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(1.1) $$P(x) = \sum_{n \geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

$P_1, P_2, P_3 \quad \dots$

(1.2) ...

(1.3) (i)
$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| \leq \frac{C}{(N+1)!}|x-0|^{N+1} = \frac{C}{(N+1)!}x^{N+1}$$

w/ $C \ni \quad |f^{(n)}(x)| = \cancel{f^{(n)}} |e^x| = e^x \leq C \quad \forall x \in I$

$\therefore$ pick $C = 1$

Then

$$\left| e^x - \sum_{n=0}^{N} \frac{x^n}{n!} \right| \leq \frac{1}{(N+1)!} \leq .05$$

pick N ..

(ii) $$\left| e^x - \sum_{n=0}^{8} \frac{x^n}{n!} \right| \leq \frac{1}{9!} = 2.75 \cdot 10^{-6} \leq 2.8 \cdot 10^{-6}$$ ✓

(ii) Now for $x \in [0,1]$

$$-x^2 \in [-1,0]$$

So

$$\left| e^{-x^2} - \sum_{n=0}^{8} \frac{(-1)^n x^{2n}}{n!} \right| \le 2.8 \cdot 10^{-6}$$

$$\therefore \sum_{n=0}^{8} \quad - 2.8 \cdot 10^{-6} \le e^{-x^2} \le \sum_{n=0}^{8} \quad + 2.8 \cdot 10^{-6}$$

$$\therefore \quad " \quad - 2.8 \cdot 10^{-6} \le \int_0^1 e^{-x^2} dx \le \sum_{n=0}^{8} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \Big|_0^1 + 2.8 \cdot 10^{-6}$$

$$\| $$

$$\sum_{n=0}^{8} \frac{(-1)^n}{n!(2n+1)}$$

$\| \leftarrow$ check

$.7468 \ldots$

(1.4)
$$\left| \sin x - \sum_{n=0}^{N} a_n x^n \right| \le \frac{C}{(N+1)!}(x - x_0)^{N+1}$$

$$\le \frac{C\,(\pi/2)^{N+1}}{(N+1)!}$$

+ $|f^{(n)}(x)| \le C$ will be true for $C = 1$

$\therefore$ pick N +

$$\frac{(\pi/2)^{N+1}}{(N+1)!} \le .1$$

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

w/ $f(x) = \sin x$

Then

(ii) $x \in [0,1]$ $\quad x^4 \in [0,1]$

So

$$\left| \sin x^4 - \sum_{n=0}^{N} a_n x^{4n} \right| \le .1$$

$$\sum a_n \frac{x^{4n+1}}{4n+1} - .1 \le \int_0^1 \sin x^4 \, dx \le \sum_{n=0}^{N} a_n \frac{x^{4n+1}}{4n+1} + .1$$

(21)

(i) since $\frac{|\sin(n)+\cos(n)|}{n^2} \le \frac{2}{n^2}$ + $\frac{1}{n^2}$ is convergent

this series is too

(ii) $2\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n+5} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n+5}$ By alternating series test convergent

(iii) $\lim_{n\to\infty} a_n \ne \cancel{\infty}\ 0$ $\therefore$ divergent

(iv) $\frac{1}{\log n} > \frac{1}{n}$ + $a_n = \frac{1}{n}$ diverges so divergent

(v) By alternating series test this sum converges

(22) From $\int_p^{\infty} f(x)\,dx \le \sum_{n=p}^{\infty} f(n) \le \int_p^{\infty} f(x)\,dx + f(p)$ eq 26 we get

$$\int_1^{\infty} \frac{dx}{x^2+1} \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \int_1^{\infty} \frac{dx}{x^2+1} + \frac{1}{2}$$

$$\| $$

$$\tan^{-1}(x)\Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

So $\Rightarrow$ ~~[illegible]~~ Result as shown

$$\int_2^\infty \frac{dx}{x^3} \le \sum_{n=2}^\infty \frac{1}{n^3} \le \int_2^\infty \frac{dx}{x^3} + \frac{1}{2^3}$$

$$+ \quad \int_2^\infty \frac{dx}{x^3} = \left.\frac{x^{-2}}{-2}\right|_2^\infty = \frac{-1}{2}\left(0 - \frac{1}{4}\right) = \frac{1}{8}$$

$\therefore$ Result as shown

(23)

$$\sum_{n=1}^\infty \frac{1}{n^3} = S$$

consider $\quad S_N = \sum_{n=1}^N \frac{1}{n^3}$

Then ~~[illegible]~~ $\quad |S - S_N| = \left|\sum_{n=N+1}^\infty \frac{1}{n^3}\right| \le \int_{N+1}^\infty \frac{dx}{x^3} + \frac{1}{(N+1)^3}$

$$\Rightarrow |S - S_N| \le -\left.\frac{x^{-2}}{2}\right|_{N+1}^\infty + \frac{1}{(N+1)^3}$$

$$= \frac{+1}{2}\frac{1}{(N+1)^2} + \frac{1}{(N+1)^3}$$

Then to ~~[illegible]~~ ~~[illegible]~~ Pick $N \ni$

$$\frac{1}{2}\frac{1}{(N+1)^2} + \frac{1}{(N+1)^3} < .02.$$

It seems I have used $a_n = \frac{1}{n^3}$ rather than $a_n = \frac{1}{n^4}$ but the technique is the same.

(24) $$\sum_{n=N+1}^{\infty} \leq \int_{N+1}^{\infty} \frac{1}{(2x+1)^2}\,dx + \frac{1}{(2N+2+1)^2}$$

$$= \frac{(2x+1)^{-1}}{-1}\frac{1}{2}\Big|_{N+1}^{\infty} + \frac{1}{(2N+3)^2}$$

$$= \frac{1}{2}\frac{1}{2N+3} + \frac{1}{(2N+3)^2} \quad \checkmark$$

(25) intuitively $a_n \rightarrow \frac{1}{2\sqrt{n}} + \frac{1}{2\sqrt{n}} - \frac{1}{\sqrt{2}\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{2}\sqrt{n}}$

$$= \left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{n}} \quad n \to +\infty$$

which is a divergent series

From the asymptotics just performed a_n is certainly not ~~...~~ always oscillatory. The 1st few terms maybe but the tail of the sum is not. The sum $\frac{(-1)^{n-1}}{\sqrt{n}}$

is ossilatory in the tail

(26) ... ~~easy~~ ~~easy~~

(27) $a_n = n x^n$

$\left|\frac{a_{n+1}}{a_n}\right| \to |x| < 1 \Rightarrow \rho = 1$

$$f(x) = \sum_{n=1}^{\infty} n x^n = x^{+1} \sum_{n=1}^{\infty} n x^{n-1} = x \frac{d}{dx} \sum_{n=0}^{\infty} x^n = x \frac{d}{dx}\left(\frac{1}{1-x}\right) = \cdots$$

(28) $\left|\frac{a_{n+1}}{a_n}\right| = |x| < 1 \Rightarrow \rho = 1$

$$= \sum_{n \geq 0} 1 x^{n+1} + \sum_{n \geq 0} \frac{x^{n+1}}{n+1} = x \frac{1}{1-x} + \int_0^x \sum x^n dx$$

$$= \frac{x}{1-x} + \int_0^x \frac{1}{1-x} dx$$

$$= \frac{x}{1-x} + -\ln(1-x)\Big|_0^x$$

$$= \frac{x}{1-x} - \ln(1-x)$$

2.9

(i) $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+2}{(n+1)(n+1)!} x^{n+1} \cdot \frac{n\, n!}{n+1} \frac{1}{x^n}\right|$

$$\Rightarrow \frac{|x|}{n+1} \to 0 < 1 \quad \forall x \quad \therefore \rho = +\infty$$

(ii) $h(x) = \int_0^x t f'(t)\,dt$ $\qquad f(t) = \sum_{n=1}^{\infty} (n+1)$

$$f'(t) = \sum_{n=1}^{\infty} \frac{n+1}{n!} x^{n-1}$$

$$h(x) = \int_0^x \sum_{n\geq 1} \frac{n+1}{n!} t^n \, dt$$

$$= \sum_{n\geq 1} \frac{x^{n+1}}{n!} = \cancel{x} \quad \cancel{xe^x}$$

(iii) $h(x) = xe^x$

2.10 $f_n(x) = \dfrac{3\cos(nx) - \sin^2(nx)}{3^n - 1}$ is continuous $\forall n$

If $\sum_{n\geq 1} f_n(x)$ converges uniformly $\sum_{n\geq 1}^{\infty} f_n(x)$ will be continuous.

Since $|f_n(x)| \leq \dfrac{4}{3^n - 1} \leq \cancel{\dfrac{4}{3^n - 3}}$ 其 By the Weierstrass M-test this sum is uniformly convergent.

& the limiting fn of $S_n(x)$ is convergent.

2.11 $\quad f_n \equiv \dfrac{\cos(2^n x)}{3^n}$

(ii) Then $|f_n| \le \dfrac{1}{3^n}$. By the Weierstrass-M test this sum converges uniformly.

Since f_n is continuous $\forall n$ the limit f_n is continuous.

(i) $f(x)$ is well defined since $f = \lim_{N \to \infty} S_N(x)$ & this sum converges absolutely.

(iii) f_n is differentiable $\forall n$

$$f_n' = -\sin(2^n x)\left(\frac{2}{3}\right)^n$$

$|f_n'| \le \left(\frac{2}{3}\right)^n$ which converges thus

$$f'(x) = \sum_{n \ge 0} f_n'(x) = \sum_{n \ge 0} -\sin(2^n x)\left(\frac{2}{3}\right)^n$$

$f'(0) = 0$.

2.12 Require $\dfrac{1}{|2+x^2|} < 1 \Rightarrow |2+x^2| > 1$

$2 + x^2 > 1$

$\Rightarrow x^2 > -1$ true $\forall x$

$$+\ S(x) = \frac{1}{1 - \frac{1}{2+x^2}} = \cdots$$

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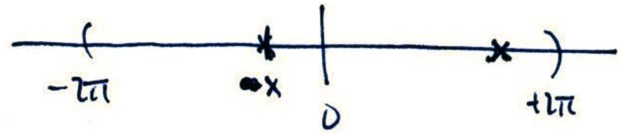
(3,1) =

$$\left\{ f(2\pi) = \frac{1}{4}(4\pi^2) - \frac{\pi}{2}(2\pi) = \pi^2 - \pi^2 = 0 \checkmark \right.$$

Check $f(-x) = f(x)$

So $x \in [0, 2\pi] \Rightarrow -x \in [-2\pi, 0]$ which maps to $2\pi - x$

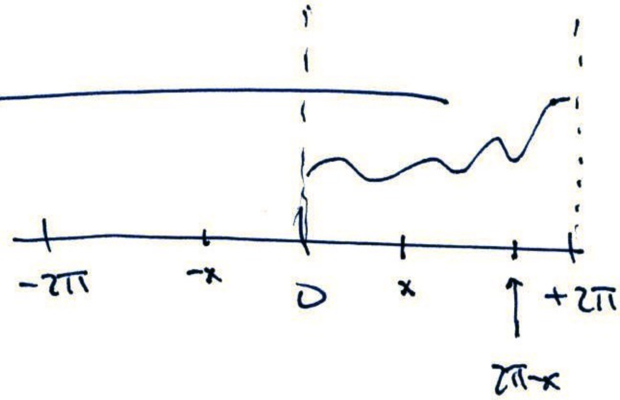
$\therefore$



$\cancel{f(-x) = f(2\pi + x)}$ for $\cancel{x \in (0, 2\pi)}$

Now $\cancel{f(2\pi - x) = \frac{1}{4}(2\pi - x)^2 - \frac{\pi}{2}(2\pi - x) = \frac{1}{4}(4\pi^2 - 4\pi x + x^2) - \pi + x}$

$= \cancel{\pi - \pi x + \frac{1}{4}x^2 - \pi + x}$



So $f(-x) = f(x)$ becomes

$\parallel$

$f(2\pi - x) = f(x)$ for $x \in [0, 2\pi)$. The RHS. is $\frac{x^2}{4} - \frac{\pi}{2}x$

& $f(2\pi - x) = \frac{1}{4}(2\pi - x)^2 - \frac{\pi}{2}(2\pi - x)$

$$= \frac{1}{4}(4\pi^2 - 4\pi x + x^2) - \pi^2 + \frac{\pi}{2}x$$

$$= \cancel{\pi^2} - \pi x + \frac{x^2}{4} - \cancel{\pi^2} + \frac{\pi}{2}x = -\frac{\pi}{2}x + \frac{x^2}{4}$$

yes they are equal ✓

(1a) Since $f(\cdot)$ is an even fn we have that only cos coefficients are nonzero

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx \qquad n \geq 0$$

for $n \geq 1$ we have

$$= \frac{2}{\pi}\int_0^\pi \left(\frac{x^2}{4} - \frac{\pi}{2}x\right)\cos(nx)\,dx \underset{\uparrow}{=} \frac{2}{\pi}\left[-\frac{\pi\cos(nx)}{2n^2} + \frac{x\cos(nx)}{2n^2} - \frac{\pi x\sin(nx)}{2n} + \frac{(-2+n^2x^2)\sin(nx)}{4n^3}\right]\Bigg|_0^\pi \qquad n \geq 1$$

using MMA

$$= \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{-\cancel{\pi}(-1)^n}{\cancel{2}n^2} + \frac{\pi(-1)^n}{2n^2} - \left(-\frac{\pi}{2n^2}\right)\right]$$

$$= \left[\underbrace{\frac{(-1)(-1)^n}{n^2} + \frac{(-1)^n}{n^2}}_{\cancel{\frac{(-1)^n(-\cancel{1}+1)}{n^2}}\ 0} + \frac{1}{n^2}\right] = \frac{1}{n^2}$$

$\therefore$

$$f(x) \sim \sum_{n\geq 1}\frac{\cos(nx)}{n^2} + \underbrace{\frac{2}{\pi}\int_0^\pi f(x)\,dx}$$

$$= \frac{2}{\pi}\left[-\frac{\pi x^2}{4} + \frac{x^3}{12}\right|_0^\pi = \frac{2}{\pi}\Bigg[\underbrace{\underbrace{-\frac{\pi^3}{4}}_{-\frac{3\pi^3}{12}} + \frac{\pi^3}{12}}_{-\frac{2\pi^3}{12}}\Bigg] = -\frac{\pi^3}{6}$$

✓

$\therefore \quad a_0 = \frac{-\pi^2}{6}$

so $f(x) \sim -\frac{\pi^2}{6} + \sum_{n\geq 1} \frac{\cos(nx)}{n^2}$

(iii) Now if $x \equiv 0$ $\quad f(0) \sim -\frac{\pi^2}{6} + \sum_{n\geq 1} \frac{1}{n^2} = 0$

$\Rightarrow \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$ ✓

to obtain $\sum_{n\geq 1} \frac{1}{n^4}$ we would integrate the Fourier expansion for $f(x)$ twice. giving (for the 1st integration)

$$\int_0^{x'} f(x)\,dx \sim -\frac{\pi^2}{6}x' + \sum_{n\geq 1} \frac{\sin(nx)}{n^3}\Bigg|_0^{x'} + C_1 = -\frac{\pi^2}{6}x' + \sum_{n\geq 1} \frac{\sin(nx)}{n^3} + C_1$$

evaluating at $x = 0$ gives $C_1 = 0$. since $\int^x f(\cdot)\,d\cdot = -\frac{\pi x^2}{4} + \frac{x^3}{12}$ ✓

$$\therefore \quad \int f(\cdot)\,d\cdot = -\frac{\pi^2}{6}x + \sum_{n\geq 1} \frac{\sin(nx)}{n^3}$$

Integrating a second time gives:

$$\iint f = -\frac{\pi^2}{12}x^2 + \sum_{n\geq 1} -\frac{\cos(nx)}{n^4} + C_2$$ ✓

Where as

$$\iint f = -\frac{\pi x^3}{12} + \frac{x^4}{48} \quad ✓$$

Thus $$-\frac{\pi x^3}{12} + \frac{x^4}{48} \sim -\frac{\pi^2 x^2}{18} + \sum_{n \geq 1} -\frac{\cos(nx)}{n^4} + C_2$$

To evaluate C_2 we recognize this term as part of the "a_0" term in the Fourier expansion of $-\frac{\pi x^3}{12} + \frac{x^4}{48}$, since this has

$$a_0^{(LHS)} = \frac{2}{\pi}\int_0^\pi \left(-\frac{\pi x^3}{12} + \frac{x^4}{48}\right) dx = \frac{2}{\pi}\left[-\frac{\pi x^4}{48} + \frac{x^5}{240}\right|_0^\pi \quad ✓$$

$$= \frac{2}{\pi}\left[-\frac{\pi^5}{48} + \frac{\pi^5}{240}\right] = 2\pi^4\left[\frac{-5}{240} + \frac{1}{240}\right] \quad ✓$$

$$= -\frac{8\pi^4}{240} = \cancel{-30\pi^4}$$

$$= -\frac{\pi^4}{30} \quad ✓$$

So

$$a_0^{(LHS)} = \cancel{-30\pi^4} \quad -\frac{\pi^4}{30}$$

& $$a_0^{(RHS)} = \frac{2}{\pi}\int_0^\pi -\frac{\pi^2 x^2}{18}\, dx = -\frac{\pi}{36}\frac{x^3}{3}\Big|_0^\pi = -\frac{\pi}{18}(\pi^3) = -\frac{\pi^4}{18} \quad ✓$$

So equating constant terms gives

$$-\frac{\pi^4}{30} = -\frac{\pi^4}{18} + C_2 \Rightarrow \cancel{C_2 = \left(-30 + \frac{1}{9}\right)\pi^4}$$

$$= \cancel{\frac{(-270+1)\pi^4}{9}} = \cancel{\frac{-269\pi^4}{9}}$$

$$C_2 = \pi^4\left(\frac{-1}{30} + \frac{1}{18}\right) = \frac{\pi^4}{3}\left(\frac{-1}{10} + \frac{1}{6}\right) = \frac{\pi^4}{6}\left(\frac{-1}{5} + \frac{1}{3}\right)$$

$$= \frac{\pi^4}{6}\left(\frac{-3+5}{15}\right) = \frac{\pi^4 \cdot 2}{6 \cdot 15} = \frac{\pi^4}{45}$$

Thus

$$\frac{-\pi x^3}{12} + \frac{x^4}{48} \sim \frac{-\pi^2 x^2}{12} + \sum_{n \geq 1} \frac{-\cos(nx)}{n^4} + \frac{\pi^4}{45}$$ Now evaluating at $x=0$

giving $\displaystyle\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{45}$ For some reason this is two times too large. Using Matlab, I compute that

$\displaystyle\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$. I don't see the error in my logic...

(iv) From the 1st integral of 7 we obtain

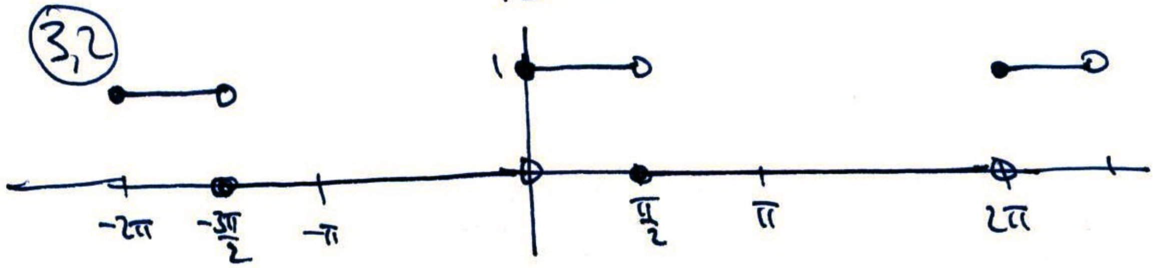
$$-\frac{\pi^2 x}{6} + \sum_{n\geq 1} \frac{\sin(nx)}{n^3} \sim -\frac{\pi x^2}{4} + \frac{x^3}{12}$$

So $$\sum_{n\geq 1} \frac{\sin(nx)}{n^3} \sim -\frac{\pi x^2}{4} + \frac{x^3}{12} + \frac{\pi^2 x}{6}$$

1

MJ 30 Christmas

3,2



(i) Since f is neither odd nor even we have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx \qquad n \geq 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx \qquad n \geq 1$$

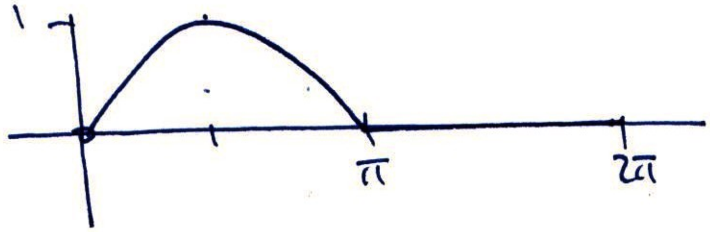
one evaluates these integrals & the F.S. is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1} a_n\cos(nx) + b_n\sin(nx)$$

(ii) No. since $a_n = O(\frac{1}{n})$

(iii) Since $f(x)$ has a discontinuity in the interval its Fourier coefficients decay at the rate of $O(\frac{1}{n})$

3,3



By Proposition 3,2,5

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} (|a_n| + |b_n|) \qquad \forall x \in \mathbb{R} \qquad \text{true } a_n, b_n = O(\tfrac{1}{n^2})$$

In this case

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\,dx = \frac{1}{\pi}\left.\frac{\cos(x)\cos(nx) + n\sin(x)\sin(nx)}{-1+n^2}\right|_0^{\pi}$$

$n \neq 1$

$$= \cancel{\frac{1}{\pi}\frac{1}{n^2-1}(-1)(-1)^n} \qquad \cancel{n}\neq 1$$

$$a_n = \frac{1}{\pi}\frac{1}{n^2-1}\left((-1)^{n+1} - 1\right) \qquad n \neq 1 \qquad a_0 = \frac{1}{\pi}\frac{1}{(-1)}\cdot(-2) = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi}\left.\frac{-1}{2}\cos(x)^2\right|_0^{\pi} = \frac{-1}{2\pi}(1-1) = 0$$

$$\& \quad b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx = \frac{1}{\pi}\left[\left.\frac{-n\cos(nx)\sin(x) + \cos(x)\sin(nx)}{-1+n^2}\right|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\frac{1}{n^2-1}\cdot 0 = 0 \qquad n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(x)\,dx = \frac{1}{\pi}\left[\frac{x}{2} - \frac{1}{4}\sin(2x)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2}$$

$$\therefore\ f(x) \sim \frac{a_0}{2} + \sum_{n\geq 1} a_n\cos(nx) + b_n\sin(nx)$$

$$= \frac{1}{\pi} + \cancel{\sum_{n\geq 1} b_n} \sum_{n\geq 2} \frac{1}{\pi}\frac{1}{(n^2-1)}\left((-1)^{n+1}-1\right)\cos(nx) + \frac{1}{2}\sin(x)$$

$$= \frac{1}{\pi} + \frac{1}{2}\sin(x) + \sum_{\substack{n \text{ even} \\ \geq 2}} \frac{1}{\pi}\frac{(-2)}{n^2-1}\cos(nx)$$

$\{$ Since $(-1)^{n+1}-1 = 0$ if n is odd

So $$f(x) \sim \frac{1}{2}\sin(x) + \frac{1}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n)^2-1}\cos(2nx)$$

So

$$|f(x) - S_N(x)| \leq \sum_{n=N+1}^{\infty} \frac{2}{\pi}\frac{1}{(2n)^2+1} \leq \frac{2}{\pi}\int_{N+1}^{\infty} \frac{dx}{(2x)^2+1} + \frac{1}{(2(N+1))^2+1}$$

$$= \frac{2}{\pi} \quad \ldots$$ evaluate these integrals & bound by $\leq .1$

(3.4) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-x} e^{-inx}\, dx = \frac{1}{2\pi} \frac{e^{-(1+in)x}}{-1-in}\Bigg|_0^{2\pi}$$

$$= \frac{(-1)}{2\pi}\frac{1}{1+in}\left(e^{-(1+in)2\pi} - 1\right)$$

~~Now~~ $S_2 = \sum_{n=-2}^{+2} c_n e^{inx} =$ Note sine f is rel $c_{-n} = c_n^*$

w/ $c_2 = \frac{-1}{2\pi}\frac{1}{1+i(2)}(\ldots)$

(3.5)

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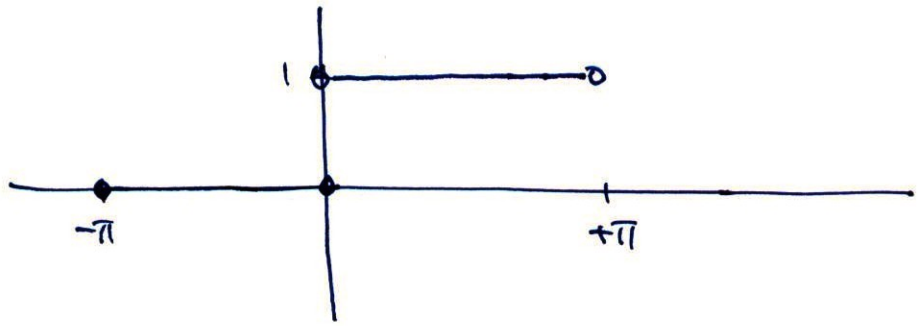
(3.6)



$$f \sim \sum_n c_n e^{inx} \quad \text{w/} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(x) e^{-inx}\,dx \qquad n \in \mathbb{Z}$$

$$= \frac{1}{2\pi}\int_0^{\pi} 1 \cdot e^{-inx}\,dx = \frac{1}{2\pi}\left.\frac{e^{-inx}}{-in}\right|_0^{\pi} \qquad n \neq 0$$

$$= \frac{1}{2\pi(-in)}\left(e^{-in\pi} - 1\right)$$

so $c_n = \frac{1}{2\pi(-in)}\left((e^{-i\pi})^n - 1\right)$

$$= \frac{1}{2\pi(-in)}\left((-1)^n - 1\right) = \frac{1}{2\pi(-in)} \cdot \begin{cases} 0 & n \text{ even} \\ -2 & n \text{ odd} \end{cases}$$

$$\therefore \quad c_n = \begin{cases} 0 & n \text{ even} \\ \frac{1}{\pi i n} & n \text{ odd} \end{cases} \qquad n \neq 0.$$

$$= \begin{cases} 0 & n \text{ even} \neq 0 \\ \frac{-1}{\pi n} & n \text{ odd} \end{cases}$$

& $c_0 = \frac{1}{2\pi}\int_0^{\pi} dx = \frac{1}{2}$

so

$$f \sim \frac{1}{2} + \sum_{n \text{ odd}} \frac{1}{\pi i n} e^{inx}$$

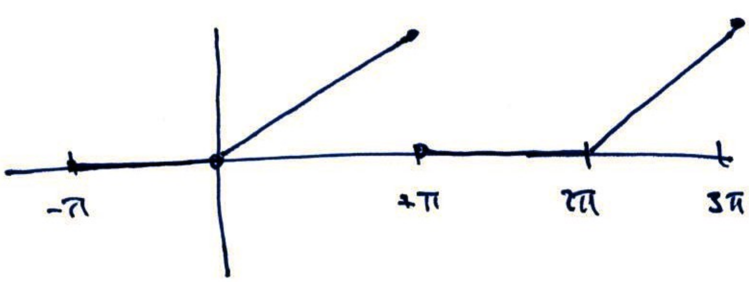
(ii) $$f \sim \frac{1}{2} + \sum_{n \geq 0} \frac{1}{\pi i(2n+1)} e^{i(2n+1)x}$$

~~Evaluating~~ at ~~x=0~~ integrating f once gives

$$\int^x f = \frac{x}{2} + \sum_{n \geq 0} \frac{1}{\pi i(2n+1)^2 i} e^{i(2n+1)x} + C_1$$

|| (evaluating the integral)

$$\begin{cases} 0 & -\pi < x < 0 \\ x & 0 \leq x \leq \pi \end{cases} = \frac{x}{2} - \sum_{n \geq 0} \frac{e^{i(2n+1)x}}{\pi(2n+1)^2} + C_1$$



to specify the constant C_1 we will integrate the function f over specified limits say 0 to x^* giving

$$\int_0^{x^*} f(t)\,dt = x^* = \int_0^{x^*} \frac{1}{2}\,dt + \int_0^{x^*} \sum_{n \geq 0} \frac{1}{\pi i(2n+1)} e^{i(2n+1)x}$$

$$\Rightarrow \quad x^* = \frac{x^*}{2} + \sum_{n\geq 0} \frac{1}{\pi i^2 (2n+1)^2} e^{i(2n+1)x} \Big|_0^{x^*}$$

$$\Rightarrow \quad x^* = \frac{x^*}{2} + -\sum_{n\geq 0} \frac{\left(e^{i(2n+1)x^*} - 1\right)}{\pi (2n+1)^2}$$ replacing x^* w/ x

$$\Rightarrow \quad x = \frac{x}{2} - \sum_{n\geq 0} \frac{e^{i(2n+1)x}}{\pi(2n+1)^2} + \sum_{n\geq 0} \frac{1}{\pi(2n+1)^2} \qquad 0 < x < \pi$$

... I can not see how to specify an x such that the 1st summation will disappear. Instead consider perseval's theorm:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x)|^2 dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2 = \sum_{n=-\infty}^{+\infty} |c_n|^2$$

in this case

$$\sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{1}{4} + \frac{1}{\pi^2(-1)}\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2}$$

$\therefore$ solving for $\sum_{n \text{ odd}}$ gives our result

(3.6) No. $a_n = (-1)^n$ + Parseval's theorem requires a fn in $L^2(-\pi, +\pi)$ to satisfy $\sum_n |c_n|^2 < \infty$ or equivalently

$$\frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2) \leq \infty$$

Since this latter sum is $\frac{1}{2}\sum_{n=1} 1 \not< \infty$ $f \notin L^2(-\pi, +\pi)$

(3.7)

(i) let $x = \frac{\pi}{2}$ then $\sin(n\frac{\pi}{2}) = \begin{cases} +1 & n = \{1, 5, 9, \dots\} \\ 0 & n = \{2, 4, 6, \dots\} \\ -1 & n = \{3, 7, \dots\} \end{cases}$

so

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin(n\tfrac{\pi}{2})$$

$$= \sum_{n \in \{1,5,9,\dots\}} \frac{2(-1)^n}{n} + \sum_{n \in \{3,7,11,\dots\}} \frac{2(-1)^{n+1}}{n}$$

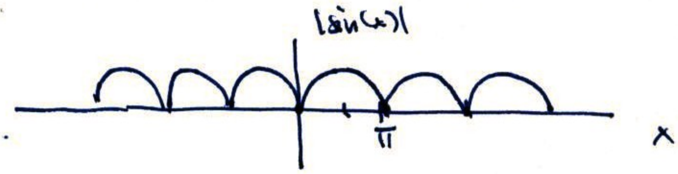
$$= -\sum_{n \in \{1,5,9,\dots\}} \frac{2}{n} + \sum_{n \in \{3,7,11,\dots\}} \frac{2}{n}$$

$$= 2\sum_{n \in \{1,3,5,7,9,11,\dots\}} \frac{(-1)^{p(n)}}{n} = 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)}$$

(ii) No: Thm 26.6 requires $f_n(x)$ to be differentiable, which we have but also $|f_n'(x)| \le k_n$ & $\sum k_n < \infty$

Since in this case $|f_n'(x)| \le |2| = 2$ ~~we have~~ which is NOT convergent. We cannot use Thm 26.6 to ~~test~~ ~~convergence~~. insure the validity of this ~~oppel~~ operation

(3,8) (i) I'll assume this result is correct. Just integrate to obtain it.



(ii) Let $x = \pi$ ...

(iii) Using Parseval's Thm:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} |f(x)|^2\,dx = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n\ge 1} |a_n|^2 + |b_n|^2$$

(iv) ...

(v) ...

(vi) $$\Big|\,|\sin x| - f_N(x)\Big| \le \sum_{n=N+1}^{\infty} |a_n|^2 + |b_n|^2 = \sum_{n=N+1}^{\infty} \frac{4}{\pi}\frac{1}{(2n-1)(2n+1)}$$

$$\le \frac{4}{\pi}\int_{N+1}^{\infty} \frac{dx}{(2x-1)(2x+1)} + \frac{4}{\pi}\frac{1}{(2(N+1)-1)(2(N+1)+1)}$$

$$\le .1$$

3.9

(i) ...

(ii) ...

(iii) $$\Big|f(x) - \underset{\substack{\| \\ S_1(x)}}{\sin x}\Big| \le \sum_{n=2}^{\infty} \left(|a_n| + |b_n|\right) = \sum_{n=2}^{\infty} \frac{1}{(2n-1)^5}$$

So $$|f - \sin x| \le \int_2^{\infty} \frac{dx}{(2x-1)^5} + \frac{1}{(2(2)-1)^5} = \dots$$

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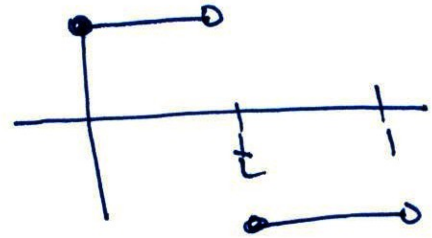
(4.1) . . .

(4.2) $\psi(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1) \\ 0 & \text{else} \end{cases}$



$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) \qquad (j,k) \text{ integers.}$$

$$f(x) = \begin{cases} \sin(2\pi x) & x \in [0,1) \\ 0 & \text{else} \end{cases}$$

$$c_{j,k} = \int_{-\infty}^{+\infty} f(x)\overline{\psi_{j,k}(x)}\,dx = \int_0^1 \sin(2\pi x)\, 2^{j/2}\psi(2^j x - k)\,dx = \begin{cases} 0 & k \neq 0 \\ \text{since} \\ f + \psi_{j,k} \end{cases}$$

do not overlap.

$$= 2^{j/2}\int_0^1 \sin(2\pi x)\,\psi(2^j x)\,dx$$

Now

$$\psi(2^j x) = \begin{cases} 1 & 2^j x \in [0, 1/2] \\ -1 & 2^j x \in [\frac{1}{2}, 1) \end{cases}$$

$$= \cancel{2^{j/2}\int}$$

$$\Rightarrow \psi(2^j x) = \begin{cases} 1 & x \in [0, 2^{-j-1}) \\ -1 & x \in [2^{-j-1}, 2^{-j}) \end{cases}$$

So

$$c_{j,0} = 2^{j/2}\left[\int_0^{2^{-j-1}} \sin(2\pi x)\,dx + \int_{2^{-j-1}}^{2^{-j}} (-1)\sin(2\pi x)\,dx\right] \quad \text{for } j \in \{0,1,2,3,\dots\}$$

$= \dots$ (do the integral ...)

In addition since for $j \in \{-1,-2,-3,\dots\}$ the interval of definition of $f(2^j x)$ grows in x. (For instance if $j=-1$

$$f(2^{-1}x) = \begin{cases} 1 & x \in [0,1) \\ -1 & x \in [1,2) \end{cases}$$

$$f(2^{-2}x) = \begin{cases} 1 & x \in [0,2) \\ -1 & x \in [2,4) \end{cases}$$

So

$$c_{j,0} = 2^{j/2} \int_0^{2^{j-1}} \sin(2\pi x)\,dx = 0 \quad \text{for } j \leq -1.$$

Thus only $c_{j,0}$ for $j \geq 0$ are non zero. & given by the integral above.

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5.1 $f(x) = \sin(\pi x)\chi_{[0,1]}(x)$

(i) From eq 5.8 $\quad f_k(x) \equiv \sum_{n=0}^{2^k-1} a_n \chi_{I_n}(x) = \sum_{n=0}^{2^k-1} a_n \chi_{[n2^{-k},(n+1)2^{-k})}(x)$

(k = level or scale) w/ $a_n = 2^k \int_{n2^{-k}}^{(n+1)2^{-k}} f(x)\,dx$

so in this problem:

$$a_n = 2^k \int_{n2^{-k}}^{(n+1)2^{-k}} \sin(\pi x)\chi_{[0,1]}(x)\,dx \qquad 0 \le n \le 2^k-1$$

when $n \in [0, 2^k-1]$ $\quad n2^{-k}$ + $(n+1)2^{-k} \in [0,1]$

$\{$ $n = 2^k-1$ has $n2^{-k} = (2^k-1)(2^{-k}) = 1-2^{-k}$

+ $(n+1)2^{-k} = 2^k \cdot 2^{-k} = 1$ $\}$

So

$$a_n = 2^k(-\pi)\cos(\pi x)\Big|_{n2^{-k}}^{(n+1)2^{-k}} = -2^k\pi\left(\cos(\pi(n+1)2^{-k}) - \cos(\pi n 2^{-k})\right)$$

Using Mathematica, one could easily compute & plot

$f_k(x)$ for various k's since

$$\chi_{[n2^{-k},(n+1)2^{-k})}(x) = \begin{cases} 1 & n2^{-k} \le x \underset{\neq}{<} (n+1)2^{-k} \\ 0 & \text{otherwise.} \end{cases}$$

(b) ... this is a trivial exercise

5.2 As in 5.1

(i) $$f_k(x) = \sum_{n=0}^{2^k-1} a_n \chi_{[n2^{-k},(n+1)2^{-k})}(x) \quad \text{w/} \quad a_n = 2^k \int_{n2^{-k}}^{(n+1)2^{-k}} f(x)\,dx$$

so

$$a_n = 2^k \int_{n2^{-k}}^{(n+1)2^{-k}} e^x\,dx = \cdots$$

(ii) Easy to do ...

(iii) " " " ...