(1) (a) linear, semilinear, quasilinear, or fully non linear
(b) order.

(1.2.1)

(1)(A) linear, ord = 2
(2)(B) linear, ord = 2
(3)(C) linear, order = 1
(4)(D) linear, ord 1
(5)(E) linear, ord 2
(6) linear, order 2
(7) linear, ord 2
(8) linear, ord 2
(9) lin ord 2
(10) linear, ord 2
(11) linear, ord 2
(12) linear, ord 3
(13) linear, order 4

(1) Non linear, fully, order 1
(2) semi linear, ord 2
(3) Fully non linear, ord 2
(4) Fully non lin, order 2
(5) Fully nonlinear, order 2
(6) Fully nonlinear, ord 1
(7) ~~Fully nonlin~~, ord 1, quasilinear
(8) Quasilinear, order 1
(9) semilinear, ord 2
(10) quasilinear, order 2
(11) semilinear, ord 2
quasilinear, ord 2
(12) semilinear, ord 3

(1) linear, ord 2
(2) lin, ord 2
(3) linear, order 1

(1) quasilinear, order 1
(2) semilinear, order 2
(3) quasilinear, ord 1
(4) quasilinear, order 2

(2)
$$(x_1+x_2+\cdots+x_n)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha} x^\alpha$$

$$\binom{|\alpha|}{\alpha} \equiv \frac{|\alpha|!}{\alpha!} = \frac{|\alpha|!}{\alpha_1!\,\alpha_2!\,\alpha_3!\cdots\alpha_n!}$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}\cdots x_n^{\alpha_n}$$

For $k=2$

$$(x_1+x_2+\cdots+x_n)(x_1+x_2+\cdots+x_n)$$

$$= \cancel{x_1}\; x_1^2+x_2^2+\cdots+x_n^2 + 2x_1x_2+2x_1x_3+\cdots+2x_1x_n$$
$$+ \cancel{2x_2x_3} + 2x_2x_3 + 2x_2x_4 + \cdots 2x_2x_n$$
$$2x_3x_4 + \cdots 2x_3x_n$$

$=$

$$x_1^2+2x_1x_2+2x_1x_3+2x_1x_4+\cdots+2x_1x_n$$
$$+x_2^2+2x_2x_3+2x_2x_4+\cdots+2x_2x_n$$
$$+x_3^2+2x_3x_4+\cdots+2x_3x_n$$

$$\ddots \quad x_{n-1}^2 \quad x_{n-1}x_n$$
$$x_n^2$$

for $|\alpha| = 2$

one component must be 2 which happens in n locations

giving rise to terms like x_j^2 w/ coefficient like

$$\frac{2!}{2!} = 1$$ these correspond to the diagonal terms

expressed above.

or two components are 1's this gives $\frac{n(n-1)}{2}$ which is the #

of nonzero off diagonal elts in an upper Δ matrix

$$\frac{n(n-1)}{2} + \frac{n(n-1)}{2} + n \stackrel{?}{=} n^2$$

$$n^2 - n + n = n^2 \quad \text{yes}$$

Thus this formula is good for $k=2$.

Assume true for k up to k. Try to prove for $k = k+1$

$$(x_1 + x_2 + x_3 + \cdots + x_n)^{k+1}$$

$$= (x_1 + x_2 + \cdots + x_n)^k (x_1 + x_2 + x_3 + \cdots + x_n)$$

$$=$$

$$\left(\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x^{\alpha}\right)(x_1+x_2+\cdots+x_n)$$

$$=\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x^{\alpha}x_1+\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x^{\alpha}x_2+\cdots+\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x^{\alpha}x_n$$

$$=\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x_1^{\alpha_1+1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}+\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x_1^{\alpha_1}x_2^{\alpha_2+1}\cdots x_n^{\alpha_n}+\cdots+$$

$$\sum_{|\alpha|=k}\binom{|\alpha|}{\alpha}x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n+1}$$

- - -

$$\binom{|\alpha|}{\alpha}=\frac{|\alpha|!}{\alpha_1!\alpha_2!\cdots\alpha_n!}$$

$$\cancel{|\alpha|+1}\qquad \cancel{\frac{(|\alpha|+1)!}{(\alpha_1+1)!\alpha_2!\cdots}}\qquad \frac{(|\alpha|+1)!}{\alpha_1!\alpha_2!\cdots(\alpha_i+1)!\cdots\alpha_n!}$$

$$=\frac{(|\alpha|+1)\ |\alpha|!}{(\alpha_i+1)\ \alpha_1!\alpha_2!\cdots\alpha_n!}=\frac{(|\alpha|+1)\ |\alpha|!}{(\alpha_i+1)\ \alpha_1!\alpha_2!\cdots\alpha_n!}$$

$$\rightarrow \cancel{\binom{|\alpha|}{\alpha}} = \frac{(|\alpha|+1)}{(\alpha_i+1)}\binom{|\alpha|}{\alpha} = \binom{|\alpha'|}{\alpha'} \quad \ldots$$

Not sure ... how do?

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(3) $D^\alpha(uv) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta u D^{\alpha-\beta} v$ ... ? How show?

Also what does $\beta \leq \alpha \Rightarrow \beta_i \leq \alpha_i$ cont ~~~~ $\alpha + \beta$ scalars?

(4) ~~g~~ consider $g(t) = f(tx)$

The $g(t)$ has a taylor series expansion

$$g(t) = \sum_{k=0}^{+\infty} \frac{g^{(k)}(0)}{k!} t^k$$

Now $g(t) = f(tx)$ $x \in \mathbb{R}^n$

so ~~$g(\ldots)$~~ = ~~$D\ldots$~~

$g^{(0)}(0) = f(0)$ $g^{(1)}(t) = Df(tx) \cdot x$

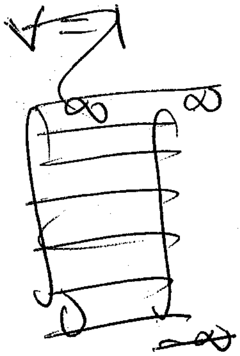
$g^{(1)}(0) = Df(0) \cdot x$

$g^{(2)}(t) =$ ~~$\ldots$~~ $D(Df(tx) \cdot x)$

$\therefore$ I'm not comfortable enough w/ the multiindex notation ...

Pg 161 Error

$$u_t + F(u)_x = 0 \qquad \mathbb{R}\times(0,\infty)$$



Show: $\displaystyle\int_{-\infty}^{\infty} u(\cdot,t)\,dx = \int_{-\infty}^{\infty} g\,dx$

consider $\displaystyle\int_{-\infty}^{\infty} \big(u(\cdot,t)\,dx - g\,dx\big) = 0$

$$\frac{d}{dt} = \int_{-\infty}^{\infty} u_t(\cdot,t)\,dx$$

$$= -\int_{-\infty}^{\infty} F(u(\cdot,t))_x\,dx = -F(u(\cdot,t))\Big|_{-\infty}^{\infty}$$

$= 0$ As $F(0) = 0$

$\therefore \displaystyle\int_{-\infty}^{\infty} u(\cdot,t)\,dx - g\,dx = C_1$.

But at $t=0$ gives $C_1 = 0$.

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MA 542

02/22/99

Prof. Liu

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PROBLEM SET #2

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#9.) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1,\ x_n > 0\}$. [illegible]
$u \in C(\bar{U}^+)$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$V(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

$x = (x_1, x_2, \dots, x_n)$

$\widetilde{(x_1, \dots, x_n)} = (x_1, x_2, \dots, x_{n-1}, -x_n)$

Show V harmonic on $B(0,1)$

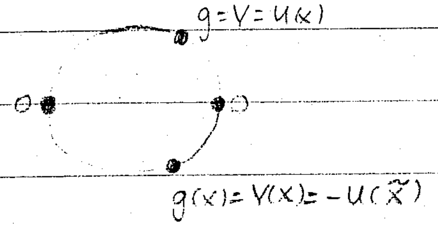
Know:

u harmonic on U^+

u continuous on $\bar{U}^+$

$u = 0$ on $\partial U^+ \cap \{x_n = 0\}$
$\qquad\qquad\quad \overset{\shortparallel}{\bar{U}^+ \cap \{x_n = 0\}}$

Method: Let $g = V\big|_{\partial B(0,1)}$



define $f(x)$ by Poisson Integral Formula.

$f(x)$ harmonic.

$$f(x) = \frac{1 - |x|^2}{n\,\alpha(n)} \int_{S^{n-1}} \frac{g(y)}{|x-y|^n}\, dS(y)$$

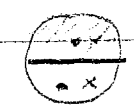
- Show $f(\tilde{x}) = -f(x)$ $\quad (\Rightarrow f = 0$ on $\partial U^+ \cap \{x_n = 0\})$

$\Rightarrow$ f agrees with u on $\overline{\partial U^+}$

∂U^+

$\Rightarrow$ Since f, u harmonic in U^+,

$\Rightarrow$ $f = u$ in $\bar{U}^+$



for $x \in U^-$

$f(x) = -f(\tilde{x}) = -u(\tilde{x})$

$v(x) = -u(\tilde{x})$

$\Rightarrow$ $f(x) = v(x) \quad \forall\, x \in B(0,1)$

$\Rightarrow$ $\boxed{v(x) \text{ harmonic}}$

To show:

$$f(\tilde{x}) = -f(x)$$

$$f(x) = \frac{1-|x|^2}{n\,\alpha(n)} \int_{S^{n-1}} \frac{g(y)}{|x-y|^n}\, dS(y)$$

$$f(\tilde{x}) = \frac{1-|\tilde{x}|^2}{n\,\alpha(n)} \int_{S^{n-1}} \underbrace{\frac{g(y)}{|\tilde{x}-y|^n}}_{=f(y)}\, dS(y) = \frac{1-|x|^2}{n\,\alpha(n)} \int_{S^{n-1}} \frac{-g(\tilde{y})}{|x-\tilde{y}|^n}\, dS(y) =$$

$$= -\frac{1-|x|^2}{n\,\alpha(n)} \int_{S^{n-1}} \underbrace{\frac{g(\tilde{y})}{|x-\tilde{y}|^n}}_{=f(\tilde{y})}\, dS(y) = -\frac{1-|x|^2}{n\,\alpha(n)} \int_{S^{n-1}} \frac{g(y)}{|x-y|^n}\, dS(y) = -f(x$$

✓

10/10 #10.) Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$

(i) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(ii) Use (i) to show $v(x,t) := x \cdot Du(x,t) + 2t u_t(x,t)$ solves the heat equation as well.

(i)

$U_\lambda(x,t) = U(\lambda x, \lambda^2 t)$

$\partial_t U = \lambda^2 (\partial_t U)\Big|_{(\lambda x, \lambda^2 t)}$

$\Delta U = \lambda^2 [\, \partial_{x_1}^2 + \partial_{x_2}^2 + \dots + \partial^2 x_n \,]$

$\partial_t U - \Delta U = -\lambda^2 \cdot (\underbrace{\partial^2 x_1 + \dots + \partial^2 x_n - \partial_t (U)}_{= 0})$

$\therefore \partial_t U - \Delta U = 0$

(ii)

$V(x,t) = \breve{x} \cdot \nabla u(x,t) + 2t\, U_t(x,t)$

Note: $U(\lambda x, \lambda^2 t)$ is a solution to the heat equation means

$(\partial_t - \Delta)(U(\lambda x, \lambda^2 t)) = 0$

Now, $\partial_\lambda \Big[(\partial_t - \Delta)(U(\lambda x, \lambda^2 t)) \Big] = \partial_\lambda \cdot 0 = 0$

$\Rightarrow (\partial_t - \Delta)\big(\partial_\lambda (U(\lambda x, \lambda^2 t))\big) = 0$

which means $\partial_\lambda (U(\lambda x, \lambda^2 t))$ is a solution to the Equation

$\phi(\lambda, x, t) = \left(x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + \dots + x_n \frac{\partial u}{\partial x_n} + 2\lambda t \frac{\partial u}{\partial t}\right)\Big|_{(\lambda x, \lambda^2 t)}$

$(x_1, \dots, x_n) \cdot \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) + 2\lambda t \frac{\partial u}{\partial t}$

For $\lambda = 1$ (since true for all $\lambda \in \mathbb{R}$)

$$(x_1, \ldots, x_n) \cdot \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) + 2t \frac{\partial u}{\partial t}$$

$$\parallel$$

$$\vec{x} \cdot Du + 2t\, u_t$$

10/10 #11.) Assume $n=1$ and $u(x,t) = v(\frac{x^2}{t})$.

(a) Show

$$u_t = u_{xx}$$

iff

(*) $4zv''(z) + (2+z)v'(z) = 0 \qquad (z>0)$

(b) Show that the general solution of (*) is

$$v(z) = c\int_0^z e^{-s/4} s^{-1/2}\, ds + d$$

(c) Differentiate $v(\frac{x^2}{t})$ with respect to x and select the constant c properly, so as to obtain the fundamental solution Φ for $n=1$.

(a)

$$u_t = -\left(\frac{x}{t}\right)^2 \cdot v'\left(\frac{x^2}{t}\right)$$

$$u_x = \frac{2x}{t} v'\left(\frac{x^2}{t}\right)$$

$$u_{xx} = \frac{2}{t} v'\left(\frac{x^2}{t}\right) + \frac{2x}{t} v''\left(\frac{x^2}{t}\right) \cdot \frac{2x}{t} =$$

$$= \frac{2}{t} v'\left(\frac{x^2}{t}\right) + \frac{4x^2}{t^2} v''\left(\frac{x^2}{t}\right)$$

So, $-\left(\frac{x}{t}\right)^2 v'\left(\frac{x^2}{t}\right) = \frac{2}{t} v'\left(\frac{x^2}{t}\right) + \frac{4x^2}{t^2} v''\left(\frac{x^2}{t}\right)$

Let $z = \frac{x^2}{t}$ Then, $-\frac{z}{t} v'(z) = \frac{2}{t} v'(z) + \frac{4z}{t} v''(z) \Rightarrow$

$$\Rightarrow -z v'(z) = 2v'(z) + 4zv''(z) \Rightarrow$$

$$\Rightarrow 4zv''(z) + (2+z)v'(z) = 0$$

(b)

Let $y(z) = v'(z)$ Then $4zy'(z) + (2+z)y(z) = 0$

$$\frac{dy}{dz} = -\frac{(2+z)}{4z} y \Rightarrow \frac{dy}{y} = -\frac{(2+z)}{4z} dz = \left(-\frac{1}{2z} - \frac{1}{4}\right) dz$$

$$\int \frac{dy}{y} = \int \left(-\frac{1}{2z} - \frac{1}{4}\right) dz \Rightarrow$$

$$\Rightarrow \ln(y) = -\frac{1}{2}\ln(z) - \frac{1}{4}z + C_1$$

$$\Rightarrow y = \frac{e^{-z/4}}{z^{1/2}} C_2$$

$$\frac{dv}{dz} = C_2 \frac{e^{-z/4}}{z^{1/2}}$$

$$dv = C_2 \frac{e^{-z/4}}{z^{1/2}} dz$$

$$\int dx = \int C_2 \frac{e^{-z/4}}{z^{1/2}} dz$$

$$\boxed{v(z) = C_2 \int_0^z s^{-1/2} e^{-s/4} ds + C_3}$$

(c) $$\frac{dv}{dx}\left(\frac{x^2}{t}\right) = v'\left(\frac{x^2}{t}\right) \cdot \frac{2x}{t} = C_2 \left(\frac{x^2}{t}\right)^{-1/2} e^{-\left(\frac{x^2}{t}\right)/4} \cdot \frac{2x}{t} =$$

$$= C_2 \frac{t^{1/2}}{x} \cdot \frac{2x}{t} e^{-x^2/4t} = \boxed{C_4 \frac{e^{-\frac{x^2}{4t}}}{t^{1/2}}}$$

FUNDAMENTAL SOLUTION TO THE HEAT EQUATION (FOR $n=1$) is: $\frac{1}{2\sqrt{\pi}\sqrt{t}} e^{-\frac{x^2}{4t}}$

$$\therefore \boxed{C_4 = \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{4\pi}}}$$

Note: $\int_{-\infty}^{\infty} e^{-z^2}\, dz = \sqrt{\pi}$

or

$$\int_{-\infty}^{+\infty} C_4 \frac{e^{-\frac{x^2}{4t}}}{t^{1/2}}\, dx = 1$$

let $z = \dfrac{x}{\sqrt{4t}}$

$dz = \dfrac{dx}{\sqrt{4t}}$

$\Rightarrow \sqrt{4t}\, dz = dx$

$$\int_{-\infty}^{+\infty} C_4 \frac{e^{-z^2}}{t^{1/2}}\, 2t^{1/2}\, dz = 2C_4\left(\int_{-\infty}^{\infty} e^{-z^2}\, dz\right) = 2C_4\sqrt{\pi}$$

So,

$$2C_4\sqrt{\pi} = 1 \Rightarrow \boxed{C_4 = \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{4\pi}}}$$

#12) Write down an explicit formula for a solution of

10/10

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

where $c \in \mathbb{R}$.

Consider

$$u_t + cu = 0 \Rightarrow u = e^{-ct}$$

look at $u = e^{-ct} U$

$$u_t = -ce^{-ct} U + e^{-ct} U_t$$

$$+ \quad -\Delta u = -e^{-ct} \Delta U$$

$$+ \quad cu = ce^{-ct} U$$

$$f = e^{-ct}(U_t - \Delta U) \Rightarrow U_t - \Delta U = fe^{ct}$$

and

$$u(x,0) = U(x,0) = g(x)$$

$\therefore$ pick U solution to $U_t - \Delta U = fe^{ct}$, $U(x,0) = g(x)$ ✓

$$U(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)$$

where $\tilde{f}(x,t) = e^{ct} f(x,t)$

$$U(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s)\, dy\, ds$$

So, $$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\, e^{cs} f(y,s)\, dy\, ds$$

$$\boxed{u(x,t) = e^{-ct}\left[\frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy + \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} e^{cs} f(y,s)\, dy\, ds\right.}$$

#13.) Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula,

10/10

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\, ds$$

for a solution of the initial/boundary-value problem.

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t=0\}, \\ u = g & \text{on } \{x=0\} \times [0,\infty) \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x<0\}$ by odd reflection.)

$u_t - u_{xx} = 0 \qquad x>0,\ t>0$

$u\big|_{t=0} = 0 \qquad x>0$

$u\big|_{x=0} = g(t) \qquad t>0 \qquad \boxed{g(0)=0}$

Let $v(x,t) = u(x,t) - g(t)$

Then $v_t - v_{xx} = u_t - u_{xx} - g'(t) + 0 = -g'(t)$

So $v_t - v_{xx} = -g'(t) \qquad x>0,\ t>0$

$v\big|_{t=0} = 0 \qquad x>0 \qquad$ (as $u(x,0) + g(0) = 0 + 0 = 0$)

$v\big|_{x=0} = 0 \qquad t>0$

Now,

$$\text{Let } W(x,t) = \begin{cases} v(x,t) = u(x,t) - g(t) & x \geq 0 \\ -v(-x,t) = -u(-x,t) + g(t) & x \leq 0 \end{cases}$$

$\boxed{\text{Will assume} \to W(0,t) = 0 \text{ boundary condition}}$

(note: $W(0,t)$ is well-defined as $V(0,t) = 0$)

$$\text{Then} \quad W_t - W_{xx} = \boxed{\underbrace{\begin{cases} -g'(t) & x > 0 \\ g'(t) & x < 0 \end{cases}}_{h(t)}} \qquad \underline{t > 0}$$

$$W\Big|_{t=0} = 0 \qquad y \in \mathbb{R}$$

This is solved by $\int_0^t \int_{-\infty}^{\infty} \Phi(x-y, t-s)\, h(t)\, d\tau$

$$W(x,t) = \int_0^t \left[\int_{-\infty}^0 \Phi(x-y,t-s)\, g'(s) - \int_0^{\infty} \Phi(x-y,t-s)\, g'(s) \right] dy\, ds$$

$$W = \frac{1}{\sqrt{\pi}} \int_0^t g'(s) \left[\int_{-\infty}^0 \frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{\sqrt{4(t-s)}}\, dy - \int_0^{\infty} \frac{e^{\frac{-(x-y)^2}{4(t-s)}}}{\sqrt{4(t-s)}}\, dy \right] ds$$

$$\frac{1}{\sqrt{\pi}} \int_0^t g'(s) \left[\left(\int_{-\infty}^0 - \int_0^{\infty} \right) \left(\frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{\sqrt{4(t-s)}} \right) dy \right] ds$$

$\boxed{\begin{array}{l} \text{given } f = e^{-u^2} \\ \text{Note: } \int_{-\infty}^{-a} f - \int_{-a}^{\infty} f = \\ = 2\int_{-\infty}^{-a} f - \int_{-\infty}^{\infty} f \end{array}}$

$\boxed{\text{assume } x > 0}$ $\left(a = \frac{x}{\sqrt{4(t-s)}} \right.$

$$\|$$

let $u = \frac{y-x}{\sqrt{4(t-s)}}$

$du = \frac{dy}{\sqrt{4(t-s)}}$

$$\frac{1}{\sqrt{\pi}} \int_0^t g'(s) \left[\int_{-\infty}^{-\frac{x}{\sqrt{4(t-s)}}} e^{-u^2}\, du - \int_{-\frac{x}{\sqrt{4(t-s)}}}^{\infty} e^{-u^2}\, du \right] =$$

$$= \frac{1}{\sqrt{\pi}} \int_0^t g'(s) \left(2 \int_{-\infty}^{\frac{-x}{\sqrt{4(t-s)}}} e^{-u^2} \, du - \sqrt{\pi} \right) ds =$$

$$= \frac{-\sqrt{\pi}}{\sqrt{\pi}} \int_0^t g'(s) \, ds + \frac{2}{\sqrt{\pi}} \int_0^t g'(s) \, F\left(\frac{-x}{\sqrt{4(t-s)}}\right) ds =$$

$$= -\frac{\sqrt{\pi}}{\sqrt{\pi}} \left(g(t) \right) + \frac{2}{\sqrt{\pi}} \left[F\left(\frac{-x}{\sqrt{4(t-s)}}\right) g(s) \Big|_0^t - \int_0^t g(s) \frac{-2x}{(4(t-s))^{3/2}} e^{\frac{-x^2}{4(t-s)}} \right] ds =$$

$$= -g(t) + \frac{2}{\sqrt{\pi}} \left[\cancelto{0}{F(-\infty) g(t)} - \cancelto{0}{F\left(\frac{-x}{4t}\right) g(0)} + \frac{2x}{8} \int_0^t \frac{g(s) \, e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} ds \right] =$$

$$\boxed{W = -g(t) + \frac{x}{\sqrt{4\pi}} \int_0^t g(s) \frac{e^{\frac{-x^2}{4(t-s)}}}{(t-s)^{3/2}} ds}$$ ✓

$$\frac{d}{ds} F\left(\frac{-x}{\sqrt{4(t-s)}}\right) = F'\left(\frac{-x}{\sqrt{4(t-s)}}\right) \frac{(-x)(-\tfrac{1}{2})}{(4(t-s))^{3/2}} (-4) = -e^{\frac{-x^2}{4(t-s)}} \frac{2x}{(4(t-s))^{3/2}}$$

$$F' = e^{-u^2}$$

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PROBLEM SET #1
02/26/99
PROF. LIU.

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#1.) 10/10

$$u_t + \vec{b}\cdot Du + cu = 0 \qquad c = A_1 + \dots + A_n$$

$$u_t = -(\vec{b}\cdot Du + cu)$$

$$= -(b_1 u_{x_1} + b_2 u_{x_2} + \dots + b_n u_{x_n} + cu)$$

$$= -(b_1 u_{x_1} + b_2 u_{x_2} + \dots + b_n u_{x_n} + (A_1 + \dots + A_n)u)$$

$$= -\left[b_1\left(u_{x_1} + \frac{A_1}{b_1}u\right) + b_2\left(u_{x_2} + \frac{A_2}{b_2}u\right) + \dots + b_n\left(u_{x_n} + \frac{A_n}{b_n}u\right)\right]$$

$$e^{\frac{A_1}{b_1}x_1} u_t = -\left[b_1\left(e^{\frac{A_1}{b_1}x_1} u_{x_1} + \frac{A_1}{b_1} u e^{\frac{A_1}{b_1}x_1}\right) + b_2 e^{\frac{A_1}{b_1}x_1}\left(u_{x_2} + \frac{A_2}{b_2}u\right) + \dots\right]$$

$$\left(e^{\frac{A_1}{b_1}x_1} u\right)_t = -\left[b_1 \frac{\partial}{\partial x_1}\left(e^{\frac{A_1}{b_1}x_1} u\right) + b_2\left(\left(e^{\frac{A_1}{b_1}x_1} u\right)_{x_2} + \frac{A_2}{b_2}\left(e^{\frac{A_1}{b_1}x_1} u\right)\right) + \dots\right]$$

Now multiply by $e^{\frac{A_2}{b_2}x_2}$

$$\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2} u\right)_t = -\left[b_1 \frac{\partial}{\partial x_1}\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2} u\right) + b_2\left(e^{\frac{A_2}{b_2}x_2}\left(e^{\frac{A_1}{b_1}x_1} u\right)_{x_2} + \frac{A_2}{b_2} e^{\frac{A_2}{b_2}x_2}\left(e^{\frac{A_1}{b_1}x_1} u\right)\right) + \dots\right]$$

$$\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2} u\right)_t = -\left[b_1 \frac{\partial}{\partial x_1}\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2} u\right) + b_2 \frac{\partial}{\partial x_2}\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2} u\right) + \dots\right]$$

$$\vdots$$

$$\left(e^{\frac{A_1}{b_1}x_1 + \frac{A_2}{b_2}x_2 + \dots + \frac{A_n}{b_n}x_n} u\right)_t = -\left[b_1 \frac{\partial}{\partial x_1}\left(e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u\right) + b_2 \frac{\partial}{\partial x_2}\left(e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u\right) + \dots + b_n \frac{\partial}{\partial x_n}\left(e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u\right.\right.$$

$$\left(e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u\right)_t = -\vec{b}\cdot D\left(e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u\right)$$

$$v = e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u \qquad A_1 + \dots + A_n = c$$

$$V_t = -\vec{b} \cdot DV \qquad V(x,0) = e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n} u(x,0)$$

$$= g(x)\, e^{\frac{A_1}{b_1}x_1 + \dots + \frac{A_n}{b_n}x_n}$$

$$V = G(\vec{x} - t\vec{b}) \qquad \boxed{u = e^{-\frac{A_1}{b_1}x_1 - \dots - \frac{A_n}{b_n}x_n} G(\vec{x} - t\vec{b})}$$

$$\Rightarrow \vec{u} = e^{\left(\frac{A_1}{b_1}, \dots, \frac{A_n}{b_n}\right)\cdot(-t\vec{b})} g(\vec{x} - t\vec{b})$$

$$G(x) = g(x)\, e^{\left(\frac{A_1}{b_1}, \dots, \frac{A_n}{b_n}\right)\cdot \vec{x}}$$

$$G(\vec{x} - t\vec{b}) = g(\vec{x} - t\vec{b})\, e^{\left(\frac{A_1}{b_1}, \dots, \frac{A_n}{b_n}\right)(\vec{x} - t\vec{b})}$$

$$= e^{-t(A_1 + \dots + A_n)} g(\vec{x} - t\vec{b})$$

$$= e^{-tc} g(\vec{x} - t\vec{b})$$

or

$$u_t = -[b_1 u_{x_1} + \dots + b_n u_{x_n} + cu]$$

$$= -\left[b_1\left(u_{x_1} + \frac{c}{b_1}u\right) + b_2 u_{x_2} + \dots + b_n u_{x_n}\right]$$

$$\left(e^{\frac{c}{b_1}x_1}u\right)_t = -\left[b_1\left(e^{\frac{c}{b_1}x_1}u_{x_1} + \frac{c}{b_1}e^{\frac{c}{b_1}}u\right) + b_2\left(e^{\frac{c}{b_1}x_1}u\right)_{x_2} + \dots\right]$$

$$= -\left[b_1\left(e^{\frac{c}{b_1}x_1}u\right)_{x_1} + \dots + b_n\left(e^{\frac{c}{b_n}x_n}u\right)_{x_n}\right]$$

$$= -\vec{b}\cdot\left(D e^{\frac{c}{b_1}x_1}u\right)$$

$$V = e^{\frac{c}{b_1}x_1}u \qquad V(x,0) = e^{\frac{c}{b_1}x_1}u(x,0) = e^{\frac{c}{b_1}x_1}g(x)$$

$$V_t = -\vec{b}\cdot DV$$

$$V = G(\vec{x} - t\vec{b}) = e^{\frac{c}{b_1}(x_1 - tb_1)} g(\vec{x} - t\vec{b})$$

$$u = e^{-\frac{c}{b_1}x_1} V = e^{-\frac{c}{b_1}x_1} e^{\frac{c}{b_1}x_1} e^{-\frac{ctb_1}{b_1}} g(\vec{x} - t\vec{b})$$

$$u = e^{-ct} g(\vec{x} - t\vec{b})$$

or $\quad e^{\frac{c}{b}x+A}$

$$u_t = -b\left(u_x + \frac{c}{b}u\right)$$

$$\left(e^{\frac{c}{b}x+A}u\right)_t = -b\left(e^{\frac{c}{b}x+A}u_x + \frac{c}{b}e^{\frac{c}{b}x+A}u\right)$$

$$= -b\left(e^{\frac{c}{b}x+A}\right)_x$$

$$v = e^{\frac{c}{b}x+A}u(x,t) \qquad e^{\frac{c}{b}x+A}u = G(x-tb)$$

$$v(x,0) = e^{\frac{c}{b}x+A}g(x) = G(x) \qquad e^{\frac{c}{b}x+A}u = e^{\frac{c}{b}(x-tb)+A}g(x-tb)$$

$$u = e^{-ct}g(x-tb)$$

1.)

$u_t + \vec{b} \cdot Du + cu = 0$ in $\mathbb{R}^n \times (0,\infty)$

$u = g$ on $\mathbb{R}^n \times (0,\infty)$ where $c \in \mathbb{R}$ and $\vec{b} \in \mathbb{R}^n$ are constants

try

$$u(x,t) = e^{\phi(x)} g(\vec{x} - t\vec{b})$$

$$e^{\phi(x)} Dg \circ (-\vec{b}) + \vec{b} [e^{\phi(x)} D\phi \, g(\vec{x} - t\vec{b}) + e^{\phi(x)} Dg] +$$

$$+ c [e^{\phi(x)} g(\vec{x} - t\vec{b})] = 0$$

$$\Rightarrow Dg \circ (-\vec{b}) + \vec{b} \circ [D\phi \, g(\vec{x} - t\vec{b}) + Dg] + c \, g(\vec{x} - t\vec{b}) = 0$$

$$\Rightarrow g(\vec{x} - t\vec{b}) \cdot [\vec{b} \circ D\phi + c] = 0$$

$$\vec{b} \circ D\phi + c = 0$$

$$(b_1, \ldots, b_n) \circ (\phi_{x_1}, \ldots, \phi_{x_n}) + c = 0$$

$$b_1 \phi_{x_1} + b_2 \phi_{x_2} + \cdots + b_n \phi_{x_n} + c = 0$$

$$b_1 \phi_{x_1} + \cdots + b_n \phi_{x_n} = -c$$

let $\phi(\vec{x}) = \dfrac{A_1 x_1}{b_1} + \dfrac{A_2 x_2}{b_2} + \cdots + \dfrac{A_n x_n}{b_n}$

Then

$$b_1 \phi_{x_1} + b_2 \phi_{x_2} + \cdots + b_n \phi_{x_n}$$

$$= A_1 + A_2 + \cdots + A_n = -c$$

$\therefore$ pick n values $A_1, \ldots, A_n$ s.t $\sum_{k=1}^{n} A_k = -c$

So, a solution to the D.E is

$$u(x,t) = e^{\left(\frac{A_1 x_1}{b_1} + \cdots + \frac{A_n x_n}{b_n}\right)} \cdot G(\vec{x} - t\vec{b}) = e^{\sum \frac{A_k}{b_k} x_k} G(\vec{x} - t\vec{b})$$

but $U(x,0) = e^{\sum \frac{A_k}{b_k} x_k} G(x) = g(x)$

$\Rightarrow G(x) = g(x) e^{-\sum \frac{A_k}{b_k} x_k}$

$\therefore U(x,t) = e^{\sum \frac{A_k}{b_k} x_k} \; e^{-\sum \frac{A_k}{b_k}(x_k - t b_k)} \; g(\vec{x} - t\vec{b})$

$$u(x,t) = g(\vec{x} - t\vec{b}) e^{\sum_{k=1}^{n} A_k t}$$

$$= g(\vec{x} - t\vec{b}) e^{-tc} = e^{-tc} g(\vec{x} - t\vec{b})$$

Check:

$u(x,0) = g(x)$ ✓

$u_t = -c e^{-ct} g(\vec{x} - t\vec{b}) + e^{-ct} Dg \circ (-\vec{b})$

$+ \; b \circ Du = e^{-ct} b \circ Dg = e^{-ct} b \circ Dg$

$+ \; cu = c e^{-ct} g(\vec{x} - t\vec{b})$

$$0$$

✓

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#2) $\Delta u = 0$

A $n \times n$ orthogonal matrix

$$V(\vec{y}) = u(A\vec{y})$$

$\nabla^2 u = 0$

$V(\vec{y}) = u(A\vec{y})$

a_{ji} $\quad$ δ_{ij}

$A^T A = I$ ✓

Want to show $\nabla^2 V = 0$

$D^k u := \{ D^\alpha u(x) \mid |\vec{\alpha}| = k \}$

$|\vec{\alpha}| = \alpha_1 + \dots + \alpha_n$

$D^1 u = \{ \partial x_1 u, \partial x_2 u, \dots, \partial x_n u \}$

$\Delta V = \operatorname{div}(DV) = \nabla \circ (DV)$

$= \partial^2_{x_1} V + \partial^2_{x_2} V + \dots + \partial^2_{x_n} V = \sum_{i=1}^{n} V_{x_i x_i}$

$V_{y_i} = \partial_{y_i} u(A\vec{y}) = \partial_{y_i} u \left(A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right)$

Note: $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} = x$

$\partial_{y_i} u = \begin{bmatrix} a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ \vdots \\ a_{n1}y_1 + a_{n2}y_2 + \dots + a_{nn}y_n \end{bmatrix}$

$V_{y_i} = \partial x_1 u \underbrace{\partial_{y_i} x_1}_{a_{1i}} + \partial x_2 u \underbrace{\partial_{y_i} x_2}_{a_{2i}} + \dots + \partial_{x_n} u \underbrace{\partial_{y_i} x_n}_{a_{ni}}$ ✓

$= \partial_{x_1} u \, a_{1i} + \partial x_2 u \, a_{2i} + \dots + \partial_{x_n} u \, a_{ni}$

$V_{y_i y_i} = \partial_{y_i}(\quad) + \partial_{y_i}(\quad) + \dots + \partial_{y_i}(\quad)$

$= a_{1i} \partial_{y_i}(\partial_{x_1} u)$

$= a_{1i} \left[\partial_{x_1}(\partial_{x_1} u) a_{1i} + \partial_{x_2}(\partial x_1 u) a_{2i} + \dots + \partial_{x_n}(\partial_{x_1} u) a_{ni} \right] +$

$$+ a_{2i}\left[\partial_{x_1}(\partial_{x_2}u)\, a_{1i} + \partial_{x_2}(\partial_{x_2}u)\, a_{2i} + \dots + \partial_{x_n}(\partial_{x_n}u)\, a_{ni} \right] +$$

$$\vdots$$

$$+ a_{ni}\left[\partial_{x_1}(\partial_{x_n}u)\, a_{1i} + \partial_{x_2}(\partial_{x_n}u)\, a_{2i} + \dots + \partial_{x_n}(\partial_{x_n}u)\, a_{ni} \right]$$

$$= \partial_{x_1}^2 u \; a_{1i}^2 + \partial_{x_1 x_2}\, a_{1i}\, a_{2i} + \dots + \partial_{x_1 x_n} u \;\; a_{1i}\, a_{ni}$$

$$V_{y_i y_i} = a_{1i} \sum_{j=1}^{n} a_{ji}\, \partial_j(\partial_1 u) + a_{2i} \sum_{j=1}^{n} a_{ji}\, \partial_j \partial_2 u + \dots$$

$$\dots + a_{ni} \sum_{j=1}^{n} a_{ji}\, \partial_j(\partial_n u)$$

$$= \sum_{j=1}^{n} a_{ji}\left[a_{1i}\, \partial_{j1} u + a_{2i}\, \partial_{j2} u + \dots + a_{ni}\, \partial_{jn} u \right] =$$

$$= \sum_{j=1}^{n} a_{ji} \sum_{k=1}^{n} a_{ki}\, \partial_{jk} u = \boxed{\sum_{j,k=1}^{n} (\partial_{jk} u)\, a_{ji}\, a_{ki}}$$

$$\therefore\; V_{y_i y_i} = \sum_{j,k=1}^{n} (\partial_{jk} u)\, a_{ji}\, a_{ki}$$

So,

$$\Delta V_{y_i y_i} = \sum_{j,k=1}^{n} (\partial_{jk} u) \sum_{i=1}^{n} a_{ji}\, a_{ki}$$

$$= \sum_{j,k=1} (\partial_{jk} u)\, \delta_{jk}$$

$$= \sum_{k=1}^{n} (\partial_{kk} u) = 0$$

<u>Note</u>: $\underbrace{\sum_{i=1}^{n} a_{ji}\, a_{ki}}$

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

#2.) Prove that Laplace's equation

$$\Delta u = 0$$

is rotation invariant; that is, if A is an orthogonal $n \times n$ matrix and we define

$$v(\vec{y}) = u(A\vec{y})$$

then $\Delta v = 0$

$Dv = A^T Du$ ✓ $\qquad Dv := (\partial_{y_1} v, \partial_{y_2} v, \ldots, \partial_{y_n} v)$

$D \cdot Dv = D \cdot (A^T Du)$

$D(u|_{A\vec{y}}) = A^T (Du)|_{A\vec{y}}$

LEMMA (1): $Dv = A^T Du$

LEMMA (2): $D \cdot (\underbrace{C}_{\text{constant}} \underbrace{\vec{f}}_{\text{function}}) = tr\left(\begin{bmatrix} -Df_1- \\ \vdots \\ -Df_n- \end{bmatrix} \begin{bmatrix} C \end{bmatrix}\right) = tr\ C^T \begin{bmatrix} | & & | \\ Df_1 & \cdots & Df_n \\ | & & | \end{bmatrix}$

$$Du := \begin{bmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_n} u \end{bmatrix} = \begin{bmatrix} u_{x_1}(A\vec{y}) \\ \vdots \\ u_{x_n}(A\vec{y}) \end{bmatrix}$$

Now, $D^2 v = D \cdot Dv = D \cdot (A^T Du|_{A\vec{y}}) =$

$$= tr\left(\begin{bmatrix} -Du_{x_1}- \\ \vdots \\ -Du_{x_n}- \end{bmatrix} A^T\right) =$$

$$= tr\left(\begin{bmatrix} (Du_{x_1}) \cdot A \\ \vdots \\ (Du_{x_n}) \cdot A \end{bmatrix} A^T\right) =$$

$$= tr\begin{bmatrix} -Du_{x_1}- \\ \vdots \\ -Du_{x_n}- \end{bmatrix} = \sum_{i=1}^{n} u_{x_i x_i} = 0$$

✓

$$B\vec{f} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \qquad D\begin{bmatrix} b_1 f_1 + b_2 f_2 \\ b_3 f_1 + b_4 f_2 \end{bmatrix}$$

$$Dg = \begin{bmatrix} \partial_{x_1} g \\ \vdots \\ \partial_{x_n} g \end{bmatrix} \qquad D\cdot\vec{g} \overset{\text{definition}}{=} \partial_{x_1} g_1 + \partial_{x_2} g_2 + \dots + \partial_{x_n} g_n \longrightarrow tr(D\vec{g})$$

$$D\vec{g} = \begin{bmatrix} | & & | \\ Dg_1 & \dots & Dg_n \\ | & & | \end{bmatrix}$$

$$\vec{g} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

$V = u(Ay) \qquad \vec{V} = \vec{u}(Ay)$

RULE(1): $DV = A^T Du\big|_{Ay} \qquad D\vec{V} = A^T D\vec{u}\big|_{Ay}$

const.

RULE(2): $D\cdot(B\vec{f}) = tr(Df\cdot B^T)$

RULE(2)': $D(B\vec{f}) = D\vec{f}\cdot B^T$

PROOF OF RULE(2) FROM RULE(2)':

$$D\cdot(B\vec{f}) \overset{def.}{=} tr(DB\vec{f}) \overset{(2)'}{=} tr(B^T D\vec{f})$$

$$D\cdot DV = D\cdot\left(A^T Du\big|_{Ay}\right) = tr\left(D(Du)\big|_{Ay} A\right) = tr\left(A^T D(Du)_{Ay} A\right) =$$

$$= tr(A^{-1} DDu A) = tr(DDu)_{Ay} = \sum_{i=1}^{n} u_{x_i x_i}\big|_{Ay} = 0$$

$$Du = \begin{bmatrix} u_{y_1} \\ \vdots \\ u_{y_n} \end{bmatrix}$$

$$DV\big|_{\vec{y}} = D\,u(Ay) = \begin{bmatrix} \partial_{y_1}(u(Ay)) \\ \vdots \\ \partial_{y_n}(u(Ay)) \end{bmatrix} = \begin{bmatrix} c_1\cdot Du \\ \vdots \\ c_n\cdot Du \end{bmatrix} = \begin{bmatrix} -c_1- \\ \vdots \\ -c_n- \end{bmatrix}\begin{bmatrix} Du \end{bmatrix} = \boxed{A^T Du}$$

$$Ay = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_n & - \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{y} \\ \vdots \\ \vec{r}_n \cdot \vec{y} \end{bmatrix}$$

$\partial_{y_i} u(Ay)$

$$\partial_{y_i} u(\vec{r}_1 \cdot \vec{y}, \vec{r}_2 \cdot \vec{y}, \dots, \vec{r}_n \cdot \vec{y}) = u(a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n, a_{21}y_1 + \dots + a_{2n}y_n, \dots, a_{n1}y_1 + \dots + a_{nn}y_n)$$

$$(\partial_1 u) \cdot \partial_{y_i}(\vec{r}_1 \cdot \vec{y}) + (\partial_2 u)\, \partial_{y_i}(\vec{r}_2 \cdot \vec{y}) + \dots + (\partial_n u)\, \partial_{y_i}(\vec{r}_n \cdot \vec{y})$$

$$\partial_1 u \cdot a_{1i} + \partial_2 u\, a_{2i} + \dots + \partial_n u\, a_{ni}$$

$$[a_{1i} \quad a_{2i} \quad \dots \quad a_{ni}] \cdot Du$$

Rule 1 ✓

$$A = \begin{bmatrix} | \\ c_i \\ | \end{bmatrix} \qquad c_i \cdot Du$$

$D\vec{f}$ $\qquad f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$

$$\begin{bmatrix} | & | & & | \\ Df_1 & Df_2 & \dots & Df_n \\ | & | & & | \end{bmatrix}$$

$$\begin{bmatrix} \partial_{x_1} f_1 & \dots & \partial_{x_1} f_n \\ \partial_{x_2} f_1 & \dots & \partial_{x_2} f_n \\ \vdots & & \\ \partial_{x_n} f_1 & \dots & \partial_{x_n} f_n \end{bmatrix}$$

$$DDu = D\begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \end{bmatrix} = \begin{bmatrix} u_{x_1x_1} & u_{x_2x_1} & \dots & u_{x_nx_1} \\ u_{x_1x_2} & u_{x_2x_2} & \dots & \vdots \\ \vdots & & \ddots & \vdots \\ u_{x_1x_n} & u_{x_2x_n} & \dots & u_{x_nx_n} \end{bmatrix}$$

$\Delta u = D \cdot Du = \mathrm{tr}(DDu)$

$$D\breve{f}(Ay) = \begin{bmatrix} | & & | \\ Df_1(Ay) & \cdots & Df_n(Ay) \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ A^T(Df_1)\big|_{Ay} & \cdots & A^T(Df_n)\big|_{Ay} \\ | & & | \end{bmatrix}$$

$$= A^T \begin{bmatrix} | & & | \\ Df_1\big|_{Ay} & \cdots & Df_n\big|_{Ay} \\ | & & | \end{bmatrix}$$

$$D\breve{f}(A\breve{y}) = \left(A^T D\breve{f}\big|_{A\breve{y}}\right)$$

Also $\underline{\text{Need this}}$ $\boxed{D(\breve{a} \cdot \vec{f}) = [D\breve{f}]\, a^T}$ for Rule (2)'

$$D([-a-][\breve{f}]) = \begin{bmatrix} D\breve{f} \end{bmatrix} \begin{bmatrix} | \\ a \\ | \end{bmatrix}$$

$D(B\breve{f}) =$

$$= D \begin{bmatrix} b_{ij} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} =$$

$$= D \begin{bmatrix} - b_1 - \\ \vdots \\ - b_n - \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = D \begin{bmatrix} \breve{b}_1 \cdot f \\ \vdots \\ \breve{b}_n \cdot f \end{bmatrix} = \begin{bmatrix} | & & | \\ D(\breve{b}_1 \cdot \breve{f}) & \cdots & D(\breve{b}_n \cdot \breve{f}) \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ D\breve{f}(\breve{b}_1^{\,T}) & \cdots & D\breve{f}(\breve{b}_n^{\,T}) \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ D\breve{f}(\breve{b}_1^T) & \cdots & D\breve{f}(\breve{b}_n^T) \\ | & & | \end{bmatrix}$$

$$= \underbrace{\left[D\breve{f} \right]}_{n\times n} \begin{bmatrix} | & & | \\ \breve{b}_1^T & \cdots & \breve{b}_n^T \\ | & & | \end{bmatrix} \Rightarrow^{(2')} \quad D(B\breve{f}) = \left[D\breve{f} \right] B^T$$

$$= \underbrace{\left[Df \right]}_{n\times n} \left[B \right]^T$$

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#3.) Using the Green's function to calculate the solution of the following BVP and plot the solution

$$-\Delta u = 4 \quad \text{in} \quad B(0,2)$$
$$u = 3+x \quad \text{on} \quad \partial B(0,2)$$

$-\Delta u = 4$ on $B(0,2)$

$u = 3+x$ on $\partial B(0,2)$

know solution on $B(0,r)$

$-\nabla^2 u = 4$

$(-\partial_x^2 + \partial_y^2) U = 4$

$-\partial_x^2 V = 4$

$V(x) = -\frac{4x^2}{2} = -2x^2$

$-(\partial_x^2 + \partial_y^2) U = -(\partial_x^2 + \partial_y^2)(U + 2x^2) = 0$

$-\nabla^2 U - 4 = 0$

let $V = U + 2x^2$

then

$-\nabla^2 U = 4 \iff \boxed{-\nabla^2 V = 0}$

$-\nabla^2 (U + f(x)) = 0$

$\iff -\nabla^2 U - \nabla^2 f = 0$

$\iff -\nabla^2 U = \nabla^2 f(x) = 4$

$\hookrightarrow \frac{\partial^2 f}{\partial x^2} = 4$

$f(x) = 2x^2, \; 4x, \; 4$

Know $\left. U(x,y) \right|_{\partial B(0,2)} = 3+x$

$\therefore \left. V(x,y) \right|_{\partial B(0,2)} \; = \; 3 + x + 2x^2$

∴ Equation becomes

$-\nabla^2 V = 0$ on $B(0,2)$

$V = 3 + x + 2x^2$ on $\partial B(0,2)$

Sol (45) w/ $\alpha(2)$ =

∴ $V(x) = \dfrac{4-|x|^2}{2\pi\cdot 2}\displaystyle\int_{\partial B(0,2)} \dfrac{3+y_1+\cancel{2y_1^2}}{|\vec{x}-\vec{y}|^2}\, r_1\, d\theta$

$g(y_1,y_2) = 3+y_1 \cancel{+2y_1^2}$

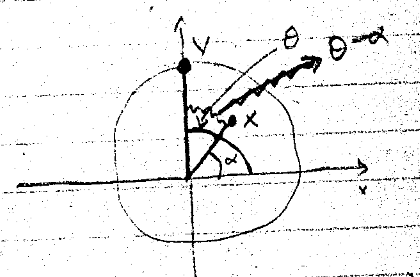
$y_1 = r\cos(\theta) = 2\cos(\theta)$

$V(x_1,x_2) = \dfrac{4-|x|^2}{2\pi\cdot\cancel{2}}\cdot\cancel{2}\displaystyle\int_0^{2\pi}\left(3+2\cos(\theta)+8\cos^2(\theta)\right)d\theta$

$|\vec{x}-\vec{y}|^2 = (x_1-y_1)^2+(x_2-y_2)^2$

∴ $V(x_1,x_2) = \dfrac{4-x_1^2-x_2^2}{2\pi}\displaystyle\int_0^{2\pi}\dfrac{3+2\cos(\theta)+\cancel{8\cos^2(\theta)}}{(x_1-2\cos(\theta))^2+(x_2-2\sin(\theta))^2}\,d\theta$

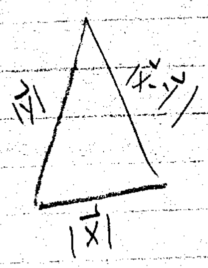
∴ $U = V - 2x^2 = V - 2\rho^2\cos^2(\alpha)$



Note:

$\dfrac{4-|x|^2}{2\pi\cdot 2}\displaystyle\int_0^{2\pi}\dfrac{3+y_1+2y_1^2}{|\vec{x}-\vec{y}|^2}\, r_1\, d\theta$

$\dfrac{4-\rho^2}{2\pi\cdot 2}\displaystyle\int_0^{2\pi}\dfrac{3+y_1+2y_1^2}{4+\rho^2-4\rho\cos(\theta-\alpha)}\cdot 2\, d\theta$



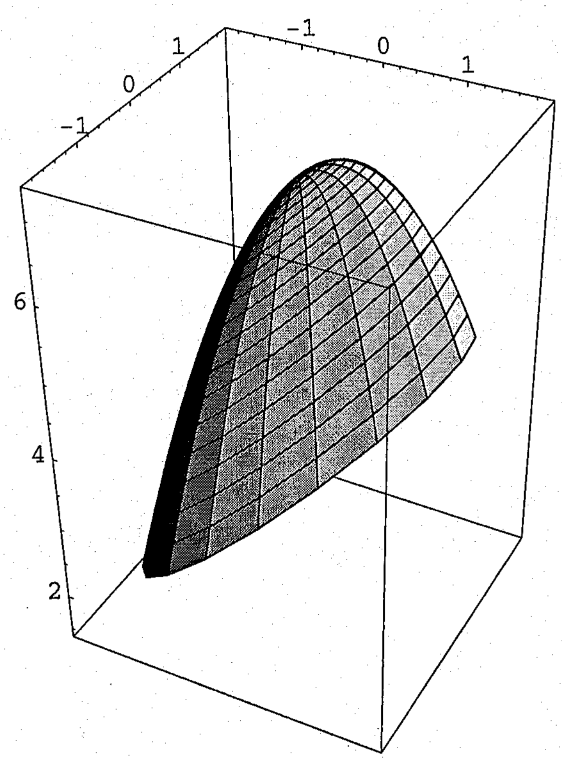
$|\vec{x}-\vec{y}|^2 = |x|^2+|y|^2-2|x||y|\cos(\theta-\alpha)$

Homework 1

■ Solution to Poisson Problem.

In[60]:=
$$v[\rho\_, \alpha\_] := \frac{(4-\rho^2)}{2\pi} \text{NIntegrate}\Big[\text{Evaluate}\Big[\text{SetPrecision}\Big[\frac{(3 + 2\,\text{Cos}[\theta] + 8\,(\text{Cos}[\theta])^2)}{(\rho\,\text{Cos}[\alpha] - 2\,\text{Cos}[\theta])^2 + (\rho\,\text{Sin}[\alpha] - 2\,\text{Sin}[\theta])^2}, 18\Big]\Big], \{\theta, 0, 2\pi\}\Big]$$

```
ParametricPlot3D[
 {ρ Cos[α], ρ Sin[α], v[ρ, α] - 2 ρ^2 (Cos[α])^2},
 {ρ, 0, 1.9}, {α, 0, 2 π}];
```



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$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3. \end{cases} \qquad |x|^{-n+2}$$

$$|D\Phi| \leq \begin{cases} C\frac{1}{|x|} & n=2 \\ C\cancel{\frac{1}{|x|^{n-3}}}\frac{1}{|x|^{n-1}} & n \geq 3 \end{cases} \qquad x^p \to px^{p-1}$$

$$|D\Phi| \leq \frac{C}{|x|^{n-1}}$$

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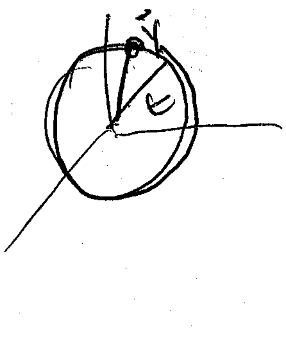
$$\left| \int_{\partial B(0,\epsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y) \right| \qquad \frac{\partial f}{\partial \nu} := \nabla_x f \cdot \nu$$

$$\le \|Df\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\epsilon)} |\Phi(y)|\, dS(y)$$

$$|\Phi(y)| = \begin{cases} C_1 \,|\log|y|| & n = 2 \\ C_2 \dfrac{1}{|y|^{n-2}} & n \geq 3. \end{cases}$$

$n \geq 3$.

$$\int_{\partial B(0,\epsilon)} \frac{1}{|y|^{n-2}} dS(y) = \frac{1}{\epsilon^{n-2}} \int_{\partial B(0,\epsilon)} dS(y) = \frac{1}{\epsilon^{n-2}} n\alpha(n)\epsilon^{n-1} = n\alpha(n)\epsilon.$$



$$\alpha(n) := \int_{B(0,1)} dy \qquad \int_{\partial B(0,\epsilon)} dS(y) = n\alpha(n)\epsilon^{n-1}$$

$$\alpha(n)\epsilon^n = \int_{B(0,\epsilon)} dy.$$

$n=2$

$$\left| \int_{\partial B(0,\epsilon)} \big|\log|y|\big| \, dS(y) \right| = |\log \epsilon| \int_{\partial B(0,\epsilon)} dS(y) = |\log \epsilon| \, n\alpha(n)\epsilon^{n-1}$$

$$= |\log \epsilon| \, 2\alpha(2)\epsilon^{1} = C_1 |\log \epsilon| \epsilon$$

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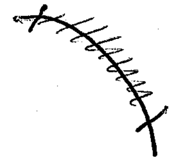
$$\phi(r) := \frac{1}{n r^{n-1}\alpha(n)} \int_{\partial B(x,r)} u(y)\, dS(y)$$

$\cancel{dS(y) = nr^{n-1}\alpha(n)\, d\Omega(y)}$

let $y = x + rz$

$$dS(y) = dS(z)\, r^{n-1}$$

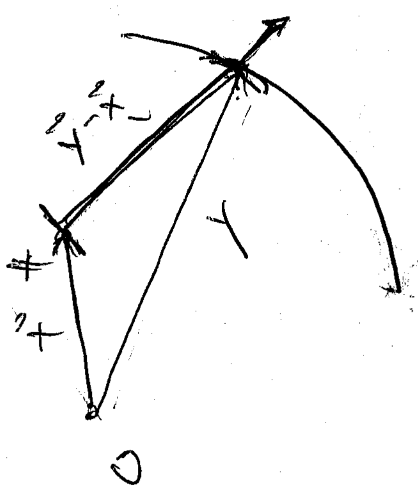


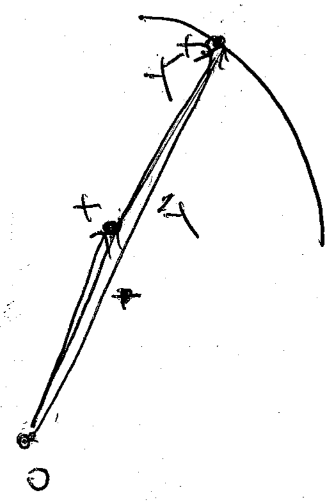
$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

$u(\underbrace{x_1+rz_1}, \underbrace{x_2+rz_2}, \dots, x_n+rz_n)$

$$\phi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz)\cdot z\, dS(z)$$

$$= \frac{1}{n r^{n-1}\alpha(n)} \int_{\partial B(x,r)} Du(y)\cdot\left(\frac{y-x}{r}\right) dS(y)$$





(2) $y'' - xy' - y = 0$

$x = 0$ ordinary pt $\quad y = \sum_{k=0}^{\infty} a_k x^k$

$$\sum_{k=2}^{\infty} a_k (k)(k-1) x^{k-2} - \sum_{k=1}^{\infty} a_k k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - \sum_{k=1,0}^{\infty} a_k k x^k - \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} \left(a_{k+2}(k+2)(k+1) - a_k(k+1) \right) x^k + a_2 = 0$$

$\therefore \quad a_{k+2}(k+1)(k+2) = a_k(k+1) \qquad k = 0, 1, 2, \ldots$

$$a_{k+2} = \frac{a_k}{(k+2)}$$

$$= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \, dS(y)$$

$$= \frac{r}{n\,\alpha(n) r^{n}} \int_{B(x,r)} \Delta u \, dy$$

$$= \frac{r}{n} \fint_{B(x,r)} \Delta u \, dy \qquad \Delta u(\cancel{x})$$

$$\lim_{r \to 0^+} U_r = 0.$$

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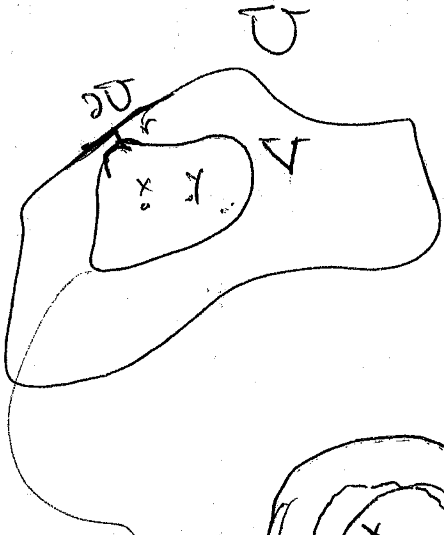
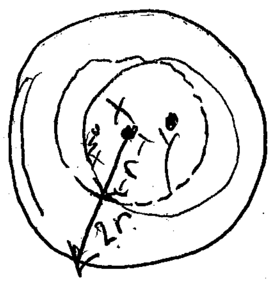


$$r = \frac{1}{4}\operatorname{dist}(V, \partial U)$$

$$u(x) = \fint_{B(x,2r)} u\,dz = \frac{1}{\alpha(n)2^n r^n}\int_{B(x,2r)} u\,dz$$

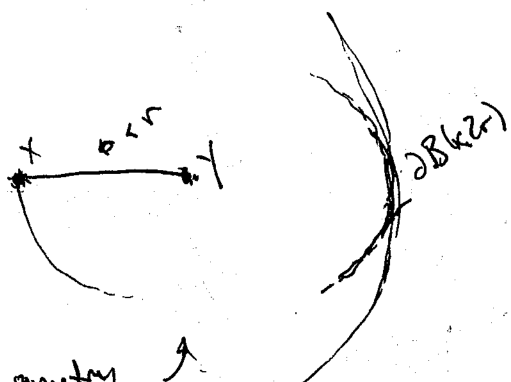


$$\geq \frac{1}{\alpha(n)2^n r^n}\int_{B(y,r)} u\,dz$$

if

$$B(y,r) \subset B(x,2r)$$

$$= \frac{1}{2^n}\frac{1}{\alpha(n) r^n}\int_{B(y,r)} u\,dz$$



$$= \frac{1}{2^n}\fint_{B(y,r)} u\,dz = \frac{1}{2^n}u(y)$$

Yes By geometry

$$u(x) \geq \frac{1}{2^n}u(y)$$

$u(y) \geq \frac{1}{2^n}u(x)$ By switching x & y in above proof.

$\frac{1}{2^n}u(y) \leq u(x) \leq 2^n u(y)$ by combining 2 inequalities

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1

If $u \in B(0,r)$

Then $\hat{u} \in B(0,1)$ & if $\Delta \hat{u} = 0$ on $B(0,1)$ (*)

w/ $\hat{u} = \hat{g}$ on $\partial B(0,1)$

Then $\Delta u = 0$ on $B(0,r)$

$\Longleftrightarrow$ $u = g$ on $\partial B(0,r)$

The sol to (*) By Poisson's int. formula is

$$\hat{u}(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{\hat{g}(y)}{|x-y|^n} \, dS(y)$$

$$u(xr) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(ry)}{|x-y|^n} \, dS(y)$$

$v = ry \Rightarrow y = \frac{v}{r}$

$dS(y) = \frac{dS(v)}{r^{n-1}}$

$$u(x) = \frac{1-\frac{|x|^2}{r^2}}{n\alpha(n)} \int_{\partial B(0,r)} \frac{g(v)}{|\frac{x}{r}-\frac{v}{r}|^n} \frac{dS(v)}{r^{n-1}}$$

$$= \frac{r^2-|x|^2}{r^2 n\alpha(n) \frac{1}{r^n} r^{n-1}} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) = \frac{r^2-|x|^2}{n\alpha(n) r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS$$

eq (45)

Our problem is $\qquad \Delta u = f \quad B(0,r)$

$\qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad \partial B(0,r)$

If $\quad u \in B(0,r) \qquad \hat{u} = u(rx)$

$\qquad \hat{u} \in B(0,1) \qquad \hat{f} = r^2 f(rx)$

Then $\hat{u}$ satisfies $\quad \Delta \tilde{u} = \cancel{r^2} \Delta u = \tilde{f} = \cancel{r^2} f(rx)$

$$\Delta u = f(rx)$$

$$+ \quad \underset{\partial B(0,1)}{\tilde{u}|} = \underset{\partial B(0,r)}{u|} = 0.$$

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$u(0,t) = -u(0,t)$

$\Rightarrow u(0,t) \equiv 0.$

~~$\tilde{v}(x,t) = \frac{1}{2}[$~~

$$u(x,t) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \cdots & \underline{x-t} > 0 \Rightarrow \underline{x > t} \\ \frac{1}{2}\left(g(x+t) - g(t-x)\right) + \cdots & \underline{x-t} \le 0. \end{cases}$$

$$= \begin{cases} + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \\ + \frac{1}{2}\left[\int_{x-t}^{0} -h(-y)\,dy + \int_{0}^{x+t} h(y)\,dy\right] & x \le t \end{cases}$$

2nd integral $\Rightarrow$ $v = -y$

$dv = -dy$

$$= \frac{1}{2}\left[\int_{t-x}^{0} h(v)\,dv + \int_{0}^{x+t} h(y)\,dy\right]$$

$$= \frac{1}{2}\left[\int_{-x+t}^{x+t} h(y)\,dy\right]$$

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$$r^{n-1} U_r = \frac{1}{n\alpha(n)} \int_{B(x,r)} \Delta u(y)\,dy$$

$$\frac{\partial}{\partial r}\left(r^{n-1} U_r\right) = \frac{1}{n\alpha(n)} \frac{\partial}{\partial r}\left[\int_0^r \int_{\partial B(x,\rho)} \Delta u(y)\,dS(y)\,d\rho\right]$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} \Delta u(y)\,dS(y)$$

If $u_{tt} - \Delta u = 0$.

$$\therefore \frac{\partial}{\partial r}\left(r^{n-1} U_r\right) = \frac{1}{n\alpha(n)} \int_{\partial B(x,r)} u_{tt}\,dS(y)$$

$$= \frac{r^{n-1}}{n\alpha(n)\,r^{n-1}} \int_{\partial B(x,r)} u_{tt}\,dS(y)$$

$$= r^{n-1} ⨍_{\partial B(x,r)} u_{tt}\,dS(y) = r^{n-1}\, U_{tt}.$$

$$(n-1)\cancel{r^{n-2}}^{\,-1} U_r + \cancel{r^{n-1}}\, U_{rr} = \cancel{r^{n-1}}\, U_{tt}$$

$$\Rightarrow U_{tt} = U_{rr} + \frac{(n-1)}{r} U_r$$

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10.15

$$U(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y)$$

$$\tilde{U} = rU \Rightarrow U = \frac{\tilde{U}}{r}$$

$$u(x,t) = \lim_{r\to 0^+} U = \lim_{r\to 0^+} \frac{\tilde{U}}{r}$$

$r+t-(-r+t) = 2r$

$$= \lim_{r\to 0^+}\left[\frac{1}{2}\,\frac{\tilde{G}(r+t)-\tilde{G}(t-r)}{r} + \frac{1}{2r}\int_{-r+t}^{r+t}\tilde{H}(y)\,dy\right]$$

$$= \tilde{G}'(t) + \tilde{H}(t)$$

$$\Rightarrow u(x,t) = \frac{\partial}{\partial t}\left(r \cancel{\fint_{\partial B(x,r)}} g(y)\,dS(y)\right) + r\cancel{\fint_{\partial B(x,r)}} h(y)\,dS(y)$$

$\cancel{\tilde{G}(t) = r\tilde{U}(}$

$$u(x,t) = \frac{\partial}{\partial t}\left(t\fint_{\partial B(x,t)} g(y)\,dS(y)\right) + t\fint_{\partial B(x,t)} h(y)\,dS(y).$$

$$\fint_{\partial B(x,t)} g(y)\,dS(y) = \fint_{\partial B(0,1)} g(x+zt)\,dS(z)$$

let $y = x+zt$

$$dS(y) = t^{3-1}\,dS(z) = t^2\,dS(z)$$

But $\displaystyle ⨍_{\partial B(x,t)} dS(y) = \underset{\substack{| \\ n=3}}{\frac{1}{n\alpha(n)t^{n-1}}} \int_{\partial B(x,t)} \cdots \, dS(y)$

$$= \frac{1}{3\alpha(3)t^2} \int_{\partial B(x,t)} \cdots \, dS(y) \qquad \text{let } y = x + zt$$

$$= \frac{1}{3\alpha(3)} \int_{\partial B(0,1)} \cdots \, dS(z) = ⨍_{\partial B(0,1)} \cdots \, dS(z).$$

$\therefore$

$$\frac{\partial}{\partial t} ⨍_{\partial B(x,t)} g(y)\, dS(y) = \frac{\partial}{\partial t} ⨍_{\partial B(0,1)} g(x+zt)\, dS(z)$$

$$= ⨍_{\partial B(0,1)} Dg(x+zt) \cdot z \, dS(z)$$

$$= ⨍_{\partial B(x,t)} Dg(y) \cdot \left(\frac{y-x}{t}\right) dS(y)$$

$\therefore$ (21)

$$\Rightarrow u(x,t) = ⨍_{\partial B(x,t)} g(y)\, dS(y) + ⨍_{\partial B(x,t)} Dg(y) \cdot \left(\frac{y-x}{t}\right) dS(y) + t ⨍_{\partial B(x,t)} h \, dS$$

$$u(x,t) = \fint_{\partial B(x,t)} \left(t h(y) + g(y) + Dg(y) \cdot \frac{(y-x)}{t} \right) dS(y)$$

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$$u_{tt} - \Delta u = 0 \qquad \mathbb{R}^2 \times (0,\infty)$$
$$u = g \quad u_t = h \qquad \text{on } \mathbb{R}^2 \times \{t=0\}.$$

$$\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$$

$$\bar{u}_{tt} = u_{tt}$$

$$\Delta_3 \bar{u} = \Delta_2 u.$$

$$(21) \Rightarrow \bar{u}(x,t) = \frac{\partial}{\partial t}\left(t \fint_{\partial \bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} \right) + t \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S}$$

$$u(x,t) =$$

$$r(y) = (t^2 - |y-x|^2)^{1/2} \qquad y = (y_1, y_2)$$

$$Dr(y) = \frac{1}{2}(t^2 - |y-x|^2)^{-1/2} \cdot \underbrace{D_y(-|y-x|^2)}_{D_y(-(y_1-x_1)^2 - (y_2-x_2)^2)}$$

$$= \frac{1}{2\sqrt{t^2 - |y-x|^2}} \, (-2(y_1-x_1), -2(y_2-x_2))$$

$$= -\frac{(y_1-x_1, y_2-x_2)}{\sqrt{t^2 - |y-x|^2}}$$

$$\therefore \quad (1+|Dr|^2)^{1/2} = \left(1 + \frac{(y_1-x_1)^2 + (y_2-x_2)^2}{t^2-|y-x|^2}\right)^{1/2}$$

$$= \left(1 + \frac{|y-x|^2}{t^2-|y-x|^2}\right)^{1/2}$$

$$= \left(\frac{t^2 - \cancel{|y-x|^2} + \cancel{|y-x|^2}}{t^2-|y-x|^2}\right)^{1/2} = \quad t(t^2-|y-x|^2)^{-1/2}$$

$$\therefore \quad \int_{\partial\bar{B}(\bar{x},t)} \bar{g}\, d\bar{S} = \frac{1}{2\pi t^2} \cdot t \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\, dy$$

"n" $= 2$

$\alpha(1) =$

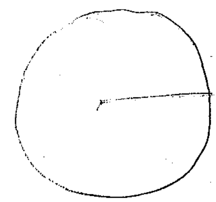
$$= \frac{1}{2\pi t} \cdot \alpha(n) r^n \Big|_{\substack{n=2 \\ r=t}} \int_{B(x,t)} \frac{g(y)}{(\quad)^{1/2}}\, dy.$$

$\alpha(2) = \pi$

$$= \frac{t}{2} \int_{B(x,t)} \frac{g(y)}{}$$

$$u(x,t) = \frac{1}{2}\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy \right)$$

$$+ \frac{t^2}{2} \fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,dy$$



Now:

$$t^2 \fint_{\substack{B(x,t)\\ \text{in } \mathbb{R}^2}} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy = \frac{t^2}{\pi t^2} \int_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy.$$

let $y = x + tz$

$dy = t^2\,dz = t^2\,dz$

on $B(x,t)$

$$= \frac{t^2}{\pi} \int_{B(0,1)} \frac{g(x+tz)}{(t^2-t^2|z|^2)^{1/2}}\,dz = \frac{t^2}{\pi t} \int_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz = \frac{t}{\pi} \int_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

$$= t \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz$$

$\therefore$

$$\frac{\partial}{\partial t}\left(t^2 \fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,dy \right) = \frac{\partial}{\partial t}\left(t \fint_{B(0,1)} \frac{g(x+tz)}{(1+|z|^2)^{1/2}}\,dz \right)$$

$$= \fint_{B(0,1)} \frac{g(x+tz)}{(1-|z|^2)^{1/2}}\,dz + t \fint_{B(0,1)} \frac{Dg(x+tz)\cdot z}{(1-|z|^2)^{1/2}}\,dz.$$

$$= \frac{1}{\pi} \int_{B(0,1)} \cdots \quad +$$

$$= \frac{t^2}{\pi t^2} \int_{B(x,t)} \frac{g(y)}{\left(1 - \frac{|y-x|^2}{t^2}\right)^{1/2}} \frac{dy}{t^2} + \frac{t\, t^2}{\pi t^2} \int_{B(x,t)} \frac{Dg(y) \cdot \left(\frac{y-x}{t}\right)}{\left(1 - \frac{|y-x|^2}{t^2}\right)^{1/2}} \frac{dy}{t^2}$$

$$y = x + tz \qquad dz = \frac{1}{t^2}\, dy$$

$$z = \frac{y-x}{t}$$

$$= \frac{t^2}{\pi t^2 \cdot t} \int_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy + \frac{t \cdot t^2}{\pi t^2 t^2} \int \frac{Dg(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy.$$

$$= t \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\, dy + t \fint_{B(x,t)} \frac{Dg(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{1/2}}\, dy$$

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$g \equiv 0$

$$\Rightarrow u(x,t) = \cancel{\frac{1}{\gamma_n} \frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2} \fint_{\partial B(x,t)} g\,dS\right)}$$

$$= \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-3}{2}}\left(t^{n-2} \underbrace{\fint_{\partial B(x,t)} h\,dS}_{H(x;t)}\right)$$

$$u_{tt} = \frac{d^2}{dt^2}\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-3}{2}}$$

$$\frac{n}{2} - 1 - \frac{1}{2} = \frac{n-1}{2} - 1$$

$$\Rightarrow u_{tt} = \frac{d^2}{dt^2}\frac{1}{\gamma_n}\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-1}{2}-1}\left(t^{2\left(\frac{n-1}{2}\right)-1} H(x;t)\right)$$

$$\overset{\text{Lemma 2 (i)}}{=} \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{d}{dt}\right)^{\frac{n-1}{2}}\left(t^{n-1} H_t(x;t)\right)$$

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(8) $u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$.

Solves $u_{tt} - u_{xx} = 0$

$u = g \quad u_t = h$.

Duhamel's principle $\Rightarrow$ $g = 0. \quad u_t = h(\cdot, s)$

Then to solve $u_{tt} - u_{xx} = 0$

$u = 0 \quad u_t = h(\cdot, s) \qquad \mathbb{R} \times \{t = s\}$

$\Leftrightarrow$ $\cancel{u_{tt} - u_{xx} = 0}$

$\cancel{u = 0 \quad u_t = h(\cdot, t+s)} \qquad \cancel{\mathbb{R} \times \{t = 0\}}$

~~soln is then~~

$\tau = t - s \Rightarrow t = \tau + s$

$d\tau = dt$.

$\cancel{u(x,t) = \frac{1}{2}\int}$

$\therefore$ problem becomes.

$\cancel{u(x,t) = \frac{1}{2}\int_{x-t+s}}$

$u_{\tau\tau} - \Delta u = 0 \qquad \mathbb{R}^n \times (0, \infty)$

$u = 0 \quad u_\tau = f \qquad \mathbb{R}^n \times \{\tau = 0\}$

$\Rightarrow u(x,\tau) = \frac{1}{2}\int_{x-\tau}^{x+\tau} f(y,t)\,dy \Rightarrow u(x,t) = \frac{1}{2}\int_{x-t+s}^{x+t-s} f(y,s)\,dy$

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-\cancel{t+s}}^{x+t-s} f(y,s)\,dy\,ds \qquad \tau = t-s$$

$$= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds \qquad \text{Add } 2s \text{ to } t$$

let $v = t-s \Rightarrow s = t-v$

$dv = -ds$

$$= \frac{1}{2}\int_0^t \int_{x-v}^{x+v} f(y,t-v)\,dy\,(-dv)$$

$$= \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)\,dy\,ds$$

$n = 3$ $\quad u = 0 \quad u_t = f(\cdot\,;s)$

$\Rightarrow$ $\cancel{U = 0}$ $\quad U_t = H(\cdot\,;s)$

$G = 0$

$$\Rightarrow u(x,t,s) = \fint_{\partial B(x,t)} t h(y)\,dS(y)$$

But starting at $t = s$

$$\Rightarrow u(x,t;s) = (t-s)\fint_{\partial B(x,t-s)} f(y,s)\,dS(y)$$

$$\therefore \quad u(x,t) = \int_0^t (t-s) ⨍_{\partial B(x,t-s)} f(y,s)\, dS(y)\, ds$$

$$= \int_0^t (t-s) \frac{1}{4\pi(t-s)^2} \int_{\partial B(x,t-s)} f(y,s)\, dS(y)\, ds$$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,t-s)} \frac{f(y,s)}{(t-s)}\, dS(y)\, ds.$$

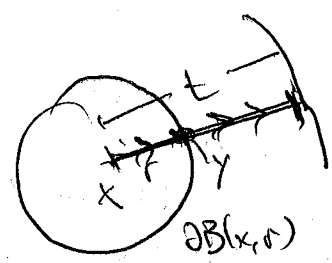
let ~~...~~ $r = t-s \qquad dr = -ds$

$dr = -ds$

$$= \frac{1}{4\pi} \int_0^t \int_{\partial B(x,r)} \frac{f(y,t-r)}{r}\, dS(y)\,(-dr)$$

$r =$

$$\therefore \quad u(x,t) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y, t-|y-x|)}{|y-x|}\, dy^3$$



$r = |x-y|$

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$$\mathring{e}(t) = \int_U u_t(x,t)u_{tt}(x,t) + D(x,t) \cdot Du_t \; dx.$$

$$\partial_t |Dw|^2 = 2$$

$$\partial_t \, Dw \cdot Dw = 2Du_t \cdot Dw$$

$$\mathring{e}(t) = \int_U u_t(u_{tt}) + \int_U (Dw \cdot Du_t)\,dx$$

$$\int_{\partial U} w \frac{\partial w_t}{\partial \nu}\,dS - \int_U w \Delta u_t \,dx$$

Not right way

$$= \cancel{\int_{\partial U} u_t \frac{\partial w}{\partial \nu}\,dS}^{\,0} - \int_U u_t \Delta w \,dx$$

$$\therefore \; \dot{e} = \int_U u_t(u_{tt} - \Delta w)\,dx = 0$$

APPENDIX B: INEQUALITIES

B.1. Convex functions.

Definition. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* convex *provided*

$$f(\tau x + (1-\tau)y) \leq \tau f(x) + (1-\tau)f(y) \tag{1}$$

for all $x, y \in \mathbb{R}^n$ and each $0 \leq \tau \leq 1$.

THEOREM 1 (Supporting hyperplanes). *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is convex. Then for each $x \in \mathbb{R}^n$ there exists $r \in \mathbb{R}^n$ such that the inequality*

$$f(y) \geq f(x) + r \cdot (y - x) \tag{2}$$

holds for all $y \in \mathbb{R}^n$.

Remarks. (i) The mapping $y \mapsto f(x) + r \cdot (y-x)$ determines the *supporting hyperplane* to f at x. Inequality (2) says the graph of f lies above each supporting hyperplane. If f is differentiable at x, $r = Df(x)$.

(ii) If f is C^2, then f is convex if and only if $D^2 f \geq 0$. The C^2 function f is *uniformly convex* if $D^2 f \geq \theta I$ for some constant $\theta > 0$: this means

$$\sum_{i,j=1}^{n} f_{x_i x_j}(x)\xi_i \xi_j \geq \theta|\xi|^2 \quad (x, \xi \in \mathbb{R}^n).$$

□

THEOREM 2 (Jensen's inequality). *Assume $f : \mathbb{R} \to \mathbb{R}$ is convex, and $U \subset \mathbb{R}^n$ is open, bounded. Let $u : U \to \mathbb{R}$ be summable. Then*

$$f\left(⨍_U u\,dx\right) \leq ⨍_U f(u)\,dx. \tag{3}$$

Remember from §A.3 the notation $⨍_U u\,dx = \frac{1}{|U|}\int_U u\,dx$ = *average* of u over U.

Proof. Since f is convex, for each $p \in \mathbb{R}$ there exists $r \in \mathbb{R}$ such that

$$f(q) \geq f(p) + r(q-p) \quad \text{for all } q \in \mathbb{R}.$$

Let $p = ⨍_U u\,dx$, $q = u(x)$:

$$f(u(x)) \geq f\left(⨍_U u\,dx\right) + r\left(u(x) - ⨍_U u\,dx\right).$$

Integrate with respect to x over U.

□

Convex functions are discussed more fully in §3.3.2 and §9.6.1.

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$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} \left(u_t^2(x,t) + |Du(x,t)|^2\right) dx \qquad 0 \le t \le t_0$$

$$\cancel{\dot{e}(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} 2u_t u_{tt}\, dx + \frac{1}{2}\int_{B(x_0,t_0-t)} Du \circ Du_t \, dx}$$

let $\quad B(x_0, t_0 - t) \longleftrightarrow B(0,1)$

let $x = x_0 + (t_0 - t) z$

$$e(t) = \frac{1}{2}\int_{B(0,1)}$$

Then $dx = (t_0 - t)^n dz$.

$$\therefore \; e(t) = \frac{1}{2}\int_{B(0,1)} \left(u_t\left(\frac{x-x_0}{t_0-t}, t\right)^2 + \left|Du\left(\frac{x-x_0}{t_0-t}, t\right)\right|^2\right)(t_0-t)^n dz$$

$\dot{e}$

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} \left(u_t u_{tt} + Du_t \circ Du\right) dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(u_t^2 + |Du|^2\right) dS$$

By analogy w/ 1D

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$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2(x,t) + |Du(x,t)|^2\,dx$$

let $x = x_0 + (t_0-t)z \Rightarrow dx = (t_0-t)^n\,dz$

$\quad \& \quad z = \dfrac{x-x_0}{t_0-t}$.

Then

$$e(t) = \frac{1}{2}\int_{B(0,1)} \left(u_t^2(x_0+(t_0-t)z,t) + |Du(x_0+(t_0-t)z,t)|^2\right)(t_0-t)^n\,dz$$

$$\dot{e}(t) = \frac{1}{2}\int_{B(0,1)} \cancel{2u_t} + 2u_t(x_0+(t_0-t)z,t)\Big[u_{tt}(\_\_,\_) + Du_t\cdot(-z)\Big]$$

$$+ 2Du(x_0+(t_0-t)z,t)\cdot\Big[D^2$$

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} u_t u_{tt}\,dx + \underbrace{\int_{B(x_0,t_0-t)} Du \cdot Du_t\,dx}_{\displaystyle \int_{\partial B(x_0,t_0-t)} u_t \frac{\partial u}{\partial \nu}\,dS - \int_{B(x_0,t_0-t)} u_t D^2 u\,dx} - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\,dS$$

$$\therefore \dot{e}(t) = \int_{B(x_0,t_0-t)} u_t(u_{tt} - D^2 u)\,dx$$

$$+ \int_{\partial B(x_0,t_0-t)} \frac{\partial u}{\partial \nu} u_t\,dS - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du|^2)\,dS$$

$$= \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial \nu} u_t - \frac{1}{2}u_t^2 - \frac{1}{2}|Du|^2\right) dS$$

By C-S. ineq

$$\left|\frac{\partial u}{\partial \nu} u_t\right| = |u_t|\left|\frac{\partial u}{\partial \nu}\right| \le |u_t|\,|Du|\cdot\underset{1}{\underset{\shortparallel}{|\nu|}}$$

$$\shortparallel$$

$Du \circ \nu$ w/ $|\nu| = 1$

& $|u_t||Du| \le \frac{|u_t|^2}{2} + \frac{|Du|^2}{2}$ Cauchy ineq.

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Use Fourier transform of scalar to see idea

① $u_t + b\cdot Du + cu = 0$ ✱
$u = g$

Solved $u_t + b\cdot Du = 0$
$u = g$ ↑g

~~g~~ sol $u(x,t) = g(x-tb)$ (1)

let $u(x,t) = g(x-tb) + v(x,t)$ & put in our eq ✱

$$v_t + b\cdot Dv + cg(x-tb) + cv(x,t) = 0.$$

$$v(x,0) = 0.$$

"imagine $b=0$" $u_t + ~~b\cdot D~~ \; cu = 0$
$U(x,t) = A(x)e^{-ct}$ (2)

Try sol of form $u(x,t) = A(x)e^{-ct}$

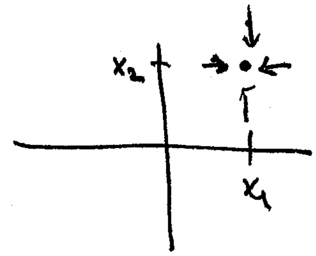
$$-c\cancel{Ae^{-ct}} + b\cdot D A(x)e^{-ct} + c\cancel{Ae^{-ct}} = 0$$

$$A(x) = g(x)$$

(9) $U = U(x_1, x_2, \cdots x_n)$.

$$\partial_{x_i} U \equiv \lim_{h\to 0} \frac{U(\vec{x}+he_i) - U(\vec{x})}{h}$$

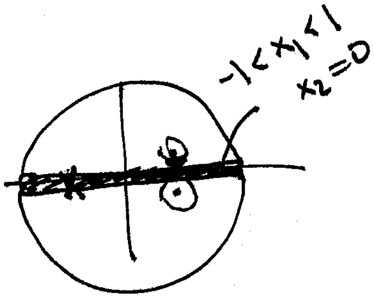
$$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ i spot.}$$



$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Delta V = \Delta U = 0.$

If $\vec{x} \in U$ say w/ $x_n \gtrless 0$.



if $x_n < 0$

$$\Delta V = -\left[\partial^2_{x_1} U + \partial^2_{x_2} U + \cdots + \partial^2_{x_n} U\right]$$

$$= -\Delta U = 0.$$

If $\vec{x} \ni x_n = 0$

1st consider $\partial_{x_i} V \equiv \lim_{h\to 0} \frac{V(\vec{x}+he_i) - V(\vec{x})}{h}$ $i \neq n$

$$= \lim_{h\to 0} \frac{U(\vec{x}+he_i) - U(\vec{x})}{h} = \cancel{\partial_{x_i} U}.$$

$$= 0$$

$\cancel{\partial^2_{x_1} V = \partial^2_{x_i} U}$

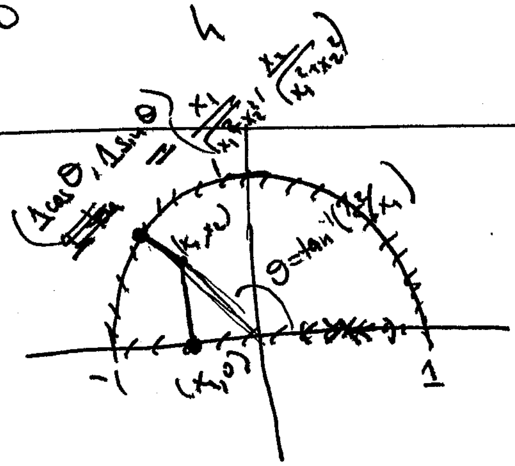
$\cancel{i \neq n}$ $U(\vec{x}) \equiv 0$

Because $\&$ $U(\vec{x}+he_i) \equiv 0$.

$\Rightarrow \partial^2_{x_i} U = 0.$ $i \neq n$

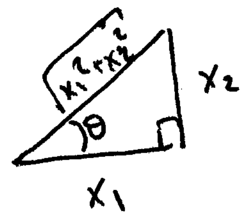
$$\partial_{x_n} v = \lim_{h\to 0} \frac{v(\vec{x}+he_n) - v(\vec{x})}{h}$$

2D:

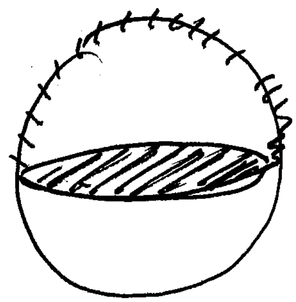


$x_1^2+x_2^2<1$

$x_1^2<1$



3D



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look at $x_n=0$

$$\partial_{x_n} v = \lim_{h\to 0} \frac{v(\vec{x}+he_n) - v(\vec{x})}{h}$$

consider $\lim_{h\to 0^+} \frac{v(\vec{x}+he_n) - v(\vec{x})}{h} + \lim_{h\to 0^-} \frac{v(\vec{x}+he_n) - v(\vec{x})}{h}$

$$\lim_{h\to 0^+} \frac{u(\vec{x}+he_n) - \cancelto{0}{u(\vec{x})}}{h}$$

$$\lim_{h\to 0^+} \frac{u(\vec{x}+he_n)}{h} = \lim_{h\to 0^+} \frac{\cancelto{0}{u(\vec{x})} + h\partial_{x_n}u(\vec{x}) + O(h^2)}{h}$$

$$\lim_{h\to 0^-} \frac{-u(x_1,\ldots,x_{n-1},-h) - \cancelto{0}{u(\vec{x})}}{h}$$

~~$\lim_{h\to 0^+}$~~

Consider $\partial_{x_n} V = ?$

$$\partial_{x_n} V \equiv \lim_{h \to 0} \frac{V(\vec{x}+he_n) - V(\vec{x})}{h} \quad ; \quad \lim_{h \to 0} \frac{V(\vec{x}) - V(\vec{x}-he_n)}{h}$$

$$\partial_{x_n}^2 V = \partial_{x_n}(\partial_{x_n} V)$$

$$= \lim_{h_1 \to 0} \frac{\partial_{x_n} V(\vec{x}+h_1 e_n) - \partial_{x_n} V(\vec{x})}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{1}{h_1} \left[\lim_{h_2 \to 0} \left(\frac{V(\vec{x}+h_1 e_n + h_2 e_n) - V(\vec{x}+h_1 e_n)}{h_2} \right) - \lim_{h_3 \to 0} \left(\frac{V(x+h_3 e_n) - V(x)}{h_3} \right) \right]$$

$h_1 = h_2 = h_3 = h$

$$= \lim_{h \to 0} \frac{1}{h^2} \left[V(\vec{x}+2he_n) - V(\vec{x}+he_n) - V(x+he_n) + V(x) \right]$$

$$= \lim_{h \to 0} \frac{1}{h^2} \left[V(\vec{x}+2he_n) - 2V(x+he_n) + V(x) \right]$$

$$\cong \lim_{h\to 0} \frac{1}{h^2}\left[v(x+he_n) - 2v(x) + v(x-he_n)\right]$$

$$= \lim_{h\to 0} \frac{1}{h^2}\left[U(x_1,\dots,x_{n-1},h) - 0 - U(x_1,\dots,x_{n-1},h)\right] =$$

$$= \lim_{h\to 0} \frac{1}{h^2}\left[0 \right] = 0$$

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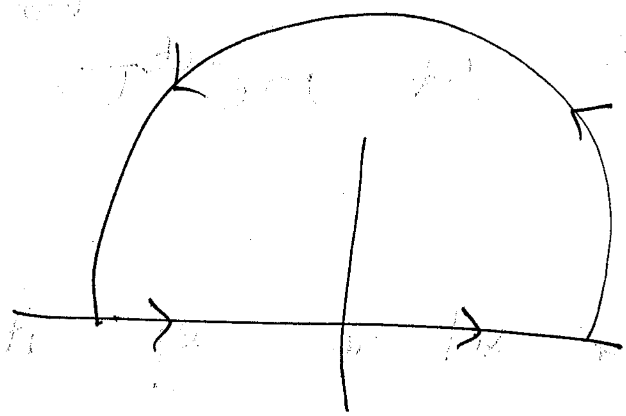
12 let $u = e^{-ct} v$. from dropping Δu term + f + integrating.

$\Rightarrow$ sol to eq iff

$v_t - \Delta v = e^{-ct} f$. get by putting in.

$v =$ [illegible]

$$\int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)}\,dy$$



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(13)

$$u_t - u_{xx} = 0 \qquad \mathbb{R}_+ \times (0,\infty)$$
$$u = 0 \qquad \mathbb{R}_+ \times \{t=0\}$$
$$u = g \qquad \text{on } \{x=0\} \times [0,\infty)$$

$$v = u(x,t) - g$$

$$v_t = -g' + u_t \, ; \quad v_{xx} = u_{xx}$$

$$\Rightarrow \quad v_t - \cancel{v}_{xx} = -g'(t) \qquad \mathbb{R}_+ \times (0,\infty)$$
$$v(0,t) = 0. \qquad \{x=0\} \times [0,\infty)$$
$$v(x,0) = u(x,0) - g(0) = 0 \qquad \mathbb{R}^+ \times \{t=0\}$$

Inhomogeneous heat problem

$$v_t - v_{xx} = -g'(t) \qquad \mathbb{R} \times (0,\infty) \qquad v \text{ odd extended in } x$$
$$v = 0 \qquad \mathbb{R} \times \{t=0\}$$

by eq (13) pg 49

$$v(x,t) = -\int_0^t \frac{1}{(4\pi(t-s))^{1/2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)} g'(s)\, dy\, ds$$

$$v(x,t) = -\frac{1}{\sqrt{4\pi}} \int_0^t \frac{g'(s)}{(t-s)^{1/2}} \underbrace{\int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)}\, dy}_{F(s)}\, ds$$

$$\sigma = \frac{(x-y)^2}{4(t-s)}$$

$$= \frac{-1}{\sqrt{4\pi}} \int_0^t \frac{g'(s)}{(t-s)^{1/2}} F(s)\, ds$$

$$= -\frac{1}{\sqrt{4\pi}} \left[\frac{F(s)}{(t-s)^{1/2}} g(s) \Big|_0^t - \int_0^t g(s) \left[\frac{F'(s)}{(t-s)^{1/2}} + \frac{1}{2}(-1) \frac{F(s)}{(t-s)^{3/2}} \right] ds \right]$$

$$F'(s) = \int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)} \left[+\frac{(x-y)^2}{4(t-s)^2}(-1) \right] dy.$$

$$\frac{F'(s)}{(t-s)^{1/2}} = \int_{-\infty}^{\infty} e^{-(\ \ )} \frac{(x-y)^2}{4(t-s)^{5/2}} (-1)\, dy = \frac{1}{4(t-s)^{5/2}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4(t-s)}$$

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$$(D_a u, D^2_{xa} u) = \begin{pmatrix} u_{a_1} & u_{x_1 a_1} & \cdots & u_{x_n a_1} \\ \vdots & & & \\ u_{a_n} & & & \end{pmatrix}$$

$$\operatorname{rank}(D_a u, D^2_{xa} u) = n$$

$$u(x;a) = v(x; f(a))$$

$$u_{x_i a_j} = \sum_{k=1}^{n-1} v_{x_i b_k}(x; f(a))\, f^k_{a_j}(a)$$

~~the~~ Matrix for $D^2_{xa} u$

$$\det(D^2_{xa} u) = \sum_{k_1, \dots, k_n = 1}^{n-1} v_{x_1 b_{k_1}} \cdots v_{x_n b_{k_n}} \det \begin{pmatrix} f^{k_1}_{a_1} & \cdots & f \\ & & \end{pmatrix}$$

$$D^2_{xa} u = \begin{matrix} u_{x_1 a_1} & u_{x_2 a_1} & \cdots & u_{x_n a_1} \\ u_{x_1 a_2} & u_{x_2 a_2} & \cdots & u_{x_n a_2} \\ \vdots & & & \\ u_{x_1 a_n} & \cdots & & u \end{matrix}$$

$$= v_{x_1 b_1} f^1_{a_1} + v_{x_1 b_2} f^2_{a_1} + \cdots$$

$$U_{a_j}(x;a) = \sum_{k=1}^{n-1} v_{b_k} \psi^k_{a_j}$$

Ex 1: $\quad U(x;a) = a \circ x + f(a)$

$\quad Du = a$

Then $\quad x \circ Du + f(Du) = x \circ a + f(a) = u$ ✓

$D_a U = x + Df$

$D_x U = a$

$D^2_{xa} U = \delta_{ij}$

Then $\quad (D_a U, D^2_{xa} U) = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & & \\ \vdots & 0 & 0 & 1 & & \\ a_n & 0 & & & \cdots & 1 \end{pmatrix}$

Ex 2: $\quad |Du| = 1$

$u = a \circ x + b$

$Du = a \qquad |a| = 1 \quad$ as $\quad a \in \partial B(0,1)$ ✓

$D_a U = D_{(\vec{a},b)} U = (x_1, x_2, \ldots, x_n, 1)$

Then $D_x U = a$

$$D^2_{x(\bar{a},b)} U = \begin{pmatrix} D_{(\bar{a},b)} U_{x_1} = a_1 \\ D_{(\bar{a},b)} U_{x_2} = a_2 \\ \vdots \\ D_{(\bar{a},b)} U_{x_n} = a_n \end{pmatrix}$$

Then
$$D^2_{x(\bar{a},b)} U = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & \\ & & & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots 0 & 1 & 0 \end{pmatrix}$$

$$\therefore \; (D_a U, D^2_{xa} U) = \underbrace{\begin{pmatrix} x_1 & 1 & 0 & 0 & \cdots & & 0 \\ x_2 & 0 & 1 & 0 & \cdots & & 0 \\ \vdots & 0 & & & & & \\ \vdots & \vdots & & & & & \\ x_n & \vdots & & & 1 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}}_{n+2}$$

Rank must be $n+1$ yes ✓

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Ex 3: $u_t + H(Du) = 0$

$u(x,t;a,b) = a \cdot x - tH(a) + b$

$u_t = -H(a)$ $\quad D_a u = D_{(\bar{a},b)} u = \cancel{x}\,(x_1 - tH_{a_1}, x_2 - tH_{a_2}, \ldots x_n - tH_{a_n}, 1)$

$Du = a$

1. $u_t + H(Du) = -H(a) + H(a) = 0$ ✓

$D_{(\bar{a},b)} Du = D_{(\bar{a},b)} \bar{a} = \cancel{1\,0\,0}$

$=$

$$= \begin{matrix} \partial_{a_1} a_1 & \partial_{a_1} a_2 & \partial_{a_1} a_3 & \cdots & \partial_{a_1} b \\ \partial_{a_2} a_1 & \partial_{a_2} a_2 & \partial_{a_2} a_3 & \cdots & \partial_{a_2} b \\ \vdots & & & & \\ \partial_b a_1 & \partial_b a_2 & \partial_b a_3 & \cdots & \partial_b b \end{matrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ & & & & 1 \end{pmatrix}_{n+1 \times n+1}$$

$$\therefore (D_a u, D^2_{xa} u) = \begin{pmatrix} x_1 - tH_{a_1} & 1 & 0 & \cdots & \\ x_2 - tH_{a_2} & 0 & 1 & 0 & \cdots \\ \vdots & & & & \\ x_n - tH_{a_n} & & & & \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

~~$(n+2)\times(n+1)$~~

$(n+1)\times(n+2)$

$\text{Rank}(\quad , \quad) = n+1$ ✓.

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$V(x) = u(x; \phi(x))$ envelope of

$v(x) = u(x; \phi(x))$

$$v_{x_i} = u_{x_i}(x; \phi(x)) + \sum_{j=1}^{n} u_{a_j}(x; \phi(x))\, \phi^j_{x_i}(x)$$

each $u_{a_j} \equiv 0$

$\therefore$ $v_{x_i} = u_{x_i}$

$\therefore$ $F(Dv, v, x) = F(Du, u, x)$

Ex 4: $u^2(1 + |Du|^2) = 1$

Claim $u(x,a) = \pm(1 - |x-a|^2)^{1/2}$ $|x-a| < 1$

is a complete integral

$$Du = \pm\frac{1}{2}(1 - |x-a|^2)^{-1/2}\left(-\cancel{2} D(x-a)\cdot(x-a)\right)$$

$$= \mp \cancel{\frac{1\cdot(x-a)}{\sqrt{1-|x-a|^2}}}$$

Pf $D\left((x-a)\circ(x-a)\right)$

$$= D\left((x_1-a_1)^2 + (x_2-a_2^2) + \cdots (x_n-a_n)^2\right)$$

$$= \left(2(x_1-a_1), 2(x_2-a_2), \ldots, 2(x_n-a_n)\right)$$

$$= 2(x-a)$$

$$\therefore \quad Du = \mp \frac{(x-a)}{\sqrt{1-|x-a|^2}}$$

$$\therefore \quad |Du|^2 = \frac{|x-a|^2}{1-|x-a|^2}$$

$$1+|Du|^2 = \frac{1}{1-|x-a|^2}$$

$$\therefore \quad u^2(1+|Du|^2) = 1 \quad ✓.$$

$(x-a)^2$

$\to 2(x-a)$

Check Rank condition

$$D_a u \overset{?}{=} \pm\frac{1}{2}\frac{-D_a\left((x-a)\circ(x-a)\right)}{\sqrt{1-|x-a|^2}} = \pm\frac{1}{2}\frac{(x-a)}{\sqrt{1-|x-a|^2}}$$

for some smooth function K. Consequently

$$
\begin{aligned}
|D_x^k D_t^l u(x,t)| &\le \iint_{C(1)} |D_t^l D_x^k K(x,t,y,s)||u(y,s)|\,dyds \\
&\le C_{kl}\|u\|_{L^1(C(1))}
\end{aligned}
\tag{39}
$$

for some constant C_{kl}.

2. Now suppose the cylinder $C(r) := C(0,0;r)$ lies in U_T. Let $C(r/2) = C(0,0;r/2)$. We rescale by defining

$$v(x,t) := u(rx, r^2 t).$$

Then $v_t - \Delta v = 0$ in the cylinder $C(1)$. According to (39),

$$|D_x^k D_t^l v(x,t)| \le C_{kl}\|v\|_{L^1(C(1))} \quad ((x,t) \in C(\tfrac{1}{2})).$$

But $D_x^k D_t^l v(x,t) = r^{2l+k} D_x^k D_t^l u(rx, r^2 t)$ and $\|v\|_{L^1(C(1))} = \frac{1}{r^{n+2}}\|u\|_{L^1(C(r))}$. Therefore

$$\max_{C(r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{2l+k+n+2}}\|u\|_{L^1(C(r))}.$$

□

Remark. If u solves the heat equation within U_T, then for each fixed time $0 < t \le T$, the mapping $x \mapsto u(x,t)$ is analytic. (See Mikhailov [**M**].) However the mapping $t \mapsto u(x,t)$ is not in general analytic. □

2.3.4. Energy methods.

a. Uniqueness.

Let us investigate again the initial/boundary-value problem

$$
\begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{on } \Gamma_T. \end{cases}
\tag{40}
$$

We earlier invoked the maximum principle to show uniqueness, and now—by analogy with §2.2.5—provide an alternative argument based upon integration by parts. We assume as usual that $U \subset \mathbb{R}^n$ is open, bounded and that ∂U is C^1. The terminal time $T > 0$ is given.

THEOREM 10 (Uniqueness). *There exists at most one solution $u \in C_1^2(\bar{U}_T)$ of* (40).

$$D^2_{xa} U = ?$$

$$D_x U = \mp \frac{(x-a)}{\sqrt{1-|x-a|^2}}$$

$$D^2_{xa} U \stackrel{?}{=} \mp \frac{(-1)}{\sqrt{1-|x-a|^2}} \mp (x-a)\left(-\frac{1}{2}\right)\frac{+2(x-a)}{(1-|x-a|^2)^{3/2}}$$

$$= \pm \frac{1}{\sqrt{1-|x-a|^2}} \pm \frac{(x-a)^2}{(1-|x-a|^2)^{3/2}}$$

Check:

$$D_x U = \mp \frac{1}{\sqrt{1-|x-a|^2}} (x_1-a_1, x_2-a_2, x_3-a_3, \dots, x_n-a_n)$$

$$D^2_{xa} U = (D_a U_{x_1} ; D_a U_{x_2} ; D_a U_{x_3} \cdots ; D_a U_{x_n})$$

$$= \Big(\underbrace{D_a \frac{\pm (x_1-a_1)}{\sqrt{1-|\ \ |^2}}}_{\text{colums}} ; \underbrace{D_a \frac{\pm (x_2-a_2)}{\sqrt{1-|\ \ |^2}}}_{\text{columns}} ; D_a \frac{\pm (x_3-a_3)}{\sqrt{1-|\ \ |^2}} ; \cdots ;$$

Note in this expression that the expression in the square brackets vanishes in some region *near* the singularity of Φ. Integrate the last term by parts:

$$
(37)\qquad \begin{aligned} u(x,t) = \iint_C [\Phi(x-y,t-s)(\zeta_s(y,s) + \Delta\zeta(y,s)) \\ + 2D_y\Phi(x-y,t-s)\cdot D\zeta(y,s)]u(y,s)\,dyds. \end{aligned}
$$

We have proved this formula assuming $u \in C^\infty$. If u satisfies only the hypotheses of the theorem, we derive (37) with $u^\varepsilon = \eta_\varepsilon * u$ replacing u, η_ε being the standard mollifier in the variables x and t, and let $\varepsilon \to 0$.

3. Formula (37) has the form

$$
(38)\qquad u(x,t) = \iint_C K(x,t,y,s)u(y,s)\,dyds \quad ((x,t)\in C''),
$$

where

$$
K(x,t,y,s) = 0 \quad \text{for all points } (y,s)\in C',
$$

since $\zeta \equiv 1$ on C'. Note also K is smooth on $C - C'$. In view of expression (38), we see u is C^∞ within $C'' = C(x_0,t_0;\frac{1}{2}r)$. □

c. Local estimates for solutions of the heat equation.

Next we record some estimates on the derivatives of solutions to the heat equation, paying attention to the differences between derivatives with respect to x_i $(i = 1,\dots,n)$ and with respect to t.

THEOREM 9 (Estimates on derivatives). *There exists for each pair of integers $k,l = 0,1,\dots,$ a constant $C_{k,l}$ such that*

$$
\max_{C(x,t;r/2)} |D_x^k D_t^l u| \le \frac{C_{kl}}{r^{k+2l+n+2}}\|u\|_{L^1(C(x,t;r))}
$$

for all cylinders $C(x,t;r/2)\subset C(x,t;r)\subset U_T$, and all solutions u of the heat equation in U_T.

Proof. 1. Fix some point in U_T. Upon shifting the coordinates, we may as well assume the point is $(0,0)$. Suppose first that the cylinder $C(1) := C(0,0;1)$ lies in U_T. Let $C\left(\frac{1}{2}\right) := C\left(0,0;\frac{1}{2}\right)$. Then, as in the proof of Theorem 8,

$$
u(x,t) = \iint_{C(1)} K(x,t,y,s)u(y,s)\,dyds \quad ((x,t)\in C(\tfrac{1}{2}))
$$

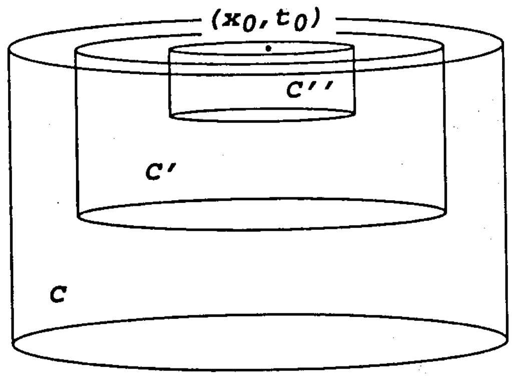
$$D_a v(x;a) = 0$$

$$\Longleftrightarrow \quad \pm\frac{1}{2}\frac{(x-a)}{\sqrt{1-|x-a|^2}} = 0$$

$$\Rightarrow \quad x = a$$

$$a = \phi(x) = x$$

Then $v(x, \phi(x)) = \pm 1$



2. Assume temporarily that $u \in C^\infty(U_T)$ and set

$$v(x,t) := \zeta(x,t)u(x,t) \quad (x \in \mathbb{R}^n,\, 0 \le t \le t_0).$$

Then

$$v_t = \zeta u_t + \zeta_t u, \;\; \Delta v = \zeta \Delta u + 2D\zeta \cdot Du + u\Delta\zeta.$$

Consequently

$$v = 0 \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \tag{33}$$

and

$$v_t - \Delta v = \zeta_t u - 2D\zeta \cdot Du - u\Delta\zeta =: \tilde{f} \tag{34}$$

in $\mathbb{R}^n \times (0, t_0)$. Now set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds.$$

According to Theorem 2

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} = \tilde{f} & \text{in } \mathbb{R}^n \times (0, t_0) \\ \tilde{v} = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{35}$$

Since $|v|, |\tilde{v}| \le A$ for some constant A, Theorem 7 implies $v \equiv \tilde{v}$; that is,

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\tilde{f}(y,s)\, dyds. \tag{36}$$

Now suppose $(x,t) \in C''$. As $\zeta \equiv 0$ off the cylinder C, (34) and (36) imply

$$\begin{aligned} u(x,t) = \iint_C \Phi(x-y, t-s)[(\zeta_s(y,s) - \Delta\zeta(y,s))u(y,s) \\ - 2D\zeta(y,s) \cdot Du(y,s)]\, dyds. \end{aligned}$$

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Ex 5: $|Du| = 1$ $n = 2$

$\Rightarrow \sqrt{u_x^2 + u_y^2} = 1$

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(1) $F(Du, u, x) = 0$

$$\partial_{x_i} \Rightarrow \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) p^j_{x_i} + \underline{\qquad} + \partial_{x_i}(Du, u, x) = 0$$

$$+ \frac{\partial F}{\partial u} u_{x_i}$$

$\Rightarrow$ As $p^j = u_{x_j}$

$$\Rightarrow \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du, u, x) u_{x_j x_i} + \frac{\partial F}{\partial u} u_{x_i} + \frac{\partial F}{\partial x_i} = 0$$

Set $\dot{x}^j = \frac{\partial F}{\partial p_j}(\bar{p}, z, \bar{x})$ at.

$$\Rightarrow \sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(\bar{p}(s), z(s), \bar{x}(s)) u_{x_i x_j}(\bar{x}(s))$$

$$+ \frac{\partial F}{\partial z}(\bar{p}(s), z(s), \bar{x}(s)) p^i(s) + \frac{\partial F}{\partial x_i}(\bar{p}(s), z(s), \bar{x}(s)) = 0.$$

(a) $\Rightarrow \quad \dot{p}^i = \sum_{j=1}^{n} u_{x_i x_j} \dot{x}^j$

if define $\quad \dot{x}^j = \dfrac{\partial F}{\partial p_j}$

(b) $\Rightarrow \quad \dot{p}^i = \underbrace{\sum_{j=1}^{n} u_{x_i x_j} \dfrac{\partial F}{\partial p_j}}$ But

$$-\frac{\partial F}{\partial z} u_{x_i} - \frac{\partial F}{\partial x_i} = -\frac{\partial F}{\partial z} p^i - \frac{\partial F}{\partial x_i}$$

Diff (3)

$$\Rightarrow \quad \dot{z} = \sum_{i=1}^{n} u_{x_i} \dot{x}_i = \sum_{i=1}^{n} p^i \dot{x}_i = \sum_{i=1}^{n} p^i \frac{\partial F}{\partial p_i}$$

$\therefore$ Finally

$$\dot{\bar{p}}(s) = - D_x F - D_z F \, \bar{p}$$

$$\dot{z}(t) = D_p F(\bar{p}(s), z(s), \bar{x}(s)) \cdot \bar{p}(s)$$

$$\dot{\bar{x}} = D_p F(\bar{p}, z, \bar{x})$$

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for all points $x, y \in V$. These inequalities assert that *the values of a non-negative harmonic function within V are all comparable*: u cannot be very small (or very large) at any point of V unless u is very small (or very large) everywhere in V. The intuitive idea is that since V is a positive distance away from ∂U, there is "room for the averaging effects of Laplace's equation to occur".

Proof. Let $r := \frac{1}{4}\operatorname{dist}(V, \partial U)$. Choose $x, y \in V$, $|x-y| \leq r$. Then

$$\begin{aligned} u(x) = ⨍_{B(x,2r)} u\,dz &\geq \frac{1}{\alpha(n)2^n r^n}\int_{B(y,r)} u\,dz \\ &= \frac{1}{2^n} ⨍_{B(y,r)} u\,dz = \frac{1}{2^n}u(y). \end{aligned}$$

Thus $2^n u(y) \geq u(x) \geq \frac{1}{2^n}u(y)$ if $x, y \in V$, $|x-y| \leq r$.

Since V is connected and $\bar{V}$ is compact, we can cover $\bar{V}$ by a chain of finitely many balls $\{B_i\}_{i=1}^N$, each of which has radius r and $B_i \cap B_{i-1} \neq \emptyset$ for $i = 2, \ldots, N$. Then

$$u(x) \geq \frac{1}{2^{nN}}u(y)$$

for all $x, y \in V$. □

2.2.4. Green's function.

Assume now $U \subset \mathbb{R}^n$ is open, bounded, and ∂U is C^1. We propose next to obtain a general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U,$$

subject to the prescribed boundary condition

$$u = g \quad \text{on } \partial U.$$

a. Derivation of Green's function.

Suppose first of all $u \in C^2(\bar{U})$ is an arbitrary function. Fix $x \in U$, choose $\varepsilon > 0$ so small that $B(x,\varepsilon) \subset U$, and apply Green's formula from §C.2 on the region $V_\varepsilon := U - B(x,\varepsilon)$ to $u(y)$ and $\Phi(y-x)$. We thereby compute

$$\begin{aligned} &\int_{V_\varepsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\,dy \\ &= \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y), \end{aligned} \tag{24}$$

(a) $F(Du, u, x) = b(x) \cdot Du(x) + c(x)u(x) = 0$

$F(p, z, x) = b(x) \cdot p + c(x)z$

$D_pF = \bar{b}(x)$

$\dot{x}(s) =$

$\dot{z}(s) = b(\bar{x}(s)) \cdot \bar{p}(s) \qquad \bar{p}(s) = Du(s)$

$\Rightarrow \quad b(\bar{x}(s)) \cdot \cancel{Dx} \; D \quad = -c(x)u(x)$

$\dot{z}(s) = -c(\bar{x}(s))z(s)$

the sum taken over all multiindices. We assert this power series converges, provided

$$|x - x_0| < \frac{r}{2^{n+2}n^3 e}. \tag{23}$$

To verify this, let us compute for each N the remainder term:

$$\begin{aligned} R_N(x) &:= u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x-x_0)^\alpha}{\alpha!} \\ &= \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x-x_0))(x-x_0)^\alpha}{\alpha!} \end{aligned}$$

for some $0 \le t \le 1$, t depending on x. We establish this formula by writing out the first N terms and the error in the Taylor expansion about 0 for the function of one variable $g(t) := u(x_0 + t(x - x_0))$, at $t = 1$. Employing (22), (23), we can estimate

$$\begin{aligned} |R_N(x)| &\le CM \sum_{|\alpha|=N} \left(\frac{2^{n+1}n^2 e}{r}\right)^N \left(\frac{r}{2^{n+2}n^3 e}\right)^N \\ &\le CMn^N \frac{1}{(2n)^N} = \frac{CM}{2^N} \\ &\to 0 \quad \text{as } N \to \infty. \end{aligned}$$

□

See §4.6.2 for more on analytic functions and partial differential equations.

f. Harnack's inequality.

Recall from §A.2 that we write $V \subset\subset U$ to mean $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact.

THEOREM 11 (Harnack's inequality). *For each connected open set $V \subset\subset U$, there exists a positive constant C, depending only on V, such that*

$$\sup_V u \le C \inf_V u$$

for all nonnegative harmonic functions u in U.

Thus in particular

$$\frac{1}{C} u(y) \le u(x) \le C u(y)$$

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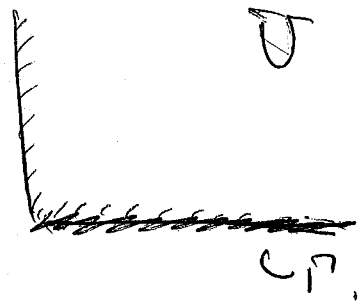
$x_1 u_{x_2} + x_2 u_{x_1} = u$ on U

$u = g$ on Γ



$= \underbrace{\cancel{(x_1, -x_2)\circ Du}}_{F(Du,u,x)} - u = 0$

$= \underbrace{(-x_2, x_1)\circ Du}_{F(Du,u,x)} - u = 0$

$\dot{\vec{x}}(s) = \vec{b}(\vec{x}(s))$; $\dot{z}(s) = -c(\vec{x}(s))z(s)$

$\Leftrightarrow \quad \dot{x}^1 = -x^2 \qquad \dot{z} = z.$

$\dot{x}^2 = x^1$

$\Leftrightarrow \quad \cancel{\ddot{x}^1 = -\dot{x}^2 = -x^1 = x^1(s) = A\cos s + B\sin s}$

$\cancel{x^1(0) = x_0}$

$\frac{d}{ds}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \qquad \begin{pmatrix} x^1(0) \\ x^2(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$

$\lambda^2 - \lambda \cdot 0 + 1 = 0$

$\lambda = \pm i$

$$x^1(s) = A\cos s + B\sin s$$
$$\dot{x}^1(s) = -A\sin s + B\cos s = -x^2(s)$$

$$\Rightarrow \quad x^2(s) = A\sin s - B\cos s.$$

Then $\quad x^1(s) = A\cos s + B\sin s$
$\quad x^2(s) = A\sin s - B\cos s$

$$\begin{matrix} x^1(0) \\ x^2(0) \end{matrix} = \begin{matrix} x_0 \\ 0 \end{matrix} \qquad \begin{matrix} A = x_0 \\ -B = 0 \end{matrix}$$

$$\Rightarrow \quad x^1(s) = x_0 \cos s$$
$$x^2(s) = x_0 \sin s$$

$$z(s) = z_0 e^s = u(x_1(s), x_2(s))\Big| \quad \therefore u(\ldots)\Big|_{s=0} = u(x_0, 0)$$

$$= g(x_0) = z_0$$

pick $0 < s < \pi/2$

so $x^1(s) > 0 \ \forall s.$

$$\Rightarrow \quad z(s) = g(x^0)e^s$$

Then given $(x_1, x_2) \in U$ pick $x_0 \ni$

$$(x_0 \cos s, x_0 \sin s) = (x_1, x_2)$$

$$\Rightarrow \quad x_0 = \sqrt{x_1^2 + x_2^2} \quad \& \quad s = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$u(x_1,x_2) = g\left(\sqrt{x_1^2+x_2^2}\right)e^{\tan^{-1}(x_2/x_1)}$$

(b) $F(Du,u,x) = \vec{b}(x,u(x))\cdot Du + c(x,u) = 0$

$F(p,z,x) = \vec{b}(x,z)\cdot p + c(x,z)$

$D_pF = b(x,z)$

$\therefore$ (11c) $\Rightarrow$ $\dot{x}(s) = \vec{b}(\underline{x}(s),\underline{z(s)})$

$+$ (11b) $\dot{z}(s) = b(\bar{x}(s),z(s))\cdot\bar{p}(s)$

$= -c(x,z)$

$\therefore$ (a) $\dot{x}(s) = \vec{b}(\bar{x}(s),z(s))$

$\dot{z}(s) = -c(\bar{x}(s),z(s))$

$u_{x_1} + u_{x_2} = u^2$ in U $\quad u = g$ on Γ.

U

$\vec{b} = (1,1)$

$c(x,u) = -u^2$

~~$\Gamma\ldots$~~ $\Gamma \subset \partial U$.

$\dot{\vec{x}} = (1,1) \rightarrow \dot{x}^1 = 1 \quad \dot{x}^2 = 1$

$\dot{z} = +z^2.$

$\hookrightarrow$ $x^1(s) = s + x^0$

$x^2(s) = s$

$\Rightarrow \frac{dz}{z^2} = ds$

$\frac{1}{z_0} + -\frac{1}{z} = s + s_0$

$\Rightarrow (s_0 = 0,\ z_0 = v(x^1(0), x^2(0)))$

$\Rightarrow \frac{1}{z^0} - \frac{1}{z(s)} = s$

$\frac{1}{z^0} - s = \frac{1}{z(s)} \Rightarrow z(s) = \frac{1}{\frac{1}{z^0} - s} = \frac{z^0}{1 - z^0 s} = \frac{g(x^0)}{1 - s g(x^0)}$

Fix point $(x_1, x_2) \in U$. pick s & x^0 $\ni$.

$(x^1(s), x^2(s)) = (x^0 + s, s) = (x_1, x_2)$

$\Rightarrow s = x_2$

& $x^0 = x_1 - x_2$

$\Rightarrow v(x_1, x_2) = \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}$

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$u^1_t + -u^2_x = 0$

$u^2_t - p(u^1)_x = 0$

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_t + \begin{pmatrix} -u^2 \\ -p(u^1) \end{pmatrix}_x = 0$$

$\therefore \quad F(z) = (-z^2, -p(z^1))$

$u_{tt} - (p(u_x))_x = 0$

let $u^1 = u_x$ Th $u^2_t - (p(u^1))_x = 0$

$u^2 = u_t$ $\quad u^2_t - (p(u^1))_x = 0.$

$u^1_t = u_{xt} = u^2_x$ compatibility condition ✓.

$(u^2)_t - (p(u^1))_x = 0$

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$$F^2(\cancel{u}) = \rho v^2 + p(\rho, e) = u^1\left(\frac{u^2}{u^1}\right)^2 + p\left(u^1, \frac{u^3}{u^1} - \left(\frac{u^2}{u^1}\right)^2 \frac{1}{2}\right)$$

$$= \frac{(u^2)^2}{u^1} + p\left(u^1, \frac{u^3}{u^1} - \frac{1}{2}\left(\frac{u^2}{u^1}\right)^2\right)$$

$$F^3(u) = \cancel{\rho E} = \cancel{u^1 (u^3)} \quad \rho E v + p v$$

$$= u^3\left(\frac{u^2}{u^1}\right) + \frac{u^2}{u^1} p\left(u^1, \frac{u^3}{u^1} - \frac{1}{2}\left(\frac{u^2}{u^1}\right)^2\right)$$

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$z_1 = \phi$

$z_2 = v$

$$F = \left(z_1 z_2,\ z_2^2/2 + z_1\right)$$

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$$\int_0^{\infty}\int_{-\infty}^{\infty}\left(v\circ u_t + v\circ F(u)_x\right) dx\, dt = 0$$

$$\Rightarrow \int_0^{\infty}\int_{-\infty}^{\infty} v\circ u_t \, dx\, dt + \int_0^{\infty}\int_{-\infty}^{\infty} v\circ F(u)_x \, dx\, dt = 0$$

$$= \int_{-\infty}^{\infty}\left(v\circ u \Big|_0^{\infty} - \int_0^{\infty} v_t\circ u \, dt \right) dx + \int_0^{\infty}\left(v\circ F(u)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v_x\circ F(u)\, dx\right) dt$$

$$= 0$$

v compact support

$$= \int_{-\infty}^{\infty} v(x,0)\circ u(x,0)\, dx - \int_0^{\infty}\int_{-\infty}^{\infty} v_t\circ u \, dt\, dx$$

$$+ - \int_0^{\infty}\int_{-\infty}^{\infty} v_x\circ F(u)\, dx\, dt = 0$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty}\left(u\circ v_t + F(u)\circ v_x\right) dx\, dt - \int_{-\infty}^{\infty} g(x)\circ v(x)\Big|_{t=0} dx = 0$$

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why vanish?

$$\iint_{V_\ell} (\vec{U} \circ \vec{V}_t + \vec{F}(\vec{U}) \circ V_x)\, dx\, dt = \int_{-\infty}^{\infty} (\vec{U} \circ \vec{V} \Big|_0^{\infty}\, dx - \iint_{V_\ell} \vec{U}_t \circ \vec{V}\, dx\, dt$$

yes for time integr.

$$+ \int_0^{\infty} \vec{F}(\vec{U}) \circ \vec{V} \Big|_{-\infty}^{\infty}\, dt - \iint_{V_\ell} \vec{F}(\vec{U}).$$

NO For space integral.

$= \cdots$

$$0 = \int_C ((F(U_\ell) - F(U_r))\nu^1 + (U_\ell - U_r)\nu^2) \circ V\, dl$$

$x = s(t) \Rightarrow F(x,t) = x - s(t) = 0$

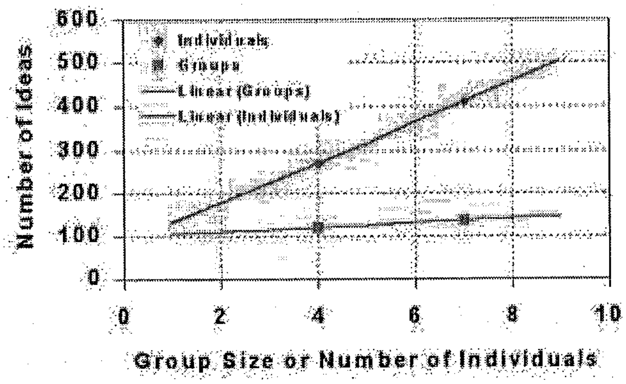
The Normal to this curve is given by ∇F

& unit normal given by $\frac{\nabla F}{|\nabla F|} = \frac{(1, -\dot{s})}{\sqrt{1 + \dot{s}^2}} = (\nu^1, \nu^2)$

Thus (18)

$$(\vec{F}(\bar{U}_\ell) - F(\bar{U}_r))\, 1 - \dot{s}(\bar{U}_\ell - \bar{U}_r) = 0$$

Creativity of Groups and Individuals



From Bouchard, et. al.)

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$$-\sigma v'(x-\sigma t) + B(v(x-\sigma t))\, v'(x-\sigma t) = 0$$

— — — —

$$\lambda_k(z)\,(\vec{l}_\ell(z)\circ \vec{r}_k(z)) = l_k(z)\circ(\underbrace{\lambda_k(z)\, r_k(z)}_{(B(z)\, r_k)})$$

$$= l_k^T (B\, r_k)$$

$$= r_k^T \underbrace{B^T l_k}_{\lambda_k} = \lambda_\ell\, r_k^T l_k = (l_k \circ r_k)\,\lambda_\ell$$

$$\tilde{u} = \Phi(u)$$

$$\tilde{u}_t = D\Phi(u)\, u_t \qquad \tilde{u}_x = D\Phi(u)\, u_x$$

Then $u_t + B(u)\, u_x = 0$

$\Rightarrow$ multiply by $D\Phi(u)$

$$\underbrace{D\Phi(u)\, u_t}_{\tilde{u}_t} + D\Phi(u) B(u) (D\Phi(u))^{-1}\, D\Phi(u)\, u_x = 0$$

$$\tilde{u}_t + (D\Phi(u) B(u) (D\Phi(u))^{-1})\, \tilde{u}_x = 0$$

Monday, 1 November 1999 14:30:39

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$$\begin{pmatrix} B_\varepsilon & & 0 \\ & & \vdots \\ & & -1 \\ 0 \cdots & 2 & 0 \end{pmatrix}\begin{pmatrix} I & & 0 \\ & & \vdots \\ & & (-\varepsilon)^{-1} \\ 0 \cdots & 0 & 1 \end{pmatrix} = \cancel{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} & 0 \\ \vdots & & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} & -1 \\ 0 & 0 & \cdots & 2 & 0 \end{pmatrix}}$$

$$\cancel{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & \\ 0 & & & 1 & (-\varepsilon)^{-1} \\ 0 & & & 0 & 1 \end{pmatrix}} = \cancel{\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} & (-\varepsilon)^{-1} b_{1m} \\ b_{21} & \cdots & & b_{2m} & (-\varepsilon)^{-1} b_{2m} \\ \vdots & & & & \\ b_{m1} & b_{m2} & \cdots & b_{mm} & + b_{mm}(-\varepsilon)^{-1} - 1 \\ & & & & (-\varepsilon)^{-1}(-1) \\ 0 & 0 & \cdots & 2 & \end{pmatrix}}$$

Block matrix multiplications

$$\cancel{B_\varepsilon \begin{pmatrix} 0 \\ -1 \end{pmatrix}}$$

$$\begin{pmatrix} B_\varepsilon & B_\varepsilon \begin{pmatrix} 0 \\ \vdots \\ (-\varepsilon)^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix} \\ \underset{1\times m}{(0 \cdots 2)} \, \underset{m\times m}{I} + 0 & (0 \cdots 2)\begin{pmatrix} 0 \\ \vdots \\ (-\varepsilon)^{-1} \end{pmatrix} + 0 \end{pmatrix}$$

$$= \begin{pmatrix} B_\varepsilon & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots & 2 & 2(-\varepsilon)^{-1} \end{pmatrix}$$

Because $B_\varepsilon \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = -\varepsilon \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Thus

$$B_\varepsilon \begin{pmatrix} 0 \\ \vdots \\ (-\varepsilon)^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$+ \therefore$

$$B_\varepsilon \begin{pmatrix} 0 \\ \vdots \\ (-\varepsilon)^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\|u+v\|_{L^p}^p = \int_U |u+v||u+v|^{p-1}\,dx \le \int_U |u|\cdot|u+v|^{p-1}\,dx + \int_U |v||u+v|^{p-1}\,dx$$

By Δ ineq for $|\cdot|$. Apply hölders eq to each integral:

$$\left(\int_U |u|^p\,dx\right)^{\frac{1}{p}}\left(\int_U |u+v|^{(p-1)q}\,dx\right)^{1/q} + \left(\int_U |v|^p\,dx\right)\left(\int_U |u+v|^{(p-1)q}\right)^{\frac{1}{q}}$$

But q is the conjugate variable of p - $q = \frac{p}{p-1}$

So $$\|u+v\|_{L^p}^p \le \|u\|_{L^p}\left(\int_U |u+v|^p\,dx\right)^{\frac{p-1}{p}} + \|v\|_{L^p}\left(\int_U |u+v|^p\,dx\right)^{\frac{p-1}{p}}$$

$$= \left(\|u\|_{L^p} + \|v\|_{L^p}\right)\left(\int_U |u+v|^p\,dx\right)^{\frac{p-1}{p}}$$

$$\|u+v\|_{L^p}^p \le \left(\|u\|_{L^p} + \|v\|_{L^p}\right)\|u+v\|_{L^p}^{p-1}$$ continuing agan w/

$$\|u+v\|_{L^p}^{p-1} \le \left(\|u\|_{L^p} + \|v\|_{L^p}\right)\|u+v\|_{L^p}^{p-2} \quad \dots$$

$$\Rightarrow \|u+v\|_{L^p}^p \le \left(\|u\|_{L^p} + \|v\|_{L^p}\right)^p. \rightarrow \|u+v\|_{L^p} \le \|u\|_{L^p} + \|v\|_{L^p}$$