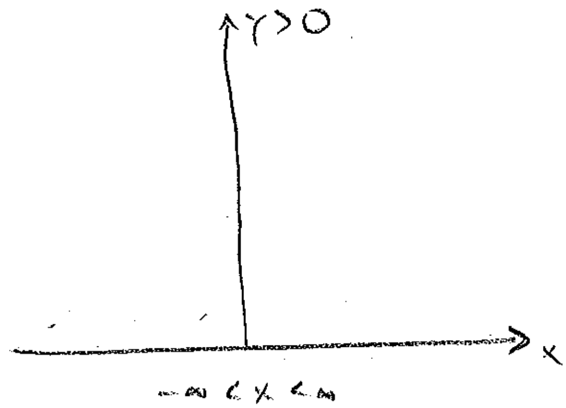
Pg 6 Debnath



$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

w/ $u(x,0) = 0$

$u_y(x,0) = \frac{1}{n}\sin(nx)$

Solving:

Separation of variables gives $Y(0) = 0$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = C \quad \text{constant}$$

$C = 0 \quad X(x) = A + Bx \qquad Y(y) = C + Dy$

$Y(0) = C = 0.$

$u(x,y) = Dy(A + Bx)$

$u_y(x,y) = D(A+Bx) = \frac{1}{n}\sin(nx)$ not possible.

$C > 0$ Let $C = \lambda^2$

$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad Y(y) = C\sin(\lambda y) + D\cos(\lambda y)$

$Y(0) = 0 = D$

$u_y = Y'(y)\,X(x) = \frac{1}{n}\sin(nx)$ not possible

let $C<0$ let $C = -\lambda^2$

$X(x) = A\cos(\lambda x) + B\sin(\lambda x)$ $\quad Y(y) = Ce^{\lambda y} + De^{-\lambda y}$

$Y(0) = 0 = C + D$

$\Rightarrow$ change sol to hyperbolics

$Y(y) = C\sinh(\lambda y) + D\cosh(\lambda y)$

$Y(0) = 0 = D$ $\quad \Rightarrow Y(y) = C\sinh(\lambda y)$

$\therefore \quad u_y(x,y) = X(x)\,Y'(y)$

$= (A\cos(\lambda x) + B\sin(\lambda x))\,\lambda C\cosh(\lambda y)$

$u_y(x,0) = (A\cos(\lambda x) + B\sin(\lambda x))\,\lambda C = \frac{1}{n}\sin(nx)$

take $C = 1$ $\quad$ pick $\lambda = n$ $\quad A = 0$

Then $u_y(x,0) = nB\sin(nx) = \frac{1}{n}\sin(nx)$

take $B = \frac{1}{n^2}$

Full solution

$$U(x,y) = \frac{1}{n^2} \sin(nx) \sinh(ny)$$

Thus one knows that if $U_y(x,0) = 0$ the sol is $U(x,y) \equiv 0$

But if we perturb of this state by a very small amount

$$U_y(x,0) = \frac{1}{n} \sin nx \quad \sim \quad \|U_y(x,0)\| \sim O(\tfrac{1}{n}) \qquad \tfrac{1}{n} \ll 1$$

The change in the solution is infinitely large

$$U(x,y) = \frac{1}{n^2} \sinh(ny) \sin(nx) \longrightarrow \infty \qquad \text{fixed } x \quad y \to +\infty$$

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$$U_x = U_\xi \xi_x + U_\eta \eta_x$$

$$U_{xx} = U_{\xi\xi} \xi_x^2 + \underline{U_{\xi\eta} \xi_x \eta_x} + U_\xi \xi_{xx}$$

$$+ \underline{U_{\eta\xi} \eta_x \xi_x} + U_{\eta\eta} \eta_x^2 + U_\eta \eta_{xx}$$

$$= U_{\xi\xi} \xi_x^2 + 2U_{\xi\eta} \xi_x \eta_x + U_{\eta\eta} \eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx}$$

&

$$U_{xy} = U_{\xi\xi} \xi_x \xi_y + U_{\xi\eta} \xi_x \eta_y + U_\xi \xi_{xy}$$

$$+ U_{\eta\xi} \xi_y \eta_x + U_{\eta\eta} \eta_y \eta_x + U_\eta \eta_{xy}$$

$$= U_{\xi\xi} \xi_x \xi_y + U_{\xi\eta}(\xi_x \eta_y + \xi_y \eta_x) + U_{\eta\eta} \eta_x \eta_y$$

$$+ U_\xi \xi_{xy} + U_\eta \eta_{xy}$$

put into 1.5.1

$$\Rightarrow \underline{A \xi_x^2 U_{\xi\xi}} + 2A \xi_x \eta_x U_{\xi\eta} + A \eta_x^2 U_{\eta\eta} + A \xi_{xx} U_\xi + A \eta_{xx} U_\eta$$

$$+ \underline{B \xi_x \xi_y U_{\xi\xi}} + B(\xi_x \eta_y + \xi_y \eta_x) U_{\xi\eta} + B \eta_x \eta_y U_{\eta\eta}$$

Pg 10 Debnath

$$A u_{xx} + B u_{xy} + C u_{yy} + D u_x + E u_y + F u = G$$

putting in u_{xx} obtained by transformations.

$$A \xi_x^2 \underline{u_{\xi\xi}} + 2A \xi_x \eta_x u_{\xi\eta} + A \eta_x^2 u_{\eta\eta} + A \xi_{xx} u_\xi + A \eta_{xx} u_\eta$$

$$+ B \xi_x \xi_y \underline{u_{\xi\xi}} + B(\xi_x \eta_y + \xi_y \eta_x) u_{\xi\eta} + B \eta_x \eta_y u_{\eta\eta} + B \xi_{xy} u_\xi$$

$$+ B \eta_{xy} u_\eta$$

$$+ C \xi_y^2 \underline{u_{\xi\xi}} + 2C \xi_y \eta_y u_{\xi\eta} + C \eta_y^2 u_{\eta\eta} + C \xi_{yy} u_\xi + C \eta_{yy} u_\eta$$

$$+ D \xi_x u_\xi + D \eta_x u_\eta + E \xi_y u_\xi + E \eta_y u_\eta + F u = G$$

$$= (A \xi_x^2 + B \xi_x \xi_y + C \xi_y^2) u_{\xi\xi} + \left(2A \xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C \xi_y \eta_y\right) u_{\xi\eta}$$

$$+ (A \eta_x^2 + B \eta_x \eta_y + C \eta_y^2) u_{\eta\eta}$$

$$+ (A \xi_{xx} + B \xi_{xy} + C \xi_{yy} + D \xi_x + E \xi_y) u_\xi$$

$$+ (A \eta_{xx} + B \eta_{xy} + C \eta_{yy} + D \eta_x + E \eta_y) u_\eta$$

$$+ F u = G$$

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$

$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$

$$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$

$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$

$$F^* = F$$

$$G^* = G$$

The true "guts" of this transformation is that we obtain a Non-linear eq for ξ + η but ~~one~~ that we can solve Exactly.
Due to the quadratic formula.

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$\alpha = \xi + \eta \qquad \beta = \xi - \eta$$

$$u_\xi = u_\alpha \alpha_\xi + u_\beta \beta_\xi =$$

$$0 = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = A\xi_x^2 + 2\sqrt{A}\sqrt{C}\,\xi_x\xi_y + C\xi_y^2$$

{ Assuming $A > 0$ + $C > 0$ what if not? }

$$(\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y)^2$$

$$B^* = 2A\xi_x\eta_x + \underset{\substack{\| \\ 4AC}}{B}(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(A\xi_x\eta_x + AC(\xi_x\eta_y + \xi_y\eta_x) + C\xi_y\eta_y)$$

$$= 2(\underbrace{\sqrt{A}\,\xi_x + \sqrt{C}\,\xi_y}_{=0.})(\sqrt{A}\,\eta_x + \sqrt{C}\,\eta_y)$$

$\therefore \quad B^* = 0$ w/ ~~...~~ parabolic eqs

if $\eta = y$.

Then $D = \text{Jacobian} = J = \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix} = \begin{vmatrix} \xi_x & 0 \\ \xi_y & 1 \end{vmatrix} = \xi_x \neq 0.$

only $A^* = 0$ because we have only one solution to the quadratic equation.

eq 1.5.3

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^* \qquad (1)$$

let $\alpha = \frac{1}{2}(\xi + \eta)$

$\beta = \frac{1}{2i}(\xi - \eta)$

Then

$$U_\xi = U_\alpha \alpha_\xi + U_B B_\xi = \frac{1}{2}U_\alpha + \frac{1}{2i}U_B$$

$$U_\eta = U_\alpha \alpha_\eta + U_B B_\eta = \frac{1}{2}U_\alpha - \frac{1}{2i}U_B$$

$$U_{\xi\xi} = (U_\xi)_\alpha \alpha_\xi + (U_\xi)_B B_\xi$$

$$= \left(\frac{1}{2}U_{\alpha\alpha} + \frac{1}{2i}U_{B\alpha}\right)\frac{1}{2} + \left(\frac{1}{2}U_{\alpha B} + \frac{1}{2i}U_{BB}\right)\frac{1}{2i}$$

$$= \frac{1}{4}U_{\alpha\alpha} + \frac{1}{4i}U_{\alpha B} + \frac{1}{4i}U_{\alpha B} + \frac{-1}{4}U_{BB}$$

$$= \frac{1}{4}U_{\alpha\alpha} + \frac{1}{2i}U_{\alpha B} - \frac{1}{4}U_{BB}$$

$$U_{\xi\eta} = (U_\xi)_\alpha \alpha_\eta + (U_\xi)_B B_\eta$$

$$= \frac{1}{2}\left(\frac{1}{2}U_{\alpha\alpha} + \frac{1}{2i}U_{\alpha B}\right) + \frac{-1}{2i}\left(\frac{1}{2}U_{\alpha B} + \frac{1}{2i}U_{BB}\right)$$

$$= \frac{U_{\alpha\alpha}}{4} + \cancel{\frac{1}{4i}U_{\alpha B}} - \cancel{\frac{1}{4i}U_{\alpha B}} + \frac{1}{4}U_{BB}$$

$$= \frac{1}{4}(U_{\alpha\alpha} + U_{BB})$$

$$U_{\eta\eta} = (U_\eta)_\alpha \alpha_\eta + (U_\eta)_B B_\eta$$

$$= \left(\frac{1}{2}U_{\alpha\alpha} - \frac{1}{2i}U_{B\alpha}\right)\frac{1}{2} + \left(\frac{U_{\alpha B}}{2} - \frac{U_{BB}}{2i}\right)\left(\frac{-1}{2i}\right)$$

$$\cancel{B^*} \; U_{\eta\eta} = \frac{U_{\alpha\alpha}}{4} - \frac{1}{2i} U_{\alpha\beta} - \frac{U_{\beta\beta}}{4}$$

$\therefore$ eq (1) becomes.

$$\frac{A^*}{4}\left(U_{\alpha\alpha} + \frac{2}{i} U_{\alpha\beta} - U_{\beta\beta}\right) + \frac{B^*}{4}\left(U_{\alpha\alpha} + U_{\beta\beta}\right)$$

$$+ \frac{C^*}{4}\left(U_{\alpha\alpha} - \frac{2}{i} U_{\alpha\beta} - U_{\beta\beta}\right) + \frac{D^*}{2}\left(U_\alpha + \frac{1}{i} U_\beta\right)$$

$$+ \frac{E^*}{2}\left(U_\alpha - \frac{1}{i} U_\beta\right) + F^* U = G^*$$

$$\Rightarrow \left(\frac{A^*}{4} + \frac{B^*}{4} + \frac{C^*}{4}\right) U_{\alpha\alpha} + \left(\frac{A^*}{2i} - \frac{C^*}{2i}\right) U_{\alpha\beta}$$

$$+ \left(\frac{-A^*}{4} + \frac{B^*}{4} - \frac{C^*}{4}\right) U_{\beta\beta} + \left(\frac{D^*}{2} + \frac{E^*}{2}\right) U_\alpha$$

$$+ \left(\frac{D^*}{2i} - \frac{E^*}{2i}\right) U_\beta + F^* U = G^*.$$

$\cancel{A^* = 0 = C^*}$ Define $A^{**} = \frac{1}{4}(A^* + B^* + C^*)$

$$B^{**} = \frac{A^*}{2i} - \frac{C^*}{2i}$$

$\Rightarrow$

$$C^{**} = \frac{1}{4}\left(-A^* + B^* - C^*\right)$$

Solving for $A^*, \dots, C^*$ gives

$$A^* = 2i B^{**} + C^* \quad \text{put in eq (1) + (3)}$$

$$A^{**} = \frac{1}{4}(2iB^{**} + B^* + 2C^*)$$

$$+ \quad C^{**} = \frac{1}{4}(-2iB^{**} - 2C^* + B^*) \qquad \text{Adding}$$

$$\rightarrow A^{**} + C^{**} = \frac{2B^*}{4} \Rightarrow B^* = 2(A^{**} + C^{**})$$

Then subtracting $A^{**} - C^{**} = \frac{1}{4}(4iB^{**} + 4C^*)$

$$\rightarrow \quad C^* = A^{**} - C^{**} - iB^{**}$$

Then $A^* = \overset{2iB^{**}}{\cancel{4i(A^{**} + C^{**})}} + A^{**} - C^{**} - iB^{**}$

$$= A^{**} - C^{**} + iB^{**}$$

$\therefore$ summary.

$$A^* = A^{**} + iB^{**} - C^{**} \qquad \text{if} = 0$$

$\cancel{B^*} \geq$

$$B^* = 2A^{**} + 2C^{**} \qquad \text{if} = 0$$

$$C^* = A^{**} - iB^{**} - C^{**}$$

PS 13 Debnath

Ex 1.5.1

(a) $A = 1;\ B = 0;\ C = -c^2$

$B^2 - 4AC = +4c^2 > 0$ hyperbolic.

(b) $A = -k$ one way; $B = 0;\ C = 0$ | another $A = 0;\ B = 0;\ C = -k$

$B^2 - 4AC = 0$ parabolic. $B^2 - 4AC = 0$

(c) $A = 1;\ B = 0;\ C = 1$

$B^2 - 4AC = -4 < 0$ elliptic

(d) $A = 1;\ B = 0;\ C = x$

$B^2 - 4AC = 0 - 4x$ > 0 if $x < 0$ hyperbolic

$= 0$ if $x = 0$ parabolic

< 0 if $x > 0$ elliptic

Ex 1.5.2

$x u_{xx} + u_{yy} = x^2$

$A = x;\ B = 0;\ C = 1$

$B^2 - 4AC = -4x$ > 0 $x < 0$ hyperbolic

$= 0$ $x = 0$ parabolic

< 0 $x > 0$ elliptic.

Characteristic eqs are:

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2-4AC}\right)$$

$$\frac{dy}{dx} = \frac{1}{2x}\left(0 \pm \sqrt{-4x}\right) = \frac{\pm 2\sqrt{-x}}{2x} = \frac{\pm\sqrt{-x}}{x}$$

$$x = (-x)^2$$

$$= \pm\frac{1}{\sqrt{-x}}$$

$$dy = \pm\frac{1}{\sqrt{-x}}dx = \pm(-x)^{-1/2}dx$$

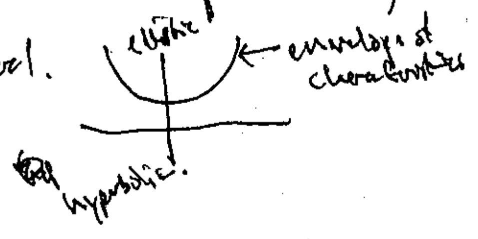
$$y = \pm\frac{(-x)^{1/2}}{1/2} + C_1$$

$$y = \pm 2\sqrt{-x} + C_1 .$$

$$\xi = y + 2\sqrt{-x} \mp \text{const.} \qquad (y-y_0)^2 = -4x .$$

$$\eta = y - 2\sqrt{-x} \mp \text{const.} \qquad (y-y_0)^2 =$$

Note: The characteristics only exist in the region of the xy plane where the determinant $B^2-4AC>0$ + The envelope of the characteristics gives the boundary between hyperbolic regions + elliptic regions. That this is true in general.



pg 14-15 Debnath

$\xi = y + 2\sqrt{-x}$

$\eta = y - 2\sqrt{-x}$

To transform to cannonical form we only need to compute A^*, B^*, C^*.

$\xi_x = \dfrac{-1}{\sqrt{-x}}$ $\quad \xi_y = 1 \quad \xi_{xy} = 0.$

$\eta_x = \dfrac{1}{\sqrt{-x}}$ $\quad \eta_y = 1. \quad \eta_{xy} = 0$

$$\xi_{xx} = -\left(\tfrac{-1}{2}\right)(-x)^{-3/2}(-1) \qquad \xi_{yy} = 0$$
$$= \left(-\tfrac{1}{2}\right)(-x)^{-3/2}$$

$\eta_{yy} = 0 \; ; \quad \eta_{xx} = \frac{1}{2}(-x)^{-3/2}.$

$\therefore \quad \left.\begin{array}{l} (\xi-\eta)^2 = 4(-x) \\ (\xi-\eta)^4 = (16x)^2. \end{array}\right\}$ how to change $x + y$'s to $\eta + \xi$'s

$$x u_{xx} + u_{yy} = x^2$$

$$\Rightarrow x\left(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi\xi_{xx} + u_\eta\eta_{xx}\right)$$

$$+ \left(u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\eta\eta_{yy} + u_\xi\xi_{yy}\right) = x^2$$

$$\Rightarrow x\left[u_{\xi\xi}\left(\frac{1}{-x}\right) + 2u_{\xi\eta}\left(\frac{-1}{\sqrt{-x}}\right)\left(\frac{1}{\sqrt{-x}}\right)\right.$$

$$+ u_{\eta\eta}\left(\frac{1}{-x}\right) + u_\xi\left(\frac{-1}{2}\right)(-x)^{-3/2} + u_\eta \frac{1}{2}(-x)^{-3/2}\Big)$$

$$+ \left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) = x^2$$

$$= -\cancel{u_{\xi\xi}} + 2u_{\xi\eta} - \cancel{u_{\eta\eta}} + \frac{1}{2}u_\xi \frac{\cancel{(-x)}}{\cancel{(-x)}(-x)^{1/2}} - \frac{u_\eta}{2}\frac{\cancel{(-x)}}{\cancel{(-x)}(-x)^{1/2}}$$

$$+ \cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}} = x^2$$

$$= 4u_{\xi\eta} + \frac{1}{2}u_\xi \underbrace{\frac{1}{\sqrt{-x}}}_{= \frac{4}{\xi-\eta}} - \frac{u_\eta}{2}\frac{1}{\sqrt{-x}} = x^2$$

$$= \frac{1}{16^2}(\xi-\eta)^4.$$

$$u_{\xi\eta} = \frac{1}{4}\frac{1}{16^2}(\xi-\eta)^4 - \frac{1}{2}\frac{u_\xi}{(\xi-\eta)} + \frac{u_\eta}{2(\xi-\eta)}$$

$x > 0$ Then

$$\xi = y + 2i\sqrt{x} \qquad \eta = y - 2i\sqrt{x}$$

$$\alpha = y \qquad \beta = \frac{1}{2i}(\xi-\eta) = 2\sqrt{x}$$

one shall then know enough to make the $(x,y) \to (\alpha, \beta)$ transformation directly.

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$\alpha_x = 0$ $\quad\quad$ $\beta_y = 0 = \beta_{xy} = \beta_{yy}$

$\alpha_y = 1$ $\quad\quad$ $\beta_x = \frac{1}{\sqrt{x}}$; $\beta_{xx} = \frac{-1}{2x^{3/2}}$

$\alpha_{xy} = \alpha_{xx} = \alpha_{yy} = 0$

eq $\quad xU_{xx} + U_{yy} = 0 \quad\quad + \; x = (\beta/2)^2.$

$$\Rightarrow x\left(U_{\alpha\alpha}\alpha_x^2 + 2U_{\alpha\beta}\alpha_x\beta_x + U_{\beta\beta}\beta_x^2 + U_\alpha\alpha_{xx} + U_\beta\beta_{xx}\right)$$

$$+\left(U_{\alpha\alpha}\alpha_y^2 + 2U_{\alpha\beta}\alpha_y\beta_y + U_{\beta\beta}\beta_y^2 + U_\alpha\alpha_{yy} + U_\beta\beta_{yy}\right)$$

$$= \left(\frac{\beta}{2}\right)^4$$

$$\Rightarrow x\left(U_{\beta\beta}\frac{1}{x} - \frac{U_\beta}{2x^{3/2}}\right) + U_{\alpha\alpha} + \cancel{2U_{\alpha\beta}}^{\,0} = \frac{\beta^4}{2^4}$$

$$U_{\beta\beta} + U_{\alpha\alpha} = \frac{U_\beta}{2\sqrt{x}} + \frac{\beta^4}{2^4} = \frac{U_\beta}{\beta} + \frac{\beta^4}{2^4}$$

in Region $x > 0$ & $\frac{\beta^2}{4} > 0$.

$U_{yy} = 0 \quad\quad A = 0 = B \neq C$

eqs: $\quad A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$

$\Rightarrow C\left(\frac{\xi_y}{\xi_x}\right)^2 = 0.$ are the characteristic eqs.

$$\frac{dx}{dy} = -\frac{\xi_y}{\xi_x} = 0 \Rightarrow \frac{dx}{dy} = 0$$

$\Rightarrow x = \text{const}$ But this is only valid at $x = 0 \Rightarrow$ $\xi = 0$

pg 16 Debnath

$A = -c^2, \; B = 0, \; C = 1$ $\qquad y \to t$

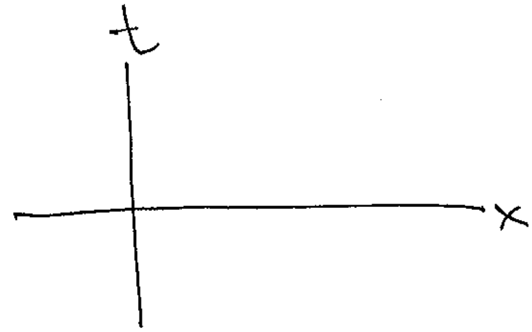
$$\frac{dy}{dx} = \frac{1}{2(-c^2)}\left(\pm\sqrt{4c^2}\right) = \mp\frac{1}{c}$$

$$\Rightarrow \frac{dt}{dx} = \pm\frac{1}{c}$$

$$t = \pm\frac{x}{c} + \text{const.}$$

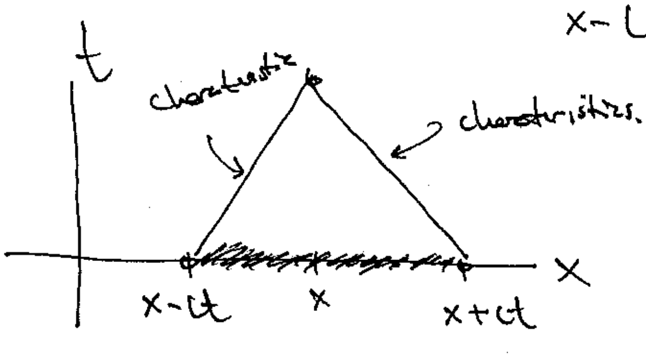
$$\pm x - ct = \text{const}'$$

$$x \pm ct = \text{const}''.$$



$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\,d\tau$$

Thus



Thus info at x,t depends on values at x-ct & x+ct.

(Also the integral in between for I.V.P. If $g \equiv 0$ this integral is not present but idea is same.

info to det. value of u(x,t) comes along characteristic $\pm$ values between characteristics. i.e. interval $(x-ct, x+ct)$

Pg 16 Debnath

$\xi = x - ct \qquad \eta = x + ct$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_{xx} = (u_x)_x = (u_\xi + u_\eta)_\xi + (u_\xi + u_\eta)_\eta$

$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$+ \; u_{tt} = ?$

$u_t = u_\xi(-c) + cu_\eta$

$u_{tt} = (u_\xi(-c) + cu_\eta)_\xi(-c) + (u_\xi(-c) + cu_\eta)_\eta c$

$= c^2 u_{\xi\xi} - c^2 u_{\eta\xi} - c^2 u_{\xi\eta} + c^2 u_{\eta\eta}$

$= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$

$\cancel{u_{xx} = c^2} \quad \therefore \quad u_{xx} = c^2 u_{tt}$ is not wave eq $u_{tt} - c^2 u_{xx} = 0$!!

$\Rightarrow \cancel{c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})} =$

$\Rightarrow \cancel{c^2(u_{tt} - c^2 u_{xx})} = c^2$

$\cancel{c^2}(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = \cancel{c^2}(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}})$

$4u_{\xi\eta} = 0$

$u_{\xi\eta} = 0 \qquad \cancel{u_\eta = f(\xi)}$

~~$v = \int f(\xi)\,\eta + g(\xi)$~~

$$\frac{\partial^2}{\partial\xi\,\partial\eta} v = 0$$

$$\frac{\partial v}{\partial\eta} = f(\eta)$$

$$v = \int^{\eta} f(\eta')\,d\eta' + g(\xi)$$

$$\therefore \quad v(\xi,\eta) = \phi(\xi) + \psi(\eta)$$

$$v(x,t) = \phi(x-ct) + \psi(x+ct)$$

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$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x-ct}^{0} g(\tau)\,d\tau + \frac{1}{2c}\int_{0}^{x+ct} g(\tau)\,d\tau$$

$$= \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{0}^{x-ct} g(\tau)\,d\tau + \cdots$$

— — — — —

pg 19 Debnath

$A = -c^2(x)$; $B = 0$; $C = 1$

$B^2 - 4AC = 4c^2(x) > 0$ hyperbolic.

$$\left(\frac{dx}{dt}\right)^{-1} = \frac{1}{2A}\left(B \pm \sqrt{B^2-4AC}\right)$$

$$= \frac{-1}{2c^2}\left(\pm 2\sqrt{c^2}\right) = \pm\frac{1}{\sqrt{c^2(x)}} = \pm\frac{1}{a}$$

$$\frac{dt}{dx} = \pm\frac{1}{a(x)}$$

$$t = \pm\int^{x}\frac{d\tau}{a(\tau)} + \text{const}$$

want ξ to be positive slopes & η to be negative slopes

$$t \pm \int^{x}\frac{d\tau}{a(\tau)} = \text{const}$$

$$\xi = \underbrace{t - \int^{x}\frac{d\tau}{a(\tau)}}_{\text{positive slopes}} \quad \& \quad \eta = t + \int^{x}\frac{d\tau}{a(\tau)}$$

Transform to characteristic eqs:

$\cancel{U_x = \frac{1}{a(x)}}$ $\quad \cancel{U_{xx} = \frac{1}{a(x)^2} a'(x)}$

$$U_x = U_\xi \xi_x + U_\eta \eta_x \qquad U_t = U_\xi \xi_t + U_\eta \eta_t$$

$$= -\frac{1}{a(x)} U_\xi + \frac{1}{a(x)} U_\eta \qquad = U_\xi + U_\eta$$

$$\cancel{U_{xx} = (U_x)_\xi \xi_x + (U_x)_\eta \eta_x}$$

$$\cancel{= \left(-\frac{1}{a(x)} U_\xi + \frac{1}{a(x)} U_\eta\right)_\xi \left(-\frac{1}{a(x)}\right) + \left(-\frac{1}{a(x)} U_\xi + \frac{1}{a(x)} U_\eta\right)_\eta \frac{1}{a(x)}}$$

$$\cancel{= +\frac{1}{a(x)^2} U_{\xi\xi} + \frac{-1}{a(x)^2} U_{\eta\xi} - \frac{1}{a(x)^2} U_{\xi\eta} + \frac{1}{a(x)^2} U_{\eta\eta}}$$

Must change $a(x)$ to $\boxed{a(x(\eta,\xi))}$

$\therefore$

$$U_{xx} = (U_x)_\xi \xi_x + (U_x)_\eta \eta_x$$

$$= \left(-\frac{1}{a(x)} U_\xi + \frac{1}{a(x)} U_\eta\right)_\xi \left(-\frac{1}{a(x)}\right) + \left(-\frac{1}{a(x)} U_\xi + \frac{1}{a(x)} U_\eta\right)_\eta \frac{1}{a(x)}$$

But:

$$= \left(+\frac{1}{a^2(x)} \frac{\partial}{\partial \xi} a(x(\eta,\xi)) U_\xi - \frac{1}{a(x)} U_{\xi\xi} + \frac{-1}{a^2(x)} \frac{\partial}{\partial \xi} a(x(\eta,\xi)) U_\eta \right.$$

$$\left. + \frac{1}{a(x)} U_{\eta\xi}\right)\left(\frac{-1}{a(x)}\right)$$

$$\left(+\frac{1}{a(x)^2}\frac{\partial a}{\partial \eta}u_\xi + \frac{-1}{a(x)}u_{\xi\eta} + \frac{-1}{a^2}\frac{\partial a}{\partial \eta}u_\eta + \frac{1}{a(x)}u_{\eta\eta}\right)\left(\frac{1}{a(x)}\right)$$

$$= -u_\xi \frac{1}{a^3} a' x_\xi + \frac{1}{a^2}u_{\xi\xi} + \frac{1}{a^3} a' x_\xi u_\eta - \frac{1}{a^2}u_{\eta\xi}$$

$$u_\xi\left(+\frac{1}{a^3}\right)a' x_\eta - \frac{1}{a^2}u_{\xi\eta} - \frac{1}{a^3}a' x_\eta u_\eta + \frac{1}{a^2}u_{\eta\eta}$$

$$+ \quad x_\xi = \frac{1}{\xi_x} = \frac{1}{(-1/a)} = -a$$

$$x_\eta = \frac{1}{\eta_x} = \frac{1}{(1/a)} = a$$

$\therefore$

$$u_{xx} = \cancel{\frac{u_\xi}{a^2}a'} + \frac{1}{a^2}u_{\xi\xi} - \frac{a'}{a^2}u_\eta - \frac{1}{a^2}u_{\eta\xi}$$

$$+ \cancel{\frac{a'}{a^2}u_\xi} - \frac{u_{\xi\eta}}{a^2} - \frac{a'}{a^2}u_\eta + \frac{1}{a^2}u_{\eta\eta}$$

Got a two?

$$= \frac{1}{a^2}\left(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}\right) - 2\left(u_\xi + u_\eta\right)\frac{a'}{a^2}$$

$$u_{tt} = (u_t)_\xi \xi_t + (u_t)_\eta \eta_t$$

$$= (u_{\xi\xi} + u_{\eta\xi}) + (u_{\xi\eta} + u_{\eta\eta})$$

$$= u_{\xi\xi} + 2u_{\eta\xi} + u_{\eta\eta}$$

$$u_{tt} - c^2 u_{xx} = 0$$

$$c^2 u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - 2(u_\eta - u_\xi)\,a'(x)$$

$$u_{tt} - c^2 u_{xx} = 2u_{\xi\eta} + 2u_{\xi\eta} + 2(u_\eta - u_\xi)\,a'(x) = 0$$

$$\Rightarrow \quad 4u_{\xi\eta} + 2(u_\eta - u_\xi)\,a'(x) = 0$$

If $a(x) = Ax^n$

Then $\eta - \xi = 2\int^x \frac{d\tau}{A\tau^n}$

$$\eta - \xi = \frac{2}{A}\,\frac{\tau^{-n+1}}{-n+1}\Bigg|^x = \frac{-2}{A}\,\frac{1}{(n-1)}\,\frac{1}{x^{n-1}}$$

$$= \frac{-2n}{(n-1)\,A n x^{n-1}} = \frac{-2n}{(n-1)\,a'(x)}$$

$$\Rightarrow \quad a'(x) = \frac{-2n}{n-1}\,\frac{1}{(\eta-\xi)}$$

$\therefore$ wave eq becomes following Debnath w/o my two.

$$4u_{\xi\eta} + (u_\eta - u_\xi)\left(\frac{-2n}{(n-1)}\,\frac{1}{(\eta-\xi)}\right) = 0$$

$$U_{\xi\eta} = \frac{n(U_\eta - U_\xi)}{2(n-1)(\eta-\xi)} \quad \text{pg } 20$$

If $n=1$ $\quad a(x) = Ax \quad a' = A \quad$ eq 1.5.46

$$4U_{\xi\eta} + A(U_\eta - U_\xi) = 0$$

$$\xi = \alpha/A \quad \eta = \beta/A$$

$$\Rightarrow 4\cancel{A^2}U_{\alpha\beta} + \cancel{A^2}(U_\beta - U_\alpha) = 0$$

$$U_{\alpha\beta} = \frac{1}{4}(U_\alpha - U_\beta)$$

$(x-y)U_{xy} + U_y - U_x$ $\quad$ eq 1.5.52

$m=1$ $\quad$ eq 1.5.51

$$(x-y)U_{xy} = U_x - U_y$$

$$\therefore \underbrace{(x-y)U_{xy} - U_x + U_y}_{} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}((x-y)U) = 0$$

$$(x-y)U = \phi(x) + \psi(y)$$

Note:
Idea behind $\frac{\partial^2}{\partial x \partial y}((x-y)U)$ is that $x-y$ is almost a constant wrt. $\frac{\partial^2}{\partial x \partial y}$.

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$$(x-y)u_{xy} = m(u_x - u_y)$$

$\frac{\partial^2}{\partial x \partial y} \Rightarrow$ By eq 1.5.52

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy}) + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)u_{xy} = \underbrace{m\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u_{xy}}$$

changing order of operators

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) + \frac{\partial^2}{\partial x \partial y}$$

$$(x-y)\frac{\partial^2}{\partial x \partial y}(\underline{u_{xy}}) = (m+1)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(\underline{u_{xy}})$$

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/

$$U(x,t) = X(x)T(t)$$

$$\ddot{T}X = c^2 X'' T$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{\ddot{T}}{T} = \lambda$$

$$\frac{d^2X}{dx^2} - \lambda X = 0$$

$$\frac{d^2X}{dx^2} - \alpha^2 X = 0.$$

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x}.$$

$$X(0) = A = 0.$$

$$X(l) = B\sin\alpha l = 0.$$

$$\alpha_n l = n\pi$$

$$\alpha_n = \frac{n\pi}{l}$$

$$U(x,t) = \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi ct}{l}\right) + b_n\sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$U_t(x,t) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{l}\right)\left(-a_n\sin\left(\frac{n\pi ct}{l}\right) + b_n\cos\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$U_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{n\pi c}{l}\right)b_n\sin\left(\frac{n\pi x}{l}\right) = \frac{\pi c}{l}\sum_{n=1}^{\infty} n b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$\cancel{\phi(v,x_*) = \frac{x_* f(v) - v f(x_*)}{f(v) - f(x_*)}}$$

$$U(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$$

Multiply by $\sin\left(\frac{m\pi x}{\ell}\right) \Rightarrow$

$$\int_0^{\ell} \sin\left(\frac{m\pi x}{\ell}\right) U(x,0) = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) \sin\left(\frac{m\pi x}{\ell}\right) dx}_{\frac{\ell}{2} \quad n=m}$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx$$

$$b_n = \frac{2}{\cancel{\ell}} \cdot \frac{\cancel{\ell}}{\pi c} \frac{1}{n} \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx = \frac{2}{n\pi c} \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx.$$

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$$\omega_n = \frac{n\pi c}{l}$$

$$\omega_1 = \frac{\pi c}{l}$$

$$\nu_1 = \frac{\omega_1}{2\pi} = \frac{c}{2l} = \frac{1}{2l}\sqrt{\frac{T^*}{\rho}}$$

$$T_1 = \frac{2\pi}{\omega_1} = \frac{1}{\nu_1} = 2l\sqrt{\frac{\rho}{T^*}} = \frac{2l}{c}$$

(1.6.25)

$$u_n(x,t) = \left(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right)\sin\left(\frac{n\pi x}{l}\right)$$

if velocity is zero initially $b_n = 0$ then +

$$u_n(x,t) = a_n\cos(\omega_n t)\sin\left(\frac{n\pi x}{l}\right) = a_n \overset{\cos}{\cancel{\sin}}\left(\frac{n\pi c}{l}t\right)\sin\left(\frac{n\pi x}{l}\right)$$

$= 0$ at

$$\frac{n\pi x}{l} = m\pi \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$x = \frac{l m \cancel{\pi}}{n\cancel{\pi}} = l\left(\frac{m}{n}\right) = m\left(\frac{l}{n}\right)$$

to have $x \in [0, l]$ $m = 0, +1, +2, \cdots n$

$\therefore$ $x = 0, \frac{l}{n}, \frac{2l}{n}, \cdots, l$

Note:

~~[illegible]~~n $\cos(\theta_1)\sin(\theta_2) =$

$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ (in volves only products of trig fns of themselves)

$\therefore \cos\theta_1 \sin\theta_2 \overset{?}{=} \frac{1}{2}\left(\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)\right)$

$\frac{1}{2}\left(\sin\theta_1\cos\theta_2 + \cancel{\sin\theta_2\cos\theta_1} + \sin\theta_1\cos\theta_2 - \cancel{\sin\theta_2\cos\theta_1}\right)$

$= \sin\theta_1\cos\theta_2$ yes

$$\therefore u_n(x,t) = a_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right)$$

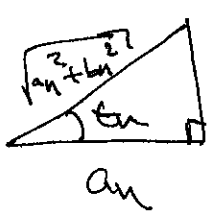
$$= \frac{a_n}{2}\left(\sin\left(\frac{n\pi}{l}(x+ct)\right) + \sin\left(\frac{n\pi}{l}(x-ct)\right)\right) \quad \text{eq } 1.6.28$$

From eq 1.6.19

$$u_n = \left[a_n\cos\left(\frac{n\pi ct}{l}\right) + b_n \sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

let $c_n = (a_n^2 + b_n^2)^{1/2}$

$$u_n = c_n\left[\frac{a_n}{(a_n^2+b_n^2)^{1/2}}\cos\left(\frac{n\pi ct}{l}\right) + \frac{b_n}{(a_n^2+b_n^2)^{1/2}}\sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$



b_n then $\frac{a_n}{\sqrt{a_n^2+b_n^2}} = \cos\epsilon_n$ & $\frac{b_n}{\sqrt{a_n^2+b_n^2}} = \sin\epsilon_n$

$$\therefore\ u_n(x,t) = C_n \underbrace{\left[\cos\epsilon_n \cos\left(\frac{n\pi ct}{l}\right) + \sin\epsilon_n \sin\left(\frac{n\pi ct}{l}\right)\right]}_{\cos\left(\frac{n\pi ct}{l} - \epsilon_n\right)} \sin\left(\frac{n\pi x}{l}\right)$$

$$u_n(x,t) = C_n \cos\left(\frac{n\pi ct}{l} - \epsilon_n\right) \sin\left(\frac{n\pi x}{l}\right) \qquad \text{eq } 1.6.29$$

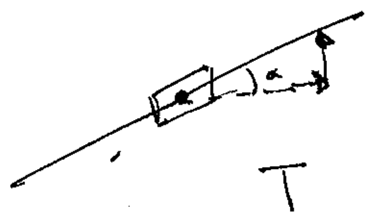
$$k_n = \int_0^l \frac{1}{2}\rho \left(\frac{\partial u_n}{\partial t}\right)^2 dx$$

$$\rho g h = \rho g\, u_n(x,t)$$

$$V_n = \frac{1}{2} T^* \int_0^l \left(\frac{\partial u_n}{\partial x}\right)^2 dx$$ How get?

Newtons law:

Horizontal forces cancel
tangential forces don't.



$$k_n = \frac{\rho}{2}\int_0^l \left(\frac{n\pi c}{l}\right)^2 c_n^2 \sin^2\left(\frac{n\pi x}{l}\right) \sin^2\left(\frac{n\pi ct}{l} - \epsilon_n\right) dx$$

$$= \left(\frac{n\pi c}{l}\right)^2 c_n^2 \frac{\rho}{2} \sin^2\left(\frac{n\pi ct}{l} - \epsilon_n\right) \underbrace{\int_0^l \sin^2\left(\frac{n\pi x}{l}\right) dx}_{\frac{l}{2} \text{ by eq } 1.6.24}$$

$$= \frac{\rho l}{4}\left(\frac{n\pi c}{l}\right)^2 C_n^2 \sin^2\left(\frac{n\pi c t}{l} - \epsilon_n\right)$$

$$= \frac{\rho}{4} \frac{n^2\pi^2 c^2}{l} C_n^2 \sin^2\left(\frac{n\pi c t}{l} - \epsilon_n\right) \qquad \text{w/ } \omega_n = \frac{n\pi c}{l}$$

$$= \cancel{\rho c^2 \pi}$$

$$= \frac{1}{4}\rho l \omega_n^2 C_n^2 \sin^2(\omega_n t - \epsilon_n)$$

$$V_n = \frac{T^*}{2}\int_0^l C_n^2 \left(\frac{n\pi}{l}\right)^2 \cos^2\left(\frac{n\pi x}{l}\right)\cos^2\left(\frac{n\pi c t}{l} - \epsilon_n\right) dx$$

$$= \frac{T^*}{2} C_n^2 \left(\frac{n\pi}{l}\right)^2 \cos^2\left(\frac{n\pi c t}{l} - \epsilon_n\right) \underbrace{\int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx}_{\frac{l}{2}}$$

$$c^2 = \frac{T^*}{\rho}$$

$$\Rightarrow V_n = C_n^2 \frac{\rho c^2}{4}\left(\frac{n\pi}{l}\right)^2 l \cos^2(\omega_n t - \epsilon_n)$$

$$= C_n^2 \frac{\rho}{4} l \omega_n^2 \cos^2(\omega_n t - \epsilon_n)$$

$$E_n = K_n + V_n = \frac{1}{4}\cancel{\rho}\rho (\omega_n c_n)^2 \, 1 \quad \text{eq } 1.6.34.$$

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$$u_t = k\, u_{xx}$$

$$\underbrace{\frac{1}{T}\frac{dT}{dt}}_{\text{units } 1/\text{time} \,=\, s^{-1}} = \frac{k}{X}\frac{d^2X}{dx^2} = \lambda$$

$$\frac{[k]}{\cancel{\ }L^2} = \frac{L^2/T}{L^2} \sim \frac{1}{T}$$

~~[illegible]~~ How know / why put k on left hand side of the equation? Try it w/o that

$$\frac{dT}{dt} - \lambda T = 0 \quad \& \quad \frac{d^2X}{dx^2} - \frac{\lambda}{k} X = 0$$

It is easier to solve this eq w/o having to worry about k.

~~[illegible]~~ i.e. let $\lambda = k\beta$

Then check which ~~solutions~~ ~~of~~ signs of β give solutions.

Writing eqs a)

$$\frac{1}{kT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda$$

$$\frac{d^2X}{dx^2} - \lambda X = 0 \quad \& \quad \frac{dT}{dt} - \lambda k T = 0$$

$$\frac{d^2 X}{dx^2} - \lambda X = 0$$

let $\lambda < 0$. $\lambda = -\alpha^2$.

$$X(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

$X(0) = X(l) = 0$

$\Rightarrow A = 0$

$\alpha l = n\pi$

$\alpha_n = \frac{n\pi}{l}$

$\lambda = 0$

$$\frac{d^2 X}{dx^2} = 0$$

$X = A + Bx$

$X(0) = X(l) = 0$

$\Rightarrow A = B = 0$

$\lambda > 0 \Rightarrow \lambda = \alpha^2$

$$\frac{d^2 X}{dx^2} - \alpha^2 X = 0$$

$X(x) = Ae^{\alpha x} + Be^{-\alpha x}$.

$X = A'\cosh(\alpha x) + B'\sinh(\alpha x)$

$X(0) = X(l) = 0$

$\Rightarrow A' = B' = 0$

+ T eq is $T(t) = T_0 e^{-\alpha^2 k t}$

$$\therefore \quad u_n(x,t) = \sin\left(\frac{n\pi}{l}\right)\exp\left[-\left(\frac{n\pi}{l}\right)^2 k t\right] a_n$$

- - -

$$U(x,y,t) = S(x,y)T(t)$$

$$\frac{T'}{kT} = \frac{S_{xx} + S_{yy}}{S} = -\lambda$$

$$T' + \lambda k T = 0 \qquad S_{xx} + S_{yy} + \lambda S = 0$$

If $\lambda < 0$

let $S = X(x)Y(y)$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \lambda = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \lambda = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \alpha$$

$$\Rightarrow \frac{d^2Y}{dy^2} + \alpha Y = 0$$

$\alpha \leq 0$ get zero solution for $Y(y)$, as before. For $\alpha > 0$ let $\alpha = \mu^2$

$$\frac{d^2Y}{dy^2} + \mu^2 Y = 0$$

$$Y(y) = C_1 \sin \mu y + C_2 \cos(\mu y)$$ putting in BC's give

$$Y_n(y) = C_n \sin\left(\frac{n\pi}{b}y\right)$$

Then eq for x becomes

$$\frac{d^2X}{dx^2} + (\lambda - \alpha)X = 0$$

$$\frac{d^2X}{dx^2} + \left(\lambda - \left(\frac{\pi n}{b}\right)^2\right)X = 0$$

$+ \lambda < 0 \therefore \lambda = -\eta^2$

$$\therefore \frac{d^2X}{dx^2} - \left(\eta^2 + \left(\frac{\pi n}{b}\right)^2\right)X = 0$$

Sol $X(x) = D_1 \cosh\left(\sqrt{\eta^2 + (\ )^2}\, x\right) + D_2 \sinh\left(\sqrt{\eta^2 + (\ )^2}\, x\right)$

$X(0) = X(a) = 0 \Rightarrow \quad X(x) \equiv 0.$

$\therefore$ ~~≡~~ $\lambda > 0.$ or else we obtain a trivial solution

$$\frac{X''}{X} = -\left(\frac{Y''}{Y} + \lambda\right) = -\mu$$

$X'' + \mu X = 0 \qquad Y'' + (\lambda - \mu)Y = 0$

$\Rightarrow X(x) \propto \sin\left(\frac{m\pi x}{a}\right)$:

$$\mu_m = \left(\frac{m\pi}{a}\right)^2$$

$$\therefore Y'' + (\lambda - \mu_m)Y = 0$$

$$Y'' + \left(\lambda - \left(\frac{m\pi}{a}\right)^2\right)Y = 0$$

Now $\cancel{\lambda > \left(\frac{m\pi}{a}\right)^2} \Rightarrow Y(y) \propto \sin\left(\frac{n\pi}{b}y\right)$

$$\therefore \left(\frac{n\pi}{b}\right)^2 = \left(\lambda - \left(\frac{m\pi}{a}\right)^2\right) \Rightarrow \lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

$$S_{mn}(x,y) \propto \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$\therefore \quad T(t) \propto \exp\{-k\lambda_{mn}t\}.$$

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\exp(-k\lambda_{mn}t)$$

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Mult by $\sin\left(\frac{r\pi x}{a}\right)$ + integrate from 0 to a.

$$\Rightarrow \quad \int_0^a f(x,y)\sin\left(\frac{r\pi x}{a}\right)dx = \sum_{n=1}^{\infty} a_{rn}\frac{a}{2}\sin\left(\frac{n\pi y}{b}\right)$$

$$= \frac{a}{2}\sum_{n=1}^{\infty} a_{rn}\sin\left(\frac{n\pi y}{b}\right)$$

Multiply by $\sin\left(\frac{p\pi y}{b}\right)$ + integrate from 0 to b

$$\int_0^b\int_0^a f(x,y)\sin\left(\frac{\pi r x}{a}\right)\sin\left(\frac{p\pi y}{b}\right)dx\,dy = \frac{b}{2}\cdot\frac{a}{2}a_{rp}$$

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dx\,dy.$$

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$\lambda = 0$

$r^2 R'' + rR' = 0$

$rR'' + R' = 0$

$\frac{d}{dr}(rR') = 0$

$rR' = C_1$

$R' = \frac{C_1}{r}$

$R(r) = C_1 \ln r + D$

$\therefore U(r,\theta) = (D + C\ln r)(A + B\theta)$

$\lambda = 0 \Rightarrow U =$ constant solution.

$\Theta'' + \lambda\Theta = 0$

$\lambda = 0$

$\Theta'' = 0$

$\Theta(\theta) = A + B\theta$

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$a^n a_n = \left(\frac{2}{2\pi}\right) \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$\Rightarrow a_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$\& \quad a^n b_n = \left(\frac{2}{2\pi}\right) \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta$$

$$b_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta.$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \int_0^{2\pi} f(\phi)\big(\cos(n\phi)\cos(n\theta) + \sin(n\theta)\sin(n\phi)\big)\,d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n\phi \mp n\theta)\right\} d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\left\{1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\,d\phi\right\}$$

$$2\cos t = e^{it} + e^{-it}$$

<u>Now</u>:

$$1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n(\theta-\phi))\,d\phi = 1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{in(\theta-\phi)\} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \exp\{-in(\theta-\phi)\}$$

$$= 1 + \frac{(r/a)\exp\{i(\theta-\phi)\}}{1-\left(\frac{r}{a}\right)\exp\{i(\theta-\phi)\}} + \frac{(r/a)\exp\{-i(\theta-\phi)\}}{1-(r/a)\exp\{-i(\theta-\phi)\}}$$

$$= 1 + \frac{r\exp\{i(\theta-\phi)\}}{a - r\exp\{i(\theta-\phi)\}} + \frac{r\exp\{-i(\theta-\phi)\}}{a - r\exp\{-i(\theta-\phi)\}}$$

$$= 1 + \frac{r e^{i(\theta-\phi)}\left(a - re^{-i(\theta-\phi)}\right)}{\left(a - r\exp\{i(\theta-\phi)\}\right)\left(a - r\exp\{-i(\theta-\phi)\}\right)} + \frac{r\exp\{-i(\theta-\phi)\}\left(a - re^{i(\theta-\phi)}\right)}{\left(a - re^{-i(\theta-\phi)}\right)\left(a - re^{i(\theta-\phi)}\right)}$$

$$= 1 + \frac{r e^{i(\theta-\phi)}\left(a - r e^{-i(\theta-\phi)}\right)}{a^2 - ra\exp\{i(\theta-\phi)\} - ra\exp\{-i(\theta-\phi)\} + r^2}$$

$$+ \frac{r e^{-i(\theta-\phi)}\left(a - r e^{i(\theta-\phi)}\right)}{a^2 - ar\exp\{i(\theta-\phi)\} - ar\exp\{-i(\theta-\phi)\} + r^2}$$

$$= 1 + \frac{r e^{i(\theta-\phi)}\left(a - r e^{-i(\theta-\phi)}\right) + r e^{-i(\theta-\phi)}\left(a - r e^{i(\theta-\phi)}\right)}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$

$$= \frac{a^2 - 2ar\cos(\theta-\phi) + r^2 + r\left\{a e^{i(\theta-\phi)} - r + a e^{-i(\theta-\phi)} - r\right\}}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$

$$= a^2 - \cancel{2ar\cos(\theta-\phi)} + r^2 + -2r^2 + \cancel{2ra\cos(\theta-\phi)}$$

$$= \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}$$

$\therefore$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2 - r^2)\, f(\phi)\, d\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2}.$$

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$$\mathcal{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} e^{-ax^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x^2+\frac{ik}{a}x)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(x^2+\frac{ik}{a}x+(\frac{ik}{2a})^2)+a(\frac{ik}{2a})^2}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x+\frac{ik}{2a})^2}\,dx$$

let $v = x + \frac{ik}{2a}$ How does one justify this rigorously?

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-av^2}\,dv \cdot e^{-k^2/4a}$$

let $\eta = av^2$; $\left(\frac{\eta}{a}\right)^{1/2} = v$

$$dv = \frac{1}{2}\left(\frac{\eta}{a}\right)^{-1/2}\frac{1}{a}\,d\eta$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-av^2}\,dv\, e^{-k^2/4a} = \frac{\cancel{2}}{\sqrt{a}}\,\frac{\sqrt{2}}{2\sqrt{\pi}}\int_{0}^{\infty} e^{-\eta}\eta^{-1/2}\,d\eta\, e^{-k^2/4a}$$

$$e^{-k^2/4a}\frac{1}{\sqrt{2\pi a}}\int_0^\infty \eta^{-1/2}e^{-\eta}\,d\eta = e^{-k^2/4a}\frac{\Gamma(1/2)}{\sqrt{2\pi a}} = \frac{1}{\sqrt{2a}}e^{-k^2/4a} \quad \text{eq } 1.7.3$$

(b) $$\mathcal{F}\{\exp(-a|x|)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}e^{-a|x|}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{-ikx}e^{ax}\,dx + \int_0^\infty e^{-ikx}e^{-ax}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{(-ik+a)x}\,dx + \int_0^\infty e^{(-ik-a)x}\,dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left.\frac{e^{(-ik+a)x}}{(-ik+a)}\right|_{-\infty}^{0} + \left.\frac{e^{(-ik-a)x}}{-ik-a}\right|_0^\infty\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{(-ik+a)} + \frac{-1}{-ik-a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{-ik+a} + \frac{1}{ik+a}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\,2\,\mathrm{Re}\left(\frac{1}{ik+a}\right) = \frac{\sqrt{2}}{\sqrt{\pi}}\,\mathrm{Re}\left(\frac{-ik+a}{(ik+a)(-ik+a)}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \operatorname{Re}\left(\frac{-ik+a}{k^2+a^2} \right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{a}{a^2+k^2} .$$

– – – – –

$$\mathcal{F}\{u_x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u_x \, dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ikx} u \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} (-ik) \int_{-\infty}^{\infty} e^{-ikx} u \, dx$$

$$= 0 + ik \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,t) \, dx}_{\mathcal{F}\{u\}}$$

$$\mathcal{F}\{u_x\} = ik \mathcal{F}\{u\}.$$

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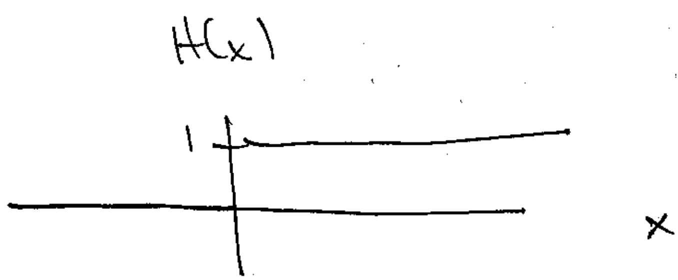
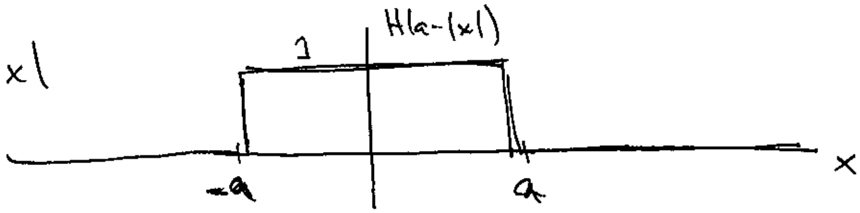


$H(a-|x|) \neq 0$ iff

$a-|x| > 0$

$\Rightarrow a > |x|$

~~[illegible]~~ $a > |x|$



$$\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$$

$$\|$$

$$\cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} F(k)G(k)e^{ikx}\,dk = \cancel{\frac{1}{\sqrt{2\pi}}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

$$\int_{-\infty}^{\infty} F(k)G(k)e^{ikx}\,dk = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

let $v = x - \xi \qquad \xi = x - v$

$dv = -d\xi$

$$= -\frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty} f(v)\,g(x-v)\,dv$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x-v)\,f(v)\,dv = g(x) * f(x) \, .$$

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$\sqrt{2\pi}\,\delta(x)$ is the identity w.r.t. convolution

$$f(x) * \sqrt{2\pi}\,\delta(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)\sqrt{2\pi}\,\delta(\xi)\,d\xi$$

$$= f(x).$$

$u_{tt} = c^2 u_{xx}$

$u(x,0) = f(x) \qquad u_t(x,0) = g(x)$

$\Rightarrow \quad \frac{d^2 U}{dt^2} = c^2 (ik)^2 U = -c^2 k^2 U.$ is F.T. of eq

$\frac{d^2 U}{dt^2} + c^2 k^2 U = 0.$

$U(k,0) = F(k)$

$U_t(k,0) = G(k)$

$U(t,k) = Ae^{ickt} + Be^{-ickt}.$

Then w/ I.C. Fourier transform.

$U(0,k) = A + B = F(k)$

$U_t(k,t) = Aick\,e^{ickt} - Bick\,e^{-ickt}.$

$U_t(k,0) = Aick - Bick = G(k) \Rightarrow A - B = \frac{G(k)}{ick}$

$\therefore \quad \cancel{A = F(k) - B}$

$\cancel{(F(k) - B)ick - Bick = G(k)}$

$$\cancel{ickF(k) - ickB - Bick = G(k)}$$

$$\cancel{2ickB(k) = G(k) - ickF(k)}$$

$$\cancel{B(k) = \frac{G(k) - ickF(k)}{2ick}}$$

Then $\cancel{A = F(k) - \left(\frac{G}{2ick} - \frac{F(k)}{2}\right)}$

Adding 2 eqs gives

$$2A = F(k) + \frac{G(k)}{ick} \Rightarrow A = \frac{F(k)}{2} + \frac{G(k)}{2ick}$$

Subtracting

$$B = \frac{1}{2}\left(F(k) - \frac{G(k)}{ick}\right)$$

Then

$$U(k,t) = \left(\frac{F(k)}{2} + \frac{G(k)}{2ick}\right)e^{ickt} + \left(\frac{F(k)}{2} - \frac{G(k)}{2ick}\right)e^{-ickt}$$

$$= \frac{F(k)}{2}\left(e^{ickt} + e^{-ickt}\right) + \frac{G(k)}{2ick}\left(e^{ickt} - e^{-ickt}\right)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,\frac{F(k)}{2}\left(e^{ickt} + e^{-ickt}\right)dk$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{G(k)}{2ick}\left(e^{ikx + ickt} - e^{ikx - ickt}\right)dk$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{F(k)}{2} \left(e^{ik(x+ct)} + e^{ik(x-ct)} \right) dk$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{2ick} \left(e^{ik(x+ct)} - e^{ik(x-ct)} \right) dk$$

$$= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \left\{ e^{ik(x+ct)} + e^{ik(x-ct)} \right\} dk \right]$$

$$+ \frac{1}{2c} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{ik} \left\{ e^{ik(x+ct)} - e^{ik(x-ct)} \right\} dk \right]$$

$$f(x) = \mathcal{F}^{-1}\{F(k)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} F(k)\, dk$$

$$\therefore\ u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) \left(\int_{\cancel{x+ct}\ x-ct}^{x+ct} e^{ik\xi}\, d\xi \right) dk$$

exchange order of integration

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct) \right] + \frac{1}{2c} \int_{\cancel{x+ct}\ x-ct}^{x+ct} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k) e^{ik\xi}\, dk \right) d\xi$$

$$+ \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

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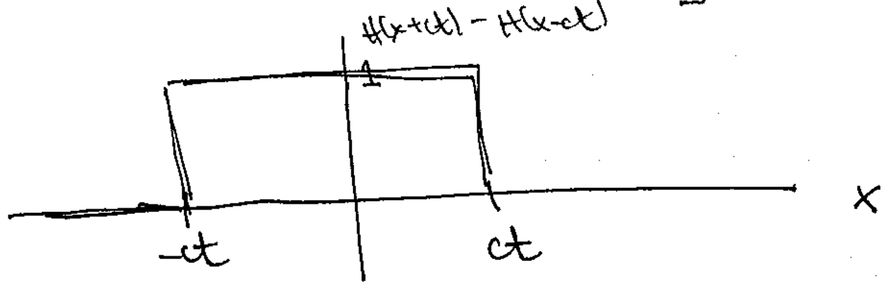
$f=0 \quad g(x) = \delta(x)$

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(\xi)\, d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} H'(\xi)\, d\xi$$

$$= \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right]$$



$$c^2t^2 - x^2 > 0$$

$$\Leftrightarrow \quad ct > |x|$$

$$\therefore \quad u(x,t) = \frac{1}{2c} H(c^2t^2 - x^2)$$

$$\eta_t + c\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0 \qquad i^3 = -i$$

$$E_t + c(ik)E + \frac{ch^2}{6}(ik)^3 E = 0$$

$$E_t + \left(c(ik) - \frac{ich^2k^3}{6}\right)E = 0$$

$$\Rightarrow E(k,t) = E_0(k)\exp\left(\left(-cik + \frac{ich^2k^3}{6}\right)t\right)$$

$$= F(k)\exp\left\{ikct\left(\frac{k^2h^2}{6} - 1\right)\right\}$$ w/ $F(k)$ the F.T. of $f(x)$.

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)\, e^{ikx} \exp\left\{ ikct\left(\frac{k^2h^2}{6} - 1\right)\right\} dk$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp\left\{ ik\left[(x-ct) + ct\frac{k^2h^2}{6}\right]\right\} dk$$

If $f(x) = \delta(x)$

$$F(k) = \frac{1}{\sqrt{2\pi}}$$

Then

$$\eta(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[ik\left\{(x-ct) + ct\frac{k^2h^2}{6}\right\}\right] dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \exp\left[ik\left\{(x-ct) + ct\frac{k^2h^2}{6}\right\}\right] dk$$

$$+ \frac{1}{2\pi} \int_{0}^{\infty} \exp\left[ik\left\{(x-ct) + \frac{ctk^2h^2}{6}\right\}\right] dk$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} \exp\left[-ik\left\{(x-ct) + \frac{ctk^2h^2}{6}\right\}\right] dk + \frac{1}{2\pi} \int_{0}^{\infty} \exp\left[ik\left\{(x-ct) + \frac{ctk^2h^2}{6}\right\}\right] dk$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \cos\left(k\left(x - ct + \frac{ctk^2h^2}{6}\right)\right) dk$$

$$\eta(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos\left(k(x-ct) + \left(\frac{cth^2}{6}\right)k^3\right) dk$$

w/ df of Airy fn

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty} \cos\left(kz + \frac{k^3}{3}\right) dk \qquad \text{want} \quad \left(\frac{cth^2}{6}\right)k^3 = \frac{v^3}{3}$$

This is the substitution needed

$$\Rightarrow \quad v^3 = \left(\frac{cth^2}{2}\right)k^3$$

$$v = \left(\frac{cth^2}{2}\right)^{1/3} k$$

$$dk = \left(\frac{2}{cth^2}\right)^{1/3} dv \quad \& \quad k = \left(\frac{2}{cth^2}\right)^{1/3} v$$

$$\therefore \quad \eta(x,t) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\left(\frac{2}{cth^2}\right)^{1/3}(x-ct)v + \frac{v^3}{3}\right)\left(\frac{2}{cth^2}\right)^{1/3} dv$$

$$= \frac{1}{\pi}\left(\frac{2}{cth^2}\right)^{1/3}\int_0^{\infty} \cos\left(\alpha v + \frac{v^3}{3}\right) dv$$

$$= \left(\frac{2}{cth^2}\right)^{1/3} Ai\left[\left(\frac{2}{cth^2}\right)^{1/3}(x-ct)\right]$$

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$$\frac{d^2U}{dy^2} + (ik)^2 U = 0 \qquad U(k,0) = F(k)$$

$$\frac{d^2U}{dy^2} - k^2 U = 0 \qquad U(k,y) \to 0 \quad y \to \infty$$

$$\Rightarrow \quad U(y) = Ae^{ky} + Be^{-ky}$$

$$U(k,0) = A + B = F(k)$$

$$U(k,y) \to 0 \Rightarrow A = 0. \quad k > 0 \qquad (\text{Remember } -\infty < k < \infty)$$

$$+ \; U(k,y) \to 0 \Rightarrow B = 0 \quad k < 0.$$

$$\therefore \quad U(k,y) \sim e^{-ky} \quad \text{when } k > 0 \text{ + } y \to +\infty$$

$$U(k,y) \sim e^{ky} \quad \text{when } k < 0 \text{ + } y \to +\infty$$

$$\Rightarrow \quad U(k,y) \sim e^{-|k|y}$$

$$+ \text{ I.C.} \quad U(k,0) = F(k)$$

$$\Rightarrow \quad U(k,y) = \underbrace{F(k)e^{-|k|y}}_{\text{product of 2 fns of } k}.$$

Then to take inverse fourier transform

$$\mathcal{F}^{-1}\{F(k)G(k)\} = f(x) * g(x)$$

Then w/ $f(x) = \mathcal{F}^{-1}\{F(k)\}$ + $g(x) = \mathcal{F}^{-1}\{G(k)\}$

Now $\mathcal{F}^{-1}\{e^{-|k|y}\} = ?$ once known

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi$$

w/ $g(x) = \mathcal{F}^{-1}\{e^{-|k|y}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|k|y} e^{ikx}\, dk$ computing the inverse fourier transform.

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{ky} e^{ikx}\, dk + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ky} e^{ikx}\, dk$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(y+ix)k}\, dk + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(-y+ix)k}\, dk$$

$$= \frac{1}{\sqrt{2\pi}} \left.\frac{e^{(y+ix)k}}{y+ix}\right|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left.\frac{e^{(-y+ix)k}}{(-y+ix)}\right|_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{y+ix} + \frac{1}{\sqrt{2\pi}} \frac{(-1)}{(-y+ix)}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{y+ix} + \frac{1}{y-ix}\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{y-ix}{(y+ix)(y-ix)} + \frac{y+ix}{(y-ix)(y+ix)}\right]$$

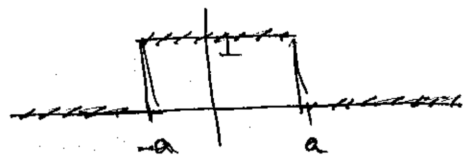
$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2y}{y^2+x^2}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{y}{x^2+y^2}\right)$$

$$\therefore \quad u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) \sqrt{\frac{2}{\pi}} \frac{y}{(x-\xi)^2 + y^2} \, d\xi$$

$$= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2} \, d\xi$$

If $f(x) = T_0 H(a-|x|)$ $\qquad a-|x|>0$

$\Rightarrow |x|<a$



$$u(x,y) = \frac{y T_0}{\pi} \int_{-\infty}^{\infty} \frac{H(a-|\xi|)\, d\xi}{(\xi-x)^2 + y^2}$$

$$= \frac{y T_0}{\pi} \int_{-a}^{a} \frac{d\xi}{(\xi-x)^2 + y^2}$$

$$= \frac{y T_0}{y^2 \pi} \int_{-a}^{a} \frac{d\xi}{1 + \left(\frac{\xi - x}{y}\right)^2} =$$

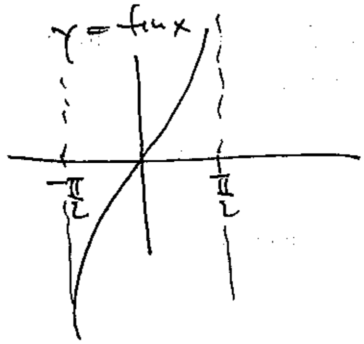
let $v = \dfrac{\xi - x}{y} \qquad dv = \dfrac{1}{y} d\xi$

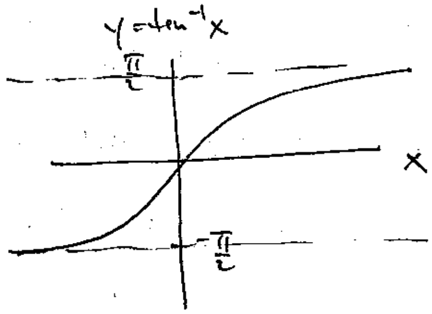
$$= \frac{T_0}{y\pi} \int_{\frac{-a-x}{y}}^{\frac{a-x}{y}} \frac{y\, dv}{1 + v^2} = \frac{T_0}{\pi} \int_{\frac{-a-x}{y}}^{\frac{a-x}{y}} \frac{dv}{1+v^2} = \frac{T_0}{\pi} \tan^{-1} v \Big|_{\frac{-a-x}{y}}^{\frac{a-x}{y}}$$

$$= \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{a-x}{Y}\right) - \tan^{-1}\left(\frac{-a-x}{Y}\right)\right)$$

$$\tan^{-1}(\theta) \stackrel{?}{=} -\tan^{-1}(-\theta) \quad \text{check}$$

~~Tan~~





But $\tan^{-1}x$ is an odd fn.

$\therefore$

$$= \frac{T_0}{\pi}\left(-\tan^{-1}\left(\frac{x-a}{Y}\right) + \tan^{-1}\left(\frac{a+x}{Y}\right)\right)$$

$$= \frac{T_0}{\pi}\left(\tan^{-1}\left(\frac{a+x}{Y}\right) - \tan^{-1}\left(\frac{x-a}{Y}\right)\right)$$

But $\tan^{-1}\alpha - \tan^{-1}\beta =$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$$

$\div \cos(\alpha)\cos(\beta)$

$$= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$\therefore$ $\tan(\alpha-\beta) = \dfrac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

$\therefore \quad \alpha - \beta = \tan^{-1}\left(\frac{\tan(\alpha) - \tan\beta}{1 + \tan\alpha\tan\beta}\right)$

let ~~$\alpha = \tan^{-1}(\alpha)$~~ $\alpha = \tan^{-1}(\hat{\alpha})$ + $\beta = \tan^{-1}(\tilde{\beta})$

The

$$\tan^{-1}(\hat{\alpha}) - \tan^{-1}(\tilde{\beta}) = \tan^{-1}\left(\frac{\tilde{\alpha} - \tilde{\beta}}{1 + \tilde{\alpha}\hat{\beta}}\right)$$

Thus

$$u(x,y) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{\frac{a+x}{y} - \left(\frac{x-a}{y}\right)}{1 + \left(\frac{x+a}{y}\right)\left(\frac{x-a}{y}\right)}\right)$$

$$= \frac{T_0}{\pi}\tan^{-1}\left(\frac{2a/y}{\frac{y^2 + (x^2 - a^2)}{y^2}}\right) = \frac{T_0}{\pi}\tan^{-1}\left(\frac{2ay}{y^2 + x^2 - a^2}\right)$$

$u(x,y) \equiv \text{const}$

$$\Rightarrow \quad \frac{2ay}{y^2 + x^2 - a^2} = c_1$$

$$\frac{2a}{c_1}y = x^2 + y^2 - a^2$$

$$\alpha y = x^2 + y^2 - a^2$$

$$a^2 = x^2 + y^2 - \alpha y$$

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$f(x) = \delta(x)$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\delta(\xi)\,d\xi}{(x-\xi)^2+y^2} = \frac{y}{\pi}\frac{1}{x^2+y^2}$$

---

let $\quad u(x,y) = \int^{y} v(x,\eta)\,d\eta$

$$u_y = v(x,y)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u\right) = 0.$$

$$v(x,0) = u_y(x,0) = f(x)$$

$$\therefore \quad v(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(x-\xi)^2+y^2}$$

Then $\quad u(x,y) = \int^{y} v(x,\eta)\,d\eta = \frac{1}{\pi}\int^{y}\eta\,d\eta\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(x-\xi)^2+\eta^2}$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi\int^{y}\frac{\eta\,d\eta}{(x-\xi)^2+\eta^2}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\,d\xi\left(\frac{1}{2}\ln\left((x-\xi)^2+\eta^2\right)\right)\Big|^{y} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\log\left[(x-\xi)^2+y^2\right]d\xi$$

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$$U_t = \kappa(ik)^2 U = -\kappa k^2 U$$

$$U(k,0) = F(k)$$

$$U(k,t) = F(k)e^{-\kappa k^2 t}$$ (product of 2 fourier transformed fns)

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{-\kappa k^2 t} e^{ikx} \, dk$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx - \kappa k^2 t} \, dk}_{} = \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi}$$

Seperate into

$$\cancel{e^{ikx}} F(k) + e^{-\kappa k^2 t}$$

Then ~~By~~ By inverse convolution thm $= f(x) * g(x)$

w/ $f(x) = \mathcal{F}^{-1}\{F(k)\}$ + $g(x) = \mathcal{F}^{-1}\{e^{-\kappa k^2 t}\}$.

$$g(x) = \sqrt{2a} \exp(-a x^2)$$ w/ $\frac{1}{4a} = \kappa t$ $\Rightarrow$ $a = \frac{1}{4\kappa t}$

From eq 1.7.3
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$$\therefore g(x) = \sqrt{\frac{2}{4\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right) = \frac{1}{\sqrt{2\kappa t}} \exp\left(\frac{-x^2}{4\kappa t}\right)$$

Then $\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi$.

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)\,\frac{1}{\sqrt{2\kappa t}}\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]d\xi$$

$$= \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} f(\xi)\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]d\xi$$

let $\cancel{\xi}$ $\lambda = \frac{\xi - x}{2\sqrt{\kappa t}} \Rightarrow \xi = \lambda(2\sqrt{\kappa t}) + x$

$$d\lambda = \frac{d\xi}{2\sqrt{\kappa t}}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty} f(x + 2\sqrt{\kappa t}\,\lambda)\exp\left[-\lambda^2\right]2\sqrt{\kappa t}\,d\lambda$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2\sqrt{\kappa t}\,\lambda)\exp(-\lambda^2)\,d\lambda$$

(c) $f(x) = \delta(x)$ 1.7.88 gives

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\int_{-\infty}^{\infty}\delta(\xi)\exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right]d\xi = \frac{1}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right)$$

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(b) $f(x) = T_0 H(x)$

Then $u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} T_0 H(\xi) \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi$

$$= \frac{T_0}{\sqrt{4\pi\kappa t}} \int_0^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] d\xi$$

let $\cancel{\eta = \frac{x-\xi}{\sqrt{4\kappa t}} = \frac{x-\xi}{2\sqrt{\kappa t}}}$ $\qquad \eta = \frac{\xi - x}{\sqrt{4\kappa t}} = \frac{\xi - x}{2\sqrt{\kappa t}}$

$\cancel{d\eta = -\frac{d\xi}{2\sqrt{\kappa t}}}$ $\qquad d\eta = \frac{d\xi}{2\sqrt{\kappa t}}$

Then $\cancel{u(x,t) = \frac{T_0}{\sqrt{4\pi\kappa t}} \int_{\frac{x}{2\sqrt{\kappa t}}}^{-\infty}}$ Then

$$u(x,t) = \frac{T_0}{2\sqrt{\pi\kappa t}} \int_{\frac{-x}{2\sqrt{\kappa t}}}^{\infty} \exp\left[-\eta^2\right] \cancel{2\sqrt{\kappa t}}\, d\eta$$

$$u(x,t) = \frac{T_0}{\sqrt{\pi}} \int_{\frac{-x}{2\sqrt{\kappa t}}}^{\infty} \exp\left[-\eta^2\right] d\eta = \cancel{\frac{T_0}{\sqrt{\pi}}}$$

$\mathrm{Erf}(x) = \int_0^x e^{-\eta^2} d\eta \qquad \mathrm{Erfc}(x) =$

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$$\Psi(\bar{x},t) = A\exp\left[i(\bar{x}\cdot\bar{k} - \omega t)\right]$$ putting into time-dependent Schrödinger eq.

$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}\left(i^2(k^2+l^2+m^2)\right)$$

$$\hbar\omega = V + \frac{\hbar^2}{2m}(k^2+l^2+m^2)$$

$$\hbar\omega = V + \frac{(\hbar k)^2}{2m}$$

$$c_g = \frac{\partial\omega}{\partial k} = \cancel{\hbar}\,\frac{1}{\cancel{\hbar}}\,\frac{2(\hbar k)}{2m}\cancel{\hbar}$$

$$= \frac{\hbar k}{m} = \frac{p}{m} = V.$$

$\hbar k = p$

$\frac{h}{2\pi} k = p$

look up relationships between $p = \hbar k$

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi$$

$$\Psi(x,0) = \Psi_0(x)$$

$$\Psi(x,t) \to 0 \quad \text{as } |x| \to \infty$$

$$i\hbar\bar{\Psi}_t = -\frac{\hbar^2}{2m}(ik)^2\bar{\Psi}. = \frac{\hbar^2 k^2}{2m}\bar{\Psi}$$

$$\Rightarrow \bar{\Psi}_t = -\frac{i\hbar k^2}{2m}\bar{\Psi} \quad + \quad \bar{\Psi}(k,0) = \bar{\Psi}_0(k)$$

sol $\bar{\Psi}(k,t) = \bar{\Psi}_0(k)\exp\left[\frac{-i\hbar k^2}{2m}t\right]$

$$\Rightarrow \Phi(k,t) = \Phi_0(k) \exp\left[-i\alpha k^2 t\right] \qquad \alpha = \frac{\hbar}{2m}$$

Inverting

Note: This can't be just inverted as the heat equation Because the heat eq has the inverse of $\exp[-\kappa k^2 t]$ pg 32 Debnath this has inverse of $\exp[-i\alpha k^2 t]$ which is harder to compute?

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_0(k) \exp\Big[\underbrace{-i\alpha k^2 t + ikx}_{ik(x-\alpha k t)}\Big]\, dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y,0) e^{-iky}\, dy \int_{-\infty}^{\infty} \exp\left[ik(x-\alpha k t)\right] dk$$

$$= \frac{1}{2\pi} \int_{y=-\infty}^{\infty} \int_{k=-\infty}^{\infty} f(y,0)\, dy \cdot \exp\left[ik(x-y-\alpha k t)\right] dk$$

~~Now $\exp[ik(x-y-\alpha k t)] = \exp[$~~

$$= \cancel{\exp\left[-i\alpha t\left\{\frac{(x-y)k}{i\alpha t} + \frac{k^2}{i}\right\}\right]}$$

$$= \cancel{\exp\left[i\alpha t\left[k^2 i - \frac{k(x-y)}{\alpha t}\right]\right] = \exp\left[i\alpha t\left\{k^2\right.\right.}$$

Now: $$ik(x-y-\alpha k t) = -i\alpha t\left(k^2 - \frac{(x-y)k}{\alpha t}\right) = -i\alpha t\left(k^2 - 2k\left(\frac{x-y}{2\alpha t}\right)\right)$$

$$= -i\alpha t\left(k^2 - 2k\left(\frac{x-y}{2\alpha t}\right) + \left(\frac{x-y}{2\alpha t}\right)^2 - \left(\frac{x-y}{2\alpha t}\right)^2\right)$$

Then

$$\exp\left[ik(x-y-\alpha k t)\right] = \exp\left[-i\alpha t\left(k - \left(\frac{x-y}{2\alpha t}\right)\right)^2\right] \exp\left[i\alpha t\left(\frac{x-y}{2\alpha t}\right)^2\right]$$

$$= \exp\left[-i\alpha t\left(k - \frac{x-y}{2\alpha t}\right)^2\right] \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]$$

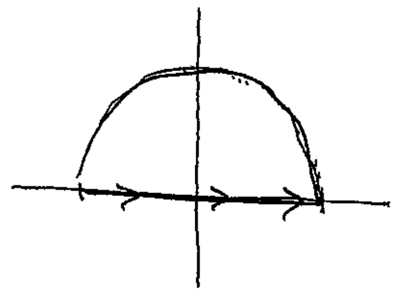
let $\xi = k - \dfrac{x-y}{2\alpha t}$

$$= \exp\left[-i\alpha t \xi^2\right] \exp\left[\frac{i(x-y)^2}{4\alpha t}\right]$$

Then 1.7.77 becomes

$$T(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(y,0)\,dy \int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right] \exp\left[-i\alpha t \xi^2\right] dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\frac{i(x-y)^2}{4\alpha t}\right] T(y,0)\,dy \int_{-\infty}^{\infty} \exp\left[-i\alpha t \xi^2\right] d\xi$$



Now $\displaystyle\int_{-\infty}^{\infty} \exp\left[-i\alpha t \xi^2\right] d\xi = 2\int_{0}^{\infty} \exp\left[-i\alpha t \xi^2\right] d\xi$

let $v = i\alpha t \xi^2 \Rightarrow \xi = \left(\dfrac{v}{i\alpha t}\right)^{1/2}$

$dv = 2i\alpha t \xi\, d\xi$

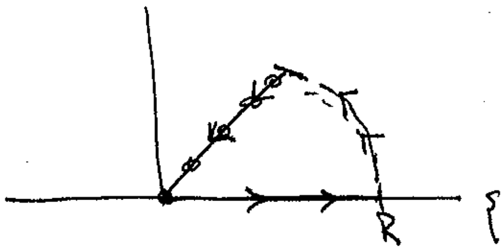
$$\overset{?}{=} \cancel{2}\int_0^{\infty} \exp[-v] \frac{dv}{\cancel{2} i\alpha t \left(\frac{v}{i\alpha t}\right)^{1/2}} = \frac{\sqrt{i\alpha t}}{i\alpha t}\underbrace{\int_0^{\infty} e^{-v} v^{-1/2}\, dv}_{\Gamma(1/2) = \sqrt{\pi}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{i\alpha t}}$$

Now $\sqrt{i} = \left(e^{i(\frac{\pi}{2} + 2\pi n)}\right)^{1/2} = e^{i(\frac{\pi}{4} + \pi n)} = \begin{cases} e^{i\frac{\pi}{4}} \\ e^{i\frac{5\pi}{4}} \end{cases}$

$$\left\{\begin{array}{l} = \cos\frac{\pi}{4} + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1+i}{\sqrt{2}}. \\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \end{array}\right.$$

pick transformation ~~such that~~ $\ni$ we can transform out the imaginary i in the exponential



$$\xi = re^{i\theta}$$

$$e^{\left(-i\alpha t r^2 e^{2i\theta}\right)} = \text{integrand.}$$

$$= e^{\left(-i\alpha t r^2(\cos 2\theta + i\sin 2\theta)\right)}$$

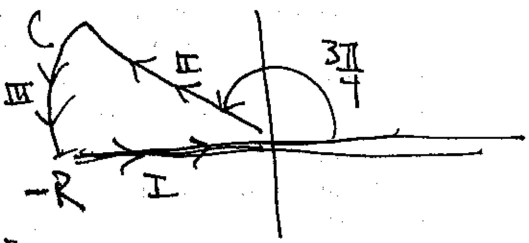
$$= \exp\left[\alpha t r^2\left(-i\cos(2\theta) + \sin(2\theta)\right)\right]$$

This decays to zero as $r \to \infty$ iff $\sin(2\theta) < 0$

$$\Rightarrow \quad \pi < 2\theta < 2\pi \iff \frac{\pi}{2} < \theta < \pi \quad \text{pick } \theta \ni$$

$$-i\cos(2\theta) = 0 \quad \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

Then do integral like



Then $\int_C F(\xi)\,d\xi = 0$ ~~since~~ $F(\xi)$ is analytic

$$= \int_{-R}^{0} + \int_{\mathrm{II}} + \int_{\mathrm{III}}^{\nearrow 0 \;\; r\to+\infty} = 0$$

$$\overset{\shortparallel}{\frac{1}{2}\int_{-\infty}^{\infty} F(\xi)\,d\xi} \quad \text{As } R \to +\infty \qquad \therefore \; \frac{1}{2}\int_{-\infty}^{\infty} \exp\left[-i\alpha t\xi^2\right] d\xi = -\int_{\mathrm{II}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \exp[-i\alpha t\xi^2]\, d\xi = -2\int_{\mathrm{II}} \exp(-i\alpha t\xi^2)\, d\xi$$

$$\xi = re^{i\frac{3\pi}{4}} = \cancel{r(\cos\frac{3\pi}{4} +}$$

$$\xi^2 = r^2 e^{i\frac{3\pi}{2}} = r^2(\cancel{\cos\frac{3\pi}{2}}^{0} + i\sin\frac{3\pi}{2})$$

$$= r^2 i(-1)$$

$$\therefore \; d\xi = e^{i\frac{3\pi}{4}}\, dr.$$

$$= -2\int_0^{+\infty} e^{+i\frac{3\pi}{4}} \exp[\alpha t r^2(-i)^2]\, dr$$

$$= -2e^{\frac{3\pi}{4}i}\int_0^{\infty} e^{-\alpha t r^2}\, dr$$

let $\eta = \alpha t r^2 \qquad r = +\frac{\eta^{1/2}}{\sqrt{\alpha t}}$

$$d\eta = 2\alpha t r\, dr$$

$$= 2\frac{\alpha t}{\sqrt{\alpha t}}\eta^{1/2} dr = \eta^{1/2} 2\sqrt{\alpha t}\, dr$$

$$= -2e^{\frac{3\pi}{4}i}\int_0^{\infty} e^{-\eta}\frac{d\eta}{2\sqrt{\alpha t}}\eta^{-1/2}$$

$$= -\frac{e^{\frac{3\pi}{4}i}}{\sqrt{\alpha t}}\sqrt{\pi} = \frac{-\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\sqrt{\pi}}{\sqrt{\alpha t}} = \frac{(1-i)\sqrt{\pi}}{\sqrt{2\alpha t}}$$

$$\therefore\quad t(x,t) = \frac{1}{2\pi}\left(\frac{1-i}{\sqrt{2\alpha t}}\right)\sqrt{\pi}\int_{-\infty}^{\infty}\exp\left[\frac{i(x-y)^2}{4\alpha t}\right]t(y,0)\,dy$$

$$= \frac{1-i}{2\sqrt{2\alpha\pi t}}\int_{-\infty}^{\infty}\exp\left[\frac{i(x-y)^2}{4\alpha t}\right]t(y,0)\,dy \qquad \text{eq } 1.7.78$$

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Let $\cancel{u(x,y,z) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}}$

Use double fourier transform

$$U(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathcal{U}(k,l,z)\, e^{i(kx+ly)}\, dk\, dl$$ Then eq 1.8.7 becomes

$$\frac{d^2\mathcal{U}}{dz^2} + (ik)^2\mathcal{U} + (il)^2\mathcal{U} = 0 \Rightarrow \frac{d^2\mathcal{U}}{dz^2} - \underbrace{(k^2+l^2)}_{\kappa^2}\mathcal{U} = 0$$

$$\mathcal{U}(z) = Ae^{|\kappa|z} + Be^{-|\kappa|z}$$

Requiring that $\lim_{z\to+\infty} \mathcal{U}(z) = 0 \Rightarrow A = 0 \Rightarrow \mathcal{U}(z) = Be^{-|\kappa|z}$

Then B.C. $\mathcal{U}(0) = B = F(k,l)$

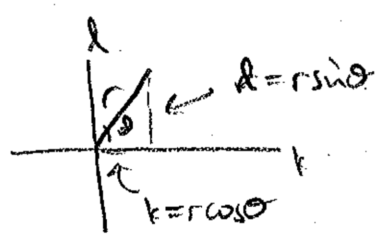
$\therefore \mathcal{U}(k,l,z) = F(k,l)\, e^{-|\kappa|z} = F(k,l)\, G(k,l)$ w/ $G(k,l) = e^{-|\kappa|z}$

Then

$$g(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(k,l)\, e^{i(kx+ly)}\, dk\, dl = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\sqrt{k^2+l^2}\,z}\, e^{i(kx+ly)}\, dk\, dl$$

Try changing to polar let $r = \sqrt{k^2+l^2}$

$$\begin{matrix} -\infty \le k \le +\infty \\ -\infty \le l \le +\infty \end{matrix} \Leftrightarrow \begin{matrix} 0 \le r \le +\infty \\ 0 \le \theta \le 2\pi \end{matrix}$$



Then

$$g(x,y) = \frac{1}{2\pi}\int_0^{+\infty}\int_0^{2\pi} e^{-zr} e^{i(r\cos\theta x + r\sin\theta y)} r\, d\theta\, dr$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} r\, e^{r(-z + i(\cos\theta x + \sin\theta y))} dr\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}$$ ... Don't think this will work the best.

Now consider

$$\frac{z}{(x^2+y^2+z^2)^{3/2}} = -\frac{d}{dz}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

So If I can show that

$$\mathcal{F}^{-1}\left\{ \frac{e^{-|k|z}}{|k|} \right\} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

or $\mathcal{F}^{-1}\left\{ \frac{e^{-|k|z}}{|k|} \right\} = \frac{1}{\sqrt{x^2+y^2+z^2}}$ then the z derivative of both sides gives the desired expression.

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{e^{-\sqrt{k^2+l^2}\,z}}{\sqrt{k^2+l^2}} e^{+i(kx+ly)} dk\, dl \overset{?}{=}$$

Changing to polar as suggested above gives

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} e^{-rz} e^{i(r\cos\theta x + r\sin\theta y)} dr\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left. \frac{e^{r(-z+i(\cos\theta x+\sin\theta y))}}{-z+i(\cos\theta x+\sin\theta y)} \right|_0^{+\infty} d\theta$$ ✓

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{z-i(\cos\theta x+\sin\theta y)}$$ ✓

$$\left\{ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \right\}$$

$$= \frac{1}{2\pi}\oint \frac{d\theta}{z-\frac{i}{2}\left[x(e^{i\theta}+e^{-i\theta}) + y\left(\frac{e^{i\theta}-e^{-i\theta}}{i}\right)\right]}$$ * ✓

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{z-i(\cos\theta x+\sin\theta y)} \; \frac{z+i(\cos\theta x+\sin\theta y)}{z+i(\cos\theta x+\sin\theta y)}$$

$$z^2+(\cos\theta x+\sin\theta y)^2$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{z - \frac{ix}{2}(e^{i\theta}+e^{-i\theta}) - \frac{1}{2}y(e^{i\theta}-e^{-i\theta})}$$ ✓

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}\,d\theta}{ze^{i\theta} - \frac{ix}{2}(e^{2i\theta}+1) - \frac{y}{2}(e^{2i\theta}-1)}$$ ✓

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}\,d\theta}{(-\frac{y}{2}-\frac{ix}{2})e^{2i\theta} + ze^{i\theta} - \frac{ix}{2} + \frac{y}{2}}$$ ✓

$$\eta = e^{i\theta}$$
$$d\eta = ie^{i\theta}d\theta$$

$$= \frac{1}{2\pi i}\oint \frac{d\eta}{-\frac{i}{2}(x-iy)\eta^2 + z\eta - \frac{i}{2}(x+iy)}$$

$$(2/{-i}) = +2i$$

$$= \frac{1}{2\pi i}\oint \frac{(2/{-i})\,d\eta}{(x-iy)\eta^2 + 2iz\eta + (x+iy)}$$

$$= \frac{1}{\pi}\oint \frac{d\eta}{(x-iy)\eta^2 + 2iz\eta + (x+iy)}$$ ✓

$$= \frac{1}{\pi(x-iy)}\oint \frac{d\eta}{\eta^2 + \frac{2iz}{(x-iy)}\eta + \frac{x+iy}{x-iy}}$$ ✓

$$= \frac{1}{\pi(x-iy)}\oint \frac{d\eta}{(\eta-\lambda_1)(\eta-\lambda_2)}$$

roots of denominator are

$$\eta = \frac{\frac{-2iz}{x-iy} \pm \sqrt{\left(\frac{2iz}{(x-iy)}\right)^2 - 4\left(\frac{x+iy}{x-iy}\right)}}{2}$$

$$\oint_C f(z)\,dz$$
$$= 2\pi i \sum \text{Res } C$$

$$\lambda_1 + \lambda_2 = \frac{x+iy}{x-iy}$$

$$\lambda_1 \cdot \lambda_2 = \frac{2iz}{x-iy}$$

$$\eta = \frac{-iz}{x-iy} \pm \sqrt{\left(\frac{iz}{x-iy}\right)^2 - \frac{x+iy}{x-iy}}$$

$$|\eta_+|^2 = \eta_+ \cdot \eta_+^*$$

$$= \frac{-iz}{x-iy} + \sqrt{\left(\frac{iz}{x-iy}\right)^2 -}$$

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Assuming $\mathcal{F}^{-1}\{\exp(-|x|z)\} = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$ {How show?}

Then

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{-\infty}\int_{-\infty}^{+\infty} f(\xi,\eta)\, g(x-\xi, y-\eta)\, d\xi\, d\eta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}$$

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$$d\rho = \frac{\partial \rho}{\partial p}\Big|_s dp$$

$$\frac{D\rho}{Dt} = \frac{1}{c^2}\frac{Dp}{Dt}$$

$$\frac{1}{c^2}\frac{Dp}{Dt} + \rho(\nabla \cdot u) = 0$$

$$\Rightarrow \frac{1}{c^2}(p_t + u p_x) + \rho u_x = 0$$

$$\rho(u_t + u u_x) + p_x = 0$$

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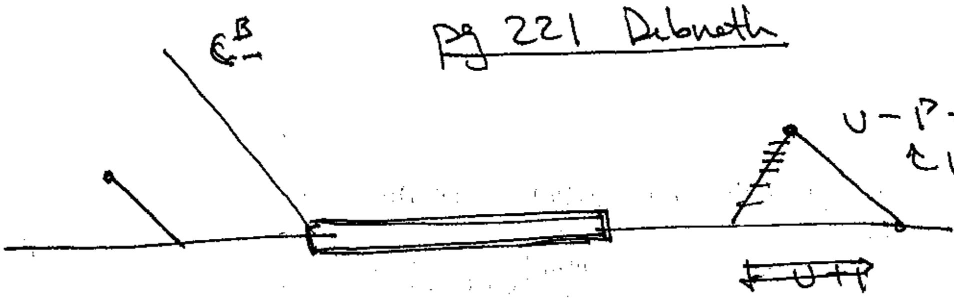
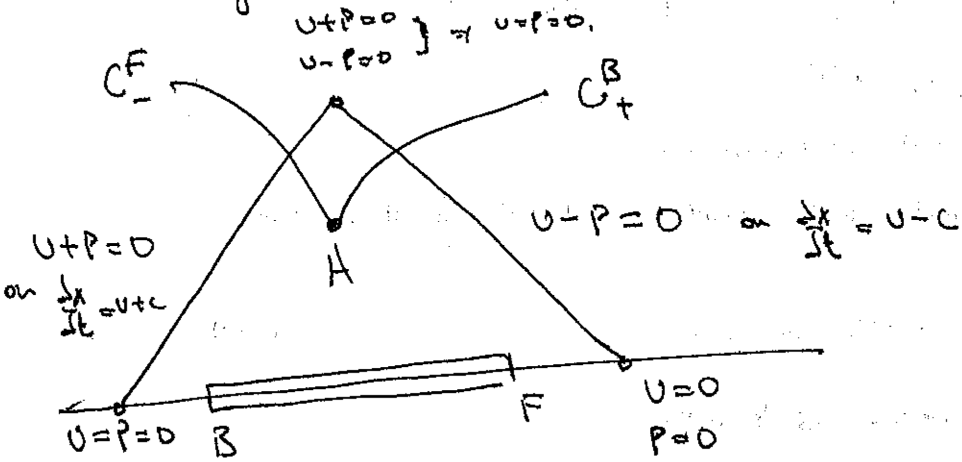


$U - P = 0 - 0 = 0$.
$\therefore$ left R.I. is constant

At $\frac{dx}{dt} = u - c$ intersects
w/ x-axis $U = 0$
$P = 0$

$\therefore U - P = 0$
$\Rightarrow U = P$

In Region between C_-^F, A, C_+^B



Cons. Mass is:

$$\rho_t + u\rho_x + \rho u_x = 0$$

Mmt:

$$\rho(u_t + uu_x) + p_x = 0$$

$$\rho(u_t + uu_x) + \underbrace{p'(\rho)\rho_x}_{c^2 \rho_x} = 0$$

$$\rho_t + u\rho_x + \rho u_x = 0$$

$$u_t + uu_x + \frac{c^2}{\rho}\rho_x = 0$$

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{pmatrix} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \vec{0}.$$

$$\xi = \xi(x,t) \qquad \tau = t.$$

$$\Rightarrow \cancel{I \frac{\partial U}{\partial \tau} + A} \quad I\left(\frac{\partial}{\partial \tau} + \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi}\right)U + A \frac{\partial \xi}{\partial x}\frac{\partial U}{\partial \xi} = 0$$

$$\Rightarrow \quad I \frac{\partial U}{\partial \tau} + \left(\frac{\partial \xi}{\partial t} I + \frac{\partial \xi}{\partial x} A\right)\frac{\partial U}{\partial \xi} = 0$$

$\frac{\partial U}{\partial \tau}$ is continuous.

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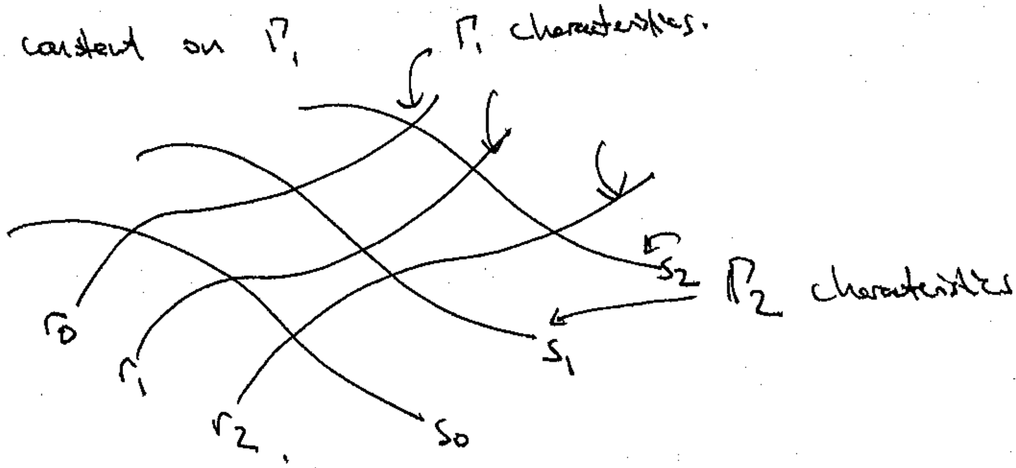
r constant on Γ_1



Notice: Along Γ_1 s changes $\Rightarrow$ let s parametrize Γ_1 characteristics
+ Along Γ_2 r changes $\Rightarrow$ let r parametrize Γ_2 characteristics.

$$\frac{dx}{ds} = (u+c)\frac{dt}{ds} \quad \text{on } \Gamma_1$$

$$\frac{dx}{dr} = (u-c)\frac{dt}{dr} \quad \text{on } \Gamma_2.$$

- - -

$$F(\rho) = s + r \qquad \text{eq } 6.8.49a,b$$

$$2u = 2r - 2s \quad \Rightarrow \quad u = r - s$$

- - -

$$x_s = (u+c)\,t_s$$

$$x_r = (u-c)\,t_r$$

$$x_{sr} = (u_r + c_r)t_s + (u+c)t_{sr}$$

set equal.

$$+ \quad x_{rs} = (u_s - c_s)t_r + (u-c)t_{rs}$$

$$\Rightarrow (\upsilon_r + c_r)t_s + (\cancel{\upsilon} + c)t_{sr} = (\cancel{\upsilon} - c)t_{rs} + (\upsilon_s - c_s)t_r$$

$$\Rightarrow \quad 2c\,t_{rs} + (\upsilon_r + c_r)t_s - (\upsilon_s - c_s)t_r = 0$$

$$\Rightarrow 2c\,t_{rs} + (1 + c_r)t_s - (-1 - c_s)t_r = 0.$$

$$F'(\rho)\rho_r = F'(\rho)\rho_s = 1 \qquad w/ \; F'(\rho) = \frac{c}{\rho}$$

$$\Rightarrow \rho_r = \rho_s = \frac{\rho}{c}$$

$$c_r = c'(\rho)\rho_r = c'(\rho)\frac{\rho}{c}$$

$$\& \; c_s = c'(\rho)\rho_s = c'(\rho)\frac{\rho}{c}.$$

$\therefore$ Above becomes

$$2c\,t_{rs} + \left(1 + \frac{\rho}{c}c'(\rho)\right)t_s + \left(1 + \frac{\rho}{c}c'(\rho)\right)t_r = 0$$

$$t_{rs} + \underbrace{\frac{1}{2c}\left(1 + \frac{\rho}{c}c'(\rho)\right)}_{\text{only depends on } \rho.}(t_s + t_r) = 0$$

$= \text{fn of } r+s. \Rightarrow f(r+s)$

Pg 22B Debnath

$$p = k\rho^{\gamma}$$

$$c^2 = p'(\rho) = k\gamma\rho^{\gamma-1}$$

Then $\frac{\partial}{\partial t} \Rightarrow$ $2cc_t = k\gamma(\gamma-1)\rho^{\gamma-2}\rho_t$

$$= \frac{k\gamma\rho^{\gamma-1}}{\rho}(\gamma-1)\rho_t$$

$$= \frac{c^2}{\rho}(\gamma-1)\rho_t$$

$$\rightarrow \rho_t = \frac{2\rho}{(\gamma-1)c}c_t$$

$+ \frac{\partial}{\partial x}$ ... same for $\frac{\partial}{\partial x}$

$$\rho_x = \frac{2\rho}{(\gamma-1)c}c_x$$

Then eqs 6.8.16 + 6.8.17 $\Rightarrow$

$$\frac{2\cancel{\rho}c_t}{(\gamma-1)c} + u\frac{2\cancel{\rho}c_x}{(\gamma-1)c} + \cancel{\rho}u_x = 0$$

$$+ \quad u_t + uu_x + \frac{c^2}{\rho}\frac{2\rho}{(\gamma-1)c}c_x = 0$$

$$\Rightarrow \quad c_t + u c_x + \frac{(\gamma-1)}{2} c u_x = 0$$

$$\& \quad u_t + u u_x + \frac{2c}{(\gamma-1)} c_x = 0$$

Following books hints.

$$\frac{1}{(\gamma-1)} c_t + \frac{u}{\gamma-1} c_x + \frac{1}{2} c u_x = 0$$

$$\frac{1}{2} u_t + \frac{u}{2} u_x + \frac{1}{\gamma-1} c c_x = 0$$

Adding.

$$\frac{1}{2} u_t + \frac{1}{\gamma-1} c_t + \frac{u_x}{2}(u+c) + \frac{1}{\gamma-1}(u+c) c_x = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2} u + \frac{c}{\gamma-1}\right)}_{r} + (u+c)\frac{\partial}{\partial x}\left(\frac{u}{2} + \frac{c}{\gamma-1}\right) = 0.$$

$$\Rightarrow \quad r_t + (u+c) r_x = 0$$

Subtracting

$$\frac{1}{2} u_t - \frac{1}{\gamma-1} c_t + \frac{1}{\gamma-1}(c-u) c_x + \frac{1}{2}(u-c) u_x = 0$$

$$\frac{\partial}{\partial t}\left(\frac{u}{2}-\frac{c}{\gamma-1}\right)+(u-c)\frac{\partial}{\partial x}\left(-\frac{c}{\gamma-1}+\frac{u}{2}\right)=0$$

$$\frac{\partial}{\partial t}\left(\frac{u}{2}-\frac{c}{\gamma-1}\right)+(u-c)\frac{\partial}{\partial x}\left(\frac{u}{2}-\frac{c}{\gamma-1}\right)=0.$$

$$\Longleftrightarrow \frac{\partial}{\partial t}\left(\frac{c}{\gamma-1}-\frac{u}{2}\right)-(c-u)\frac{\partial}{\partial x}\left(\frac{c}{\gamma-1}-\frac{u}{2}\right)=0$$

- - . -

$$\frac{2}{\gamma-1}c=r+s \qquad c=\frac{(\gamma-1)}{2}(s+r)$$

$$F(\rho)=\int_{\rho_0}^{\rho}\frac{c(\rho)}{\rho}\,d\rho \qquad \text{w/} \quad c(\rho)=\sqrt{\left.\frac{dp}{d\rho}\right|_s}=\sqrt{k\gamma}\,\rho^{\frac{1}{2}(\gamma-1)}$$

$$F(\rho)=\int_{\rho_0}^{\rho}\sqrt{k\gamma}\,\rho^{\frac{\gamma}{2}-\frac{3}{2}}\,d\rho=\sqrt{k\gamma}\left.\frac{\rho^{\frac{\gamma}{2}-\frac{1}{2}}}{\frac{\gamma}{2}-\frac{1}{2}}\right|_{\rho_0}^{\rho}$$

$$=\sqrt{k\gamma}\,\frac{\rho^{\frac{\gamma-1}{2}}}{\frac{\gamma-1}{2}}+\text{const.}$$

$$=\frac{2}{\gamma-1}\underbrace{\sqrt{\gamma k}\,\rho^{\frac{\gamma-1}{2}}}_{c}=\frac{2c}{\gamma-1}. \qquad \therefore \text{R.I.}$$

$$s,r=\frac{2c}{\gamma-1}\pm u$$

$u + c = \,?$

$c = -\frac{(r-1)}{2}u + \text{const.}$

$u + c = u - \frac{ru}{2} + \frac{u}{2} + \text{const}$

$= $ How get 6.8.64?

If talking about backwards progressing wave

$c = \frac{(r-1)}{2}u + \text{const.}$

Then $u + c = \frac{r}{2}u + \frac{1}{2}u + \text{const} = \frac{(r+1)}{2}u + c_0$

$\frac{\partial u}{\partial x} = f'(\xi)\xi_x$

& $\xi_x = 1 - (u_x + c_x)t$

But $c_x = \frac{1}{2}(r-1)u_x$ put in Above

$u_x = f'(\xi)\left(1 - \frac{t}{2}(r+1)u_x\right)$

$\left(1 + \frac{t}{2}(r+1)\right)u_x = f'(\xi)$

$\Rightarrow u_x = \frac{f'(\xi)}{\left(1 + \frac{1}{2}(r+1)t f'(\xi)\right)}$

$$u_t = f'(\xi)\xi_t$$

$$\xi_t = -(u_t + c_t)t - (u+c)$$

$$-\frac{(r+1)}{2}u_t t - \frac{(r+1)}{2}u$$

$$= -\frac{(r+1)}{2}\left[u_t t + f(\xi)\right]$$

Then

$$u_t = -f'(\xi)\frac{(r+1)}{2}\left(u_t t + f(\xi)\right)$$

$$\left(1 + f'(\xi)\frac{(r+1)}{2}t\right)u_t = -f'(\xi)f(\xi)\left(\frac{r+1}{2}\right)$$

$$u_t = \frac{-f'(\xi)f(\xi)\left(\frac{r+1}{2}\right)}{\left(1 + f'(\xi)\frac{(r+1)}{2}t\right)}$$

$$1 + f'(\xi)\frac{(r+1)}{2}t = 0$$

$$t = \frac{-2}{f'(\xi)(r+1)}$$

$$t = \frac{2}{(r+1)\left[-f'(\xi)\right]_{max}}$$

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$$c'(\rho) = \sqrt{kr'}\left(\frac{r-1}{2}\right)\rho^{\frac{1}{2}(r-3)} = \sqrt{kr'}\,\frac{(r-1)}{2}\,\frac{\rho^{\frac{1}{2}(r-1)}}{\rho}$$

$$= \frac{c}{\rho}\,\frac{(r-1)}{2}$$

$$F(\rho) = \int_{\rho_0}^{\rho} \frac{c(\rho)}{\rho}\,d\rho = \cdots$$ see pg 3 my ~~notes~~ notes from pg 228 Debnath

$$f(r+s) = \frac{1}{2c}\left(1 + \frac{\cancel{\rho}}{\cancel{c}}\,\frac{1}{2}(r-1)\,\frac{\cancel{c}}{\cancel{\rho}}\right)$$

$$= \frac{1}{2c}\left(1 + \frac{r}{2} - \frac{1}{2}\right) = \frac{1}{2c}\left(\frac{r}{2} + \frac{1}{2}\right) = \frac{1}{4c}(r+1)$$

Now $c = \cancel{\rho F'(\rho)} = \cancel{\frac{2\rho c}{r-1}}\ \cancel{\rho\frac{2}{(r-1)}\frac{1}{2}(r-1)\frac{c}{\rho}}$

$$F(\rho) = r+s$$

$$\frac{2c}{r-1} = r+s \Rightarrow c = \frac{(r-1)}{2}(r+s)$$

$$\Rightarrow f(r+s) = \frac{1}{4}(r+1)\,\frac{2}{(r-1)}\,\frac{1}{(r+s)} = \frac{1}{2}\,\frac{(r+1)}{(r-1)}\,\frac{1}{(r+s)}$$

$$\therefore \quad t_{rs} + \frac{n}{(r+s)}(t_r + t_s) = 0$$

let $t(r,s) = g(r+s)\,w(r,s)$

$$t_r = g'(r+s)\,w(r,s) + g(r+s)\,w_r$$

$$t_s = g'(r+s)\,w(r,s) + g(r+s)\,w_s$$

$$t_{rs} = g''(r+s)\,w(r,s) + g'(r+s)\,w_s + g'(r+s)\,w_r + g(r+s)\,w_{rs}$$

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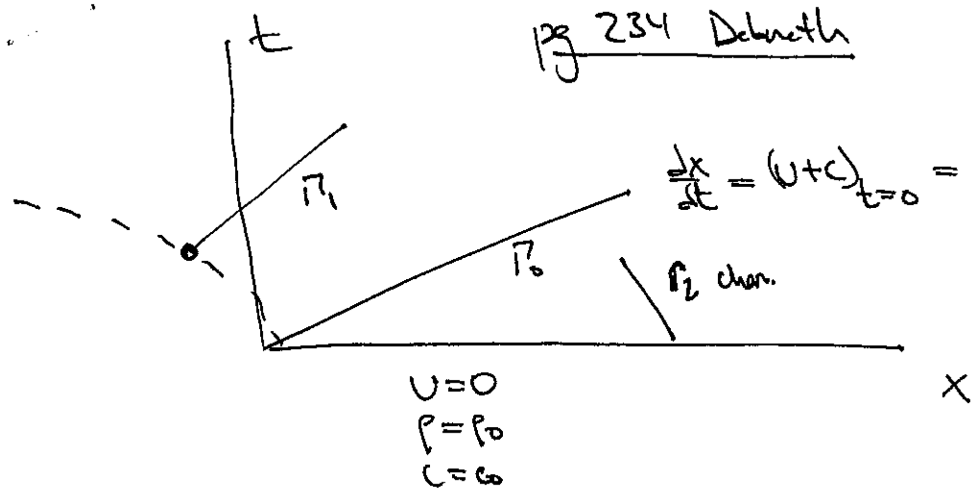


$$\frac{dx}{dt} = (U+c)_{t=0} = c_0 \Rightarrow x = c_0 t.$$

All Γ_2 characteristic start on x axis + progress left.

$$\frac{2c}{\gamma-1} - U = \left.\frac{2c}{\gamma-1} - U\right|_{t=0} = \frac{2c_0}{\gamma-1}$$

$$\Rightarrow U = \frac{2}{\gamma-1}(c-c_0)$$

eq 6.8.97ab.

$$c = c_0 + \frac{(\gamma-1)}{2}U$$

Γ_1 character

(B) $$\frac{2c}{\gamma-1} + U = \left.\frac{2c}{\gamma-1} + U\right|_{\text{evaluate on piston}} = 2r$$

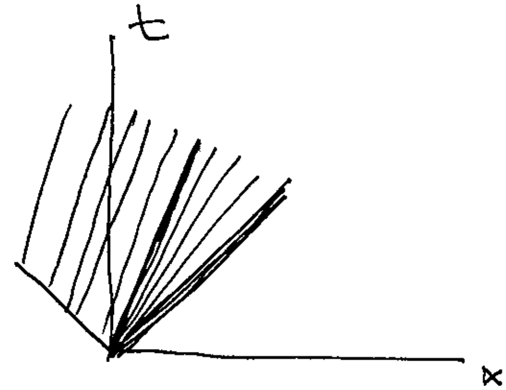
$2r$ = constant. initial value on pistons surface at (x_0, t_0)

on $$\frac{dx}{dt} = U + c$$

~~BC~~ $\Rightarrow$ $\cancel{U = 2r - \frac{2c}{\gamma-1}}$

But ⇒ By $u = \frac{2}{\gamma-1}(c - c_0)$ which holds everywhere

(These are given by Γ_2 characteristics)



$$\frac{2c_0}{\gamma-1} + \frac{2}{\gamma-1}\left(\frac{\gamma-1}{2}u\right) = 2r$$

$$\Rightarrow \quad u = \text{constant on } \Gamma_1$$

$$\therefore \quad \frac{dx}{dt} = u + c_1 \qquad = c_0 + \frac{(\gamma+1)}{2}u.$$

$$c = c_0 + \frac{(\gamma-1)}{2}u$$

$$c = c_0 + \frac{(\gamma-1)}{2}u \quad \& \quad \xi = u + c$$

$$\xi - u = c_0 + \frac{(\gamma-1)}{2}u$$

$$\xi - c_0 = \frac{(\gamma+1)}{2}u \quad \Rightarrow \quad u = \frac{2}{\gamma+1}(\xi - c_0)$$ ⇐ value of velocity in fan

$$\Rightarrow \quad c = c_0 + \frac{\gamma-1}{\gamma+1}\xi - \left(\frac{\gamma-1}{\gamma+1}\right)c_0$$

$$= \left(\frac{\gamma+1}{\gamma+1} - \frac{(\gamma-1)}{\gamma+1}\right)c_0 + \frac{\gamma-1}{\gamma+1}\xi$$

$$= \frac{2}{\gamma+1}c_0 + \frac{\gamma-1}{\gamma+1}\xi$$ ⇐ value of sound speed in fan.

Get ρ in fan

$$c^2 = \gamma k \rho^{\gamma-1}$$

$$c = c_0 + \frac{\gamma-1}{2} u$$

$$\sqrt{\gamma k}\, \rho^{\frac{\gamma-1}{2}} = \sqrt{k\gamma}\, \rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{2} u$$

$$\rho^{\frac{\gamma-1}{2}} = \rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{2\sqrt{k\gamma}} u$$

$$\rho = \left(\rho_0^{\frac{\gamma-1}{2}} + \frac{\gamma-1}{2\sqrt{k\gamma}} u \right)^{\frac{2}{\gamma-1}} = \rho_0 \left(1 + \frac{(\gamma-1)\rho_0^{-\left(\frac{\gamma-1}{2}\right)}}{2\sqrt{k\gamma}} u \right)^{\frac{2}{\gamma-1}}$$

$$= \rho_0 \left(1 + \frac{\gamma-1}{2c_0} u \right)^{\frac{2}{\gamma-1}}$$

Then $u = \frac{2}{\gamma+1}(\xi - c_0)$

gives $\rho = \rho_0 \left(1 + \frac{\gamma-1}{\cancel{2}c_0} \frac{\cancel{2}}{\gamma+1} (\xi - c_0) \right)^{\frac{2}{\gamma-1}}$

$$= \rho_0 \left(1 + \left(\frac{\gamma-1}{\gamma+1}\right) \frac{\xi}{c_0} - \frac{\gamma-1}{\gamma+1} \right)^{\frac{2}{\gamma-1}}$$

$$= \rho_0 \left(\frac{\gamma+1-\gamma+1}{\gamma+1} + \left(\frac{\gamma-1}{\gamma+1}\right) \frac{\xi}{c_0} \right)^{\frac{2}{\gamma-1}}$$

$$= \rho_0 \left(\frac{2}{\gamma+1} + \left(\frac{\gamma-1}{\gamma+1}\right) \frac{\xi}{c_0} \right)^{\frac{2}{\gamma-1}}$$

$$\frac{dx}{dt} = 6 + \frac{1}{2}(r+1)\dot{X}(\tau) \qquad x(\tau) = \text{known}$$

$$x - X(\tau) = \left(6 + \frac{1}{2}(r+1)\dot{X}(\tau)\right)(t-\tau)$$

$$x = X(\tau) + \left[6 + \frac{r+1}{2}\dot{X}(\tau)\right](t-\tau)$$