pg 182 Courant / John

$$|\vec{A}\times\vec{B}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_3a_1b_1b_3 + a_1^2b_3^2$$

$$+ a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$$

$$= a_1^2b_2^2 + a_1^2b_3^2 + a_2^2b_1^2 + a_2^2b_3^2 + a_3^2b_1^2 + a_3^2b_2^2$$

$$- 2a_2a_3b_2b_3 - 2a_1a_3b_1b_3 - 2a_1a_2b_1b_2$$

$$= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2)$$

$$-2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a_2b_1b_2)$$

$$= a_1^2(b_1^2 + b_2^2 + b_3^2) - a_1^2b_1^2 + a_2^2(b_1^2 + b_2^2 + b_3^2) -$$

$$- 2(a_2a_3b_2b_3$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2$$

$$- 2(a_2a_3b_2b_3 + a_1a_3b_1b_3 + a_1a_2b_1b_2)$$

$\overset{?}{=} |\bar{A}|^2 |\bar{B}|^2 - (\underbrace{a_1 b_1 + a_2 b_2 + a_3 b_3})^2$

$-2a_1a_2b_1b_2 - 2a_1a_3b_1b_3 - 2a_2b_2a_3b_3 +$ sqr terms

Yes

As $|\bar{A} \times \bar{B}|^2 \geq 0$

$\rightarrow |A||B| \geq |A \circ B|$

pg 189 Carroll / John

$z_{n-1} = \det(\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_{n-1}, \vec{E}_{n-1})$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & a_{22} & & a_{2n-1} & 0 \\ a_{31} & a_{32} & & \vdots & \vdots \\ \vdots & \vdots & & & 0 \\ a_{n-2\,1} & a_{n-2\,2} & & a_{n-2,n-1} & 1 \\ a_{n-1\,1} & a_{n-1\,2} & & a_{n-1\,n-1} & \\ a_{n1} & a_{n2} & & a_{n\,n-1} & 0 \end{vmatrix}$$

switch rows $n-1$ + n

$$= - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & 0 \\ a_{21} & \vdots & & \vdots & \vdots \\ a_{31} & & & & \\ \vdots & & & & \\ a_{n-2\,1} & a_{n-2\,2} & & a_{n-2\,n-1} & 0 \\ a_{n1} & a_{n2} & & a_{n\,n-1} & 0 \\ a_{n-1\,1} & a_{n-1\,2} & & a_{n-1\,n-1} & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} \\ a_{21} & \vdots & & \\ a_{31} & & & \\ \vdots & & & \\ a_{n-2} & & & \\ a_{n1} & a_{n} & & \end{vmatrix}$$

= minor of element 1 at $n-1, n$ in original matrix.

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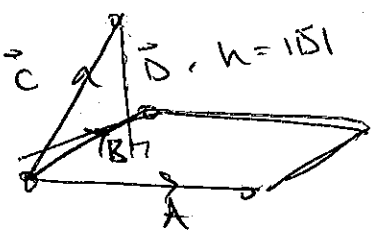
$$\alpha^2 = |A|^2 |B|^2 - (A\circ B)^2$$

$$= (A\circ A)(B\circ B) - (A\circ B)(A\circ B) =$$

pg 192 Courant (John



$$A\circ C = \lambda A\circ A + \mu A\circ B \qquad B\circ C = \lambda B\circ A + \mu B\circ B$$

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$$P(\vec{B},\vec{A},\vec{C}) = \begin{vmatrix} B\circ B & B\circ A & B\circ C \\ A\circ B & \cancel{B\circ B}\, A\circ A & A\circ C \\ \cancel{B\circ C}\ C\circ B & C\circ A & C\circ C \end{vmatrix}$$

$$= - \begin{vmatrix} A\circ B & A\circ A & A\circ C \\ B\circ B & B\circ A & B\circ C \\ C\circ B & C\circ A & C\circ C \end{vmatrix} = + \begin{vmatrix} A\circ A & A\circ B & A\circ C \\ B\circ A & B\circ B & B\circ C \\ C\circ A & C\circ B & C\circ C \end{vmatrix}$$

<u>Pg 194 Courant / John</u>

(74c) $V = |A \times B|\, h = |\det(A,B,C)|$

$$V^2 = \det(A,B,C)\det(A,B,C) =$$

$$= \begin{vmatrix} A\circ A & A\circ B & A\circ C \\ B\circ A & B\circ B & B\circ C \\ C\circ A & C\circ B & C\circ C \end{vmatrix}$$

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$$[\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n; \vec{A}_1', \vec{A}_2', \dots, \vec{A}_n'] =$$

$$\begin{vmatrix} A_1 \circ A_1' & A_1 \circ A_2' & \cdots & A_1 \circ A_n' \\ A_2 \circ A_1' & A_2 \circ A_2' & \cdots & A_2 \circ A_n' \\ \vdots & & & \\ A_n \circ A_1' & A_n \circ A_2' & \cdots & A_n \circ A_n' \end{vmatrix}$$

$$= \det(A_1, A_2, \dots, A_n) \circ \det(A_1', A_2', \dots, A_n') \quad \text{By eq 68f.}$$

By eq 81b

$$\epsilon V \; \epsilon' V' =$$

Now $\operatorname{Sgn}[A_1, \dots, A_n; E_1, E_2, \dots, E_n]$

taking transpose

$$= \operatorname{Sgn} \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & & & a_{nn} \end{vmatrix} = \operatorname{Sgn} |A_1, A_2, \dots, A_n|$$

(8) (a) From Pg 181 (71c)

$$\det(\vec{A},\vec{B},\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

Thus

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} = |\vec{A}\times\vec{B}||\vec{C}|\cos\phi_{\vec{A}\times\vec{B},\vec{C}}$$

$$= |\vec{A}||\vec{B}|\sin\phi_{\vec{A},\vec{B}}\ |\vec{C}|\cos\phi_{\vec{A}\times\vec{B},\vec{C}}$$

$$\leq \sqrt{(a^2+a'^2+a''^2)(b^2+b'^2+b''^2)(c^2+c'^2+c''^2)}$$

$$= |\vec{A}||\vec{B}||\vec{C}|$$

(b) equality of magnitude iff $\sin\phi_{\vec{A},\vec{B}} = \pm 1 \Rightarrow \phi_{\vec{A},\vec{B}} = \pm\frac{\pi}{2}$

& $\cos\phi_{\vec{A}\times\vec{B},\vec{C}} = 0,\pi \Rightarrow \vec{C} \parallel$ to $\vec{A}\times\vec{B}$.

Thus equality if $\vec{A},\vec{B},\vec{C}$ form a right/left hand trio.

As $\det(\vec{A},\vec{B},\vec{C})$ represents geometrically the volume of the parallelopiped $\Rightarrow$ the volume equals the product of the lengths if the parallelopiped is a rectangle.

(9) (a) As this expression is linear in $\vec{A}, \vec{B}, + \vec{C}$ we can show it by considering $\vec{A} = \hat{i}$

$= \hat{j}$

$= \hat{k}$

$\vec{B} = \hat{i}, \hat{j}, \hat{k}$ + $\vec{C} = \hat{i}, \hat{j}, \hat{k}$ in turn.

pg 205 Caract / John

$$y_j = \sum_{k=1}^{3} a_{jk} x_k + b_j$$

$$y_j - b_j = \sum_{k=1}^{3} a_{jk} x_k$$

~~Mult by $a^T \Rightarrow a_{kj}$ + sum over~~ k.

$$\cancel{x_k = \sum_{k=1}^{3} a_{kj}(y_k - b_k)} \quad \cancel{jk}$$

$$x_k = \sum_{j=1}^{3} a_{jk}(y_j - b_j)$$

+ $v_j = \sum_{k=1}^{3} a_{jk} u_k$

$$u_k = \sum_{k=1}^{3} a_{jk} v_j$$

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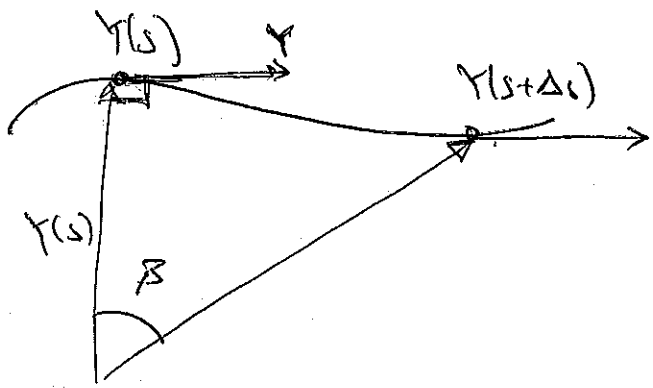
1

$$\frac{\partial}{\partial x_j} \frac{c}{\sqrt{(\xi_1 - x_1)^2 + (\xi_2 - x_2)^2 + (\xi_3 - x_3)^2}}$$

$$= \frac{c}{2} \frac{2(\xi_j - x_j)}{(\xi_1 - x_1)^2 + (\xi_2 - x_2) + (\xi_3 -}$$

Pg 213 Courant / John



$$|\Delta\gamma|^2 = |\gamma(s+\Delta s) - \gamma(s)|^2$$

Assuming $|\gamma(s)| = 1$ & $|\gamma(s+\Delta s)| = 1$

$$= \gamma(s+\Delta s)^2 - 2\gamma(s+\Delta s)\gamma(s) + \gamma(s)^2$$

$$= 1 - 2\gamma(s+\Delta s)\circ\gamma(s) + 1 = 2 - 2\gamma(s)\circ\gamma(s+\Delta s)$$

$$\Rightarrow |\Delta\gamma| = \pm\sqrt{2(1-\cos\beta)}$$

$\beta \measuredangle$ Between $\vec{\gamma}(s)$ & $\vec{\gamma}(s+\Delta s)$

$0 < \beta < \pi$.

But $\sin^2\beta = \frac{1}{2}(1-\cos 2\beta)$

$\therefore$ $1 - \cos 2\beta = 2\sin^2\beta$

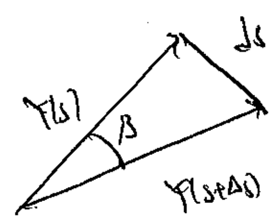
$\therefore$

$$|\Delta\gamma| = \pm 2\sin\beta/2$$

take + As $|\Delta\gamma| > 0$

$$\frac{d^2 X}{ds^2} = \frac{\frac{dX}{ds}(s+\Delta s) - \frac{dX}{ds}(s)}{\Delta s} = \frac{\gamma(s+\Delta s) - \gamma(s)}{\Delta s}$$

$$= \frac{2\sin \beta/2}{\Delta s}$$



$\Delta s =$

$$\left|\frac{d^2 \bar{X}}{ds^2}\right| = \left|\frac{d\gamma}{ds}\right|$$

$$= \lim_{\Delta s \to 0} \left|\frac{\Delta \gamma}{\Delta s}\right| = \lim_{\Delta s \to 0} \frac{2\sin \beta/2}{\beta} \cdot \frac{\beta}{\Delta s}$$ As $\Delta s \to 0$, $\beta \to 0$.

$$= \lim_{\Delta s \to 0} \frac{\beta}{\Delta s}$$

$$\frac{d^2 \bar{X}}{dt^2} = \frac{d^2 s}{dt^2} \cdot \underbrace{\frac{d\vec{X}}{ds}}_{\text{tangent to path}} + \left(\frac{ds}{dt}\right)^2 \underbrace{\frac{d^2 \bar{X}}{ds^2}}_{\perp \text{ to tangent}}$$

$\perp$ to tangent directed inwards of magnitude i.e. This entire force

$$\left|\left(\frac{ds}{dt}\right)^2 \frac{d^2 \bar{X}}{ds^2}\right| = K\left(\frac{ds}{dt}\right)^2$$

Pg 219 Courant / John

3.1a

1) $f(a,b)=0$

a known

pick b_0

compute $b_1=f(a,b_0)$

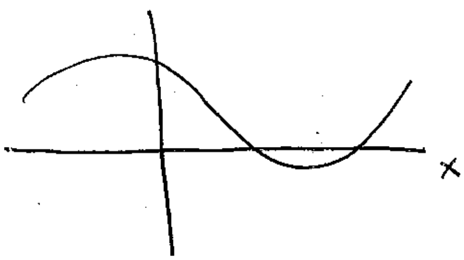
compute $b_2=f(a,b_1)$

$\vdots$

in general $b_n=f(a,b_{n-1})$

This is a difference eq for b_n. desire $b_n \to b$. As $n\to\infty$

graph $f(a,x)$ expand $f(a,b+\Delta b_n)$



$= f(a,b) + f_{\xi_2}(a,b)\Delta b + \cancel{O(\Delta b^2)}\ f_{\xi_2\xi_2}(a,b)\frac{\Delta b^2}{2} + O(\Delta b^3)$

Assuming $b \ni f(a,b)=0$.

$\Delta b = b_n - b$.

Then $\cancel{b} = \cancel{f_{\xi_2}(a,b)\Delta b} =$

$\cancel{f_{\xi_2}(a,b)\Delta b = -f_{\xi_2\xi_2}(a,b)\frac{\Delta b^2}{2} + O(\Delta b^2)}$

$\Rightarrow \cancel{\Delta b = -\frac{2f_{\xi_2}}{f_{\xi_2\xi_2}} + O(\Delta b^2)}$

$$b_{n+1} = f(a, b + \Delta b_n)$$

$$= \cancel{f(a,b)} + f_{\xi_2}(a,b)\,\Delta b_n + f_{\xi_2\xi_2}(a,b)\frac{\Delta b_n^2}{2} + O(\Delta b_n^3)$$

0

$$\underbrace{b_{n+1} - b} = -b + f_{\xi_2}(a,b)\,\Delta b_n + O(\Delta b_n^2)$$

$$\Delta b_{n+1} = \frac{d}{d\xi_2}\left(f(a,\xi_2) \cancel{- \frac{\xi_2^2}{2}} \right)\Bigg|_{\xi_2 = b} \Delta b_n + O(\Delta b_n^2)$$

diff. eq ~~in~~ ~~Δb_{n+1}~~ derivative ~~of~~ is constant

if $\cancel{\frac{\partial}{\partial \xi_2}}$ f(a

$$\Delta b_{n+1} = -b + f_{\xi_2}(a,b)\,\Delta b_n + O(\Delta b_n^2)$$

$$\sim \Delta b_{n+1} - f_{\xi_2}(a,b)\,\Delta b_n = -b. \quad *$$

desire $\Delta b_n \to 0$. sol to * is

$$\Delta b_n = + \frac{b}{f_{\xi_2}(a,b)}$$

$y_{n+1} = y_n + c f(a, y_n)$ Changed the problem from zero to a fixed point problem of the fn

$y_n \to b \Rightarrow f(a,b) = 0$ ✓. $y - cf(a,y)$

No we have a sol to this fixed point problem if As Above

$$\left| \frac{d}{dy}\left(y + cf(a,y)\right) \right|_{y=b} \le q < 1$$ Smaller $q \Rightarrow$ faster convergence

$\boxed{\text{pick } c \ni q \approx 0.}$

$$\Rightarrow \left| 1 + c f_y(a,y) \right|_{y=b} \le q < 1$$

$\Rightarrow$ pick $c \approx \dfrac{-1}{f_y(a,b)}$

$$\Rightarrow \quad -1 < 1 + c f_y(a,b) < 1$$

$$-2 < c f_y(a,b) < 0$$

$$-2/c < f_y(a,b) < 0$$

Thus we require $f_y(a,b)$

But in practice b is not known so we could generalize the above to $c = c_n \approx \dfrac{-1}{f_y(a, y_n)}$

Then the method becomes $y_{n+1} = y_n + c_n f(a, y_n)$

$$= y_n - \frac{f(a, y_n)}{f_y(a, y_n)}$$

or Newtons Method!

picking $c_n = c_0 = \dfrac{-1}{f_y(a,y_0)}$

one gets for the convergence relationship

$$\left| 1 - \frac{f_y(a,y_n)}{f_y(a,y_0)} \right| \leq q < 1 \quad \forall n$$

$$\Leftrightarrow \left| 1 - \frac{f_y(a,y)}{f_y(a,y_0)} \right| \leq q < 1 \quad \forall y \text{ in a Neighborhood of } b$$

$$\left| \frac{f_y(a,y_0) - f_y(a,y)}{f_y(a,y_0)} \right| \leq q < 1 \quad \text{"}$$

Assume $f_y(x,y)$ is Lipshitz in variable y on some neighborhood of b.

$$\to |f_y(a,y_1) - f_y(a,y_2)| \leq K|y_1 - y_2|$$

Then $\dfrac{K|y_0 - y|}{|f_y(a,y_0)|} \leq q < 1$

i.e. req on all iterates is that

$$|y - y_0| \leq \frac{|f_y(a,y_0)|}{K} q < \frac{mq}{K} \quad \text{if } |y-b| < \epsilon$$

pick a neighborhood of b $\ni$ $f_y(a,y)$ is, radius δ bounded away from 0

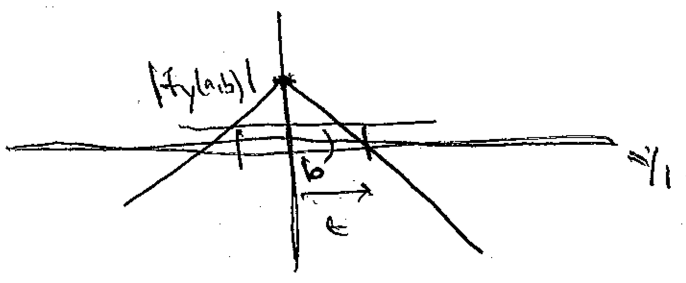
i.e $|f_y(a,y)| > m > 0$

~~Thus for y~~. We know that a Neighborhood of b exists by fact of Lipschitz condition

$$\big|\,|f_y(a,y_1)| - |f_y(a,b)|\,\big| \leq K|y_1 - b|$$

$$|f_y(a,b)| - K|y_1 - b| \leq |f_y(a,y_1)|$$

$\cancel{x}_0$



region where $|f_y(a,y)|$ is bounded away from 0.

Thus pick y_0 $\ni$

$$|y_0 - b| \leq \cancel{\text{Max}}\ \cancel{\beta}\ \text{Min}\left\{\epsilon, \frac{m\gamma}{2K}\right\}$$

Then ~~$f(a,b)=$~~ $y_n = b + (y_n - b) = b + \Delta y_n$

$$y_{n+1} - b = y_n - b + cf(a,y_n) = \underbrace{y_n - b}_{\Delta y_n} + c\Big[\underbrace{\cancel{f(a,b)}}_{0} + f_y(a,b)\,\Delta y_n + O(\Delta y_n^2)\Big]$$

$$\Delta y_{n+1} = \Delta y_n + c f_y(a,b)\,\Delta y_n + O(\Delta y_n^2)$$

$$= \left| 1 + c f_y(a,b) \right| \Delta y_n$$

$$\therefore\ |\Delta y_{n+1}| \approx |1 + c f_y(a,b)|\,|\Delta y_n|$$

But for $c = \dfrac{-1}{f_y(a, y_0)}$

pg 4 gives $|1 + c f_y(a,b)| \le q$

$$\therefore\ |\Delta y_{n+1}| \le q\,|\Delta y_n|$$

$$\Rightarrow\ |\Delta y_n| \le q^n\,|\Delta y_0| \xrightarrow{n \to \infty} 0$$

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$z = x^2 + y^2$ has tangent plan normal $\nabla z = 2x\hat{i} + 2y\hat{j}$

$\nabla z\big|_{(0,0)} = 0\hat{i} + 0\hat{j}$!! tangent plan horizontal

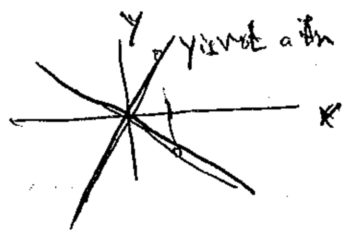
pg 221 Courant / John

Ex 3.1b Tangent plan to $z = f(x,y)$ is $\zeta - z = (\xi - x) f_x + (\eta - y) f_y$

① (a) $f_x = 2x$ $f_x(0,0) = 0$

$f_y = -2y$ $f_y(0,0) = 0$ $\Rightarrow$ tangent plan is horizontal.

Also. $f(x,y) = 0 \Rightarrow x = y$ or $y = -x$



y is not a fn of x near (0,0).
for has 2 values.

(b) $f_x(x,y) = \dfrac{1}{2\sqrt{\log(x+y)}} \dfrac{1}{(x+y)}$

$f_x(1.5, -.5) = \infty$

$f_y(x,y) = \dfrac{1}{2\sqrt{\log(x+y)}} \dfrac{1}{(x+y)}$; $f_y(1.5, -.5) = \infty$. } strongly not zero.

$f(x,y) = 0 = \sqrt{\log(x+y)}$

$\Rightarrow \log(x+y) = 0 \Rightarrow x + y = 1$

(c) $f(x,y) = \sin(\pi(x+y)) - 1 = 0$

$\Rightarrow \quad \sin(\pi(x+y)) = 1$

$\pi(x+y) = \frac{\pi}{2} + 2\pi n$

$x+y = \frac{1}{2} + 2n$

init $x = \frac{1}{4}, y = \frac{1}{4} \Rightarrow n = 0. \quad \Rightarrow \quad y = \frac{1}{2} - x$

$f_x = \pi\cos(\pi(x+y))\Big|_{x=1/4, y=1/4} = 0$ Tangent line is horizontal.

$f_y = \pi\cos(\pi(x+y))\Big| = 0$ No conclusion.

(d) $f(x,y) = 0$

$x^2 + y^2 - y = 0$

$x^2 + (y^2 - y + 1/4) - 1/4 = 0$

$x^2 + (y - 1/2)^2 - 1/4 = 0$

$(x^2 - 1/4) + (y - 1/2)^2 = 0$

$f_x = 2x\Big|_0 = 0$

$f_y = 2y - 1\Big|_0 = -1$

Yes can be inverted to yield $y = y(x)$

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$$y' = -\frac{F_x(x, f(x))}{F_y(x, f(x))}$$

$$y'' = -\frac{\left(F_{xx}(x, f(x)) + F_{xy} \cdot f'(x)\right)}{F_y(x, f)} + \frac{F_x \cdot (F_{xy} + F_{yy} f')}{F_y^2}$$

$$= -\frac{\left(F_y F_{xx} + F_{xy} f' F_y - F_x F_{xy} - F_x F_{yy} f'\right)}{F_y^2}$$

$$= \quad \text{But } f' = -\frac{F_x}{F_y}$$

$$\Rightarrow y'' = -\frac{\left(F_y F_{xx} - F_x F_{xy} - F_x F_{xy} + F_x^2 F_{yy}/F_y\right)}{F_y^2}$$

$$= -\frac{\left(F_y^2 F_{xx} - F_x F_y F_{xy} - F_x F_y F_{xy} + F_x^2 F_{yy}\right)}{F_y^3}$$

$$= -\underline{\left(F_y^2 F_{xx} - 2 F_x F_y F_{xy} + F_x^2 F_{yy}\right)}$$

$F = 0 \qquad F_x = 2(x^2+y^2)\cdot 2x - 4a^2x$

$F_y = 2(x^2+y^2)(2y) + 4a^2y \neq 0$

unless

$y = 0$

Max $\quad y' = 0$

$\Rightarrow \quad x = 0 \qquad$ or $\qquad x^2+y^2 = a^2$

eqs solve them and to be

$\begin{cases} F_x(x,y) = 0 \\ + \; F(x,y) = 0 \end{cases}$

$\Rightarrow$ From $F(x,y) = 0$

$\Rightarrow \quad y^4 + 2a^2y^2 = 0$

$y = 0.$

$+ \quad F(x,y) = 0$

$\Rightarrow$

$$\underbrace{(x^2+y^2)^2}_{} - 2a^2(x^2-y^2) = 0$$

$$a^4 - 2a^2(x^2-y^2) = 0$$

$$\Rightarrow \quad x^2 - y^2 = \frac{1}{2}a^2$$

Add to $x^2+y^2 = a^2$

$$\Rightarrow \quad 2x^2 = \frac{3a^2}{2} \quad \Rightarrow \quad x = \pm\frac{\sqrt{3}\,a}{2}$$

Then $\quad y^2 = a^2 - \frac{3}{4}a^2 = \frac{a^2}{4} \quad \Rightarrow \quad y = \pm\frac{a}{2}$

Eq 2: Folium of Descarte   Pg 224 Conrad / John

$F_x = 3x^2 - 3ay$

$F_y = 3y^2 - 3ax$

$$y' = -\frac{F_x}{F_y} = -\frac{(x^2 - ay)}{y^2 - ax}$$

$$y' = 0 \Rightarrow x^2 - ay = 0 \Rightarrow y = \frac{1}{a}x^2$$

$$\& \ F = 0 \Rightarrow x^3 + y^3 - 3axy = 0$$

$$x^3 + \frac{1}{a^3}x^6 - \frac{3ax^3}{a} = 0$$

$$x^3\left(1 + \frac{x^3}{a^3} - 3\right) = 0$$

$$x^3\left(-2 + \frac{x^3}{a^3}\right) = 0$$

$$\Rightarrow x = 0 \Rightarrow y = 0 \quad \text{not included}$$

$$x^3 = 2a^3 \Rightarrow x = a\sqrt[3]{2}$$

$$y = \frac{1}{a}a^2 2^{2/3} = a2^{2/3} = a\sqrt[3]{4}$$

① (a) $\underbrace{x^2 + xy + y^2 - 7}$ $= 0$

$F(x,y) = 0 \qquad (x,y) = (2,1)$

$F(2,1) = \cancel{4+2+1}\ 4+2+1-7 = 0$ ✓

$F_y = x + 2y \quad ; F_x = 2x + y$

$F_y(2,1) = 2 + 2 = 4 \neq 0.$

∴ we can invert $F(x,y) = 0$ locally to get $y = f(x)$

By implicit function thm.

Then $f'(x) = y' = -\dfrac{F_x}{F_y} = -\dfrac{(2x+y)}{(x+2y)}\Big|_{(2,1)} = -\dfrac{(5)}{4} = -\dfrac{5}{4}.$

+ 2nd derivative

$$f'' = y'' = -\frac{F_{xx}}{F_y} - \frac{1}{F_y} F_{xy} y' + \frac{F_x}{F_y^2} F_{yx} + \frac{F_x}{F_y^2} \cdot F_{yy} y'$$

$F_{xx} = 2 \quad ; F_{xy} = 1 \quad ; F_{yy} = 2$

$$\therefore y'' = -\frac{2}{4} - \frac{1}{4}(1)\left(-\frac{5}{4}\right) + \frac{5}{4^2} 1 + \frac{5}{4^2} 2 \left(-\frac{5}{4}\right)$$

$$= -\frac{1}{2} + \frac{5}{16} + \frac{5}{16} - \frac{80}{4\cdot 16}$$

$$= -\frac{1}{2} + \frac{5}{8} - \frac{25}{2\cdot 16}$$

$$= \frac{-16}{32} + \frac{4\cdot 5}{32} - \frac{25}{32} = \frac{-21}{32} < 0$$ to be convex $f''(x) > 0$

$\therefore$ not convex.

(b) $\underbrace{x \cos xy}_{F(x,y)} = 0$

$F(x,y) = 0$ near the pt $(x,y) = (1, \frac{\pi}{2})$

Th $F(1,\frac{\pi}{2}) = 1 \cos\frac{\pi}{2} = 0$ ✓

To solve $F(x,y) = 0$ for y we need $F_y \neq 0$ at $(1,\frac{\pi}{2})$

$F_y = -x^2 \sin xy$; $F_x = \cos(xy) - xy\sin(xy)$ $F_{xx} = -y\sin(xy) - y\sin(xy) - xy^2\sin(xy)$

$F_{yy} = -x^3\cos(xy)$; $F_{xy} = \cancel{} - 2x\sin xy - x^2 y\cos(xy)$;

$F_y(1,\pi/2) = -\sin\frac{\pi}{2} = -1 \neq 0$. $\therefore$ unique sol for $y = f(x)$ locally.

By implicit fn thm $y(x) = y$ is differentiable

$\therefore \frac{d}{dx} F(x,y) = 0$

$\Rightarrow F_x + F_y \cdot y' = 0 \Rightarrow y' = -\frac{F_x}{F_y}$

$\Rightarrow y'' =$ eq (5) $F_x(1,\frac{\pi}{2}) = -\frac{\pi}{2}$

From above:

$F_x(1,\frac{\pi}{2}) = -\frac{\pi}{2}$

$F_{yy}(1,\frac{\pi}{2}) = 0$

$F_{xy}(1,\pi/2) = -2 - \frac{\pi}{2}\cdot 0$

$F_{xx}(1,\pi/2) = -\pi - \frac{\pi^2}{4}$

F

Now from the implicit value theorem. $\exists\ y = f(x)$ in some interval about pt $x=1$.
+ f is continuous + has a continuous derivative in I $\{$ ie $f \in C^1$ $\}$

$$y' = f' = -\frac{F_x}{F_y}$$

$$\therefore f'(1) = -\frac{(-\pi/2)}{-1} = -\frac{\pi}{2}$$

As $F(x,y)$ has cont. partial derivatives of all orders $f(x)$ has ~~?~~ cont partial derivatives of all orders + from eq (5)

$$y'' = -\frac{(F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy})}{F_y^3}$$

$$y''(1) = -\frac{\left(1\left(-\pi - \frac{\pi^2}{4}\right) - 2\left(-\frac{\pi}{2}\right)(-1)(-2)\right)}{(-1)}$$

$$= -\pi - \frac{\pi^2}{4} + \pi = -\frac{\pi^2}{4} < 0$$ non convex but concave.

(c) $xy + \log(xy) = 1$ at $(1,1)$

$\Rightarrow \underbrace{xy + \log(xy) - 1}_{F(x,y)} = 0$ at $(1,1)$

The $F(1,1) = 0$ ✓ $\quad F_y(1,1) = ?$

$F_y(x,y) = x + \frac{1}{xy}(x) = x + \frac{1}{y}.$

$F_y(1,1) = 1+1 = 2 \neq 0$

$\therefore$ By the implicit fn thm $\exists$ sol for $y \ni$ ~~...~~ $f(x) \in C^1$

$F_x + F_y y' = 0$

$\Rightarrow \quad y' = -\frac{F_x}{F_y}$

For this fn

$F_y = x + \frac{1}{y} \qquad F_y(1,1) = 2$

$F_x = y + \frac{1}{x} \qquad F_x(1,1) = 2$

$F_{yy} = -\frac{1}{y^2} \qquad F_{yy}(1,1) = -1$

$F_{xy} = 1 \qquad F_{xy}(1,1) = 1$

$F_{xx} = -\frac{1}{x^2} \qquad F_{xx}(1,1) = \frac{-1}{1} = -1$

Thus 1st derivative $f'(x) = \frac{-(2)}{2} = -1$

2nd derivative exists. & is given by eq 5 $y'' = -\frac{(F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy})}{F_y^3}$

$\therefore\ y'' = f''(1) = \dfrac{-\left(4(-1) - 2(2)(2) + 2^2(-1)\right)}{8}$

$= \dfrac{-(-4-8-4)}{8} = 2 > 0$ $\therefore$ fn $f(x)$ is convex at $x=1$

(d) $x^5 + y^5 + xy = 3$ near $(1,1)$

Now: $\Rightarrow$ $\underbrace{x^5+y^5+xy-3}_{F(x,y)} = 0$

$F(x,y) = 0$

$F(1,1) = 0$ ✓ $F_y(1,1) = ?$

$F_y = 5y^4 + x$

$F_y(1,1) = 6 \neq 0$ $\therefore$ By the implicit fn thm $F(x,y) = 0$ can be solved

for y near $x=1$. $y = f(x)$ $x \in$ Nbhd of 1.

~~Thus~~ Also know the existence + differentiability of $f \in C^1$.

$f'(x) = -\dfrac{F_x}{F_y}$

$F_x = 5x^4 + y$ $F_x(1,1) = 6$

$F_y = 5y^4 + x$ $F_y(1,1) = 6$

$F_{xx} = 20x^3 + 0 = 20x^3 \qquad F_{xx}(1,1) = 20$

$F_{xy} = 1 \qquad F_{xy}(1,1) = 1$

$F_{yy} = 20y^3. \qquad F_{yy}(1,1) = 20.$

$\therefore \quad f'(x) = -\frac{6}{6} = -1$

$$f'' = -\frac{(F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy})}{F_y^3}$$

$$\therefore f''(1) = -\frac{(6^2 \cdot 20 - 2 \cdot 6 \cdot 6 \cdot 1 + 6^2 \cdot 20)}{6^3}$$

$$= -6^2 \frac{(20 - 2 + 20)}{6^3} = -\frac{38}{6} = \cancel{\frac{38}{6}} = \frac{-19}{3} \not> 0$$

$\therefore$ f is not convex at $x=1$.

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(5) $x^2 + xy + y^2 = 27.$

$\Rightarrow \underbrace{x^2 + xy + y^2 - 27}_{=F(x,y)} = 0$

Then $(x+y)^2 - xy - 27 = 0$ $\qquad 27 \to 36$ next greatest square of an integer.

$\Rightarrow (x+y)^2 - xy - (36-9) = 0$

$\Rightarrow (x+y)^2 - 36 - xy + 9 = 0$

$x = 3 \quad y = 3$ works great!!

Now:

$F_x = 2x + y$

$F_y = x + 2y$

$\}$ $\neq 0$ when $x = 3 = y$ ∴ the implicit fn thm tells us that we can solve the above eq for x or y implicitly & obtain a C^1 fn both times

Solving for y we get

$$f' = -\frac{F_x}{F_y} = -\left.\frac{(2x+y)}{x+2y}\right|_{\substack{x=3\\y=3}} = -1$$

But ~~f(x)~~ ⇒ $f'(x) = -\frac{(2x+y)}{(x+2y)} = 0$

$f'(x) = 0$

$\rightarrow \quad y = -2x.$

Thus w/ the eq originally $x^2 + xy + y^2 = 27$

pts where y is max & min are

$$x^2 - 2x^2 + 4x^2 = 27$$
$$3x^2 = 27$$
$$x^2 = 9 \qquad x = \pm 3.$$

computation of the other derivatives:

$F_{xx} = 2$

$F_{xy} = 1$

$F_{yy} = 2$

if $x = +3 \quad y = -6$
$\quad x = -3 \quad y = 6$

Whether these are max/or min's could be determined by the second derivative test.

$$y'' = -\frac{(F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy})}{F_y^3}$$

$$y'' = -\frac{((x+2y)^2 2 - 2(2x+y)(x+2y) + (x+2y)^2 \cdot 2)}{(x+2y)^3}$$

$$y''(x=+3) = -\frac{(((3-12)^2 \cdot 2 - 2\cancel{(}^{\,0} + (3+-12)^2 \cdot 2)}{(3-12)^3}$$

$$y''(+3) = -\left(\frac{81\cdot 2 + 81\cdot 2}{-81}\right) = +4 \overset{>0}{\cancel{<0}} \Rightarrow \overset{\text{min.}}{\cancel{\text{max.}}}$$

This could be obtained more quickly from the fact ~~that~~ you know ~~they~~ the y values of ~~3~~ Both pts obvious $+6 > -6$ & $+6$ must be the max & -6 the min.

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(6)
$$y = y_0 + \int_{x_0}^{x} f(\xi, y)\, d\xi$$

sol to D.E
$$\frac{dy}{dx} = f(x,y)$$

Then
$$\underbrace{y - y_0 - \int_{x_0}^{x} f(\xi, y)\, d\xi}_{F(x,y)} = 0.$$

$$F(x,y) = 0.$$

Note $F(x_0, y_0) = 0$

$$\& \quad F_y(x,y) = 1 - \int_{x_0}^{x} f_y(\xi, y)\, d\xi$$

$F_y(x_0, y_0) = 1 \neq 0$. $\therefore$ By the implicit fn thm $\exists\ y = y(x)$

fn w/ $\in C^1$

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Bottom of pg:

$$-\left[F(x,y_0) - F(x,y_0 - \beta)\right] + F(x,y_0)$$

~~MVT~~ M.V.T.

$$= -\left[F_y(\xi)(\beta)\right] + F(x,y_0) \qquad \text{Now } |F(x_0,y_0)| \le M\alpha$$

$$< -\frac{m}{2}\beta + M\alpha$$

pick α ∋ $-\frac{1}{2}m\beta + M\alpha < 0 \Rightarrow \alpha < \frac{m\beta}{2M}$

~~pg 229~~

Pg 228 Courant / John

(1) $f(x,y) = 0 \Rightarrow \{ y = \underline{f^{-1}(x,0)}$

$f_y(x_0,y_0) = 0$

$f(x,y) = x - y^2 = 0$

$f(0,0) = 0$. ; $f_y = -2y$ $\quad f_y(0,0) = 0$.

$y = \pm\sqrt{x}$ taking power of 3 removes this ambiguity

Check Books on $f = x + y^3$.

(2) Same as problem 1.

(3) $F(x,y) = \underbrace{y^3 - y^2 + (1+x^2)y - \phi(x) = 0}$

$y \overset{?}{=} f(x)$

at ~~x=0~~ ~~y=φ(0)~~

F(Need By Implicit fn thm $F_y(x_0,y_0) \neq 0$.

$F_y = 3y^2 - 2y + (1+x^2) = \underbrace{3y^2 - 2y + 1}_{\overset{?}{=} 0} + x^2$

$y = +2 \pm \sqrt{4 - 4(3)} < 0$ No real solutions

$\therefore \; 3y^2 - 2y + 1 > 0 \quad \forall y$.

$\therefore\ F_y > 0\ \forall\ x \,\&\, y \in \mathbb{R}^2$.

$\therefore\ \forall x$ F_y is strictly increasing in y. Thus $F(x,y) = 0$ can have no more than one solution $\forall x$. Such a solution must exist

Because $\forall x$ $\underbrace{y^3 - y^2 + (1-x^2)y}_{G(x,y)}$ ~~$- \phi(x)$~~ takes on both + & negative values. $\therefore$ $G(x,y)$ takes on all real values, thus in particular some value of y $G(x,y) = \phi(x)$ & Thus

$$\underbrace{G(x,y) - \phi(x)}_{= F(x,y)} = 0$$

$= F(x,y) = 0.$

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$$F_x + F_u f_x = 0 \quad , \; F_y + F_u f_y = 0 \; , \; \ldots$$

Adding:

$$F_x\,dx + F_y\,dy + F_z\,dz + \ldots + \underbrace{F_u f_x\,dx + F_u f_y\,dy + F_u f_z\,dz + \ldots}_{F_u \underbrace{(f_x\,dx + f_y\,dy + \ldots)}_{du.}} = 0$$

$$F = x^2 + y^2 + u^2 - 1 = 0$$

$$u_x = -\frac{F_x}{F_u} = \frac{-2x}{2u} = -\frac{x}{u} \; ; \quad u_y = -\frac{F_y}{F_u} = -\frac{y}{u}$$

$$u_{xx} = -\frac{1}{u} + \frac{x}{u^2} u_x = \frac{-u + x(-x/u)}{u^2} = \frac{-u^2 - x^2}{u^3} =$$

$$u_{xy} = +\frac{x}{u^2} u_y = \frac{x}{u^2}\left(\frac{-y}{u}\right) = \frac{-xy}{u^3}$$

$$u_{yy} = -\frac{1}{u} \qquad\qquad = -\frac{(y^2 + u^2)}{u^3}$$

Pg 230 Concept Ithm

(1) $x+y+z=\sin(xyz)$

$\Leftrightarrow \underbrace{x+y+z-\sin(xyz)}_{F(x,y,z)}=0$

At $(x,y,z)=\vec{0}$

$F(0,0,0)=0.$

$F_z = 1-\cos(xyz)\,xy$

$F_z(0,0,0) = 1 \neq 0.$ $\therefore$ By Implicit function of more than 2 independent variables

$z = f(x,y)$ in a region around $(0,0,0)$

$z_x = ?$ $\quad F_x + F_z z_x = 0$

$$z_x = -\frac{F_x}{F_z} = \frac{-(1-\cos(xyz)(yz))}{1-xy\cos(xyz)}$$

$$= \frac{-(1-yz\cos(xyz))}{1-xy\cos(xyz)}$$

$F_y + F_z z_y = 0 \Rightarrow z_y = \frac{-F_y}{F_z}$

$$z_y = -\frac{(1 - xz\cos(xyz))}{1 - xy\cos(xyz)}$$

② (a) let $F(x,y,z) = \sin x + \cos y + \tan z = 0$

$F(0, \frac{\pi}{2}, \pi) = 0 + 0 + 0 = 0$

$F_z(x,y,z) = \sec^2 z$

$F_z(0, \frac{\pi}{2}, \pi) = 1 \neq 0$ $\exists$ a unique, continuous, differentiable

solution for $z = z(x,y)$

(b) let $F(x,y,z,w) = x^2 + 2y^2 + 3z^2 - w$

$F(1,2,-1,8) = 1 + 8 + 3 - 8 = 4.$

How does 4 change things if at all?

let $G(x,y,z,w) = F(x,y,z,w) - 4$

Then $G(x,y,z,w) = 0$ at $(1,2,-1,8)$.

$G_z = 6z$

$\underbrace{G_z(1,2,-1,8)}$ $= -6 \neq 0 \Rightarrow \exists$ unique sol to $G(x,y,z,w) = 0$
$\Rightarrow$ unique sol to

$\Rightarrow F_z(1,2,-1,8) = -6.$

(e) let $F(x,y,z) = 1+x+y-\cosh(x+z)-\sinh(y+z)$

$F(0,0,0) = 1-1 = 0$

$F_z = -\sinh(x+z) - \cosh(y+z)$

$F_z(0,0,0) = -1 \neq 0$

can be solved for $z = z(x,y)$ near this point.

③ $\underbrace{x+y+z+xyz^3}_{F(x,y,z)} = 0$

$F(0,0,0) = 0$

$F_z = 1 + 3xyz^2$

$F_z(0,0,0) = 1 \neq 0 \Rightarrow z = z(x,y)$ exists near $(0,0,0)$.

$z = z(x,y) = \cancel{z(0,0)}^{0} + x z_x + y z_y + \frac{x^2}{2} z_{xx} + xy\, z_{xy} + \frac{y^2}{2} z_{yy}$

$\underbrace{+ \frac{1}{3!}\left\{ x^n f_{x^n}(0,0) + \binom{n}{1} f_{x^{n-1},y}(0,0) + \right.}$

$\frac{1}{3!}\left[x^3 f_{xxx}(0,0,0) + 3 \underset{x^2y}{f_{xxy}(0,0)} + 3 \underset{xy^2}{f_{xyy}(0,0)} + \underset{y^3}{f_{yyy}(0,0)} \right]$

Y

$$+ \frac{1}{4!}\Big\{ x^4 f_{xxxx}(0,0) + \binom{4}{1} f_{xxxy}(0,0)\, x^3 y + \binom{4}{2} f_{xxyy}\, x^2 y^2$$

$$+ \binom{4}{3} f_{xyyy}\, x y^3 + f_{yyyy}\, y^4 \Big\} + R(5)$$

$z_x = ?$

$1 + \cancel{z_x} + 3\cancel{x}z^2 z_x \cancel{=0}$ $\qquad 1 + z_x + yz^3 + 3xyz^2 z_x = 0$

$\cancel{z_x = \dfrac{-1}{(1+3xyz^2)}}$ $\qquad 1 + yz^3 + (3xyz^2 + 1)z_x = 0$

$$z_x = -\frac{(1+yz^3)}{(1+3xyz^2)}$$

$z_y = ?$

$1 \cancel{+ z_y + xz^3 = 0}$

$1 + z_y + xz^3 + 3xyz^2 z_y = 0$; $\qquad z_x(0,0) = -1$, $\quad z_y(0,0) = -1$

$(1+3xyz^2)z_y = -(1+xz^3)$

$$z_y = -\frac{(1+xz^3)}{(1+3xyz^2)}$$

.....

$$z_{yx} = -\frac{(z^3 + 3xz^2 z_x)}{(1+3xyz^2)} + \frac{(1+xz^3)}{(1+3xyz^2)^2}(3yz^2 + 6xyz\,z_x)$$

$z_{yx}(0,0) = 0 + 0 = 0$

$$z_{xx} = +\frac{(1+yz^3)}{(1+3xyz^2)^2}(3yz^2 + 6xyz\,z_x) \quad ; \quad z_{xx}(0,0) = 0.$$

$$z_{yy} = \frac{+(1+xz^3)}{(1+3xyz^2)}\left(3xz^2+3xy-2zz_y\right)$$

$$z_{yy}(0,0) = 0.$$

continuing in this manner we compute $z_{xxx}, z_{xxy}, z_{xyy}, z_{yyy},$

$z_{xxxx}, z_{xxxy}, z_{xxyy}, z_{xyyy}, z_{yyyy}$. Remembering that to compute one value of z. we need ~~compute~~ know the values of ~~~~ ~~~~ at the previous values at z_{000}.

Again: as in find all derivatives up to 4th order we form

$$z = -x - y + \cdots$$

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$(\eta - y) - (\xi - x) f'(x) = 0$ tangent

$$f'(x) = -\frac{F_x}{F_y}$$

$(\eta - y) F_y + (\xi - x) F_x = 0$

Normal:

$(\eta - y) f'(x) + (\xi - x) = 0$

$-(\eta - y) F_x + F_y (\xi - x) = 0$

If $F_y = 0$ Assume $F_x \neq 0$ then

$F(x, y) = 0$ can be solved for $x = g(y)$ in a neighborhood

g is cont. + diff w/ derivative =

$g'(y) =$

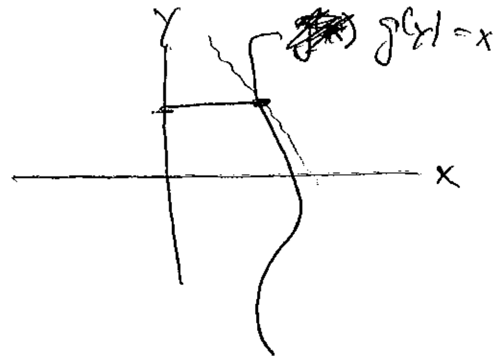
$F(g(y), y) = 0$

$\frac{d}{dy} \Rightarrow \quad F_x g'(y) + F_y = 0$

$\Rightarrow \quad g'(y) = -\frac{F_y}{F_x}$ Then eq for tangent to curve

$x = g(y)$ is ~~the~~ same ~~as before~~

$$(\eta - \cancel{y}) - (\xi - \cancel{x}) f'(x) = 0$$



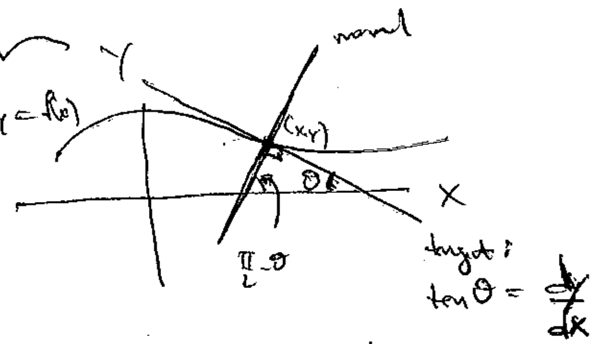
Tangent line is:

$$g'(y) = \frac{dx}{dy} = \frac{(\xi - x)}{(\eta - y)}$$

$$\Rightarrow (\xi - x) - g'(y)(\eta - y) = 0$$

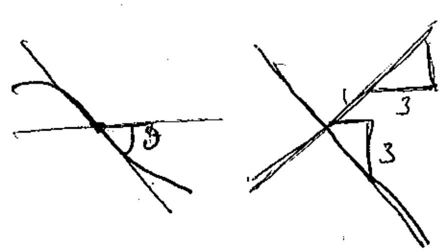
$$\Rightarrow (\xi - x) + \frac{F_y}{F_x}(\eta - y) = 0 \Rightarrow F_y(\eta - y) + F_x(\xi - x) = 0$$ same ✓.

eq for the normal was:



slope of normal is $-\frac{1}{f'(x)}$

$$\frac{\eta - y}{\xi - x} = \frac{-1}{f'(x)}$$



$$(\eta - y) f'(x) + (\xi - x) = 0$$ ✓

check.

eq for the normal becomes.

$$\frac{\xi - x}{\eta - y} = \frac{dx}{dy} = \frac{-1}{g'(y)} \Rightarrow \frac{\xi - x}{\eta - y} = \frac{F_x}{F_y}$$

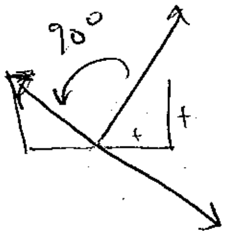
$$(\xi - x) F_y - (\eta - y) F_x = 0$$ same ✓.

Normal to eq $F(x,y)=0$

is given by $\vec{n} = \nabla F$ ≤ ②

This has unit normal given by 12c.

then the tangent

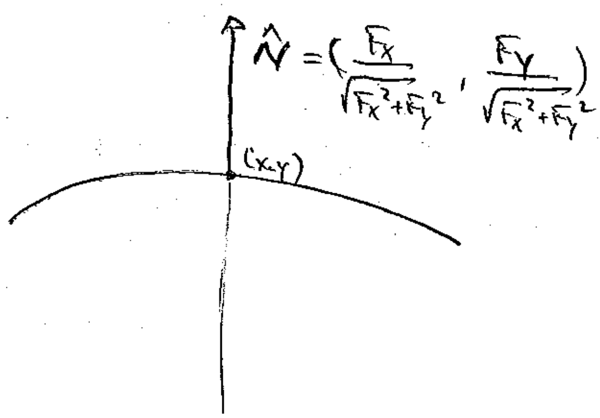


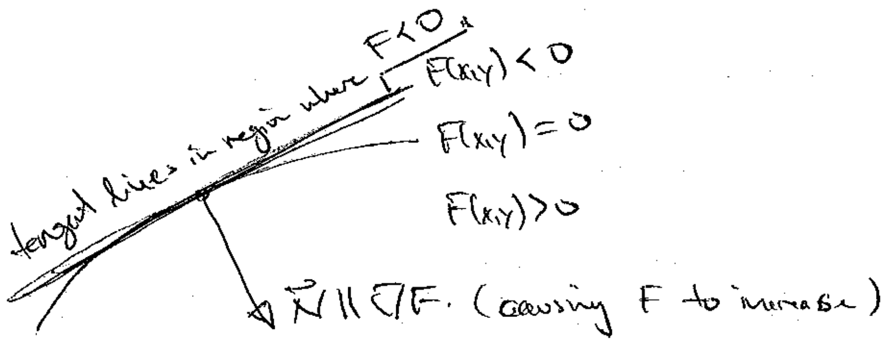
eq 10c $\quad k = \dfrac{f''}{(1+f'^2)^{3/2}} = -\dfrac{(F_y^2 F_{xx} - 2F_xF_yF_{xy} + F_x^2 F_{yy})}{F_y^3 \left(1 + F_x^2/F_y^2\right)^{3/2}}$

$$= -\frac{(F_y^2 F_{xx} - 2F_xF_yF_{xy} + F_x^2 F_{yy})}{\cancel{F_y^2}\left(F_y^2 + F_x^2\right)^{3/2}}$$

— — —

$\xi = x - \rho \dfrac{F_x}{\sqrt{F_x^2 + F_y^2}}$





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$F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad F_x = \frac{2x}{a^2} \quad F_{xx} = \frac{2}{a^2}$

$F_y = \frac{2y}{b^2} \quad F_{yy} = \frac{2}{b^2}$

tangent:

$(\xi - x)F_x + (\eta - y)F_y = 0$ $\qquad F_{xy} = 0.$

$$(\xi - x)\frac{2x}{a^2} + (\eta - y)\frac{2y}{b^2} = 0$$

$$\Rightarrow (\xi - x)\frac{x}{a^2} + (\eta - y)\frac{y}{b^2} = 0$$

$$\frac{\xi x}{a^2} - \underline{\frac{x^2}{a^2}} + \frac{\eta y}{b^2} - \underline{\frac{y^2}{b^2}} = 0$$

combine to give: -1

$$\frac{\xi x}{a^2} + \frac{\eta y}{b^2} = 1$$

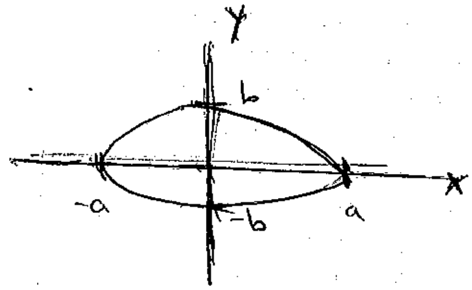
$$k = \frac{F_y^2 F_{xx} - 2F_x F_y F_{xy} + F_x^2 F_{yy}}{(F_x^2 + F_y^2)^{3/2}}$$

$$= \frac{\frac{4y^2}{b^4}\cdot\frac{2}{a^2} + \frac{4x^2}{a^4}\frac{2}{b^2}}{\left(\frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^{3/2}} = \frac{\cancel{8}\,1}{a^2b^2\,\cancel{2^3}}\frac{\left(\frac{y^2}{b^2} + \frac{x^2}{a^2}\right)}{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{3/2}}$$

$$= \frac{\ }{\cancel{(x^2 b^4 + y^2 a^4)}^{1/2}} \qquad \frac{1}{a^2 b^2} \frac{1}{\left(\frac{1}{a^4 b^4}\right)^{3/2} (b^4 x^2 + a^4 y^2)^{3/2}}$$

$$= \frac{(ab)^{2\cdot 2}}{(\qquad)^{3/2}}$$

$\boxed{a > b}$ ⇒ ellipse



$$k(x,y) = \frac{a^4 b^4}{(a^4 y^2 + b^4 x^2)^{3/2}} \qquad k(x,y) =$$

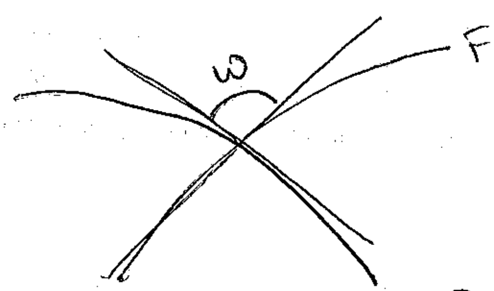
pick $y = 0$ & $x = \pm a$

$$k(x,0) = \frac{a^4 b^4}{b^6 x^3} = \frac{a^4}{b^2 a^3} = \frac{a}{b^2}$$

$$k(0, \pm b) = \frac{a^4 b^4}{(a^4 b^2)^{3/2}} = \frac{a^4 b^4}{a^6 b^3} = \frac{b}{ab^2}$$

$$\cos\omega = \frac{\vec{N}_1 \circ \vec{N}_2}{\sqrt{N_1 \circ N_1}\sqrt{N_2 \circ N_2}} = \frac{F_x G_x + F_y G_y}{\ } \qquad F>0$$

$$N_1 \circ N_2 = |N_1||N_2| \cos\omega.$$



Confocal parabolas:

$$F(x,y,c_1) = y^2 - 2c_1\left(x + \frac{c_1}{2}\right) = 0$$

$$F(x,y,c_2) = y^2 - 2c_2\left(x + \frac{c_2}{2}\right) = 0$$

? Can one use confocal parabolas as a nice grid generator? Give square grids.

Subtract $\Rightarrow$ $-c_1\left(x + \frac{c_1}{2}\right) + c_2\left(x + \frac{c_2}{2}\right) = 0$

$$x(c_2 - c_1) \qquad -\frac{c_1^2}{2} + \frac{c_2^2}{2} = 0$$

$$x = -\frac{1}{2}\frac{(c_2^2 - c_1^2)}{c_2 - c_1} = -\frac{(c_2 + c_1)}{2}$$

Then $y^2 = -\frac{2c_1}{2}\left(c_2 + \cancel{x} + \cancel{x_1}\right) = -c_1(c_2 + \cancel{2c_1}) = -c_1c_2$.

Notice that

$F_x(x,y,c_1)F_x(x,y,c_2) + F_y(x,y,c_1)F_y(x,y,c_2) =$

$$-2c_1(-2c_2) + 2y(2y) \overset{?}{=} 0$$

$$c_1c_2 + y^2 \overset{?}{=} 0 \qquad \text{yes}$$

Curvature of confocal parabolas:

$$F_x = -2c \qquad F_{xx} = 0$$
$$F_y = 2y \qquad F_{yy} = 2$$
$$F_{xy} = 0$$

$$(14a) \quad k = \frac{F_y^2F_{xx} - 2F_xF_yF_{xy} + F_x^2F_{yy}}{(F_x^2 + F_y^2)^{3/2}}$$

$$= \frac{2(-2c)^2}{(4c^2 + 4y^2)^{3/2}} = \frac{8c^2}{8(c^2+y^2)^{3/2}} = \frac{c^2}{(c^2+y^2)^{3/2}}$$

$$k = \frac{c^2}{c^3} = \frac{1}{|c|} = \frac{1}{\rho} \Rightarrow \rho = |c|$$

Center of curvature:

$$\xi = -\frac{c}{2} - \rho\,(\hat{\nabla} F)_1$$

← 1st component of unit normal evaluated at $(-\frac{c}{2}, 0)$

$-\frac{c}{2}$: pt want to take centers of curvature about

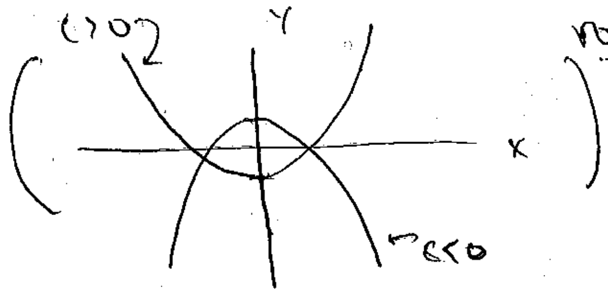
ρ: radius of curvature

$-$: direction of decreasing F

$$F = y^2 + 2c\left(x + \frac{c}{2}\right)$$

$\nabla F = (-2c, 2y)$ pts in direction of increasing F.

$\therefore F = 0 \Rightarrow y^2 = 2c\left(x + \frac{c}{2}\right)$

$\Bigg($  $\Bigg)^{\text{rotate}}$

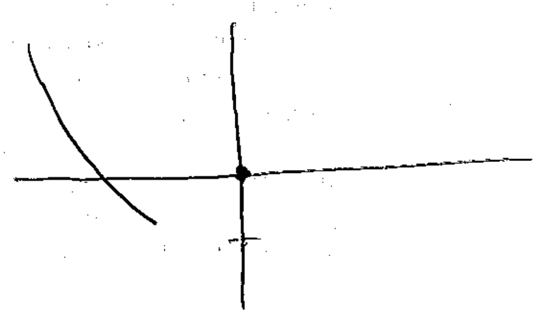
$$\therefore \xi = -\frac{c}{2} - \frac{|c|(-2c)}{\sqrt{4c^2+4y^2}}\Bigg|_{y=0}$$

$$= -\frac{c}{2} + \frac{2c|c|}{\sqrt{4c^2}}$$

$$= -\frac{c}{2} + \frac{\cancel{2}c}{\cancel{2}}\frac{\cancel{|c|}}{\cancel{|c|}} = \frac{c}{2}$$

$$\eta = 0 - \rho\,\hat{\nabla} F_1\Big|_{x=-\frac{c}{2},\, y=0} = 0$$

c>0



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$F_x = 2(y-x)(-1)$

$F_y = 2(y-x)$

$F_x^2 + F_y^2 = 4(y-x)^2 =$

pg 239 Courant / John

$F(x,y,z) = x^2 + y^2 + z^2 = r^2$

tangent plane $(F_x, F_y, F_z) \circ \underbrace{(\vec{r} - \vec{r}_0)}_{(\xi - x,\, \eta - y,} = 0$ $\quad \vec{r} = (\xi, \eta, \zeta)$, $\vec{r}_0 = (x, y, z)$

$2x$

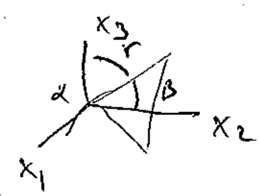
$2x(\xi - x) + 2y(\eta - y) + 2z(\zeta - z) = 0$

$2x\xi + 2y\eta + 2z\zeta - 2\underbrace{(x^2+y^2+z^2)}_{r^2} = 0$

$x\xi + y\eta + z\zeta = r^2$

Directional cosigns:

$\cos\alpha = \dfrac{2x}{\sqrt{4x^2+4y^2+4z^2}}$



$\cos\beta = \dfrac{2y}{2|r|}$

$\cos\gamma = \dfrac{z}{r}$

$$\frac{\cancel{2}x}{a^2}(\xi - x) + \frac{\cancel{2}y}{b^2}(\eta - y) + \frac{\cancel{2}z}{c^2}(\zeta - z) = 0$$

$$= \frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} = 1$$

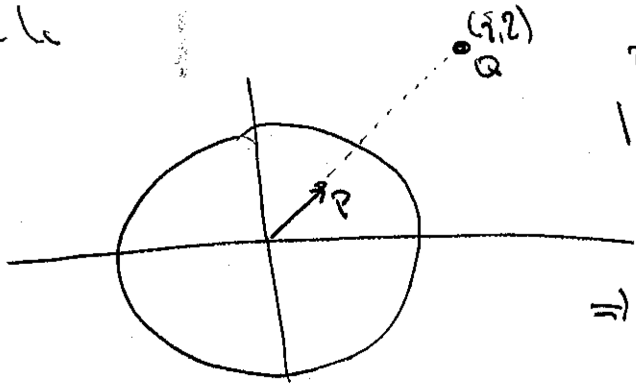
pg 244 Courant / John

Find inverse mapping for

$$\xi = \frac{x}{x^2+y^2} \qquad \eta = \frac{y}{x^2+y^2}$$

$$\xi^2 + \eta^2 = \frac{x^2+y^2}{(x^2+y^2)^2} = \frac{1}{(x^2+y^2)}$$

Note: This mapping corresponds to the reflection w/ respect to the unit circle



$\vec{OP} = (x,y)$

$|\vec{OQ}| = \dfrac{1}{|\vec{OP}|}$ in the same direction

$\Rightarrow$

$$\vec{OQ} = \frac{1}{|\vec{OP}|}\,\hat{OP} = \frac{1}{|\vec{OP}|}\frac{\vec{OP}}{|\vec{OP}|} = \frac{(x,y)}{|\vec{OP}|^2} = \frac{(x,y)}{x^2+y^2}$$

Then to get point P from given pt P.

$$\vec{OP} = \frac{1}{|OQ|}\,\hat{OQ} = \frac{\vec{OQ}}{|OQ|^2}$$

$$(x,y) = \frac{(\xi,\eta)}{\xi^2+\eta^2} \Rightarrow x = \frac{\xi}{\xi^2+\eta^2}$$

$$y = \frac{\eta}{\xi^2+\eta^2}$$

$$\xi = r = \frac{x}{x^2+y^2}$$

$$-\frac{1}{r}x + x^2+y^2 = 0$$

$x^2-y^2 = \text{const}$

$y = \pm\sqrt{x^2 - \text{const}}$

$\lim_{x\to\infty} y = \pm\sqrt{x^2-\text{const}} = \pm|x| = \pm x$

The level lines of $P_{\underline{r}}$: $\xi(x,y) = x^2-y^2$ & $\eta(x,y) = 2xy$ intersect each other at Right ∡s.

Pf: $\xi(x,y) = \xi_0 = x^2-y^2$ has normal $\nabla\xi = (2x,-2y)$

& $\eta(x,y) = \eta_0 = 2xy$ has normal $\nabla\eta = (2y,2x)$

Then $\nabla\xi\cdot\nabla\eta = 4(xy-xy) = 0$ ✓. $\Rightarrow$

$\parallel$ to x axis $\Rightarrow y = k \Rightarrow \xi = x^2-k^2$

$+ \eta = 2xk \Rightarrow x = \dfrac{\eta}{2k}$

$$\Rightarrow \xi = \frac{\eta^2}{4k^2} - k^2 \Rightarrow \eta^2 = 4k^2(\xi+k^2)$$

$\parallel$ to y axis $\Rightarrow$ $x = c$

Then $\xi = c^2 - y^2$

$\eta = 2cy \Rightarrow y = \frac{\eta}{2c}$

$$-(\xi - c^2) = \frac{\eta^2}{4c^2}$$

$$\eta^2 = 4c^2(c^2 - \xi)$$

pg 248 Current ([illegible]

$$y^2 = 2c\left(x + \frac{c}{2}\right)$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y^2 = 2cx + c^2$$

$$c^2 + 2cx - y^2 = 0$$

$$c = \frac{-2x \pm \sqrt{4\text{[illegible]}^2 - 4(-y^2)}}{2(\text{[illegible]})} = -x \pm \sqrt{x^2 + y^2}.$$

Consider

$$\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\phi_x & \phi_y\\ t_x & t_y\end{pmatrix}\begin{pmatrix}g_\xi\\ h_\xi\end{pmatrix} \quad + \quad \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\phi_x & \phi_y\\ t_x & t_y\end{pmatrix}\begin{pmatrix}g_\eta\\ h_\eta\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}g_\xi\\ h_\xi\end{pmatrix} = \frac{1}{\begin{vmatrix}\phi_x & \phi_y\\ \phi_x & t_y\end{vmatrix}}\begin{pmatrix}t_y & -\phi_y\\ -t_x & \phi_x\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{1}{(\phi_x t_y - t_x\phi_y)}\begin{pmatrix}t_y\\ -t_x\end{pmatrix} = \frac{1}{D}\begin{pmatrix}t_y\\ -t_x\end{pmatrix}$$

$$\therefore \quad g_\xi = \frac{t_y}{D} \quad \cancel{\&} \; h_\xi = -\frac{t_x}{D}$$

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other eq gives

$$\begin{pmatrix}g_\eta\\ h_\eta\end{pmatrix} = \frac{1}{D}\begin{pmatrix}t_y & \cancel{\phi_x}\,-\phi_y\\ \cancel{\phi_y}\,-t_x & \phi_x\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{D}\begin{pmatrix}\cancel{\phi_x}\,-\phi_y\\ \phi_x\end{pmatrix}$$

$$g_\eta = \frac{\cancel{\phi_x}\,-\phi_y}{D} \quad + \quad h_\eta = \frac{\phi_x}{D}$$

~~But Also from the defs $x = g(\xi,\eta)$ $y = h(\xi,\eta)$~~

$\cancel{x_\xi = g_\xi}$

My D is

$$\phi_x f_y - f_x \phi_y = \xi_x \eta_y - \eta_x \xi_y$$

$\cancel{\xi = \phi(x,y)}$

$\cancel{1 = \phi_x x_\xi} +$

From 24c we get $g_\xi = \dfrac{f_y}{D}$

But $g(\xi,\eta) = x$ & $f(x,y) = \eta$

$$\Rightarrow \quad x_\xi = \frac{\eta_y}{D}$$

From $g_\eta = -\dfrac{\phi_y}{D}$ we get

$$x_\eta = -\frac{\xi_y}{D}$$

From $h_\xi = -\dfrac{f_x}{D}$ we get

$$y_\xi = -\frac{\eta_x}{D}$$

From $h_\eta = \dfrac{\phi_x}{D}$ we get

$$y_\eta = \frac{\xi_x}{D}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\theta_x = \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\theta_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$D = \xi_x\eta_y - \xi_x\eta_x = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{1}{r^3}\left(x^2+y^2\right) = \frac{1}{r}$$

get partial derivatives of the inverse fns

$$x_r = \frac{\eta_y}{D} = \frac{\theta_y}{D} = \frac{x/r^2}{1/r} = \frac{x}{r}$$

$$x_\eta = x_\theta = -\frac{\xi_y}{D} = -\frac{r_y}{D} = \frac{-y/r}{1/r} = -y$$

$$y_\xi = y_r = -\frac{\eta_x}{D} = -\frac{\theta_x}{D} = -\frac{(-y/r^2)}{1/r} = \frac{y}{r}$$

$$y_\xi = y_\theta = \frac{\xi_x}{D} = \frac{r_x}{D} = \frac{x/r}{1/r} = x$$

$$\frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} = x_\xi y_\eta - x_\eta y_\xi$$

$$= \frac{\eta_y}{D}\left(+\frac{\xi_x}{D}\right) - \left(-\frac{\xi_y}{D}\right)\left(-\frac{\eta_x}{D}\right)$$

$$= +\frac{1}{D^2}\left(\eta_y \xi_x - \xi_y \eta_x\right)$$

$$= \frac{1}{D^2}\frac{\partial(\xi,\eta)}{\partial(x,y)} = \frac{1}{D} = \left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right)^{-1}$$

2nd & Higher derivatives of the inverse mapping. $\xi = \xi(x,y) = \xi(x(\xi,\eta), y(\xi,\eta))$

say want $g_{\xi\xi}$ & $h_{\xi\xi}$ then

$x_{\xi\xi}$ $\quad 1 = \phi_x g_\xi + \phi_y h_\xi \quad \Rightarrow \quad 1 = \xi_x x_\xi + \xi_y y_\xi \quad (1)$

$y_{\xi\xi}$ $\quad 0 = \psi_x g_\xi + \psi_y h_\xi \qquad\quad 0 = \eta_x x_\xi + \xi_y y_\xi$

$$\partial_\xi \Rightarrow \quad 0 = \cancel{\xi_x \; \partial_x(\xi_x x_\xi + \xi_y y_\xi)\, \xi_x x_\xi + \partial_y(\xi_x x_\xi + \xi_y y_\xi)}\, y_\xi$$

$$+ \quad 0 = \cancel{\partial_x(\eta_x x_\xi + \xi_y y_\xi)\, x_\xi + \partial_y(\eta_x x_\xi + \xi_y} y_\xi)\, y_\xi$$

$$\Rightarrow \quad 0 = \cancel{\xi_{xx} x_\xi^2 + \xi_{xy} x_\xi y_\xi + \xi_{xy} x_\xi y_\xi + \xi_{yy} y_\xi^2}$$

In taking the ∂_ξ derivative of eq (1) For some terms use the ~~chain~~ chain rule & on some terms you do not.

$$0 = \partial_\xi(\xi_x)\, x_\xi + \xi_x x_{\xi\xi} + \partial_\xi(\xi_y)\, y_\xi + \xi_y y_{\xi\xi}$$

$$= \partial_x(\xi_x)\, x_\xi^2 + \partial_y(\xi_x)\, y_\xi x_\xi + \xi_x x_{\xi\xi} + \partial_x(\xi_y)\, x_\xi y_\xi + \partial_y(\xi_y)\, y_\xi^2$$

$$+ \xi_y y_{\xi\xi}$$

$$= \xi_{xx} x_\xi^2 + \underline{\xi_{xy} x_\xi y_\xi} + \xi_x x_{\xi\xi} + \underline{\xi_{xy} x_\xi y_\xi} + \xi_{yy} y_\xi^2 + \xi_y y_{\xi\xi}$$

$$= \xi_{xx} x_\xi^2 + 2\xi_{xy} x_\xi y_\xi + \xi_{yy} y_\xi^2 + \xi_x x_{\xi\xi} + \xi_y y_{\xi\xi}$$

Ask students: Whats wrong w/

$$\partial_\xi(\xi_y) = (\partial_\xi \xi)_y = (1)_y = 0$$ eq to saying

$$\xi = f(\xi, \eta) \qquad \frac{d}{dx}(x) = 0$$

∂_ξ of $\left(0 = \eta_x x_\xi + \eta_y y_\xi \right)$

$$\Rightarrow \; 0 = \eta_{xx} x_\xi^2 + \eta_{xy} x_\xi y_\xi + \eta_x x_{\xi\xi} + \eta_{xy} y_\xi x_\xi + \eta_{xy} x_\xi y_\xi + \eta_y y_{\xi\xi}$$

$$= \eta_{xx} x_\xi^2 + 2\eta_{xy} x_\xi y_\xi + \eta_{yy} y_\xi^2 + \eta_x x_{\xi\xi} + \xi_y y_{\xi\xi}$$

$$\cancel{x_{\xi\xi}} \; \Rightarrow \; \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} x_{\xi\xi} \\ y_{\xi\xi} \end{pmatrix} = \begin{pmatrix} -\left(\xi_{xx} x_\xi^2 + 2\xi_{xy} x_\xi y_\xi + \xi_{yy} y_\xi^2\right) \\ -\left(\eta_{xx} x_\xi^2 + 2\eta_{xy} x_\xi y_\xi + \eta_{yy} y_\xi^2\right) \end{pmatrix}$$

RHS

$$= \frac{1}{D^2} \begin{pmatrix} \xi_{xx} \eta_y^2 - 2\xi_{xy} \eta_y \eta_x + \xi_{yy} \eta_x^2 \\ \eta_{xx} \eta_y^2 - 2\eta_{xy} \eta_y \eta_x + \eta_{yy} \eta_x^2 \end{pmatrix}$$

$$\therefore \quad x_{\xi\xi} = -\frac{1}{D^3} \begin{vmatrix} \xi_{xx} \eta_y^2 - 2\xi_{xy} \eta_y \eta_x + \xi_{yy} \eta_x^2 & \xi_y \\ \eta_{xx} \eta_y^2 - 2\eta_{xy} \eta_y \eta_x + \eta_{yy} \eta_x^2 & \eta_y \end{vmatrix}$$

$$\& \quad y_{\xi\xi} = -\frac{1}{D^3} \begin{vmatrix} \xi_x & \xi_{xx} \eta_y^2 - 2\xi_{xy} \eta_y \eta_x \;\; \xi_{yy} \eta_x^2 \\ \eta_x & \end{vmatrix}$$

Pg 259 Courant/John

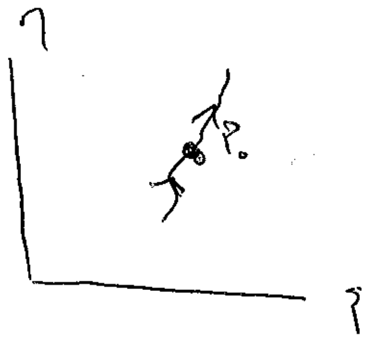
$$\xi = \phi(x_0,y_0) + a(x-x_0) + b(y-y_0)$$

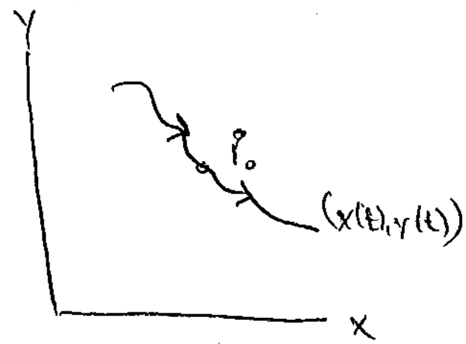
$$\eta = f(x_0,y_0) + c(x-x_0) + d(y-y_0)$$

$$\frac{d\eta/dt}{d\xi/dt} = \frac{c\dot{x} + d\dot{y}}{a\dot{x} + b\dot{y}} =$$

$$\frac{d\mu}{dm} = \frac{d}{a+bm} - \frac{b(c+dm)}{(a+bm)^2}$$

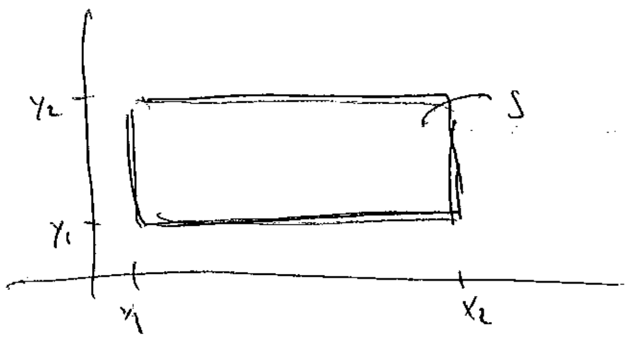
$$= \frac{d(a+\cancel{b}m) - bc - d\cancel{b}m}{(a+bm)^2} = \frac{ad-bc}{(a+bm)^2}$$





$$m = \frac{g'(t)}{f'(t)} = \text{slope in } xy \text{ plane}$$

Pr: Rectangle w/ sides $\parallel$ to the axis has area = product of length of sides.



Let x_1, x_2, y_1, y_2 be arbitrary.

A_0 uses ~~rectangles~~ squares of size 1×1 to cover region S.

Now: $A_n^+(S) =$

A_1 uses ~~rectangles~~ squares of size $\frac{1}{2} \times \frac{1}{2}$ to cover region S

$A_n^-(S)$

A_2 " squares of size $\frac{1}{4} \times \frac{1}{4}$ to cover region

⋮

$$\left\lfloor \frac{x_2 - x_1}{(\frac{1}{2})^n} \right\rfloor$$

A_n uses squares of size $(\frac{1}{2})^n \times (\frac{1}{2})^n$ to cover region S.

$$A_n^-(S) \geq \left\lfloor \frac{x_2 - x_1}{(\frac{1}{2})^n} \right\rfloor \left\lfloor \frac{y_2 - y_1}{(\frac{1}{2})^n} \right\rfloor \left(\left(\tfrac{1}{2}\right)^2\right)^n$$

floor fn

$$A_n^+(S) \geq \left\lceil \frac{x_2 - x_1}{(\frac{1}{2})^n} \right\rceil \cdot \left\lceil \frac{y_2 - y_1}{(\frac{1}{2})^n} \right\rceil \left(\left(\tfrac{1}{2}\right)^2\right)^n$$

ceiling fn

Check $\lceil nx \rceil \stackrel{?}{=} n\lceil x \rceil \quad n \in \mathbb{Z}^+$

if yes then

$$A_n^-(s) \geq \lfloor x_2 - x_1 \rfloor \lfloor y_2 - y_1 \rfloor$$

$$A_n^+(s) \geq \lceil x_2 - x_1 \rceil \lceil y_2 - y_1 \rceil$$

Then $A_n^-(s) \neq A_n^+(s) \quad n \to \infty$

$\Rightarrow \quad A^-(s) > \lfloor x_2 - x_1 \rfloor \lfloor y_2 - y_1 \rfloor$

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$$A^+\left(\bigcup_{i=1}^{N} S_i\right) \geq A^-\left(\bigcup_{i=1}^{N} S_i\right)$$

$+$ $$A^*\left(\bigcup_{i=1}^{N} S_i\right) \leq \sum_{i=1}^{N} A^*(S_i) \geq A^-\left(\bigcup_{i=1}^{N} S_i\right)$$

Pg 874 correct/2dm

1

(1) $S + T$ are Jordan measurable & $S \subset T$

$A(S) \leq A(T)$

Note $\bar{A_n}(\cancel{T}) \geq \bar{A_n}(S)$

Since

pg 544 Carat / Ilm

If $L = f(x,y)\,dy - g(x,y)\,dx$

$dL = df\,(dy) - dg\,dx$ taking ~~[illegible]~~ the exterior derivative from pg 313 eq 5B6.

$= (f_x\,dx + f_y\,dy)\,dy$

$\quad - (g_x\,dx + g_y\,dy)\,dx$

$= f_x\,dx\,dy + \cancel{f_y\,dy^2}^{\,0} - \cancel{g_x\,dx^2}^{\,0} - g_y\,dy\,dx$

$= f_x\,dx\,dy + g_y\,dx\,dy$ changing the order of the last 2 differentials.

w/

$a(x,y) = -g(x,y)$

$b(x,y) = f(x,y)$

$$\iint_R (f_x + g_y)\,dx\,dy = \int_{+C} [f(x,y)\,dy - g(x,y)\,dx] \quad \text{gives:}$$

$$\iint_R (b_x - a_y)\,dx\,dy = \int_{+C} (b\,dy + a\,dx) = \int_{+C} (a\dot{x} + b\dot{y})\,ds$$

$$= \int_{+C} a\,dx + b\,dy$$

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3 If $L = \underbrace{(f y_u - g x_u)}_{A} du + \underbrace{(f y_v - g x_v)}_{B} dv$

$dL = dA\, du + dB\, dv$

$dA = A_u\, du + A_v\, dv$

$dB = B_u\, du + B_v\, dv$

$\therefore\ dL = (A_u du + A_v dv) du + (B_u du + B_v dv) dv$

$= A_u \cancel{du} du + A_v dv du + B_u du dv + B_v \cancel{dv} dv$

$= (B_u - A_v) du dv \underset{\text{from before}}{=} (f_x + g_y) dx dy.$

$\therefore\quad f_x + g_y = (B_u - A_v) \dfrac{\partial(u,v)}{\partial(x,y)}$

or $\quad B_u - A_v = (f_x + g_y) \dfrac{\partial(x,y)}{\partial(u,v)}$

check directly.

$= f_u y_u + f y_{vu} - g_u x_v - g x_{vu}$

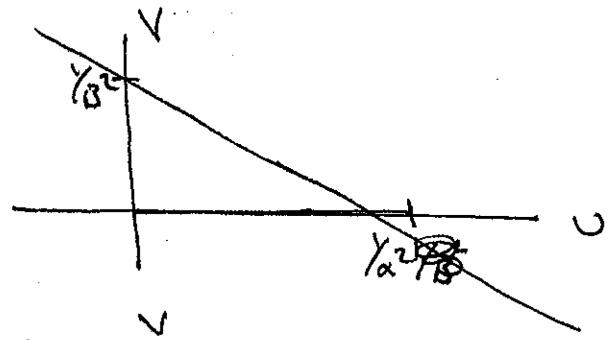
(1)

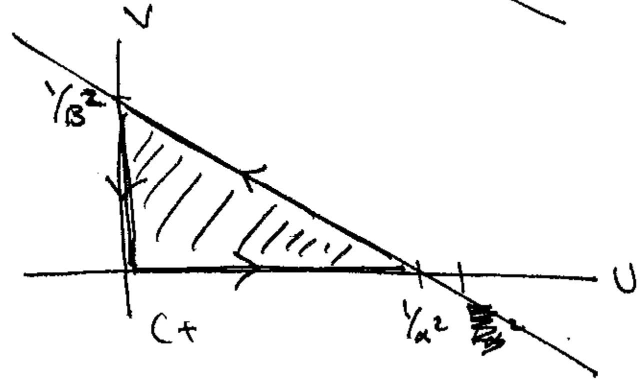
(a) $A = au + bv$ $\qquad B = 0$

$u \geq 0 \quad v \geq 0 \qquad \alpha^2 u + \beta^2 v < 1$

Lin $\alpha^2 u + \beta^2 v = 1$

u	v
0	$1/\beta^2$
$1/\alpha^2$	0





C+ keeps region on the "left" side. Also counter clockwise.

Diagram them in 2D

$$= \int_{C+} A\,du + B\,dv$$

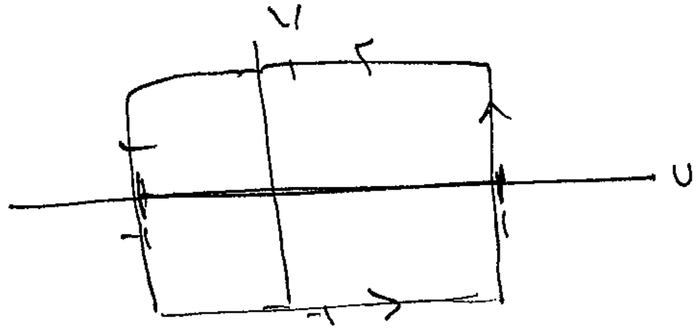
$$= \iint_R \left(\frac{\partial B}{\partial u} - \frac{\partial A}{\partial v}\right) du\,dv \quad ? \text{ From pg 551}$$

Check:

$$= \iint_R (0 - b)\,du\,dv = -b\left(\frac{1}{2}\left(\frac{1}{\alpha^2}\right)\left(\frac{1}{\beta^2}\right)\right) = \frac{-b}{2\alpha^2\beta^2}$$

(b) $A = u^2 - v^2$ $\quad B = 2uv \quad |u| < 1$

$|v| < 1$



$$\int_{C+} A\,du + B\,dv$$

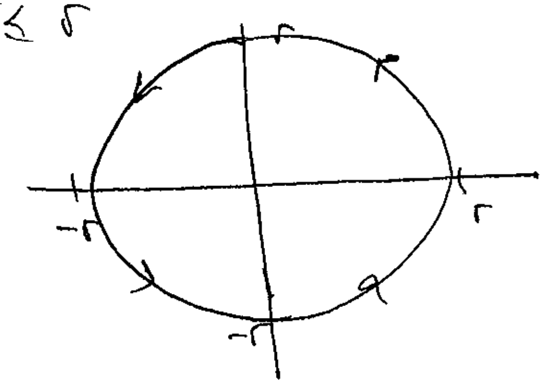
Divergence Thm in 2D (Greens thm)

$$= \iint_R \left(\frac{\partial B}{\partial u} - \frac{\partial A}{\partial v} \right) du\,dv$$

$$= \iint_R (2v - (-2v))\, du\,dv = 4 \iint_R v\, du\,dv$$

$$= 4(2) \int_{-1}^{1} v\,dv = 0.$$

(c) $A = v^n \quad B = u^n \quad u^2 + v^2 \leq r^2$



Divergence thm in 2D (Greens thm)

$$= \int_{C+} (A\,du + B\,dv) = \iint_R (B_u - A_v)\, du\,dv$$

$$= n \iint_R (u^{n-1} - v^{n-1})\, \cancel{du\,dv}\ dA$$

For general n. one can expand Fourier series

$$\cos^{n-1}\theta = \sum_{k=0}^{n-1} a_k \cos(k\theta) \qquad n \geq 4$$

& $$\sin^{n-1}\theta = \sum_{k=0}^{n-1} b_k \sin(k\theta)$$

As $\int_0^{2\pi} \begin{Bmatrix} \cos(k\theta) \\ \sin(k\theta) \end{Bmatrix} d\theta = 0$ The only term to worry about is

a_0 & b_0.

This may depend on what power n is. & then one must show that

$$a_0 \equiv \frac{1}{2\pi} \int_0^{2\pi} \cos^{n-1}\theta \cdot \cos\theta \, d\theta = 0$$

$$= n \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \left(r^{n-1}\cos^{n-1}\theta - r^{n-1}\sin^{n-1}\theta\right) r\,dr\,d\theta$$

let $u = r\cos\theta$

$v = r\sin\theta$

$dA = r\,d\theta\,dr$

$$= n \int_{\theta=0}^{2\pi} \left(\cos^{n-1}\theta - \sin^{n-1}\theta\right)d\theta \underbrace{\int_{r=0}^{R} r^{n}\,dr}_{\left.\frac{r^{n+1}}{n+1}\right|_0^R = \frac{R^{n+1}}{n+1}}$$

$$= \frac{n}{n+1} R^{n+1} \int_{\theta=0}^{2\pi} \left(\cos^{n-1}\theta - \sin^{n-1}\theta\right)d\theta$$

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ yes

$\| $

$\sin\frac{\pi}{2}\cos\theta - \sin\theta\cancel{\cos\frac{\pi}{2}}^{\,0}$

$= \cos\theta$

$$= \frac{n}{n+1} R^{n+1} \int$$

Now for $n=1$ integral is zero

for $n=2$ " " " as both terms integrate separately to zero

for $n=3$ integral becomes

$$\int_0^{2\pi} \left(\cos^2\theta - \sin^2\theta\right) d\theta = \int_0^{2\pi} \cos 2\theta \, d\theta = 0$$

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(2) Derive $\int_{+C^*} f(r,\theta)\,dr + g(r,\theta)\,d\theta = \iint_{R^*} \frac{1}{r}\left\{\frac{\partial g}{\partial r} - \frac{\partial f}{\partial \theta}\right\} dS$

consider

$$\int_{+C^*} A\,du + B\,dv = \iint_{R^*} (B_u - A_v)\,dS$$

$u = u(r,\theta)$
$v = v(r,\theta)$

$du = u_r\,dr + u_\theta\,d\theta$

$dv = v_r\,dr + v_\theta\,d\theta$

LHS

$$= \int_{+C^+} (\underbrace{Au_r + Bv_r}_{f(r,\theta)})\,dr + (\underbrace{Au_\theta + Bv_\theta}_{g(r,\theta)})\,d\theta$$

$$= \iint_{R^*} (B_u - A_v)\,du\,dv = \cancel{\iint_{R^*} \left(B_r \frac{\partial r}{\partial u} + B_\theta\right.}$$

$B_u = \frac{\partial B}{\partial r} r_u + \frac{\partial B}{\partial \theta}\theta_u$ put in Above

$+ A_v = \frac{\partial A}{\partial r} r_v + \frac{\partial A}{\partial \theta}\theta_v$

$$= \iint_{R^*} \left(\frac{\partial B}{\partial r} r_u + \frac{\partial B}{\partial \theta} \theta_u - \frac{\partial A}{\partial r} r_v - \frac{\partial A}{\partial \theta} \theta_v \right) du\, dv.$$

Now:

$$= \quad du\, dv = \frac{\partial(u,v)}{\partial(r,\theta)} \, dr\, d\theta = r\, dr\, d\theta$$

$$\Rightarrow \iint_{R^*}$$ Now $f(r,\theta) = A u_r + B v_r$

$$\therefore \frac{\partial f}{\partial \theta} = \frac{\partial A}{\partial \theta} u_r + A u_{r\theta} + \frac{\partial B}{\partial \theta} v_r + B v_{r\theta}$$

& $g(r,\theta) = A u_\theta + B v_\theta$

Thus $$\frac{\partial g}{\partial r} = \frac{\partial A}{\partial r} u_\theta + A u_{r\theta} + \frac{\partial B}{\partial r} v_\theta + B v_{r\theta}$$

Then $$\frac{\partial g}{\partial r} - \frac{\partial f}{\partial \theta} = \frac{\partial A}{\partial r} u_\theta + \cancel{A u_{r\theta}} + \frac{\partial B}{\partial r} v_\theta + \cancel{B v_{r\theta}}$$

$$- \frac{\partial A}{\partial \theta} u_r - \cancel{A u_{r\theta}} - \frac{\partial B}{\partial \theta} v_r - \cancel{B v_{r\theta}}$$

$$=$$

(3) Pg 556 Courant / John

From problem 2.

$$\int_{+C^*} f(r,\theta)\,dr + g(r,\theta)\,d\theta = \iint_{R^*} \frac{1}{r}\left\{\frac{\partial g}{\partial r} - \frac{\partial f}{\partial \theta}\right\} dS$$

$\Rightarrow$ let ~~$g = r^2$~~ $g = r^2$ & $f = 0$.

The RHS = $\iint_{R^*} \frac{1}{r}\{2r\}\,dS = 2\iint_{R^*} dS$

& LHS = $\int_{+C^*} r^2\,d\theta$ $\therefore$ Area $= \frac{1}{2}\int_{+C^*} r^2\,d\theta$

let $g = 0$ & $f = -r\theta$

The RHS is $\iint_{R^*} \frac{1}{r}\{0 + r\}\,dS = \iint_{R^*} dS$

& LHS = $-\int_{+C^*} r\theta\,dr$

$\therefore$ Area $= -\int_{+C^*} r\theta\,dr$

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(4) Stokes thm in the plane is

$$\iint_R (\nabla \times \vec{B}) \circ \hat{k} \, dx\,dy = \int_{+C^*} \vec{B} \circ \hat{t} \, ds$$

$v = v(x,y)$
$u = u(x,y)$.

let $\vec{B} = u \nabla v$

Then $\nabla \times (u \nabla v) =$

$$\left\{ \text{But } \nabla \times (\alpha \vec{U}) = \nabla \alpha \times \vec{U} + \alpha \nabla \times \vec{U} \right\}$$

$$\therefore \quad \nabla \times (u \nabla v) = \nabla u \times \nabla v + u \, \cancel{\nabla \times \nabla v}^{\,0} = \nabla u \times \nabla v$$

$$\therefore \quad \int_{+C^*} u \nabla v \circ \hat{t} \, ds = \iint_{R^*} (\nabla u \times \nabla v) \circ \hat{k} \, du\,dv$$

$$\nabla u \times \nabla v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & 0 \\ v_x & v_y & 0 \end{vmatrix} = \hat{k}(u_x v_y - v_x u_y)$$

$$= \hat{k} \, \frac{d(u,v)}{d(x,y)}$$

$$\therefore \quad \int_{+C^*} u \nabla v \circ \hat{t} \, ds = \iint_{R^*} \frac{d(u,v)}{d(x,y)} \, dS$$