pg 3 Courant / Friedrichs

$đQ = T\,dS$ If $đQ = 0$ adiabatic change

$\Rightarrow S = S_0$ constant entropy

$$\frac{dp}{d\rho} = f_\rho = f_\tau \left(\frac{d\rho}{d(\rho^{-1})}\right)^{-1} = f_\tau \left(\underline{-\rho^2}\right)^{-1}$$

$$= -f_\tau$$

$$\frac{dp}{d\rho} = c^2$$

$$\boxed{\frac{d\rho^{-1}}{d\rho} = -1\rho^{-2}}$$

$$\frac{c^2}{\rho^2} = \cancel{-g_\tau} \quad ?$$

$$c^2 \rho^2 = -g_\tau$$

If

$g_{\tau\tau} > 0 \Rightarrow$

$$\frac{d\tau}{d\rho} = \frac{d(\rho^{-1})}{d\rho} = -\rho^{-2}$$

$$\Rightarrow \frac{d\rho}{d\tau} = -\rho^2$$

$$\Leftarrow \; g_{\rho\tau} \frac{d\rho}{d\tau} = g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 = f_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2$$

$$= f_{\rho\rho}\, \rho^4 > 0$$

$$\Rightarrow f_{\rho\rho} > 0$$

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$f_{\rho\rho} \geq 0$

$$f_{\rho\rho} = \frac{\partial f_\rho}{\partial \rho} = \frac{\partial}{\partial \tau} f_\rho \frac{\partial \tau}{\partial \rho} = \frac{\partial^2}{\partial \tau^2} f \left(\frac{\partial \tau}{\partial \rho}\right)^2 = f_{\tau\tau} \left(\frac{\partial \rho^{-1}}{\partial \rho}\right)^2$$

$$= f_{\tau\tau}\left(-1\rho^{-2}\right)^2 = f_{\tau\tau}\,\rho^{-4}.$$

$\therefore$ if $f_{\rho\rho} \geq 0 \Rightarrow f_{\tau\tau} = g_{\tau\tau} \geq 0$.

$$g_s(\tau, S) > 0$$

$$\Rightarrow \quad p_s > 0$$

$$\Rightarrow \quad -e_{\tau s} > 0 \qquad \text{eq } 2.02.$$

$$\Rightarrow \quad -T_\tau > 0$$

$$\frac{\partial \rho^{-1}}{\partial \rho} = -\rho^{-2}$$

$$\Rightarrow \quad -T_\rho \frac{d\rho}{d\tau} > 0 \qquad \therefore \ \frac{\partial \rho}{\partial \tau} = \frac{-1}{\rho^2}$$

$$\Rightarrow \quad -T_\rho(-\rho^{-2}) > 0 \quad \Rightarrow \quad T_\rho > 0$$

$$R = \frac{R_0}{M}$$

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s = \gamma A \rho^{\gamma - 1} = \frac{\gamma}{\rho} A \rho^\gamma = \frac{\gamma}{\rho} p = \gamma \tau p$$

$$= \gamma R T$$

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What is wrong w/ the following dimensional argument?

$$c^2 = \frac{dp}{d\rho} = -v^2 \frac{dp}{dv}$$

$$[c^2] = -[v]^2 \frac{[p]}{[v]} = -[v][p] = -\frac{L^3 \left(\frac{LM}{T^2}\right)}{L^2}$$

$$= -\frac{L^2 M}{T^2} = -\left(\frac{L^2}{T^2}\right) M$$

How do we account for the mass in this expression, i.e.

$$[c^2] = \frac{L^2}{T^2}$$

This is because v is specific volume i.e. volume per unit mass

so $[v] = \frac{L^3}{M}$ + the above becomes:

$$[c^2] = [v][p] = \frac{\cancel{L^3}^{\,1}}{\cancel{M}} \cdot \frac{\left(\frac{LM}{T^2}\right)}{\cancel{L^2}} = \left(\frac{L}{T}\right)^2 \checkmark$$

pg 7 C/F

B

$$e = \frac{1}{\gamma-1} p\tau = \frac{A\rho^{\gamma}\tau}{\gamma-1} = \frac{A\tau^{-\gamma}\tau}{\gamma-1} = \frac{A\tau^{-(\gamma-1)}}{\gamma-1}$$

$$= \frac{A\rho^{\gamma-1}}{\gamma-1} \qquad \text{eq } (3,07)$$

$$RT = \underbrace{p\tau}_{\text{eq } 3.01} = \underbrace{\tau A\rho^{\gamma}}_{\text{eq } 3.03} = A\tau^{-(\gamma-1)} = A\rho^{\gamma-1}$$

$$p_\rho(\rho, S) = A\gamma\rho^{\gamma-1} > 0 \quad \text{if } \rho > 0 \therefore \text{eq } 2.04 \checkmark$$

$$p_S(\tau, S) = A_S \tau^{-\gamma} = \frac{(\gamma-1)e^{\frac{S-S_0}{c_v}}}{c_v} \tau^{-\gamma} > 0 \quad \text{yes} \quad \gamma > 1$$

$$\text{eq } 3.09 \qquad A = p\tau^{\gamma} = (\gamma-1)\exp\left\{\frac{S-S_0}{c_v}\right\}$$

$$\Rightarrow c_v \log\left(\frac{p\tau^{\gamma}}{\gamma-1}\right) + S_0 = S$$

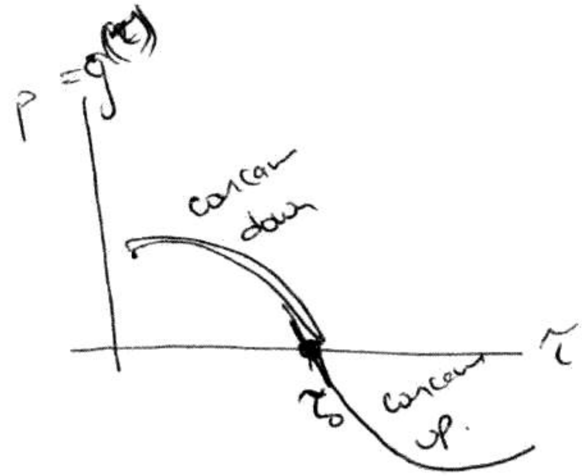
$$de = T\,dS - p\,d\tau$$

$$e = e^{(1)}(\tau) + e^{(2)}(S)$$

$$p = -e_\tau \qquad T = e_S$$

$$p = +A\left(\frac{\rho}{\rho_0}\right)^\gamma - B$$

$$p = (\tau_0-\tau)\frac{E}{\tau_0}$$



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pg 14

# eqs		# unknowns
~~7.01 ⇒ 3~~		
7.02 ⇒ 1		7.02
7.03 ⇒ 3		
7.04 ⇒ 1 } 1		
7.05 ⇒ 1		

7.08.2	1	4	$p, \vec{u}$
7.09.1	3	1	p
7.10	1	1	S
7.11	1	0	

$e = C_v T$ def of polytropic

$+ \quad C_v = \frac{R}{\gamma - 1}$ $\qquad p\tau = RT$

$$e = \frac{RT}{\gamma-1} = \frac{p\tau}{\gamma-1}$$

If $\quad p = f(\rho) \quad$ or $\quad p = g(\tau)$

Then from 7,02

$$p = -e_\tau \Rightarrow \quad g(\tau) = -e_\tau \Rightarrow \quad e = \underbrace{-\int_{\tau_0}^{\tau} g(\tau')\,d\tau'}_{e^{(1)}(\tau)} + \underbrace{C(S)}_{e^{2}(S)}$$

+ energy is seperable

If energy is seperable $\quad e = e^{(1)}(\tau) + e^{(2)}(S)$

$$p = -\frac{\partial e^{(1)}(\tau)}{\partial \tau} = g(\tau)$$

If T depends only on $S \Rightarrow \quad T = T(S) = e_S$

$$\Rightarrow \quad e = \int_{S_0}^{S} T(S')\,dS' + C(\tau)$$

$\rightarrow$ energy is seperable $\Rightarrow \quad p = g(\tau)$ see above

$$de = T\,dS - p\,d\tau$$

$p = -e_\tau \qquad T = e_S \qquad$ ideal gas $T = \dfrac{p\tau}{R}$

$$\frac{p\tau}{R} = e_S$$

$\Rightarrow$ mult 1st eq by $\dfrac{\tau}{R}$ + subtract from 2nd eq

$$\frac{p\tau}{R} = -\frac{\tau}{R}e_\tau$$

$$-\;\frac{p\tau}{R} = e_S$$

$$0 = \frac{\tau}{R}e_\tau + e_S \quad\Rightarrow\quad Re_S + \tau e_\tau = 0 \qquad \text{eq } 4.01$$

solve eq 4.01 by method of characteristics, let t be coord along a characteristic + t_2 parametrize the initial curve in S,τ space

when $t=0$ we land on the initial curves: $S = S(t_2)$, $\tau = \tau(t_2)$

$$\frac{dS}{dt} = R \qquad \frac{d\tau}{dt} = \tau \qquad \frac{de}{dt} = 0$$

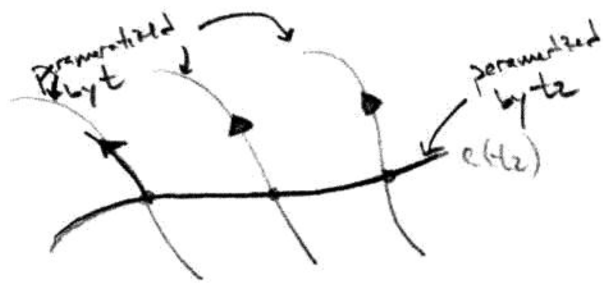
$$S = Rt + S_0 \qquad \frac{d\tau}{\tau} = dt \qquad e = e_0 = e(t_2) \qquad e = e(t_2)$$

$S(t=0) = S_0 = S(t_2)$

$$\ln\tau = t + t_0$$

$$\tau = e^{t+t_0} = \tau_0 e^{t}$$

$$\tau(t=0) = \tau_0 = \tau(t_2)$$



$\Rightarrow \quad e = e(t_2) \qquad S = Rt + S(t_2) \Rightarrow \dfrac{S - S(t_2)}{R} = t$

$$\tau = \tau(t_2)e^{t}$$

$$\tau = \tau(t_2)\exp\left[\frac{S - S(t_2)}{R}\right] = \tau(t_2)\exp\left[-\frac{S(t_2)}{R}\right]\exp\left[\frac{S}{R}\right]$$

$$\Rightarrow \underbrace{\tau(t_2)\exp\left(-\frac{S(t_2)}{R}\right)} = \tau\exp\left(-\frac{S}{R}\right)$$

$$F(t_2) = \tau\exp\left(-\frac{S}{R}\right)$$

$$t_2 = F^{-1}\left(\tau\exp\left(-\frac{S}{R}\right)\right)$$

$$\therefore \quad e = e\left(F^{-1}\left(\tau\exp\left(-\frac{S}{R}\right)\right)\right) = e\left(\tau\exp\left(-\frac{S}{R}\right)\right) \quad \text{eq 4.02}$$

& eq 4.03

$$\equiv h(\tau H)$$

$$p = -e_\tau = -h'(\tau H)H = -h'(\rho^{-1}H)H \quad \text{eq 4.04}$$

Req $f_p(\rho, S) > 0$ $\quad g_\tau(\tau, S) < 0$

& $g_{\tau\tau}(\tau, S) > 0$

$$p_\rho = -h''(\rho^{-1}H)H^2(-\rho^{-2}) = v^2H^2h''(\rho^{-1}H) > 0$$

$$h''(\tau H) > 0$$

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$p = -e_\tau \qquad T = \frac{p\tau}{R}$

$T = e_s$

$p = -e_\tau$ stays same +

$\frac{p\tau}{R} = +e_s \Rightarrow p = +\frac{R}{\tau} e_s$

$\therefore \; -e_\tau = +\frac{R}{\tau} e_s$

$R e_s + \tau e_\tau = 0$ linear eq for e_τ.

Now sol w/ method of char

$\xi = \xi(s,\tau)$

$\frac{de}{d\xi} = 0$ Along $\frac{d\xi}{ds} = R \qquad \frac{d\xi}{d\tau} = \tau$

$\Rightarrow \xi = Rs + \xi(\tau)$

$\therefore \; \xi = Rs + \frac{\tau^2}{2} + \xi_0$

$e = e(\xi)$

$\frac{d}{d\xi} e(s(\xi), \tau(\xi)) = e_s \frac{ds}{d\xi} + e_\tau \frac{d\tau}{d\xi} =$

Not

$\Rightarrow \frac{ds}{d\xi} = R \;+\; \frac{d\tau}{d\xi} = \tau$

$R e_S + \tau e_\tau = 0$ $\qquad$ $e = e(S(\xi), \tau(\xi)) = e(\xi)$

if: $\frac{dS}{d\xi} = R$ $\qquad$ $\frac{de}{d\xi} = 0$ $\qquad$ $\frac{de}{d\xi} = e_S S_\xi + e_\tau \tau_\xi$
$= R e_S + \tau e_\tau$

$\frac{d\tau}{d\xi} = \tau$

$$\frac{dS}{d\tau} = \frac{R}{\tau}$$

$\therefore$ $\cancel{S = R\xi + S(\tau)}$

& $\cancel{\frac{d\tau}{\tau} = d\xi}$ $\qquad \Rightarrow dS = \frac{R\,d\tau}{\tau}$

$\cancel{\ln\tau = \xi + \xi_0}$ $\qquad \Rightarrow S = R\ln\tau + C_1$

$\Rightarrow$ $\qquad \tau = C_2 e^{S/R} \Rightarrow C_2 = e^{-S/R}\cdot\tau.$

$\therefore$ $\cancel{S = R\ln\left(\frac{\tau}{\tau_0}\right) + S_0}$

$S = R\xi + C_1 \Rightarrow \xi = \frac{S}{R} - C_1$

$\ln\tau = \xi + C_2$

$\ln\tau = \frac{S}{R} - C_1 + C_2 = \frac{S}{R} + C_3$

$\tau = C_4 e^{S/R}$

$\tau e^{-S/R} = C_4$. a constant of

$\therefore$ sol $e = e(\tau e^{-S/R})$

$$p = -e_\tau = -h'(\tau H)H$$

$$= -h'(\rho^{-1}H)H$$

monotonicity of 2.04 $f_\rho > 0$ + $g_\tau < 0$

+ 2.05

convexity condition $g_{\tau\tau} > 0$

require ~~the~~ $p_\tau = -h''(\tau H)H^2 < 0$

$$\Rightarrow \quad h'' > 0$$

$$g_\tau = -h''(\tau H)H^2$$

$$g_{\tau\tau} = -h'''(\tau H)H^3 \overset{\text{required}}{>} 0$$

$$\Rightarrow \quad h''' < 0$$

$$T = e_s = h'(\tau H)\frac{(-1)}{R}H = \frac{-1}{R}h'(\tau H)H\tau$$

For physical systems T decreases as $\tau H \nearrow$.

$\therefore \quad T = \hat{T}(\tau H)$ is invertible. $\therefore \quad \tau H = \mathcal{F}(T)$

$\therefore \quad e = h(\tau H) = h(\mathcal{F}(T))$ + energy is a fn of temp. only.

$$c^2 = f_\rho \qquad c^2 = \frac{-1}{\rho^2} g_\tau$$

$$c^2 = \frac{-1}{\rho^2}\left(-h''(\tau H)\,H^2\right) = h''(\tau H)\,\tau^2 H^2$$

$$e_\tau + RT_\tau = h'(\cancel{\tau H})H + R\left(\frac{-1}{R}\right)\left[h''(\tau H)\tau H^2 + \cancel{h'(\tau H)}H\right]$$

$$= -\frac{h''(\tau H)\tau^2}{\tau}H^2 = -h''(\tau H)\tau H^2$$

$$+\ RT = -h'(\tau H)\tau H = \underbrace{-h'(\tau H)H}\cdot \tau$$

$$p\cdot\tau = -\tau e_\tau.$$

Then $\displaystyle c^2 = \frac{\partial p}{\partial \rho} = \frac{\partial(\rho^{-1})}{\partial\rho}\frac{\partial p}{\partial \tau} = -\rho^{-2}\frac{\partial p}{\partial \tau} = -\tau^2\frac{\partial p}{\partial \tau} = -\tau^2\frac{\partial}{\partial \tau}(-e_\tau)$

$$= +\tau^2 e_{\tau\tau}$$

$$e = C_v T$$

$$\| \quad = -\frac{C_v}{R} h'(\tau H)\tau H \quad = -\frac{1}{\gamma-1} h'(\tau H)\tau H$$

$h(\tau H)$

$$\therefore \quad h'(\tau H) = -(\gamma-1)(\tau H)^{-1} h(\tau H)$$

$$\therefore \quad \frac{dh}{h} = -\frac{(\gamma-1)}{(\tau H)}$$

$$\ln h = -(\gamma-1)\ln(\tau H) + C_1$$

$$h = C_2 \cancel{B} (\tau H)^{-(\gamma-1)}$$

$$h = \left(\frac{\tau H}{H_0}\right)^{-(\gamma-1)}$$

Then $p = -h'(\tau H) H$

$$= +\frac{(\gamma-1)}{H_0}\left(\frac{\tau H}{H_0}\right)^{-\gamma+1-1} H \quad = \frac{\gamma-1}{H_0} \tau^{-\gamma} \left(\frac{H}{H_0}\right)^{-\gamma} H$$

$$= A \tau^{-\gamma} \qquad w/ \qquad A = \frac{\gamma-1}{H_0}\left(\frac{H}{H_0}\right)^{-\gamma} H$$

$$= \gamma-1 \left(\frac{H}{H_0}\right)^{-(\gamma-1)}$$

$$H = \exp\{-S/R\}$$

$$= A(S)^{-(\gamma-1)} = \left[\exp\left(-\frac{S}{R}+\frac{S_0}{R}\right)\right]^{-(\gamma-1)}$$

$$= (\gamma-1)\exp\left[\underbrace{\frac{(\gamma-1)}{R}}_{= C_v^{-1}}(S-S_0)\right]$$

$$\rho \dot{e}_{tot} + (pu)_x + (pv)_y + (pw)_z = 0$$ by B.01

$$\Rightarrow \rho\left[\frac{q^2}{2} + \cancel{\beta}e\right] + (pu)_x + (pv)_y + (pw)_z = 0$$

~~w~~

$$\rho\left[\frac{q^2}{2} + e\right]_t + \rho U \circ \nabla\left[\frac{q^2}{2} + e\right] + (pu)_x + (pv)_y + (pw)_z = 0$$

From $\dot{F} = F_t + (\vec{U} \circ \nabla) F$ (By Reynolds transport thm)

$\Rightarrow$ $i \equiv e + p\tau$

$$\rho\left[\tfrac{1}{2}q^2\right]_t + \rho e_t + \rho\vec{U} \circ \nabla[e]$$

$$+ \rho\bar{U} \circ \nabla\left[\frac{q^2}{2}\right] + (pu)_x + (pv)_y + (pw)_z = 0$$

$\Rightarrow$ ~~[illegible]~~

$de = \tau^2 p\,dp + T\,dS$ so the chace becomes

$$\rho\left(\tfrac{1}{2}q^2\right)_t + \rho\tau^2 p\,p_t + \rho T S_t$$

$$\rho\vec{U}\circ\nabla\left[i - p\tau\right] + \rho\vec{U}\circ\nabla\left[\frac{q^2}{2}\right] + \nabla\circ(p\vec{U}) = 0 \checkmark$$

$$\Rightarrow \rho\left(\tfrac{1}{2}q^2\right)_t + \rho\vec{U}\circ\nabla\left[i + \frac{q^2}{2}\right] - \rho\vec{U}\circ\nabla(p\tau)$$

$$+ \tau p p_t + \rho T S_t \qquad + \nabla\circ(p\vec{U}) = 0 \checkmark$$

$$\Rightarrow \rho\left(\tfrac{1}{2}q^2\right) + \rho T S_t + \rho\vec{U}\circ\nabla\left[i + \frac{q^2}{2}\right]$$

$$+ \tau p p_t - \rho\vec{U}\circ\nabla(p\tau) + \nabla\circ(p\vec{U}) = 0$$

From eq ~~7.26.2~~

$$\nabla(p\tau) = \tau\nabla p + p\nabla\tau$$

$$\Rightarrow \rho\left(\tfrac{1}{2}q^2\right)_t + \rho T S_t + \rho\vec{U}\circ\nabla\left[i + \frac{q^2}{2}\right]$$

$$+ \tau p \rho_t - \rho\tau\, \cancel{\bar{u}\circ}\nabla p - p\rho\, \bar{u}\circ\nabla\tau$$

$$+ \cancel{\vec{u}\circ}\nabla p + p\nabla\circ\vec{u} = 0$$

$$\Rightarrow \rho\left(\tfrac{1}{2}q^2\right)_t + \rho T S_t + \rho\bar{U}\circ\nabla\left[i + \frac{q^2}{2}\right]$$

$$+ \tau p \rho_t - p\rho\left(-\tau^2\right)\bar{u}\circ\nabla\rho$$

$$+ p\nabla\circ\bar{u} = 0$$

$$\nabla\tau = -\rho^{-2}\nabla\rho$$
$$= -\tau^2\nabla\rho$$

$\Rightarrow$

$$\rho\left(\tfrac{1}{2}q^2\right)_t + \rho T S_t + \rho\vec{U}\circ\nabla\left[i + \frac{q^2}{2}\right]$$

$$+ \underbrace{\tau p \rho_t + \tau p\, \bar{u}\circ\nabla\rho + p\nabla\circ u} = 0$$

$$\cancel{p(\tau\rho_t + \tau}$$

$$p\tau\left(\underbrace{\rho_t + \bar{u}\circ\nabla\rho + \rho\nabla\circ u}\right)$$

$$= 0$$

pg 23 Lowcut

$$c_* = \mu \hat{q}$$

$$q^2 + \frac{2}{r-1}c^2 = \hat{q}^2$$

define $\mu^2 = \frac{r-1}{r+1}$ we get

$$\frac{(r-1)}{r+1}q^2 + \frac{2\,\cancel{[illegible]}\,c^2}{r+1} = \left(\frac{r-1}{r+1}\right)\hat{q}^2$$

$$\Rightarrow \mu^2 q^2 + 2(r+1)c^2 = \mu^2\hat{q}^2$$

Now from $\mu^2 = \frac{r-1}{r+1} \Rightarrow \mu^2(r+1) = r-1$

$$\Rightarrow (\mu^2-1)r = -\mu^2-1$$

$$\Rightarrow r = \frac{1+\mu^2}{1-\mu^2}$$

So $r+1 = \frac{1+\mu^2+1-\mu^2}{1-\mu^2} = \frac{2}{1-\mu^2}$

So $\frac{2}{r+1} = 2\left(\frac{1-\mu^2}{2}\right) = 1-\mu^2$

$\therefore$

$$\mu^2 q^2 + (1-\mu^2)c^2 = \mu^2\hat{q}^2 \quad \text{or} \quad \text{eq}$$

$$\mu^2 q^2 + (1-\mu^2) c^2 = c_*^2$$

$$\Rightarrow \quad q^2 - (\mu^2 q^2 + (1-\mu^2) c^2) = q^2 - c_*^2$$

$$\Rightarrow \quad (1-\mu^2) q^2 + (\mu^2 - 1) c^2 = \text{"}$$

$$\Rightarrow \quad (1-\mu^2)(q^2 - c^2) = q^2 - c_*^2$$

4 25 Ciurlt

$$c^2 - q^2 = 2(i + \tfrac{1}{2}c^2) - \hat{q}^2$$

Since $\rho \to 0 \Rightarrow \tau \to \infty$ from eq 9.04 $i \to 0$ as $\rho \to 0$

+ $c \to 0$ as $\rho \to 0$

$\therefore$ $c^2 - q^2 \longrightarrow -\hat{q}^2$ as $\rho \to 0$.

So $c^2 - q^2 < 0$ when $\tau \to +\infty$

+ ~~xxxxx~~ if $2i(\tau, S) = \hat{q}$ $c^2 - q^2 > 0$

(How know $i(\tau, S) = \frac{\hat{q}}{2}$)

pg 25 Courant

$$q^2 + 2i = \hat{q}^2$$

$\frac{d}{dq} \Rightarrow \quad 2q\,dq + 2di = 0$

$\Rightarrow \quad q\,dq + di = 0$

$\Rightarrow$

$$q = \sqrt{u^2 + v^2}$$

$$dq = \frac{1}{2}\frac{(2u\,du + 2v\,dv)}{\sqrt{u^2+v^2}}$$

$$\Rightarrow \quad q\,dq = u\,du + v\,dv$$

So $\Rightarrow \quad u\,du + v\,dv + di = 0$

w/ $\quad di = c^2 \tau\, dp \qquad$ from $\quad di = c^2 \tau\, dp$

we get $\quad u\,du + v\,dv = -c^2 \tau\, dp \qquad$ eq 16.04

· Irrotational flow:

$$v_x - u_y = 0$$

given continuity: $\quad (\rho u)_x + (\rho v)_y = 0$

+ use 16.04 to eliminate $d\rho$.

$$\rho_x u + \rho u_x + \rho_y v + v_y \rho = 0$$

$$\Rightarrow \quad -\frac{1}{c^2\tau}(u u_x + v v_x) u + \rho u_x + \frac{-1}{c^2\tau}(u u_y + v v_y) v + \rho v_y = 0$$

$$\Rightarrow \quad -u^2 u_x - vu v_x + c^2 u_x - vu u_y - v^2 v_y + c^2 v_y = 0$$

$$\Rightarrow \quad (c^2 - u^2) u_x - vu(v_x + u_y) + (c^2 - v^2) v_y = 0 \qquad \text{eq 16.08} \checkmark$$

why is c^2 to be a function of flow speed q?

polytropic gas $\qquad p = A(S) \rho^\gamma \qquad 1 \leq \gamma \leq \frac{5}{3}$

$$q^2 + \frac{2}{\gamma - 1} c^2 = \hat{q}^2 \Rightarrow \mu^2 q^2 + (1 - \mu^2) c^2 = c_*^2 = (\mu \hat{q})^2$$

$$\Downarrow$$

$$c^2 = \frac{\gamma - 1}{2} (\hat{q}^2 - q^2)$$

& w/ $\quad \gamma = \dfrac{1 + \mu^2}{1 - \mu^2} \qquad \gamma - 1 = \dfrac{1 + \mu^2 - (1 - \mu^2)}{1 - \mu^2}$

$$= \frac{2\mu^2}{1 - \mu^2}$$

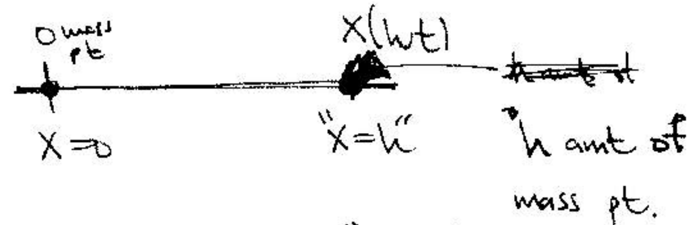
so $\quad \dfrac{\gamma - 1}{2} = \dfrac{\mu^2}{1 - \mu^2} \qquad 0 < \mu^2 < 1$

$$c^2 = \frac{1}{1 - \mu^2} \left((\mu \hat{q})^2 - (\mu q)^2 \right) = \frac{1}{(1 - \mu^2)} \left(c_*^2 - \mu^2 q^2 \right)$$

eq 16.09 $\checkmark$

pg 30 Courant/Friedrichs

$$h = \int_{x(0,t)}^{x(h,t)} \rho\, dx$$

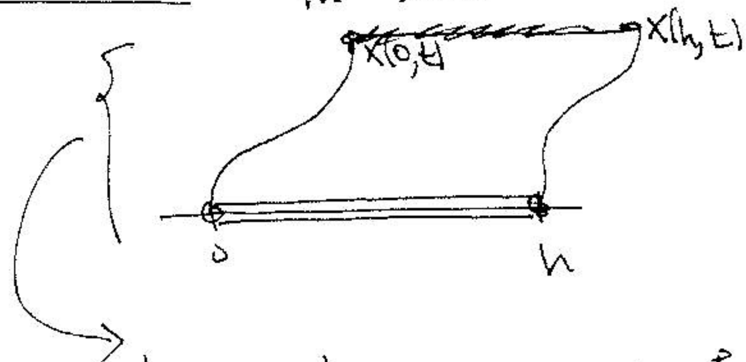


$\frac{\partial}{\partial h}$

$$\Rightarrow 1 = \rho \frac{\partial x}{\partial h} \qquad \boxed{1 = \rho x_h}$$

$$\Rightarrow x_h = 1/\rho = \tau(h,t)$$

Now as the "fluid" material moves we require that the amount of mass between 0 mass pt. + "h" mass pt stays the same



$$* \quad \underset{\substack{\uparrow \\ \text{initial}}}{h} = \int_{x(0,t)}^{x(h,t)} \rho(\hat{x},t)\, d\hat{x}$$

diagram showing movement of tagged particles. eq * Then can be interpreted as

$$\int_{x(0,t)}^{x(h,t)} \rho(\hat{x},t)\, d\hat{x} = \text{Amt of mass at time } t \text{ between tagged particles } 0 + h.$$

Set.

$=$ h = mass between 0 + "h" initially.

eq 7.02 $\quad \dot{(\rho\Delta)} = 0$

$$\Delta = \frac{\partial(x,y,z)}{\partial(a,b,c)} \overset{1D}{=} \frac{\partial x}{\partial a} = x_h$$

$$\Rightarrow (\rho x_h)_t = 0$$

eq (7.03) $\Rightarrow \quad \rho\ddot{x} + p_x = 0$

$$\Rightarrow \rho\ddot{x} = -p_x = -p_h h_x = -\rho p_h$$

$$\Rightarrow x_{tt} = -p_h$$

Now $x_{tt} = -p_h$

But by caloric eq. of state.

$\Rightarrow p = g(\tau, S)$ ~~Khe~~ + as $S = S(h)$ only

$$p = g(\tau(h,t), S(h))$$

$$p_h = g_\tau \tau_h + g_S S_h$$

But $\tau_h = x_{hh}$ by eq 18.02.2

~~then~~

$$\Rightarrow x_{tt} = -g_\tau x_{hh} \mp g_S S_h$$

$$x_{tt} = k^2 x_{hh} - g_S S_h$$

but $g_\tau = p_\rho \rho_\tau = p_\rho (\tau^{-1})_\tau$

$\therefore \quad = -\tau^{-2} p_\rho = -\rho^2 c^2$

$\therefore \quad -g_\tau = \rho^2 c^2$

$+ \quad \underline{\sqrt{-g_\tau} = \rho c}$ acoustic impedance

let $\underset{\overset{\shortparallel}{x_t}}{u}$ + $\underset{\overset{\shortparallel}{x_h}}{\tau}$ be dependent variables

then continuity of 2nd partials gives

$$u_h = \tau_t$$

+ eq $\cancel{x_{tt}}$ $x_{tt} = k^2 x_{hh} - g s S'_h$

$$u_t = k^2 \tau_h - g s S'_h$$

if $k^2 = -p_\tau$ = constant   hooks law for solid

$p = (\tau_0 - \tau) E/\tau_0$

$p_\tau = E/\tau_0$. const

Then

Aside given eqs in Eulerian – Derive the corresponding eqs in Lagrangian.

Eulerian eqs $\rho_t + (\rho u)_x = 0$ mass cnt

$(\rho u)_t + (\rho u^2 + p - \nu u_x)_x = 0$ mmt w/ viscosity.

~~Now~~

$(x, t) \longrightarrow (h, \bar{t})$

Eulerian coord Lagrangian coord under transformation

$h = h(x,t)$ Fundamental Eulerian ↔ Lagrangian relationship is

$\bar{t} = t$ $x_h = \rho^{-1}$ $\frac{\partial x}{\partial h} = \frac{1}{\rho}$

Thus $\frac{\partial}{\partial x} = \frac{\partial}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial x}$

$= \frac{\partial h}{\partial x}\frac{\partial}{\partial h} \cancel{+} \cancel{\frac{\partial}{\partial \bar{t}}}$

$= \rho\frac{\partial}{\partial h} \cancel{+} \cancel{\frac{\partial}{\partial \bar{t}}}$

$\frac{\partial}{\partial t} = \frac{\partial h}{\partial t}\frac{\partial}{\partial h} + \frac{\partial \bar{t}}{\partial t}\frac{\partial}{\partial \bar{t}}$

$= \qquad + \frac{\partial}{\partial \bar{t}}$

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If $L = \lambda_1 L_1 + \lambda_2 L_2$

$$\Rightarrow \lambda_1 A_1 u_x + \lambda_2 A_2 u_x + (\lambda_1 B_1 + \lambda_2 B_2) u_y + (\lambda_1 C_1 + \lambda_2 C_2) v_x + (\lambda_1 D_1 + \lambda_2 D_2) v_y + \lambda_1 E_1 + \lambda_2 E_2 = 0$$

$$\Rightarrow \underline{(\lambda_1 A_1 + \lambda_2 A_2)} u_x + \underline{(\lambda_1 B_1 + \lambda_2 B_2)} u_y + (\lambda_1 C_1 + \lambda_2 C_2) v_x + (\lambda_1 D_1 + \lambda_2 D_2) v_y + \lambda_1 E_1 + \lambda_2 E_2 = 0$$

~~Want~~ $= D_\xi u + D_\eta v + \lambda_1 E_1 + \lambda_2 E_2 = 0$

Derivative of u in direction ξ. Derivative of v in direction η.

want these to be in same direction

$\Rightarrow$

Multiply L by x_B

$$= A_1 U_x x_B + B_1 U_y x_B + C_1 V_x x_B + D_1 V_y x_B + E_1 x_B$$

~~Not~~ This is only expression L_1

$$x_B L = (\lambda_1 A_1 + \lambda_2 A_2) x_B U_x + (\lambda_1 B_1 + \lambda_2 B_2) x_B U_y$$

$$+ (\lambda_1 C_1 + \lambda_2 C_2) x_B V_x + (\lambda_1 B_1 + \lambda_2 B_2) x_B V_y$$

$$+ x_B(\lambda_1 E_1 + \lambda_2 E_2) = 0$$

$$\frac{\lambda_1 B_1 + \lambda_2 B_2}{\lambda_1 A_1 + \lambda_2 A_2} = \frac{y_B}{x_B}$$

$$\therefore (\lambda_1 B_1 + \lambda_2 B_2) x_B = (\lambda_1 A_1 + \lambda_2 A_2) y_B$$

$\therefore$ Above becomes

$$x_B L = (\lambda_1 A_1 + \lambda_2 A_2) x_B U_x + (\lambda_1 A_1 + \lambda_2 A_2) y_B U_y$$

$$+ (\lambda_1 C_1 + \lambda_2 C_2) x_B V_x + (\lambda_1 A_1 + \lambda_2 A_2) y_B V_y$$

$$+ (\lambda_1 E_1 + \lambda_2 E_2) x_B = 0$$

$$\Rightarrow \quad X_B L = (\lambda_1 A_1 + \lambda_2 A_2) U_B + (\lambda_1 C_1 + \lambda_2 C_2) V_B + (\lambda_1 E_1 + \lambda_2 E_2) X_B = 0.$$

eq 22.20

— — — —

$$\downarrow \quad Y_B L = \underbrace{(\lambda_1 A_1 + \lambda_2 A_2)}_{\text{change using below.}} Y_B U_x + (\lambda_1 B_1 + \lambda_2 B_2) Y_B U_Y$$

$$+ \underbrace{(\lambda_1 C_1 + \lambda_2 C_2)}_{\text{change using below.}} Y_B V_x + (\lambda_1 D_1 + \lambda_2 D_2) Y_B V_Y$$

$$+ \quad Y_B(\lambda_1 E_1 + \lambda_2 E_2) = 0$$

Also $\dfrac{\lambda_1 D_1 + \lambda_2 D_2}{\lambda_1 C_1 + \lambda_2 C_2} = \dfrac{Y_B}{X_B} = \left(\dfrac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2}\right)^{-1} = \circledast$

$$(\lambda_1 C_1 + \lambda_2 C_2) Y_B = (\lambda_1 D_1 + \lambda_2 D_2) X_B$$

$\therefore$

$$Y_B L = (\lambda_1 B_1 + \lambda_2 B_2)[X_B U_x + Y_B U_Y]$$

$$(\lambda_1 D_1 + \lambda_2 D_2)[X_B V_x + Y_B V_Y]$$

$$+ \quad (\lambda_1 E_1 + \lambda_2 E_2) Y_B$$

$$= (\lambda_1 B_1 + \lambda_2 B_2) U_B + (\lambda_1 A_1 + \lambda_2 A_2) V_B + (\lambda_1 E_1 + \lambda_2 E_2) Y_B$$

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if $L \equiv 0$. then

1) $$\lambda_1(A_1 u_\sigma + C_1 v_\sigma + E_1 x_\sigma) + \lambda_2(A_2 u_\sigma + C_2 v_\sigma + E_2 x_\sigma) = 0$$

2) $$\cancel{\lambda_1(B_1 u_\sigma + D_1 v_\sigma) + \lambda_2(B_2 u_\sigma + D_2 v_\sigma)}$$

$$\lambda_1(B_1 u_\sigma + D_1 v_\sigma + E_1 y_\sigma) + \lambda_2(B_2 u_\sigma + D_2 v_\sigma + E_2 y_\sigma) = 0$$

+ $$\frac{\lambda_1 A_1 + \lambda_2 A_2}{\lambda_1 B_1 + \lambda_2 B_2} = \frac{x_\sigma}{y_\sigma}$$ eq that derivative in direction of σ is along the one selected by the eq.

$$\lambda_1(A_1 y_\sigma - B_1 x_\sigma) + \lambda_2(A_2 y_\sigma - B_2 x_\sigma) = 0$$

+ $$\frac{\lambda_1 C_1 + \lambda_2 C_2}{\lambda_1 D_1 + \lambda_2 D_2} = \frac{x_\sigma}{y_\sigma}$$

$$\lambda_1(C_1 y_\sigma - D_1 x_\sigma) + \lambda_2(C_2 y_\sigma - D_2 x_\sigma) = 0$$ eq 22.04 ✓.

4 eqs 2 unknowns

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0$$

Why dets of submatrices vanish.

$$(A_1 y_6 - B_1 x_6)(C_2 y_6 - D_2 x_6) - (A_2 y_6 - B_2 x_6)(C_1 y_6 - D_1 x_6) = 0$$

$$(A_1 C_2 - A_2 C_1) y_6^2 + (-A_1 D_2 - B_1 C_2 + A_2 D_1 + B_2 C_1) y_6 x_6$$

$$+ (B_1 D_2 - B_2 D_1) x_6^2 = 0$$

$$\Rightarrow \quad [AC] y_6^2 - 2([AD] - [BC]) x_6 y_6 + [BD] x_6^2 = 0.$$

$$(2b)^2 - 4ac = 4(b^2 - ac) > 0$$

$a \neq 0$. & $x_6 \neq 0$ for if it did

$$2205 \rightarrow \begin{vmatrix} A_1 y_6 & A_2 y_6 \\ C_1 y_6 & C_2 y_6 \end{vmatrix} = y_6 [AC] = 0$$

$$\Rightarrow [AC] = 0 \rightarrow\leftarrow$$

$$\frac{y_6}{x_6} = \rho_{\pm}$$

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$$\lambda_1 (A_1 y_B - B_1 x_B) + \lambda_2 (A_2 y_B - B_2 x_B)$$

$$\cancel{\lambda_1 (B_1 u_B + D_1 v_B + E_1 y_B) + \lambda_2 (B_2 u_B + D_2 v_B + E_2 y_B)} = 0$$

$$\Rightarrow \begin{vmatrix} A_1 y_B - B_1 x_B & A_2 y_B - B_2 x_B \\ \cancel{B_1 u_B + D_1 v_B + E_1 y_B} & \cancel{B_2 u_B + D_2 v_B + E_2 y_B} \\ A_1 u_B + C_1 v_B + E_1 x_B & A_2 u_B + C_2 v_B + E_2 x_B \end{vmatrix} = 0$$

let $y_B = \xi x_B$

$$\Rightarrow \begin{vmatrix} (A_1 \xi - B_1) \cancel{x_B} & (A_2 \xi - B_2) \cancel{x_B} \\ A_1 u_B + C_1 v_B + E_1 x_B & A_2 u_B + C_2 v_B + E_2 x_B \end{vmatrix} = 0$$

$$\Rightarrow \quad A_1 A_2 u_B \xi + A_1 C_2 v_B \xi + A_1 E_2 x_B \xi$$

$$- B_1 A_2 u_B - B_1 C_2 v_B - B_1 E_2 x_B$$

$$- A_1 A_2 u_B \xi + A_1 B_2 u_B - C_1 A_2 v_B \xi + C_1 B_2 v_B$$

$$- E_1 A_2 \xi u_B + E_1 B_2 x_B = 0$$

$$= (A_1 A_2 \xi - B_1 A_2 - A_1 A_2 - E_1 A_2 \xi + A_1 B_2) u_6$$

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Attempting to get eq 22.15

$$\begin{vmatrix} A_1\xi - B_1 & A_2\xi - B_2 \\ A_1u_B + C_1v_B + E_1x_B & A_2u_B + C_2v_B + E_2x_B \end{vmatrix} = 0$$

$$= (\cancel{A_1A_2\xi} - A_2B_1 - \cancel{A_1A_2\xi} + A_1B_2)\,u_B$$

$$+ (C_2A_1\xi - C_2B_1 - C_1A_2\xi + C_1B_2)\,v_B$$

$$+ (A_1E_2\xi - E_2B_1 - A_2E_1\xi + E_1B_2)\,x_B = 0$$

$$= [A\,B]u_B + \left(-[C\,A]\xi + [C\,B]\right)v_B$$

$$+ \left([A\,E]\xi + [E\,B]\right)x_B = 0$$

$$= Tu_B + (a\xi - S)\,v_B + (K\xi - H)\,x_B = 0$$

w/ $T = [AB]$ $\quad K = [AE]$

$S = [BC]$ $\quad H = [BE]$

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reducible $\Rightarrow E_1 = E_2 = 0.$

$$T \frac{dU}{d\alpha} + (a\xi_+ - S)\frac{dV}{d\alpha} = 0$$

$$T \frac{dU}{d\beta} + (a\xi_- - S)\frac{dV}{d\beta} = 0$$

$$T \frac{dU}{dV} = S - a\xi_+$$

$$T \frac{dU}{dV} = S - a\xi_-$$

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$$\rho_t + u\rho_x + \rho u_x = 0$$

$$\rho(u_t + uu_x) + c^2\rho_x = 0 \Rightarrow u_t + uu_x + \frac{c^2}{\rho}\rho_x = 0$$

Mult 1st eq by λ & Add to second eq

$$u_t + (u + \lambda\rho)u_x + \left(\frac{c^2}{\rho} + u\lambda\right)\rho_x + \lambda\rho_t$$

$$\Rightarrow \underline{u_t} + \underline{(u+\lambda\rho)}u_x + \underline{\lambda}\rho_t + \underline{\left(\lambda u + \frac{c^2}{\rho}\right)}\rho_x = 0$$

set each combination to zero.

Need

$$x_\sigma = (u+\lambda\rho)t_\sigma$$

& $$\lambda x_\sigma = \left(\lambda u + \frac{c^2}{\rho}\right)t_\sigma$$

Then
$$u_\sigma = t_\sigma u_t + x_\sigma u_x$$
$$= (u+\lambda\rho)t_\sigma u_x + t_\sigma u_t$$
$$= t_\sigma\left(u_t + (u+\lambda\rho)u_x\right)$$

& $$\rho_\sigma = t_\sigma\rho_t + x_\sigma\rho_x$$
$$= t_\sigma\rho_t + \frac{1}{\lambda}\left(\lambda u + \frac{c^2}{\rho}\right)t_\sigma\rho_x$$

From $x_B = (u + \lambda\rho) t_B$

$$\lambda x_B = \left(\lambda u + \frac{c^2}{\rho}\right) t_B$$

$$\div \Rightarrow \quad \frac{1}{\lambda} = \frac{(u + \lambda\rho)}{\left(\lambda u + \frac{c^2}{\rho}\right)}$$

$$\cancel{\lambda u} + \frac{c^2}{\rho} = \cancel{\lambda u} + \lambda^2 \rho \quad \Rightarrow \quad \lambda^2 = \frac{c^2}{\rho^2} \text{ required.}$$

$$\lambda = \pm \frac{c}{\rho}$$

Thus $x_B = (u + c) t_B$

$$x_B = (u - c) t_B$$

2.3.01 w/ $\lambda = \pm c/\rho$ gives

$$u_t + (u + c) u_x + \frac{c}{\rho}\rho_t + \left(\frac{cu}{\rho} + \frac{c^2}{\rho}\right)\rho_x = 0$$

$$+ \quad u_t + (u - c) u_x - \frac{c}{\rho}\rho_t + \left(-\frac{cu}{\rho} + \frac{c^2}{\rho}\right)\rho_x = 0$$

$$= \quad \cancel{u_\alpha + \frac{c}{\rho}\left(\rho\rho_t + u +\right.}$$

$$u_\alpha + \frac{c}{\rho}\left(\rho_t + (u + c)\rho_x\right) = 0$$

$$+ \quad u_\beta - \frac{c}{\rho}\left(\rho_t + (u - c)\rho_x\right) = 0$$

$$\Rightarrow u_\alpha + \frac{c}{\rho}\rho_\alpha = 0$$

$$\& \quad u_\beta - \frac{c}{\rho}\rho_\beta = 0$$

For spherical isentropic flow:

$$\rho_t + u\rho_x + \rho u_x + \frac{2\rho u}{x} = 0$$

$$\rho(u_t + uu_x) + c^2\rho_x = 0$$

Mult 2st eq by λ & Add to the 1st eq