac{\|e^{(M)}\|}{\|e^{(0)}\|} \leq 10^{-d}$$

Assuming $\|P\|^n \underset{\substack{\uparrow \\ \text{when}}}{\leq} \rho(P)^n$

$$(\rho(P))^M \leq 10^{-d}$$

$$M \log_{10}(\rho(P)) \leq -d \log_{10} 10$$

generally $\rho(P) < 1$ $\quad \log_{10}(\rho(P)) < 0$

$$M \geq \frac{-d}{\log_{10}(\rho(P))}$$

$$P_\omega = (1-\omega)I + \omega D^{-1}(L+U) \qquad A = D - L - U$$

$$\Rightarrow -A = -D + L + U$$

so that

$$L+U = D - A$$

$$D^{-1}(L+U) = I - D^{-1}A$$

$$P_\omega = I - \cancel{\omega I} + \cancel{\omega I} - \omega D^{-1} A$$

$$= I - \omega D^{-1} A$$

$$= I - \omega \begin{pmatrix} 1/2 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/2 \end{pmatrix} A = I - \frac{\omega}{2} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \end{pmatrix}$$

$$\omega_{k,j} = \sin\left(\frac{jk\pi}{N}\right)$$

Then the grid period is

$$\omega_{k,j+p} = \sin\left(\frac{(j+p)k\pi}{N}\right) = \sin\left(\frac{jk\pi}{N} + \frac{pk\pi}{N}\right) = \sin\left(\frac{jk\pi}{N}\right)$$

req $\frac{pk\pi}{N} = 2\pi \rightarrow p = \frac{2N}{k}$ this is the # of grid points

Then actual wave length in space is

$$hp = \frac{2Nh}{k} \quad \text{if} \quad N=12$$

$$\underset{\ell}{\overset{|||}{}}$$

$$\ell = \frac{24h}{k}$$ eq on bottom of pg 22.

$$\lambda_k(P_\omega) = 1 - 2\omega\sin^2\left(\frac{k\pi}{2N}\right) \qquad \frac{N}{2} \le k \le N-1$$

$$\lambda_{N-1}(P_\omega) = 1 - 2\omega\sin^2\left(\frac{(N-1)\pi}{2N}\right) = 1 - 2\omega\sin^2\left(\frac{\pi}{2} - \frac{\pi}{2N}\right)$$

$$= 1 - 2\omega\left[\sin\frac{\pi}{2}\cos\left(\frac{\pi}{2N}\right) - \sin\left(\frac{\pi}{2N}\right)\cancelto{0}{\cos}\left(\frac{\pi}{2}\right)\right]^2$$

$$= 1 - 2\omega\cos^2\left(\frac{\pi}{2N}\right) \sim 1 - 2\omega$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$

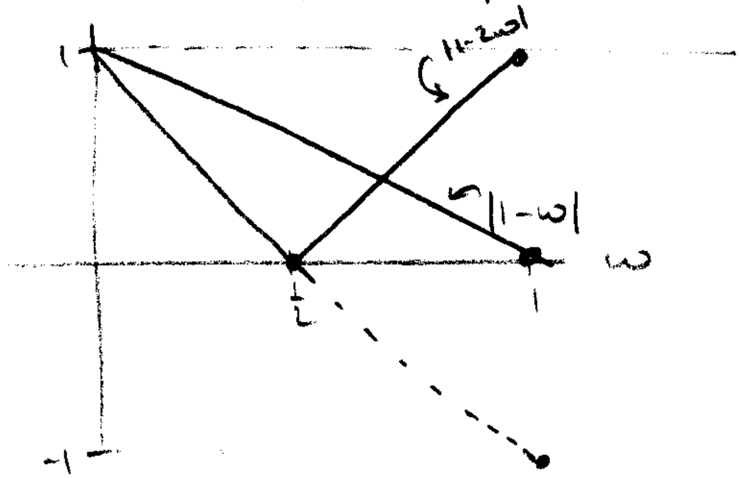
want $\text{Max}\left(|\lambda_{\frac{N}{2}}(P_\omega)|, |\lambda_N(P_\omega)|\right)$ as small as possible $\qquad\qquad \lambda_N(P_\omega)$

$$\lambda_{\frac{N}{2}}(P_\omega) = 1 - 2\omega\sin^2\left(\frac{\cancel{N}}{2}\cdot\frac{\pi}{2\cancel{N}}\right) = 1 - 2\omega\sin^2\left(\frac{\pi}{4}\right)$$

$$= 1 - \frac{2\omega}{2} = 1-\omega.$$

$$\lambda_N(P_\omega) = 1 - 2\omega\sin^2\left(\frac{\pi}{2}\right) = 1-2\omega$$

Thus want $\max(|1-\omega|, |1-2\omega|)$ as small as possible.



from $0 \le \omega \le 1$

Thus $\max(|1-\omega|, |1-2\omega|)$ is shown in the yellow. The choice of ω that makes this $\max(|1-\omega|, |1-2\omega|)$ a minimum is given by

$$1-\omega = -(1-2\omega)$$

$$1-\omega = -1+2\omega \Rightarrow 2 = 3\omega \Rightarrow \omega = \frac{2}{3}$$

(2.1) Pg 29 Briggs 05-10-01 1

$Au = f$ $e = u - v$

$\Rightarrow u = e + v$

$Ae + Av = f$

$Ae = f - Av = r$

$\Rightarrow Ae = r$

(2.3) This is a rather heuristic result. In the example shown

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \end{pmatrix} \quad \& \quad \bar{f} = h^2 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}$$

& $\bar{r} \equiv \bar{f} - A\vec{v}$

Setting the 1st component of $\bar{r} = 0$, we get for the 1st eq

$$0 = h^2 f_1 - 2v_1 + v_2 \quad \text{solving for } v_1 \Rightarrow v_1 = \frac{1}{2}(v_2 + h^2 f_1)$$

meaning that the n+1 st iterate has $v_1^{(n+1)} = \frac{1}{2}(v_2^{(n)} + h^2 f_1)$

Setting the 2nd component equal to zero gives

$$0 = h^2 f_2 + v_1 - 2v_2 + v_3 \rightarrow v_2 = \frac{1}{2}(h^2 f_2 + v_1 + v_3)$$

solving for $v_2^{(n+1)}$ in terms of $v_1^{(n+1)}$ & $v_3^{(n)}$

Setting the jth residual eq to zero gives

$$0 = h^2 f_j + v_{j-1} - 2v_j + v_{j+1} \Rightarrow v_j = \frac{1}{2}(h^2 f_j + v_{j-1} + v_{j+1})$$

or the Gauss-Seidel method in terms of current & past

iterates $v_j^{(n+1)} = \frac{1}{2}(h^2 f_j + v_{j-1}^{(n+1)} + v_{j+1}^{(n)})$

(2.4) $A = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$

$A w = \lambda w$ The jth eq looks like

$$-w_{j-1} + 2w_j - w_{j+1} = \lambda w_j$$

$$-w_{j-1} + (2-\lambda) w_j - w_{j+1} = 0 \qquad \text{middle eq}$$

1st eq $j=1$ last eq $j = N-1$

$$(2-\lambda) w_1 - w_2 = 0 \qquad -w_{N-2} + (2-\lambda) w_{N-1} = 0$$

Following the hint consider $w_j^{(k)} = \sin\left(\frac{jk\pi}{N}\right)$

Note that these are a discrete representation of the continuous eigenvectors

Then evaluating $-w_{j-1} + 2w_j - w_{j+1}$

gives $- \sin\left(\frac{(j-1)k\pi}{N}\right) + 2\sin\left(\frac{jk\pi}{N}\right) - \sin\left(\frac{(j+1)k\pi}{N}\right)$

$$= - \left[\sin\left(\frac{jk\pi}{N}\right)\cos\left(-\frac{k\pi}{N}\right) + \sin\left(-\frac{k\pi}{N}\right)\cos\left(\frac{jk\pi}{N}\right)\right]$$

$$+ 2\sin\left(\frac{jk\pi}{N}\right)$$

$$- \left[\sin\left(\frac{jk\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right) + \sin\left(\frac{k\pi}{N}\right)\cos\left(\frac{jk\pi}{N}\right)\right]$$

$\left\{ \sin(\theta_1+\theta_2) = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1 \right.$

$$= \left[-\cos\left(\frac{k\pi}{N}\right) + 2 - \cos\left(\frac{k\pi}{N}\right)\right] \sin\left(\frac{jk\pi}{N}\right)$$

$$+ \quad \sin\left(\frac{k\pi}{N}\right)\cancel{\cos\left(\frac{jk\pi}{N}\right)} - \sin\cancel{\left(\frac{k\pi}{N}\right)}\cos\left(\frac{jk\pi}{N}\right)$$

$$= \left[2\left(1-\cos\left(\frac{k\pi}{N}\right)\right)\right] \sin\left(\frac{jk\pi}{N}\right)$$

Now:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \quad \& \quad \sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\Rightarrow 1-\cos(\theta) = 2\sin^2 \theta/2$$

$\therefore$ Claim eigenvalues of A are

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2N}\right) \qquad 1 \le k \le N-1 \qquad \rightarrow N-1 \text{ distinct eigenvalues}$$

For $N = 64$

$$\lambda_1 = 4\sin^2\left(\frac{\pi}{2(64)}\right) \qquad \lambda_2 = 4\sin^2\left(\frac{\pi}{64}\right)$$

$$= 1.02\cdot 10^{-18} \qquad = 9.6\cdot 10^{-3}$$

$$\lambda_{N-2} = 4\sin^2\left(\frac{(N-2)\pi}{2N}\right) \qquad \lambda_{N-1} = 4\sin^2\left(\frac{(N-1)\pi}{2N}\right)$$

$$= 4\sin^2\left(\frac{62\pi}{2(64)}\right) \qquad = 4\sin^2\left(\frac{63\pi}{2(64)}\right)$$

$$\cong 4 \qquad \cong 3.997$$

↑ it is these large eigenvalues that make the jacobi method take such a long time converging. I am even surprised it does considering that a requirement of convergence (I thought) was

That $\rho(A) < 1$ X

Jacobi's method is $D^{-1}(L+U)$ iterations of that, You have not written down the eigenvalues of that matrix & thus cannot comment on whether or not Jacobi will converge at this time.

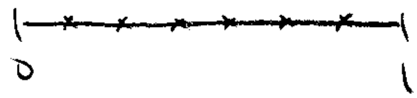
(2.5) From the hint in problem 2.4

the eigenvectors of A are $v_j^{(k)} = \sin\left(\frac{jk\pi}{N}\right)$

Note that $v_{j=0}^{(k)} = 0$ & $v_{j=N}^{(k)} = 0$

(2.6) Since $P_\omega = I - \frac{\omega}{2} A$ the eigenvalues of P_ω will be the same as our model problem A.

(2.7)



$x_j = \frac{j}{N} \quad 0 \le j \le N$

kth Fourier mode $w_{kj} = \sin\left(\frac{jk\pi}{N}\right)$ has a grid wave length (# of grid points one must look to the Right to see the same value of y)

$w_{k,j+p} = \sin\left(\frac{k\pi}{N} j + \frac{k\pi}{N} p\right)$ req $\frac{k\pi}{N} p = 2\pi$

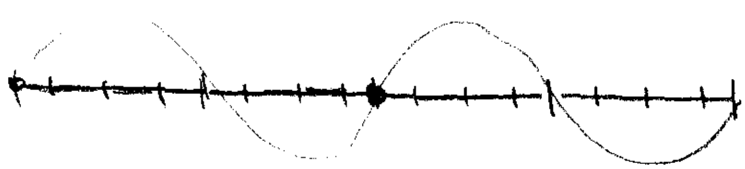
$p = \frac{2N}{k} = \frac{2}{k} N$

+ $h = \frac{1}{N} \Rightarrow N = \frac{1}{h} \quad \therefore \quad p = \frac{2}{kh}$

But actual wave length in distance $= ph = \frac{2N}{k} \cdot \frac{1}{N} = \frac{2}{k}$

which mode has $\frac{2}{k} = 8h \Rightarrow k = \frac{1}{4h}$ # of half sign waves

$\frac{1/2}{1/8h} =$



Mode has $l = \frac{1}{4} = \frac{2}{k} \Rightarrow k = \frac{1}{8}$.

(2.9) let $k \ni N \underset{\neq}{<} k \underset{\neq}{<} 2N$

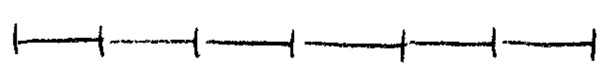
Then $k = 2N - k'$ w/ $k' \ni 1 \le k' \le N-1$

Then
$$w_{k,j} = \sin\left(\frac{jk\pi}{N}\right) = \sin\left(\frac{j\pi}{N}(2N-k')\right)$$
$$= \sin\left(2j\pi - \frac{j\pi}{N}k'\right)$$
$$= \underbrace{\sin(2j\pi)}_{0}\cos\left(\frac{j\pi k'}{N}\right) + \cos(2j\pi)\sin\left(-\frac{j\pi}{N}k'\right)$$
$$= -\sin\left(\frac{jk'\pi}{N}\right) = -w_{k',j}$$

Thus a mode w/ wave # $k \ni N < k < 2N$
Appears on the grid as $-w_{k',j}$ w/ $k' = 2N-k$

If $k = \frac{3N}{2} = 2N - \underset{\substack{\uparrow\\ \text{most}}}{\frac{N}{2}}$

Then $w_{k=\frac{3N}{2}} = -w_{k=\frac{N}{2}}$ on grid



Plot $\sin\left(\frac{3N}{2} j \frac{\pi}{N}\right) = \sin\left(\frac{3\pi}{2}j\right)$ for j continuous & then discrete $j = 1, \dots, N-1$

see how the two plots compare

& plot $-\sin\left(\frac{\pi j}{2}\right)$ discretely & see that it matches

If $l = \frac{4h}{3} = \frac{2}{k} \Rightarrow k = \frac{3}{2h}$

From problem 2.7
This is the wave length
in terms of the wave #

But $h = \frac{1}{N} \Rightarrow k = \frac{3N}{2}$ Same example as just done

take away. As I increase the wave #, i.e. $k \in N < k < 2N$ the waves get more & more oscillatory, This ~~really~~ highly oscillatory nature is "lost" on the grid due to its finite spacing & a wave form of less wave # is what is recorded, when infact a higher wave # should be, See MMa File Braggs.nb

(2.9) Smoothing factor represents the amount of damping experienced per high frequency Fourier mode.

$$\lambda_k(P_\omega) = 1 - 2\omega \sin^2\left(\frac{k\pi}{2N}\right) \qquad 1 \le k \le N-1$$

$$\left|\lambda_k(P_{2/3})\right| = \left|1 - \frac{4}{3}\sin^2\left(\frac{k\pi}{2N}\right)\right| \qquad \frac{N}{2} \le k \le N-1$$

w/

$$\frac{N}{2} \le k \le N-1$$

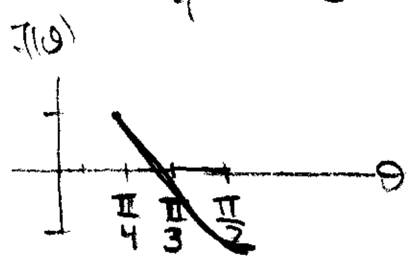
$$\frac{\pi N}{2} \le \pi k \le \pi(N-1)$$

$$\frac{\pi \cancel{N}}{4\cancel{N}} \le \frac{k\pi}{2N} \le \frac{\pi(N-1)}{2N}$$

$$\frac{\pi}{4} \le \frac{k\pi}{2N} \le \frac{\pi}{2} - \frac{\pi}{2N}$$

let $f(\theta) = 1 - \frac{4}{3}\sin^2(\theta) \qquad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

	θ	$f(\theta)$
.78	$\approx \frac{\pi}{4}$	$1 - \frac{2}{3} = \frac{1}{3}$
1.04	$\approx \frac{\pi}{3}$	$1 - \frac{4}{3}\cdot\frac{3}{4} = 0$
1.57	$\approx \frac{\pi}{2}$	$1 - \frac{4}{3} = -\frac{1}{3}$



$$f'(\theta) = -\frac{8}{3}\sin\theta\cos\theta$$

$$f'(\theta) = 0 \Rightarrow \sin\theta\cos\theta = 0$$

$$\sin 2\theta = 0 \qquad 2\theta = n\pi \quad n = 0, \pm 1, \pm 2$$

$$\theta = \frac{\pi}{2}n$$

Thus $|\mathcal{F}(\theta)|$ is bounded by $|\mathcal{F}(\frac{\pi}{4})|$ or $|\mathcal{F}(\frac{\pi}{2})|$

$$|\mathcal{F}(\theta)| \le \max\left(\left|\mathcal{F}\left(\tfrac{\pi}{4}\right)\right|, \left|\mathcal{F}\left(\tfrac{\pi}{2}\right)\right| \right)$$

$$|\mathcal{F}(\theta)| \le \max\left(\left|1 - \frac{\cancel{4}^{\,2}}{3}\frac{1}{\cancel{2}}\right|, \left|1 - \frac{4}{3}\right| \right)$$

$$= \max\left(\frac{1}{3}, \frac{1}{3} \right) = \frac{1}{3}$$

$\therefore$

$$|\lambda_k(P_{2/3})| \le \frac{1}{3}$$

Damping the smooth modes would be equivalent to attempting to make $\lambda_k(P_\omega)$ small for $1 \le k \le \frac{N}{2}$ from

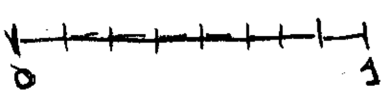
$\lambda_1 \approx 1 - \frac{\omega \pi^2 h^2}{2}$ picking ω s.t. $\lambda \sim \frac{1}{2} \Rightarrow$

$$+\frac{\omega \pi^2 h^2}{\cancel{2}} = \frac{1}{\cancel{2}}$$

$$\Rightarrow \omega = +\frac{1}{\pi^2 h^2}$$

$\omega \sim \frac{1}{h^2}$ then the eigenvalue $\lambda_k(P_\omega)$ becomes for high freq wave #'s

$$\lambda_k(P_\omega) = 1 - 2\left(+\frac{1}{\pi^2 h^2}\right)\sin^2\left(\frac{k\pi}{2N}\right) = 1 - \frac{2}{\pi^2 h^2}\sin^2\left(\frac{k\pi}{2N}\right)$$



$\therefore \lambda_k(P_\omega) = 1 - \frac{2N^2}{\pi^2}\sin^2\left(\frac{k\pi}{2N}\right)$

$h = \frac{1}{N}$

Then for high frequency components $k \approx N-1$

$$\lambda_{N-1}(P_\omega) = 1 - \frac{2N^2}{\pi^2}\sin^2\left(\frac{\pi}{2} - \frac{\pi}{2N}\right) \approx 1 - \frac{2N^2}{\pi^2} < -1 \quad \text{when}$$

$$\frac{2N^2}{\pi^2} > 2 \quad \Rightarrow N > \pi.$$

Thus $|\lambda_{N-1}(P_\omega)| > 1$ when $N > \pi$ & high frequency modes will grow.

2.10 (a) $P_G w = \lambda w$ For Gauss-Seidel $P_G = (D-L)^{-1} U$

w/ $A = D - L - U$

$\Rightarrow (D-L)^{-1} U w = \lambda w$

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & & \end{pmatrix}$$

$\Rightarrow U w = \lambda (D-L) w$

$$= \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & \ddots & \\ & & & 2 \end{pmatrix} - \underbrace{\begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & 1 & \end{pmatrix}}_{L} - \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & 1 \\ & & & 0 \end{pmatrix}}_{U}$$

(b) 1st eq in $U w = \lambda (D-L) w$

is

$w_2 = \lambda (2 w_1)$

jth eq is

$w_{j+1} = \lambda (2 w_j - w_{j-1})$ Not all constants, well? Here we are imagining a particular k for λ & w.

$w_{j+1} - 2\lambda w_j + \lambda w_{j-1} = 0$ Try $w_j = \mu^j$

$\mu^{j+1} - 2\lambda \mu^j + \lambda \mu^{j-1} = 0$

$\mu - 2\lambda + \lambda \mu^{-1} = 0$

$\mu^2 - 2\lambda \mu + \lambda = 0$

$$\mu = \frac{+2\lambda \pm \sqrt{4\lambda^2 - 4\lambda}}{2} = \lambda \pm \sqrt{\lambda^2 - \lambda}$$

Then $w = c_1 \mu_1^j + c_2 \mu_2^j$

$$= c_1 (\lambda + \sqrt{\lambda^2 - \lambda})^j + c_2 (\lambda - \sqrt{\lambda^2 - \lambda})^j$$

Then requiring $\omega_0 = \omega_N = 0$

$$c_1 + c_2 = 0 \quad \rightarrow \quad c_2 = -c_1$$

$$\& \quad c_1\left(\lambda+\sqrt{\lambda^2-1}\right)^N + c_2\left(\lambda-\sqrt{\lambda^2-1}\right)^N = 0$$

$$\Rightarrow \quad c_1\left(\lambda+\sqrt{\lambda^2-1}\right)^N - c_1\left(\lambda-\sqrt{\lambda^2-1}\right)^N = 0$$

$$\Rightarrow \quad c_1\left[\left(\lambda+\sqrt{\lambda^2-1}\right)^N - \left(\lambda-\sqrt{\lambda^2-1}\right)^N\right] = 0$$

$c_1 \neq 0$ or else the entire solution is identically zero.

$$\Rightarrow \quad \left(\frac{\lambda+\sqrt{\lambda^2-1}}{\lambda-\sqrt{\lambda^2-1}}\right)^N = 1 = e^{0+2\pi k i}$$

$$\Rightarrow \quad \frac{\lambda+\sqrt{\lambda^2-1}}{\lambda-\sqrt{\lambda^2-1}}$$ is an Nth root of unity

$$\Rightarrow \quad \frac{\lambda+\sqrt{\lambda^2-1}}{\lambda-\sqrt{\lambda^2-1}} = e^{2\pi k i/N} \qquad 0 \le k \le N-1$$

* see last page.

Now λ has an index k

$$\lambda + \sqrt{\lambda^2-1} = e^{2\pi k i/N}\left(\lambda-\sqrt{\lambda^2-1}\right) \equiv \theta\left(\lambda-\sqrt{\lambda^2-1}\right)$$

$$(1-\theta)\lambda + (1+\theta)\sqrt{\lambda^2-1} = 0$$

$$(1+\theta)^2(\lambda^2-1) = (\theta-1)^2\lambda^2$$

$$\left[(1+\theta)^2-(\theta-1)^2\right]\lambda^2 - (1+\theta)^2\lambda = 0$$

$$\left[\cancel{1}+2\theta+\cancel{\theta^2}-(\cancel{\theta^2}-2\theta+\cancel{1})\right]\lambda^2 - (1+\theta)^2\lambda = 0$$

$$4\theta\lambda^2 - (1+\theta)^2\lambda = 0$$

$$\lambda \neq 0 \Rightarrow \lambda = \frac{(1+\theta)^2}{4\theta} = \frac{1+2\theta+\theta^2}{4\theta} = \frac{1}{4}(\theta^{-1}+2+\theta)$$

$$= \frac{1}{4}(\theta^{-1/2}+\theta^{1/2})^2 = \left(\frac{\theta^{-1/2}+\theta^{1/2}}{2}\right)^2$$

$$= \left(\frac{e^{\frac{\pi k i}{N}} + e^{\frac{-\pi k i}{N}}}{2}\right)^2 = \cos^2\left(\frac{\pi k}{N}\right) \qquad 0 \le k \le N-1$$

But $k \neq 0$ for if it did

eigenvalue eq would be $U\omega = (D-L)\omega$

$$0 = (D-L-U)\omega \Rightarrow A\omega = 0 \Rightarrow \omega = 0 \rightarrow\leftarrow$$

How else can you see that $k \neq 0$, maybe before this time.

From 2nd pg if $k=0$

$$\frac{\lambda + \sqrt{\lambda^2-\lambda}}{\lambda - \sqrt{\lambda^2-\lambda}} = 1 \Rightarrow \cancel{\lambda} + \sqrt{\lambda^2-\lambda} = \cancel{\lambda} - \sqrt{\lambda^2-\lambda}$$

$$2\sqrt{\lambda^2-\lambda} = 0$$

$\lambda = 0$ or $\lambda = 1$ neither of which is possible $\rightarrow\leftarrow$

2.11 Show $w_{k,j} = \lambda_k^{j/2} \sin\left(\frac{jk\pi}{N}\right)$

given $\lambda_k = \cos^2\left(\frac{k\pi}{N}\right)$ $k = 1, 2, \ldots, N-1$

Then from Prob 2.10 $U\bar{w} = (D-L)\lambda\bar{w}$

From Problem 2.11

$$w_{k,j} = C_1\left[\left(\lambda + \sqrt{\lambda^2-\lambda}\right)^j - \left(\lambda - \sqrt{\lambda^2-\lambda}\right)^j\right]$$

w/ $\lambda_k = \cos^2\left(\frac{k\pi}{N}\right)$ or $\lambda_k = \frac{(1+\theta_k)^2}{4\theta_k}$ w/ $\theta_k = e^{\frac{2\pi i k}{N}}$

So Normalizing $C_1 = 1$

Consider $\lambda^2 - \lambda = \lambda(\lambda-1) = \cos^2\left(\frac{k\pi}{N}\right)\left(\cos^2\left(\frac{k\pi}{N}\right) - 1\right)$

$\cos^2\alpha = 1 - \sin^2\alpha$

$\cos^2\alpha - 1 = -\sin^2\alpha$

$\therefore \lambda^2 - \lambda = -\cos^2\left(\frac{k\pi}{N}\right)\sin^2\left(\frac{k\pi}{N}\right) = -\frac{1}{4}\sin^2\left(\frac{2k\pi}{N}\right)$

For $1 \le k \le N-1$

$$0 < \frac{2\pi}{N} \le \frac{2\pi k}{N} \le \frac{2\pi(N-1)}{N} < 2\pi$$

Thus $\sqrt{\lambda^2-\lambda} = \frac{i}{2}\sqrt{\sin^2\left(\frac{2k\pi}{N}\right)} = \frac{i}{2}\left|\sin\left(\frac{2k\pi}{N}\right)\right|$

$$w_{k,j} = \left(\cos^2\left(\frac{k\pi}{N}\right) + \frac{i}{2}\left|\sin\left(\frac{2k\pi}{N}\right)\right|\right)^j - \left(\cos^2\left(\frac{k\pi}{N}\right) - \frac{i}{2}\left|\sin\left(\frac{2k\pi}{N}\right)\right|\right)^j$$

Try this simplification:

$$\lambda(\lambda-1) = \frac{(1+\theta_k)^2}{4\theta_k}\left[\frac{(1+\theta_k)^2}{4\theta_k} - 1\right] \checkmark$$

$$= \frac{(1+\theta_k)^2}{4\theta_k}\left[\frac{1+2\theta_k+\theta_k^2-4\theta_k}{4\theta_k}\right] \checkmark$$

$$= \frac{(1+\theta_k)^2(1-2\theta_k+\theta_k^2)}{(4\theta_k)^2} = \frac{(1+\theta_k)^2(1-\theta_k)^2}{4^2\theta_k^2}$$

This ~~#~~ $\lambda(\lambda-1)$ is complex & thus the proper way to take its square root is to write it in polar form & ÷ by 2 & take the square root of the magnitude of the radius.

As this square root is invoked from the quadratic formula only the principle root needs to be taken, i.e. the positive one only.

$\theta_k = e^{\frac{2\pi i k}{N}}$ So that

$$\lambda(\lambda-1) = \frac{(1-\theta_k^2)^2}{4^2\theta_k^2} = \left(\frac{1-\theta_k^2}{4\theta_k}\right)^2 = \left(\frac{\theta_k^{-1}-\theta_k}{2\cdot 2}\right)^2$$

$$= \left(\frac{e^{-2\pi i k/N} - e^{2\pi i k/N}}{2\cdot 2}\right)^2 = \left(\frac{e^{2\pi i k/N} - e^{-2\pi i k/N}}{2\cdot 2}\right)^2$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

$$= \frac{1}{4}\left(i\sin\left(\frac{2\pi k}{N}\right)\right)^2$$

$$= \frac{-1}{4}\sin^2\left(\frac{2\pi k}{N}\right)$$

Then $\sqrt{\lambda(\lambda-1)} = \frac{i}{2}\sqrt{\sin^2\left(\frac{2\pi k}{N}\right)}$

As $1 \leq k \leq N-1$
when

$0 \leq \frac{2\pi}{N} \leq \pi \Rightarrow 0 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor$ k integer

$\pi \leq \frac{2\pi k}{N} \leq 2\pi \Rightarrow \left\lceil \frac{N}{2} \right\rceil \leq k \leq N$ k integer.

Thus let $k \in \{1,2,3,\ldots,\lfloor \frac{N}{2} \rfloor\}$. Then

$$\sqrt{\lambda(\lambda-1)} = \frac{i}{2}\sin\left(\frac{2\pi k}{N}\right) = \frac{i}{2}\left(\frac{\theta_k - \theta_k^{-1}}{2i}\right) = \frac{\theta_k - \theta_k^{-1}}{4}$$

If $k \in \{\lceil \frac{N}{2} \rceil, \lceil \frac{N}{2} \rceil + 1, \ldots, N-1\}$ Then

$$\sqrt{\lambda(\lambda-1)} = -i\sin\left(\frac{2\pi}{N}k\right) = -\frac{i}{2}\left(\frac{\theta_k - \theta_k^{-1}}{2i}\right) = -\frac{(\theta_k - \theta_k^{-1})}{4}$$

Now for k in 1st range ω_{kj} becomes, First term.

$$\frac{(1+\theta_k)^2}{4\theta_k} + \frac{\theta_k - \theta_k^{-1}}{4} = \frac{(1+\theta_k)^2}{4\theta_k} + \frac{\theta_k^2 - 1}{4\theta_k} = \frac{1}{4\theta_k}(1+\theta_k)\left[1+\theta_k + \theta_k - 1\right]$$

$$= \frac{(1+\theta_k)}{4\theta_k} 2\theta_k$$

& Second term becomes:

$$\frac{(1+\theta_k)^2}{4\theta_k} - \left(\frac{\theta_k - \theta_k^{-1}}{4}\right) = \frac{(1+\theta_k)^2}{4\theta_k} - \frac{(\theta_k^2 - 1)}{4\theta_k} = \frac{(1+\theta_k)}{4\theta_k}\left[1+\theta_k - (\theta_k - 1)\right]$$

$$= \frac{(1+\theta_k)}{4\theta_k} 2$$

Now for k in 2nd Range the terms change sign but since each term changes sign one gets the same expression w/ each term reversed.

Thus the only work we need to do is on the 1st range.

$$w_{k,j} = C\left[\left(\frac{1+\theta_k}{4\theta_k}\right)^j 2^j \theta_k^j - 2^j \left(\frac{1+\theta_k}{4\theta_k}\right)^j\right]$$

$$= C\left[\frac{(1+\theta_k)}{2\theta_k^{1/2}}\right]^j \left(\theta_k^{j/2} - \theta_k^{-j/2}\right)$$

$$= C\left[\frac{\theta_k^{1/2} + \theta_k^{-1/2}}{2}\right]^j \left(\theta_k^{j/2} - \theta_k^{-j/2}\right) \qquad \text{w/ } \theta_k = e^{2\pi i k/N}$$

$$= C\left[\frac{e^{\pi i k/N} + e^{-\pi i k/N}}{2}\right]^j \left(e^{\pi i k j/N} - e^{-\pi i k j/N}\right)$$

$$= C\left(\cos\left(\frac{\pi k}{N}\right)\right)^j 2i \sin\left(\frac{\pi k j}{N}\right)$$

why is this complex? The eigenvectors of a real matrix should be real?

$$\propto \cos\left(\frac{\pi k}{N}\right)^j \sin\left(\frac{\pi k j}{N}\right)$$

$$= \lambda_k^{j/2} \sin\left(\frac{\pi k j}{N}\right)$$

(2.12)

$$Au = f \qquad r = f - Av$$

$$\Leftrightarrow \quad Ae = r \qquad e = u - v$$

Thus consider $Au = 0$

A Stationary linear iteration is defined by $v^{(1)} = v^{(0)} + Br^{(0)}$

where $B \approx A^{-1}$ $\qquad \vec{r}^{0} = f - Av^{(0)}$

Then to solve $Au = f$ Assume we have an initial guess $v^{(0)}$

Then the residual eq

$$Au = f$$
$$Av = \tilde{f}$$
$$Ae = f - \tilde{f} = r$$

Solving the system

How the ~~vector~~ Approximate vectors to $Au = 0$ approach zero is the same way that the error vectors to $Au = f$ will approach zero

$$v^{(1)} = v^{(0)} + B(f - Av^{(0)})$$

$$= Bf - BAv^{(0)} + v^{(0)} = v^{(0)} - BAv^{(0)} + Bf$$

To Analize the error let u be the exact solution $u = A^{-1}f$

$e^{(i)} = u - v^{(i)}$

$$u - v^{(1)} = u - v^{(0)} + BAv^{(0)} - Bf$$

$$e^{(1)} = e^{(0)} + BA(u - e^{(0)}) - Bf$$

$$= e^{(0)} + \cancel{B}f - BAe^{(0)} - \cancel{B}f$$

$$e^{(1)} = e^{(0)} - BAe^{(0)}$$

This represents the stationary iteration scheme for the error. in the solution to $Au = f$.

+ The stationary iteration scheme for $Av = 0$ is

$$v^{(1)} = v^{(0)} + B(r^{(0)}) = v^{(0)} + B(0 - Av^{(0)})$$

$$= v^{(0)} - BAv^{(0)}$$ this is exactly

the same eq as above, Thus since the initial error $e^{(0)}$ is unknown we can watch how it decays by studying the solution to $Av = 0$ w/ a random initial condition.

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(2.13) Given a norm on $\mathbb{R}^n$ we can induce a matrix norm on A

{ norm satisfies $\|x\| > 0$ if $x \neq 0$ $\|0\| = 0$ (positive definite prop)
vector

$\|\alpha x\| = |\alpha| \, \|x\|$ absolute homogeneity,

$\|x+y\| \leq \|x\| + \|y\|$ triangle inequality }

Matrix norm

$\|A\| > 0$ if $A \neq 0$

$\|\alpha A\| = |\alpha| \, \|A\|$

$\|A+B\| \leq \|A\| + \|B\|$ $\quad \underline{\|AB\| \leq \|A\| \, \|B\|}$ consistency

new factor

By

$$\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_V}{\|x\|_V}$$

(a) If A has a constant diagonal, then

$$v^{(1)} = v^{(0)} - \frac{\omega}{\|A\|_2} A(A^{-1}(f - r^{(0)}))$$

$\cancel{r = Av}$

$r = f - Av$

$Av = f - r$

$v = A^{-1}(f-r)$

$$= v^{(0)} - \frac{\omega}{\|A\|_2}(f - Ar^{(0)})$$

$$= v^{(0)} + \frac{\omega A}{\|A\|_2} r^{(0)} - \frac{\omega f}{\|A\|_2}$$

$$v^{(1)} = v^{(0)} - \frac{\omega f}{\|A\|_2} + \frac{\omega A r^{(0)}}{\|A\|_2} = -\frac{\omega f}{\|A\|_2} + v^{(0)} + \frac{\omega A r^{(0)}}{\|A\|_2}$$

Weighted Jacobi

$$v^{(1)} = v^{(0)} + \omega D^{-1} r^{(0)} \qquad r^{(0)} = f - A v^{(0)}$$

$$= v^{(0)} + \omega D^{-1} f - \omega D^{-1} A v^{(0)}$$

$$= \omega D^{-1} f + v^{(0)} - \omega D^{-1} A v^{(0)}$$

$$\|A\|_2 v^{(1)} = \|A\|_2 v^{(0)} - \omega A v^{(0)}$$

Weighted Jacobi is $v^{(1)} = v^{(0)} + \omega D^{-1} r^{(0)}$

or $v^{(1)} = \left((1-\omega)I + \omega D^{-1}(L+U)\right) v^{(0)} + D^{-1} f$

$A = D - L - U$ But if A has a constant diagonal $D = dI$ (a constant times the identity matrix).

Then using 2nd representation of weighted Jacobi we get

$$v^{(1)} = \left[(1-\omega) I + \frac{\omega}{d}(L+U)\right] v^{(0)} + D^{-1} f$$

$$= v^{(0)} + \left[-\omega + \frac{\omega}{d}(L+U)\right] v^{(0)} + D^{-1} f$$

$$= v^{(0)} - \frac{\omega}{d}\left[d - L - U\right] v^{(0)} + D^{-1} f$$

$$= v^{(0)} - \frac{\omega}{d}\left[D - L - U\right] v^{(0)} + D^{-1} f$$

$$= v^{(0)} - \frac{\omega}{d} A v^{(0)} + D^{-1} f = v^{(0)} - \frac{\omega}{d} A v^{(0)} + \frac{1}{d} f$$

if $f \equiv 0$ As would not be a bad Assumption & since nothing is stated in the problem ~~about~~ about this, then if $\|A\|_2 = d$ we are done

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\|x\|=1} \|Ax\|_2 \geq d$$

Can I show $\max\limits_{\|x\|=1} \|Ax\|_2 \le d$

let

(c) smallest eigenvalues → smooth modes ?

smoothing property → error decreases geometrically w/ every iteration

The $e^{(0)} = \sum_{j=1}^{n} c_j w_j$

(b) $$v^{(1)} = v^{(0)} - \frac{\omega}{\|A\|_2} A v^{(0)} \qquad e^{(1)} = u - v^{(1)} \Rightarrow v^{(1)} = u - e^{(1)}$$

$$\to \cancel{u} - e^{(1)} = \cancel{u} - e^{(0)} - \frac{\omega}{\|A\|_2} A(u - e^{(0)})$$

$$e^{(1)} = e^{(0)} + \frac{\omega}{\|A\|_2} Au - \frac{\omega}{\|A\|_2} Ae^{(0)}$$

$$e^{(1)} = e^{(0)} - \frac{\omega}{\|A\|_2} Ae^{(0)} + \frac{\omega}{\|A\|_2} Au$$

Take inner product w/ $e^{(0)}$

$$(e^{(1)}, e^{(0)}) = \|e^{(0)}\|_2^2 - \frac{\omega}{\|A\|_2} \frac{(Ae^{(0)}, e^{(0)})}{(e^{(0)}, e^{(0)})} \|e^{(0)}\|_2^2 + \frac{\omega (Au, e^{(0)})}{\|A\|_2}$$

$$\{ 1 + 1 - \omega = 2 - \omega \}$$

$$= \|e^{(0)}\|_2^2 - \frac{\omega}{\|A\|_2} \frac{(Ae^{(0)}, e^{(0)})}{(e^{(0)}, e^{(0)})} \|e^{(0)}\|_2^2 \frac{(1-\omega)}{\|A\|_2} \frac{(Ae^{(0)}, e^{(0)})}{(e^{(0)}, e^{(0)})} \|e^{(0)}\|_2^2$$

$$+ \frac{(1-\omega)}{\|A\|_2} \frac{(Ae^{(0)}, e^{(0)})}{(e^{(0)}, e^{(0)})} \|e^{(0)}\|_2^2 + \omega (Au, e^{(0)})$$

Prob 2.13

(c) plotting the eigenvectors ~~of~~ A one can determine if they correspond to high or low frequency. Here we are told that, the ~~eigen~~ smallest eigenvectors correspond to the smooth modes.

to see if Richardson's iteration damps high frequency modes

let $e^{(0)} = w_N$. Then

$$\|e^{(1)}\|_2^2 \le \underbrace{\left[1 - \frac{\omega(2-\omega)}{\cancel{\lambda_N}} \cancel{\lambda_N}\right]}\|e^{(0)}\|_2^2$$

$$1 - 2\omega + \omega^2$$

$$(1-\omega)^2 < 1$$

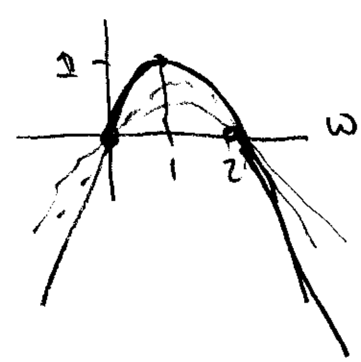
$0 < \omega < 2$

$-1 < 1-\omega < 1$

$(1-\omega)^2 < 1$

Thus the high frequency modes

plot of $\omega(2-\omega)$ v.s ω



If $e^{(0)} = w_{N-1}$

$$\|e^{(1)}\|_2^2 \le \left[1 - \frac{\omega(2-\omega)}{\lambda_N} \lambda_{N-1}\right]\|e^{(0)}\|_2^2$$

$$1 - 2\left(\frac{\lambda_{N-1}}{\lambda_N}\right)\omega + \frac{\lambda_{N-1}}{\lambda_N}\omega^2$$

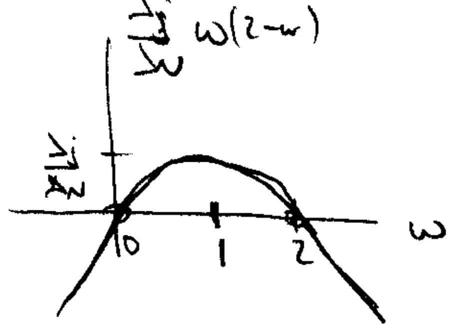
maximum/minimum of $\omega(2-\omega)$ happens at : $2 - 2\omega = 0$ or $\omega = 1$

Note: The function $\frac{\lambda_j}{\lambda_N}(\omega)(2-\omega)$

is a scaling of the one graphed on the previous page.

Thus the ~~larger~~ smaller the eigenvalue ratio $\frac{\lambda_j}{\lambda_N}$ the more "flat"

The curve becomes



+ the more $1 - \frac{\lambda_j}{\lambda_N} w(2-w)$ looks like 1 giving very poor convergence of the smooth modes (as would be expected)

(2.14) $(u,v)_A = (Au, v) = u^T A v$ $\quad \|u\|_A^2 = (u,u)_A = (Au,u) = (Au)^T u = u^T A^T u = u^T A u$

Then fact that A symmetric positive def.

(a) definition of an inner product space is that

(i) $(x,y) = \overline{(y,x)}$

(ii) $(c_1 x_1 + c_2 x_2, y) = c_1(x_1,y) + c_2(x_2,y)$

(iii) $(x,x) \geq 0 \ \forall x$ + $(x,x) = 0$ iff $x \equiv 0$

$(x,y) = x^+ y$

$\overline{(y,x)} = \overline{y^+ x} = x^+ y = (x,y)$

Thus Assuming A symmetric possitive def on reals

(i) $(x,y)_A = (Ax,y) = (Ax)^T y = x^T A^T y = x^T A y = (x^T A y)^T$

$= y^T A^T x \neq (Ay)^T x = (Ay,x) = (y,x)_A$

(ii) $(c_1 x_1 + c_2 x_2, y)_A = (A(c_1 x_1 + c_2 x_2))^T y = (c_1 x_1^T + c_2 x_2^T) A y$

$= c_1 x_1^T A y + c_2 x_2^T A y = c_1 (A^T x_1)^T y + c_2 (A^T x_2)^T y$

$= c_1 (A x_1)^T y + c_2 (A x_2)^T y$

$= c_1 (x_1, y)_A + c_2 (x_2, y)_A$

(iii) $(x,x)_A = (Ax)^T x = x^T A^T x \geq 0 \ \forall x \neq 0$
$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad '' \ x^T A x \geq 0$

& $x^T A x = 0$ iff $x = 0$. These are properties of symmetric positive definite matrices.

A norm $\|u\|_A = \sqrt{(u,u)_A}$

(i) $\|u\|_A = \sqrt{(u,u)_A} = \sqrt{(Au)^T u} = \sqrt{u^T A^T u} = \sqrt{u^T A u} \geq 0$

$\|u\|_A = 0$

$\Rightarrow u^T A u = 0 \Rightarrow u = 0.$

$$\|\alpha u\|_A = \sqrt{(\alpha u, \alpha u)_A} = \sqrt{(A\alpha u)^T(\alpha u)} = \sqrt{u^T A^T u \, \alpha^2} = |\alpha|\sqrt{u^T A u}$$

$$= |\alpha|\sqrt{(Au)^T u} = |\alpha|\sqrt{(u,u)_A} = |\alpha| \, \|u\|_A$$

$$\|x+y\|_A = \sqrt{(A(x+y))^T(x+y)}$$

$$\Rightarrow \|x+y\|_A^2 = (x^T+y^T)A^T(x+y) = (x^T+y^T)A(x+y)$$

$$= x^T A x + y^T A y + x^T A y + y^T A x$$

$$= \|x\|_A^2 + \|y\|_A^2 + x^T A y + y^T A x$$

$$= \|x\|_A^2 + \|y\|_A^2 + x^T A y + x^T A y$$

$$= \|x\|_A^2 + \|y\|_A^2 + 2x^T A y \leq \|x\|_A^2 + \|y\|_A^2 + 2|x^T A y|$$

Now show $|x^T A y| \le \|x\|_A \|y\|_A$

$$\|x\|_A \|y\|_A = \sqrt{(x^T A x)(y^T A y)}$$

if this is true then

$$\|x+y\|_A^2 \le \|x\|_A^2 + \|y\|_A^2 + 2\|x\|_A\|y\|_A = (\|x\|_A + \|y\|_A)^2$$

$$\Rightarrow \|x+y\|_A \le \|x\|_A + \|y\|_A$$

Since A is symmetric positive definite it has n eigenvectors w/ n real, positive eigenvalues

let $x = \sum_{i=1}^n a_i v_i$ & $y = \sum_{j=1}^n b_j v_j$

The $x^T A y = \left(\sum_{i=1}^n a_i v_i\right)^T A \left(\sum_{j=1}^n b_j v_j\right)$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i b_i v_i^T A v_j = \sum_{i=1}^n \sum_{j=1}^n a_i b_i \lambda_j v_i^T v_j$$

This inequality is tricky & called the

Cauchy-inequality $\quad |x^T A y| \leq \sqrt{x^T A x}\,\sqrt{y^T A y}$

Then if $x \equiv 0$ the statement is true, ~~then~~ assume $x \neq 0$

$z = y - \alpha x$ w/ $\alpha = \dfrac{(y,x)_A}{(x,x)_A}$ Then

$$|z|^2 = (z,z)_A = (z, y-\alpha x)_A = (z,y)_A - \alpha(z,x)_A$$

But $(z,x)_A = (y-\alpha x, x)_A = (y,x)_A - \alpha(x,x)_A = (y,x)_A - (y,x)_A = 0$

∴ Above equals

$$= (z,y)_A = (y-\alpha x, y)_A = (y,y)_A - \alpha(x,y)_A$$

$$= \|y\|_A^2 - \frac{(y,x)_A(x,y)_A}{\|x\|_A^2}$$

$$= \frac{\|y\|_A^2\|x\|_A^2 - |(x,y)_A|}{\|x\|_A^2} \geq 0$$

$$\Rightarrow \quad |(x,y)_A| \leq \|x\|_A^2\|y\|_A^2$$

(b) $\|r\|_2 = \|e\|_{A^2}$

$e = u - v \Rightarrow v = u - e$

$r = f - Av = f - A(u-e) = Ae$

$\|r\|_2^2 = r^T r = (Ae)^T(Ae)$

$= e^T A^2 e = [(A^2)^T e]^T e = [A^2 e]^T e$

$= \|e\|_{A^2}$

(c) $\|e\|_2 = e^T e$ to know this e one must know u, which is generally not known.

$\|e\|_A = (Ae)^T e = e^T A e$

But $Ae = Au - Av = f - Av = r$

$\therefore \|e\|_A = e^T r$ Not unless one knows e

$\|e\|_{A^2} = \|r\|_2$ from Above

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(a) Start w/ initial guess $\vec{v}$ for the solution to $Au = f$, then

The idea behind the gauss seidel method is to look at the jth eq

$$a_{j1}v_1 + a_{j2}v_2 + \cdots + a_{jj}v_j + \cdots a_{jn-1}v_{n-1} + a_{jn}v_n = f_j$$

& solve for v_j.

$$v_j = \frac{1}{a_{jj}}\left[f_j - \sum_{p=1}^{j-1} a_{jp}v_p - \sum_{p=j+1}^{n} a_{jp}v_p\right]$$

Then iterations are

$$v_j^{(n)} = \frac{1}{a_{jj}}\left[f_j - \sum_{p=1}^{j-1} a_{jp}v_p^{(n)} - \sum_{p=j+1}^{n} a_{jp}v_p^{(n-1)}\right]$$

$\vec{r} = f - Av$

$f = \vec{r} + Av$

$$= \frac{1}{a_{jj}}\left[r_j^{(n-1)} + \sum_{p=1}^{j-1} a_{jp}v_p^{(n-1)} + a_{jj}v_j^{(n-1)} + \cancel{\sum_{p=j+1}^{n} a_{jp}v_p^{(n-1)}} - \sum_{p=1}^{j-1} a_{jp}v_p^{(n)} - \cancel{\sum_{p=j+1}^{n} a_{jp}v_p^{(n-1)}}\right]$$

$$= v_j^{(n-1)} + \frac{1}{a_{jj}}\left[r_j^{(n-1)} + \sum_{p=1}^{j-1} a_{jp}v_p^{(n-1)} - \sum_{p=1}^{j-1} a_{jp}v_p^{(n)}\right]$$

$$= v_j^{(n-1)} + \frac{1}{a_{jj}}\left[r_j^{(n-1)} + \underbrace{\sum_{p=1}^{j-1} a_{jp}}_{\text{lower } \triangle \text{ portion of matrix } A.}\left[v_p^{(n-1)} - v_p^{(n)}\right]\right]$$

$j \to (a_{j1}\, a_{j2} \ldots a_{jj})$

Assume that residual is updated at the exact instant that each component of v is changed. Yes. Thus we change v_j we recompute our residual w/ our new approximate value of v (only one component is different each time) Yes.

Gauss-Seidel,

$$v_j^{(n)} = \frac{1}{a_{jj}}\left[f_j - \sum_{p=1}^{j-1} a_{jp} v_p^{(n)} - \sum_{p=j+1}^{n} a_{jp} v_p^{(n-1)}\right]$$

$$r = f - Av$$
$$f = r + Av$$

pseudocode:

given $v^{(n-1)}$ & $r^{(n-1)}$

update $v_1^{(n)}$ compute $r_1^{(n)}$

update $v_2^{(n)}$ compute $r_1^{(n)}, r_2^{(n)}$

update $v_3^{(n)}$ compute $r_1^{(n)}, r_2^{(n)}, r_3^{(n)} \ldots$

Thus r is a function of current v.

$$v_j = \frac{1}{a_{jj}}\left[f_j - \sum_{p=1}^{j-1} a_{jp} v_p - \sum_{p=j+1}^{n} a_{jp} v_p\right]$$

What is r_j when v is composed of $1, 2, \ldots, j-1$ new components & $j, j+1, \ldots, n$ old components?

$$\bar{v} = \begin{pmatrix} v_1^{(n)} \\ v_2^{(n)} \\ \vdots \\ v_{j-1}^{(n)} \\ v_j^{(n-1)} \\ v_{j+1}^{(n-1)} \\ \vdots \\ v_N^{(n-1)} \end{pmatrix}$$

$$r_j = f_j - \sum_{p=1}^{j-1} a_{jp} v_p^{(n)} - a_{jj} v_j^{(n-1)} \sum_{p=j+1}^{n} a_{jp} v_j^{(n-1)} \qquad 1 \le j \le n$$

$$\therefore f_j = r_j + \sum_{p=1}^{j-1} a_{jp} v_p^{(n)} + a_{jj} v_j^{(n+1)} + \sum_{p=j+1}^{n} a_{jp} v_j^{(n-1)}$$

Then

$$v_j = \frac{1}{a_{jj}}\left[r_j + \cancel{\sum_{p=1}^{j-1} a_{jp} v_p^{(n)}} + a_{jj} v_j^{(n+1)} + \cancel{\sum_{p=j+1}^{n} a_{jp} v_j^{(n-1)}} - \cancel{\sum_{p=1}^{j-1} a_{jp} v_p^{(n)}} - \cancel{\sum_{p=j+1}^{n} a_{jp} v_p^{(n-1)}}\right]$$

$$= v_j^{(n+1)} + \frac{r_j}{a_{jj}}$$

Thus in a vector form

$$\vec{v} \leftarrow \vec{v} + \frac{(\vec{r}, \hat{e}_i)}{(A\hat{e}_i, \hat{e}_i)} \hat{e}_i \qquad 1 \le i \le n$$

Think the Book has a sign mistake

(b) $e = u - v \qquad e_i = u_i - v_i$

$$e_i \leftarrow u_i - \left(v_i + \frac{r_i}{a_{ii}}\right) = e_i - \frac{r_i}{a_{ii}}$$

$\therefore \quad e_i \leftarrow e_i - \frac{r_i}{a_{ii}}$ or in vector form

$$\vec{e} \leftarrow \vec{e} - \frac{(\vec{r}, \hat{e}_i)}{(A\hat{e}_i, \hat{e}_i)} \hat{e}_i$$

(c) Show $\|e - s\hat{e}_i\|_A$ is minimized $\forall$ i w/ $\vec{e}$ updated w/ each update of i.

consider $\|e - s\hat{e}_i\|_A^2 = (A(e - s\hat{e}_i), e - s\hat{e}_i)$

$$= (e^T - s\hat{e}_i^T) A^T (e - s\hat{e}_i)$$

$$= (e^T - s\hat{e}_i^T) A (e - s\hat{e}_i)$$

(3.1) If $\frac{N}{2} < k < N$ then plot kth mode
(highly osscilatory modes)

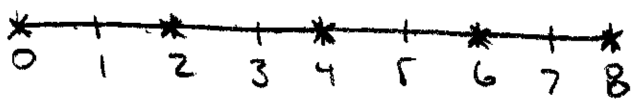
$V_{k_j} = \sin\left(\frac{jk\pi}{N}\right)$ + then sample on Ω^{2h} get $N-k$
or smooth modes on Ω^h.

Plot w/ MMA

on Ω^h $V_k = \text{Table}[\sin(\frac{jk\pi}{N}), \{j, 1, N-1\}]$

on Ω^{2h} V_k appears as $\text{Table}[\sin(\frac{jk\pi}{N}), \{j, 2, N-1, 2\}]$

$N = 8$



Sample at 2, 4, 6

$N = 64$ $\frac{N}{2} = 32$

for $k \cong \frac{N}{2}$ aliasing effect is not so great,
Aliased kth mode is $N-k$.

$\therefore$ $k = N/2$ should go to $N-k = \frac{N}{2}$

when $k \approx N$ effects are much more prominent.
$k \approx N$ mode is aliased to $N-k \approx 0$.

(3.2) (i) Relaxation on $Au = f$ w/ arbi initial guess $\iff$

(ii) relaxation on $Ae = r$ w/ zero initial guess

consider (i) Then $v \leftarrow v + B^{-1}(f - Av)$ is iterated w/ initial guess

$$v^{(n+1)} = v^{(n)} + B^{-1}(f - Av^{(n)})$$ Multiply this scheme by B

$r = f - Av$

$v = A^{-1}(f - r)$

$$Bv^{(n+1)} = Bv^{(n)} + r^{(n)}$$

$$BA^{-1}(\cancel{f} - r^{(n+1)}) = BA^{-1}(\cancel{f} - r^{(n)}) + r^{(n)}$$

$e = u - v$

$v = u - e$

But from $Ae = r$ $\quad A^{-1}r = e$

$\Rightarrow$

$$-e^{(n+1)} = -e^{(n)} + B^{-1}r^{(n)}$$

$$e^{(n+1)} = e^{(n)} - B^{-1}r^{(n)}$$

$B(\cancel{u} - e^{(n+1)})$

$= B(\cancel{u} - e^{(n)}) + r^{(n)}$

$\Rightarrow e^{(n+1)} = e^{(n)} - B^{-1}r^{(n)}$

Claim

This is relaxation on $Ae = r$ w/ zero initial guess

$$e \leftarrow e + B^{-1}(r - Ae)$$

relaxation on $Ae=r$ w/ zero initial guess

$$e \leftarrow e + B^{-1}(r - Ae)$$

$e^{(0)} = 0$ $\quad e^{(1)} = e^{(0)} + B^{-1}(r - Ae^{(0)}) = B^{-1}r$

$$e^{(2)} = e^{(1)} + B^{-1}(r - Ae^{(1)}) = B^{-1}r + B^{-1}(r - AB^{-1}r)$$

$$= 2B^{-1}r - B^{-1}AB^{-1}r$$

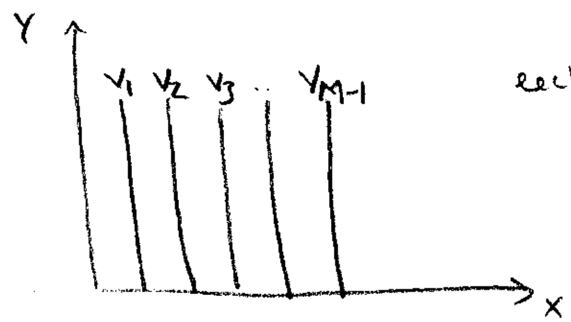
$$e^{(3)} = \cdots$$

(3.3) I_{2h}^{h} has a matrix representation of (in 1D)

$$\frac{1}{2}\begin{bmatrix} 1 & & & & \\ 2 & & & & \\ 1 & 1 & & & \\ & 2 & & & \\ & 1 & 1 & & \\ & & 2 & & \\ & & 1 & \ddots & \\ & & & & 1 \\ & & & & 2 \\ & & & 1 & 1 \\ & & & & 2 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{2h}$$

This is certainly linear & as each column is independent full rank.

For 2D problem



each v_i has $N-1$ components

It is obvious that linear interpolation is linear.

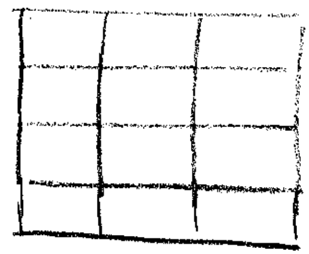
$$\vec{v} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \\ \vdots \\ \bar{v}_{M-1} \end{pmatrix} = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ \vdots \\ v_{1N-1} \\ v_{21} \\ v_{22} \\ \vdots \\ v_{2N-1} \end{pmatrix} \begin{pmatrix} \vdots \\ v_{M-1\,1} \\ v_{M-1\,2} \\ \vdots \\ v_{M-1\,N-1} \end{pmatrix}$$

$$I_{2h}^{h} v^{2h} = v^{h}$$

From pg 3 Briggs $\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{M-1} \end{pmatrix}$

w/ $\vec{v}_i$ = vector of y values at fixed x_i.

Write the 2D operator I_{2h}^{h} in terms of this vector $\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{M-1} \end{pmatrix}$



$$\begin{pmatrix} & & \\ A_{i-1} & A_i & A_{i+1} \\ & & \end{pmatrix} \begin{pmatrix} \vec{v}_{i-1} \\ \vec{v}_i \\ \vec{v}_{i+1} \end{pmatrix} = \begin{pmatrix} \vdots \\ \left.\begin{matrix} v_{2i,2j} \\ v_{2i,2j+1} \end{matrix}\right\} \bar{v}_{2i} \\ \vdots \\ \left.\begin{matrix} v_{2i+1,2j} \\ v_{2i+1,2j+1} \end{matrix}\right\} \bar{v}_{2i+1} \\ \vdots \end{pmatrix}$$

$$I_{2h}^{h} \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{i-1} \\ \vec{v}_i \\ \vec{v}_{i+1} \\ \vdots \\ \vec{v}_{M-1} \end{pmatrix} = \begin{pmatrix} \vec{v}_1^{\,2h} \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \\ \vdots \\ \vec{v}_{2i-2} \\ \vec{v}_{2i-1} \\ \vec{v}_{2i} \\ \vec{v}_{2i+1} \\ \vec{v}_{2i+2} \\ \vec{v}_{2i+3} \\ \vdots \end{pmatrix}$$

Pg. 50 Briggs

(3.4) Rank of I_h^{2h} based on (a) Full weighting (b) injection

For 1D:

Rank = # independent columns or rows & as we have $\frac{N}{2}-1$

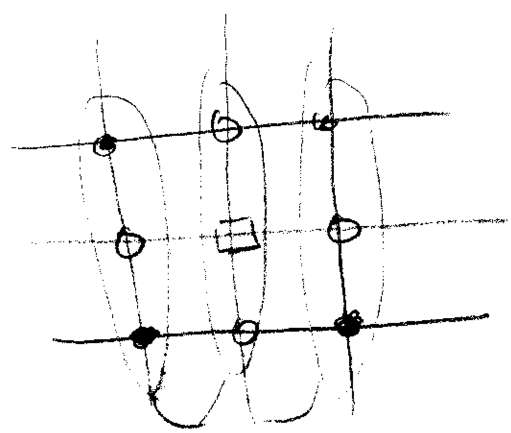
nodes remaining after a I_h^{2h} opperation we see that $\frac{N}{2}-1$ is

the rank for Both Full weighting & injection the injection matrix

just looks different. For $N = 8$

$$I_h^{2h} v^h = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & & 0 & 1 & 0 & & \\ & & & & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

For 2D:



Same vector opperations

$$v_{ij}^{2h} = \frac{1}{16}\left[\left(v^h_{2i-1,2j-1} + v^h_{2i-1,2j+1}\right) + 2v_{2i-1,2j}\right]$$

$$+ \frac{1}{16}\left[2\left(v^h_{2i,2j-1} + v^h_{2i,2j+1}\right) + 4v^h_{2i,2j}\right]$$

$$+\frac{1}{16}\left[\left(v^h_{2i+1,2j-1} + v^h_{2i+1,2j+1}\right) + 2v^h_{2i+1}\right]$$

$4 + 2(4) + 4$

$4 + 4 + 8 = 16$ ✓

$$v^{2h}_{ij} = \frac{1}{16}\left[v^h_{2i-1,2j-1} + 2v^h_{2i-1,2j} + v^h_{2i-1,2j+1} \right]$$

$$+ \frac{1}{8}\left[v^h_{2i,2j-1} + 2v^h_{2i,2j} + v^h_{2i,2j+1} \right]$$

$$+ \frac{1}{16}\left[v^h_{2i+1,2j-1} + 2v^h_{2i+1,2j} + v^h_{2i+1,2j+1} \right]$$

$$= \frac{1}{16}\begin{bmatrix} \\ \\ 1\ 2\ 1\ \cdots\ 2\ 4\ 2\ \cdots\ 1\ 2\ 1 \\ \\ \\ \end{bmatrix}
\begin{bmatrix} \vdots \\ \left.\begin{matrix} v_{2i-1,2j-1} \\ v_{2i-1,2j} \\ v_{2i-1,2j+1} \end{matrix}\right\} \bar{v}_{2i-1} \\ \vdots \\ \left.\begin{matrix} v_{2i,2j-1} \\ v_{2i,2j} \\ v_{2i,2j+1} \end{matrix}\right\} \bar{v}_{2i} \\ \vdots \\ \left.\begin{matrix} v_{2i+1,2j-1} \\ v_{2i+1,2j} \\ v_{2i+1,2j+1} \end{matrix}\right\} \vec{v}_{2i+1} \\ \vdots \end{bmatrix}$$

The pattern 1 2 1 ... 2 4 2 ... 1 2 1 will be repeated up the matrix. There should be $\left(\frac{N}{2}-1\right)\left(\frac{N}{2}-1\right)$ rows in this matrix each w/ the choice row of #'s. Thus the Rank of this matrix is $\left(\frac{N}{2}-1\right)\left(\frac{N}{2}-1\right)$. injection would have a much simpler row all zeros except for one location but the rank is still the same.

Now for choosing the update to the ith error $\bar{e}$ is held fixed $\hat{e}_i$ is held fixed but s is the unknown

$$\|e - s\hat{e}_i\|_A^2 = e^T A e - s e^T A \hat{e}_i - s \hat{e}_i^T A e + s^2 \hat{e}_i^T A \hat{e}_i$$

$$'' \quad '' \quad '' \quad + s^2 a_{ii}$$

MD 80 TWA

Then to minimize the above

$$\frac{d}{ds}\|e - s\hat{e}_i\|_A^2 = -e^T A \hat{e}_i - \hat{e}_i^T A e + 2 s a_{ii} \overset{\text{set}}{=} 0$$

$$-2 e^T A \hat{e}_i + 2 s a_{ii} = 0$$

$$s = \frac{e^T A \hat{e}_i}{a_{ii}}$$

then $\frac{d^2}{ds^2}\|e - s\hat{e}_i\|_A^2 = 2a_{ii} > 0$ {Think positive definite matrices must have positive diagonals}

$\Rightarrow$ location of a minimum.

But remembering $Ae = r$ $\quad e^T A = r^T$ as A is symmetric

$$\Rightarrow \quad s = \frac{r^T \hat{e}_i}{a_{ii}} = \frac{(r, \hat{e}_i)}{(A\hat{e}_i, \hat{e}_i)}$$

(3.5)

(a) As each relaxation step requires the even nodes to be updated & the odds both, I don't see why it would be better or worse to do one before the other. Thus I think that it does not matter

(b) Red Black Gauss-Seidel :
1st update even components $v_{2j} \leftarrow \frac{1}{2}(h^2 f_{2j} + v_{2j-1} + v_{2j+1})$

then odd components $v_{2j+1} \leftarrow \frac{1}{2}(h^2 f_{2j+1} + v_{2j} + v_{2j+2})$

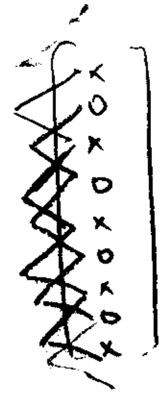
$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix} h^2$$

What are they asking?

Consider the update of the error after doing a red-Black sweep certainly the exact solution would leave

$$\left.\begin{aligned} u_{2j} &\leftarrow \frac{1}{2}(h^2 f_{2j} + u_{2j-1} + u_{2j+1}) \\ u_{2j+1} &\leftarrow \frac{1}{2}(h^2 f_{2j+1} + u_{2j} + u_{2j+2}) \end{aligned}\right\} \text{ subtracting from each gives}$$

$e_j = u_j - v_j$



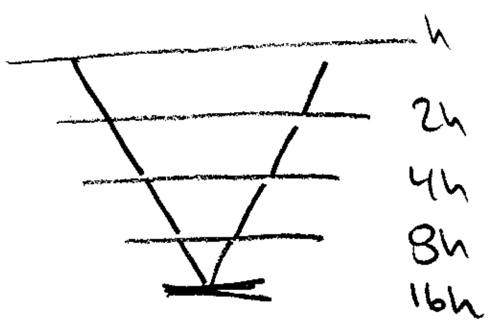
$e_{2j} \leftarrow \frac{1}{2}(e_{2j-1} + e_{2j+1})$

$e_{2j+1} \leftarrow \frac{1}{2}(e_{2j} + e_{2j+2})$

Can this error $\bar{e}$ be represented by $\bar{e}^h = I_{2h}^{h} \tilde{e}^{2h}$

Yes, seems odd components are average of even components & even components are average of odd components

(c) 1 v cycle



$$\text{Storage} = 2N^d\left\{1 + 2^{-d} + 2^{-2d} + 2^{-3d} + \cdots + 2^{-nd}\right\}$$

$$= 2N^d\left\{\sum_{k=0}^{n}(2^{-d})^k\right\} \le 2N^d\left\{\sum_{k=0}^{\infty}(2^{-d})^k\right\}$$

$$= 2N^d\left(\frac{1}{1-2^{-d}}\right)$$

$\underline{d=1}$

$$\text{Storage} \le \frac{2N}{1-2^{-1}} \le \frac{2N}{1/2} \le 2(2N)$$

$d=2$

$$\text{Storage} \le 2N^2\left(\frac{1}{1-2^{-2}}\right) = 2N^2\,\frac{1}{\left(1-\frac{1}{4}\right)} = \left(\frac{4}{3}\right)2N^2$$

– – – –

$$2\left\{1 + 2^{-d} + 2^{-2d} + \cdots + 2^{-nd}\right\}WU \le 2\sum_{k=0}^{\infty}(2^{-d})^k \; WU$$

$$2\,\frac{1}{1-2^{-d}} \; WU$$

$d=1 \quad 2\,\dfrac{W\upsilon}{(1-1/2)} = 4W\upsilon$

$d=2 \quad 2\,\dfrac{W\upsilon}{1-1/4} = 2\,\dfrac{W\upsilon}{3/4} = \dfrac{8}{3}W\upsilon$

$d=3 \quad 2\,\dfrac{W\upsilon}{1-1/8} = \dfrac{16}{7}W\upsilon$

$$\left(\frac{2}{1-2^{-d}}W\upsilon\right)\left(1+2^{-d}+(2^{-d})^2+(2^{-d})^3+\cdots(2^{-d})^n\right)$$

$$\le \left(\frac{2}{1-2^{-d}}W\upsilon\right)\frac{1}{1-2^{-d}} = \frac{2}{(1-2^{-d})^2}W\upsilon$$

$d=1 \quad \dfrac{2}{(1-1/2)^2}W\upsilon = \dfrac{2}{(1/4)}W\upsilon = 8W\upsilon$

$d=2 \quad \dfrac{2W\upsilon}{(1-1/4)^2} = \dfrac{2W\upsilon}{9/16} = \dfrac{32}{9}W\upsilon \cong 3.5W\upsilon \approx \dfrac{7}{2}W\upsilon$

$d=3 \quad \dfrac{2}{(1-1/8)^2}W\upsilon = \dfrac{2W\upsilon}{(7/8)^2} = \dfrac{64.2}{49}W\upsilon = \dfrac{128}{49}W\upsilon. \cong 2.61W\upsilon$

$\cong 2.5W\upsilon$

$\|E^h\| < \frac{\epsilon}{2}$ will be true if

we require $kh^p \le \frac{\epsilon}{2}$

$h^p \le \frac{\epsilon}{2k}$

$h \le \left(\frac{\epsilon}{2k}\right)^{1/p} = h^*$

# of V-cycles

$r^\nu = O(N^{-p})$

$\nu \log r = O(-p \log N)$ $\quad r < 1 \therefore \log r < 0$

$\nu = O\left(\frac{p \log N}{|\log r|}\right) = O(\log N)$

Single V cycle costs $O(N^d)$

$\|e^{2h}\| \le k(2h)^p = 2^p k h^p$

$\|e^h\| \le 2^{-p} \|e^{2h}\| \le kh^p$ ✓.

$\therefore$ reduce Algebraic error by factor 2^{-p}

req $r^\nu = 2^{-p}$

$\nu = \frac{-p \log 2}{-|\log r|} = \frac{p \log 2}{|\log r|} = O(1)$

in 15 WU error reduced by 10^{-3}. If we assume a geometric decrease in error

$$r^{15} = 10^{-3}$$

$$15 \log r = -3 \log 10$$

$$r = (10^{-3})^{1/15}$$

$$\log r = -\frac{1}{5} \log 10 \Rightarrow r = \exp\left[-\frac{1}{5} \log 10\right]$$

$$= .63$$

when entering 10^{p} into the calculator do it as 1.E p

each work unit then produces a .63 factor decrease in the error.

Get about 3 decimal digits in 15 WU. How get .2 decimal digits per work unit?

~~1 ... = .85 dlr~~
~~1 dollar = 1.17~~

$N=16: \|e\|_\infty = .2 \cdot 10^{-3}$

doubling # of nodes = ½ size of cell
error should go down by ¼

$N=32: \|e\|_\infty = .49 \cdot 10^{-4}$

should be ~ $.2 \cdot 10^{-3}/4 = 5 \cdot 10^{-5} = .5 \cdot 10^{-4}$ yes ✓

$N=64: \|e\|_\infty = .11 \cdot 10^{-4}$

should be $.49 \cdot 10^{-4}/4 = 1.2 \cdot 10^{-5}$
$= .12 \cdot 10^{-4}$ yes ✓

Question does the $\|\cdot\|_h$ error measure decrease as the $\|\cdot\|_\infty$ error measure?

$N=16: \|e\|_h = .41 \cdot 10^{-3}$

÷ by 4 gives $= 1.02 \cdot 10^{-4}$ yes

$N=32: \|e\|_h = .15 \cdot 10^{-3}$

$N=64: \|e\|_h = .47 \cdot 10^{-4}$ ÷ by 4 gives $3.75 \cdot 10^{-5}$ yes

Factor column is obtained by computing λ of

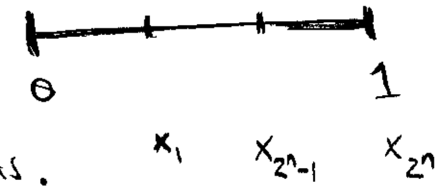
$$\|r\|_h^{(n+1)} = \lambda \|r\|_h^{(n)}$$ i.e. how much the residual error shrinks at each iteration

or might be factor of $\|e\|_\infty$

$$\|e\|_\infty^{(n+1)} = \lambda \|e\|_\infty^{(n)}$$

(4.1) Finest grid has $N-1 = 2^n - 1$ interior points

$h = \frac{1}{N}$ spacing on Ω^h



level ℓ has spacing $2^{\ell-1}h$ between grid points.

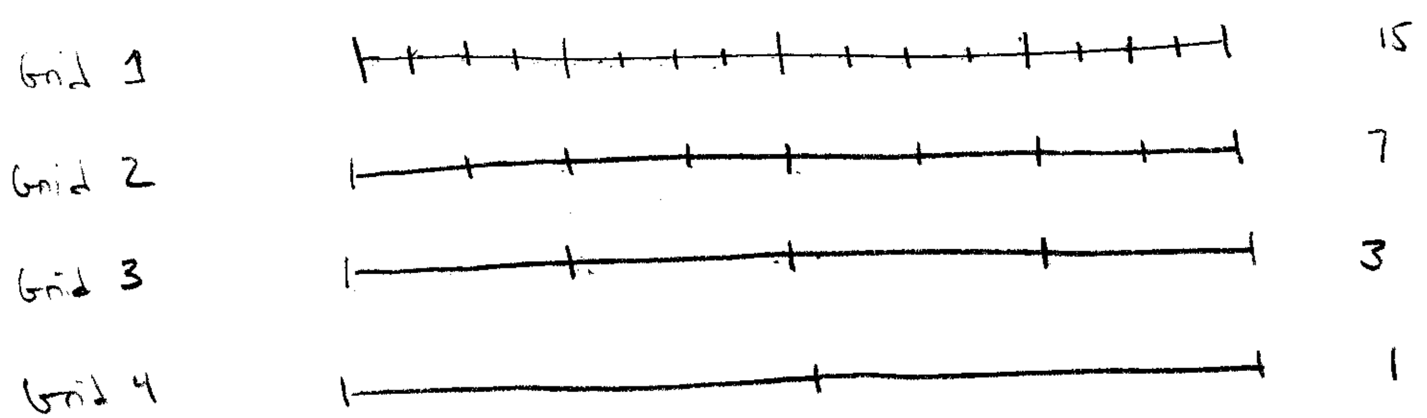
Then level ℓ has $\frac{1}{2^{\ell-1}h}$ many cells from 0 to 1

$= \frac{N}{2^{\ell-1}} = \frac{2^n}{2^{\ell-1}} = 2^{n-\ell+1}$ cells from 0 to 1. Then the # of nodes total (internal + external) is $2^{n-\ell+1}+1$, Then # internal nodes is

$$2^{n-\ell+1}+1-2 = 2^{n-\ell+1}-1$$

Lets do an example to be sure this is correct: $N=16=2^4$ = # of cells in 1D grid.

internal nodes



$n=4$ in formulation above +

ℓ	$2^{5-\ell}-1$
1	15
2	7
3	3
4	1

correct !!

. Thus the data structures needed require. Given n as input (indirect measure of # points of grid) To just store internal points

1st grid requires a vector of size $2^n - 1$

2nd grid requires a vector of size $2^{n-1} - 1$

3rd $2^{n-2} - 1$

.
.
.

$n-1$ th $2^{n-(n-1)+1} - 1 = 2^2 - 1 = 3$

n th 1

$$\vec{v} : (v_1 \, v_2 \cdots v_{p_1} \; v_1 \, v_2 \cdots v_{p_2} \; v_1 \, v_2 \cdots v_{p_3} \cdots)$$

If we want to store the boundaries also then

1st grid requires a vector of size $2^n + 1$

2nd grid requires a vector of size $2^{n-1} + 1$

3rd $2^{n-2} + 1$

⋮

ℓth $2^{n-\ell+1} + 1$

⋮

$n-1$ th 5

nth 3

Storing the components of all grids in 1 array in the following order:

$$\underbrace{v_1, v_2, v_3}_{\text{nth grid}}, \underbrace{v_4, v_5, v_6, v_7, v_8}_{\text{n-1 th grid}}, \cdots, \underbrace{\qquad\qquad}_{\text{1st grid}}$$

total # of storage elements required for all grids, (Arrays of this size are needed for both $\vec{v}$ & $\vec{f}$.) is.

$$\sum_{l=1}^{n} (2^{n-l+1}+1) = n + 2^{n+1} \sum_{l=1}^{n} 2^{-l} = n + 2^{n+1} \sum_{l=1}^{n} (\tfrac{1}{2})^{l}$$

$$= n + 2^{n+1} \frac{\left[(\frac{1}{2})^{n+1} - \frac{1}{2}\right]}{(1/2 - 1)}$$

$$\Delta a^n = a^{n+1} - a^n = a^n(a-1)$$

$$a^n = \frac{\Delta a^n}{a-1}$$

$$\therefore \sum_n a^n = \frac{1}{(a-1)} \sum \Delta a^n = \frac{1}{(a-1)} (a^{N+1} - a^0)$$

$$= n + 2^{n+1} \frac{\left[(\frac{1}{2})^{n} - 1\right]}{-1}$$

$$= n + 2^{n+1}(1 - (1/2)^n)$$

$$= n + 2^{n+1} - 2$$

$$= \underline{\underline{2^{n+1} + n - 2}}$$

We will need the following sum $\quad 1 \le \ell \le n$

$$S_\ell = \sum_{k=\ell}^{n} (2^{n-k+1} + 1) = (n+1-\ell) + 2^{n+1} \sum_{k=\ell}^{n} (2^{-1})^k$$

$$\sum_{k=k_1}^{k_2} a^k = \frac{(a^{k_2+1} - a^{k_1})}{(a-1)}$$

$$= (n+1-\ell) + 2^{n+1} \left[\frac{(2^{-1})^{n+1} - (2^{-1})^{\ell}}{(2^{-1} - 1)} \right]$$

$$= (n+1-\ell) + 2^{n+1}(2^{-\ell} - 2^{-n-1})\,2$$

$$= (n+1-\ell) + 2^{n+2-\ell} - 2$$

$$= n-1-\ell + 2^{n+2-\ell}$$

Check

$S_n = -1 + 2^2 = 3 \quad$ # elements in grid n

$S_{n-1} = 2^{2+1} = 8 \quad$ # elements in grid n + $n-1$.

$\vdots$

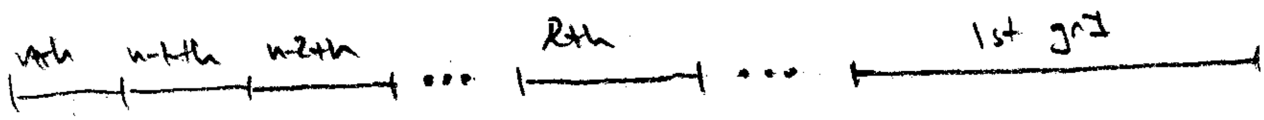
nth grid has $2^{n-l+1}+1\Big|_{l=n} = 3$ locations

n-1 th grid has $2^{n-l+1}+1\Big|_{l=n-1} = 5$.

$\vdots$

1 th grid has $2^{n}+1$ locations ✓

← grids increase



lth grid starts at

$$\sum_{k=l+1}^{n}\left(2^{n-k+1}+1\right)+1 = n-\cancel{1}-(l+1)+2^{n+2-l-1}\cancel{+1}$$

$$= n-l-1+2^{n+1-l}$$

lth grid stops at

$$\sum_{k=l}^{n}\left(2^{n-k+1}+1\right) = n-1-l+2^{n+2-l}$$

· Check:

Grid	Start	Stop
n	$-1+2=1$ ✓	$-1+2^2=3$ ✓
$n-1$	$0+2^2=4$ ✓	$0+2^3=8$ ✓
$\vdots$		
1	$n-2+2^n$	$n-2+2^{n+1}$

Check ℓth grid stop $-$ ℓth grid start $+1$

$= \#$ elements in the ℓth grid

$$= \cancel{n}-\cancel{1}-\cancel{\ell}+2^{n+2-\ell} - (\cancel{n}-\cancel{\ell}+\cancel{1}+2^{n+1-\ell}) + 1$$

$$= 2^{n+1-\ell}(2-1)+1$$

$$= 2^{n+1-\ell}+1$$

Summary -

Store multigrid vectors as (including boundary values) one array
"|" denotes a new grids

$$\underbrace{v_1 v_2 v_3}_{n\text{th grid}} \mid \underbrace{v_4\, v_5\, v_6\, v_7\, v_8}_{n-1\text{ th grid}} \mid \cdots \mid \underbrace{\qquad\qquad}_{1\text{st finest grid}}$$

w/ $n \ni N = \#$ of cells $= 2^n$

Then total size of this array is $2^{n+1} + n - 2$

& indices for the lth grid $1 \le l \le n$ are

finest grid   coarsest grid

$$Start_l = n - l - 1 + 2^{n+1-l}$$

{Matlab functions}

$$End_l = n - l - 1 + 2^{n+2-l}$$

Level l has $2^{n+1-l} + 1$ values in it.

pg 65 Briggs

06-18-01 ①

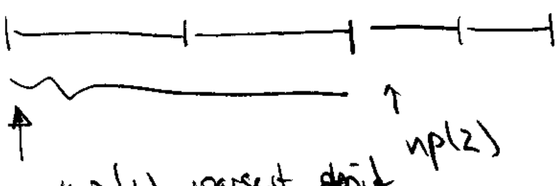
Following requirements in Briggs, 1 Array.

For Node centered Multi-Grid to

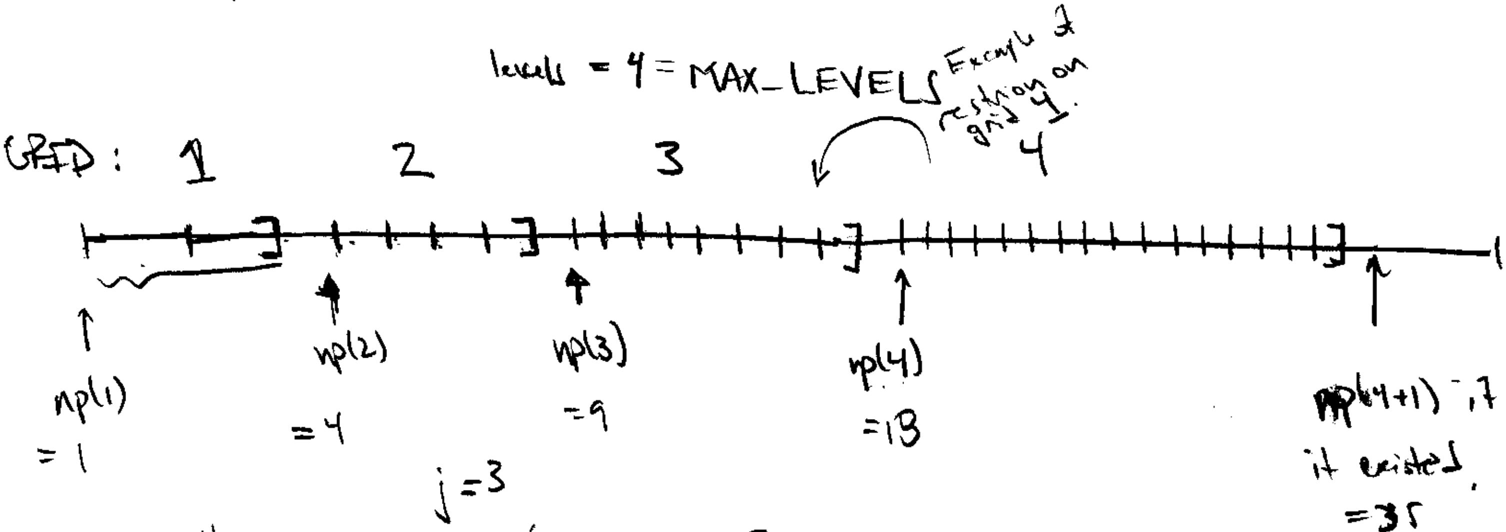
$$2^{MAX\_LEVEL} + 1 = \#$$

once the user picks the depth of the level, this then determines the # of nodes at this current level.

As the level # increases the grid is refined. Restriction takes [illegible] from [illegible] to a coarser grid



levels = 4 = MAX_LEVELS



nc = 3

$j = 3$

$j = 2(3) + -1 = 5$

$np(3) = 4 + 5 = 9$ ✓

$j = 2(5) - 1 = 9$

$np(4) = 18$ ✓

$j = 2(9) - 1 = 17$

$np(5) = 18 + 17 = 35$ ✓

pg 65 Briggs

(4.2)

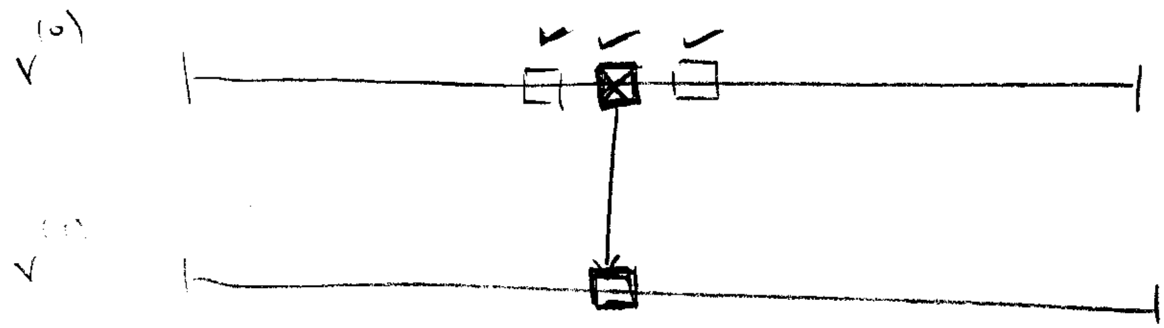
(a) A weighted Jacobi sweep represents iterating

$$v^{(n+1)} = \left[(1-\omega)I + \omega P_J\right] v^{(n)} + \omega D^{-1} f \qquad \nu \text{ times.}$$

Following on pg 10. For our test problem,

$$v_j^* = \frac{1}{2}\left(v_{j-1}^{(0)} + v_{j+1}^{(0)} + h^2 f_j\right) \qquad 1 \le j \le N-1$$

Then $v_j^{(1)} = (1-\omega) v_j^{(0)} + \omega v_j^*$



Might be good idea to have an array w/ # elements in each grid at the level l.

If there are 4 levels the ~~size of the~~ array np is

$np(1) = 1$ $+3$
$np(2) = 4$ $+5$
$np(3) = 9$ $+9$
$np(4) = 18$

(4.3) derivation of red-Black Gauss-Seidel

iterate $v \leftarrow P_G v + (D-L)^{-1} f$ *

w/ $P_G = (D-L)^{-1} U$ & $Au = f$

w/ $A = D - L - U$.

But w/ red-Black we 1st update the even components then the odd components in the iteration *.

From $A = D - L - U$ $Au = f$

$D^{-1} A = I - D^{-1} L - D^{-1} U$

$Au = f$ multiply by D^{-1}

$D^{-1}(D - L - U) u = D^{-1} f$

$(I - D^{-1} L - D^{-1} U) u = D^{-1} f$

$(I - D^{-1} L) u = D^{-1} U u + D^{-1} f$

$u = (I - D^{-1} L)^{-1} (D^{-1} U) u + (I - D^{-1} L)^{-1} D^{-1} f$

Iterate

$u^{(n+1)} = (I - D^{-1} L)^{-1} (D^{-1} U) u^{(n)} + (I - D^{-1} L)^{-1} D^{-1} f$

$= (I - D^{-1} L)^{-1} \left[(D^{-1} U) u^{(n)} + D^{-1} f \right]$

If we multiply A by D^{-1} immediately we have the matrices $D^{-1}L$ + $D^{-1}U$ + $D^{-1}f$.

Procedure 1) scale A to get $D^{-1}L$ + $D^{-1}U$ + $D^{-1}f$

2) Multiply $U^{(n)}$ by $D^{-1}U$

3) Add vector $D^{-1}f$

4) Multiply by $(I - D^{-1}L)^{-1}$ or solve

$$(I - D^{-1}L)U^{(n+1)} = \underbrace{D^{-1}U\,U^{(n)}}_{\text{computed already + stored in}} + D^{-1}f$$

$$L = \begin{pmatrix} 0 & & & & 0 \\ l_{21} & 0 & & & \\ l_{31} & \ddots & 0 & & \\ \vdots & & \ddots & \ddots & \\ & & & 0 & 0 \end{pmatrix}$$

$$D^{-1}L = \begin{pmatrix} 0 & & & & 0 \\ l_{21}/d_2 & 0 & & & \\ l_{31}/d_3 & l_{32}/d_3 & 0 & & \\ \vdots & & & & \\ l_{n1}/d_n & l_{n2}/d_n & \cdots & \frac{l_{n,n-1}}{d_n} & 0 \end{pmatrix}$$

lower diagonal matrix w/ zeros on the diagonal.

$$I - D^{-1}L = \begin{pmatrix} 1 & & & & 0 \\ \frac{l_{21}}{d_2} & 1 & & & \\ \frac{l_{31}}{d_3} & \frac{l_{32}}{d_3} & 1 & & \\ \vdots & & & & \\ l_{n1}/d_n & l_{n2}/d_n & \cdots & \frac{d_{n,n-1}}{d_n} & 1 \end{pmatrix}$$

Note $I - D^{-1}L$ is unit lower diagonal.

Thus this is a simple Back substitution to compute $U^{(n+1)}$

Back substitution routine would look like for regular gauss-seidel

$$v_j = -\sum_{i=1}^{j-1} L_{ji} v_i + f_j$$

$$= f_j - \sum_{i=1}^{j-1} L_{ji} v_i$$

$$\begin{pmatrix} L_{j1} L_{j2} \cdots L_{jj-1} \, 1 & \end{pmatrix} \begin{pmatrix} v_j \end{pmatrix} = \begin{pmatrix} f_j \end{pmatrix}$$

Thus to code this one simply

passing $A = -L$

```
do i=1, N
  if (i=1)
     V(i) = F(i)
  else
```

$$A = (D-L)-U$$

$$D^{-1}A = \underbrace{I - D^{-1}L}_{\text{stored in vector/matrix called } A} - \underbrace{D^{-1}U}_{\text{stored w/ minus sign in vector/matrix called } U}$$

Total Algorithm from point of view of code written is

$$A = D + L + U$$

$$D^{-1}A = \underbrace{I + D^{-1}L}_{\text{stays on this side}} + \underbrace{D^{-1}U}_{\text{goes on other side take negative of it, as is done}}$$

$$Au = f$$

write $A = D + L + U$ w/ D diagonal, L lower triangle, U upper triangle

$$(D+L+U)u = f$$

$$D^{-1} \rightarrow$$

$$(I + D^{-1}L + D^{-1}U)u = D^{-1}f$$

$$(I + D^{-1}L)u = -D^{-1}Uu + D^{-1}f \quad \text{iterate then}$$

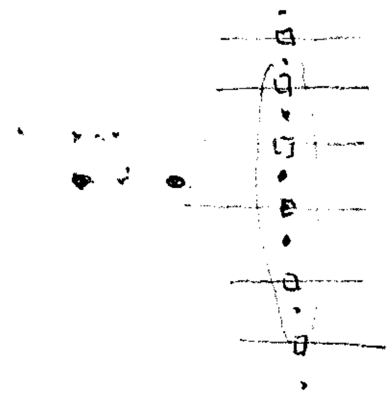
$$\underbrace{(I + D^{-1}L)}_{\text{stored in matrix/vector } A} u^{(n+1)} = -\underbrace{D^{-1}U}_{\text{stored in matrix vector } U} u^{(n)} + \underbrace{D^{-1}f}_{\text{stored in vector } B}$$

06-03-01

To update in terms of Red-Black want to 1st update even components using

$$L_{ii} V_i = f_i - \sum_{j=1}^{i-1} L_{ij} v_j$$

i even. then do the same for i odd.



$$L_{2i2i} V_{2i} = f_{2i} - \sum_{j=1}^{2i-1} L_{2ij} v_j$$

$$L_{2i+1,2i+1} V_{2i+1} = f_{2i+1} - \sum_{j=1}^{2i} L_{2i+1,j} v_j$$

If input matrix is of size $n=5$

looping over the odd indices 1,3,5

even indices 2,4.

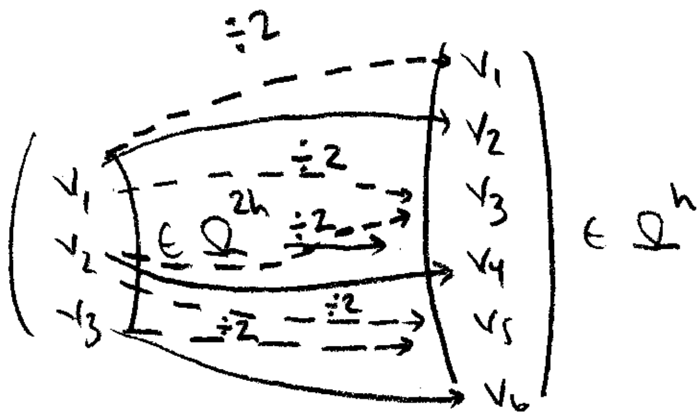
$n=6$

odd indices 1,3,5

even indices 2,4,6

Idea is this given u^{2h} $I_{2h}^{h} u^{2h}$ obtains by ~~extrapolation~~ extrapolation the vector u^h. $A^h u^h$ would then correspond physically to a second derivative operator acting on u^h. But at the extrapolated points we have piecewise linear fns of which the 2nd derivative is always zero. This gives an explanation for Fig 22 (c).

$$I_{2h}^{h} v^{2h} = \frac{1}{2}\begin{bmatrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \\ & & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$



$$v_{2j}^{h} = v_j^{2h}$$

$$v_{2j+1}^{h} = \frac{1}{2}\left(v_j^{2h} + v_{j+1}^{2h}\right)$$

Thus given $$\hat{e}_i^{2h} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$I_{2h}^{h} \hat{e}_i^{2h} = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ 2i-1 \\ 2i \\ 2i+1 \\ \\ \\ \\ \end{array}$$

} Might be the ith, i-1, i+1th element of a vector in Ω^h, but this is the locations in $1, \dots, 2n$.

Then $A^h I_{2h}^h \hat{e}_i^{2h} =$

$$A^h = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} \frac{1}{h^2}$$

$$A^h I_{2h}^h \hat{e}_i^{2h} = \frac{1}{h^2} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \\ 2(\frac{1}{2}) - 1 \\ -\frac{1}{2} + 2 - \frac{1}{2} \\ -1 + 2(\frac{1}{2}) \\ -\frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{h^2} \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \\ -\frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ 2i-2 \\ 2i-1 \\ 2i \\ 2i+1 \\ 2i+2 \\ \\ \\ \\ \end{array}$$

Then $I_h^{2h} A^h I_{2h}^h \hat{e}_i^{2h} = \frac{1}{h^2} \begin{pmatrix} \frac{1}{4}(0 + -\frac{1}{2}(2) + 0) \\ \frac{1}{4}(0 + 2(1) + 0) \\ \frac{1}{4}(0 + 2(-\frac{1}{2}) + 0) \end{pmatrix} \begin{array}{l} i-1 \\ i \\ i+1 \end{array} = \frac{1}{h^2} \begin{pmatrix} -\frac{1}{4} \\ \frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}$

using full weighting to compute values gives.

$$\underbrace{I_h^{2h} A^h I_{2h}^h}_{A^{2h}} \hat{e}_i^{2h} = \begin{pmatrix} \frac{-1}{4h^2} \\ \frac{1}{2h^2} \\ \frac{-1}{4h^2} \end{pmatrix} = \frac{1}{(2h)^2} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = \text{ith column of } A^{2h}.$$

How know its ith row of A^{2h}?

$$(I_h^{2h} A^h I_{2h}^h)^T = (I_{2h}^h)^T (A^h)^T (I_h^{2h})^T$$

if. (*)

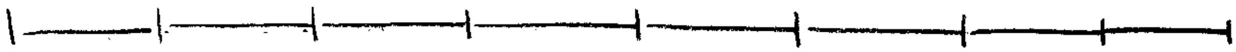
$$= \cancel{\tfrac{1}{c}} I_h^{2h} A^h \cancel{c} I_{2h}^h$$

$$= I_h^{2h} A^h I_{2h}^h \quad \text{same}$$

know $I_h^{2h} = c(I_{2h}^h)^T$ taking transpose of both sides

$$I_{2h}^h = \frac{1}{c}(I_h^{2h})^T \quad \rightarrow \quad (I_{2h}^h)^T = \frac{1}{c} I_h^{2h}$$

$$(I_h^{2h})^T = c I_{2h}^h$$



8 cells
9 nodes
7 internal nodes

coarsen this grid $\Rightarrow$ 4 cells, 5 nodes total, 3 internal nodes

$$3 = \frac{8}{2} - 1$$

size of space we are leaving from in I_h^{2h} is $N-1$

As we have a row of $\frac{1}{4}(1 \quad 2 \quad 1)$ $\forall$ <u>even</u> index of

$v^h \Rightarrow O(\frac{N}{2})$ rank of I_h^{2h}. Specifically given N #'s

from $1, 2, 3, \ldots, N$ there are how many even #s.

$1, \underline{2}, 3, \underline{4}, 5, \underline{6}$ $\quad \frac{6}{2} = 3$

N even $= \frac{N}{2}$

$1, \underline{2}, 3, \underline{4}, 5$

N odd $\Rightarrow \frac{N}{2} - 1$

really want to count # of pairs of 3's $= \dfrac{N - \mathrm{Mod}(N, 3)}{3}$

If I know Rank $I_h^{2h} = \frac{N}{2} - 1$ then nullity must be

$$\frac{N}{2} - 1 + \underline{X} = N - 1 = \text{dimension of space leaving.}$$

$$\Rightarrow \underline{X} = \frac{N}{2} = \text{nullity.}$$

This is why we must relax again after we solve on the coarser grid. Solving on the coarse grid is good but when we project back to the fine grid using I_{2h}^{h} we can generate fine modes also. These fine modes then need to be relaxed away.

$$I_{2h}^{h} w_k^{2h} = \cos^2\left(\frac{k\pi}{2N}\right) w_k^h - \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h \qquad k' = N-k$$

For N large $1 \le k \ll \frac{N}{2}$ $\quad \cos(\theta) \approx 1 - \frac{\theta^2}{2} + O(\theta^4) \quad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$

$$\sin\theta \approx \theta - \frac{\theta^3}{3!} + O(\theta^5)$$

$$I_{2h}^{h} w_k^{2h} = \left(1 - O\left(\frac{k^2}{N^2}\right)\right)^2 w_k^h + O\left(\frac{k^2}{N^2}\right) w_{k'}^h$$

$$= \left(1 - O\left(\frac{k^2}{N^2}\right)\right) w_k^h + O\left(\frac{k^2}{N^2}\right) w_{k'}^h$$

$$f^{2h} \leftarrow I_h^{2h}(f^h - A^h P^\nu v^h)$$

$$v^{2h} = (A^{2h})^{-1} I_h^{2h}(f^h - A^h P^\nu v^h)$$

$$v^h \leftarrow P^\nu v^h + I_{2h}^h (A^{2h})^{-1} I_h^{2h}(f^h - A^h P^\nu v^h)$$

Putting u^h in this right hand side we must remain unchanged!

$$u^h = P^\nu u^h + I_{2h}^h (A^{2h})^{-1} I_h^{2h}(f^h - A^h P^\nu u^h)$$

$$e^h = u^h - v^h$$

$$e^h \leftarrow P^\nu(u^h - v^h) + I_{2h}^h (A^{2h})^{-1} I_h^{2h}(-A^h P^\nu u^h + A^h P^\nu v^h)$$

$$= P^\nu e^h - I_{2h}^h (A^{2h})^{-1} I_h^{2h} P^\nu e^h$$

$$= \left[I - I_{2h}^h (A^{2h})^{-1} I_h^{2h}\right] P^\nu e^h \qquad \text{eq } 5.2$$

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$k \ll N$

$$CG\omega_k = O\left(\frac{k^2}{N^2}\right)\omega_k + O\left(\frac{k^2}{N^2}\right)\omega_{k'}$$

$$CG\omega_{k'} = \left(1 - O\left(\frac{k^2}{N^2}\right)\right)\omega_k + \left(1 - O\left(\frac{k^2}{N^2}\right)\right)\omega_{k'}$$

In the diagram Fig 29 (c), After we have relaxed out all the fine components one obtains a point on $(R,0)$ in the L,H plane then this point can be represented in the $R(I_{2h}^{h}) - N(I_h^{2h}A^h)$ plane as

$$s^h + t^h$$

s^h: element of $R(I_{2h}^{h})$

t^h: element of $N(I_h^{2h}A^h)$

Since $CG\,s^h = 0$ <u>one</u> application of CG

& $CG\,t^h = t^h$

takes us to the point e^h on the $N(I_h^{2h}A^h)$ axis.

The diagram in Fig 29 (c) seems to indicate that several iterations are required.

(5.1) Show $I_h^{2h} w_k^h = \cos^2\left(\frac{k\pi}{2N}\right) w_k^{2h}$

$w_{k,j}^h = \sin\left(\frac{jk\pi}{N}\right)$ $\quad 1 \le k \le \frac{N}{2}$ $\quad I_h^{2h}$ is full weighting.

consider element j of $I_h^{2h} w_k^h$

I_h^{2h} for full weighting looks like $\longrightarrow$ $\frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & \\ & & 1 & 2 & 1 & & & \\ & & & & 1 & 2 & 1 & \\ & & & & & & & \ddots \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_7 \end{bmatrix}$ $\vdots$

element j corresponds to opperating

$$v_j^{2h} = \frac{1}{4}\left(v_{2j-1}^h + 2v_{2j}^h + v_{2j+1}^h\right)$$

$$\Rightarrow \left(I_h^{2h} w_k^h\right)_j = \frac{1}{4}\left(\sin\left(\frac{(2j-1)k\pi}{N}\right) + 2\sin\left(\frac{2jk\pi}{N}\right) + \sin\left(\frac{(2j+1)k\pi}{N}\right)\right)$$

$$= \frac{1}{4}\left[\sin\left(\frac{2jk\pi}{N} - \frac{k\pi}{N}\right) + 2\sin\left(\frac{2jk\pi}{N}\right) + \sin\left(\frac{2jk\pi}{N} + \frac{k\pi}{N}\right)\right]$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$$

$$= \frac{1}{4}\left[\sin\left(\frac{2jk\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right) - \cancel{\sin\left(\frac{k\pi}{N}\right)\cos\left(\frac{2jk\pi}{N}\right)} + 2\sin\left(\frac{2jk\pi}{N}\right) + \sin\left(\frac{2jk\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right) + \cancel{\sin\left(\frac{k\pi}{N}\right)\cos\left(\frac{2jk\pi}{N}\right)}\right]$$

....

$$= \frac{1}{4} \sin\left(\frac{jk\pi}{N/2}\right)\left[2\cos\left(\frac{k\pi}{N}\right) + 2\right]$$

$$= \frac{1}{2} \sin\left(\frac{jk\pi}{N/2}\right)\left[\cos\left(\frac{k\pi}{N}\right) + 1\right] \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$= \cos^2\left(\frac{k\pi}{2N}\right) \sin\left(\frac{jk\pi}{N/2}\right)$$

$$= \cos^2\left(\frac{k\pi}{2N}\right) w_j^{2h}$$

5.2 If $k' = N - k \quad 1 \le k < \frac{N}{2}$

Then from problem 5.1 all the algebra is the same except where we make the last assignment, i.e

$$(I_h^{2h} w_{k'}^h)_j = \cos^2\left(\frac{k'\pi}{2N}\right)\sin\left(\frac{jk'\pi}{N/2}\right)$$ But if $k' > N/2$

$$\sin\left(\frac{jk'\pi}{N/2}\right) \neq w_{k',j}^{2h}$$ from $k' = N-k$ we get

$$\sin\left(\frac{jk'\pi}{N/2}\right) = \sin\left(\frac{j(N-k)\pi}{N/2}\right)$$

$$= \sin\left(\frac{jN\pi}{N/2} - \frac{jk\pi}{N/2}\right)$$

$$= \cancel{\sin(2j\pi)}^{\,0}\cos\left(\frac{jk\pi}{N/2}\right) - \cos(2\pi j)\sin\left(\frac{jk\pi}{N/2}\right)$$

$$= -\sin\left(\frac{jk\pi}{N/2}\right) = -w_{k,j}^{2h}$$

& $$\cos\left(\frac{k'\pi}{2N}\right) = \cos\left(\frac{N\pi}{2N} - \frac{k\pi}{2N}\right)$$

$$= \cancel{\cos\left(\frac{\pi}{2}\right)}^{\,0}\cos\left(\frac{k\pi}{2N}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{k\pi}{2N}\right) = \sin\left(\frac{k\pi}{2N}\right)$$

$$\therefore \quad (I_h^{2h} w_{k'}^h)_j = -\sin^2\left(\frac{k\pi}{2N}\right) w_{k,j}^{2h}$$

What happens to $w_{\frac{N}{2}}^h$ should be the same thing in either formula from prob 5.1 or 5.2., thus I hypothesis that

$$\therefore \quad I_h^{2h} w_{\frac{N}{2}}^h = -\sin^2\left(\frac{N/2\,\pi}{2N}\right) w_{\frac{N}{2}}^{2h}$$

$$= -\sin^2\left(\frac{\pi}{4}\right) w_{\frac{N}{2}}^{2h} = -\frac{1}{2} w_{\frac{N}{2}}^{2h}$$

From Formula from problem 5.1

$$I_h^{2h} w_{\frac{N}{2}}^{h} = \cos^2\left(\frac{\pi}{4}\right) w_{\frac{N}{2}}^{2h} = \frac{1}{2} w_{\frac{N}{2}}^{2h}$$

Thus these answers are different lets investigate further what happens under 111 weighting to the $\frac{N}{2}$th mode

$$\left(I_h^{2h} w_{\frac{N}{2}}^{h}\right)_j = \left(\frac{1}{2}\right) \sin\left(\frac{j\pi \, N/2}{N/2}\right) \equiv 0$$

$\frac{N}{2}$th mode is an element of the nullity of I_h^{2h}.

(5.3) $\{w_k^h, w_{k'}^h\}$ $k' = N-k$

$$w_{k',j}^h = \sin\left(\frac{jk'\pi}{N}\right) = \sin\left(\frac{jN\pi}{N} - \frac{jk\pi}{N}\right)$$

$$= \cancel{\sin(j\pi)}^{0}\cos\left(\frac{jk\pi}{N}\right) - \cos(j\pi)\sin\left(\frac{jk\pi}{N}\right)$$

$$= -(-1)^j \sin\left(\frac{jk\pi}{N}\right) = (-1)^{j+1}\sin\left(\frac{jk\pi}{N}\right) = (-1)^{j+1} w_{k,j}^h$$

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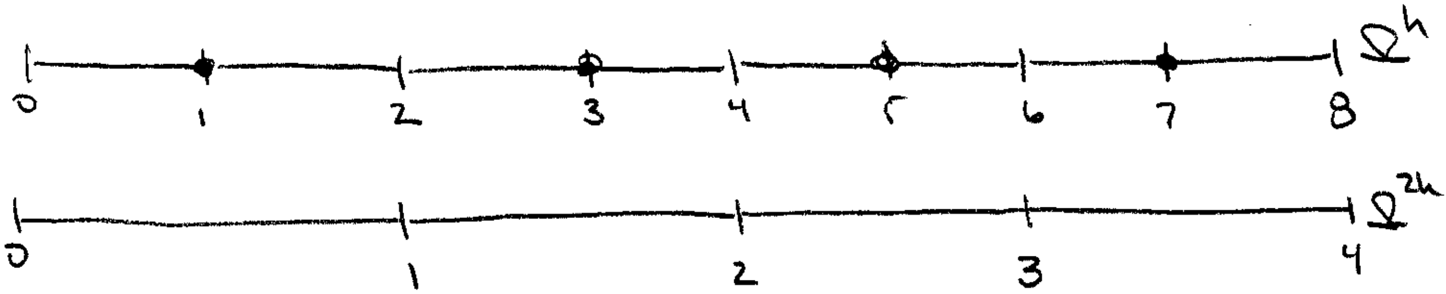
(5.4) $n_i = A^h \hat{e}_i^h$ $\quad$ i odd.

To show that $\{n_i, i=1,\dots,\frac{n}{2}\}$ is a basis for Nullspace of I_h^{2h}

we must show that each $n_i \in Null\{I_h^{2h}\}$ & each

n_i is linearly independent.

Example:



1st claim $\{n_i\}$ is linearly independent.

Assume

$$c_1 n_1 + c_2 n_2 + \cdots + c_{\frac{n}{2}} n_{\frac{n}{2}} = 0 \qquad \forall \text{ not all } c_i = 0$$

$$\Rightarrow c_1 A^h \hat{e}_1 + c_2 A^h \hat{e}_3 + \cdots \; c_{\frac{n}{2}} A^h \underbrace{\hat{e}_{n-1}} = 0$$

last odd index of Ω^h grid $\quad$ n even $\quad n-1$
n odd $\quad n-2$

$$\Rightarrow A^h\left[c_1 \hat{e}_1 + c_2 \hat{e}_3 + \cdots \; c_{\frac{n}{2}} \hat{e}_{n-1}\right] = 0$$ But A^h is

invertible thus

$$c_1 \hat{e}_1 + c_2 \hat{e}_3 + \cdots + c_{\frac{n}{2}} \hat{e}_{n-1} = 0$$

Hence since $\{\hat{e}^h_i\}$ for i odd is linearly independent $\Rightarrow$ all c's $\equiv 0$

$\rightarrow\leftarrow$ $\therefore$ The set $\{n_i\}$ is linearly independent.

Show each $n_i \in \text{Null}\{I_h^{2h}\}$

$$I_h^{2h} n_i = I_h^{2h} A^h \hat{e}_i^h$$ Here I assume that A^h is of the tridiagonal form show at the beginning (can I show this without this condition?)

$$A^h = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & -1 & \\ & & & & \ddots & \end{pmatrix}$$

& I_h^{2h} full weighting $= \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & \\ & & & & 1 & 2 & 1 \\ & & & & & & \ddots \end{bmatrix} \in \mathbb{R}^{(\frac{n}{2}-1)\times(n-1)}$

$A^h \in \mathbb{R}^{(n-1)\times(n-1)}$

product is defined
& is in $(\frac{n}{2}-1)\times(n-1)$

Then product of

$$I_h^{2h} A^h = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & & \\ & & 1 & 2 & 1 & & & & \\ & & & & 1 & 2 & 1 & & \\ & & & & & & \ddots & & \\ & & & & & & 1 & 2 & 1 \\ & & & & & & & & 1 \;\; 2 \;\; 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & & \ddots \\ & & -1 & 2 \\ & & & -1 \;\; 2 \end{bmatrix}$$

Then the

$$= \frac{1}{4}\left(\begin{array}{ccccc} \overset{?}{0} & \overset{?}{2} & \overset{?}{0} & \overset{?}{-1} & 0 \;\cdots \\ 0 & \underset{?}{-1} & \underset{?}{0} & \underset{?}{x} & x \end{array} \right.$$

element $(I_h^{2h} A^h)_{ij} = \sum_{k=1}^{n-1} (I_h^{2h})_{ik} (A^h)_{kj}$

in row i of I_h^{2h} the only non zero elements are in columes

$2i-1, 2i, + 2i+1$

in column j of (A^h) the only non zero elements are in

rows $j-1, j, + j+1$

$\therefore$ The product of row i will only have possibly non-zero product w/ the following columns.

$\vdots$ This method is by way of multipling the two matrices

$I_h^{2h} + A^h$.

Try instead to see what the product of A^h w/ $\hat{e}_i$

i odd lots later

$$A^h \hat{e}_i^h = -1(\hat{e}_{i-1}^h) + 2(\hat{e}_i^h) - (\hat{e}_{i+1}^h)$$

Then what does I_h^{2h} do to $\hat{e}_i$ vectors?

$$I_h^{2h} \hat{e}_i^h = \frac{1}{4}\left[\hat{e}^{2h}_{\left(\frac{i-1}{2}\right)} + \hat{e}^{2h}_{\left(\frac{i+1}{2}\right)}\right]$$

i odd

$$I_h^{2h} \hat{e}_i^h = \frac{1}{4}\left[2\hat{e}_{(i/2)}^{2h} \right]$$

i even

$\therefore \quad I_h^{2h} A^h \hat{e}_i^h$ w/ i odd becomes

$$= -I_h^{2h} \underset{\substack{\uparrow \\ \text{even}}}{\hat{e}_{i-1}^h} + 2 I_h^{2h} \hat{e}_i^h - I_h^{2h} \underset{\substack{\uparrow \\ \text{even}}}{e_{i+1}^h}$$

$$= \frac{1}{4}\left[-2\cancel{\hat{e}_{(\frac{i-1}{2})}^{2h}} + 2\left[\cancel{\hat{e}_{(\frac{i-1}{2})}^{2h}} + \cancel{\hat{e}_{(\frac{i+1}{2})}^{2h}} \right] - 2\cancel{\hat{e}_{(\frac{i+1}{2})}^{2h}} \right] = \vec{0}$$

Thus $\hat{e}_i^h$ for i odd is in the null space & we have a basis.

Prob 5.5 Show:

$$I_{2h}^{h} w_k^{2h} = c_k w_k^h - s_k w_{k'}^h \qquad c_k = \cos^2\left(\frac{k\pi}{2N}\right)$$

$$s_k = \sin^2\left(\frac{k\pi}{2N}\right)$$

$$w_{k,j}^{2h} = \sin\left(\frac{kj\pi}{N/2}\right)$$

$$I_{2h}^{h} = \frac{1}{2}\begin{array}{c|l} \left[\begin{matrix} 1 & & \\ 2 & & \\ 1 & 1 & \\ & 2 & \\ & 1 & 1 \\ & & 2 \\ & & 1 & \ddots \end{matrix}\right] & \begin{matrix} j=1 \\ j=2 \\ j=3 \\ j=4 \\ j=5 \\ \\ j=\frac{N}{2}-1 \end{matrix} \end{array}$$

So that for j even

$$\left(I_{2h}^{h} w_k^{2h}\right)_j = \frac{1}{2}\, 2 \sin\left(\frac{k(j/2)\pi}{N/2}\right)$$

j odd

$$\left(I_{2h}^{h} w_k^{2h}\right)_j = \frac{1}{2}\left[1 \sin\left(\frac{k\left(\frac{j-1}{2}\right)\pi}{N/2}\right) + 1 \sin\left(\frac{k\left(\frac{j+1}{2}\right)\pi}{N/2}\right)\right]$$

Then j even

$$\left(I_{2h}^{h} w_k^{2h}\right)_j = \sin\left(\frac{kj\pi}{N}\right)$$

& j odd

$$\left(I_{2h}^{h} w_k^{2h}\right)_j = \frac{1}{2}\left[\sin\left(\frac{k(j-1)\pi}{N}\right) + \sin\left(\frac{k(j+1)\pi}{N}\right)\right]$$

j odd then becomes

$$= \frac{1}{2}\Big[\sin\left(\frac{kj\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right) - \cancel{\sin\left(\frac{k\pi}{N}\right)\cos\left(\frac{kj\pi}{N}\right)}$$

$$+ \sin\left(\frac{kj\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right) + \cancel{\sin\left(\frac{k\pi}{N}\right)\cos\left(\frac{kj\pi}{N}\right)} \Big]$$

$$= \sin\left(\frac{kj\pi}{N}\right)\cos\left(\frac{k\pi}{N}\right)$$

$$\left\{ \cos^2\theta = \frac{1+\cos(2\theta)}{2} \Rightarrow 2\cos^2\theta - 1 = \cos(2\theta) \Rightarrow \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \right\}$$

$$= \sin\left(\frac{kj\pi}{N}\right)\left(2\cos^2\left(\frac{k\pi}{2N}\right) - 1\right)$$

$k = N - k'$

$$= \cos^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{kj\pi}{N}\right) - \left(1 - \cos^2\left(\frac{k\pi}{2N}\right)\right)\sin\left(\frac{kj\pi}{N}\right)$$

$$= \cos^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{kj\pi}{N}\right) - \sin^2\left(\frac{k\pi}{2N}\right)\underline{\sin\left(\frac{(N-k')j\pi}{N}\right)}$$

$$\sin\left(j\pi - \frac{k'j\pi}{N}\right) = \underbrace{\cancel{\sin(j\pi)}}_{0}\cos\left(\frac{k'j\pi}{N}\right) - \cos(j\pi)\sin\left(\frac{k'j\pi}{N}\right)$$

$$= (-1)^{j+1}\sin\left(\frac{k'j\pi}{N}\right)$$

But j is odd $\Rightarrow (-1)^{j+1} = (-1)^{\text{even power}} = \sin\left(\frac{k'j\pi}{N}\right)$

Thus for j odd

$$(I_{2h}^{h} w_k^{2h})_j = \cos^2\left(\frac{k\pi}{2N}\right) w_{k,j}^{h} - \sin^2\left(\frac{k\pi}{2N}\right) w_{k',j}^{h}$$

Now for j even

$$(I_{2h}^{h} w_k^{2h})_j = \sin\left(\frac{kj\pi}{N}\right)$$ But $1 = \cos^2\theta + \sin^2\theta$

$$= \cos^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{kj\pi}{N}\right) + \sin^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{kj\pi}{N}\right)$$

But from problem 5.3

$$w_{k,j}^{h} = \sin\left(\frac{kj\pi}{N}\right) = (-1)^{j+1} w_{k',j}^{h}$$

But as j is even $= -w_{k',j}^{h}$ & for j even we get

$$(I_{2h}^{h} w_k^{2h})_j = \cos^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{kj\pi}{N}\right) - \sin^2\left(\frac{k\pi}{2N}\right)\sin\left(\frac{k'j\pi}{N}\right)$$

$$= \cos^2\left(\frac{k\pi}{2N}\right) w_{k,j}^{h} - \sin^2\left(\frac{k\pi}{2N}\right) w_{k',j}^{h}$$

(5.6) $$CG = I - I_{2h}^{h} (A^{2h})^{-1} I_h^{2h} A^h$$

on modes $\bar{w}_k$ & $\bar{w}_{k'}$ $\quad 1 \le k \le \frac{N}{2}$ $\quad k' = N - k$

$$CG\,\bar{w}_k = ?$$

$$A^h w_k^h = \lambda_k^h w_k^h \quad \checkmark$$

$$I_h^{2h} A^h w_k^h = \lambda_k^h I_h^{2h} w_k^h = \lambda_k^h \cos^2\left(\frac{k\pi}{2N}\right) w_k^{2h} \quad \checkmark$$

Thus $$(A^{2h})^{-1} I_h^{2h} A^h w_k^h = \lambda_k^h \cos^2\left(\frac{k\pi}{2N}\right) \underbrace{(A^{2h})^{-1} w_k^{2h}}_{\lambda_k^{(2h)^{-1}} w_k^{2h}} \quad \checkmark$$

$$= \frac{\lambda_k^h}{\lambda_k^{(2h)}} \cos^2\left(\frac{k\pi}{2N}\right) w_k^{(2h)}$$

Then $$I_{2h}^h (A^{2h})^{-1} I_h^{2h} A^h w_k^h = \frac{\lambda_k^h}{\lambda_k^{(2h)}} \cos^2\left(\frac{k\pi}{2N}\right) \underbrace{I_{2h}^h w_k^{(2h)}}_{\cos^2\left(\frac{k\pi}{2N}\right) w_k^h - \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h}$$

Then

$$CG\, w_k^h = w_k^h - \frac{\lambda_k^h}{\lambda_k^{(2h)}} \cos^2\left(\frac{k\pi}{2N}\right)\left[\cos^2\left(\frac{k\pi}{2N}\right) w_k^h - \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h\right]$$

But for the example problem considered here

$$\lambda_k(A^h) = 4\sin^2\left(\frac{k\pi}{2N}\right) = 4\sin^2\left(\frac{k\pi}{2}h\right)$$

$\left\{ N = \#\text{ cells on fine grid} \right.$ $\therefore$ $N \cdot h = 1$

$$\lambda_k(A^{2h}) = 4\sin^2(k\pi h) = 4\sin^2\left(\frac{k\pi}{2(\frac{N}{2})}\right) = 4\sin^2\left(\frac{k\pi}{N}\right)$$

So that

$$CG\,\omega_k^h = \omega_k^h - \frac{\cancel{4}\sin^2\left(\frac{k\pi}{2N}\right)}{\cancel{4}\sin^2\left(\frac{k\pi}{N}\right)}\cos^2\left(\frac{k\pi}{2N}\right)\left[\cos^2\left(\frac{k\pi}{2N}\right)\omega_k^h - \sin^2\left(\frac{k\pi}{2N}\right)\omega_{k'}^h\right]$$

$$= \left[1 - \frac{\sin^2\left(\frac{k\pi}{2N}\right)\cos^4\left(\frac{k\pi}{2N}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}\right]\omega_k^h$$

$$+ \frac{\sin^4\left(\frac{k\pi}{2N}\right)\cos^2\left(\frac{k\pi}{2N}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}\omega_{k'}^h$$

But $\sin(2\theta) = 2\sin\theta\cos\theta$ so $\sin\theta\cos\theta = \dfrac{\sin(2\theta)}{2}$

$\Rightarrow$

$$CG\,\omega_k^h = \left[1 - \frac{\left(\sin\left(\frac{k\pi}{2N}\right)\cos\left(\frac{k\pi}{2N}\right)\right)^2\cos^2\left(\frac{k\pi}{2N}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}\right]\omega_k^h$$

$$+ \left[\frac{\left(\sin\left(\frac{k\pi}{2N}\right)\cos\left(\frac{k\pi}{2N}\right)\right)^2\sin^2\left(\frac{k\pi}{2N}\right)}{\sin^2\left(\frac{k\pi}{N}\right)}\right]\omega_{k'}^h$$

$$CGw_k^h = \left[1 - \frac{\sin^{\cancel{2}}\cancel{\left(\frac{k\pi}{2N}\right)} \cos^2\left(\frac{k\pi}{2N}\right)}{4 \sin^{\cancel{2}}\left(\frac{k\pi}{N}\right)}\right] w_k^h$$

$$+ \left[\frac{1}{4}\sin^2\left(\frac{k\pi}{2N}\right)\right] w_{k'}^h$$

$$= \left[1 - \frac{\cos^2\left(\frac{k\pi}{2N}\right)}{4}\right] w_k^h + \frac{1}{4}\sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$= \left[1 - \frac{1}{4}\left(1 - \sin^2\left(\frac{k\pi}{2N}\right)\right)\right] w_k^h + \frac{1}{4}\sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$= \left[\frac{3}{4} + \frac{1}{4}\sin^2\left(\frac{k\pi}{2N}\right)\right] w_k^h + \frac{1}{4}\sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$= \sin^2\left(\frac{k\pi}{2N}\right) w_k^h + \left[\frac{3}{4} - \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right)\right] w_k^h + \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$- \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$= \sin^2\left(\frac{k\pi}{2N}\right) w_k^h + \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

$$+ \left[\frac{3}{4} - \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right)\right] w_k^h - \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

Adding the two vectors component wise gives

$$\left[\frac{3}{4} - \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right)\right] w_{k,j}^h - \frac{3}{4}\sin^2\left(\frac{k\pi}{2N}\right)(-1)^{j+1} w_{k,j}^h$$

$$CG\,w_{k'}^h = ?$$

$$A^h w_{k'}^h = \lambda_{k'}^h w_{k'}^h$$

$$I_h^{2h} A^h w_{k'}^h = \lambda_{k'}^h \left(-\sin^2\left(\frac{k\pi}{2N}\right)\right) w_k^{2h}$$

$$(A^{2h})^{-1} w_k^{2h} = \frac{1}{\lambda_k^{(2h)}} w_k^{2h}$$

$$I_{2h}^h w_k^{2h} = \cos^2\left(\frac{k\pi}{2N}\right) w_k^h - \sin^2\left(\frac{k\pi}{2N}\right) w_{k'}^h$$

Then $$CG\,w_{k'}^h = w_{k'}^h - \frac{\lambda_{k'}^h}{\lambda_k^{(2h)}}\left(-\sin^2\left(\frac{k\pi}{2N}\right)\right)\left[c_k w_k^h - s_k w_{k'}^h\right]$$

$$= \frac{\lambda_{k'}^h}{\lambda_k^{(2h)}} \sin^2\left(\frac{k\pi}{2N}\right) c_k w_k^h + \left[1 - \frac{\lambda_{k'}^h}{\lambda_k^{(2h)}}\left(\sin^2\left(\frac{k\pi}{2N}\right)\right)\right]$$

$$\frac{3}{4}\Big[\underbrace{1-\sin^2\left(\tfrac{k\pi}{2N}\right)}_{\cos\left(\frac{k\pi}{2N}\right)}\Big]\omega_{k,j}^h - \frac{3}{4}\sin^2\left(\tfrac{k\pi}{2N}\right)(-1)^{j+1}\omega_{k,j}^h$$

$$\lambda_k^h = 4\sin^2\left(\tfrac{k\pi}{2N}\right)$$

$$\left[1-\frac{\lambda_k^h}{\lambda_k^{(2h)}}\cos^4\left(\tfrac{k\pi}{2N}\right)\right]\omega_k^h + \frac{\lambda_k^h}{\lambda_k^{(2h)}}\cos^2\left(\tfrac{k\pi}{2N}\right)\sin^2\left(\tfrac{k\pi}{2N}\right)\omega_{k'}^h$$

$$-\sin^2\left(\tfrac{k\pi}{2N}\right)\omega_k^h \quad - \sin^2\left(\tfrac{k\pi}{2N}\right)\omega_{k'}^h$$

$$= \left[1-\frac{4\sin^2\left(\frac{k\pi}{2N}\right)\cos^4\left(\frac{k\pi}{2N}\right)}{\lambda_k^{(2h)}} - \sin^2\left(\tfrac{k\pi}{2N}\right)\right]\omega_k^h$$

$$+\left[\frac{4\sin^2\left(\frac{k\pi}{2N}\right)}{\lambda_k^{(2h)}}\cos^2\left(\tfrac{k\pi}{2N}\right)\sin^2\left(\tfrac{k\pi}{2N}\right) - \sin^2\left(\tfrac{k\pi}{2N}\right)\right]\omega_{k'}^h$$

$$= \sin^2\left(\tfrac{k\pi}{2N}\right)\left[\frac{1}{\sin^2\left(\frac{k\pi}{2N}\right)} - \frac{4\cos^4\left(\frac{k\pi}{2N}\right)}{\lambda_k^{(2h)}} - 1\right]\omega_k^h$$

$$+\sin^2\left(\tfrac{k\pi}{2N}\right)\left[\frac{4\sin^2\left(\frac{k\pi}{2N}\right)\cos^2\left(\frac{k\pi}{2N}\right)}{\lambda^{(2h)}} - 1\right]\omega_{k'}^h$$

$$= \sin^2\left(\frac{k\pi}{2N}\right)\left[\frac{1}{\sin^2\left(\frac{k\pi}{2N}\right)} - \frac{\cos^4\left(\frac{k\pi}{2N}\right)}{\sin^2\left(\frac{k\pi}{N}\right)} - 1\right]\omega_k^h$$

$$+ \sin^2\left(\frac{k\pi}{2N}\right)\left[\underbrace{\underbrace{\frac{\sin^2\left(\frac{k\pi}{N}\right)}{\lambda_k^{(2h)}} - 1}_{\frac{1}{4}-1}}_{-\frac{3}{4}}\right]\omega_k^h$$

1st term becomes

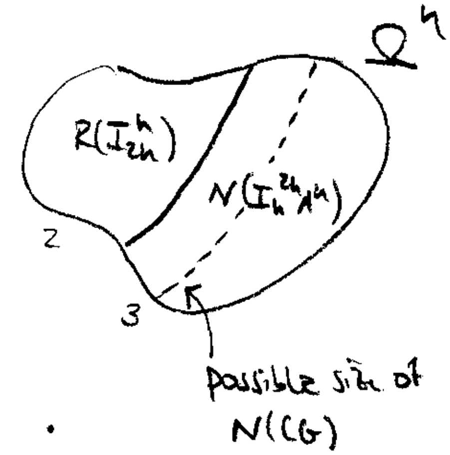
$$\frac{1}{\sin^2\left(\frac{k\pi}{2N}\right)} - \frac{\cos^{\cancel{4}\,2}\left(\frac{k\pi}{2N}\right)}{4\sin^2\left(\frac{k\pi}{2N}\right)\cancel{\cos^2}\left(\frac{k\pi}{2N}\right)} - 1$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\frac{1}{\sin^2\left(\frac{k\pi}{2N}\right)}\left[1 - \left(\frac{1}{4}\right)\cos^2\left(\frac{k\pi}{2N}\right) - \sin^2\left(\frac{k\pi}{2N}\right)\right]$$

(5.8) We know from page 79 Briggs that

$N(CG) \supset R(I_{2h}^h)$ & Also that



$$\Omega^h = R(I_{2h}^h) \oplus N(I_h^{2h}A^h)$$

& for $t^h \in N(I_h^{2h}A^h)$ $\quad CG t^h = t^h \quad \therefore$

given a basis B for $N(I_h^{2h}A^h)$ $\quad CGB = B$ if the dimension of

$\dim(R(I_{2h}^h)) = n_1$
& $\dim(N(I_h^{2h}A^h)) = n_2$ & $n_1 + n_2 = N-1$ then

The $\dim(N(CG)) \leq N-1-\dim(B) = N-1-n_2 = n_1 = \dim(R(I_{2h}^h))$

But since $\dim(N(CG)) \geq \dim(R(I_{2h}^h))$ from subspace arguments above we have $\dim(N(CG)) = \dim(R(I_{2h}^h))$

This is enough to give us that $N(CG) = R(I_{2h}^h)$

for to pick a basis for $N(CG)$ pick the same basis as $R(I_{2h}^h)$

Then this Basis has enough elements to span $N(CG)$ by dimension argument. Then this Basis is also a Basis for $N(CG)$

$\therefore \quad N(CG) = R(I_{2h}^h)$

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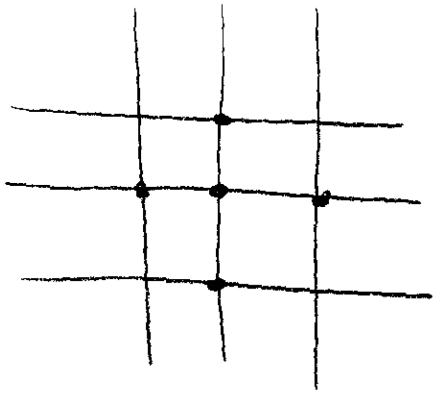
5.9

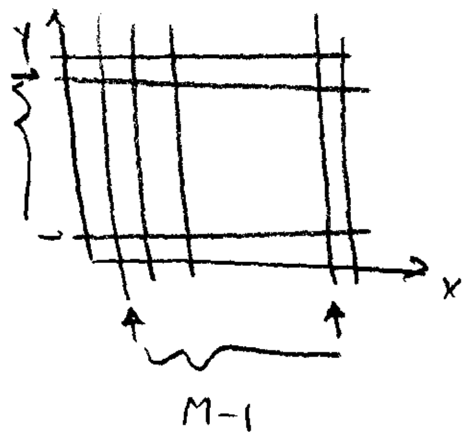


Represent the 5 points

What does the stencil for $A^{2h} = I_h^{2h} A^h I_{2h}^h$ look like?

The full matrix for a stencil like this looks like

$$\begin{bmatrix} A & -I & & & \\ -I & A & -I & & \\ & & \ddots & & \\ & & & -I & A \end{bmatrix} \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_{M-1} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_{M-1} \end{bmatrix}$$



w/ $A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & -1 & 2 \end{bmatrix}$

What does I_{2h}^h (bilinear interpolation) look like in matrix form?

$$v^h_{2i,2j} = v^{2h}_{ij}$$

$$v^h_{2i+1,2j} = \frac{1}{2}\left(v^{2h}_{ij} + v^{2h}_{i+1,j}\right)$$