Worked Examples and Solutions for the Book: A Text-Book of Convergence by William Leonard Ferrar

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To my brother Doug.

Introduction

Here you'll find solutions to the problems that I wrote up as I worked through this excellent book. I would say that the problems you will find here are more challenging than the ones found in a typical a first year calculus course. They are a great introduction to some more advanced techniques. For some of the problems I used `R` to perform any needed calculations or plots. Any code snippets for various exercises can be found at the following location:

`http://www.waxworksmath.com/Authors/A_F/Ferrar/ferrar.html`

I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Chapter 1 (Preliminary Discussion)

Examples I

Exercise 1

Using partial fractions we can write the terms u_n as

$$u_n = \frac{1}{n(n+1)(n+2)(n+3)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} + \frac{D}{n+3}\,.$$

Multiplying by the denominator of the fraction on the left-hand-side

$$1 = A(n+1)(n+2)(n+3) + Bn(n+2)(n+3) + Cn(n+1)(n+3) + Dn(n+1)(n+2)\,.$$

If we set $n = 0$ in the above we get $A = \frac{1}{6}$. If we set $n = -1$ in the above we get $B = -\frac{1}{2}$. If we set $n = -2$ in the above we get $C = \frac{1}{2}$. Finally if we set $n = -3$ in the above we get $D = -\frac{1}{6}$. Thus we can write the terms of our series u_n as

$$\begin{aligned}
u_n &= \frac{1}{6n} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} - \frac{1}{6(n+3)} \\
&= \frac{1}{6n} - \frac{3}{6(n+1)} + \frac{3}{6(n+2)} - \frac{1}{6(n+3)} \\
&= \frac{1}{6n} - \frac{1}{6(n+1)} - \frac{2}{6(n+1)} + \frac{2}{6(n+2)} + \frac{1}{6(n+2)} - \frac{1}{6(n+3)} \\
&= \frac{1}{6}\left[\frac{1}{n} - \frac{1}{n+1}\right] - \frac{1}{3}\left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{1}{6}\left[\frac{1}{n+2} - \frac{1}{n+3}\right]\,.
\end{aligned}$$

This last expression is in a form that we can sum and we find

$$\begin{aligned}
\sum_{n=1}^{N} u_n &= \frac{1}{6}\left[1 - \frac{1}{N+1}\right] - \frac{1}{3}\left[\frac{1}{2} - \frac{1}{N+2}\right] + \frac{1}{6}\left[\frac{1}{3} - \frac{1}{N+3}\right] \\
&= \frac{1}{18} - \frac{1}{6}\left[\frac{1}{N+1} - \frac{2}{N+2} + \frac{2}{N+3}\right]\,.
\end{aligned}$$

Letting $N \to \infty$ we get

$$\sum_{n=1}^{\infty} u_n = \frac{1}{18}\,.$$

Which is the conclusion reached in the book.

Exercise 2

Using partial fractions we can write the terms u_n as

$$u_n = \frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}\,.$$

Multiplying by the denominator of the fraction on the left-hand-side gives

$$1 = A(n+2) + Bn\,.$$

If we set $n = 0$ in the above we get $A = \frac{1}{2}$. If we set $n = -2$ in the above we get $B = -\frac{1}{2}$. Thus we have

$$\begin{aligned}
\frac{1}{n(n+2)} &= \frac{1}{2n} - \frac{1}{2(n+2)} \\
&= \frac{1}{2n} - \frac{1}{2(n+1)} + \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+2}\right].
\end{aligned}$$

This last expression we can sum to get

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+2)} &= \frac{1}{2}\left[1 - \frac{1}{N+1}\right] + \frac{1}{2}\left[\frac{1}{2} - \frac{1}{N+2}\right] \\
&= \frac{1}{2} + \frac{1}{4} - \frac{1}{2}\left[\frac{1}{N+1} + \frac{1}{N+2}\right] \\
&= \frac{3}{4} - \frac{1}{2}\left[\frac{1}{N+1} + \frac{1}{N+2}\right].
\end{aligned}$$

Thus we find

$$\sum_{n=1}^{\infty} u_n = \frac{3}{4}\,.$$

Exercise 3

Using partial fractions we can write the terms u_n as

$$\frac{n}{(n+1)(n+2)(n+3)} = \frac{A}{n+1} + \frac{B}{n+2} + \frac{C}{n+3}\,.$$

Multiplying by the denominator of the fraction on the left-hand-side gives

$$n = A(n+2)(n+3) + B(n+1)(n+3) + C(n+1)(n+2)\,.$$

If we set $n = -1$ in the above we get $A = -\frac{1}{2}$. If we set $n = -2$ in the above we get $B = 2$. Finally, if we set $n = -3$ in the above we get $C = -\frac{3}{2}$. Thus we have shown that we can write u_n as

$$u_n = -\frac{1}{2(n+1)} + \frac{2}{n+2} - \frac{3}{2(n+3)}\,.$$

Note that using this we can also write u_n as

$$\begin{aligned}
u_n &= -\frac{1}{2}\frac{1}{n+1} + \frac{3}{2}\frac{1}{n+2} + \frac{1}{2}\frac{1}{n+2} - \frac{3}{2}\frac{1}{n+3} \\
&= -\frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{n+2} - \frac{1}{n+3}\right].
\end{aligned}$$

This last expression is in a form we can sum easily. We find

$$\begin{aligned}\sum_{n=1}^{N} u_n &= -\frac{1}{2}\left[\frac{1}{2} - \frac{1}{N+2}\right] \\ &= -\frac{1}{4} + \frac{1}{2(N+2)} + \frac{1}{2} - \frac{3}{2(N+3)} \\ &= \frac{1}{4} + \frac{1}{2}\left[\frac{1}{N+2} - \frac{3}{N+3}\right].\end{aligned}$$

Letting $N \to \infty$ we get $\sum_{n=1}^{\infty} u_n = \frac{1}{4}$.

Exercise 4

Part (i): Using partial fractions we can write the terms u_n as

$$u_n = \frac{2n+3}{(n+1)(n+2)(n+3)} = \frac{A}{n+1} + \frac{B}{n+2} + \frac{C}{n+3}.$$

Multiplying by the denominator of the fraction on the left-hand-side gives

$$2n+3 = A(n+2)(n+3) + B(n+1)(n+3) + C(n+1)(n+2).$$

If we set $n = -1$ in the above we get $A = \frac{1}{2}$. If we set $n = -2$ in the above we get $B = 1$. If we set $n = -3$ in the above we get $C = -\frac{3}{2}$. Thus we have

$$\begin{aligned}u_n &= \frac{1}{2(n+1)} + \frac{1}{n+2} - \frac{3}{2(n+3)} \\ &= \frac{1}{2(n+1)} \quad \frac{1}{2(n+2)} + \frac{1}{2(n+2)} + \frac{1}{n+2} - \frac{3}{2(n+3)} \\ &= \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{3}{2}\left[\frac{1}{n+2} - \frac{1}{n+3}\right].\end{aligned}$$

This last expression is in the form that we can sum and we find

$$\begin{aligned}\sum_{n=1}^{N} u_n &= \frac{1}{2}\left[1 - \frac{1}{N+2}\right] + \frac{3}{2}\left[\frac{1}{3} - \frac{1}{N+3}\right] \\ &= 1 - \frac{1}{2}\left[\frac{1}{N+2} + \frac{3}{N+3}\right].\end{aligned}$$

If we let $N \to \infty$ then we can conclude that the sum of u_n is one.

Part (ii): To start with we will cancel n on the top and bottom of the given fraction and write it out using partial fractions to get

$$u_n = \frac{n}{(n-1)(n+1)(n+2)} = \frac{A}{n-1} + \frac{B}{n+1} + \frac{C}{n+2}.$$

If we multiply by denominator of the fraction on the left-hand-side gives

$$n = A(n+1)(n+2) + B(n-1)(n+2) + C(n-1)(n+1)\,.$$

If we let $n = 1$ we get $A = \frac{1}{6}$. If we let $n = -1$ then we get $B = \frac{1}{2}$. If we let $n = -2$ then we get $C = -\frac{2}{3}$. Thus we have shown that we can write u_n as

$$\begin{aligned}
u_n &= \frac{1}{6(n-1)} + \frac{1}{2(n+1)} - \frac{2}{3(n+2)} \\
&= \frac{1}{6}\left[\frac{1}{n-1} - \frac{1}{n}\right] + \frac{1}{6n} + \frac{1}{2(n+1)} - \frac{2}{3(n+2)} \\
&= \frac{1}{6}\left[\frac{1}{n-1} - \frac{1}{n}\right] + \frac{1}{6n} - \frac{1}{6(n+1)} + \frac{1}{6(n+1)} + \frac{1}{2(n+1)} - \frac{2}{3(n+2)} \\
&= \frac{1}{6}\left[\frac{1}{n-1} - \frac{1}{n}\right] + \frac{1}{6}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{1}{6(n+1)} + \frac{3}{6(n+1)} - \frac{4}{6(n+2)} \\
&= \frac{1}{6}\left[\frac{1}{n-1} - \frac{1}{n}\right] + \frac{1}{6}\left[\frac{1}{n} - \frac{1}{n+1}\right] + \frac{2}{3}\left[\frac{1}{n+1} - \frac{1}{n+2}\right].
\end{aligned}$$

This last expression is in a form that we can sum. Note that we cannot start our summation at $n = 1$ since that is a singularity of the fraction we are summing. Thus we will start our summation at $n = 2$. When we do this we find

$$\sum_{n=2}^{N} \frac{1}{6}\left[1 - \frac{1}{N}\right] + \frac{1}{6}\left[\frac{1}{2} - \frac{1}{N+1}\right] + \frac{2}{3}\left[\frac{1}{3} - \frac{1}{N+2}\right].$$

If we let $N \to \infty$ then we see

$$\sum_{n=2}^{\infty} u_n = \frac{1}{6} + \frac{1}{12} + \frac{2}{9} = \frac{17}{36}\,.$$

Part (iii): Using partial fractions we can write u_n as

$$u_n = \frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}\,.$$

If we multiply both sides by the denominator of the fraction on the left-hand-side we get

$$2n + 1 = An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2\,.$$

If we let $n = 0$ we get $B = 1$. If we let $n = -1$ then we get $D = -1$. Thus we have just argued that

$$\begin{aligned}
2n + 1 &= An(n+1)^2 + (n+1)^2 + Cn^2(n+1) - n^2 \\
&= An(n+1)^2 + 2n + 1 + Cn^2(n+1) \\
&= A(n^3 + 2n^2 + n) + 2n + 1 + C(n^3 + n^2)\,.
\end{aligned}$$

If we take the derivative of this expression with respect to n we get

$$2 = A(3n^2 + 4n + 1) + 2 + C(3n^2 + 2n)\,.$$

If we let $n = 0$ we get $A = 0$. If we let $n = -1$ we get $C = 0$. Thus we have shown

$$u_n = \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}\,.$$

This is a form that we can sum and we have

$$\sum_{n=1}^{N} u_n = 1 - \frac{1}{(N+1)^2}\,.$$

Thus $\sum_{n=1}^{\infty} u_n = 1$.

Part (iv) : Using partial fractions we can write u_n as

$$u_n = \frac{3n+5}{(n+1)(n+2)(n+3)} = \frac{A}{n+1} + \frac{B}{n+2} + \frac{C}{n+3}\,.$$

If we multiply both sides by the denominator of the fraction on the left-hand-side we get

$$3n+5 = A(n+2)(n+3) + B(n+1)(n+3) + C(n+1)(n+2)\,.$$

If we let $n = -1$ we get $A = 1$. If we let $n = -2$ then we get $B = 1$. Finally if we let $n = -3$ to get $C = -2$. Thus we have shown that

$$\begin{aligned} u_n &= \frac{1}{n+1} + \frac{1}{n+2} - \frac{2}{n+3} \\ &= \frac{1}{n+1} - \frac{1}{n+2} + \frac{2}{n+2} - \frac{2}{n+3} \\ &= \frac{1}{n+1} - \frac{1}{n+2} + 2\left[\frac{1}{n+2} - \frac{1}{n+3}\right]\,. \end{aligned}$$

This is in a form that we can sum explicitly. We find

$$\sum_{n=1}^{N} u_n = 1 - \frac{1}{N+2} + 2\left[\frac{1}{3} - \frac{1}{N+3}\right] = \frac{5}{3} - \frac{1}{N+2} - \frac{2}{N+3}\,.$$

Using this we find $\sum_{n=1}^{\infty} u_n = \frac{5}{3}$.

Chapter 2 (Formal Definitions)

Note that in the first few of these solutions I will present the proofs in more detail. In later solutions I will present less details as the general ideas should be understood the more problems the student works.

Examples II

Exercise 1

Part (i): To show α_n converges to zero for any given $\epsilon > 0$ we must be able to find a value of N such that $|\alpha_n| < \epsilon$ for all $n \geq N$. In this case this is the statement that

$$\left|(-1)^n \frac{1}{n}\right| < \epsilon\,,$$

for all $n \geq N$. The above will be true for any n such that $n > \frac{1}{\epsilon}$. Thus we have found a value of N such that $|\alpha_n| < \epsilon$ if we take N to be any integer larger than $\frac{1}{\epsilon}$.

Part (ii): For this problem to show α_n converges to zero for any given $\epsilon > 0$ we must be able to find a value of N such that $|\alpha_n| < \epsilon$ for all $n \geq N$. In this case this is the statement that

$$\left|\frac{n+1}{n^2+2}\right| < \epsilon\,,$$

for all $n \geq N$. To show this we first note that

$$n + 1 < 2n\,,$$

for $n > 1$ and that

$$n^2 + 2 > n^2\,,$$

for all n. Thus combining these two inequalities we have

$$\frac{n+1}{n^2+2} < \frac{2n}{n^2} = \frac{2}{n}\,.$$

Thus given any $\epsilon > 0$ we can obtain $|\alpha_n| < \epsilon$ when

$$\frac{2}{n} < \epsilon\,.$$

This later inequalities will happen for $n > \frac{2}{\epsilon}$. Thus we need to take N to be an integer larger than the value of

$$\frac{2}{\epsilon}\,,$$

and in that case we will have $|\alpha_n| < \epsilon$ for all $n \geq N$.

Part (iii): For this part, as before, we assume that we are given $\epsilon > 0$ then we seek to find a value of N such that $|\alpha_n| < \epsilon$ for all $n \geq N$. We will have $|\alpha_n| < \epsilon$ if

$$\frac{1}{\sqrt{n}} < \epsilon\,,$$

or

$$n > \frac{1}{\epsilon^2}\,.$$

Thus if we pick N larger than $\frac{1}{\epsilon^2}$ we will have $|\alpha_n| < \epsilon$ for all $n \geq N$.

Part (iv): Assume that we are given $\epsilon > 0$ then we seek to find a value of N such that $|\alpha_n| < \epsilon$ for all $n \geq N$. To find such a N we note that we can find an upper bound for the numerator as

$$n^2 + 3 < n^2 + 3n^2 = 4n^2\,,$$

for $n > 1$. For the denominator we can find a lower bound by noting that we can take

$$n^3 - 1 > n^3 - \frac{n^3}{2} = \frac{n^3}{2}$$

Which will happen if

$$-1 > -\frac{n^3}{2} \quad \text{or} \quad \frac{n^3}{2} > 1 \quad \text{or} \quad n > \sqrt[3]{2} \approx 2.828427\,.$$

Thus if we pick n larger than the maximum of these two numbers (one and 2.828427) say the number 3 then we have

$$|\alpha_n| < \frac{4n^2}{\frac{n^3}{2}} = \frac{8}{n}\,.$$

We can make this smaller than any value of ϵ if we take $n > \frac{8}{\epsilon}$. Thus in summary if we have N larger than the maximum of the three numbers $\{1, \sqrt[3]{2}, \frac{8}{\epsilon}\}$ we will have $|\alpha_n| < \epsilon$ for all $n \geq N$.

Part (v): Assume that we are given $\epsilon > 0$ then we seek to find a value of N such that $|\alpha_n| < \epsilon$ for all $n \geq N$. To find such a N we first note that $n + 5 < n + 5n = 6n$ for $n > 1$. Then using that result we have

$$|\alpha_n| < \frac{6n}{n^{3/2}} = \frac{6}{\sqrt{n}}\,.$$

We can make this less than ϵ if we take $n > \frac{36}{\epsilon^2}$. If we take N larger than one and $\frac{36}{\epsilon^2}$ we will have $|\alpha_n| < \epsilon$ for all $n \geq N$.

Exercise 2

Part (i): We need to consider $|\alpha_n - 1|$ and show that this can be made smaller than any ϵ if n is large enough. The difference in the absolute value is

$$\frac{n}{n+1} - 1 = \frac{-1}{n+1}\,.$$

Thus $|\alpha_n - 1| = \frac{1}{n+1}$. Since $n+1 > n$ for all $n \geq 1$ we have that

$$|\alpha_n - 1| < \frac{1}{n}\,,$$

for all $n \geq 1$. We can make this right-hand-side less than any $\epsilon > 0$ if we take $n > \frac{1}{\epsilon}$.

Part (ii): We need to consider $|\alpha_n - 3|$ and show that this can be made smaller than ϵ if n is made large enough. Note that we can write the expression for α_n as

$$\alpha_n = \frac{3n^2+1}{n^2-5n} = \frac{3n^2 - 3(5n) + 3(5n) + 1}{n^2-5n} = \frac{3(n^2-5n) + 15n + 1}{n^2-5n} = 3 + \frac{15n+1}{n^2-5n}\,.$$

From the given expression if we can show that the fraction $\frac{15n+1}{n^2-5n}$ converges to zero then by using the result from Example 6 in this section (that result is the statement that if β_n converges to β then $\beta_n + c$ converges to $\beta + c$) we will have that $\alpha_n \to 3$ as we desired to show.

To show that

$$\frac{15n+1}{n^2-5n} \to 0\,,$$

we first bound the numerator above as

$$15n + 1 < 15n + n = 16n\,,$$

for all positive $n > 1$. Next for the denominator we have

$$n^2 - 5n > \frac{1}{2}n^2\,,$$

when

$$-5n > -\frac{1}{2}n^2 \quad \text{or} \quad 5 < \frac{n}{2} \quad \text{or} \quad n > 10\,.$$

Thus using these two inequalities we have

$$\frac{15n+1}{n^2-5n} < \frac{32n}{n^2} = \frac{32}{n}\,.$$

Thus if given $\epsilon > 0$ we can make our fraction less than ϵ if we take $N > \frac{32}{\epsilon}$.

Part (iii): Note that we can write the expression for α_n as the following

$$\begin{aligned}
\alpha_n &= \frac{4(n^3 - 2n^2 + 1) + 8n^2 - 4 + 6n - 7}{n^3 - 2n^2 + 1} \\
&= 4 + \frac{8n^2 + 6n - 11}{n^3 - 2n^2 + 1}\,. \qquad (1)
\end{aligned}$$

Thus if we can show that

$$\frac{8n^2 + 6n - 11}{n^3 - 2n^2 + 1} \to 0\,,$$

the original expression for α_n has a limit of four. To show this note that we can find an upper bound on the numerator as

$$8n^2 + 6n - 11 < 8n^2 + 6n < 8n^2 + 6n^2 < 14n^2 \, ,$$

for all positive $n > 1$. Next we can find a lower bound on the denominator as

$$n^3 - 2n^2 + 1 > n^3 - 2n^2 > n^3 - \frac{1}{2}n^3 = \frac{1}{2}n^3 \, .$$

Which is true when

$$-2n^2 > -\frac{1}{2}n^3 \quad \text{or} \quad 4 < n \, .$$

Thus we have

$$\frac{8n^2 + 6n - 11}{n^3 - 2n^2 + 1} < \frac{28n^2}{n^3} = \frac{28}{n} \, .$$

Given an $\epsilon > 0$ we can make this fraction less than ϵ if we take $N > \frac{28}{\epsilon}$.

A more heuristic way of making this same argument is the following. Consider the fraction

$$f(n) \equiv \frac{8n^2 + 6n - 11}{n^3 - 2n^2 + 1} = \frac{8}{n} \left(\frac{n^2 + \frac{3}{4}n - \frac{11}{8}}{n^2 - 2n + \frac{1}{n}} \right) \, . \tag{2}$$

Then as $n \to +\infty$ we have

$$\frac{n^2 + \frac{3}{4}n - \frac{11}{8}}{n^2 - 2n + \frac{1}{n}} \to 1 \, ,$$

and $\frac{8}{n} \to 0$ as $n \to +\infty$ we expect the total fraction $f(n) \to 0$ as $n \to \infty$.

Exercise 3

Part (i): For the given expression for α_n we have

$$\alpha_n = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1} \, ,$$

which shows that the value of α_n is one minus something that gets smaller as n increases. Thus we have that $\alpha_n < 1$ giving an upper bound (that is never obtained) for α_n. This form also makes us expect that $\alpha_{n+1} > \alpha_n$ since in α_{n+1} we are subtracting a smaller fraction from the constant one than in α_n. We can "test" if $\alpha_{n+1} > \alpha_n$ is true by assuming it is and seeing if using reversible transformations we can end up with an obviously true statement. The statement $\alpha_{n+1} > \alpha_n$ is equivalent to

$$\frac{n+1}{n+2} > \frac{n}{n+1} \, ,$$

or

$$(n+1)^2 > n(n+2) \, .$$

If we expand both sides and cancel common expressions on both sides we get $1 > 0$ which we know to be true. Thus $\alpha_{n+1} > \alpha_n$ and so we have that $\alpha_n \geq \alpha_1 = \frac{1}{2}$ for all $n \geq 1$ showing the lower bound for α_n.

Part (ii): As suggested in the hint in Equation 1 we have decomposed α_n into a constant plus a function of n and we have shown that $f(n) \to 0$ as n increases. From this we have that $\alpha_n > 4$ and we have no least element. From the above arguments we might expect that $f(n+1) < f(n)$ for n large enough. We can attempt to show this by assuming that it is true and then performing reversible transformations on the inequality until we end up with an expression known to be true. The expression $f(n+1) < f(n)$ is then equivalent to

$$\frac{8(n+1)^2 + 6(n+1) - 11}{(n+1)^3 - 2(n+1)^2 + 1} < \frac{8n^2 + 6n - 11}{n^3 - 2n^2 + 1} .$$

One would need to verifty this by clearing denominators and simplifying. Then if this is true we have that $\alpha_n \leq 4 + f(1)$ showing that α_n has a greatest value.

Exercise 4

Let $x < 1$ then in this example we want to show that x^n converges to zero as $n \to \infty$. Let $\epsilon > 0$ be given. Since ϵ is typically thought of as "small", we can assume that $\epsilon < 1$. For this problem we want to find a value of N such that

$$x^n < \epsilon \, ,$$

for all $n \geq N$. If we take the logarithm of both sides of the above we have

$$n \log(x) < \log(\epsilon) \, .$$

As $x < 1$ we have $\log(x) < 0$ so if we divide both sides of the above inequality by $\log(x)$ we must reverse the direction of the inequality to get

$$n > \frac{\log(\epsilon)}{\log(x)} \, .$$

Now as both $\log(\epsilon)$ and $\log(x)$ are negative the above fraction is positive (as it must be). Thus if we pick a value of N that is larger than this fraction we will have $|\alpha_n| < \epsilon$ when $n \geq N$.

Exercise 5

Now as y is larger than one we can write it as $y = 1 + p$ where p is a positive number. In taking the nth power of y we note that

$$y^n = (1 + p)^n \, ,$$

and the right-hand-side of the above must be *larger* than any *single* term we choose taken from the binomial expansion of the expression $(1+p)^n$. This is because by the binomial expansion $(1+p)^n$ is equal to the sum of $n+1$ positive terms. Lets consider this inequality written with the $k+1$st binomial expansion term which gives

$$\begin{aligned} y^n &> \binom{n}{k+1} p^{k+1} \\ &= \frac{n!}{(k+1)!(n-k-1)!} p^{k+1} \\ &= \frac{n(n-1)(n-2)\cdots(n-k)}{(k+1)!} p^{k+1} \\ &> \frac{(n-k)^{k+1} p^{k+1}}{(k+1)!} \,. \end{aligned}$$

Thus using this we have that

$$\begin{aligned} \frac{n^k}{y^n} &< \frac{n^k (k+1)!}{(n-k)^{k+1} p^{k+1}} = \frac{(k+1)!}{p^{k+1}} \frac{n^k}{(n-k)^{k+1}} = \frac{(k+1)!}{p^{k+1}} \frac{n^k}{n^{k+1}\left(1-\frac{k}{n}\right)^{k+1}} \\ &= \frac{(k+1)!}{p^{k+1}\left(1-\frac{k}{n}\right)^{k+1}} \left(\frac{1}{n}\right) . \end{aligned}$$

Now as

$$1 - \frac{k}{n} > 1 - \frac{k}{k+1} = \frac{1}{k+1} \,,$$

when $n > k+1$. Thus we have

$$\frac{n^k}{y^n} < \left(\frac{(k+1)!(k+1)^{k+1}}{p^{k+1}}\right) \frac{1}{n} < \epsilon \,,$$

If we take

$$n > \frac{(k+1)!(k+1)^{k+1}}{p^{k+1}\epsilon} \,.$$

This shows that the fraction $\frac{n^k}{y^n}$ converges to zero as $n \to \infty$.

Exercise 6

Since we are told that α_n converges to α given any $\epsilon > 0$ we can find a N such that

$$|\alpha_n - \alpha| < \epsilon \,,$$

for all $n \geq N$. We can write the above inequality (by adding and subtracting c) as

$$|(\alpha_n - c) - (\alpha - c)| < \epsilon \,.$$

for all $n \geq N$. This is the same as the statement that the sequence $\alpha_n - c$ converges to $\alpha - c$.

To show convergence of $c\alpha_n$ to $c\alpha$ let $\epsilon > 0$ be given and find a value of N such that

$$|\alpha_n - \alpha| < \frac{\epsilon}{|c|},$$

for all $n \geq N$. If we multiply both sides by $|c|$ this becomes

$$|c\alpha_n - c\alpha| < \epsilon,$$

which shows that $c\alpha_n$ converges to $c\alpha$.

Exercise 7

To start with recall that it is a fact of absolute values that

$$||\alpha_n| - |\alpha|| \leq |\alpha_n - \alpha|.$$

Now because we are told that α_n converges to α if we are given a value of $\epsilon > 0$ we can find a N such that $|\alpha_n - \alpha| < \epsilon$ for all $n \geq N$. By the above inequality we have *also* found a value of N such that $||\alpha_n| - |\alpha|| < \epsilon$ for all $n \geq N$ since the same value of N works.

Exercise 8

To show that a sequence diverges through negative values we have to show that the sequence $-\alpha_n$ diverges though positive values.

Part (i): As $-\alpha_n = n^2 - 6$ if we are given a value of $A > 0$ then we can make $-\alpha_n$ greater than this value if we take

$$n^2 - 6 > A \quad \text{or} \quad n > \sqrt{A+6}.$$

Part (ii): Here we have $-\alpha_n = 2^n$ so we will have $-\alpha_n > A$ if $2^n > A$ or $n > \log_2(A)$.

Part (iii): Here we have $\alpha_n = (-3)^{2n+1} = -3(-3)^{2n} = -3 \cdot 9^n$. Thus $-\alpha_n = 3 \cdot 9^n$. If we take $A > 0$ then we can find n such that $-\alpha_n > A$ if we take

$$n > \frac{\log(A/3)}{\log(9)}.$$

Chapter 3 (Bounds: Monotonic Sequences)

Examples III

Exercise 1

Part (i): If a sequence is monotonic decreasing then

$$\alpha_{n+1} < \alpha_n \,,$$

or squaring both sides we must have

$$\alpha_{n+1}^2 < \alpha_n^2 \,.$$

For the given expression for α_n this is

$$1 + \frac{1}{n+1} < 1 + \frac{1}{n} \,,$$

or

$$\frac{1}{n+1} < \frac{1}{n} \,,$$

which we know to be true.

Part (ii): As this sequence is just the one from Part (i) but shifted down by the constant one, if the first sequence is a monotonic decreasing sequence then this one must be is also.

Part (iii): We start by writing α_n as

$$\alpha_n = \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}} - 1\right)}{\sqrt{n^3}} = \frac{\sqrt{1+\frac{1}{n}} - 1}{n} \,.$$

Next we use Bernoulli's inequality [1] one version of which is

$$(1+x)^r \leq 1 + rx \,, \tag{3}$$

when $0 \leq r \leq 1$ and x is a real number $x \geq -1$ to bound our numerator above as

$$\sqrt{1+\frac{1}{n}} = \left(1 + \frac{1}{n}\right)^{1/2} \leq 1 + \frac{1}{2n} \,.$$

This means that we can bound α_n as

$$\alpha_n \leq \frac{1 + \frac{1}{2n} - 1}{n} = \frac{1}{2n^2} \,.$$

As this bounds α_n by a monotonically decreasing sequence it can be shown that α_n itself is a monotonically decreasing sequence.

[1] https://en.wikipedia.org/wiki/Bernoulli's_inequality

Exercise 2

Part (i): To work this exercise we first recall Stirling's approximation or

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + O\left(\frac{1}{n}\right)\right) .$$

Thus using that we have

$$\frac{1}{n!} = \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n \left(\frac{1}{1 + O\left(\frac{1}{n}\right)}\right) \approx \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n \left(1 - O\left(\frac{1}{n}\right)\right) .$$

Thus using this we have

$$\frac{n^n}{n!} = \frac{e^n}{\sqrt{2\pi n}}\left(1 - O\left(\frac{1}{n}\right)\right) ,$$

which gets larger as n increase and thus α_n is increasing.

Part (ii): The sequence α_n will be increasing if $\alpha_{n+1} > \alpha_n$ or for this exercise that is

$$(n+1)^2 - (n+1) + 2 > 3n^2 - n + 2 .$$

Expanding the left-hand-side and simplifying, we find that this inequality is equivalent to

$$6n + 2 > 0 ,$$

which is itself equivalent to $n > -\frac{1}{3}$ which we know is true. As every step is reversible the original statement is true and α_n is increasing.

Exercise 3

For this sequence to be increasing we must have

$$(n+1)^2 + 2b(n+1) + c > an^2 + 2bn + c .$$

On expanding and canceling common terms this becomes

$$2an + a - 2b > 0 ,$$

which is equivalent to

$$n > \frac{2b - a}{2a} ,$$

as we were to show.

Exercise 4

If $b_n > 0$ and $a_{n+1} > a_n$ then we have

$$a_{n+1}(b_1 + b_2 + \cdots + b_n) > a_n(b_1 + b_2 + \cdots + b_n) \, .$$

Now as a_n is increasing we have

$$a_n > a_{n-1} > a_{n-2} > \cdots > a_2 > a_1 \, ,$$

and thus the left-hand-side of the above is bounded below by

$$a_{n+1}(b_1 + b_2 + \cdots + b_n) > a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \, .$$

Now if our sequence u_n is defined as

$$u_n = \frac{\sum_{i=1}^n a_i b_i}{\sum_{i=1}^n b_i} \, ,$$

or

$$u_n(b_1 + b_2 + \cdots + b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \, .$$

Now if $b_n > 0$ and a_n is monotonically increasing from what we proved above we have that

$$u_{n+1}(b_1 + b_2 + \cdots + b_n) > a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \, ,$$

or

$$u_{n+1} > \frac{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n}{b_1 + b_2 + \cdots + b_n} = u_n \, ,$$

Thus u_n is an increasing sequence.

Exercise 6

If we assume that a_n converges to a limit (say the number l) then l must satisfy

$$l = \frac{k}{1 + l} \, ,$$

which simplifies to

$$l^2 + l - k = 0 \, .$$

We can show this if we let a_n be given by $a_n = l + b_n$ with $b_n \to 0$ as $n \to \infty$. In this case from the definition of a_n we have that b_n must satisfy

$$(l + b_{n+1})(1 + b_n + l) = k \, ,$$

or expanding and moving all terms with b to the left-hand-side we get

$$l b_n + (l + b_n) b_{n+1} + b_n b_{n+1} = k - l - l^2 \, .$$

Now the left-hand-side goes to zero as n goes to infinity which means that the right-hand-side must be *equal* to zero. The book has a nice argument for that in this section of the text. Thus we have that

$$l^2 + l - k = 0 \, ,$$

as we were to show.

Exercise 7

If a_n tends to a finite limit then this again means we can write it as $a_n = l + b_n$ where $b_n \to 0$ as $n \to \infty$. Then from the definition of a_n we have that b_n must satisfy

$$l + b_{n+1} = l^2 + 2lb_n + b_n^2 + k - k^2 \,,$$

or expanding and moving all terms with b to the left-hand-side we get

$$b_n^2 + 2lb_n - b_{n+1} = k^2 - k - l^2 + l \,.$$

Now the left-hand-side goes to zero as n goes to infinity which means that the right-hand-side must be *equal* to zero. This means that

$$l^2 - l - k(k-1) = 0 \,,$$

or using the quadratic equation to solve for l this give

$$l = \frac{l \pm \sqrt{4k^2 - 4(k+1)}}{2} = \frac{1 \pm (2k-1)}{2} \,,$$

assuming that $2k - 1 \geq 0$. The minus sign in the above expression gives

$$l = \frac{2-2k}{2} = 1 - k \,,$$

while the plus sign in the above expression gives

$$k = \frac{1 + 2k - 1}{2} = k \,,$$

as we were to show.

Chapter 5 (The Comparison Test; The Ratio Test)

Examples IV

Exercise 1

As we have the bounds

$$\frac{1}{\left(n+\frac{1}{2}\right)^2} < \frac{1}{n^2},$$

and the sum of the series with terms $\frac{1}{n^2}$ converges then the sum of the series with the smaller terms must converge also by the comparison test.

Next as

$$3n - 1 < 3n,$$

for all n we have that

$$\frac{1}{3n-1} > \frac{1}{3n}.$$

As $\sum \frac{1}{3n} = \frac{1}{3}\sum \frac{1}{n}$ and the sum on the right-hand-side or $\sum \frac{1}{n}$ diverges so we must have that $\sum \frac{1}{3n-1}$ also diverges by the comparison test.

Exercise 2

For the terms in the first sum notice that

$$\frac{1}{(2n+1)^3} < \frac{1}{8n^3},$$

and using the comparison test as $\sum \frac{1}{n^3}$ converges we know that $\sum \frac{1}{(2n+1)^3}$ also converges.

For the terms of the second sum notice that

$$\frac{n}{(3n+2)^3} < \frac{n}{(3n)^3} = \frac{n}{27n^3} = \frac{1}{27n^2}.$$

Then using the comparison test as $\sum \frac{1}{27n^2}$ converges so must $\sum \frac{n}{(3n+2)^3}$.

Exercise 3

For the terms of the first sum we have that $4n - 1 < 4n$ for all n we have

$$\frac{1}{4n-1} > \frac{1}{4n},$$

squaring both sides of that inequality this means that

$$\frac{1}{(4n-1)^2} > \frac{1}{(4n)^2},$$

and multiplying both sides by n we get that

$$\frac{n}{(4n-1)^2} > \frac{n}{(4n)^2} = \frac{1}{16n}.$$

Now using the comparison test we see that as $\sum \frac{1}{16n} = \frac{1}{16}\sum \frac{1}{n}$ diverges so must $\sum \frac{n}{(4n-1)^2}$.

For the terms of the second series as $2n-1 < 2n$ for all n by taking the square root of that inequality we have that

$$(2n-1)^{1/2} < (2n)^{1/2},$$

or equivalently that

$$\frac{1}{(2n)^{1/2}} < \frac{1}{(2n-1)^{1/2}}.$$

Now as $\sum \frac{1}{(2n)^{1/2}} = \frac{1}{2^{1/2}} \sum \frac{1}{n^{1/2}}$ diverges by the comparison test we must have that $\sum \frac{1}{(2n-1)^{1/2}}$ diverges also.

Examples V

Exercise 1

For each of the given examples I will present in the order (left-to-right and top-to-bottom) the terms v_n of a series that converges and has

$$\frac{u_n}{v_n} \to L > 0,$$

as $n \to \infty$. One can then use Theorem 9 in the text to prove convergence. These results are obtained by considering the numerator and the denominator of the terms u_n and retaining only the leading order terms in each. We have

$$v_n = \frac{1}{n^2}$$
$$v_n = \frac{1}{n^2}$$
$$v_n = \frac{1}{n^2}$$
$$v_n = \frac{1}{n^2}$$
$$v_n = \frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$$
$$v_n = \frac{n^2}{n^6} = \frac{1}{n^2}.$$

Exercise 2

As in the previous exercise for each of the given examples I will present in the order (left-to-right and top-to-bottom) the terms v_n of a series that diverges. See the previous exercise for some discussion on this. We have

$$
\begin{aligned}
v_n &= \frac{1}{n} \\
v_n &= \frac{1}{n} \\
v_n &= \frac{1}{n} \\
v_n &= \frac{1}{n} \\
v_n &= \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}} \\
v_n &= \frac{1}{n}\,.
\end{aligned}
$$

Exercise 3

The statement that $\frac{u_n}{v_n} \to 0$ means that for any $\epsilon > 0$ we can find a N_0 such that

$$
\left|\frac{u_n}{v_n}\right| < \epsilon\,,
$$

for all $n \geq N_0$. The above is equivalent to the statement that

$$
|u_n| < \epsilon |v_n|\,,
$$

for $n \geq N_0$. As both u_n and v_n are positive we have that $u_n < \epsilon v_n$ for $n \geq N_0$. We can extend this statement to all n by finding a value of C such that each expression

$$
\frac{u_n}{v_n} < C\,,
$$

for $1 \leq n \leq N_0$ is true. Since there are a finite number of such inequalities we can find such a C. Then let $C' = \max(C, \epsilon)$ and we have

$$
u_n < C' v_n \quad \text{for all } n\,.
$$

Then we can use Theorem 8 in the book to prove that $\sum u_n$ converges.

Exercise 4

The first example of u_n and v_n has the limit

$$
\frac{u_n}{v_n} = \frac{n^{-2}}{n^{-1}} = \frac{1}{n} \to 0\,,
$$

as $n \to \infty$. Here $\sum u_n$ is convergent but $\sum v_n$ is not.

For the second example keep u_n as n^{-2} and take $v_n = n^{-3/2}$ then we have

$$\frac{u_n}{v_n} = \frac{n^{-2}}{n^{-3/2}} = \frac{1}{n^{1/2}} \to 0\,,$$

as $n \to \infty$. For this example we see that both $\sum u_n$ and $\sum v_n$ are convergent.

Examples VI

Exercise 1

We will use d'Alembert's test on the absolute value of u_n. We have

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{(n+1)|x|^n}{(n+2)|x|^{n+1}} = \frac{1}{|x|}\,.$$

Our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$ or $-1 < x < +1$. For the case where $x = 1$ then our sum is $\sum(n+1)$ which has terms that increase as n gets larger (they go to infinity) and thus this sum cannot converge. If $x = -1$ then the terms of the sum are $u_n = (n+1)(-1)^n$ and the sequence of partial sums will osscilate between positive and negative values and thus the infinite sum will not converge.

For the second example, d'Alembert's test would compute

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{(n+1)|x|^n}{n+2} \times \frac{n+3}{(n+2)|x|^{n+1}} \right) = \frac{1}{|x|}\,.$$

Again our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$ or $-1 < x < +1$. If $x = 1$ then our sum is $\sum \frac{n+1}{n+2}$. This sum has terms that limit to one (and not zero) as $n \to \infty$ and thus this sum cannot converge. If $x = -1$ then $|u_n| \to 1$ (which is not zero) and thus this sum cannot converge.

For the third example, d'Alembert's test would compute

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{(n+1)|x|^n}{(n+2)(n+3)} \times \frac{(n+3)(n+4)}{(n+2)|x|^{n+1}} \right) = \frac{1}{|x|}\,.$$

Our sum will again converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$ or $-1 < x < +1$. If $x = 1$ then our sum is $\sum \frac{n+1}{(n+2)(n+3)}$. This sum converges as can be shown by using Theorem 9 by comparing it with the terms from the convergent series $\sum \frac{1}{n^2}$. If $x = -1$ then we have

$$u_n = \frac{(n+1)(-1)}{(n+2)(n+3)}\,,$$

which are the terms of a convergence sum by using the alternating series test.

Exercise 2

We can write this sum as

$$1 + \sum_{n=1}^{\infty} u_n x^n \,,$$

with

$$u_n = \frac{a(a+1)(a+2)\cdots(a+n-2)(a+n-1)b(b+1)(b+2)\cdots(b+n-2)(b+n-1)}{c(c+1)(c+2)\cdots(c+n-2)(c+n-1)d(d+1)(d+2)\cdots(d+n-2)(d+n-1)} \,.$$

Now to check this note that if $n = 1$ the products in the above become

$$u_1 = \frac{ab}{cd} \,.$$

If $n = 2$ the products in the above become

$$u_2 = \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} \,.$$

Thus from the above representation we when we compute the ratio $\frac{u_n}{u_{n+1}}$ we would get

$$\frac{u_n}{u_{n+1}} = \frac{(c+n)(d+n)}{(a+n)(b+n)x} \,. \tag{4}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{x}$. By d'Alembert's test our sum will converge if $\frac{1}{x} > 1$ which happens if $x < 1$.

If $x = 1$ then from Equation 4 our limit is one and d'Alembert's test is inconclusive. To study convergence at $x = 1$ we can use Raabe's test. To use Raabe's test we need to compute

$$\lim_{n\to\infty} n \left(\frac{u_n}{u_{n+1}} - 1 \right) \,.$$

From Equation 4 when $x = 1$ this expression is

$$\begin{aligned}
R &= \lim_{n\to\infty} n \left(\frac{(c+n)(d+n)}{(a+n)(b+n)} - 1 \right) \\
&= \lim_{n\to\infty} n \left(\frac{(c+n)(d+n) - (a+n)(b+n)}{(a+n)(b+n)} \right) \\
&= \lim_{n\to\infty} n \left(\frac{cd - ab + (c+d-a-b)n}{(a+n)(b+n)} \right) \\
&= c + d - a - b \,.
\end{aligned}$$

From this expression we see that our sum will converge if $c + d - a - b > 1$ and diverge if $c + d - a - b < 1$.

Exercise 3

For the first series we have
$$u_n = \frac{x^n}{n!}\,,$$
for $n \geq 0$. Will use d'Alembert's test so we need to compute
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{|x|^n}{n!}\right) \times \left(\frac{(n+1)!}{|x|^{n+1}}\right) = \lim_{n\to\infty} \frac{n+1}{|x|} = \infty\,.$$
As this is larger than one for any x our series converges for any value of x.

For the second series we have
$$u_n = \frac{x^{2n+1}}{(2n+1)!}\,,$$
for $n \geq 0$. For this series we will also use d'Alembert's test where we find
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \left(\frac{|x|^{2n+1}}{(2n+1)!}\right) \times \left(\frac{(2n+3)!}{|x|^{2n+3}}\right) = \lim_{n\to\infty} \frac{(2n+3)(2n+2)}{|x|^2} = \infty\,.$$
As this is larger than one for any x our series converges for any value of x.

Exercise 4

Using d'Alembert's test on this sum we need to compute
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n!|x|^n}{(n+1)!|x|^{n+1}} = \frac{1}{|x|} \lim_{n\to\infty} \frac{1}{n+1} = 0\,.$$
As this is smaller than one for any x our series diverges for all values of x.

Exercise 5

On this sum to use d'Alembert's test we need to compute
$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{n^k|x|^n}{(n+1)^k|x|^{n+1}} = \frac{1}{|x|} \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = \frac{1}{|x|}\,.$$
To compute this we used the fact that
$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^k = \lim_{n\to\infty} 1^k = 1\,.$$
Thus this sum will converge if $\frac{1}{|x|} > 1$ or when $|x| < 1$.

Exercise 6

Now the first sum can be written as

$$1+\sum_{n=1}^{\infty}\frac{a+(n-1)}{2^n n!}x^n\,.$$

To study convergence by using d'Alembert's test we need to compute

$$\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|}=\lim_{n\to\infty}\left(\frac{(a+n-1)|x|^n}{2^n n!}\right)\times\left(\frac{2^{n+1}(n+1)!}{(a+n)|x|^{n+1}}\right)=\frac{2}{|x|}\lim_{n\to\infty}\left(\frac{a+n-1}{a+n}\right)(n+1)=\infty\,,$$

for any x. If this limit is larger than one for any x our series converges for any value of x.

Now this second sum can be written as

$$1+\sum_{n=1}^{\infty}\frac{n(a+(n-1))}{2^n n!}x^n\,.$$

To study convergence by using d'Alembert's test we need to compute

$$\begin{aligned}\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|}&=\lim_{n\to\infty}\left(\frac{n(a+n-1)|x|^n}{2^n n!}\right)\times\left(\frac{2^{n+1}(n+1)!}{(n+1)(a+n)|x|^{n+1}}\right)\\&=\frac{2}{|x|}\lim_{n\to\infty}\left(\frac{n(a+n-1)}{(n+1)(a+n)}\right)(n+1)=\infty\,,\end{aligned}$$

for any x. If this limit is larger than one for any x our series converges for any value of x.

Exercise 7

Now this sum can be written as

$$1+\sum_{n=1}^{\infty}\frac{n(a+(n-1))}{b^n}x^n\,.$$

Using d'Alembert's test on this series we compute

$$\begin{aligned}\lim_{n\to\infty}\frac{|u_n|}{|u_{n+1}|}&=\lim_{n\to\infty}\left(\frac{n(a+n-1)|x|^n}{b^n}\right)\times\left(\frac{b^{n+1}}{(n+1)(a+n)|x|^{n+1}}\right)\\&=\frac{b}{|x|}\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{a+n-1}{a+n}\right)=\frac{b}{|x|}\,.\end{aligned}$$

This series will converge if this expression is greater than one or $|x|<b$ and diverges if it is less than one. If $x=b$ then this limit becomes one and d'Alembert's test is inconclusive. When $x=b$ the series is

$$1+\sum_{n=1}^{\infty}n(a+(n-1))\,,$$

which cannot converge as the terms in the sum increase to infinity as $n \to \infty$. We can also show that this series diverges using Raabe's test. In that case the limit we need to evaluate is

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\,(1-1) = 0\,.$$

As this is less than one our series diverges.

Exercise 8

Using d'Alembert's test for this sum we have

$$\lim_{n\to\infty} \frac{|u_n|}{|u^{n+1}|} = \lim_{n\to\infty} \left(\frac{(a+n)|x|^n}{(b+n)}\right) \times \left(\frac{(b+n+1)}{(a+n+1)|x|^{n+1}}\right) = \frac{1}{|x|}\,.$$

This series will converge if this is greater than one or when $|x| < 1$ and diverge if this smaller than one. If $x = 1$ the sum is

$$\sum \frac{a+n}{b+n}\,,$$

which does not converge as

$$\lim_{n\to\infty} \frac{a+n}{b+n} = 1 \neq 0\,.$$

We can also show that this series diverges using Raabe's test. In that case the limit we need to evaluate is

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\,(1-1) = 0\,.$$

As this is less than one our series diverges.

Exercise 9

If we are told that $u_n \leq u_{n+1}$ for all $n \geq N_0$ (and strictly "less than" for at least one value of n) then

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}}\,,$$

must be less than one and therefore by d'Alembert's test the sum $\sum_n u_n$ diverges. If instead we are told that

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1\,,$$

then by Raabe's test we can conclude that $\sum u_n$ diverges.

Exercise 10

These statements are the converses of the previous exercise in that by using d'Alembert's or Raabe's test we can show that the needed expressions for those test satisfy the needed conditions for the series to converge.

Exercise 11

Assume this series can be written as $\sum_{n=1}^{\infty} u_n$. It is not too difficult to see that it takes the form

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-4) \cdot (3n-1)} x^n \quad \text{for} \quad n \geq 1\,,$$

Using this we can write down an expression for the fraction

$$\frac{|u_n|}{|u_{n+1}|}\,.$$

Notice that the changes in going from one term in the sum u_n to the next term in the sum u_{n+1} results in the addition of some factors into the numerator and denominator. Thinking in this way we have that

$$\frac{|u_n|}{|u_{n+1}|} = \frac{1}{|x|}\left(\frac{3n-1}{2n-1}\right)\,.$$

Thus from this we see that

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \frac{3}{2|x|}\,.$$

The sum will converge if this is larger than one and diverge is this is less than one. Thus the sum converges if $-\frac{3}{2} < x < \frac{3}{2}$. If $x = \frac{3}{2}$ then to use Raabe's test we need to compute

$$\begin{aligned}
\lim_{n\to\infty} n\left(\frac{|u_n|}{|u_{n+1}|} - 1\right) &= \lim_{n\to\infty} n\left(\frac{2}{3}\left(\frac{3n-1}{2n-1}\right) - 1\right) \\
&= \lim_{n\to\infty} \frac{n}{6n-3} = \frac{1}{6}\,.
\end{aligned}$$

As this is less than one the sum must diverge when $x = \frac{3}{2}$.

Exercise 12

From the form given for u_n in the first sum we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(2n+3)}{(n+1)|x|} \to \frac{2}{|x|}\,,$$

as $n \to \infty$. Thus by d'Alembert's test this series will converge if this expression is greater than one or $|x| < 2$ or $-2 < x < +2$ and diverge if it is larger than one. To determine convergence when $x = 2$ we can use Raabe's test where we need to compute

$$\begin{aligned}
\lim_{n\to\infty} n\left(\frac{|u_n|}{|u_{n+1}|} - 1\right) &= \lim_{n\to\infty} n\left(\frac{2n+3}{2(n+1)} - 1\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{2(n+1)}\right) = \frac{1}{2}\,.
\end{aligned}$$

As this is less than one the sum must diverge when $x = 2$.

From the form given for u_n in the second sum we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(2n+1)}{(n+2)|x|} \to \frac{2}{|x|},$$

as $n \to \infty$. Thus by d'Alembert's test this series will converge if this expression is greater than one or $|x| < 2$ or $-2 < x < +2$ and diverge if it is larger than one. To determine convergence when $x = 2$ we can use Raabe's test where we need to compute

$$\begin{aligned}\lim_{n\to\infty} n\left(\frac{|u_n|}{|u_{n+1}|} - 1\right) &= \lim_{n\to\infty} n\left(\frac{2n+1}{2(n+2)} - 1\right) \\ &= \lim_{n\to\infty} n\left(\frac{-3}{2(n+2)}\right) = -\frac{3}{2}.\end{aligned}$$

While this is less than one which indicates divergence, it is negative so I'm not fully sure that Raabe's test holds in this case and thus we will try to answer the question of convergence for this sum using different methods.

To study convergence at $x = 2$ the first thing we will do is to try to write an explicit expression for u_n. Notice that we can put even factors into the denominator of u_n (and then we have to put them into the numerator also) to write u_n as

$$\begin{aligned}u_n &= \frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n-3)\cdot(2n-1)}x^n = \frac{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)(n+1)!}{1\cdot 2\cdot 3\cdot 4\ldots(2n-2)\cdot(2n-1)(2n)}x^n \\ &= \frac{2^n n!(n+1)!}{(2n)!}x^n.\end{aligned}$$

If $x = 2$ this simplifies to

$$u_n = \frac{4^n n!(n+1)!}{(2n)!}.$$

Now we cannot just apply d'Alembert's test to this expression since it must give us an indeterminate result (as it did above). To see this note that

$$\begin{aligned}\lim_{n\to\infty}\frac{u_n}{u_{n+1}} &= \lim_{n\to\infty}\left(\frac{4^n n!(n+1)!}{(2n)!} \times \frac{(2n+2)!}{4^{n+1}(n+1)!(n+2)!}\right) \\ &= \frac{1}{4}\lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+2)} = \frac{1}{4}(4) = 1.\end{aligned}$$

As another method we will use Sterling's approximation[2] or

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{5}$$

to study the behavior of u_n for large n. In this case we have

$$\begin{aligned}u_n &\sim \frac{4^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}}{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}} \\ &= \frac{\sqrt{\pi}}{e}\sqrt{n+1}\frac{(1+n)^{n+1}}{n^n}.\end{aligned}$$

[2]https://en.wikipedia.org/wiki/Stirling's_approximation

when we simplify. If we recall that

$$\left(1 + \frac{1}{n}\right)^n \to e\,,$$

as $n \to \infty$. We see that the above limit becomes

$$\frac{\sqrt{\pi}}{e}(n+1)^{3/2} \cdot e = \sqrt{\pi}(n+1)^{3/2} \to \infty\,.$$

As this means that the *terms* of our sum diverge to infinity our series must *diverge* when $x = 2$.

Exercise 13

From the form of u_n given we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(2n+5)}{|x|(n+1)} \to \frac{2}{|x|}\,,$$

Thus by d'Alembert's test this series will converge if this expression is greater than one or $|x| < 2$ or $-2 < x < +2$ and diverge if it is larger than one. If $x = 2$ then $\frac{|u_n|}{|u_{n+1}|} \to 1$ and our test is indeterminate. To determine convergence at $x = 2$ we can use Raabe's test where the limit we need to consider in this case is

$$n\left(\frac{2n+5}{2(n+1)} - 1\right) = n\left(\frac{3}{2(n+1)}\right) \to \frac{3}{2} > 1\,,$$

and our series converges when $x = 2$.

Note that the terms of the second sum are equal to five times the terms of the first sum the convergence region for this sum is the same as the convergence region for the first sum.

Exercise 14

From the form for u_n given we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(3n+4)}{|x|(n+1)} \to \frac{3}{|x|}\,,$$

as $n \to \infty$. Thus by d'Alembert's test this series will converge if this expression is greater than one or $|x| < 3$ or $-3 < x < +3$ and diverge if it is larger than one. If $x = 3$ then $\frac{|u_n|}{|u_{n+1}|} \to 1$ and our test is indeterminate. To determine convergence at $x = 3$ we can use Raabe's test where the limit we need to consider in this case is

$$n\left(\frac{3n+4}{3(n+1)} - 1\right) = n\left(\frac{1}{3(n+1)}\right) \to \frac{1}{3} < 1\,,$$

and our series diverges when $x = 3$.

For the next sum from the form for u_n given we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(3n+7)}{|x|(n+1)} \to \frac{3}{|x|}\,.$$

By d'Alembert's test this series will converge if this expression is greater than one or when $-3 < x < +3$ and diverge if it is larger than one. If $x = 3$ then $\frac{|u_n|}{|u_{n+1}|} \to 1$ and our test is indeterminate. To determine convergence at $x = 3$ we can use Raabe's test where the limit we need to consider in this case is

$$n\left(\frac{3n+7}{3(n+1)} - 1\right) = n\left(\frac{4}{3(n+1)}\right) \to \frac{4}{3} > 1\,,$$

and our series converges when $x = 3$.

Exercise 15

Part (i): For this part we will use the first sum in Exercise 14 as a guide. Namely we will consider the sum

$$\sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdots (n-1) \cdot n}{5 \cdot 9 \cdot 13 \cdots (4n-3) \cdot (4n+1)}$$

From the form for u_n given we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(4n+5)}{|x|(n+1)} \to \frac{4}{|x|}\,,$$

as $n \to \infty$. Thus by d'Alembert's test this series will converge if this expression is greater than one or $|x| < 4$ or $-4 < x < +4$ and diverge if it is larger than one. If $x = 4$ then $\frac{|u_n|}{|u_{n+1}|} \to 1$ and our test is indeterminate. To determine convergence at $x = 4$ we can use Raabe's test where the limit we need to consider in this case is

$$n\left(\frac{4n+5}{4(n+1)} - 1\right) = n\left(\frac{1}{4(n+1)}\right) \to \frac{1}{4} < 1\,,$$

and our series diverges when $x = 4$.

Part (ii): For this part we will use the second sum in Exercise 14 as a guide. Namely we will consider the sum

$$\sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdots (n-1) \cdot n}{9 \cdot 13 \cdots (4n-3) \cdot (4n+5)}\,.$$

Notice that this sum has a denominator "incremented by one" from the one considered in the previous part. This makes the denominator larger and the terms of u_n smaller and will allow the sum to converge at $x = 4$ as we will now show.

The same arguments in the previous part of this exercise will show that the sum converges for $-4 < x < +4$ and that if $x = 4$ our test is indeterminate. To determine convergence at $x = 4$ we can use Raabe's test where the limit we need to consider in this case is

$$n\left(\frac{4n+9}{4(n+1)} - 1\right) = n\left(\frac{5}{4(n+1)}\right) \to \frac{5}{4} > 1\,,$$

and our series converges when $x = 4$.

Chapter 6 (Theorems On Limits)

Examples VII

Exercise 4

Using partial fractions we can write

$$\frac{1}{\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)} = \frac{A}{n+\frac{1}{2}} + \frac{B}{n+\frac{3}{2}} + \frac{C}{n+\frac{5}{2}}\,.$$

If we multiply by the denominator of the fraction on the left-hand-side we get

$$1 = A\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right) + B\left(n+\frac{1}{2}\right)\left(n+\frac{5}{2}\right) + C\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\,.$$

If we let $n = -\frac{1}{2}$ we get

$$1 = A(1)(2) \quad \text{so} \quad A = \frac{1}{2}\,.$$

If we let $n = -\frac{3}{2}$ we get

$$1 = B(-1)(1) \quad \text{so} \quad B = -1\,.$$

If we let $n = -\frac{5}{2}$ we get

$$1 = 0 + 0 + C(-2)(-1) \quad \text{so} \quad C = \frac{1}{2}\,.$$

Using these values we have just shown that

$$\begin{aligned}
\frac{1}{\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)} &= \frac{1}{2\left(n+\frac{1}{2}\right)} - \frac{1}{n+\frac{3}{2}} + \frac{1}{2\left(n+\frac{5}{2}\right)} \\
&= \frac{1}{2\left(n+\frac{1}{2}\right)} - \frac{1}{2\left(n+\frac{3}{2}\right)} - \frac{1}{2\left(n+\frac{3}{2}\right)} + \frac{1}{2\left(n+\frac{5}{2}\right)}\,. \qquad (6)
\end{aligned}$$

As an identity we will use notice that if we sum "differences" we get

$$\sum_{n=0}^{N}(a_n - a_{n+1}) = (a_0 - a_1) + (a_1 - a_2) + \cdots + (a_{N-1} - a_N) + (a_N - a_{N+1}) = a_0 - a_{N+1}\,. \quad (7)$$

If we use this twice in Equation 6 when we sum the original expression from $n = 0$ to $n = N$ we have

$$\begin{aligned}
\sum_{n=0}^{N} \frac{1}{\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)} &= \frac{1}{2}\left(\frac{1}{\frac{1}{2}} - \frac{1}{N+\frac{3}{2}}\right) - \frac{1}{2}\left(\frac{1}{\frac{3}{2}} - \frac{1}{N+\frac{5}{2}}\right) \\
&= \frac{2}{3} - \frac{1}{2N+3} + \frac{1}{2N+5}\,.
\end{aligned}$$

If we take $N \to \infty$ we find that this sum is

$$\sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)} = \frac{2}{3}\,.$$

Exercise 5

Lets consider the partial fractions expansion

$$\frac{1}{n^2(n+1)} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} \,.$$

If we multiply by the denominator of the fraction on the left-hand-side we get

$$1 = An(n+1) + B(n+1) + Cn^2 \,.$$

If we let $n = 0$ in the above we get $B = 1$. If we let $n = -1$ in the above we get $C = 1$. If we let $n = 1$ in the above we get $A = -1$. Using all of these we can conclude that

$$\frac{1}{n^2(n+1)} = -\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n+1} = -\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{1}{n^2} \,.$$

If we sum both sides from $n = 1$ to $n = N$ and use the fact that sums of differences telescopes i.e. using Equation 7 we get

$$\sum_{n=1}^{N} \frac{1}{n^2(n+1)} = -\left(1 - \frac{1}{N+1}\right) + \sum_{n=1}^{N} \frac{1}{n^2} \,. \tag{8}$$

If we now take $N \to \infty$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = -1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \,,$$

which is the expression we were wanting to prove.

Exercise 6

As a high level overview of how we will try to solve this problem we will use partial fractions and Equation 7 to simplify the partial sums of the given expression. To start we note that using partial fractions we can show

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \,.$$

Recalling that

$$(a-b)^3 = a^3 - 3a^2b + 3ab^3 - b^3 \,,$$

if we cube this fraction we get

$$\begin{aligned}\left(\frac{1}{n(n+1)}\right)^3 &= \frac{1}{n^3} - \frac{3}{n^2(n+1)} + \frac{3}{n(n+1)^2} - \frac{1}{(n+1)^3} \\ &= \frac{1}{n^3} - \frac{1}{(n+1)^3} - \frac{3}{n^2(n+1)} + \frac{3}{n(n+1)^2} \,.\end{aligned}$$

Note that summing the first two terms will give a sum of the form $a_n - a_{n+1}$ for which we can use Equation 7. Using this insight we have

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{1}{n(n+1)}\right)^3 &= \left(1 - \frac{1}{(N+1)^3}\right) \\ &\quad - 3\sum_{n=1}^{N}\frac{1}{n^2(n+1)} + 3\sum_{n=1}^{N}\frac{1}{n(n+1)^2}\,. \end{aligned} \tag{9}$$

Note that we can use Equation 8 to simplify the third term in the above. To simply the fourth term above we will use the same arguments as in the previous problem. We start by using partial fractions to write

$$\frac{1}{n(n+1)^2} = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\,.$$

Notice summing the first two terms will be the sum of a difference and thus if we sum this from $n = 1$ to $n = N$ we get

$$\begin{aligned}\sum_{n=1}^{N}\frac{1}{n(n+1)^2} &= 1 - \frac{1}{N+1} - \sum_{n=1}^{N}\frac{1}{(n+1)^2} \\ &= 1 - \frac{1}{N+1} - \sum_{n=2}^{N+1}\frac{1}{n^2} \\ &= 1 - \frac{1}{N+1} - \left(\sum_{n=1}^{N+1}\frac{1}{n^2} - 1\right) \\ &= 2 - \frac{1}{N+1} - \sum_{n=1}^{N+1}\frac{1}{n^2}\,. \end{aligned} \tag{10}$$

With this and Equation 8 we can simplify Equation 9. We have

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{1}{n(n+1)}\right)^3 &= 1 - \frac{1}{(N+1)^3} \\ &\quad - 3\left(-1 + \frac{1}{N+1} + \sum_{n=1}^{N}\frac{1}{n^2}\right) \\ &\quad + 3\left(2 - \frac{1}{N+1} - \sum_{n=1}^{N+1}\frac{1}{n^2}\right)\,. \end{aligned}$$

Lets take the limit where $N \to \infty$ and we get

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{1}{n(n+1)}\right)^3 &= 1 + 3 - 3\sum_{n=1}^{\infty}\frac{1}{n^2} + 6 - 3\sum_{n=1}^{\infty}\frac{1}{n^2} \\ &= 10 - 6\sum_{n=1}^{\infty}\frac{1}{n^2} = 10 - \pi^2\,, \end{aligned}$$

when we use the identity

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}\,. \tag{11}$$

Exercise 7

The ratio we want to consider to use d'Alembert's test is

$$\frac{u_n}{u_{n+1}} = \frac{-(x-n)}{n+1} = \frac{n-x}{n+1} \to 1\,,$$

when $n \to \infty$. Thus this test is inconclusive and we can't tell for which x values the sum converges or diverges. Notice that for any fixed x this fraction is eventually positive. To attempt to determine convergence we will use Raabe's test. To use that test we need to compute

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)\,.$$

In this case we find

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{-x-1}{n+1}\right) \to -x-1\,.$$

We will have convergence if this expression is greater than one. Thus we have convergence if $x < -2$ and divergence if $x > -2$. For visualization this function is plotted in the `R` code `examples_vii_exercise_7.R`.

Examples VIII

Exercise 4

If $a_n \to 0$ as $n \to \infty$ then we also have $a_n + 1 \to 1$ as $n \to \infty$. Then the ratio of these two sequences will converge to

$$\frac{a_n}{1+a_n} \to \frac{0}{1} = 0\,,$$

by using the limit theorems from this section of the book.

Exercise 5

To start we assume that $\sum a_n$ converges. Then as $a_n > 0$ we have

$$\frac{a_n}{1+a_n} < a_n\,,$$

so we can conclude that

$$\sum \frac{a_n}{1+a_n}\,,$$

converges using the comparison theorem. To prove the other direction we will assume that

$$\sum \frac{a_n}{1+a_n}\,,$$

converges. Then we know that

$$\lim_{n\to\infty}\left(\frac{a_n}{a_n+1}\right)=0\,,$$

or else the series of these terms cannot converge. We can show that this means that $a_n \to 0$ as $n \to \infty$. This means that we can find an N such that a_n is as small as we like for $n \geq N$. Select the N such that

$$a_n < \frac{1}{2}\,,$$

for $n \geq N$. This means that for $n \geq N$ we have

$$1 + a_n < \frac{3}{2}\,,$$

so

$$\frac{2}{3} < \frac{1}{1+a_n}\,.$$

If we multiply by a_n this is

$$\frac{2}{3}a_n < \frac{a_n}{1+a_n}\,.$$

As the sum of the terms on the right-hand-side converges so must the sums of the terms on the left-hand-side by the comparison theorem. As $\sum a_n$ is a multiple of this left-hand-side sum it too must converge.

Exercise 6

If one of the sums converged but the other did not then this situation would be a contradiction to the result from the previous exercise. Thus these two sums either both converge or both diverge.

Exercise 7

We can use the same arguments from Exercise 5 to argue this case. The only things that at change would be the numerical bounds (like the number $\frac{2}{3}$).

Exercise 8

If a_n^{-1} converges then since

$$\frac{1}{a_n+c} < \frac{1}{a_n}\,,$$

the sum of the terms $(a_n+c)^{-1}$ converges by the comparison test. In the other direction if

$$\sum \frac{1}{a_n+c}\,,$$

converges then

$$\lim_{n\to\infty}\left(\frac{1}{a_n+c}\right)=0\,,$$

thus $a_n \to \infty$. Note that we can write

$$\frac{1}{a_n+c}=\frac{1}{a_n\left(1+\frac{c}{a_n}\right)}\,.$$

Then as $a_n \to \infty$ there exists a N such that

$$\frac{c}{a_n}<2\,,$$

for $n \geq N$. This means that

$$\frac{1}{a_n+c}>\frac{1}{a_n(1+2)}=\frac{1}{3a_n}\,.$$

Using this and the comparison theorem as $\sum(a_n+c)^{-1}$ converges we have that $\sum\frac{1}{3a_n}$ converges (and so will $\sum a_n^{-1}$ as it is a constant multiple of the previous sum).

Exercise 9

Consider the sums of the first N terms

$$s_N=\sum_{n=1}^{N}u_n\,,$$

Thus

$$s_N^2=\sum_{n=1}^{N}\sum_{m=1}^{N}u_nu_m=\sum_{n=1}^{N}u_n^2+\sum_{n=1}^{N}\sum_{m=1;m\neq n}^{N}u_nu_m=S_N+\sum_{n=1}^{N}\sum_{m=1;m\neq n}^{N}u_nu_m\,. \tag{12}$$

As $u_n>0$ for all n we have that

$$s_N^2>S_N\,,$$

for all N. If we take the limit $N\to\infty$ we get $s^2>S$.

In a similar way (but being a bit loose about the the upper most index N in these sums) we have

$$\begin{aligned}
s_N^2&=\sum_{n=1}^{N}\sum_{m=1}^{N}u_nu_m\\
&=\sum_{n=1;m=n+1}^{N}u_nu_m+\sum_{n=m+1;n=1}^{N}u_nu_m+\sum_{n,m=1;|n-m|\geq 2}^{N}u_nu_m\\
&=2\sum_{n=1}^{N}u_nu_{n+1}+\sum_{n,m=1;|n-m|\geq 2}^{N}u_nu_m\geq 2\sum_{n=1}^{N}u_nu_{n+1}\,.
\end{aligned}$$

Taking the limit $N\to\infty$ we get $s^2>2\sigma$.

Exercise 10

Here we can take $u_n = \frac{1}{n}$. Then $\sum u_n$ diverges but $\sum u_n^2$ converges.

Exercise 11

As the sum converges we can find a value of N such that the tail of the sum is less than any number we specify. For example we will find n such that

$$\sum_{m=n}^{\infty} u_m < \frac{s}{2}\,.$$

Then since

$$u_1 + u_2 + \cdots + u_n + \sum_{m=n}^{\infty} u_m = s\,,$$

we must have that

$$u_1 + u_2 + \cdots + u_n > \frac{s}{2}\,.$$

Using this result we have

$$\frac{u_n}{u_1 + u_2 + \cdots + u_n} < \frac{2u_n}{s}\,.$$

Now from the above inequality and the comparison theorem as $\sum u_n$ converges we have that

$$\sum \frac{u_n}{u_1 + u_2 + \cdots + u_n}\,,$$

converges.

Examples IX

Exercise 1

We write our sum as the following

$$\sum \frac{1}{n^k} = \sum \frac{1}{e^{k\log(n)}}\,.$$

To use the condensation test in this later sum we will take

$$\phi(n) = \frac{1}{e^{k\log(n)}}\,.$$

Then we have

$$h^n \phi(h^n) = \frac{h^n}{e^{kn\log(h)}} = \left(\frac{h}{e^{k\log(h)}}\right)^n\,.$$

For this later sum to converge we will need to have the fraction that we are taking the powers of less than one or

$$\frac{h}{e^{k\log(h)}} < 1\,.$$

This later fraction means that

$$h < e^{k\log(h)} \quad \text{or} \quad h < h^k\,.$$

As h is greater than or equal to 2 we can divide by h to get

$$1 < h^{k-1}\,.$$

Taking the logarithm of both sides we get

$$0 < (k-1)\log(h)\,.$$

Dividing by $\log(h) > 0$ we get $k-1 > 0$ or $k > 1$. Thus this later sum will converge if $k > 1$. By the condensation theorem then the original sum will converge under the same condition.

Exercise 2

We will use the condensation theorem on this sum. To do that we will let

$$\phi(n) = \frac{1}{n\log(n)(\log(\log(n)))^k}\,.$$

With this function we see that

$$\phi(h^n) = \frac{1}{h^n n\log(h)\log(n\log(h))^k} = \frac{1}{h^n n\log(h)(\log(n)+\log(h))^k}\,.$$

With this we see that

$$h^n\phi(h^n) = \frac{1}{n\log(h)(\log(n)+\log(h))^k}\,.$$

Now the sum $\sum h^n\phi(h^n)$ is a series of positive terms which we will compare with the series

$$\sum \frac{1}{n\log(n)^k}\,,$$

using the ratio test. Ignoring the constant $\frac{1}{\log(h)}$ (which does not affect convergence of any series) we have

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{\frac{1}{n\log(n)^k}}{\frac{1}{n(\log(n)+\log(h))^k}}\right) &= \lim_{n\to\infty}\left(\frac{(\log(n)+\log(h))^k}{\log(n)^k}\right) \\
&= \lim_{n\to\infty}\left(1+\frac{\log(\log(h))}{\log(n)}\right)^k = 1 > 0\,.
\end{aligned}$$

Thus by the comparison test the two series converge and diverge in tandem. As the series with terms $\frac{1}{n\log(n)^k}$ converges when $k > 1$ and diverges when $k \le 1$ the sum $\sum h^n\phi(h^n)$ converges/diverges under the same conditions. Using that result and the condensation theorem we have that the original sum will converge when $k > 1$ and diverge when $k \le 1$.

Exercise 3

As a first step note that we must have $a > 0$ or else the terms $h^n \phi(h^n)$ won't limit to zero. For simplicity we will assume that $b > 0$ and $c > 0$ also. Next note that the terms of the original sum satisfy

$$\frac{1}{n^a \log(n)^b \log(\log(n))^c} < \frac{1}{n^a},$$

and so by the comparison theorem the sum in this exercise will converge where the sum $\sum n^{-a}$ does. This later sum will converge if $a > 1$ and thus we have convergence of the original sum when $a > 1$.

At this point, when $a \leq 1$, we have not answered the question of convergence. To continue to study this sum we will try to use the condensation theorem. Towards that end we let

$$\phi(n) = \frac{1}{n^a \log(n)^b \log(\log(n))^c}.$$

Then we have

$$\phi(h^n) = \frac{1}{h^{an} n^b \log(h)(\log(\log(h^n)))^c},$$

so that

$$h^n \phi(h^n) = \frac{1}{\log(h) h^{(a-1)n} n^b (\log(n) + \log(\log(h)))^c}. \tag{13}$$

The factor $\frac{1}{\log(h)}$ will not affect convergence of the sum $\sum h^n \phi(h^n)$ and so we can ignore it in what follows. As h is an integer such that $h \geq 2$ when $a < 1$ we have $a - 1 < 0$ and

$$h^{a-1} = \frac{1}{h^{1-a}} < 1.$$

Thus the factor

$$\frac{1}{h^{(a-1)n}} = \left(\frac{1}{h^{a-1}}\right)^n,$$

in $h^n \phi(h^n)$ is the nth power of an expression less than one. As this term is an upper bound on the terms of $h^n \phi(h^n)$ by the comparison theorem we have that the series $\sum h^n \phi(h^n)$ converges when $a < 1$.

Thus we have shown that our sum converges for all $a > 0$ except perhaps $a = 1$. We now consider this case. If $a = 1$ then Equation 13 (ignoring the constant $\log(h)$ factor) is

$$h^n \phi(h^n) = \frac{1}{n^b (\log(n) + \log(\log(h)))^c}.$$

Now since

$$\frac{1}{n^b (\log(n) + \log(\log(h)))^c} < \frac{1}{n^b},$$

then by the comparison theorem the given series will converge if $\sum n^b$ does. This later series will converge if $b > 1$. If $b \leq 1$ then by Theorem 9 in the book (studying the limit of the terms of two series $\frac{u_n}{v_n} \to L$) the above series will converge or diverge like

$$\frac{1}{n^b \log(n)^c}.$$

Applying the condensation test to this series when $\phi(n) = \frac{1}{n^b \log(n)^c}$ we need to consider

$$h^n \phi(h^n) = \frac{h^n}{h^{nb} n^c \log(h)} = \frac{1}{h^{n(b-1)} n^c \log(h)} \,.$$

To see if a series with terms like $h^n \phi(h^n)$ converges we will use a ratio test by computing

$$\begin{aligned}\lim_{n\to\infty} \frac{h^n \phi(h^n)}{h^{n+1}\phi(h^{n+1})} &= \lim_{n\to\infty} \frac{h^{(n+1)(b-1)}(n+1)^c}{h^{n(b-1)} n^c} \\ &= h^{b-1} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^c = h^{b-1} \,.\end{aligned}$$

Our series will converge if this limit is greater than one or $h^{b-1} > 1$. As $h \geq 2$ this will happen if $b > 1$ and diverge if $b < 1$.

If we are in the case where $a = 1$ and $b = 1$ then our series has terms that look like

$$\frac{1}{\log(\log(n))^c} \,.$$

Applying the condensation test one more time with $\phi(n) = \frac{1}{\log(\log(n))^c}$ we have

$$h^n \phi(h^n) = \frac{h^n}{\log(n \log(h)^c} = \frac{h^n}{(\log(n) + \log(h))^c} \,.$$

As $h \geq 2$ a series of terms with this form diverges (as the limit does not go to zero).

Thus in summary it looks like this series converges

- if $a \neq 1$ or
- if $a = 1$ and $b > 1$.

Chapter 7 (Alternating Series)

Examples X

Exercise 1

For the first example we have $u_n = \frac{(-1)^n}{n+1}$ for $n \geq 0$. For this expression we have $\lim_{n\to\infty} |u_n| = 0$ and so our alternating series will converge.

For the second example we have $u_n = \frac{(-1)^n}{2n+1}$ for $n \geq 0$. For this expression we have $\lim_{n\to\infty} |u_n| = 0$ and so our alternating series will converge.

Exercise 2

For this example we have $u_n = \frac{(-1)^{n+1}}{n^p}$ for $n \geq 1$. For this expression we will have $\lim_{n\to\infty} |u_n| = 0$ as long as $p > 0$ and in that case our alternating series will converge.

Exercise 3

I was not sure how to show this expression. If anyone sees how to show it please contact me.

Exercise 4

For this example we have $u_n = \frac{(-1)^{n+1}}{x+n}$ for $n \geq 1$. For this expression we will have $\lim_{n\to\infty} |u_n| = 0$ and our alternating series will converge.

Exercise 5

For this example we have $u_n = \frac{(-1)^{n+1} x^n}{n}$ for $n \geq 1$. For this expression we will have $\lim_{n\to\infty} |u_n| = 0$ if $|x| < 1$ and our alternating series would converge in that case. If $x = -1$ then the terms of the series become

$$u_n = \frac{(-1)^{n+1}(-1)^n}{n} = -\frac{1}{n}\,,$$

and the series will diverge. If $x = 1$ then the terms of the series become

$$u_n = \frac{(-1)^{n+1}}{n}\,,$$

and the series will converge.

Exercise 6

Lets consider the partial sum of this series i.e.the sum of the first N terms. We then have

$$S_N = 1 + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n(n+1)} .$$

Note that we can write the fraction in the sum as

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} .$$

Then we have

$$\begin{aligned} S_N &= 1 + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n+1} \\ &= 1 + \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} - \sum_{n=2}^{N+1} \frac{(-1)^{n}}{n} \\ &= 1 + 1 + \sum_{n=2}^{N} \frac{(-1)^{n+1}}{n} - \sum_{n=2}^{N+1} \frac{(-1)^{n}}{n} \\ &= 2 + 2\sum_{n=2}^{N} \frac{(-1)^{n+1}}{n} + \frac{(-1)^N}{N+1} \\ &= 2\sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} + \frac{(-1)^N}{N+1} . \end{aligned}$$

If then take the limit where $N \to \infty$ we get the desired expression.

Miscellaneous Examples On Chapters I-VII

Exercise 1

Part (a): Note that if we let the sequence of b_{n+1}'s be given by the geometric mean of a_n and b_n i.e. $b_{n+1} = \sqrt{a_n b_n}$ than this *exact* result (and many others related to the arithmetic-geometric mean) is discussed in great detail the paper [1]. This result is also mentioned in a problem in [2]. What follows initially is mostly taken from [1] where we assume that $b_{n+1} = \sqrt{a_n b_n}$. This "warm-up" will suggest what we can do for the problem at hand.

In the case of the arithmetic-geometric mean if we consider an initial value for a and b and then iterate the given sequence we will find that the larger initial number b will start to decrease and the smaller initial number a will start to increase. Thus

$$a < a_1 < b_1 < b .$$

Another iteration will give the same trend

$$a < a_1 < a_2 < b_2 < b_1 < b\,.$$

Thus the sequence $\{a_n\}$ is increasing and bounded above and the sequence $\{b_n\}$ is decreasing and bounded below. Thus each must progress a finite limit say β and α. We will now show that the limit of each sequence is the same. Notice that we can compute

$$\begin{aligned}
\frac{b_1 - a_1}{b - a} &= \frac{b_1 - a_1}{b - a}\left(\frac{b_1 + a_1}{b_1 + a_1}\right) = \frac{b_1^2 - a_1^2}{(b-a)(b_1 + a_1)} \\
&= \frac{\frac{1}{4}(b+a)^2 - ba}{(b-a)(b_1 + a_1)} = \frac{\frac{1}{4}(b-a)^2}{(b-a)(b_1 + a_1)} \\
&= \frac{1}{4}\left(\frac{b-a}{b_1 + a_1}\right) = \frac{b-a}{4\left(\frac{1}{2}(b+a) + a_1\right)} \\
&= \frac{b-a}{2(b+a) + 4a_1} < \frac{b-a}{2(b+a)} < \frac{1}{2}\,.
\end{aligned}$$

Thus we have shown that

$$b_1 - a_1 < \frac{1}{2}(b-a)\,.$$

This in tern implies that

$$b_2 - a_2 < \frac{1}{2}(b_1 - a_1) < \left(\frac{1}{2}\right)^2 (b-a)\,.$$

Continuing this process we have that

$$b_n - a_n < \left(\frac{1}{2}\right)^n (b-a) \quad \text{for} \quad n \geq 1\,.$$

Thus as $n \to \infty$ we see that the limits β and α must be equal.

For the arithmetic-harmonic mean sequence (the iteration sequence given here) we can derive a similar bound on the distance between a_n and b_n as a function of n in terms of the initial distance between the two starting values a and b. The idea is the same as above but the algebra is a bit different. Towards that end we have

$$\begin{aligned}
\frac{b_1 - a_1}{b - a} &= \frac{\frac{2ab}{a+b} - \frac{a+b}{2}}{b-a} = \frac{\frac{4ab}{2(a+b)} - \frac{(a+b)^2}{2(a+b)}}{b-a} \\
&= \frac{4ab - (a+b)^2}{2(a+b)(b-a)} = \frac{4ab - (a^2 + 2ab + b^2)}{2(a+b)(b-a)} \\
&= -\frac{a^2 - 2ab + b^2}{2(a+b)(b-a)} = -\frac{(a-b)^2}{2(a+b)(b-a)} \\
&= -\frac{b-a}{2(a+b)}\,.
\end{aligned}$$

Using this we have that

$$\frac{|b_1 - a_1|}{|b-a|} = \frac{|b-a|}{2(a+b)} < \frac{1}{2}\,.$$

Thus

$$|b_1 - a_1| < \frac{1}{2}|b - a|\,,$$

the same type of expression we had before. Generalizing this to arbitrary n we have

$$|b_n - a_n| < \left(\frac{1}{2}\right)^n |b - a| \quad \text{for} \quad n \geq 1\,.$$

Thus as $n \to \infty$ we see that the limits of b_n and a_n must be equal. One can use the `R` code in `misc_examples_i_vii_exercise_1.R` to numerically experiment with iterating this sequence.

Exercise 2

We start with $a_1 = \cos(\theta)$ and $b_1 = 1$ and we will iterate

$$\begin{aligned} a_{n+1} &= \frac{1}{2}(a_n + b_n) \\ b_{n+1} &= \sqrt{a_{n+1} b_n}\,, \end{aligned}$$

for $n \geq 2$ "by hand". For $n = 1$ we get

$$a_2 = \frac{1}{2}(1 + \cos(\theta)) = \cos^2\left(\frac{\theta}{2}\right)\,,$$

and

$$b_2 = \sqrt{\cos^2\left(\frac{\theta}{2}\right)} = \cos\left(\frac{\theta}{2}\right)\,.$$

Next for $n = 2$ we have

$$a_3 = \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) + \cos^2\left(\frac{\theta}{2}\right)\right) = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\left(1 + \cos\left(\frac{\theta}{2}\right)\right) = \cos\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{4}\right)\,.$$

Using the above we have

$$b_3 = \sqrt{\cos^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right)} = \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\,.$$

For a_4 we have

$$\begin{aligned} a_4 &= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)^2 + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\right) = \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\left(\cos\left(\frac{\theta}{4}\right) + 1\right) \\ &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)^2\,. \end{aligned}$$

Then for b_4 we have

$$
\begin{aligned}
b_4 &= \sqrt{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)^2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)} \\
&= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right) .
\end{aligned}
$$

While some might be able to see the pattern at this point I'm going to compute another set of terms by hand as its not fully clear yet to me what the full pattern is. For a_5 I get

$$
\begin{aligned}
a_5 &= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)^2 + \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\right) \\
&= \frac{1}{2}\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\left(\cos\left(\frac{\theta}{8}\right)+1\right) \\
&= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\cos\left(\frac{\theta}{16}\right)^2 .
\end{aligned}
$$

For b_5 we get

$$
\begin{aligned}
b_5 &= \sqrt{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\cos\left(\frac{\theta}{16}\right)^2\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)} \\
&= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\cos\left(\frac{\theta}{16}\right) .
\end{aligned}
$$

From these expressions it looks like the sequence a_n takes the form

$$
\begin{aligned}
a_1 &= \cos(\theta) \\
a_2 &= \cos\left(\frac{\theta}{2}\right)^2 \\
a_3 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)^2 \\
a_4 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)^2 \\
a_5 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\cos\left(\frac{\theta}{16}\right)^2 .
\end{aligned}
$$

Thus the general pattern looks like

$$
\begin{aligned}
a_1 &= \cos(\theta) \\
a_n &= \left(\prod_{k=1}^{n-1}\cos\left(\frac{\theta}{2^k}\right)\right)\cos\left(\frac{\theta}{2^{n-1}}\right)^2 ,
\end{aligned}
$$

for $n \geq 2$.

For b_n the terms we compute were

$$\begin{aligned}
b_1 &= 1 \\
b_2 &= \cos\left(\frac{\theta}{2}\right) \\
b_3 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right) \\
b_4 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right) \\
b_5 &= \cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{8}\right)\cos\left(\frac{\theta}{16}\right) .
\end{aligned}$$

It looks like the pattern is

$$b_n = \prod_{k=1}^{n} \cos\left(\frac{\theta}{2^k}\right) ,$$

for $n \geq 1$.

Now we want to show that a_n is monotone increasing and b_n is monotone decreasing. Consider

$$\begin{aligned}
a_{n+1} - a_n &= \left(\prod_{k=1}^{n} \cos\left(\frac{\theta}{2^k}\right)\right)\cos\left(\frac{\theta}{2^n}\right) - \left(\prod_{k=1}^{n-1} \cos\left(\frac{\theta}{2^k}\right)\right)\cos\left(\frac{\theta}{2^{n-1}}\right) \\
&= \left(\prod_{k=1}^{n-1} \cos\left(\frac{\theta}{2^k}\right)\right)\left[\cos^2\left(\frac{\theta}{2^n}\right) - \cos\left(\frac{\theta}{2^{n-1}}\right)\right] .
\end{aligned}$$

Using

$$\cos^2(x) = \frac{1+\cos(2x)}{2} ,$$

we have

$$\cos^2\left(\frac{\theta}{2^n}\right) = \frac{1}{2}\left(1 + \cos\left(\frac{\theta}{2^{n-1}}\right)\right) .$$

Thus we have that

$$\cos^2\left(\frac{\theta}{2^n}\right) - \cos\left(\frac{\theta}{2^{n-1}}\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\theta}{2^{n-1}}\right) = \frac{1}{2}\left(1 - \cos\left(\frac{\theta}{2^{n-1}}\right)\right) .$$

As $1 - \cos(x) > 0$ for all x we have that $a_{n+1} - a_n > 0$ and a_n is monotone increasing.

Next consider

$$\begin{aligned}
b_{n+1} - b_n &= \prod_{k=1}^{n} \cos\left(\frac{\theta}{2^k}\right) - \prod_{k=1}^{n-1} \cos\left(\frac{\theta}{2^k}\right) \\
&= \left(\prod_{k=1}^{n-1} \cos\left(\frac{\theta}{2^k}\right)\right)\left[\cos\left(\frac{\theta}{2^n}\right) - 1\right] .
\end{aligned}$$

As $\cos\left(\frac{\theta}{2^n}\right) < 1$ we have $\cos\left(\frac{\theta}{2^n}\right) - 1 < 0$ thus $b_{n+1} - b_n < 0$ and b_n is monotone decreasing.

Next we want to show that both a_n and b_n tend to $\frac{\sin(\theta)}{\theta}$ as $n \to \infty$. Note that from the above expressions for a_n and b_n we have that

$$a_n = b_{n-1} \cos\left(\frac{\theta}{2^{n-1}}\right) ,$$

thus as $\cos\left(\frac{\theta}{2^{n-1}}\right) \to 1$ as $n \to \infty$ if we can show that $b_n \to \frac{\sin(\theta)}{\theta}$ as $n \to \infty$ we also have that a_n has this same limit. To show the limit for b_n we start with the identity

$$\sin(x) = \sin\left(2\left(\frac{x}{2}\right)\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) .$$

If we repeat this expansion on the factor $\sin\left(\frac{x}{2}\right)$ we get

$$\begin{aligned}\sin(x) &= \sin\left(2\left(\frac{x}{2}\right)\right) = 2\left(2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\right)\cos\left(\frac{x}{2}\right) \\ &= 2^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) .\end{aligned}$$

Repeating this expansion on the factor $\sin\left(\frac{x}{4}\right)$ (and simplifying) we get

$$\sin(x) = 2^3\sin\left(\frac{x}{8}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) .$$

If we did this k times it looks like the pattern is thus

$$\sin(x) = 2^k\sin\left(\frac{x}{2^k}\right)\prod_{i=1}^{k}\cos\left(\frac{x}{2^i}\right) .$$

To evaluate this as $k \to \infty$ we note that using the approximation $\sin(x) \approx x$ for small x we have that

$$\lim_{k\to\infty}\left(2^k\sin\left(\frac{x}{2^k}\right)\right) \approx \lim_{k\to\infty}\left(2^k\left(\frac{x}{2^k}\right)\right) = x .$$

This means that we have just shown that

$$\prod_{i=1}^{\infty}\cos\left(\frac{x}{2^i}\right) = \frac{\sin(x)}{x} ,$$

as we were to show.

Exercise 3

Part (i): We can write the partial sum we want to evaluate as

$$\sum_{k=0}^{N}(k+1)^2 x^k .$$

Expanding the quadratic and breaking up the sum into parts we get

$$\begin{aligned}
\sum_{k=0}^{N}(k+1)^2x^k &= \sum_{k=0}^{N}(k^2+2k+1)x^k \\
&= \sum_{k=0}^{N}k^2x^k + 2\sum_{k=0}^{N}kx^k + \sum_{k=0}^{N}x^k \\
&= \sum_{k=1}^{N}k^2x^k + 2\sum_{k=1}^{N}kx^k + \sum_{k=0}^{N}x^k\,. \qquad (14)
\end{aligned}$$

Now as we have

$$S(x) = \sum_{k=0}^{N}x^k = \frac{1-x^{N+1}}{1-x}\,. \qquad (15)$$

We know the value of the last sum in the expression for $\sum_{k=0}^{N}(k+1)^2x^k$. If we take the first derivative of the expression for $S(x)$ we get

$$\begin{aligned}
S'(x) = \sum_{k=0}^{N}kx^{k-1} &= \frac{-(N+1)x^N}{1-x} + \frac{1-x^{N+1}}{(1-x)^2} \\
&= \frac{-(N+1)x^N(1-x)+1-x^{N+1}}{(1-x)^2} \\
&= \frac{1-(N+1)x^N-Nx^{N+1}}{(1-x)^2}\,.
\end{aligned}$$

If we multiply both sides of this expression by x we get

$$\sum_{k=0}^{N}kx^k = \frac{x-(N+1)x^{N+1}+Nx^{N+2}}{(1-x)^2}\,. \qquad (16)$$

Note that when $k=0$ the first term is identically zero. Next we take the derivative of both sides of the above to get

$$\sum_{k=0}^{N}k^2x^{k-1} = \frac{1-(N+1)^2x^N+N(N+2)x^{N+1}}{(1-x)^2} + \frac{2(x-(N+1)x^{N+1}+Nx^{N+2})}{(1-x)^3}\,.$$

If we simplify the right-hand-side of the above we get

$$\sum_{k=0}^{N}k^2x^{k-1} = \frac{1+x-(N+1)^2x^N+(2N^2+2N-1)x^{N+1}-N^2x^{N+2}}{(1-x)^3}\,.$$

If we multiply both sides by x we get

$$\sum_{k=0}^{N}k^2x^k = \frac{x+x^2-(N+1)^2x^{N+1}+(2N^2+2N-1)x^{N+2}-N^2x^{N+3}}{(1-x)^3}\,. \qquad (17)$$

We could use the expressions just derived in the right-hand-side of Equation 14 to evaluate the given sum. An easier solution however might be to consider the desired sum

$$\sum_{k=0}^{N}(k+1)^2 x^k\,,$$

and then shift the k index down by one to get

$$\sum_{k=1}^{N-1} k^2 x^{k-1} = \frac{1}{x}\sum_{k=1}^{N-1} k^2 x^k\,.$$

We can now use Equation 17 to evaluate this. We find

$$\sum_{k=1}^{N-1} k^2 x^{k-1} = \frac{1+x-N^2x^{N-1}+(2N^2-2N-1)x^N-(N-1)^2x^{N+1}}{(1-x)^3}\,.$$

Part (ii): We can write the term in the sum as two parts

$$\frac{1+r^2}{r(r+1)(r+2)(r+3)} = \frac{1}{r(r+1)(r+2)(r+3)} + \frac{r}{(r+1)(r+2)(r+3)}\,.$$

Now using partial fractions on the first term gives

$$P_1 = \frac{1}{r(r+1)(r+2)(r+3)} = -\frac{1}{2(r+1)} + \frac{1}{2(r+2)} - \frac{1}{6(r+3)} + \frac{1}{6r}\,. \tag{18}$$

Using partial fractions on the second term gives

$$P_2 = \frac{r}{(r+1)(r+2)(r+3)} = \frac{2}{r+2} - \frac{3}{2(r+3)} - \frac{1}{2(r+1)}\,. \tag{19}$$

If we sum the terms in P_1 from $r=1$ to $r=n$ we have

$$\begin{aligned}
S_1 &= -\frac{1}{2}\sum_{r=1}^{n}\frac{1}{r+1} + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r+2} - \frac{1}{6}\sum_{r=1}^{n}\frac{1}{r+3} + \frac{1}{6}\sum_{r=1}^{n}\frac{1}{r} \\
&= -\frac{1}{2}\sum_{r=2}^{n+1}\frac{1}{r} + \frac{1}{2}\sum_{r=3}^{n+2}\frac{1}{r} - \frac{1}{6}\sum_{r=4}^{n+3}\frac{1}{r} + \frac{1}{6}\sum_{r=1}^{n}\frac{1}{r} \\
&= \frac{1}{2}\left[-\frac{1}{2} + \frac{1}{n+2}\right] + \frac{1}{6}\left[1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right] \\
&= \frac{1}{18} - \frac{1}{6(n+1)} + \frac{1}{3(n+2)} - \frac{1}{6(n+3)}\,,
\end{aligned}$$

when we simplify. To evaluate the *second* term P_2 we will write it as follows

$$\frac{3}{2(r+2)} - \frac{3}{2(r+3)} + \frac{1}{2(r+2)} - \frac{1}{2(r+1)}\,.$$

If we sum these terms from $r = 1$ to $r = n$ we get

$$
\begin{aligned}
S_1 &= \frac{3}{2}\sum_{r=1}^{n}\frac{1}{r+2} - \frac{3}{2}\sum_{r=1}^{n}\frac{1}{r+3} + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r+2} - \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r+1} \\
&= \frac{3}{2}\sum_{r=3}^{n+2}\frac{1}{r} - \frac{3}{2}\sum_{r=4}^{n+3}\frac{1}{r} + \frac{1}{2}\sum_{r=3}^{n+3}\frac{1}{r} - \frac{1}{2}\sum_{r=2}^{n+1}\frac{1}{r} \\
&= \frac{3}{2}\left[\frac{1}{3} - \frac{1}{n+3}\right] + \frac{1}{2}\left[\frac{1}{n+2} + \frac{1}{n+3} - \frac{1}{2}\right] \\
&= \frac{1}{4} + \frac{1}{2(n+2)} - \frac{1}{n+3}\,.
\end{aligned}
$$

Thus in total when we add the two parts S_1 and S_2 and simplify we get

$$
\sum_{r=1}^{n}\frac{1+r^2}{r(r+1)(r+2)(r+3)} = \frac{11}{18} - \frac{1}{6(n+1)} + \frac{5}{6(n+2)} + \frac{5}{6(n+3)}\,.
$$

Exercise 4

Part (i): Lets use the condensation test (Theorem 16 in the book) with

$$
\phi(n) = \frac{1}{n\log(n)^2}\,.
$$

Then from this functional form we see that

$$
h^n\phi(h^n) = \frac{h^n}{h^n n^2\log(h)^2} = \frac{1}{n^2\log(h)^2}\,.
$$

The sum of the terms $h^n\phi(h^n)$ converges by comparing its terms with that from the convergent series $\sum n^{-2}$. As the series $\sum h^n\phi(h^n)$ converges by using the condensation test so does the series $\sum \phi(n)$.

Part (ii): For the terms of this sum note that we have

$$
\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}\,,
$$

and thus our series converges by using the comparison test with the convergence series $\sum n^{-2}$.

Exercise 5

Note that this sum has terms u_n given by

$$
u_n \equiv \frac{\prod_{k=0}^{n-1}(b+k)}{\prod_{k=0}^{n-1}(a+k+1)}\,,
$$

using the convention that $\prod_{k=0}^{n} f_k = 1$ if $n \leq 0$. Then consider

$$\frac{u_n}{u_{n+1}} = \frac{\prod_{k=0}^{n-1}(b+k)}{\prod_{k=0}^{n-1}(a+k+1)} \times \frac{\prod_{k=0}^{n}(a+k+1)}{\prod_{k=0}^{n}(b+k)} = \frac{a+n+1}{b+n} \to 1\,,$$

as $n \to \infty$. Thus d'Alembert's test does not tell us if the sum converges. To determine convergence we will use Raabe's test. With this test we need to compute

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{a+n+1}{b+n} - 1\right) = n\left(\frac{1+a-b}{b+n}\right) \to 1+a-b\,,$$

as $n \to \infty$. Now as $a > b$ we have that $a - b > 0$ and $1 + a - b > 1$. Thus by Raabe's test $\sum_n u_n$ converges.

We will now show that the terms are monotonically decreasing. To do that using what we derived for the ratio above we have that

$$u_{n+1} = \left(\frac{b+n}{a+n+1}\right) u_n\,.$$

As $b < a$ we have that

$$\frac{b+n}{a+n+1} < \frac{a+n}{a+n+1} < 1\,.$$

Thus

$$u_{n+1} = \left(\frac{b+n}{a+n+1}\right) u_n < u_n\,,$$

and our sequence is monotonically decreasing.

Now we will show that the given sum converges to $\frac{a}{a-b}$. This "proof" is much like the one given for Exercise 17 below. We start with

$$\frac{a}{a-b} = \frac{(a-b)+b}{a-b} = 1 + \frac{b}{a-b} = 1 + \frac{b}{a+1}\left(\frac{a+1}{a-b}\right)\,. \tag{20}$$

Note that this last factor is the "same" as the first fractional expression we started with but with the values of a and b increased by one. That is using Equation 20 again it becomes

$$\frac{a+1}{a-b} = 1 + \frac{b+1}{a+2}\left(\frac{a+2}{a-b}\right)\,.$$

We can keep applying this identity getting a final fraction that has it numerator increased by one from the previous numerator. This gives the pattern

$$\begin{aligned}
\frac{a}{a-b} &= 1 + \frac{b}{a+1}\left(\frac{a+1}{a-b}\right) \quad \text{once} \\
&= 1 + \frac{b}{a+1}\left(1 + \frac{b+1}{a+2}\left(\frac{a+2}{a-b}\right)\right) \\
&= 1 + \frac{b}{a+1} + \frac{b}{a+1}\left(\frac{b+1}{a+2}\right)\left(\frac{a+2}{a-b}\right) \quad \text{twice} \\
&= 1 + \frac{b}{a+1} + \frac{b}{a+1}\left(\frac{b+1}{a+2}\right)\left(1 + \left(\frac{b+2}{a+3}\right)\frac{a+3}{a-b}\right) \\
&= 1 + \frac{b}{a+1} + \frac{b}{a+1}\left(\frac{b+1}{a+2}\right) + \frac{b}{a+1}\left(\frac{b+1}{a+2}\right)\left(\frac{b+2}{a+3}\right)\left(\frac{a+3}{a-b}\right) \quad \text{three times}\,.
\end{aligned}$$

The patterns should now be clear. We can continue this expansion $n+1$ times to get the expression given in the book.

Exercise 6

If we combine the fractions in the argument of the sum we get

$$
\begin{aligned}
u_n &= \frac{2n+1-n}{n(2n+1)} - \frac{1}{2n+2} = \frac{n+1}{n(2n+1)} - \frac{1}{2n+2} \\
&= \frac{(n+1)(2n+2) - n(2n+1)}{n(2n+1)(2n+2)} = \frac{3n+2}{n(2n+1)(2n+2)} \,.
\end{aligned}
$$

A sum with these terms converges by the comparison test with a series with the terms n^{-2}.

Exercise 7

Part (i): Using partial fractions we can write

$$
\begin{aligned}
\frac{1}{n(n+1)(n+2)} &= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)} \\
&= \frac{1}{2n} - \frac{1}{2(n+1)} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)} \,.
\end{aligned}
$$

We wrote the expression above as we did to facilitate the sum we will take next. Towards evaluating the sum we have

$$
\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)} &= \frac{1}{2}\left(\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{n+1}\right) - \frac{1}{2}\left(\sum_{n=1}^{N} \frac{1}{n+1} - \sum_{n=1}^{N} \frac{1}{n+2}\right) \\
&= \frac{1}{2}\left(\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=2}^{N+1} \frac{1}{n}\right) - \frac{1}{2}\left(\sum_{n=1}^{N} \frac{1}{n+1} - \sum_{n=2}^{N+1} \frac{1}{n+1}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{N+1}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N+2}\right) \\
&= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{(N+1)(N+2)}\right) ,
\end{aligned}
$$

when we simplify. We know that an infinite sum with these terms converges using the comparison test to the series with terms n^{-3}. In addition, using the above summation formula we see that the sum specifically converges to the value $\frac{1}{4}$.

Part (ii): Using partial fractions we can write

$$
\frac{1}{n(n+1)(n+3)} = \frac{1}{3n} - \frac{1}{2(n+1)} + \frac{1}{6(n+3)} \,.
$$

Now subtract and add $\frac{1}{3(n+1)}$ between the first and second term above to get

$$
\frac{1}{n(n+1)(n+3)} = \frac{1}{3n} - \frac{1}{3(n+1)} - \frac{1}{6(n+1)} + \frac{1}{6(n+3)} \,,
$$

when we simplify. Again we wrote the expression above as we did to facilitate the sum we will take next. To sum this note that we can write the above as

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)(n+3)} &= \frac{1}{3}\left(\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\frac{1}{n+1}\right) - \frac{1}{6}\left(\sum_{n=1}^{N}\frac{1}{n+1} - \sum_{n=1}^{N}\frac{1}{n+3}\right) \\
&= \frac{1}{3}\left(\sum_{n=1}^{N}\frac{1}{n} - \sum_{n=2}^{N+1}\frac{1}{n}\right) - \frac{1}{6}\left(\sum_{n=1}^{N}\frac{1}{n+1} - \sum_{n=3}^{N+2}\frac{1}{n+1}\right) \\
&= \frac{1}{3}\left(1 - \frac{1}{N+1}\right) - \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} - \frac{1}{N+1+1} - \frac{1}{N+2+1}\right) \\
&= \frac{7}{36} - \frac{1}{3(N+1)} + \frac{1}{6(N+2)} + \frac{1}{6(N+3)}\,.
\end{aligned}$$

when we simplify. We know that an infinite sum with these terms converges using the comparison test to the series with terms n^{-3}. In addition, using the above summation formula we see that the sum specifically converges to the value $\frac{7}{36}$.

Part (iii): To sum terms with this form we consider

$$(n+1)(n+2)(n+3)(n+4) - n(n+1)(n+2)(n+3) = 4(n+1)(n+2)(n+3)\,.$$

Then note that

$$\sum_{n=1}^{N}(f_{n+1} - f_n) = \sum_{n=1}^{N} f_{n+1} - \sum_{n=1}^{N} f_n = \sum_{n=2}^{N+1} f_n - \sum_{n=1}^{N} f_n = f_{N+1} - f_1\,. \tag{21}$$

By summing first statement in this part we have

$$\sum_{n=1}^{N}\left[(n+1)(n+2)(n+3)(n+4) - n(n+1)(n+2)(n+3)\right] = 4\sum_{n=1}^{N}(n+1)(n+2)(n+3)\,.$$

Using the identity in Equation 21 we see the left-hand-side of the above is equal to

$$(N+1)(N+2)(N+3)(N+4) - 1(2)(3)(4)\,.$$

Setting this equal to $4\sum_{n=1}^{N}(n+1)(n+2)(n+3)$ we obtain

$$\sum_{n=1}^{N}(n+1)(n+2)(n+3) = \frac{1}{4}((N+1)(N+2)(N+3)(N+4) - 24)\,.$$

This is almost the sum we want. Shifting the index n down by one we get

$$\sum_{n=2}^{N+1} n(n+1)(n+2) = \frac{1}{4}((N+1)(N+2)(N+3)(N+4) - 24)\,.$$

Including the $n = 1$ term and moving the $n = N+1$ term out of the sum on the left we have

$$\sum_{n=1}^{N} n(n+1)(n+2) - 1(2)(3) + (N+1)(N+2)(N+3) = \frac{1}{4}((N+1)(N+2)(N+3)(N+4) - 24)\,.$$

If we solve for the sum of interest we get

$$\sum_{n=1}^{N} n(n+1)(n+2) = \frac{1}{4}N(N+1)(N+2)(N+3)\,.$$

Another way to work this problem is to expand the given expression and to do the sums individually. We have

$$n(n+1)(n+2) = n^3 + 3n^2 + 2n\,.$$

Then using the following sum identities

$$\sum_{n=1}^{N} n = \frac{1}{2}N(N+1) \tag{22}$$

$$\sum_{n=1}^{N} n^2 = \frac{1}{6}N(N+1)(2N+1) \tag{23}$$

$$\sum_{n=1}^{N} n^3 = \frac{N^2}{4}(N+1)^2\,. \tag{24}$$

This means that we can compute the desired sum as

$$\begin{aligned}
\sum_{n=1}^{N} n(n+1)(n+2) &= \sum_{n=1}^{N} n^3 + 3\sum_{n=1}^{N} n^2 + 2\sum_{n=1}^{N} n \\
&= \frac{N^2}{4}(N+1)^2 + \frac{3}{6}N(N+1)(2N+1) + \frac{2}{2}N(N+1) \\
&= \frac{N(N+1)}{4}\left(N(N+1) + 2(2N+1) + 4\right) \\
&= \frac{N(N+1)}{4}\left(N^2 + 5N + 6\right) = \frac{1}{4}N(N+1)(N+2)(N+3)\,,
\end{aligned}$$

the same as the above.

An infinite sum with terms like these cannot converge as the terms do not limit to zero as $n \to \infty$.

Part (iv): As before we will use Equation 15 to derive Equation 16 which by dividing both sides by x is the sum we desire in this exercise.

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n|x|^{n-1}}{(n+1)|x|^n} = \frac{n}{(n+1)|x|}\,.$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$.

Part (v): As before we will use Equation 15 to derive Equation 16 and then Equation 17. This gives the sum of terms $n^2 x^n$ to which we will have to add the result from Equation 15 to derive the desired sum.

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{(n^2+1)|x|^n}{((n+1)^2+1)|x|^{n+1}} = \frac{(n^2+1)}{((n+1)^2+1)|x|}\,.$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$.

Exercise 8

Before we begin we note that if α is a root of $f(x) = x^3 - 7x + 6 = 0$ then by inspection of the graph of $f(x)$ we see that $\alpha \in \{-3, 1, 2\}$.

Part (i): In this case lets let $s_n = 2 + \xi_n$ where $\xi_n > 0$. Then we find

$$s_{n+1} = (7s_n - 6)^{1/3} = (14 + 7\xi_n - 6)^{1/3} = (8 + 7\xi_n)^{1/3} = 2\left(1 + \frac{7}{8}\xi_n\right)^{1/3}\,.$$

Next we recall Bernoulli's inequality which has two forms. One form states

$$(1+x)^r \geq 1 + rx\,, \tag{25}$$

which is valid when $r \leq 0$ or $r \geq 1$ (and $x \geq -1$). Another form is when r is not in the previous range specified then we have

$$(1+x)^r \leq 1 + rx\,. \tag{26}$$

which is valid when $0 \leq r \leq 1$ (and $x \geq -1$).

With this background then using Bernoulli's inequality (namely Equation 26) we have that

$$s_{n+1} \leq 2\left(1 + \frac{7}{24}\xi_n\right) = 2 + \frac{7}{12}\xi_n\,.$$

As

$$\frac{7}{12}\xi_n < \xi_n\,,$$

we have that

$$s_{n+1} < 2 + \xi_n = s_n\,,$$

showing that s_n is monotonically decreasing.

As a way to prove that $s_n - \alpha$ is the same sign is to work this exercise with everything in terms of $s_n - \alpha$. From the iteration equation we have

$$(s_{n+1} - \alpha + \alpha)^3 = 7(s_n - \alpha + \alpha) - 6\,,$$

or expanding

$$(s_{n+1} - \alpha)^3 + 3\alpha(s_{n+1} - \alpha)^2 + 3\alpha^2(s_{n+1} - \alpha) + \alpha^3 = 7(s_n - \alpha) + 7\alpha - 6\,.$$

Using the fact that α is a solution of $\alpha^3 = 7\alpha - 6$ we get

$$(s_{n+1} - \alpha)^3 + 3\alpha(s_{n+1} - \alpha)^2 + 3\alpha^2(s_{n+1} - \alpha) = 7(s_n - \alpha)\,.$$

We can write this as

$$(s_{n+1} - \alpha)\left[(s_{n+1} - \alpha)^2 + 3\alpha(s_{n+1} - \alpha) + 3\alpha^2\right] = 7(s_n - \alpha)\,. \tag{27}$$

Lets consider the quadratic equation $x^2 + 3\alpha x + 3\alpha^2 = 0$ which has roots

$$x = \frac{-3\alpha \pm \sqrt{9\alpha^2 - 4(3\alpha^2)}}{2} = \frac{-3\alpha \pm \sqrt{3}|\alpha| i}{2}\,.$$

The fact that these roots are complex means that this quadratic has no real zeros. Thus the quadratic expression

$$(s_{n+1} - \alpha)^2 + 3\alpha(s_{n+1} - \alpha) + 3\alpha^2\,,$$

is either always positive or always negative. Taking $(s_{n+1} - \alpha) = 0$ we get the above equal to $3\alpha^2$ showing that it is always positive. Using that fact with Equation 27 we see that both $s_n - \alpha$ and $s_{n+1} - \alpha$ have the same sign.

Exercise 9

Part (i): For the first sum recall that

$$\sum \operatorname{sech}(nx) = \sum \frac{1}{\cosh(nx)} = \sum \frac{2}{e^{nx} + e^{-nx}}\,.$$

Now note that when $x > 0$ we have the terms of this sum bounded above by

$$\frac{2}{e^{nx} + e^{-nx}} < \frac{2}{e^{nx}} = 2(e^{-x})^n\,.$$

As $e^{-x} < 1$ when $x > 0$ this sum converges and by the comparison test our original sum converges.

When $x < 0$ the same type of arguments say

$$\frac{2}{e^{nx} + e^{-nx}} < \frac{2}{e^{-nx}} = 2(e^{x})^n\,.$$

Now $e^x < 1$ when $x < 0$ and the sum of these upper bounds converges. Again by the comparison test we have that our original sum converges when $x < 0$.

If $x = 0$ then each term of the sum is $\text{sech}(nx) = 1$ and the sum diverges.

Part (ii): Using much of the same ideas as the first sum for the second sum recall that

$$\sum x^n \text{sech}(nx) = \sum \frac{2x^n}{e^{nx} + e^{-nx}} .$$

If $x = 0$ then each term is zero and the sum converges to zero.

If $x > 0$ then we can bound the terms of our sum as

$$\frac{2x^n}{e^{xn} + e^{-xn}} < \frac{2x^n}{e^{xn}} = 2(xe^{-x})^n .$$

An infinite sum of these upper bounding terms will converge if $|xe^{-x}| < 1$ or as $x > 0$ when $x < e^x$. This is true for all $x > 0$ as can be seen by plotting the functions $y = x$ and $y = e^x$ over the domain $x > 0$.

If $x < 0$ then we can bound the terms of our sum as

$$\frac{2x^n}{e^{xn} + e^{-xn}} < \frac{2x^n}{e^{-xn}} = 2(xe^{x})^n .$$

An infinite sum of these upper bounding terms will converge if $|xe^{x}| < 1$ or as $x < 0$ when $-x < e^{-x}$. This is true for all $x < 0$ as can be seen by plotting the functions $y = -x$ and $y = e^{-x}$ over the domain $x < 0$.

Another way to argue convergence is to use d'Alembert's test. In that case if $x > 0$ we need to evaluate (ignoring the factor of two)

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}} \right) &= \lim_{n\to\infty} \left(\frac{x^n}{e^{xn} + e^{-xn}} \right) \left(\frac{e^{x(n+1)} + e^{-x(n+1)}}{x^{n+1}} \right) \\
&= \frac{1}{x} \lim_{n\to\infty} \left(\frac{e^x + e^{-x(2n+1)}}{1 + e^{-2xn}} \right) = \frac{e^x}{x} .
\end{aligned}$$

For convergence we would need to have $\frac{e^x}{x} > 1$ the same condition we had before.

Exercise 10

Note that

$$\frac{1}{n + 2 + (-1)^n} = \begin{cases} \frac{1}{n+3} & n \text{ even} \\ \frac{1}{n+1} & n \text{ odd} \end{cases} .$$

Note that both the expressions on the right-hand-side are less than or equal to $\frac{1}{n+1}$. As the absolute value of the terms of this series limits to zero as $n \to \infty$ using the alternating sequence theorem we have that our series converges.

Exercise 11

We will use the ratio test to argue that the two series either both converge or both diverge. Note that the limit of the ratio of the terms of the two series is

$$\lim_{n\to\infty}\left(\frac{u_n}{\frac{u_n}{s_n}}\right) = \lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} u_n\,.$$

Thus if $\sum u_n$ is convergent this is a finite value and the series $\sum \frac{u_n}{s_n}$ must also converge.

Warning: I was not able to argue that if $\sum_{n=1}^{\infty} u_n$ diverges then we know that $\sum_{n=1}^{\infty} \frac{u_n}{s_n}$ also diverges. If anyone knows how to do this please contact me.

Exercise 12

The terms of this series can be written as

$$u_n = \frac{n!x^n}{\prod_{i=1}^{n}(i+\alpha)}\,.$$

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n+1+\alpha}{(n+1)|x|}\,.$$

Thus the limit of this fraction as $n\to\infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$.

Exercise 13

Part (i): To show that u_n is monotonically decreasing we will group pairs of terms as

$$u_n = \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+3} - \frac{1}{n+4}\right) + \left(\frac{1}{n+5} - \frac{1}{n+6}\right) + \cdots\,.$$

Now notice that each pair of terms is of the form

$$\frac{1}{n+k} - \frac{1}{n+k+1} = \frac{1}{(n+k)(n+k+1)} > \frac{1}{(n+k+1)(n+k+2)} = \frac{1}{n+k+1} - \frac{1}{n+k+2}\,,$$

for different positive integers k. This means that we can bound each of the pairs of terms in u_n as

$$u_n > \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) + \left(\frac{1}{n+6} - \frac{1}{n+7}\right) + \cdots = u_{n+1}\,.$$

This shows that u_n decreases monotonically.

Part (ii): To start this part lets consider

$$
\begin{aligned}
u_0 - u_1 &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 + 2\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \\
&= 1 - 2\left(\frac{1}{2}\right) + 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} = 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} \\
&= 2\sum_{k=2}^{\infty} \frac{(-1)^k}{k} = 2\sum_{k=0}^{\infty} \frac{(-1)^{k+2}}{k+2} = 2u_2 \,.
\end{aligned}
$$

This shows the desired relationship for $n = 1$.

Lets assume that the given expression holds up to n and write out the given expression for $n+1$ where we would want to show

$$
u_0 - u_1 + u_2 - \cdots - u_{2n-1} + u_{2n} - u_{2n+1} = 2(n+1)u_{2(n+1)} \,.
$$

Using what we know about all but the last two terms on the left-hand-side (i.e. using the assumed relationship for n) we have that the left-hand-side of the above can be written as

$$
2nu_{2n} + u_{2n} - u_{2n+1} = (2n+1)u_{2n} - u_{2n+1} \,.
$$

Using the definition of u_n the above becomes

$$
\begin{aligned}
(2n+1)u_{2n} - u_{2n+1} &= (2n+1)\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2n+k} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2n+1+k} \\
&= (2n+1)\sum_{k=-1}^{\infty} \frac{(-1)^{k+3}}{2n+2+k} - \sum_{k=0}^{\infty} \frac{(-1)^{k+2}}{2n+2+k} \\
&= (2n+1)\sum_{k=-1}^{\infty} \frac{(-1)^{k+1}}{2n+2+k} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2n+2+k} \\
&= (2n+1)\left[\frac{(-1)^0}{2n+2-1} + \frac{(-1)^1}{2n+2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2n+2+k}\right] + \frac{(-1)^1}{2n+2} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2n+2+k} \\
&= (2n+1)\left[\frac{1}{2n+1} - \frac{1}{2n+2} + u_{2n+2}\right] - \frac{1}{2n+2} + u_{2n+2} \\
&= (2n+1)\left[\frac{1}{(2n+1)(2n+2)} + u_{2n+2}\right] - \frac{1}{2n+2} + u_{2n+2} \\
&= (2n+2)u_{2n+2} \,,
\end{aligned}
$$

when we simplify. As this is the desired expression for $n+1$ we are done.

Exercise 14

To show the first part we will consider the sequence of reciprocals

$$\beta_n \equiv \frac{1}{\sqrt{n^2+1}-n}\,.$$

Note that we can write this as

$$\beta_n = \frac{1}{\sqrt{n^2+1}-n}\left(\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}\right) = \frac{\sqrt{n^2+1}+n}{n^2+1-n^2} = \sqrt{n^2+1}+n\,.$$

For this sequence we see that β_n is monotonically increasing and thus the original sequence must be monotonically decreasing. As a final observation note that $\beta_n \to \infty$ as $n \to \infty$ and thus the original sequence tends to zero $n \to \infty$.

Because of the fact that $\sqrt{n^2+1}-n$ tends to zero by using the alternating series test we have that the given series converges.

Exercise 15

The terms of this series can be written as

$$u_n = \frac{x^{2n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{[2\cdot 4\cdot 6\cdots(2n)](2n+1)}\,,$$

for $n = 1, 2, 3, \ldots$ (here $n = 1$ is the first term). To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{|x|^{2n+1}[1\cdot 3\cdot 5\cdots(2n-3)]}{[2\cdot 4\cdot 6\cdots(2n)](2n+1)}\right)\left(\frac{[2\cdot 4\cdot 6\cdots(2n)(2n+2)](2n+3)}{|x|^{2n+3}[1\cdot 3\cdot 5\cdots(2n-3)(2n-1)]}\right)\\ &= \left(\frac{1}{|x|^2}\right)\left(\frac{(2n+2)(2n+3)}{(2n-1)(2n+1)}\right)\,.\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|^2}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|^2} > 1$ which happens if $|x|^2 < 1$ or $-1 < x < +1$.

If $x = 1$ then the above fraction has the limit of one and d'Alembert's test is indeterminate. To determine convergence we will use Raabe's test. With this test we need to compute (when $x = 1$)

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{(2n+2)(2n+3)}{(2n-1)(2n+1)}-1\right) = n\left(\frac{10n+7}{(2n-1)(2n+1)}\right) \to \frac{5}{2}\,,$$

as $n \to \infty$. As this is larger than one by Raabe's test the sum converges.

Exercise 16

Part (i): To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n(n+1)|x|^n}{(n+1)(n+2)|x|^{n+1}} = \left(\frac{1}{|x|}\right)\left(\frac{(n+2)}{n}\right).$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$. If $x = 1$ then the terms of the series don't limit to zero as $n \to \infty$ and thus the sum also diverges.

Part (ii): To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{n(n+3)(n+5)|x|^n}{(n+1)(n+2)}\right) \times \left(\frac{(n+2)(n+3)}{(n+1)(n+4)(n+6)|x|^{n+1}}\right) \\ &= \frac{n(n+3)^2(n+5)}{(n+1)^2(n+4)(n+6)|x|}.\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$. If $x = 1$ then the terms of the series don't limit to zero as $n \to \infty$ and thus the sum also diverges.

Exercise 17

The first equality is easy to show

$$\frac{m+1}{m-n+1} = \frac{(m-n+1)+n}{m-n+1} = 1 + \frac{n}{m-n+1} = 1 + \frac{n}{m}\left(\frac{m}{m-n+1}\right). \tag{28}$$

Now in this last factor note that it is the "same" as the first fractional expression we started with but with the values of m and n decreased by one. That is we write it as

$$\frac{m}{m-n+1} = \frac{(m-1)+1}{(m-1)-(n-1)+1},$$

and using Equation 28 again it becomes

$$\frac{m}{m-n+1} = 1 + \frac{n-1}{m-1}\left(\frac{m-1}{m-n+1}\right).$$

We can keep applying this identity getting a final fraction that has it numerator decremented by one from the previous numerator. This gives the pattern

$$
\begin{aligned}
\frac{m+1}{m-n+1} &= 1 + \frac{n}{m}\left(\frac{m}{m-n+1}\right) \quad \text{once} \\
&= 1 + \frac{n}{m}\left(1 + \frac{n-1}{m-1}\left(\frac{m-1}{m-n+1}\right)\right) \\
&= 1 + \frac{n}{m} + \frac{n}{m}\left(\frac{n-1}{m-1}\right)\left(\frac{m-1}{m-n+1}\right) \quad \text{twice} \\
&= 1 + \frac{n}{m} + \frac{n}{m}\left(\frac{n-1}{m-1}\right)\left(1 + \left(\frac{n-2}{m-2}\right)\frac{m-2}{m-n+1}\right) \\
&= 1 + \frac{n}{m} + \frac{n}{m}\left(\frac{n-1}{m-1}\right) + \frac{n}{m}\left(\frac{n-1}{m-1}\right)\left(\frac{n-2}{m-2}\right)\left(\frac{m-2}{m-n+1}\right) \quad \text{three times} \\
&= 1 + \frac{n}{m} + \frac{n}{m}\left(\frac{n-1}{m-1}\right) + \frac{n}{m}\left(\frac{n-1}{m-1}\right)\left(\frac{n-2}{m-2}\right)\left(1 + \frac{n-3}{m-3}\left(\frac{m-4}{m-n+1}\right)\right).
\end{aligned}
$$

The patterns should now be clear. We can continue this expansion n times and we have

$$
\begin{aligned}
\frac{m+1}{m-n+1} = 1 &+ \frac{n}{m} + \frac{n}{m}\left(\frac{n-1}{m-1}\right) + \frac{n}{m}\left(\frac{n-1}{m-1}\right)\left(\frac{n-2}{m-2}\right) + \cdots \\
&+ \frac{n(n-1)(n-2)\cdots 2(1)}{m(m-1)(m-2)\cdots(m-n+2)(m-n+1)}.
\end{aligned}
$$

Exercise 18

Part (i): Note that if $x > 1$ then as $n \to \infty$ the limit of nx^{-n} is of the "type" $\infty \times 0$ and is thus indeterminate. If we write it as $\frac{n}{x^n}$ then it is of the "type" $\frac{\infty}{\infty}$ and we can apply L'Hospital's rule. We have

$$
\lim_{n\to\infty}\left(\frac{n}{x^n}\right) = \lim_{n\to\infty}\left(\frac{1}{\ln(x)x^n}\right) \to 0\,,
$$

as $n \to \infty$.

Part (ii): For this we have

$$
\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1-(x-1)}{x^2-1} = \frac{2}{x^2-1}.
$$

Part (iii): For this we will repeatedly use the identity derived above. That is consider

$$
\frac{1}{x-1} = \frac{1}{x+1} + \frac{2}{x^2-1}. \tag{29}
$$

Then if we use this expression on the last term in the above (but with $x \to x^2$) we have

$$
\begin{aligned}
\frac{1}{x-1} &= \frac{1}{x+1} + 2\left(\frac{1}{x^2+1} + \frac{2}{x^4-1}\right) \\
&= \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4-1}.
\end{aligned}
$$

If we use Equation 29 on the last term in the above (but with $x \to x^4$) we get

$$\begin{aligned}\frac{1}{x-1} &= \frac{1}{x+1} + \frac{2}{x^2+1} + 4\left(\frac{1}{x^4+1} + \frac{2}{x^8-1}\right) \\ &= \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8-1}\,.\end{aligned}$$

Continuing this procedure as many times as needed gives the intended result.

Exercise 19

To show this we will instead consider the sequence of reciprocals

$$\beta_n \equiv \frac{1}{\alpha_n} = \frac{1}{\sqrt{n+1}-\sqrt{n}}\,.$$

Note that we can write this as

$$\beta_n = \frac{1}{\sqrt{n+1}-\sqrt{n}}\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right) = \frac{\sqrt{n+1}+\sqrt{n}}{n+1-n} = \sqrt{n+1}+\sqrt{n}\,.$$

For this sequence we have

$$\beta_n = \sqrt{n+1}+\sqrt{n} < \sqrt{n+2}+\sqrt{n+1} = \beta_{n+1}\,,$$

which means that the sequence β_n is monotonically increasing and thus α_n must be monotonically decreasing. As a final observation note that $\beta_n \to \infty$ as $n \to \infty$ and thus $\alpha_n \to 0$ when $n \to \infty$.

Exercise 20

Note that

$$\frac{2n+1+n^2-(n+1)^2}{n^2(n+1)^2} = 0\,.$$

Splitting this into parts we have

$$\frac{2n+1}{n^2(n+1)^2} + \frac{1}{(n+1)^2} - \frac{1}{n^2} = 0\,.$$

Thus if we multiply both sides by x^n and sum from $n = 1$ to $n = \infty$ we have

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{n^2(n+1)^2}\right)x^n + \sum_{n=1}^{\infty}\frac{x^n}{(n+1)^2} - \sum_{n=1}^{\infty}\frac{x^n}{n^2} = 0\,.$$

Solving for the first sum and shifting indices on the second sum we have

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{n^2(n+1)^2}\right)x^n = -\sum_{n=2}^{\infty}\frac{x^{n-1}}{n^2} + \sum_{n=1}^{\infty}\frac{x^n}{n^2}\,.$$

If we include the $n = 1$ term in the first sum on the right we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{2n+1}{n^2(n+1)^2}\right)x^n &= -\sum_{n=1}^{\infty}\frac{x^{n-1}}{n^2} + 1 + \sum_{n=1}^{\infty}\frac{x^n}{n^2}\\ &= -\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n^2} + 1 + \sum_{n=1}^{\infty}\frac{x^n}{n^2}\\ &= 1 + \left(1 - \frac{1}{x}\right)\sum_{n=1}^{\infty}\frac{x^n}{n^2},\end{aligned}$$

which simplifies to the desired expression.

Exercise 21

We start with

$$\alpha_n = \left(\frac{a+n-1}{n}\right)y^n.$$

This is the product of a sequence $x_n = \frac{a+n-1}{n}$ and y^n. Note that the sequence x_n is monotonically decreasing because

$$x_{n+1} = 1 + \frac{a-1}{n+1} < 1 + \frac{a-1}{n} = x_n.$$

As $y < 1$ we have that y^n is also monotonic decreasing as $y^{n+1} < y^n$. Then as α_n is the product of two monotonically decreasing sequences it also is monotonically decreasing.

When $n \to \infty$ we have that $x_n \to 1$ and $y^n \to 0$ so that $\alpha_n \equiv x_n y^n \to 0$ in that case.

Exercise 22

The terms of this sum take the form

$$u_n = \left(\frac{a+n-1}{n}\right)x^n.$$

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{(a+n-1)|x|^n}{n}\right)\times\left(\frac{n+1}{(a+n)|x|^{n+1}}\right)\\ &= \left(\frac{(a+n-1)(n+1)}{(a+n)n}\right)\left(\frac{1}{|x|}\right).\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$. If $x = 1$ then the terms of the series don't limit to zero as $n \to \infty$ and thus the sum also diverges. If $x = -1$ then the terms of the series are given by

$$u_n = \left(\frac{a + n - 1}{n} \right) (-1)^n = (-1)^n + \left(\frac{a - 1}{n} \right) (-1)^n \, .$$

As $n \to \infty$ the second terms becomes smaller and smaller in comparison to the first term. Thus the sum of these u_n terms like this will oscillate and the series will not converge.

Chapter 9 (Absolute and Non-Absolute Convergence)

Examples XI

Exercise 1

The regions of convergence for each sum was computed in Exercise 1 on Page 24.

Exercise 2

The terms in this sum are given by $\sum_n u_n x^n$ where

$$u_n = \frac{\left(\prod_{i=0}^{n-1}(a+i)\right)\left(\prod_{i=0}^{n-1}(b+i)\right)}{\left(\prod_{i=0}^{n-1}(c+i)\right)\left(\prod_{i=0}^{n-1}(d+i)\right)}, \tag{30}$$

for $n \geq 0$. Here we are using the convention that $\prod_{i=0}^{-1} f(i) = 1$. To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n x^n|}{|u_{n+1}x^{n+1}|}$ where we find

$$\frac{|u_n x^n|}{|u_{n+1}x^{n+1}|} = \frac{(c+n)(d+n)}{(a+n)(b+n)|x|}.$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$. If $x = \pm 1$ then the terms of the series don't limit to zero as $n \to \infty$ and thus the sum also diverges.

Exercise 3

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$. For the first series our terms take the form

$$u_n = \frac{x^n}{n!},$$

for $n \geq 0$. In that case we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^n}{n!} \times \frac{(n+1)!}{|x|^{n+1}} = \frac{n+1}{|x|}.$$

Thus the limit of this fraction as $n \to \infty$ is ∞. By d'Alembert's test our sum will converge if this limit is larger than one which happens for all x.

For the second series our terms take the form

$$u_n = \frac{x^{2n+1}}{(2n+1)!}\,,$$

for $n \geq 0$. In that case we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{|x|^{2n+1}}{(2n+1)!} \times \frac{(2n+3)!}{|x|^{2n+3}} = \frac{(2n+3)(2n+2)}{|x|^2}\,.$$

Thus the limit of this fraction as $n \to \infty$ is ∞. By d'Alembert's test our sum will converge if this limit is larger than one which happens for all x.

Exercise 4

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where for this series $u_n = n!x^n$. We find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n!|x|^n}{(n+1)!|x|^{n+1}} = \frac{1}{(n+1)|x|}\,.$$

Thus the limit of this fraction as $n \to \infty$ is the value 0. By d'Alembert's test our sum will diverge if this limit is less than one which it is for all x.

Another way to argue this is to follow the hint in the book. We will have $|u_{n+1}| > |u_n|$ when

$$(n+1)!|x|^{n+1} > n!|x|^n\,,$$

or

$$(n+1)|x| > 1\,.$$

Solving for n in the above for a given x this will happen when

$$n > \frac{1}{|x|} - 1\,,$$

as $n \to \infty$ the above inequality will eventually be satisfied and $|u_{n+1}| > |u_n|$ showing that $|u_n|$ cannot limit to zero as it must for the series to be convergent.

Exercise 5

To study convergence of the infinite series we will use d'Alembert's test. For this series the terms u_n take the form $u_n = n^k x^n$. To use d'Alembert's test we need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n^k|x|^n}{(n+1)^k|x|^{n+1}} = \left(\frac{1}{|x|}\right)\left(\frac{1}{(1+\frac{1}{n})^k}\right)\,.$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$.

If $x = 1$ then the terms of the series look like $u_n = n^k$ which will converge by the comparison test if $k < -1$.

If $x = -1$ then the terms of the series look like $u_n = (-1)^k n^k$ which will converge by the alternating series test if $k < 0$.

Exercise 6

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$.

For the first series our terms take the form

$$u_n = \frac{(a+n-1)x^n}{\prod_{i=1}^{n}(2i)},$$

for $n \geq 1$. In that case we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \frac{(a+n-1)|x|^n}{\prod_{i=1}^{n}(2i)} \times \frac{\prod_{i=1}^{n+1}(2i)}{(a+n)|x|^{n+1}} \\ &= \frac{1}{|x|}\left(\frac{a+n-1}{a+n}\right)(2n+2).\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value ∞. By d'Alembert's test our sum will converge if this limit is larger than one which it is for all x.

For the second series our terms take the form

$$u_n = \frac{n(a+n-1)x^n}{\prod_{i=1}^{n}(2i)},$$

for $n \geq 1$. This is really just n times the u_n term in the previous series. In that case we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \frac{n(a+n-1)|x|^n}{\prod_{i=1}^{n}(2i)} \times \frac{\prod_{i=1}^{n+1}(2i)}{(n+1)(a+n)|x|^{n+1}} \\ &= \frac{1}{|x|}\left(\frac{n}{n+1}\right)\left(\frac{a+n-1}{a+n}\right)(2n+2).\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value ∞. By d'Alembert's test our sum will converge if this limit is larger than one which it is for all x.

Exercise 7

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$. For this series we have terms given by

$$u_n = \frac{n(a+n-1)x^n}{b^n},$$

for $n \geq 1$. With terms like this we find

$$\begin{aligned}\frac{|u_n|}{|u_{n+1}|} &= \frac{n(a+n-1)|x|^n}{|b|^n} \times \frac{|b|^{n+1}}{(n+1)(a+n)|x|^{n+1}} \\ &= \frac{|b|}{|x|}\left(\frac{n}{n+1}\right)\left(\frac{a+n-1}{a+n}\right).\end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{|b|}{|x|}$. By d'Alembert's test our sum will converge if this limit is larger than one which happens when $|x| < |b|$.

Notes on the sums $|\sum_{k=1}^{n} \cos(r\theta)|$ and $|\sum_{k=1}^{n} \sin(r\theta)|$

Here we will prove the statements

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\sin(\frac{1}{2}n\theta)\cos(\frac{1}{2}(n+1)\theta)}{\sin(\frac{1}{2}\theta)} \tag{31}$$

$$\sum_{r=1}^{n} \sin(r\theta) = \frac{\sin(\frac{1}{2}n\theta)\sin(\frac{1}{2}(n+1)\theta)}{\sin(\frac{1}{2}\theta)}. \tag{32}$$

To do this note that (here the lower summation limit is $r = 0$)

$$\begin{aligned}\sum_{r=0}^{n} \cos(r\theta) &= \mathrm{Re}\sum_{r=0}^{n} e^{ir\theta} \\ \sum_{r=0}^{n} \sin(r\theta) &= \mathrm{Im}\sum_{r=0}^{n} e^{ir\theta}.\end{aligned}$$

Thus we will evaluate $\sum_{r=0}^{n} e^{ir\theta}$. I find

$$\begin{aligned}\sum_{r=0}^{n} e^{ir\theta} &= \frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}} = \frac{1-e^{i\theta(n+1)}}{1-e^{i\theta}} \times \frac{e^{-i\theta/2}}{e^{-i\theta/2}} \\ &= \frac{e^{i\theta(n+\frac{1}{2})}-e^{-i\theta/2}}{e^{i\theta/2}-e^{-i\theta/2}} = \frac{e^{i\theta(n+\frac{1}{2})}-e^{-i\theta/2}}{2i\sin(\theta/2)} \\ &= -\frac{i}{2\sin(\theta/2)}\left[\left(\cos\left(\theta\left(n+\frac{1}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)\right) + i\left(\sin\left(\theta\left(n+\frac{1}{2}\right)\right) + \sin\left(\frac{\theta}{2}\right)\right)\right].\end{aligned}$$

Thus the real and imaginary parts of this give

$$\sum_{r=0}^{n} \cos(r\theta) = \frac{\sin\left(\theta\left(n+\frac{1}{2}\right)\right) + \sin\left(\frac{\theta}{2}\right)}{2\sin(\theta/2)} \tag{33}$$

$$\sum_{r=0}^{n} \sin(r\theta) = -\left(\frac{\cos\left(\theta\left(n+\frac{1}{2}\right)\right) - \cos\left(\frac{\theta}{2}\right)}{2\sin(\theta/2)}\right). \tag{34}$$

Expand the first sine in the numerator (call it N_c) of Equation 33 to get

$$N_c = \sin(n\theta)\cos\left(\frac{\theta}{2}\right) + \cos(n\theta)\sin\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) = \sin(n\theta)\cos\left(\frac{\theta}{2}\right) + (\cos(n\theta)+1)\sin\left(\frac{\theta}{2}\right).$$

Use the sine double angle formula $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ for $\sin(n\theta)$ and the cosign double angle formula

$$1 + \cos(x) = 2\cos^2\left(\frac{x}{2}\right),$$

for $\cos(n\theta) + 1$ in the above to get

$$\begin{aligned}
N_c &= 2\sin\left(\frac{n}{2}\theta\right)\cos\left(\frac{n}{2}\theta\right)\cos\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{n}{2}\theta\right) \\
&= 2\cos\left(\frac{n}{2}\theta\right)\left[\sin\left(\frac{n}{2}\theta\right)\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{n}{2}\theta\right)\right] \\
&= 2\cos\left(\frac{n}{2}\theta\right)\sin\left(\frac{n}{2}\theta + \frac{\theta}{2}\right) \\
&= 2\cos\left(\frac{n}{2}\theta\right)\sin\left((n+1)\frac{\theta}{2}\right).
\end{aligned}$$

Thus we have shown that

$$\sum_{r=0}^{n} \cos(r\theta) = \frac{\cos\left(\frac{n}{2}\theta\right)\sin\left((n+1)\frac{\theta}{2}\right)}{\sin(\theta/2)}. \tag{35}$$

To evaluate the sum from $r = 1$ (and not $r = 0$) we have to subtract one from both side of Equation 33 to get

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\sin\left(\theta\left(n+\frac{1}{2}\right)\right) - \sin\left(\frac{\theta}{2}\right)}{2\sin(\theta/2)}.$$

To simplify this we get

$$N_c' = \sin(n\theta)\cos\left(\frac{\theta}{2}\right) + (\cos(n\theta) - 1)\sin\left(\frac{\theta}{2}\right).$$

We now use the cosign double angle formula this time in the form of

$$\cos(x) - 1 = -2\sin^2\left(\frac{x}{2}\right),$$

to write N_c' as

$$\begin{aligned} N_c' &= 2\sin\left(\frac{n}{2}\theta\right)\cos\left(\frac{n}{2}\theta\right)\cos\left(\frac{\theta}{2}\right) - 2\sin\left(\frac{\theta}{2}\right)\sin^2\left(\frac{n}{2}\theta\right) \\ &= 2\sin\left(\frac{n}{2}\theta\right)\left[\cos\left(\frac{n}{2}\theta\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\sin\left(\frac{n}{2}\theta\right)\right] \\ &= 2\sin\left(\frac{n}{2}\theta\right)\cos\left(\frac{n}{2}\theta + \frac{\theta}{2}\right) \\ &= 2\sin\left(\frac{n}{2}\theta\right)\cos\left((n+1)\frac{\theta}{2}\right) . \end{aligned}$$

Thus we have shown that

$$\sum_{r=1}^{n} \cos(r\theta) = \frac{\sin\left(\frac{n}{2}\theta\right)\cos\left((n+1)\frac{\theta}{2}\right)}{\sin(\theta/2)} . \tag{36}$$

Now expand the first cosine in the numerator (call it N_s) of Equation 34 to get

$$N_s = \cos(n\theta)\cos\left(\frac{\theta}{2}\right) - \sin(n\theta)\sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right) = (\cos(n\theta) - 1)\cos\left(\frac{\theta}{2}\right) - \sin(n\theta)\sin\left(\frac{\theta}{2}\right) .$$

The steps are much the same as above (using the half angle formulas). The algebra is

$$\begin{aligned} N_s &= -2\sin^2\left(\frac{n}{2}\theta\right)\cos\left(\frac{\theta}{2}\right) - 2\sin\left(\frac{n}{2}\theta\right)\cos\left(\frac{n}{2}\theta\right)\sin\left(\frac{\theta}{2}\right) \\ &= -2\sin\left(\frac{n}{2}\theta\right)\sin\left(\frac{n}{2}\theta + \frac{\theta}{2}\right) \\ &= -2\sin\left(\frac{n}{2}\theta\right)\sin\left((n+1)\frac{\theta}{2}\right) . \end{aligned}$$

Thus we have shown that (we can start the sum at $r = 0$ or $r = 1$ as $\sin(0) = 0$)

$$\sum_{r=1}^{n} \sin(r\theta) = \frac{\sin\left(\frac{n}{2}\theta\right)\sin\left((n+1)\frac{\theta}{2}\right)}{\sin(\theta/2)} . \tag{37}$$

Examples XII

Exercise 1

We can show convergence for most values of θ by noting that these sums are examples of the theorem in the book that states if v_n is a monotonic sequence that converges to zero then $\sum v_n \sin(n\theta)$ is convergent for all real values of θ and $\sum v_n \cos(n\theta)$ is convergent for all real values of θ other than zero and multiples of 2π.

For $\sum \frac{\cos(n\theta)}{n^2}$ note that its terms are bounded as

$$\left|\frac{\cos(n\theta)}{n^2}\right| < \frac{1}{n^2}\,,$$

and thus this sum converges for all θ by the comparison test.

For the sum $\sum \frac{\cos(n\theta)}{n}$ if $\theta = 2\pi m$ (for m an integer) then $\cos(n\theta) = 1$ and our sum is equivalent to $\sum \frac{1}{n}$ which diverges.

Exercise 2

Note that for $k > 0$ we have that $v_n = \frac{1}{n^k}$ is a monotonic decreasing sequence that converges to zero. Then from the "theorem" given in this section we know that $\sum n^{-k}\cos(n\theta)$ will converge for all real θ other than $\theta = 0$ or θ a multiple of 2π. If $\theta = 0$ or a multiple of 2π then the series $\sum n^{-k}\cos(n\theta)$ is equivalent to the series $\sum n^{-k}$ which converges if $k > 1$.

Exercise 3

This statement is a consequence of Dirichlet's Test (Theorem 24). This follows because as we are told that $\sum a_n$ converges we know that it is bounded so there exists a K such that

$$\left|\sum_{k=1}^{n} a_k\right| < K\,,$$

for all n. Then defining $v_n \equiv n^{-x} = \frac{1}{n^x}$ which is a monotonically decreasing to zero sequence and using Dirichlet's test tells us that the product sum $\sum a_n v_n$ converges.

Exercise 4

One way to work this exercise is to use the ratio test i.e. Theorem 9 (if $a_n > 0$ for $n \geq N$) to argue that both sums either converge together or diverge together.

Another way to work this exercise is to use Able's test. To start we assume that $\sum \frac{a_n}{n}$ converges. Next let $v_n \equiv \frac{n}{n-x}$. Note that for v_n we have

$$v_n - v_{n+1} = \frac{n}{n-x} - \frac{n+1}{n+1-x} = \frac{x}{(n-x)(n+1-x)}\,.$$

If n is large enough eventually $v_n - v_{n+1} > 0$ so v_n is monotonically decreasing. From the definition of v_n we have that $v_n \to 1$. By Abel's test the product series or

$$\sum \left(\frac{a_n}{n}\right) v_n = \sum \frac{a_n}{n-x}\,,$$

also converges.

In the same way if we assume that $\sum \frac{a_n}{n-x}$ converges and let $w_n \equiv \frac{n-x}{n}$ then we have

$$w_n - w_{n+1} = \frac{n-x}{n} - \frac{n+1-x}{n+1} = -\frac{x}{n(n+1)} .$$

From this we see that w_n is a monotonic sequence and converges to the finite limit of one. Using Abel's test we know that the product series or

$$\sum \left(\frac{a_n}{x-x} \right) w_n = \sum \frac{a_n}{n} ,$$

also converges.

Exercise 5

This is an alternating series like in Theorem 18 with $u_n = \frac{1}{n-x}$. We have

$$u_n - u_{n+1} = \frac{1}{n-x} - \frac{1}{n+1-x} = \frac{1}{(n-x)(n+1-x)} > 0 ,$$

Thus for $n \geq \lceil x \rceil$ this sequence is monotonically decreasing and $u_n \to 0$ as $n \to \infty$. The sum converges by Theorem 18.

Note using Dirchlets's test with $a_n = (-1)^{n-1}$ and $v_n = \frac{1}{n-x}$ also shows convergence.

Exercise 6

Consider the alternating sequence Theorem 18 with $u_n = \frac{1}{\sqrt{n}}$ and $u_n = \frac{1}{\sqrt{n+x}}$. Now both expressions for u_n are monotone decreasing and converges to zero. Thus the sum converges by the alternating series test.

Note using Dirchlets's test with $a_n = (-1)^n$ and $v_n = \frac{1}{\sqrt{n}}$ or $\frac{1}{\sqrt{x+n}}$ also shows convergence.

Exercise 7

If we follow the hint and take $a_n = (-1)^n$ then we see that

$$|a_1 + a_2 + \cdots + a_n| \leq 2 < 3 ,$$

for all n. If b_n tends monotonically to zero then $\sum a_n b_n$ is converges by Dirichlet's test (Theorem 24).

Exercise 8

To start this exercise note that using Mathematica (or a table of such sums) we can show that

$$\sum_{r=1}^{n} \sin\left(\left(r + \frac{1}{2}\right) x\right) = \frac{\sin\left(\frac{nx}{2}\right) \sin\left(\frac{1}{2}(n+2)x\right)}{\sin\left(\frac{x}{2}\right)} .$$

This means that this sum is bounded as

$$\left|\sum_{r=1}^{n} \sin\left(\left(r + \frac{1}{2}\right) x\right)\right| \leq \frac{1}{|\sin\left(\frac{x}{2}\right)|} .$$

Let $K = \frac{1}{|\sin\left(\frac{x}{2}\right)|}$ if $\sin\left(\frac{x}{2}\right) \neq 0$. With this bound we can use Theorem 24 (Dirichlet's test) with $v_n = \frac{1}{n+1/2}$ (which is monotonically decreasing to a limit of zero) to argue that the sum

$$\sum \frac{\sin\left(\left(n + \frac{1}{2}\right) x\right)}{n + \frac{1}{2}} ,$$

converges.

Next consider the case when $\sin\left(\frac{x}{2}\right) = 0$. That means that $\frac{x}{2} = l\pi$ so that $x = 2\pi l$ for some integer l. In these cases the sum we are trying to evaluate becomes

$$\sum \frac{\sin\left(\left(n + \frac{1}{2}\right)(2\pi l)\right)}{n + \frac{1}{2}} = \sum \frac{\sin(\pi l)}{n + \frac{1}{2}} .$$

As each term in this sum is exactly zero it also converges. Thus our sum converges for all x.

Chapter 10 (The Product of Two Series)

Examples XIV

Exercise 1

We will use the fact that

$$\sum_{n=1}^{\infty} \frac{1}{n} - \log(n) = \gamma\,, \tag{38}$$

where γ is a constant such that $\gamma \approx 0.57721$.

For this exercise we find that the sum of n terms is given by

$$\begin{aligned}
s_n &= 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1} - \frac{1}{2}\log(n) \\
&+ \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \\
&= \sum_{k=1}^{2n} \frac{1}{k} - \frac{1}{2}\log(n) - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \\
&= \log(2n) + \gamma_{2n} - \frac{1}{2}\log(n) - \frac{1}{2}(\log(n) + \gamma_n) \\
&= \log(2) + \gamma_{2n} - \frac{1}{2}\gamma_n\,.
\end{aligned}$$

Here γ_n is a sequence such that $\gamma_n \to \gamma$ as $n \to \infty$. If we take $n \to \infty$ then we get

$$s_n \to \log(2) + \frac{\gamma}{2}\,,$$

as we were to show.

Exercise 2

For this exercise we find that the sum of n terms is given by

$$\begin{aligned}
s_n &= 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{3n-2} + \frac{1}{3n-1} - \frac{1}{3n} - \frac{1}{3}\log(n) \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{3n-2} + \frac{1}{3n-1} + \frac{1}{3n} - \frac{1}{3}\log(n) - \frac{2}{3} - \frac{2}{6} - \frac{2}{9} - \cdots - \frac{2}{3n} \\
&= \sum_{k=1}^{3n} \frac{1}{k} - \frac{1}{3}\log(n) - \frac{2}{3}\sum_{k=1}^{n} \frac{1}{k} \\
&= \log(3n) + \gamma_{3n} - \frac{1}{3}\log(n) - \frac{2}{3}(\log(n) + \gamma_n) \\
&= \log(3) + \gamma_{3n} - \frac{2}{3}\gamma_n\,.
\end{aligned}$$

If we take $n \to \infty$ then we get

$$s_n \to \log(3) + \frac{\gamma}{3}\,,$$

as we were to show.

Exercise 3

For this exercise we find that the sum of the first $2n$ terms is given by

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k} \\
&= \sum_{k \text{ even}}^{2n} \frac{(-1)^{k+1}}{k} + \sum_{k \text{ odd}}^{2n} \frac{(-1)^{k+1}}{k} = -\sum_{k \text{ even}}^{2n} \frac{1}{k} + \sum_{k \text{ odd}}^{2n} \frac{1}{k} \\
&= -2\sum_{k \text{ even}}^{2n} \frac{1}{k} + \sum_{k=1}^{2n} \frac{1}{k} = -2\sum_{k=1}^{n} \frac{1}{2k} + \sum_{k=1}^{2n} \frac{1}{k} = -\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{2n} \frac{1}{k} \\
&= -\log(n) - \gamma_n + \log(2n) + \gamma_{2n} \\
&= -\gamma_n + \log(2) + \gamma_{2n}\,.
\end{aligned}$$

If we take $n \to \infty$ then we get

$$s_{2n} \to \log(2)\,,$$

as we were to show. As s_{2n+1} differs from s_{2n} by a term $O(1/n)$ the limit of s_{2n+1} as $n \to \infty$ is the same as s_{2n} and both converge to $\log(2)$.

Exercise 4

Using partial fractions we can write

$$\frac{1}{n(16n^2-1)} = -\frac{1}{n} + \frac{2}{4n-1} + \frac{2}{4n+1}\,.$$

Lets consider the sum of n terms. We would have

$$\begin{aligned}
s_n &= -\sum_{k=1}^{n} \frac{1}{k} \\
&\quad + 2\left(\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \cdots + \frac{1}{4n-1}\right) \\
&\quad + 2\left(\frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \cdots + \frac{1}{4n+1}\right) \\
&= -\sum_{k=1}^{n} \frac{1}{k} \\
&\quad + 2\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \cdots + \frac{1}{4n-1} + \frac{1}{4n+1}\right)\,.
\end{aligned}$$

Notice that this second sum is all (but the number 1) of the first $4n+1$ odd terms. Thus we can write s_n as

$$\begin{aligned}
s_n &= -\sum_{k=1}^{n} \frac{1}{k} + 2\left(\sum_{k \text{ odd}}^{4n+1} \frac{1}{k} - 1\right) \\
&= -\sum_{k=1}^{n} \frac{1}{k} + 2\left(\sum_{k=1}^{4n+1} \frac{1}{k} - \sum_{k \text{ even}}^{4n+1} \frac{1}{k} - 1\right) \\
&= -\sum_{k=1}^{n} \frac{1}{k} + 2\left(\sum_{k=1}^{4n+1} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{2k} - 1\right) \\
&= -(\log(n) + \gamma_n) + 2\left(\log(4n) + \gamma_{4n} - \frac{1}{2}\log(2n) - \frac{1}{2}\gamma_{2n} - 1\right) \\
&= -\gamma_n + 2\left(\log(4) + \gamma_{4n} - \frac{1}{2}\log(2) - \frac{1}{2}\gamma_{2n} - 1\right).
\end{aligned}$$

If we take $n \to \infty$ then we get

$$s_n \to \log(8) - 2\,,$$

as we were to show.

Exercise 5

Lets denote the infinite sum by S and the partial sum of $5n$ terms as S_{5n}. By breaking the partial sum down into $5n$ terms this sum can be written as

$$\begin{aligned}
S_{5n} = &\left[\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \left(\frac{1}{2} + \frac{1}{4}\right)\right] && (39) \\
&+ \left[\left(\frac{1}{7} + \frac{1}{9} + \frac{1}{11}\right) - \left(\frac{1}{6} + \frac{1}{8}\right)\right] && (40) \\
&+ \left[\left(\frac{1}{13} + \frac{1}{15} + \frac{1}{17}\right) - \left(\frac{1}{10} + \frac{1}{12}\right)\right] && (41) \\
&\vdots \\
&+ \left[\left(\frac{1}{6n-5} + \frac{1}{6n-3} + \frac{1}{6n-1}\right) - \left(\frac{1}{4n-2} + \frac{1}{4n}\right)\right]. && (42)
\end{aligned}$$

Thus we see that stopping this sum at five terms ($n = 1$) is the sum of the term on line 39. Stopping the sum at 10 terms ($n = 2$) is the sum of the terms on lines 39 and 40. Stopping the sum at 15 terms ($n = 3$) is the sum of the terms on lines 39, 40, and 41. By looking at these terms we can conclude that the sum of $5n$ terms (for a general $n \geq 1$) is the expression above (summed over all lines).

We will separate the above into a couple of sums of the harmonic series $\sum_{k=1}^{n} \frac{1}{k}$ by "adding and subtracting" the needed terms to make "complete" sums. Towards this end we will write

S_{5n} as

$$
\begin{aligned}
S_{5n} &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \cdots + \frac{1}{6n-5} + \frac{1}{6n-3} + \frac{1}{6n-1} \\
&+ \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{6n-6} + \frac{1}{6n-4} + \frac{1}{6n-2} + \frac{1}{6n} \\
&- \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{6n-6} + \frac{1}{6n-4} + \frac{1}{6n-2} + \frac{1}{6n} \right) \\
&- \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{4n-2} + \frac{1}{4n} \right) .
\end{aligned}
$$

By combining the first two rows we can write this as

$$
S_{5n} = \sum_{k=1}^{6n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{3n} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{2n} \frac{1}{k} ,
$$

or

$$
S_{5n} = \log(6n) + \gamma_{6n} - \frac{1}{2}(\log(3n) + \gamma_{3n}) - \frac{1}{2}(\log(2n) + \gamma_{2n}) .
$$

Taking the limit of $n \to \infty$ as $\gamma_n \to \gamma$ we get

$$
S_{5n} \to \log(6) - \frac{1}{2}\log(3) - \frac{1}{2}\log(2) = \frac{1}{2}\log(6) ,
$$

as we were to show.

Exercise 6

This exercise is a generalization of the previous exercise and as such it will help to have solved that one before solving this one.

Lets denote the infinite sum by S and the partial sum of $(p+q)n$ terms as $S_{(p+q)n}$. By breaking the partial sum down into $(p+q)n$ terms this sum can be written as

$$
\begin{aligned}
S_{(p+q)n} &= \left[\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2p-1} \right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} \cdots + \frac{1}{2q} \right) \right] && (43) \\
&+ \left[\left(\frac{1}{2p+1} + \frac{1}{2p+3} + \cdots + \frac{1}{4p-1} \right) - \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \cdots + \frac{1}{4q} \right) \right] && (44) \\
&+ \left[\left(\frac{1}{4p+1} + \frac{1}{4p+3} + \cdots + \frac{1}{6p-1} \right) - \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \cdots + \frac{1}{6q} \right) \right] && (45) \\
&\vdots \\
&+ \left[\left(\frac{1}{2(n-1)p+1} + \frac{1}{2(n-1)p+3} + \cdots + \frac{1}{2np-1} \right) - \left(\frac{1}{2(n-1)q+2} + \frac{1}{2(n-1)q+4} + \cdots + \frac{1}{2nq} \right) \right] . && (46)
\end{aligned}
$$

Thus we see that stopping this sum at $p+q$ terms ($n=1$) is the sum of the term on line 43. Stopping the sum at $2(p+q)$ terms ($n=2$) is the sum of the terms on lines 43 and 44. Stopping the sum at $3(p+q)$ terms ($n=3$) is the sum of the terms on lines 43, 44, and 45. By looking at these terms we can conclude that the sum of $(p+q)n$ terms (for a general $n \geq 1$) is the expression above (summed over all lines).

As before we will separate the above into a couple of sums of the harmonic series $\sum_{k=1}^{n} \frac{1}{k}$ by "adding and subtracting" the needed terms to make "complete" sums. Towards this end we will write $S_{(p+q)n}$ as

$$
\begin{aligned}
S_{(p+q)n} &= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + \cdots + \frac{1}{2np-5} + \frac{1}{2np-3} + \frac{1}{2np-1} \\
&+ \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2np-4} + \frac{1}{2np-2} + \frac{1}{2np} \\
&- \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2np-4} + \frac{1}{2np-2} + \frac{1}{2np} \right) \\
&- \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots + \frac{1}{2nq-4} + \frac{1}{2nq-2} + \frac{1}{2nq} \right) .
\end{aligned}
$$

By combining the first two rows we can write this as

$$
S_{(p+q)n} = \sum_{k=1}^{2np} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{np} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{nq} \frac{1}{k} ,
$$

or

$$
S_{(p+q)n} = \log(2np) + \gamma_{2np} - \frac{1}{2}(\log(np) + \gamma_{np}) - \frac{1}{2}(\log(nq) + \gamma_{nq}) .
$$

Taking the limit of $n \to \infty$ as $\gamma_n \to \gamma$ we get

$$
S_{(p+q)n} \to \log(2) + \frac{1}{2} \log(p/q) ,
$$

as we were to show.

Exercise 7

This is the statement of Theorem 27 in this section with the terms a_n and b_n in that theorem taken to be the terms of the power series.

Exercise 8

Note that the given expression is equivalent to

$$
\begin{aligned}
A(x) &= (1-x) \sum_{n=0}^{\infty} s_n x^n = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=0}^{\infty} s_n x^{n+1} = \sum_{n=0}^{\infty} s_n x^n - \sum_{n=1}^{\infty} s_{n-1} x^n \\
&= s_0 + \sum_{n=1}^{\infty} (s_n - s_{n-1}) x^n .
\end{aligned}
$$

If we write $A(x) = \sum_{n=0}^{\infty} a_n x^n$ we see that $s_0 = a_0$ and that

$$
a_n = s_n - s_{n-1} ,
$$

for $n \geq 1$. Solving the above for s_1 and s_2 we find

$$\begin{aligned}
s_1 &= a_1 + s_0 = a_1 + a_0 \\
s_2 &= a_2 + s_1 = a_2 + a_1 + a_0 \,.
\end{aligned}$$

In general the solution is

$$s_n = \sum_{k=0}^{n} a_n \,,$$

which is the statement we were to show.

Exercise 9

Recall that

$$A_n^r = \frac{(n+r)!}{n!r!} \,, \tag{47}$$

with $A_0^r = 1$. Then we want to show that

$$\sum_{\nu=0}^{n} A_\nu^r = A_n^{r+1} \,.$$

Let the sum on the left-hand-side be denoted S. We have that

$$\begin{aligned}
S &= A_0^r + A_1^r + A_2^r + \cdots + A_n^r \\
&= A_n^r \left[\frac{A_n^r}{A_n^r} + \frac{A_{n-1}^r}{A_n^r} + \cdots + \frac{A_2^r}{A_n^r} + \frac{A_1^r}{A_n^r} + \frac{A_0^r}{A_n^r} \right] \\
&= A_n^r \left[1 + \frac{(r+n-1)!}{r!(n-1)!} \frac{r!n!}{(n+r)!} + \frac{(r+n-2)!}{r!(n-2)!} \frac{r!n!}{(n+r)!} + \cdots + \frac{(r+2)!}{r!2!} \frac{r!n!}{(n+r)!} + \frac{(r+1)!}{r!1!} \frac{r!n!}{(n+r)!} + \frac{r!n!}{(n+r)!} \right] \\
&= A_n^r \left[1 + \frac{n}{n+r} + \frac{n(n-1)}{(n+r)(n+r-1)} + \frac{n(n-1)(n-2)}{(n+r)(n+r-1)(n+r-2)} + \cdots \right. \\
&\quad \left. + \frac{n(n-1)(n-2)\cdots 4(3)(2)}{(n+r)(n+r-1)(n+r-2)\cdots(r+2)} + \frac{n!}{(n+r)(n+r-1)\cdots(r+2)(r+1)} \right] .
\end{aligned}$$

Notice that the above is the statement given in the "hint" for this problem.

Next from the hint the expression in brackets above is equal to $\frac{n+r+1}{r+1}$ so we get

$$\sum_{\nu=0}^{n} A_\nu^r = A_n^r \left(\frac{n+r+1}{r+1} \right) = \frac{(n+r+1)!}{n!(r+1)!} = A_n^{r+1} \,, \tag{48}$$

as we were to show.

Here we prove the given hint. Notice that we can write

$$\frac{n+r+1}{r+1} = 1 + \frac{n}{n+r} \left(\frac{n+r}{r+1} \right) .$$

This has taken a fraction of the form $\frac{n+r+1}{r+1}$ into a fraction of the form $\frac{n+r}{r+1}$ where there n is "one less" in the numerator than it was before. If we do that procedure a second time on the term in parenthesis we get

$$\frac{n+r+1}{r+1} = 1 + \frac{n}{n+r}\left[1 + \frac{n-1}{n-1+r}\left(\frac{n-1+r}{r+1}\right)\right].$$

This gives a "new" term in parenthesis that we can we can apply our recursive relationship to. We have

$$\begin{aligned}\frac{n+r+1}{r+1} &= 1 + \frac{n}{n+r}\left[1 + \frac{n-1}{n-1+r}\left[1 + \frac{n-2}{n-2+r}\left(\frac{n-2+r}{r+1}\right)\right]\right] \\ &= 1 + \frac{n}{n+r} + \frac{n(n-1)}{(n+r)(n-1+r)} \\ &+ \frac{n(n-1)(n-2)}{(n+r)(n-1+r)(n-2+r)}\left[1 + \frac{n-3}{n-3+r}\left(\frac{n-3+r}{r+1}\right)\right],\end{aligned}$$

which is the hint given.

Exercise 10

We want to prove that

$$(1-x)^{-r-1} = \sum_{n=0}^{\infty} A_n^r x^n. \tag{49}$$

We will prove this by induction. To start this process we let $r = 0$ where since

$$A_n^r = A_n^0 = 1,$$

we have the expression

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n,$$

which is true and converges when $|x| < 1$. Next assume that Equation 49 is true for $0 \le r \le R$ and let $r = R+1$. Then using the inductive hypothesis we have

$$(1-x)^{-R-1-1} = (1-x)^{-R-1}(1-x)^{-1} = \left(\sum_{n=0}^{\infty} A_n^R x^n\right)\left(\sum_{m=0}^{\infty} x^m\right).$$

We will write this product as

$$\sum_{n=0}^{\infty} C_n x^n,$$

for some coefficients C_n. An expression for C_n, due to fact that it is the coefficient of the product series, from two other power series means that we can write it as

$$C_n = \sum_{\nu=0}^{n} A_\nu^R 1^{n-\nu} = \sum_{\nu=0}^{n} A_\nu^R = A_n^{R+1}.$$

Here we have used Equation 48 to simplify this. This means we have shown

$$(1-x)^{-(R+1)-1} = \sum_{m=0}^{\infty} A_n^{R+1} x^n \,,$$

which is the required induction step.

Exercise 11

From the previous exercise we know that the coefficients in the Taylor series for

$$(1-x)^{-r-s-2} = (1-x)^{-(r+s+1)-1} \,,$$

are A_n^{r+s+1}. Note that we can write this function as the product of the two functions $(1-x)^{-r-1}$ and $(1-x)^{-s-1}$ which have Taylor series coefficients given by A_n^r and A_n^s respectively. Then from the formula for the coefficients of the product of two power series in terms of the coefficients of the individual power series we have

$$A_n^{r+s+1} = \sum_{\nu=0}^{n} A_\nu^r A_{n-\nu}^s \,,$$

which is the expression we were trying to show.

Exercise 12

We can show (or have already shown) that $E(x)$ converges absolutely for all x. Then using the theorem on the product of two infinite series (proven in the book) we have

$$\begin{aligned} E(x)E(y) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{x^k}{k!}\right) \left(\frac{x^{n-k}}{(n-k)!}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \,. \end{aligned}$$

Now using the *Binomial theorem* we have that

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n \,, \tag{50}$$

so that the expression for $E(x)E(y)$ becomes

$$E(x)E(y) = \sum_{n=0}^{\infty} \frac{1}{n!} (x+y)^n = E(x+y) \,,$$

as we were to show.

Examples XV

Exercise 1

Part (a): Define the function $f(x)$ to be

$$f(x) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \,,$$

then we want to evaluate $f(x)^2$. Note that the above series converges absolutely when $|x| < 1$ by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. Then using Theorem 27 from the book the value of $f(x)^2$ is given by $\sum_{n=1}^{\infty} c_n$ with c_n given by

$$c_n = a_n b_1 + a_{n-1} b_2 + a_{n-2} b_3 + \cdots + a_2 b_{n-1} + a_1 b_n \,.$$

For the coefficients in the series expression for $f(x)$ this evaluates to

$$\begin{aligned}
c_n &= \frac{1}{2}\left(\frac{(-1)^{n+1}}{n} x^n \left(\frac{1}{1}x\right) + \frac{(-1)^n}{n-1} x^{n-1} \left(\frac{-1}{2}x^2\right) + \frac{(-1)^{n-1}}{n-2} x^{n-2} \left(\frac{1}{3}x^3\right) + \cdots + \left(\frac{1}{1}x\right)\left(\frac{(-1)^{n+1}}{n} x^n\right)\right) \\
&= \frac{(-1)^{n+1} x^{n+1}}{2} \left(\frac{1}{n} + \frac{1}{2(n-1)} + \frac{1}{3(n-2)} + \cdots + \frac{1}{n}\right) \\
&= \frac{(-1)^{n+1} x^{n+1}}{2} \sum_{p=1}^{n} \frac{1}{p(n-p+1)} \,.
\end{aligned}$$

Using partial fractions we can write

$$\frac{1}{p(n-p+1)} = \frac{1}{(n+1)p} + \frac{1}{(n+1)(n-p+1)} \,.$$

Using this we can simplify the sum in the expression for c_n as

$$\begin{aligned}
\sum_{p=1}^{n} \frac{1}{p(n-p+1)} &= \frac{1}{n+1} \sum_{p=1}^{n} \left(\frac{1}{p} + \frac{1}{n-p+1}\right) \\
&= \frac{1}{n+1} \sum_{p=1}^{n} \frac{1}{p} + \frac{1}{n+1} \sum_{p=1}^{n} \frac{1}{n+1-p} \\
&= \frac{1}{n+1} \sum_{p=1}^{n} \frac{1}{p} + \frac{1}{n+1} \sum_{p=1}^{n} \frac{1}{p} \\
&= \frac{2}{n+1} \sum_{p=1}^{n} \frac{1}{p} \,. \qquad (51)
\end{aligned}$$

The expression for c_n now looks like

$$c_n = \frac{(-1)^{n+1} x^{n+1}}{n+2} \sum_{p=1}^{n} \frac{1}{p} \,.$$

Thus the expression we have derived for $f(x)^2$ takes the form

$$f(x)^2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1} H_n x^{n+1} \,, \qquad (52)$$

where I have defined the numbers H_n as

$$H_n \equiv \sum_{p=1}^{n} \frac{1}{p} \,. \tag{53}$$

This is the expression we were trying to show.

Part (b): If $x = 1$ the left-hand-side of Equation 52 is the expression

$$f(1)^2 = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots\right)^2 = \frac{1}{2}(\log(2))^2 \,.$$

If $x = 1$ the right-hand-side of Equation 52 is the expression

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)\frac{1}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{H_n}{n+1} \,,$$

where H_n is defined by the sum above. As this series is obtained from the terms from the product of two convergent series if this sum converges then by an application of Theorem 30 we can conclude that the two sides are equal.

To show that this series converges from the previous section we had

$$H_n = \log(n) + \gamma_n \,,$$

so the "right-hand-side" sum is given by

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\log(n)}{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\gamma_n}{n+1} \,.$$

Each of these series converges by the alternating series test and thus the original series converges.

Exercise 2

From the **geometric series** we have

$$\sum_{n=0}^{\infty}(-r)^n = \frac{1}{1+r} \,,$$

If we take $r \to x^2$ this is

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2} \,.$$

If we integrate both sides we get

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1}(x) + C \,.$$

If we take $x = 0$ as $\tan^{-1}(0) = 0$ we have that $C = 0$ and we thus have shown that

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1}(x) \, .$$

If we write this sum in starting at $n = 1$ it is

$$\tan^{-1}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \, . \tag{54}$$

This series converges absolutely by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. Then using Theorem 27 from the book the value of $(\tan^{-1}(x))^2$ is given by $\sum_{n=1}^{\infty} c_n$ with c_n given by

$$c_n = a_n b_1 + a_{n-1} b_2 + a_{n-2} b_3 + \cdots + a_2 b_{n-1} + a_1 b_n \, . \tag{55}$$

For the coefficients in the series expression for $\tan^{-1}(x)$ this evaluates to

$$\begin{aligned}
c_n &= \left(\frac{(-1)^2 x}{1} \right) \left(\frac{(-1)^{n+1} x^{2n-1}}{2n-1} \right) + \left(\frac{(-1)^3 x^3}{3} \right) \left(\frac{(-1)^n x^{2n-3}}{2n-3} \right) + \cdots + \left(\frac{(-1)^{n+1} x^{2n-1}}{2n-1} \right) \left(\frac{(-1)^2 x}{1} \right) \\
&= (-1)^{n+1} x^{2n} \left(\frac{1}{1(2n-1)} + \frac{1}{3(2n-3)} + \cdots + \frac{1}{(2n-1)1} \right) \\
&= (-1)^{n+1} x^{2n} \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{p(2n-p)} \, .
\end{aligned}$$

Using partial fractions we can write

$$\frac{1}{p(2n-p)} = \frac{1}{2np} + \frac{1}{2n(2n-p)} \, .$$

Using this we can simplify the sum in the expression for c_n as

$$\begin{aligned}
\sum_{p=1,3,5,\ldots,2n-1} \frac{1}{p(2n-p)} &= \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{2np} + \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{2n(2n-p)} \\
&= \frac{1}{n} \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{p} \, .
\end{aligned}$$

The expression for c_n now looks like

$$c_n = \frac{(-1)^{n+1} x^{2n}}{n} \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{p} \, .$$

Thus the expression we have derived for $(\tan^{-1}(x))^2$ takes the form

$$\begin{aligned}
(\tan^{-1}(x))^2 &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n}}{n} \sum_{p=1,3,5,\ldots,2n-1} \frac{1}{p} & (56) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} \sum_{p=1,3,5,\ldots,2n+1} \frac{1}{p} \, . & (57)
\end{aligned}$$

which is the expression we were trying to show when we multiply both sides by $\frac{1}{2}$.

Following the same arguments as in the previous exercise when $x = 1$ the series expansion for $\tan^{-1}(x)$ converges by the alternating series test. The right-hand-side of Equation 57 when $x = 1$ can be written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(H_{2n} - \frac{1}{2} H_n \right) ,$$

using the fact that $H_n = \log(n) + \gamma_n$ we can show that the above series converges using the alternating series test. Then using an application of Theorem 30 we can conclude that the two expressions are equal (when $x = 1$).

Exercise 3

I think there is a typo in this exercise. I believe the statement "putting $x = -y$" should be "putting $x = -1$". The statement in Examples XIV 10 is given by Equation 49. If we take $x = -1$ in that expression we get

$$2^{-r-1} = \sum_{n=0}^{\infty} (-1)^n A_n^r ,$$

which matches the desired expression in the book.

Exercise 4

Each of these series converges absolutely by comparison with the geometric series $\sum_{n=1}^{\infty} x^n$. Then an application of Theorem 29 gives that the limit as $x \to 1^-$ tends to the expressions given.

Exercise 5

Part (a): In this case $\sum a_n$ and $\sum b_n$ are absolutely convergent so $\sum c_n$ is absolutely convergent and equal to the product of the two sums (using Theorem 27).

Part (b): In this case $\sum a_n$ and $\sum b_n$ are convergent by the alternating series test (but not absolutely convergent). The coefficients c_n in this case can be written as

$$\begin{aligned} c_n &= \sum_{p=1}^{n} \left(\frac{(-1)^p}{p} \right) \left(\frac{(-1)^{n-p+1}}{n-p+1} \right) \\ &= \frac{2(-1)^{n+1}}{n+1} H_n , \end{aligned}$$

using Equation 51. Using the fact that $H_n = \log(n) + \gamma_n$ we can show that $\sum c_n$ converges using the alternating series test. Then a use of Theorem 30 gives us that the product of $\sum a_n$ and $\sum b_n$ is $\sum c_n$.

Part (c): In this case $\sum a_n$ and $\sum b_n$ are convergent by the alternating series test (but not absolutely convergent). I claim that the sum $\sum c_n$ does not converge and thus does not equal product of $\sum a_n$ times $\sum b_n$. To show that this series does not converge note that the coefficients c_n in this case can be written as

$$\begin{aligned} c_n &= \sum_{p=1}^{n} \left(\frac{(-1)^p}{\sqrt{p}} \right) \left(\frac{(-1)^{n-p+1}}{\sqrt{n-p+1}} \right) \\ &= (-1)^{n+1} \sum_{p=1}^{n} \frac{1}{\sqrt{p}\sqrt{n-p+1}} . \end{aligned}$$

To find a bound on c_n recall the inequality

$$xy \le \frac{1}{2}(x^2 + y^2) ,$$

so that

$$\sqrt{p}\sqrt{n-p+1} \le \frac{1}{2}(n+1) .$$

Using this we get

$$\frac{1}{\sqrt{p}\sqrt{n-p+1}} \ge \frac{2}{n+1} .$$

Thus a bound for c_n is

$$|c_n| \ge \sum_{p=1}^{n} \frac{2}{n+1} = \frac{2n}{n+1} ,$$

Note that $|c_n|$ does not converge to zero as it must if $\sum c_n$ was a convergent series.

Chapter 11 (Uniform Convergence)

Examples XVI

Exercise 1

Part (i): For these we need to find bounds M_n for each of the terms where the bounding sequence $\sum M_n$ converges. We can do that with

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$$
$$\left|\frac{x^n}{n(n+1)}\right| \leq \frac{1}{n(n+1)}$$
$$\left|\frac{x^{2n}}{x^{2n}+n^2}\right| \leq \frac{1}{x^{2n}+x^2} \leq \frac{1}{n^2}\,.$$

Part (ii): In the same way as the previous part using the fact that

$$|x^n| \geq \delta^n\,,$$

and

$$|1+x^n| \geq |x^n| \geq \delta^n\,,$$

we have

$$\frac{1}{|x^n|} \leq \frac{1}{\delta^n}$$
$$\left|\frac{1}{1+x^n}\right| < \frac{1}{|x^n|} \leq \frac{1}{\delta^n}$$
$$\left|\frac{1}{x^n(1+x^n)}\right| < \frac{1}{|x^{2n}|} \leq \frac{1}{\delta^{2n}}\,.$$

As $\sum \frac{1}{\delta^n}$ and $\sum \frac{1}{\delta^{2n}}$ both converge by using Theorem 35 these sums converge uniformly.

Part (iii): As $|x^n| \leq \delta^n$ and $\sum \delta^n$ converges the original sum converges uniformly.

For the second sum we can bound the terms by those of a convergent sum as

$$\frac{|x^n|}{n+1} < |x^n| < \delta^n\,.$$

For the third sum we can bound the terms by those of a convergent sum as

$$|(n+1)x^n| \leq (n+1)\delta^n\,.$$

Here the sum with terms $(n+1)\delta^n$ can be shown to converge using the ratio test.

For the fourth sum we can bound the terms by those of a convergent sum as

$$|n^3 x^n| \leq n^3 \delta^n \,,$$

and $\sum n^3 \delta^n$ can be shown to converge using the ratio test.

Part (iv): For the first sum we can bound the terms by those of a convergent sum as

$$\frac{1}{n^4 + n^2 x^2} = \frac{1}{n^2(n^2 + x^2)} < \frac{1}{n^2(n^2)} = \frac{1}{n^4} \,.$$

For the second sum we can bound the terms by those of a convergent sum as

$$\frac{1}{n^2 + n^4 x^4} = \frac{1}{n^2(1 + n^2 x^2)} < \frac{1}{n^2} \,.$$

Exercise 2

Let $a_n(\theta) = \cos(n\theta)$ then we can show that

$$s_n(\theta) = \sum_{k=1}^{n} a_k(\theta) = \sum_{k=1}^{n} \cos(k\theta) = \frac{\sin\left(\frac{n\theta}{2}\right)\cos\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \,. \tag{58}$$

If $a_n(\theta) = \sin(n\theta)$ then we can also show that

$$s_n(\theta) = \sum_{k=1}^{n} a_k(\theta) = \sum_{k=1}^{n} \sin(k\theta) = \frac{\sin\left(\frac{n\theta}{2}\right)\sin\left(\frac{(n+1)\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \,. \tag{59}$$

In both of these cases we have that

$$|s_n(\theta)| \leq \frac{1}{\left|\sin\left(\frac{\theta}{2}\right)\right|} \,.$$

If we plot $\left|\sin\left(\frac{\theta}{2}\right)\right|$ as a function of θ over the range $[0, 2\pi]$ we see it looks like a one-hump function that starts at zero for $\theta = 0$ and ends at zero for $\theta = 2\pi$. Thus for $\theta \in [\delta, 2\pi - \delta]$ we see that

$$\left|\sin\left(\frac{\theta}{2}\right)\right| \geq \left|\sin\left(\frac{\delta}{2}\right)\right| \,,$$

and thus we have the upper bound on $s_n(\theta)$ of

$$|s_n(\theta)| \leq \frac{1}{\left|\sin\left(\frac{\delta}{2}\right)\right|} \,.$$

If we seek to apply Theorem 36 (Dirichlet's test) this is the value of K we could use. Next as we are told to assume that $v_n \to 0$ by applying Dirichlet's test we have that both of the sums $\sum v_n \cos(n\theta)$ and $\sum v_n \sin(n\theta)$ converge uniformly for $\delta \leq \theta \leq 2\pi - \delta$.

Exercise 3

For the first two sums the desired conclusion follows as a direct application of what we have shown in Exercise 2 with $v_n = \frac{1}{n}$. For the second two sums using the fact that

$$\left|\frac{\sin(n\theta)}{n^2}\right| \leq \frac{1}{n^2} \quad \text{and} \quad \left|\frac{\cos(n\theta)}{n^2}\right| \leq \frac{1}{n^2},$$

for all θ from the Weierstrass M test we can conclude that they also converge uniformly.

Exercise 4

This sum can be written as

$$1 + \sum_{k=1}^{\infty} \frac{e^{-2kx}}{(2k)^2 - 1}.$$

As

$$\frac{e^{-2kx}}{(2k)^2 - 1} \leq \frac{1}{(2k)^2 - 1},$$

when $x \geq 0$ by using the Weierstrass M test we can conclude that this sum converges uniformly.

Exercise 5

This follows from Theorem 33 in that if we differentiate term-by-term that expression is equal to the derivative of the function that is the sum of the terms (except possibly at the end points of the domain). Thus since the sum converges to $f(x)$ for $x \geq 0$ the sum of the first derivatives of the terms must converge to $f'(x)$ over $x \geq \delta_1$ and the sum of the second derivatives of the terms must converge to $f''(x)$ over $x \geq \delta_2 \geq \delta_1$.

Exercise 6

As over $0 \leq x \leq 1$ we have

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2},$$

which converges by the Weierstrass M test this series converges uniformly. Because this sum converges uniformly the integral of the sum of terms is equal to the sum of the integral of each of the terms. Computing the integral of one of the terms we get

$$\int_0^1 \frac{x^n}{n^2} dx = \frac{1}{n^2(n+1)}.$$

This with the above statement gives the desired conclusion.

Exercise 7

To use Theorem 37 (Abel's test) we need to write the sum as $\sum a_n(x)v_n(x)$ where $v_n(x)$ is monotonically decreasing/increasing, $\sum a_n(x)$ uniformly convergent, and $|v_n(x)| < K$ for all n. In this sum if we take $v_n(x) = x^n$ and $a_n(x) = a_n$ then $v_n(x)$ is monotonically decreasing for $x \in [0, 1]$, $|v_n(x)| \leq 1$ for all n and $\sum a_n(x)$ is uniformly convergent. Thus we can conclude that $\sum a_n x^n$ is uniformly convergent.

Exercise 8

From the previous exercise we know that $\sum a_n x^n$ is uniformly convergent for $0 \leq x \leq 1$. Using that, and continuity properties of uniformly convergent series, as $a_n x^n$ is continuous we have that $\sum a_n x^n$ is continuous for the domain $0 < x < 1$ and that

$$\lim_{x \to 1^-} \sum a_n x^n = \sum a_n \,,$$

which is the statement of Theorem 29.

Exercise 9

This sum can be written as (perhaps with an alternating factor of $(-1)^n$ inside the sum)

$$\frac{1}{a} + 2a \sum_{n=1}^{\infty} \frac{\cos(nx)}{a^2 - n^2} \,.$$

Following the hint we have

$$\left| \frac{\cos(nx)}{a^2 - n^2} \right| \leq \frac{1}{(1/2)n^2} = \frac{2}{n^2} \,.$$

An infinite sum of this upper bound converges and so by the Weierstrass M test this series converges uniformly.

Exercise 10

I claim that this series converges uniformly for $|x| \leq \delta < 1$. In that case there is a N such that

$$|x|^n < \frac{1}{2} \,,$$

for all $n > N$. Then for these values of n we have

$$|1 + x^n| > |1 - |x^n|| > \left| 1 - \frac{1}{2} \right| = \frac{1}{2} \,,$$

and thus
$$\left|\frac{(-1)^n x^n}{n(1+x^n)}\right| \le \frac{|x|^n}{n} < \frac{\delta^n}{n}\,.$$
As the sum with these terms converges using the Weierstrass M test we have that our original sum converges uniformly over the domain $|x| \le \delta < 1$.

Exercise 11

As $\sum u_n(x)$ converges uniformly given an $\epsilon > 0$ there exists a N such that
$$|u_N(x) + u_{N+1}(x) + \cdots + u_{N+p}(x)| < \epsilon\,,$$
for all $p > 0$. Note that for the sum $\sum_{n=1}^{\infty} u_n(x)F(x)$ given a $\epsilon > 0$ the same value of N (found above) will work in that
$$\begin{aligned}|F(x)u_N(x) + F(x)u_{N+1}(x) + \cdots + F(x)u_{N+p}(x)| &= |F(x)||u_N(x) + u_{N+1}(x) + \cdots + u_{N+p}(x)| \\ &< |u_N(x) + u_{N+1}(x) + \cdots + u_{N+p}(x)| < \epsilon\,.\end{aligned}$$
Thus $\sum_{n=1}^{\infty} u_n(x)F(x)$ also converges uniformly and because of that the integral of this sum is the sum of the integrals (showing the desired expression).

Chapter 12 (Binomial, Logarithmic, and Exponential)

Examples XVII

For this section we will need the *Binomial theorem*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + \binom{n}{r}x^r + \cdots \,, \tag{60}$$

with

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \,. \tag{61}$$

Exercise 4

If we take $n = \frac{m}{2}$ in Equation 60 we get

$$(1+x)^{\frac{m}{2}} = 1 + \frac{m}{2}x + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{m-2}{2}\right)x^2 + \frac{1}{6}\left(\frac{m}{2}\right)\left(\frac{m-2}{2}\right)\left(\frac{m-4}{4}\right)x^3 + \cdots \,,$$

which is the desired expression.

Exercise 5

From the binomial theorem with $n = -2$ and replacing $x \to -x$ we get

$$\begin{aligned}
(1-x)^{-2} &= 1 - 2(-x) + \frac{(-2)(-3)}{2}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \cdots \\
&= 1 + 2x + 3x^2 + 4x^3 + \cdots \,,
\end{aligned}$$

which is the desired expression.

From the binomial theorem with $n = -3$ and replacing $x \to -x$ we get

$$\begin{aligned}
(1-x)^{-3} &= 1 - 3(-x) + \frac{(-3)(-4)}{2}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + \cdots + \frac{(-3)(-4)(-5)\cdots(-3-r+1)}{r!}(-x)^r \\
&= 1 + 3x + \frac{1}{2}(3\cdot 4)x^2 + \frac{1}{2}(4\cdot 5)x^3 + \cdots + \frac{(-1)^r(3)(4)(5)\cdots(3+r-1)}{r!}x^r \\
&= 1 + 3x + \frac{1}{2}(3\cdot 4)x^2 + \frac{1}{2}(4\cdot 5)x^3 + \cdots + \frac{3\cdot 4\cdot 5\cdots r(r+1)(r+2)}{r!}x^r \\
&= 1 + 3x + \frac{1}{2}(3\cdot 4)x^2 + \frac{1}{2}(4\cdot 5)x^3 + \cdots + \frac{(r+1)(r+2)}{2}x^r \,.
\end{aligned}$$

Exercise 6

Recalling that

$$\frac{1}{1-x} = \sum_{l=0}^{\infty} x^l \,,$$

and using the binomial theorem we can write the given expression

$$\frac{(1-x)^m}{(1-x)} = (1-x)^{m-1} \,,$$

as

$$\left(\sum_{k=0}^{\infty} \binom{m}{k} (-1)^k x^k \right) \left(\sum_{l=0}^{\infty} x^l \right) = \sum_{n=0}^{\infty} \binom{m-1}{n} (-1)^n x^n \,. \tag{62}$$

Computing the product on the left-hand-side of Equation 62 our expression is equal to

$$\sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{m}{l} (-1)^l 1^{n-l} \right) x^n = \sum_{n=0}^{\infty} \binom{m-1}{n} (-1)^n x^n \,.$$

If we equate powers of x we see that

$$\begin{aligned}
\binom{m-1}{n}(-1)^n &= \sum_{l=0}^{n} \binom{m}{l} (-1)^l \\
&= 1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^n \binom{m}{n} \,.
\end{aligned}$$

which is the desired expression.

Exercise 7

We will use the binomial theorem to derive the power series representations of $\frac{1}{1-x}$ and $\frac{1}{(1-x)^2}$. To start recall that using the binomial theorem we can write

$$\frac{1}{1-x} = (1-x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} (-1)^k x^k \,.$$

Now the binomial coefficient above can be simplified as

$$\binom{-1}{k} = \frac{(-1)(-2)(-3)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k \,.$$

Putting this into the series above we find

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} (-1)^k (-1)^k x^k = \sum_{k=0}^{\infty} x^k \,, \tag{63}$$

the well known expansion for $\frac{1}{1-x}$.

Next we have

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} (-1)^k x^k \,.$$

In this case the binomial coefficient above can be simplified as

$$\binom{-2}{k} = \frac{(-2)(-3)(-4)\cdots(-2-k+1)}{k!} = \frac{(-1)^k 2\cdot 3\cdot 4 \cdots k(k+1)}{k!} = (-1)^k (k+1)\,.$$

Putting this into the series above we find

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k \,. \tag{64}$$

Next we evaluate the expression

$$\frac{(1-x)^m}{(1-x)^2} = (1-x)^{m-2}\,.$$

This becomes

$$\left(\sum_{j=0}^{\infty} \binom{m}{j} (-1)^j x^j\right) \left(\sum_{k=0}^{\infty} (k+1) x^k\right) = \sum_{n=0}^{\infty} \binom{m-2}{n} (-1)^n x^n \,. \tag{65}$$

Computing the product on the left-hand-side of Equation 65 our expression is equal to

$$\sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \binom{m}{l} (-1)^l (n-l+1)\right) x^n = \sum_{n=0}^{\infty} \binom{m-2}{n} (-1)^n x^n \,.$$

If we equate powers of x we see that

$$\binom{m-2}{n} (-1)^n = \sum_{l=0}^{n} \binom{m}{l} (-1)^l (n-l+1)\,,$$

or

$$\binom{m-2}{n} = \sum_{l=0}^{n} \binom{m}{l} (-1)^{n-l} (n-l+1)\,.$$

In this last sum we will let $k = n - l$ then the above becomes

$$\binom{m-2}{n} = \sum_{k=0}^{n} \binom{m}{n-k} (-1)^k (k+1)\,.$$

For the next identity

$$(1+x)^m (1+x)^2 = (1+x)^{m+2}\,,$$

using the binomial theorem gives

$$\left(\sum_{n=0}^{\infty}\binom{m}{n}x^n\right)(1+2x+x)=\sum_{n=0}^{\infty}\binom{m+2}{n}x^n\,,$$

or

$$\sum_{n=0}^{\infty}\binom{m}{n}x^n+2\sum_{n=0}^{\infty}\binom{m}{n}x^{n+1}+\sum_{n=0}^{\infty}\binom{m}{n}x^{n+2}=\sum_{n=0}^{\infty}\binom{m+2}{n}x^n\,.$$

If we shift the indices of the sums on the left and release some terms we get

$$\binom{m}{0}+\binom{m}{1}x+2\binom{m}{0}x+\sum_{n=2}^{\infty}\left[\binom{m}{n}+2\binom{m}{n-1}+\binom{m}{n-2}\right]x^n=\binom{m+2}{0}+\binom{m+2}{1}x+\sum_{n=2}^{\infty}\binom{m+2}{n}x^n\,.$$

Dropping common terms on both sides we get

$$\sum_{n=2}^{\infty}\left[\binom{m}{n}+2\binom{m}{n-1}+\binom{m}{n-2}\right]x^n=\sum_{n=2}^{\infty}\binom{m+2}{n}x^n\,.$$

If we equate powers of x we see that we have

$$\binom{m+2}{n}=\binom{m}{n}+2\binom{m}{n-1}+\binom{m}{n-2}\,,$$

for a second identity.

Exercise 8

To prove this, we start by noting that

$$\begin{aligned}
r\binom{n}{r}&=\frac{rn!}{(n-r)!r!}=\frac{n!}{(n-r)!(r-1)!}\\
&=\frac{n(n-1)!}{(n-1-(r-1))!(r-1)!}\\
&=n\binom{n-1}{r-1}\,. \qquad (66)
\end{aligned}$$

Using this we find

$$\begin{aligned}
\sum_{r=1}^{n}(-1)^r r\binom{n}{r}&=n\sum_{r=1}^{n}(-1)^r\binom{n-1}{r-1}\\
&=n\sum_{r=0}^{n-1}(-1)^{r+1}\binom{n-1}{r}\\
&=-n\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}=-n(-1+1)^{n-1}=0\,, \qquad (67)
\end{aligned}$$

using the binomial theorem to “undo” the sum.

To prove the second identity we will use mathematical induction. Thus we assume that

$$\sum_{r=1}^{n} (-1)^r r^m \binom{n}{r} = 0\,,$$

is true for all $m = 1, 2, 3, \cdots, M-1, M$. We have shown this to be true for $M = 1$ in the first part of this exercise. Now consider this expression for $m = M+1$ and we will removing one r from the factor r^{M+1} using Equation 67 as

$$\begin{aligned}
\sum_{r=1}^{n} (-1)^r r^{M+1} \binom{n}{r} &= \sum_{r=1}^{n} (-1)^r r^M n \binom{n-1}{r-1} \\
&= -n \sum_{r=1}^{n} (-1)^{r-1} r^M \binom{n-1}{r-1} \\
&= -n \sum_{r=0}^{n-1} (-1)^r (r+1)^M \binom{n-1}{r}\,. \qquad (68)
\end{aligned}$$

Now if we expand $(r+1)^M$ using the binomial theorem again we get

$$(1+r)^M = \sum_{j=0}^{M} \binom{M}{j} r^j\,,$$

Note that the powers on r in this sum are all less than or equal to M. Thus by the inductive hypothesis the sum over r of each of these terms vanishes. Thus the sum with $m = M+1$ in Equation 68 also vanishes and we have shown the induction step.

Note: there must be a way to show this identity starting with a more obvious identity and then perhaps expanding terms using the binomial theorem but I was not able to come up with such a method. If anyone knows of one please contact me.

Exercise 9

To start this exercise note that for $f(x)$ defined as

$$f(x) = \prod_{r=1}^{n} (x - a_r)\,,$$

the x derivative of this is

$$\begin{aligned}
f'(x) &= \prod_{r=2}^{n} (x-a_r) + \prod_{r=1;r\neq 2}^{n} (x-a_r) + \prod_{r=1;r\neq 3}^{n} (x-a_r) + \cdots + \prod_{r=1;r\neq n}^{n} (x-a_r) \\
&= \sum_{k=1}^{n} \prod_{r=1;r\neq k}^{n} (x-a_r)\,.
\end{aligned}$$

From this expression, if we evaluate this at $x = a_p$ for a p in the domain $1 \le p \le n$ we find

$$\begin{aligned} f'(a_p) &= 0 + \cdots + 0 + \prod_{r=1;r\neq p}^{n} (a_p - a_r) + 0 + \cdots + 0 \\ &= \prod_{r=1;r\neq p}^{n} (a_p - a_r)\,. \end{aligned} \tag{69}$$

We will need that expression later.

Next if we use partial fractions to expand $\frac{\phi(x)}{f(x)}$ we would have

$$\frac{\phi(x)}{f(x)} = \frac{\phi(x)}{\prod_{r=1}^{n}(x - a_r)} = \sum_{r=1}^{n} \frac{A_r}{x - a_r}\,,$$

where A_r a constant. To evaluate the value of A_r we multiply the above by the $f(x) = \prod_{r=1}^{n}(x - a_r)$ on both sides to get

$$\phi(x) = \sum_{r=1}^{n} A_r \prod_{j=1;j\neq r}^{n} (x - a_j)\,.$$

If we evaluate this at a_r we get

$$\phi(a_r) = A_r \prod_{j=1;j\neq r}^{n} (a_r - a_j)\,,$$

so solving for A_r we get

$$A_r = \frac{\phi(a_r)}{\prod_{j=1;j\neq r}^{n}(a_r - a_j)}\,.$$

Then from Equation 69 we can write this as

$$A_r = \frac{\phi(a_r)}{f'(a_r)}\,.$$

With this coefficient we have

$$\frac{\phi(x)}{f(x)} = \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)(x - a_r)}\,. \tag{70}$$

Now using

$$\frac{1}{x - a_r} = -\frac{1}{a_r\left(1 - \frac{x}{a_r}\right)} = -\frac{1}{a_r}\sum_{k=0}^{\infty} \frac{x^k}{a_r^k}\,,$$

which (as a truncated series) is more accurate when $|x| \ll 1$ we get

$$\begin{aligned} \frac{\phi(x)}{f(x)} &= \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} \left(-\frac{1}{a_r} \sum_{k=0}^{\infty} \frac{x^k}{a_r^k} \right) \\ &= -\sum_{k=0}^{\infty} x^k \left\{ \sum_{r=1}^{n} \frac{\phi(a_r)}{f'(a_r)} a_r^{-k-1} \right\}\,. \end{aligned} \tag{71}$$

On the other side if $|x|$ is large (i.e. $|x| \gg 1$) the series we use is given by

$$\frac{1}{x-a_r} = \frac{1}{x\left(1-\frac{a_r}{x}\right)} = \frac{1}{x}\sum_{k=0}^{\infty}\frac{a_r^k}{x^k}\,.$$

In this case the sum we get is given by

$$\begin{aligned}\frac{\phi(x)}{f(x)} &= \sum_{r=1}^{n}\frac{\phi(a_r)}{f'(a_r)}\left(\frac{1}{x}\sum_{k=0}^{\infty}\frac{a_r^k}{x^k}\right)\\ &= \sum_{k=0}^{\infty}x^{-k-1}\left\{\sum_{r=1}^{n}\frac{\phi(a_r)}{f'(a_r)}a_r^k\right\}\,. \qquad (72)\end{aligned}$$

Exercise 10

This is a practical example of the expansion performed in Exercise 9 above. Using partial fractions we would have

$$\frac{5x^2-16x+13}{(x-1)(x-2)(3x-5)} = \frac{A_1}{x-1}+\frac{A_2}{x-2}+\frac{A_3}{3x-5}\,.$$

Evaluating the coefficients A_1, A_2, and A_3 we find $A_1 = 1$, $A_2 = 1$, and $A_3 = -1$ so the expansion we have shown is

$$\frac{5x^2-16x+13}{(x-1)(x-2)(3x-5)} = \frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{3x-5}\,.$$

We can write this as

$$\begin{aligned}\frac{5x^2-16x+13}{(x-1)(x-2)(3x-5)} &= -\frac{1}{1-x}-\frac{1}{2\left(1-\frac{x}{2}\right)}+\frac{1}{5\left(1-\frac{3x}{5}\right)}\\ &= -\sum_{n=0}^{\infty}x^n-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}+\frac{1}{5}\sum_{n=0}^{\infty}\frac{3^nx^n}{5^n}\\ &= \sum_{n=0}^{\infty}\left(-1-\frac{1}{2^{n+1}}+\frac{3^n}{5^{n+1}}\right)x^n\,.\end{aligned}$$

Exercise 11

Part (i): Note that by combining terms, we can write the terms of this series as

$$\begin{aligned}a_n &\equiv \frac{1}{n}-\frac{1}{2n+1}-\frac{1}{2n+2}\\ &= \frac{1}{n}-\frac{4n+3}{(2n+1)(2n+2)}\\ &= \frac{9n+2}{n(2n+1)(2n+2)}\,.\end{aligned}$$

We can show that the above sum converges using the ratio test by comparing these terms with the terms of the convergent sum $\sum_n \frac{1}{n^2}$.

Part (ii): For this part we will break the partial sums into harmonic sums and then express these in terms of the natural log. Towards this end let

$$s_n \equiv \sum_{k=1}^{n} a_k \,,$$

and simplify s_n as

$$\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \left(\frac{1}{2k+1} + \frac{1}{2(k+1)} \right) \\
&= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=2}^{n+1} \left(\frac{1}{2k-1} + \frac{1}{2k} \right) \\
&= \sum_{k=1}^{n} \frac{1}{k} - \left[\sum_{k=1}^{n} \left(\frac{1}{2k-1} + \frac{1}{2k} \right) - \left(1 + \frac{1}{2} \right) + \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) \right] \\
&= \sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{2n} \frac{1}{k} + \frac{3}{2} - \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) \\
&= \frac{3}{2} + \log(n) + \gamma_n - (\log(2n) + \gamma_{2n}) - \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) \\
&= \frac{3}{2} - \log(2) + \gamma_n - \gamma_{2n} - \left(\frac{1}{2n+1} + \frac{1}{2n+2} \right) .
\end{aligned}$$

If we take the limit of the above as $n \to \infty$ we see that

$$s_n \to \frac{3}{2} - \log(2) = 0.8068528 \,.$$

Exercise 12

Lets define $f(x) = \log(1 + x^3)$ so that

$$f'(x) = \frac{3x^2}{1 + x^3} \,.$$

Note that we can expand $\frac{1}{1+x^3}$ in a power series to write

$$f'(x) = 3x^2 \sum_{n=0}^{\infty} (-1)^n x^{3n} = 3 \sum_{n=0}^{\infty} (-1)^n x^{3n+2} \,.$$

The above series converges uniformly on any interval $|x| \le \delta$ with $\delta < 1$. Thus for small x we can integrate term by term from 0 to x to get

$$\log(1 + x^3) = 3 \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+3}}{3n+3} \,.$$

For the next function
$$f(x) = \log(1 - x + x^2)\,,$$
there are a couple of ways to perform this exercise but none of them seemed particularly cleaver/easy so I leave it to the reader to fill in the details if desired. One way to solve this exercise is to compute the derivatives of $f(x)$ that are needed for the Taylor series. Another is to factor the polynomial $x^2 - x + 1$ into two factors obtaining the roots of this quadratic as
$$x^2 - x + 1 = (r_1 - x)(r_2 - x)\,,$$
and then perform two Taylor expansions on the final two functions in
$$\log(1 - x + x^2) = \log(r_1 - x) + \log(r_2 - x) = \log(r_1 r_2) + \log\left(1 - \frac{x}{r_1}\right) + \log\left(1 - \frac{x}{r_2}\right)\,,$$
using the known Taylor expansion for $\log(1-x)$ in Equation 73. For this quadratic the roots r_1 and r_2 are complex and thus the algebra for this is complicated. Finally, one could use the Taylor expansion of $\log(1-x)$ but then replace the x with $x - x^2$ (to be considering the function $\log(1-x+x^2)$) and expand as many powers of $(x-x^2)^n$ as needed for the accuracy desired. If anyone sees a simpler/better method please contact me.

Exercise 14

Let this ratio be given as
$$\frac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3} = F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots\,,$$
Here $F(x)$ is the function representing the given ratio and $\sum f_n x^n$ is its power series. From the above we have
$$120 + 60x + 12x^2 + x^3 = (f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots)(120 - 60x + 12x^2 - x^3)\,.$$
If we expand the right-hand-side we can equate powers of x with the left-hand-side and solve for the coefficients of $F(x)$. Expanding the right-hand-side gives
$$\begin{aligned}
120 + 60x + 12x^2 + x^3 &= 120f_0 - 60f_0 x + 12f_0 x^2 - f_0 x^3 \\
&+ 120f_1 x - 60f_1 x^2 + 12f_1 x^3 - f_1 x^4 \\
&+ 120f_2 x^2 - 60f_2 x^3 + 12f_2 x^4 - f_2 x^5 \\
&+ 120f_3 x^3 - 60f_3 x^4 + 12f_3 x^5 - f_3 x^6 + \cdots \\
&= 120f_0 + (-60f_0 + 120f_1)x + (12f_0 - 60f_1 + 120f_2)x^2 \\
&+ (-f_0 + 12f_1 - 60f_2 + 120f_3)x^3 + O(x^4)\,.
\end{aligned}$$
Setting the left-hand-side and the right-hand-side equal and equating powers of x we see that $f_0 = 1$, $f_1 = 1$, $f_2 = \frac{1}{2}$, and $f_3 = \frac{1}{6}$. Note that these are the coefficients of the power series of e^x as we were to show. One could continue this process to as high an order as desired.

Exercise 15

We will use

$$(1-x)^{-1} = \sum_{n=0}^{\infty} x^n\,,$$

and

$$\log(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x\sum_{n=0}^{\infty} \frac{x^n}{n+1}\,. \tag{73}$$

Taking the requested product we find

$$\begin{aligned}(1-x)^{-1}\log(1-x) &= -x\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k+1}\right)x^n\\ &= -\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{k+1}\right)x^{n+1}\,.\end{aligned}$$

If we integrate the left-hand-side from 0 to x we note that the left-hand-side is then

$$\int_0^x (1-\xi)^{-1}\log(1-\xi)d\xi = -\frac{1}{2}\left(\log(1-\xi)\right)^2\Big|_0^x = -\frac{1}{2}(\log(1-x))^2\,.$$

While the right-hand-side is then

$$-\sum_{n=0}^{\infty}\left(\frac{1}{n+2}\sum_{k=0}^{n}\frac{1}{k+1}\right)x^{n+2}\,.$$

Equating these two gives the desired expression.

Exercise 16

Using the two expansions

$$\log(1-x) = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \tag{74}$$

$$\log(1+x) = -\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n+1}\,, \tag{75}$$

we can compute the desired product and find

$$
\begin{aligned}
\log(1+x)\log(1-x) &= \left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{n+1}}{n+1}\right)\left(\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\right)\\
&= x^2\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{n}}{n+1}\right)\left(\sum_{n=0}^{\infty}\frac{x^{n}}{n+1}\right)\\
&= x^2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\cdot\frac{1}{n-k+1}\right)x^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^{k+1}}{(k+1)(n-k+1)}\right)x^{n+2}\,.
\end{aligned}
$$

Expanding a few terms we have

$$
\begin{aligned}
\log(1+x)\log(1-x) &= -\frac{1}{1(1)}x^2 + \left(-\frac{1}{1(2)}+\frac{1}{2(1)}\right)x^3\\
&+ \left(\frac{(-1)}{1(3)}+\frac{1}{2(2)}-\frac{1}{3(1)}\right)x^4\\
&+ \left(\frac{(-1)}{1(4)}+\frac{1}{2(3)}-\frac{1}{3(2)}+\frac{1}{4(1)}\right)x^5+\cdots\\
&= -x^2-\frac{5}{12}x^4+O(x^6)\,.
\end{aligned}
$$

From the above it looks like the above series has only even terms. We can simplify this inner sum (and the full expression) to emphasis this if we note that using partial fractions we have

$$
\frac{1}{(k+1)(n-k+1)} = \frac{1}{n+2}\left(\frac{1}{k+1}\right)+\frac{1}{n+2}\left(\frac{1}{n-k+1}\right)\,.
$$

This means that when we sum for $k=0$ to $k=n$ we can simplify the second term to look like the first. I find

$$
\begin{aligned}
\sum_{k=0}^{n}\frac{(-1)^{k+1}}{(k+1)(n-k+1)} &= \frac{1}{n+2}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}+\frac{1}{n+2}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{n-k+1}\\
&= \frac{1}{n+2}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}+\frac{(-1)^n}{n+2}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\\
&= \frac{(1+(-1)^n)}{n+2}\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\,.
\end{aligned}
$$

Note that this is zero if n is odd. Thus our product can be written as

$$
\begin{aligned}
\log(1+x)\log(1-x) &= \sum_{n=0,2,4,\dots}^{\infty}\left(\frac{2}{n+2}\right)\left(\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k+1}\right)x^{n+2}\\
&= x^2\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{2n}\frac{(-1)^{k+1}}{k+1}\right)x^{2n}\,.
\end{aligned}
$$

Expanding a few terms of this series we get the same results as derived earlier.

Exercise 17

Write this as
$$(1+x)^{-1/2} = x^{-1/2}\left(1+x^{-1}\right)^{-1/2},$$
and then use the binomial theorem on the second factor. To do that note that
$$\begin{aligned}
\binom{-1/2}{0} &= 1 \\
\binom{-1/2}{1} &= -\frac{1}{2} \\
\binom{-1/2}{2} &= \frac{(-1/2)(-1/2-1)}{2!} = \frac{(-1/2)(-3/2)}{2} = \frac{1\cdot 3}{2\cdot 4}.
\end{aligned}$$
Thus to the number of terms requested we have
$$\begin{aligned}
(1+x)^{-1/2} &= x^{-1/2}\left(1 - \frac{1}{2}x^{-1} + \frac{1\cdot 3}{2\cdot 4}x^{-2} - \cdots\right) \\
&= x^{-1/2} - \frac{1}{2}x^{-3/2} + \frac{1\cdot 3}{2\cdot 4}x^{-5/2} - \cdots.
\end{aligned}$$

Chapter 13 (Power Series)

Notes on The Behavior of a Power Series on its Circle

We have

$$
\begin{aligned}
1 + z + z^2 + \cdots + z^{n-1} &= \left.\frac{1 - z^n}{1 - z}\right|_{z=e^{i\theta}} = \frac{1 - e^{in\theta}}{1 - e^{i\theta}} \left(\frac{e^{-\frac{i}{2}\theta n}}{e^{-\frac{i}{2}\theta n}}\right)\left(\frac{e^{-\frac{i}{2}\theta}}{e^{-\frac{i}{2}\theta}}\right) \\
&= \frac{e^{-\frac{i}{2}n\theta} - e^{\frac{i}{2}n\theta}}{e^{-\frac{i}{2}\theta} - e^{\frac{i}{2}\theta}} \left(\frac{e^{-\frac{i}{2}\theta}}{e^{-\frac{i}{2}\theta n}}\right) \\
&= \frac{-2i\sin\left(\frac{n\theta}{2}\right)}{-2i\sin\left(\frac{\theta}{2}\right)} e^{\frac{i}{2}n\theta - \frac{i}{2}\theta} = \frac{\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} e^{\frac{i}{2}\theta(n-1)} \\
&= \frac{\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)} \left(\cos\left(\frac{n-1}{2}\theta\right) + i\sin\left(\frac{n-1}{2}\theta\right)\right),
\end{aligned}
$$

the expression in the book.

The fact that both

$$
\sum \frac{\cos(n\theta)}{n} \quad \text{and} \quad \sum \frac{\sin(n\theta)}{n},
$$

converge follows from a special case of "Dirichlet's test" which states that if (v_n) is any monotonic sequence that converges to zero and (a_n) is a bounded sequence of numbers such that

$$
|a_1 + a_2 + \cdots + a_n| < K,
$$

then $\sum a_n v_n$ is convergent. Here $v_n = \frac{1}{n}$ a monotonic sequence and a_n is either $\cos(n\theta)$ or $\sin(n\theta)$. The above sum of z^n (when taking the real and imaginary parts) show that $|\sum a_n|$ is bounded allowing the use of Dirichlet's test.

Examples XVIII

Exercise 1

This follows from Abel's test (for uniform convergence). Here we write

$$
\sum a_n x^n = \sum a_n v_n(x),
$$

with $v_n(x) = x^n$. Now to apply Abel's test note that $v_n(x)$ is monotonically decreasing in n when $x \in (0, 1)$. We also have that

- $\sum a_n$ is a uniformly convergent series

- $|v_n(x)| = |x^n| < 1$ when $x \in (0, 1)$.

Thus Abel's test implies that $\sum a_n x^n$ converges uniformly for $x \in (0, 1)$.

This also follows from Abel's test (for uniform convergence). Here we write

$$\sum a_n x^n = \sum (-1)^n a_n (-x)^n = \sum (-1)^n a_n v_n(x)\,,$$

with $v_n(x) = (-x)^n$. Now to apply Abel's test note that $v_n(x)$ is monotonically decreasing in n when $x \in (-1, 0)$. We also have that

- $\sum (-1)^n a_n$ is a uniformly convergent series
- $|v_n(x)| = |(-x)^n| = |(-1)^n x^n| < 1$ when $x \in (-1, 0)$.

Thus Abel's test implies that $\sum a_n x^n$ converges uniformly for $x \in (-1, 0)$.

Exercise 2

This is a consequence of power series multiplication. For example we have

$$\begin{aligned}
f(x)g(x) &= \sum_n a_n x_n \sum_m b_m x^m \\
&= \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \cdots\right) \\
&= a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + a_0 b_3 x^3 + a_0 b_4 x^4 + \cdots \\
&+ a_1 b_0 x + a_1 b_1 x^2 + a_1 b_2 x^3 + a_1 b_3 x^4 + a_1 b_4 x^5 + \cdots \\
&+ a_2 b_0 x^2 + a_2 b_1 x^3 + a_2 b_2 x^4 + a_2 b_3 x^5 + a_2 b_4 x^6 + \cdots \\
&+ a_3 b_0 x^3 + a_3 b_1 x^4 + a_3 b_2 x^5 + a_3 b_3 x^6 + a_3 b_4 x^7 + \cdots \\
&= a_0 b_0 \\
&+ (a_1 b_0 + a_0 b_1) x \\
&+ (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 \\
&+ (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) x^3 + \cdots\,.
\end{aligned}$$

The general expression for the coefficient c_n of x^n in the product for $f(x)g(x)$ is the expression given in the book.

Exercise 3

Define this expression $f_n(x)$. Then we have

$$\begin{aligned}
f_n'(x) &= \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-2}}{(n-2)!}\right) e^{-x} - \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}\right) e^{-x} \\
&= -\frac{x^{n-1}}{(n-1)!} e^{-x}\,.
\end{aligned}$$

Now using $e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$ in the above we get

$$f_n'(x) = -\frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+n-1}}{k!} .$$

Integrating this we get

$$f_n(x) = -\frac{1}{(n-1)!} \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+n}}{(k+n)k!} + C ,$$

for some constant C. Taking $x = 0$ in the original expression for $f_n(x)$ we see that $f_n(0) = 1$ and thus $C = 1$. This means that we have shown that

$$f_n(x) = 1 - \frac{x^n}{(n-1)!} \left(\frac{1}{n} - \frac{x}{n+1} + \frac{x^2}{2!(n+2)} + \frac{x^3}{3!(n+3)} + \cdots \right) ,$$

as we were to show.

Exercise 4

To study convergence of these infinite series we will use d'Alembert's ratio test. To do that we will need to compute limits of the ratio $\frac{|u_n|}{|u_{n+1}|}$.

Part (i): We find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n|z|^n}{(n+1)|z|^{n+1}} = \left(\frac{n}{n+1} \right) \frac{1}{|z|} .$$

As $n \to \infty$ this will be larger than one if $|z| < 1$. If $|z| = 1$ then the limit of the terms in the sum don't go to zero as $n \to \infty$ and so the infinite sum cannot converge. To sum this series we would recall Equation 78.

Part (ii): We find

$$\frac{|u_n|}{|u_{n+1}|} = \left(\frac{(n+1)|z|^n}{(n+2)(n+3)} \right) \times \left(\frac{(n+3)(n+4)}{(n+2)|z|^{n+1}} \right) = \frac{(n+1)(n+4)}{(n+2)^2|z|} .$$

As $n \to \infty$ this will be larger than one if $|z| < 1$. If $z = 1$ then this sum diverges by comparison with the terms of the series n^{-1}. If $z = -1$ then this sum converges by the alternating series test.

To sum this series note that using partial fractions we have

$$\frac{n+1}{(n+2)(n+3)} = -\frac{1}{n+2} + \frac{2}{n+3} .$$

This means that the sum we want to evaluate can be written as

$$\sum \frac{n+1}{(n+2)(n+3)} z^n = -\sum \frac{1}{n+2} z^n + 2 \sum \frac{1}{n+3} z^n .$$

To sum these two sums on the right-hand-side we will write them so that we can evaluate them using Equation 80. We have

$$\sum_{n=0}^{\infty} \frac{z^n}{n+2} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+2} = \frac{1}{z} \sum_{n=1}^{\infty} \frac{z^n}{n+1} = \frac{1}{z} \left(\sum_{n=0}^{\infty} \frac{z^n}{n+1} - 1 \right)$$
$$\sum_{n=0}^{\infty} \frac{z^n}{n+3} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^{n+2}}{n+3} = \frac{1}{z^2} \sum_{n=2}^{\infty} \frac{z^n}{n+1} = \frac{1}{z^2} \left(\sum_{n=0}^{\infty} \frac{z^n}{n+1} - 1 - \frac{z}{2} \right).$$

We would need to combine the two sums above to get the final sum/result.

Part (iii): We find

$$\frac{|u_n|}{|u_{n+1}|} = \frac{n^2 |z|^n}{(n+1)^2 |z|^{n+1}} = \frac{n^2}{(n+1)^2 |z|} \to \frac{1}{|z|},$$

as $n \to \infty$. This will be larger than one if $|z| < 1$. If $|z| = 1$ then the terms of this sum don't limit to zero as $n \to \infty$ and the sum must diverge. To sum this series we would recall Equation 79.

Part (iv): We find

$$\frac{|u_n|}{|u_{n+1}|} = \left(\frac{n|z|^n}{(n+1)^2} \right) \times \left(\frac{(n+2)^2}{(n+1)|z|^{n+1}} \right) = \frac{n(n+2)^2}{(n+1)^3 |z|} \to \frac{1}{|z|},$$

as $n \to \infty$. This will be larger than one if $|z| < 1$. If $z = 1$ then this sum diverges by comparison with the terms of the series n^{-1}. If $z = -1$ then this sum converges by the alternating series test. To sum this series we would recall Equation 82.

Exercise 5

Note that we can write the terms of this series as $u_n z^n$ where

$$u_n = \frac{\left(\prod_{i=0}^{n-1}(a+i)\right)\left(\prod_{i=0}^{n-1}(b+i)\right)}{n!\left(\prod_{i=0}^{n-1}(c+i)\right)}, \tag{76}$$

for $n \geq 0$. Here we are using the convention that $\prod_{i=0}^{-1} f(i) = 1$. The easiest way to "verify" this is to check that these terms are correct by evaluating the above for $n = 0$, $n = 1$, and $n = 2$.

To study convergence of the infinite series we will use d'Alembert's ratio test. We first need to compute the ratio $\frac{|u_n z^n|}{|u_{n+1} z^{n+1}|}$ where we find

$$\begin{aligned} \frac{|u_n z^n|}{|u_{n+1} z^{n+1}|} &= \left(\frac{\left(\prod_{i=0}^{n-1}(a+i)\right)\left(\prod_{i=0}^{n-1}(b+i)\right)|z|^n}{n!\left(\prod_{i=0}^{n-1}(c+i)\right)} \right) \times \left(\frac{(n+1)!\left(\prod_{i=0}^{n}(c+i)\right)}{\left(\prod_{i=0}^{n}(a+i)\right)\left(\prod_{i=0}^{n}(b+i)\right)|z|^{n+1}} \right) \\ &= \frac{(n+1)(c+n)}{(a+n)(b+n)|z|} \to \frac{1}{|z|}. \end{aligned}$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|z|}$. By d'Alembert's test our sum will converge if $\frac{1}{|z|} > 1$ which happens if $|z| < 1$. This test also tells us that the sum will diverge if $|z| > 1$.

Exercise 6

Note that we can write $F(a, b, c; z)$ as

$$F(a, b, c; z) = \sum_{n=0}^{\infty} u_n z^n \,,$$

with u_n given by Equation 76.

Part (i): Using the above we have that

$$F'(a, b, c; z) = \sum_{n=1}^{\infty} n u_n z^{n-1} = \sum_{n=0}^{\infty} (n+1) u_{n+1} z^n \,.$$

From the expression for u_n we have that

$$\begin{aligned}
(n+1)u_{n+1} &= (n+1) \times \frac{\left(\prod_{i=0}^{n}(a+i)\right)\left(\prod_{i=0}^{n}(b+i)\right)}{(n+1)!\left(\prod_{i=0}^{n}(c+i)\right)} \\
&= \frac{ab}{c} \times \frac{\left(\prod_{i=1}^{n}(a+i)\right)\left(\prod_{i=1}^{n}(b+i)\right)}{n!\left(\prod_{i=1}^{n}(c+i)\right)} \\
&= \frac{ab}{c} \times \frac{\left(\prod_{i=1}^{n}(a+1+i-1)\right)\left(\prod_{i=1}^{n}(b+1+i-1)\right)}{n!\left(\prod_{i=1}^{n}(c+1+i-1)\right)} \\
&= \frac{ab}{c} \times \frac{\left(\prod_{i=0}^{n-1}(a+1+i)\right)\left(\prod_{i=0}^{n-1}(b+1+i)\right)}{n!\left(\prod_{i=0}^{n-1}(c+1+i)\right)} \\
&= \frac{ab}{c} \times u_n' \,.
\end{aligned}$$

Here the value of u_n' is Equation 76 but evaluated at $a \to a+1$, $b \to b+1$, and $c \to c+1$ slightly different than the normal definition of u_n. Thus if we denote the dependence on a, b, and c explicitly in the expression for u_n we have just shown

$$F'(a, b, c; z) = \frac{ab}{c} \sum_{n=0}^{\infty} u_n(a+1, b+1, c+1) z^n = \frac{ab}{c} F(a+1, b+1, c+1; z) \,,$$

as we were to show.

Part (ii): This relationship is known as "Euler's transformation" of the hypergeometric function[3] and (from the research I did) seemed to be somewhat involved to prove. If anyone has an easy proof of this relationship please contact me.

[3] https://en.wikipedia.org/wiki/Hypergeometric_function

Exercise 11

The terms of this sum can be written as

$$u_n = \frac{[1 \cdot 3 \cdot 5 \cdots (2n-3)]x^{2n+1}}{[2 \cdot 4 \cdot 6 \cdots (2n)](2n+1)} .$$

for $n = 1, 2, 3, \ldots$.

Part (i): For this we have

$$\begin{aligned}
\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{[1 \cdot 3 \cdot 5 \cdots (2n-3)]|x|^{2n+1}}{[2 \cdot 4 \cdot 6 \cdots (2n)](2n+1)} \right) \times \left(\frac{[2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)](2n+3)}{[1 \cdot 3 \cdot 5 \cdots (2n-3) \cdot (2n-1)]|x|^{2n+3}} \right) \\
&= \frac{(2n+2)(2n+3)}{(2n-1)(2n+1)|x|^2} .
\end{aligned}$$

If we take the limit as $n \to \infty$ the above equals $\frac{1}{|x|^2}$. This sum will converge if this limit is larger than one which happens if $|x| < 1$.

Part (ii): If $x = 1$ then the test above is inconclusive. To determine convergence we will use Rabbe's test for which we need to compute

$$\begin{aligned}
n \left(\frac{u_n}{u_{n+1}} - 1 \right) &= n \left(\frac{(2n+2)(2n+3)}{(2n-1)(2n+1)} - 1 \right) \\
&= n \left(\frac{10n+7}{(2n-1)(2n+1)} \right) \to \frac{10}{4} = \frac{5}{2} > 1 ,
\end{aligned}$$

and thus this sum converges.

Exercise 12

The terms of this sum can be written as

$$u_n = \frac{(n-1)! x^n}{n^n} .$$

for $n = 1, 2, 3, \ldots$. For this we have

$$\begin{aligned}
\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{(n-1)!|x|^n}{n^n} \right) \times \left(\frac{(n+1)^{n+1}}{n!|x|^{n+1}} \right) \\
&= \frac{1}{|x|} \frac{(n+1)^{n+1}}{n^{n+1}} = \frac{1}{|x|} \left(1 + \frac{1}{n} \right)^{n+1} \\
&= \frac{1}{|x|} \left(1 + \frac{1}{n} \right)^n \left(1 + \frac{1}{n} \right) .
\end{aligned}$$

If we take the limit as $n \to \infty$ the above equals $\frac{1}{|x|} \times e \times 1 = \frac{e}{|x|}$. This sum will converge if this limit is larger than one which happens if $|x| < e$.

Exercise 14

To study convergence of this infinite series we will use d'Alembert's ratio test. We first need to compute the ratio $\frac{|u_n|}{|u_{n+1}|}$ where we find

$$
\begin{aligned}
\frac{|u_n|}{|u_{n+1}|} &= \left(\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)\cdot(2n-1)|x|^n}{2\cdot 5\cdot 8\cdots(3n-4)\cdot(3n-1)}\right)\times\left(\frac{2\cdot 5\cdot 8\cdots(3n-1)\cdot(3n+2)}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)(2n+1)|x|^{n+1}}\right)\\
&= \left(\frac{3n+2}{2n+1}\right)\left(\frac{1}{|x|}\right).
\end{aligned}
$$

Thus the limit of this fraction as $n\to\infty$ is the value $\frac{3}{2|x|}$. By d'Alembert's ratio test our sum will converge if $\frac{3}{2|x|}>1$ which happens if $|x|<\frac{3}{2}$.

Sums involving z^n

Recall that

$$
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\,, \tag{77}
$$

gives the sum of z^n. Taking the z derivative of this gives

$$
\sum_{n=1}^{\infty} nz^{n-1} = -\frac{1}{(1-z)^2}(-1) = \frac{1}{(1-z)^2}\,.
$$

Multiplying both sides by z gives

$$
\sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}\,. \tag{78}
$$

This give the sum of nz^n. Taking a z derivative of this gives

$$
\sum_{n=1}^{\infty} n^2z^{n-1} = \frac{1}{(1-z)^2} - \frac{2z(-1)}{(1-z)^3} = \frac{1-z+2z}{(1-z)^3} = \frac{1+z}{(1-z)^3}\,.
$$

Multiplying both sides by z gives

$$
\sum_{n=1}^{\infty} n^2z^n = \frac{z(1+z)}{(1-z)^3}\,. \tag{79}
$$

This give the sum of n^2z^n. We could continue this pattern for as long as needed to get other sums of the form n^pz^n.

Starting with Equation 77 and integrating both sides we get

$$
\sum_{n=0}^{\infty}\frac{z^{n+1}}{n+1} + C = -\ln(1-z)\,.
$$

Taking $z = 1$ we see that $C = 0$ and we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n+1} = -\frac{1}{z} \ln(1-z)\,. \tag{80}$$

This gives sums of the form $\frac{z^n}{n+1}$. If we integrate the above from 0 to z (so that the sum is only the terms $\frac{z^{n+1}}{(n+1)^2}$) we get

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2} = -\int_0^z \frac{1}{x} \ln(1-x) dx = \int_z^0 \frac{\ln(1-x)}{x} dx \equiv \mathrm{Li}_2(x)\,.$$

Where the integral on the right-hand-side is the *dilogarithm*[4]. Thus we have

$$\sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)^2} = \mathrm{Li}_2(z)\,.$$

If we divide this by z we get

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+1)^2} = \frac{1}{z}\mathrm{Li}_2(z)\,. \tag{81}$$

This gives sums of the form $\frac{z^n}{(n+1)^2}$. If we take the z derivative of this we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n z^{n-1}}{(n+1)^2} &= -\frac{1}{z^2}\mathrm{Li}_2(z) + \frac{1}{z}\frac{d}{dz}\mathrm{Li}_2(z) \\
&= -\frac{1}{z^2}\mathrm{Li}_2(z) + \frac{1}{z}\left(-\frac{1}{z}\ln(1-z)\right) \\
&= -\frac{1}{z^2}\left(\mathrm{Li}_2(z) + \ln(1-z)\right)\,.
\end{aligned}$$

If we multiply this by z we get

$$\sum_{n=1}^{\infty} \frac{n z^n}{(n+1)^2} = -\frac{1}{z}\left(\mathrm{Li}_2(z) + \ln(1-z)\right)\,. \tag{82}$$

[4]https://mathworld.wolfram.com/Dilogarithm.html

Chapter 14 (Integral Test)

Notes on the Text: Proving the s_n and I_n bounds

Starting with

$$f(n+1) < \int_n^{n+1} f(x)dx < f(n)\,, \tag{83}$$

and following the suggestion in the text we write this for $n = 1, 2, 3, \cdots, n-2, n-1$ as

$$\begin{aligned}
f(2) &< \int_1^2 f(x)dx < f(1) \\
f(3) &< \int_2^3 f(x)dx < f(2) \\
&\vdots \\
f(n) &< \int_{n-1}^n f(x)dx < f(n-1)\,.
\end{aligned}$$

If we add these we get

$$s_n - f(1) < I_n < s_n - f(n)\,, \tag{84}$$

which is Equation (2) in this section.

Notes on the Text: Bounding a sum with an integral

Here we write Equation 83 for n in $\{n, n+1, n+2, \cdots, n+k-1\}$ as

$$\begin{aligned}
f(n+1) &< \int_n^{n+1} f(x)dx < f(n) \\
f(n+2) &< \int_{n+1}^{n+2} f(x)dx < f(n+1) \\
&\vdots \\
f(n+k) &< \int_{n+k-1}^{n+k} f(x)dx < f(n+k-1)\,.
\end{aligned}$$

If we then sum these we get

$$f(n+1) + f(n+2) + \cdots + f(n+k) < \int_n^{n+k} f(x)dx < f(n) + f(n+1) + \cdots + f(n+k-1)\,. \tag{85}$$

Using the sum of $f(\cdot)$ on the left-hand-side of Equation 85 to bound the sum of $f(\cdot)$ on the the right-hand-side we get

$$f(n) + f(n+1) + \cdots + f(n+k-1) < \int_{n-1}^{n+k-1} f(x)dx\,.$$

In summary then, lower and upper bounds for the sum $\sum_{i=n}^{n+k-1} f(i)$ are given by

$$\int_n^{n+k} f(x)dx < f(n) + f(n+1) + \cdots + f(n+k-1) < \int_{n-1}^{n+k-1} f(x)dx \,. \tag{86}$$

Lets consider some examples using Equation 86.

Bounding sums of $\frac{1}{n}$

If $f(x) = \frac{1}{x}$ then Equation 86 becomes in this case

$$\int_n^{n+k} \frac{dx}{x} < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k-1} < \int_{n-1}^{n+k-1} \frac{dx}{x} \,. \tag{87}$$

Now note that

$$\int_n^{n+k} \frac{dx}{x} = \ln(x)|_n^{n+k} = \ln(n+k) - \ln(n) = \ln\left(\frac{n+k}{n}\right) = \ln\left(1 + \frac{k}{n}\right) \,.$$

This means that Equation 87 becomes

$$\ln\left(1 + \frac{k}{n}\right) < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{n+k-1} < \ln\left(1 + \frac{k}{n-1}\right) \,, \tag{88}$$

which is the statement given in the book.

Bounding sums of $\frac{1}{n^2}$

If $f(x) = \frac{1}{x^2}$ then Equation 86 becomes in this case

$$\int_n^{n+k} \frac{dx}{x^2} < \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+k-1)^2} < \int_{n-1}^{n+k-1} \frac{dx}{x^2} \,. \tag{89}$$

Now note that

$$\int_n^{n+k} \frac{dx}{x^2} = -\frac{1}{x}\Big|_n^{n+k} = -\left(\frac{1}{n+k} - \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{n+k} = \frac{k}{n(n+k)} \,.$$

This means that Equation 89 becomes

$$\frac{k}{n(n+k)} < \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+k-1)^2} < \frac{k}{(n-1)(n-1-k)} \,, \tag{90}$$

which is the statement given in the book.

Bounding sums of $\frac{1}{n^3}$

If $f(x) = \frac{1}{x^3}$ then Equation 86 becomes in this case

$$\int_n^{n+k} \frac{dx}{x^3} < \frac{1}{n^3} + \frac{1}{(n+1)^3} + \cdots + \frac{1}{(n+k-1)^3} < \int_{n-1}^{n+k-1} \frac{dx}{x^3} \,. \tag{91}$$

Now note that

$$\int_n^{n+k} \frac{dx}{x^3} = \left.\frac{x^{-2}}{-2}\right|_n^{n+k} = \frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{(n+k)^2}\right) = \frac{2nk+k^2}{2n^2(n+k)^2} \,.$$

This means that Equation 91 becomes

$$\frac{2nk+k^2}{2n^2(n+k)^2} < \frac{1}{n^3} + \frac{1}{(n+1)^3} + \cdots + \frac{1}{(n+k-1)^3} < \frac{2(n-1)k+k^2}{2(n-1)^2(n-1+k)^2} \,. \tag{92}$$

Bounding sums of $\frac{1}{n^2+1}$

If $f(x) = \frac{1}{x^2+1}$ then Equation 86 becomes in this case

$$\int_n^{n+k} \frac{dx}{x^2+1} < \frac{1}{n^2+1} + \frac{1}{(n+1)^2+1} + \cdots + \frac{1}{(n+k-1)^2+1} < \int_{n-1}^{n+k-1} \frac{dx}{x^2+1} \,. \tag{93}$$

Now note that

$$\int_n^{n+k} \frac{dx}{x^2+1} = \left.\tan^{-1}(x)\right|_n^{n+k} = \tan^{-1}(n+k) - \tan^{-1}(n) \,. \tag{94}$$

To further evaluate this we need to compute the difference on the right-hand-side. To do this recall that

$$\tan(x-y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)} \,, \tag{95}$$

or

$$x - y = \tan^{-1}\left(\frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}\right) \,.$$

Then if we take $x \to \tan^{-1}(X)$ and $y \to \tan^{-1}(Y)$ the above is

$$\tan^{-1}(X) - \tan^{-1}(Y) = \tan^{-1}\left(\frac{X-Y}{1+XY}\right) \,. \tag{96}$$

This means that Equation 94 becomes

$$\tan^{-1}(n+k) - \tan^{-1}(n) = \tan^{-1}\left(\frac{n+k-n}{1+(n+k)n}\right) = \tan^{-1}\left(\frac{k}{n^2+kn+1}\right) \,,$$

and thus Equation 93 becomes

$$\tan^{-1}\left(\frac{k}{n^2+kn+1}\right) < \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+k-1)^2} < \tan^{-1}\left(\frac{k}{(n-1)^2+k(n-1)+1}\right) \,, \tag{97}$$

which is the statement given in the book.

Examples XIX

Exercise 2

To use the integral test on this series we would need to consider the integral

$$I = \int_a^\infty \frac{1}{x \log(x)^p} dx \,.$$

If we take $u = \log(x)$ then $du = \frac{dx}{x}$ and we get that

$$I = \int_{\log(a)}^\infty \frac{du}{u^p} = \int_{\log(a)}^\infty u^{-p} du \,.$$

The above converges if and only if $p > 1$ and thus the series converges for the same p values.

Exercise 3

Now since the integral $\int \frac{dx}{x^p}$ is convergent if and only if $p > 1$ by using the integral test on this series we conclude that the series is convergent for the same p values.

To show the suggested bounds we will consider Equation 86 for $n \geq 2$ where we need to compute

$$\begin{aligned} I_n^{n+k} &= \int_n^{n+k} \frac{dx}{x^p} = \frac{1}{1-p} x^{1-p} \Big|_n^{n+k} \\ &= \frac{1}{1-p} \left(\frac{1}{(n+k)^{p-1}} - \frac{1}{n^{p-1}} \right) . \end{aligned}$$

This means that

$$I_{n-1}^{n+k-1} = \frac{1}{1-p} \left(\frac{1}{(n-1+k)^{p-1}} - \frac{1}{(n-1)^{p-1}} \right) .$$

In this notation with $f(x) = \frac{1}{x^p}$ Equation 86 is

$$I_n^{n+k} < \sum_{i=n}^{n+k-1} f(i) < I_{n-1}^{n-1+k} \,.$$

If we take $n = 2$ we have

$$I_2^{2+k} < \sum_{i=2}^{k+1} f(i) < I_1^{k+1} \,.$$

The upper bound above becomes

$$I_1^{k+1} = \frac{1}{1-p} \left(\frac{1}{(1+k)^{p-1}} - \frac{1}{1^{p-1}} \right) \to \frac{1}{p-1} \,,$$

as $k \to \infty$. This means that

$$\sum_{i=2}^{\infty} \frac{1}{i^p} < \frac{1}{p-1} \,.$$

Adding one to both sides gives

$$\sum_{i=1}^{\infty} \frac{1}{i^p} < \frac{1}{p-1} + 1 = \frac{p}{p-1} \,,$$

as we were to show.

Exercise 4

For this exercise we will first prove the "hint". Consider the function $f(x) = x - \log(1+x)$. Using the known power series of $\log(1+x)$ of

$$\log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} \,, \tag{98}$$

we will have that

$$f(x) = x - \log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1} \,.$$

From this and using Taylor series we have that

$$\begin{aligned}
x - \log(1+x) &= \frac{x^2}{2} - \frac{x^3}{3} + O(x^4) < \frac{x^2}{2} \\
x - \log(1+x) &= \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} + O(x^5) > \frac{x^2}{2} - \frac{x^3}{3} \,.
\end{aligned}$$

Thus we have just shown (the hint) that

$$\frac{x^2}{2} - \frac{x^3}{3} < f(x) < \frac{x^2}{2} \,. \tag{99}$$

Now the terms we are summing take the form $a_n = f\left(\frac{1}{n}\right)$ and we are asked to sum "p terms after a_n". This is the expression

$$S \equiv \sum_{i=1}^{p} a_{n+i} = \sum_{i=1}^{p} \left(\frac{1}{n+i} - \log\left(1 + \frac{1}{n+i}\right) \right) = \sum_{j=n+1}^{n+p} \left(\frac{1}{j} - \log\left(1 + \frac{1}{j}\right) \right) \,.$$

Then using Equation 99 we can bound the above as

$$\sum_{j=n+1}^{n+p} \left(\frac{1}{j} - \log\left(1 + \frac{1}{j}\right) \right) < \frac{1}{2} \sum_{j=n+1}^{n+p} \frac{1}{j^2} \,.$$

Then using Equation 90 with $n \to n+1$ we get

$$S < \frac{1}{2}\left(\frac{p}{n(n+p)}\right),$$

which is one of the bounds we were to show.

Now to find a lower bound for S note that we have

$$S > \sum_{j=n+1}^{n+p}\left(\frac{1}{2j^2} - \frac{1}{3j^3}\right).$$

Then using Equation 86 we have that

$$S > \int_{n+1}^{n+1+p}\left(\frac{1}{2x^2} - \frac{1}{3x^3}\right)dx\,.$$

We will now evaluate this integral. We find

$$\begin{aligned}
\int_{n+1}^{n+1+p}\left(\frac{1}{2x^2} - \frac{1}{3x^3}\right)dx &= \left(\frac{1}{2}\frac{x^{-1}}{(-1)} - \frac{1}{3}\frac{x^{-2}}{(-2)}\right|_{n+1}^{n+1+p} \\
&= -\frac{1}{2x} + \frac{1}{6x^2}\bigg|_{n+1}^{n+1+p} \\
&= \left(\frac{1}{2(n+1)} - \frac{1}{6(n+1)^2}\right) - \left(\frac{1}{2(n+1+p)} - \frac{1}{6(n+1+p)^2}\right) \\
&= \frac{1}{2}\frac{p}{(n+1)(n+1+p)}\left(1 - \frac{1}{3}\frac{2(n+1)+p}{(n+1)(n+1+p)}\right).
\end{aligned}$$

Note that this is similar (but not exactly the same) to the result given in the book. If anyone sees anything wrong with what I have done please contact me.

Exercise 5

Using Equation 88 (which has $f(x) = \frac{1}{x}$) with $n \to n+1$ we get

$$\ln\left(1 + \frac{k}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k} < \ln\left(1 + \frac{k}{n}\right).$$

If we take $k = n$ we get

$$\ln\left(1 + \frac{n}{n+1}\right) < \sum_{k=n+1}^{2n}\frac{1}{k} < \ln\left(1 + \frac{n}{n}\right) = \ln(2)\,.$$

Taking the limit of this expression as $n \to \infty$ we get that

$$\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{k} = \ln(2)\,.$$

We verify this result in the R code `examples_xix_exercise_5.R`.

For the second example we will take $f(x)$ defined as

$$f(x) \equiv \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{(x+1)(x+2)}\,.$$

Note that for this $f(x)$ we have

$$f'(x) = -\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} = \frac{-(x+2)^2 + (x+1)^2}{(x+1)^2(x+2)^2} = \frac{-2x-3}{(x+1)^2(x+2)^2} < 0\,,$$

for all $x > 0$ and thus $f(x)$ is a monotonically decreasing function.

Then the sum we seek we seek to evaluate is given by

$$f(n) + f(n+2) + f(n+4) + f(n+6) + \cdots\,.$$

Without loss of generality lets assume that n is even. Then the final term in the sum above will be when $n + 2i + 2 = 2n$ or $i = \frac{n}{2} - 1$ and the above sum is

$$\sum_{i=0}^{\frac{n}{2}-1} f(n+2i)\,. \tag{100}$$

Here there are n terms in the original sum and n terms in the sum above. In terms of $f(x)$ this is

$$\sum_{i=0}^{\frac{n}{2}-1} \left(\frac{1}{n+2i+1} - \frac{1}{n+2i+2} \right) = \sum_{i=0}^{\frac{n}{2}-1} \frac{1}{(n+2i+1)(n+2i+2)}\,.$$

We will now use Equation 86 to bound this sum. Let I_n^{n+k} be the lower bounding integral in Equation 86 such that

$$\begin{aligned}
I_n^{n+k} &\equiv \int_n^{n+k} \left(\frac{1}{x+1} - \frac{1}{x+2} \right) dx \\
&= (\ln(x+1) - \ln(x+2)|_n^{n+k} = \ln\left(\frac{n+k+1}{n+k+2}\right) - \ln\left(\frac{n+1}{n+2}\right) \\
&= \ln\left(\frac{(n+k+1)(n+2)}{(n+k+2)(n+1)}\right)\,.
\end{aligned}$$

With this our bound in Equation 86 becomes

$$\ln\left(\frac{(n+k+1)(n+2)}{(n+k+2)(n+1)}\right) < \sum_{j=n}^{n+k-1} f(j) < \ln\left(\frac{(n+k)(n+1)}{(n+k+1)n}\right)\,. \tag{101}$$

To get this to match Equation 100 we need the lower and upper limits of the sum to match. As the lower limits match, we need to have k such that the upper limits match or

$$2n - 2 = n + k - 1 \quad \text{so} \quad k = n - 1\,.$$

If we put this value for k into Equation 101 we get

$$\ln\left(\frac{(2n)(n+2)}{(2n+1)(n+1)}\right) < \sum_{j=n}^{2n-2} f(j) < \ln\left(\frac{(2n-1)(n+1)}{(2n)n}\right). \tag{102}$$

Taking the limit of this expression as $n \to \infty$ we get that

$$\lim_{n\to\infty} \sum_{j=n}^{2n-2} f(j) = 0\,.$$

We also verify this result in the `R` code `examples_xix_exercise_5.R`.

Exercise 6

It looks like our sum is given by the following expression

$$\sum_{n=0}^{\infty} \frac{x^{2(2n+1)}}{2n+1} \sum_{i=0}^{2n} \frac{(-1)^i}{2i+1}\,. \tag{103}$$

We can check this with a few cases. When $n = 0$ we get

$$\frac{x^2}{1} \cdot (1) = x^2\,,$$

which is correct. When $n = 1$ we get

$$\frac{x^6}{3} \sum_{i=0}^{2} \frac{(-1)^i}{2i+1} = \left(1 - \frac{1}{3} + \frac{1}{5}\right) \frac{x^6}{3}\,,$$

which is correct. When $n = 2$ we get

$$\frac{x^{10}}{5} \sum_{i=0}^{4} \frac{(-1)^i}{2i+1} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}\right) \frac{x^{10}}{5}\,,$$

which is correct. This means that the coefficient of our power series takes the form

$$a_n = \frac{x^{4n+2}}{2n+1} \sum_{i=0}^{2n} \frac{(-1)^i}{2i+1}\,.$$

We will use d'Alembert's ratio test. We consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{|x^{4(n+1)+2}| \left|\sum_{i=0}^{2n+2} \frac{(-1)^i}{2i+1}\right|}{2(n+1)+1} \times \frac{2n+1}{|x|^{4n+2} \left|\sum_{i=0}^{2n+2} \frac{(-1)^i}{2i+1}\right|} \\
&= \lim_{n\to\infty} \left(\frac{2n+1}{2n+3}\right) \left(\frac{|x|^{4n+6}}{|x|^{4n+2}}\right) \left(\frac{\left|\sum_{i=0}^{2n+2} \frac{(-1)^i}{2i+1}\right|}{\left|\sum_{i=0}^{2n} \frac{(-1)^i}{2i+1}\right|}\right) \\
&= |x|^4 \lim_{n\to\infty} \left(\frac{\left|\sum_{i=0}^{2n+2} \frac{(-1)^i}{2i+1}\right|}{\left|\sum_{i=0}^{2n} \frac{(-1)^i}{2i+1}\right|}\right)
\end{aligned}$$

Lets now evaluate the limit of the ratio of the two sums above. Note that

$$\lim_{n\to\infty}\left(\frac{\left|\sum_{i=0}^{2n+2}\frac{(-1)^i}{2i+1}\right|}{\left|\sum_{i=0}^{2n}\frac{(-1)^i}{2i+1}\right|}\right)=\lim_{n\to\infty}\left(\frac{\left|\sum_{i=0}^{2n}\frac{(-1)^i}{2i+1}+\frac{(-1)^{2n+1}}{2(2n+1)+1}+\frac{(-1)^{2n+2}}{2(2n+2)+1}\right|}{\left|\sum_{i=0}^{2n}\frac{(-1)^i}{2i+1}\right|}\right).$$

Now as the sum $\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}$ converges (by the alternating series test) this ratio must converge to one. Thus I have shown that

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|^4\,,$$

and thus the series converges if $|x|^4<1$ or $|x|<1$ and diverges if $|x|>1$. One would need to consider the special cases $x=\pm 1$ separately.

We now show that this sum can be represented as the given product. Recall that in Examples XV that we derived the Taylor expansion of $\tan^{-1}(x)$ given by Equation 54. Following that derivation we have

$$\frac{1}{1-x^2}=\sum_{k=0}^{\infty}(x^2)^k=\sum_{k=0}^{\infty}x^{2k}\,.$$

If we integrate both sides of this we get

$$\tanh^{-1}(x)+C=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\,.$$

Taking $x=0$ we get that $C=0$ and we have shown that

$$\tanh^{-1}(x)=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}\,. \tag{104}$$

From what we have shown above if we write the Taylor series of $\tan^{-1}(x)$ and $\tanh^{-1}(x)$ in terms of "ones based" indexing we have

$$\begin{aligned}\tan^{1}(x)&=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{2n-1}=\sum_{n=1}^{\infty}a_n\\ \tanh^{1}(x)&=\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}=\sum_{n=1}^{\infty}b_n\,.\end{aligned}$$

Then using Equation 55 we get

$$\begin{aligned}c_n&=\left(\frac{(-1)^{n+1}x^{2n-1}}{2n-1}\right)\left(\frac{x}{1}\right)+\left(\frac{(-1)^{n}x^{2n-3}}{2n-3}\right)\left(\frac{x^3}{3}\right)+\cdots+\left(\frac{(-1)^{3}x^{3}}{3}\right)\left(\frac{x^{2n-3}}{2n-3}\right)+\left(\frac{(-1)^{2}x^{1}}{1}\right)\left(\frac{x^{2n-1}}{2n-1}\right)\\ &=\frac{(-1)^{n+1}x^{2n}}{(2n-1)(1)}+\frac{(-1)^{n}x^{2n}}{(2n-3)(3)}+\cdots+\frac{(-1)^{3}x^{2n}}{3(2n-3)}+\frac{x^{2n}}{(-1)^2(2n-1)}\\ &=x^{2n}\sum_{p=1,3,\ldots,2n-3,2n-1}\frac{(-1)^{n+1-\left(\frac{p-1}{2}\right)}}{(2n-p)p}\,.\end{aligned}$$

One can check that expression for the power of -1 gives the desired terms in the target sum. We can continue to simplify this and find

$$c_n = x^{2n}(-1)^{n+1} \sum_{p=1,3,\ldots,2n-3,2n-1} \frac{(-1)^{-\left(\frac{p-1}{2}\right)}}{(2n-p)p} = x^{2n}(-1)^{n+1} \sum_{p=1,3,\ldots,2n-3,2n-1} \frac{(-1)^{\frac{p-1}{2}}}{(2n-p)p}\,.$$

In this last sum we will use partial fractions as

$$\frac{1}{(2n-p)p} = \frac{1}{2np} + \frac{1}{2n(2n-p)}\,.$$

Using this I can write

$$\frac{(-1)^{n+1}(2n)}{x^{2n}} c_n = \sum_{p=1,3,\ldots,2n-3,2n-1} \frac{(-1)^{\frac{p-1}{2}}}{p} + \sum_{p=1,3,\ldots,2n-3,2n-1} \frac{(-1)^{\frac{p-1}{2}}}{2n-p}\,.$$

If we let $q = \frac{p-1}{2}$ then $p = 2q+1$ so $2n-p = 2n-2q-1 = 2(n-q)-1$. This means that as p over the range above q ranges over $1, 3, 5, \ldots, 2n-5, 2n-3, 2n-1$ we have that q ranges over $0, 1, 2, \ldots, n-3, n-2, n-1$. Thus we can write

$$\frac{(-1)^{n+1}(2n)}{x^{2n}} c_n = \sum_{q=0}^{n-1} \frac{(-1)^q}{2q+1} + \sum_{p=0}^{n-1} \frac{(-1)^q}{2(n-q)-1}\,.$$

In the second sum let $q = n-r-1$ so that $r+1 = n-q$ then as q takes on values in $0, 1, 2, \ldots, n-3, n-2, n-1$ we find that $r+1$ takes on values in $n, n-1, n-2, \ldots, 3, 2, 1$ and we can write

$$\frac{(-1)^{n+1}(2n)}{x^{2n}} c_n = \sum_{q=0}^{n-1} \frac{(-1)^q}{2q+1} + \sum_{r=0}^{n-1} \frac{(-1)^{n-r-1}}{2r-1}\,,$$

or

$$\frac{(-1)^{n+1}(2n)}{x^{2n}} c_n = \sum_{q=0}^{n-1} \frac{(-1)^q + (-1)^n(-1)^q(-1)}{2q+1} = \sum_{q=0}^{n-1} \frac{(-1)^q[1+(-1)^{n+1}]}{2q+1}\,.$$

Now if n is even this will vanish and we have $c_n = 0$. If n is odd it will not vanish and we take $n = 2m+1$ (with $m \ge 0$). In that case the above is

$$\frac{(-1)^{2m+2}(4m+2)c_{2m+1}}{x^{4m+2}} = \sum_{q=0}^{2m} \frac{(-1)^q(2)}{2q+1}\,.$$

Solving this for c_{2m+1} we get

$$c_{2m+1} = \frac{2x^{4m+2}}{4m+2} \sum_{q=0}^{2m} \frac{(-1)^q}{2q+1} = \frac{x^{4m+2}}{2m+1} \sum_{q=0}^{2m} \frac{(-1)^q}{2q+1}\,,$$

which is the same coefficients as specified in Equation 103.

Exercise 7

When $-1 < p \leq 0$ the function x^p is monotonic decreasing as x increases. Then using theorem 49 the expression

$$\sum_{i=0}^{n} i^p - \int_0^n x^p dx = \sum_{i=0}^{n} i^p - \left.\frac{x^{p+1}}{p+1}\right|_0^n = \sum_{i=0}^{n} i^p - \frac{n^{p+1}}{p+1}\,,$$

tends to a finite limit (say C) and thus we have

$$\lim_{n\to\infty}\left(\sum_{i=0}^{n} i^p - \frac{n^{p+1}}{p+1}\right) = C\,.$$

If we "divide" this expression by n^{p+1} we get

$$\lim_{n\to\infty}\left(\frac{1}{n^{p+1}}\sum_{i=0}^{n} i^p - \frac{1}{p+1}\right) = \lim_{n\to\infty}\left(\frac{C}{n^{p+1}}\right) = 0\,,$$

and thus we have shown that

$$\lim_{n\to\infty}\left(\frac{1}{n^{p+1}}\sum_{i=0}^{n} i^p\right) = \frac{1}{p+1}\,,$$

as we were to show.

Chapter 15 (The Order Notation)

Notes on the convergence of $\frac{1}{x} + \frac{2!}{x(x+1)} + \frac{3!}{x(x+1)(x+2)} + \cdots$

It looks like the general term in this series can be written as

$$u_n = \frac{n!}{\prod_{k=0}^{n-1}(x+k)}\,,$$

for $n \geq 1$. In that case we have

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{n!}{(n+1)!} \cdot \frac{\prod_{k=0}^{n}(x+k)}{\prod_{k=0}^{n-1}(x+k)} = \frac{x+n}{n+1} \\
&= \frac{n+1+x-1}{n+1} = 1 + \frac{x-1}{n+1} = 1 + \frac{x-1}{n}\left(\frac{n}{n+1}\right) \\
&= 1 + \frac{x-1}{n}\left(\frac{n+1-1}{n+1}\right) = 1 + \frac{x-1}{n}\left(1 - \frac{1}{n+1}\right) \\
&= 1 + \frac{x-1}{n} - \frac{x-1}{n(n+1)}\,.
\end{aligned}$$

To put this in the form needed in this section we will let

$$\frac{A_n}{n^{\lambda+1}} \equiv \frac{u_n}{u_{n+1}} - 1 - \frac{x-1}{n} = -\frac{(x-1)}{n(n+1)} = -\frac{1}{n^2}\left(\frac{x-1}{1+\frac{1}{n}}\right)\,.$$

This means that $\lambda = 1$ and

$$A_n = -\frac{x-1}{1+\frac{1}{n}} \quad \text{so} \quad |A_n| < |x-1|\,.$$

Notes on Applications of Theorem 50

Part (i): In the first series we can use order notation as

$$\begin{aligned}
\frac{u_n}{u_{n+1}} = \frac{n+x}{n+1} &= \left(1 + \frac{x}{n}\right)\left(1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= 1 + \frac{x-1}{n} + O\left(\frac{1}{n^2}\right)\,,
\end{aligned}$$

which gives the same conclusion as before.

Part (ii): The terms of this series u_n can be written as

$$
\begin{aligned}
u_n &= \left(\frac{2 \cdot 4 \cdot 6 \cdot 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \right)^2 \\
&= \left(\frac{(2 \cdot 4 \cdot 6 \cdot 2n)^2}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (2n) \cdot (2n+1)} \right)^2 \\
&= \left(\frac{(2^n \cdot n!)^2}{(2n+1)!} \right)^2 = \frac{2^{4n}(n!)^4}{((2n+1)!)^2} \, .
\end{aligned}
$$

This means that we have

$$
\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{2^{4n}(n!)^4}{((2n+1)!)^2} \times \frac{((2n+3)!)^2}{2^{4n+4}((n+1)!)^4} \\
&= \frac{1}{2^4} \frac{((2n+3)(2n+2))^2}{(n+1)^4} = \frac{(2n+3)^2(n+1)^2}{2^2(n+1)^4} = \frac{(2n+3)^2}{(2n+2)^2} \, .
\end{aligned}
$$

We can write this ratio as

$$
\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{\left(1 + \frac{3}{2n}\right)^2}{\left(1 + \frac{1}{n}\right)^2} = \left(1 + \frac{3}{2n}\right)^2 \left(1 + \frac{1}{n}\right)^{-2} \\
&= \left(1 + \frac{3}{n} + O\left(\frac{1}{n^2}\right)\right) \left(1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right) \\
&= 1 + \frac{3-2}{n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right) \, .
\end{aligned}
$$

So in the notation of this section $\mu = 1$ which is $\mu \leq 1$ and this series diverges.

We can also work this example without using the order notation in that we can write the ratio $\frac{u_n}{u_{n+1}}$ above as

$$
\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{(2n+3)^2}{(2n+2)^2} = \left(\frac{2n+2+1}{2n+2} \right)^2 = \left(1 + \frac{1}{2n+2}\right)^2 \\
&= 1 + \frac{2}{2(n+1)} + \frac{1}{4(n+1)^2} \, .
\end{aligned}
$$

If we set this equal to

$$
1 + \frac{1}{n} + \frac{A_n}{n^2} \, ,
$$

we can solve for A_n to get

$$
A_n = -\frac{1}{n(n+1)} + \frac{1}{4(n+1)^2} \, .
$$

From this form we see that $|A_n| < K$ for some K and we again have that $\mu = 1$.

Examples XX

Exercise 1

The terms of this series take the form

$$u_n = \frac{\prod_{k=0}^{n-1}(a+k)}{\prod_{k=0}^{n-1}(b+k)}\,,$$

for $n \geq 1$. Thus we see that we have

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{b+n}{a+n} = \frac{n+a+b-a}{n+a} = 1 + \frac{b-a}{n+a} \\
&= 1 + \frac{b-a}{n\left(1+\frac{a}{n}\right)} = 1 + \frac{b-a}{n}\left(1+\frac{a}{n}\right)^{-1} \\
&= 1 + \frac{b-a}{n\left(1+\frac{a}{n}\right)} = 1 + \frac{b-a}{n}\left(1-\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right) \\
&= 1 + \frac{b-a}{n} - \frac{a(b-a)}{n^2} + O\left(\frac{1}{n^3}\right)\,.
\end{aligned}$$

Thus using Gauss's ratio test with $\mu = b - a$ the series will converge if $\mu > 1$ or $b > a + 1$ and diverge otherwise.

Exercise 2

The first part of this exercise is worked on Page 110 using d'Alembert's ratio test. Using the expression derived there when $|z| = 1$ we have

$$\begin{aligned}
\frac{|u_n|}{|u_{n+1}|} &= \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \\
&= \left(1+\frac{1}{n}\right)\left(1+\frac{\gamma}{n}\right)\left(1+\frac{\alpha}{n}\right)^{-1}\left(1+\frac{\beta}{n}\right)^{-1} \\
&= 1 + \frac{1+\gamma-\alpha-\beta}{n} + O\left(\frac{1}{n^2}\right)\,.
\end{aligned}$$

Thus using Gauss's ratio test with $\mu = 1 + \gamma - \alpha - \beta$ the series will converge if $\mu > 1$ or $\gamma > \alpha + \beta$ and diverge otherwise.

Exercise 3

The terms of this series take the form

$$u_n = \frac{a\prod_{k=2}^{n}(ka+1)}{b\prod_{k=2}^{n}(kb+1)}\,,$$

for $n \geq 1$ where we use the convention that $\prod_{k=2}^{1} f(k) = 1$. Note that for u_n we have that

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)a+1}{(n+1)b+1}.$$

and thus

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{a}{b} \neq 1,$$

in general. If we assume that $\frac{a}{b} \neq 1$ we can conclude the series converges using d'Alembert's test. Thus our series converges if this limit is larger than one or

$$\frac{a}{b} > 1 \quad \text{so} \quad a > b.$$

Now if $a = b$ then each term in the series is one and the series must diverge.

Exercise 4

This series has terms that take the form

$$u_n = \left(\frac{\mu(\mu-1)(\mu-2)\cdots(\mu-(n-1))}{n!}\right)^2,$$

for $n \geq 1$. For this we have

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^2}{(\mu-n)^2} = \frac{(n+1)^2}{(n-\mu)^2}.$$

Note that for this ratio we have $\frac{u_n}{u_{n+1}} \to 1$ and thus we cannot use d'Alembert's test.

Using Gauss's test we have

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= \left(1+\frac{1}{n}\right)^2 \left(1-\frac{\mu}{n}\right)^{-2} \\
&= \left(1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{2\mu}{n}+O\left(\frac{1}{n^2}\right)\right) \\
&= 1+\frac{2+2\mu}{n}+O\left(\frac{1}{n^2}\right).
\end{aligned}$$

From this section for convergence we need $2+2\mu > 1$ or $\mu > -\frac{1}{2}$.

Exercise 5

The terms of this series take the form

$$u_n = \frac{a_n}{\prod_{k=0}^{n}(x+k)},$$

for $n \geq 0$. For this we find

$$\begin{aligned}\frac{u_n}{u_{n+1}} &= \frac{a_n}{\prod_{k=0}^{n}(x+k)} \times \frac{\prod_{k=0}^{n+1}(x+k)}{a_{n+1}} \\ &= \frac{a_n(x+n+1)}{a_{n+1}} = \frac{n+x+1}{an+b+O(1/n)} \,.\end{aligned}$$

As $n \to \infty$ this goes to $\frac{1}{a}$. By taking the absolute values of the above then d'Alembert's test says this series converges when $\frac{1}{|a|} > 1$ or $|a| < 1$.

If $a = 1$ then the above is

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{x+1}{n}\right)\left(1 + \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right)^{-1} = 1 + \frac{x-b+1}{n} + O\left(\frac{1}{n^2}\right) \,.$$

From this section, this series will converge if $x - b + 1 > 1$ or $x > b$.

Exercise 6

This statement is equivalent to

$$\left(1 + \frac{\alpha}{n}\right)^{-1}\left(1 + \frac{\alpha}{n} + O\left(\frac{1}{n^{1+\lambda}}\right)\right) = 1 + O\left(\frac{1}{n^{1+\lambda}}\right) \,.$$

Now using

$$\left(1 + \frac{\alpha}{n}\right)^{k} = 1 + \frac{k\alpha}{n} + O\left(\frac{1}{n^2}\right) \,,$$

we can write the left-hand-side of the above as

$$\left(1 - \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)\right)\left(1 + \frac{\alpha}{n} + O\left(\frac{1}{n^{1+\lambda}}\right)\right)$$

Then using Theorem 52 (or multiplying out) this is

$$1 + \frac{\alpha - \alpha}{n} + O\left(\frac{1}{n^{1+\lambda}}\right) = 1 + O\left(\frac{1}{n^{1+\lambda}}\right) \,,$$

as we wanted to prove.

Taking the logarithm of both sides gives

$$\begin{aligned}\log\left(1 + \frac{\alpha}{n} + O\left(\frac{1}{n^{1+\lambda}}\right)\right) &= \log\left(1 + \frac{\alpha}{n}\right) + \log\left(1 + O\left(\frac{1}{n^{1+\lambda}}\right)\right) \\ &= \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right) + 0 + O\left(\frac{1}{n^{2+2\lambda}}\right) \\ &= \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right) \,.\end{aligned}$$

Exercise 7

Taking the logarithm of the given statement gives the second expression in Exercise 6 above.

Exercise 8

Taking powers of the first expression in Exercise 6 above gives

$$\begin{aligned}
\left(1+\frac{\alpha}{n}+\frac{1}{n^{1+\lambda}}\right)^k &= \left(1+\frac{\alpha}{n}\right)^k\left(1+O\left(\frac{1}{n^{1+\lambda}}\right)\right)^k \\
&= \left(1+\frac{\alpha k}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1+O\left(\frac{1}{n^{k+k\lambda}}\right)\right) \\
&= 1+\frac{\alpha k}{n}+O\left(\frac{1}{n^{1+\theta}}\right),
\end{aligned}$$

for $\theta > 0$.

Chapter 16 (Tannery's Theorem)

Examples XXI

Exercise 1

Part (i): Following the book we define this expression as $F(n)$ and then use the binomial theorem to expand it as

$$F(n) = 1 + n\left(\frac{a}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{a}{n}\right)^2 + \left(\begin{array}{c} n \\ 3 \end{array}\right)\left(\frac{a}{n}\right)^3 + \cdots ,$$

where there are $n+1$ terms in the above expansion. Lets write the above as

$$F(n) = 1+a+\frac{a^2}{2!}\left(1-\frac{1}{n}\right)+\frac{a^3}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\frac{a^4}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)+\cdots .$$

Define these terms as $v_r(n)$ for $r \in \{2, 3, \ldots, n-1, n\}$ where

$$v_r(n) \equiv \frac{a^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) .$$

Then

$$F(n) = 1 + a + \sum_{r=1}^{n} v_r(n) ,$$

and we have now written $F(n)$ in the form needed to apply Tannery's theorem. To apply Tannery's theorem we need to find M_r such that $|v_r(n)| \le M_r$ and $\sum_r M_r$ converges. From the above form of $v_r(n)$ we see that

$$|v_r(n)| \le \frac{a^r}{r!} .$$

Thus we should take $M_r = \frac{a^r}{r!}$. Then the limit of $F(n)$ as $n \to \infty$ is equal to $\sum_r w_r$ where

$$w_r = \lim_{n\to\infty} v_r(n) = \frac{a^r}{r!} .$$

Then by Tannery's theorem we have

$$\lim_{n\to\infty} F(n) = 1 + a + \sum_{r=2}^{\infty} \frac{a^r}{r!} = e^a ,$$

which is the desired limit.

Part (ii): We can perform all of the steps in the previous part when n is replaces by a real number x and all of the steps in the above use of Tannery's theorem will still hold. Thus the conclusion still holds.

Exercise 2

The function $F(n)$ we are considering can be written as

$$F(n) = \sum_{r=1}^{n} \frac{1}{n+r}\,.$$

For this $F(n)$ we have

$$\begin{aligned}
F(1) &= \sum_{r=1}^{1} \frac{1}{1+r} = \frac{1}{2} \geq \frac{1}{2} \\
F(2) &= \sum_{r=1}^{2} \frac{1}{2+r} = \frac{1}{2+1} + \frac{1}{2+2} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{1}{2}\,.
\end{aligned}$$

From these two evaluations of F note that $F(2) > F(1)$. Based on this observation lets compute the difference $F(n+1) - F(n)$. We find

$$\begin{aligned}
F(n+1) - F(n) &= \sum_{r=1}^{n+1} \frac{1}{n+1+r} - \sum_{r=1}^{n} \frac{1}{n+r} \\
&= \sum_{r=2}^{n+2} \frac{1}{n+r} - \sum_{r=1}^{n} \frac{1}{n+r} \\
&= \frac{1}{n+n+1} + \frac{1}{n+n+2} - \frac{1}{n+1} \\
&= \frac{1}{(2n+1)(2n+2)} > 0\,,
\end{aligned}$$

when we simplify. This means that $F(n+1) > F(n)$. Thus since $F(3) > \frac{1}{2}$ and using the above we have $F(n) > \frac{1}{2}$ for all $n \geq 3$.

Exercise 3

This is an application of Tannery's theorem. Let $v_n(x) = \frac{1}{n^2 + \frac{n^4}{x^2}}$ then if we take $M_n = \frac{1}{n^2}$ we know that $\sum_n M_n$ converges and from the expression above for $v_n(x)$ we have that

$$|v_n(x)| \leq \frac{1}{n^2} = M_n\,.$$

Next note that

$$\lim_{x \to \infty} v_n(x) = w_n = \frac{1}{n^2}\,,$$

so an application of Tannery's theorem then states that

$$\sum_{n=1}^{\infty} v_n(x) \to \sum_{n=1}^{\infty} \frac{1}{n^2}\,,$$

as $x \to \infty$.

Exercise 4

Lets let $x = 1 + y$ with $y > 0$ so that we can write our sum as

$$S \equiv \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1-y} \, .$$

Now the limit $x \to 1^+$ is the same as the limit $y \to 0^+$. Based on the above form we will write

$$S = \sum_{n=1}^{\infty} a_n n^{-y} \, ,$$

where $a_n = (-1)^{n-1} n^{-1}$. We now need to check the required conditions of Theorem 55 in the book. We have that $\sum_n a_n$ is a convergent series. The other factor $v_n(y) = n^{-y}$ is monotonically decreasing for each fixed $y > 0$ and we have that

$$|n^{-y}| < 1 \, ,$$

for n large enough. In addition

$$\lim_{y \to 0^+} n^{-y} = 1 \, .$$

Based on all of these conditions we have that

$$S \to \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-1} \, .$$

as $y \to 0^+$. The fact that the summation above evaluates to $\log(2)$ is discussed in the section of the book entitled "Further results about rearrangements of series".

Exercise 5

We can write this sum as

$$F(k) = \sum_{i=1}^{k} \left(\frac{k - (i-1)}{k} \right) a_i = \sum_{i=1}^{k} \left(\frac{k + 1 - i}{k} \right) a_i = \sum_{i=1}^{k} \left(1 - \frac{i-1}{k} \right) a_i \, .$$

To apply Theorem 55 we let

$$v_i(k) = 1 - \frac{i-1}{k} \, ,$$

if $i \leq k$ and zero otherwise. Then for a fixed k we have $|v_i(k)| < 1$ and $v_i(k) \to 1$ as $k \to \infty$. An application of Theorem 55 then gives that

$$F(k) \to \sum_{i=1}^{\infty} a_i \, ,$$

as $k \to \infty$.

I was not sure how to use this result to show that

$$\frac{s_1 + s_2 + \cdots + s_{n-1} + s_n}{n} \to s\,,$$

when $s_n \to s$. This later result is discussed (and proved) in the section in the book on Cesàro sums.

Exercise 6

To use Theorem 55 we need to factor the terms in the sum into two parts. The first is a "constant" factor (denoted by a_n in the theorem) that does not depend on the variable we will take the limit of. The second is a "variable" factor (denoted by $v_n(x)$ in the theorem) that depends on the variable we will take the limit of (denoted by x in the theorem).

In the sum we are given lets take the "constant" factor to be

$$A_r \equiv \frac{(-1)^r a_r}{x - r}\,,$$

and the "variable" expression (here the variable we take the limit of is n) to be

$$v_r(n) \equiv \frac{(n!)^2}{(n-r)!(n+r)!}\,,$$

if $1 \le r \le n$ and $v_r(n) = 0$ if $r > n$. Then the sum we are given can be written as

$$\sum_{r=1}^{n} v_r(n) A_r\,.$$

We can apply Theorem 55 if we can prove certain things about these two factors. If we can, then the conclusions of that theorem give us the desired conclusion for this exercise.

Starting with the expression for $v_r(n)$ written as

$$v_r(n) = \frac{n!}{(n+r)!} \times \frac{n!}{(n-r)!}\,,$$

we can write this as

$$\begin{aligned}
v_r(n) &= \left(\frac{n!}{(n+r)(n+r-1)\cdots(n+2)(n+1)n!}\right) \times \left(\frac{n(n-1)\cdots(n-r+2)(n-r+1)n!}{n!}\right) \\
&= \frac{n(n-1)\cdots(n-r+2)(n-r+1)}{(n+r)(n+r-1)\cdots(n+2)(n+1)} \\
&= \left(\frac{n}{n+r}\right)\left(\frac{n-1}{n-1+r}\right)\left(\frac{n-2}{n-2+r}\right)\cdots\left(\frac{n-r+3}{n+3}\right)\left(\frac{n-r+2}{n+2}\right)\left(\frac{n-r+1}{n+1}\right) \qquad (105) \\
&= \left(1 - \frac{r}{n+r}\right)\left(1 - \frac{r}{n+r-1}\right)\left(1 - \frac{r}{n+r-2}\right)\cdots\left(1 - \frac{r}{n+3}\right)\left(1 - \frac{r}{n+2}\right)\left(1 - \frac{r}{n+1}\right). \qquad (106)
\end{aligned}$$

From the expression given in Equation 105 we see that for $1 \le r \le n$ we have that for a fixed n that $v_r(n)$ is monotone decreasing as r increases to n. This is because every fraction in that product decreases as r increases.

Next using Equation 106 we see that

$$|v_r(n)| < 1 \,,$$

for $1 \leq r \leq n$.

Finally using Equation 106 we see that

$$v_r(n) \to 1 \,,$$

as $n \to \infty$. As these are the conditions required for an application of Theorem 55 we can use that theorem to conclude the statement given in this exercise.

Chapter 18 (Infinite Products)

Notes on the Text

Note that given a product like

$$\prod_{n=1}^{N} u_n \,, \tag{107}$$

we can write it the more "standard" form of $\prod_{n=1}^{N}(1+a_n)$ by taking

$$\prod_{n=1}^{N}(1+(u_n-1))\,,$$

where now we see that

$$a_n \equiv u_n - 1\,. \tag{108}$$

Recall that from the book the definition of a product $\prod(1+a_n)$ to be absolutely convergent means that the sum

$$\sum \log(1+a_n)\,,$$

is absolutely convergent. Using the above transformations we see that is equivalent to the statement that for the product $\prod(1+a_n) = \prod u_n$ to be absolutely convergent means that the sum

$$\sum \log(1+a_n) = \sum \log(u_n)\,, \tag{109}$$

is absolutely convergent. Thus we can work with sums of the terms $\log(u_n)$.

In the same way a product is uniformly convergent if $\sum \log(u_n)$ is uniformly convergent.

Examples XXIII

Exercise 1

When the first two products are written as $\prod_n(1+a_n)$ and $\prod_n(1-a_n)$ we have $a_n = \frac{1}{n^2}$. Since $\sum_n a_n$ converges by using Theorem 59 in this section we have that the first two product converge.

For the third product we have $a_n = \sin^2\left(\frac{\theta}{n}\right)$. We can show that $\sum_n a_n$ converges by applying the "limit" form of the comparison test with $b_n = \frac{1}{n^2}$. To do this we need to evaluate

$$\lim_{n\to\infty} \frac{\sin^2\left(\frac{\theta}{n}\right)}{\frac{1}{n^2}}\,.$$

If we let $x = \frac{1}{n}$ then as $n \to \infty$ we have $x \to 0^+$. This means that the limit above is equivalent to

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\sin^2(\theta x)}{x^2} &= \lim_{x\to 0^+} \frac{2\sin(\theta x)\cos(\theta x)\theta}{2x} \\
&= \theta \lim_{x\to 0^+} \frac{\sin(\theta x)}{x} \\
&= \theta \lim_{x\to 0^+} \frac{\theta\cos(\theta x)}{1} = \theta^2 \,.
\end{aligned}$$

This means that $\sum_n a_n$ and $\sum_n b_n$ either both converge or both diverge. As $\sum_n b_n$ converges so does $\sum_n a_n$. This later means that the given product converges.

Exercise 2

Matching these products to $\prod_n(1-a_n)$ we have $a_n = \frac{1}{n}$ and $a_n = \frac{x}{n}$ respectively. Then since $\sum_n a_n$ diverges in both cases, Theorem 59 from this section tells us that each product converges to zero.

Exercise 3

Now if $x < 0$ then we can write the given product as

$$\prod_{n=1}^{N}\left(1 + \frac{|x|}{n}\right) .$$

Then matching this to $\prod_n(1+a_n)$ we have $a_n = \frac{|x|}{n}$ and recall that $\sum_n a_n$ diverges. Then Theorem 59 from this section tells us that this product converges to infinity.

Exercise 4

Recall the Taylor series for $\log(1+x)$

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} \,. \tag{110}$$

This means that

$$\log(1+x) - x = \sum_{k=2}^{\infty} \frac{(-1)^{k+1}x^k}{k} \,,$$

so that the absolute value of this (evaluated at $x \to \frac{x}{n}$) is given by

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| = \left|\sum_{k=2}^{\infty} \frac{(-1)^{k+1}x^k}{kn^k}\right| = \left|-\frac{x^2}{2n^2} + \frac{x^3}{3n^3} - \frac{x^4}{4n^4} + \cdots\right| .$$

We can create an upper bound by using the triangle rule ($|x+y| \le |x| + |y|$) and changing all negatives to positives to get

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| \le \frac{|x|^2}{n^2}\left|\frac{1}{2} + \frac{|x|}{3n} + \frac{|x|^2}{4n^2} + \cdots\right| .$$

As each of $\frac{1}{k} \le 1$ for $k \ge 1$ a looser upper bound is

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| \le \frac{|x|^2}{n^2}\left|1 + \frac{|x|}{n} + \frac{|x|^2}{n^2} + \cdots\right| , \tag{111}$$

which is the given expression in the book. As the right-most sum is a geometric sum we can write the above as

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| \le \frac{|x|^2}{n^2}\frac{1}{1-\left(\frac{|x|}{n}\right)} = \frac{|x|^2}{n}\left(\frac{1}{n-|x|}\right) . \tag{112}$$

Now if $|x| < A$ then Equation 111 becomes

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| \le \frac{A^2}{n^2}\left|1 + \frac{|x|}{n} + \frac{|x|^2}{n^2} + \cdots\right| . \tag{113}$$

Now as $|x| < A$ we have $\frac{|x|}{n} < \frac{A}{n}$ and if $n > 2A$ then

$$\frac{A}{n} < \frac{A}{2A} = \frac{1}{2} .$$

This means that Equation 113 becomes

$$\left|\log\left(1+\frac{x}{n}\right) - \frac{x}{n}\right| \le \frac{A^2}{n^2}\left|1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right| = \frac{A^2}{n^2}\left|\frac{1}{1-\frac{1}{2}}\right| = \frac{2A^2}{n^2} . \tag{114}$$

as we were to show.

Exercise 5

Part (i): From the comments above on this section we need to show that $\sum \log(u_n)$ is absolutely convergent. We have that

$$\log(u_n) = \log\left[\left(1+\frac{x}{n}\right)e^{-x/n}\right] = \log\left(1+\frac{x}{n}\right) - \frac{x}{n} .$$

Now using Equation 112 we see that $\sum |\log(u_n)|$ is convergent by comparing it to the series $\sum \frac{1}{n^2}$.

Part (ii): Using Part (i) and Equation 114 we can see that $\sum \log(u_n)$ is uniformly convergent by using the Weierstrass M test with $M_n = \sum \frac{2A^2}{n^2}$.

Exercise 6

Note that we can define $b_n = \frac{x^2}{n^2\pi^2}$ and then that

$$\prod(1 - b_n) = \prod(1 + (-b_n)),$$

so we can take $a_n = -b_n$.

Part (i): From the text for this product to be absolutely convergent we need $\sum |\log(1+a_n)|$ to be convergent which will happen if and only if $\sum |a_n|$ is convergent. In this case as $|a_n| = O\left(\frac{1}{n^2}\right)$ we have that $\sum |a_n|$ is convergent.

Part (ii): From the text for this product to be uniform convergent we need $|a_n(x)| \leq M_n$ where M_n are terms of a positive convergent series. Note that when $|x| \leq A$ we have

$$|a_n(x)| = \left|\frac{x^2}{n^2\pi^2}\right| \leq \frac{A^2}{n^2\pi^2}.$$

If we define $M_n = \frac{A^2}{n^2\pi^2}$ we have that the product is uniformly convergent.

Exercise 7

Define P_n as

$$P_n = \prod_{k=1}^{n} \left(1 - \frac{x}{k}\right).$$

Then we have

$$P_1 = 1 - \frac{x}{1} = 1 - x,$$

and

$$P_2 = \left(1 - \frac{x}{1}\right)\left(1 - \frac{x}{2}\right) = 1 - \frac{x}{2} - \frac{x}{1} + \frac{x^2}{2} = 1 - x + \frac{x^2 - x}{2} = 1 - x + \frac{x(x-1)}{2}.$$

Thus we have shown that the given product on the left-hand-side equals the expansion on the right-hand-side for $n \in \{1, 2\}$. Assume that the given expansion holds for $n \leq N$. Consider then constructing P_{N+1} from $\left(1 - \frac{x}{N+1}\right)$ and P_N. Call that expression E. We have

$$E = \prod_{k=1}^{N+1} \left(1 - \frac{x}{k}\right) = P_N \left(1 - \frac{x}{N+1}\right) = P_N + \frac{x}{N+1}(-P_N).$$

We now seek to evaluate $\frac{x}{N+1}(-P_N)$. To do that we start with the definition of P_N. We have

$$\begin{aligned}
P_N &= \left(1 - \frac{x}{1}\right)\left(1 - \frac{x}{2}\right)\left(1 - \frac{x}{3}\right)\cdots\left(1 - \frac{x}{N}\right) \\
&= \frac{1}{N!}(1-x)(2-x)(3-x)\cdots(N-x) \\
&= \frac{(-1)^N}{N!}(x-1)(x-2)(x-3)\cdots(x-N).
\end{aligned}$$

This means that

$$\frac{x}{N+1}(-P_N) = \frac{(-1)^{N+1}x(x-1)(x-2)(x-3)\cdots(x-N)}{(N+1)!},$$

and so we have

$$E = P_N + \frac{(-1)^{N+1}x(x-1)(x-2)(x-3)\cdots(x-N)}{(N+1)!},$$

which is the right-hand-side evaluated at $n \to N+1$. Thus by induction we have shown the given expression.

Exercise 8

Consider the product

$$\prod_{n=1}^{m}\left(1-\frac{x}{n}\right).$$

Then matching this to $\prod(1-a_n)$ we have $a_n = \frac{x}{n}$. Now as $\sum a_n$ is divergent from Theorem 59 we have that the product above tends to the limit of zero. From the equivalence just shown in Exercise 7 we have that the given polynomial sum (the one given in this exercise) tends to zero as $m \to \infty$. Note also this same conclusion is reached on Page 102 Eq. 6 in the book.

Exercise 9

All of these are products of the form $\prod(1 \pm a_n)$ with $a_n = O(q^{2n})$. A necessary and sufficient condition for these products to be absolutely convergent is that $\sum |a_n|$ be convergent. In this case $\sum |a_n|$ a geometric series and so is absolutely convergent. This means that we can evaluate these products by taking the factors in the product in any order.

As we have absolute convergence we can change the order of factors the products. Thus

$$\begin{aligned}
q_0 q_3 &= \prod_{n\geq 1}(1-q^{2n})\prod_{n\geq 1}(1-q^{2n-1}) \\
&= (1-q^2)(1-q^4)(1-q^6)(1-q^8)\cdots(1-q)(1-q^3)(1-q^5)(1-q^7)\cdots \\
&= (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)(1-q^7)(1-q^8)\cdots \\
&= \prod_{n\geq 1}(1-q^{2n-1})(1-q^{2n}) = \prod_{n\geq 1}(1-q^n).
\end{aligned}$$

In the same way

$$
\begin{aligned}
q_1 q_2 &= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1+q^{2n-1}) \\
&= (1+q^2)(1+q^4)(1+q^6)(1+q^8)\cdots(1+q)(1+q^3)(1+q^5)(1+q^7)\cdots \\
&= (1+q)(1+q^2)(1+q^3)(1+q^4)(1+q^5)(1+q^6)(1+q^7)(1+q^8)\cdots \\
&= \prod_{n\geq 1}(1+q^{2n-1})(1+q^{2n}) = \prod_{n\geq 1}(1+q^n)\,.
\end{aligned}
$$

To show the product $q_1q_2q_3 = 1$ we will derive some "telescoping" factors and then argue that the procedure we show can be continued an infinite number of times. Doing so will give the required result. Towards that end we have

$$
\begin{aligned}
q_1 q_2 q_3 &= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1+q^{2n-1})\prod_{n\geq 1}(1-q^{2n-1}) \\
&= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1+q^{2n-1})(1-q^{2n-1}) \\
&= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1-q^{2(2n-1)})\,. \qquad (115)
\end{aligned}
$$

Now split the factors in the first product above into "even" and "odd" values for n say $n \to 2n$ and $n \to 2n-1$. This gives

$$
q_1 q_2 q_3 = \prod_{n\geq 1}(1+q^{2(2n)})\prod_{n\geq 1}(1+q^{2(2n-1)})\prod_{n\geq 1}(1-q^{2(2n-1)})\,.
$$

Now multiply the two "right-most" products together term by term to get

$$
q_1 q_2 q_3 = \prod_{n\geq 1}(1+q^{2^2 n})\prod_{n\geq 1}(1-q^{2^2(2n-1)})\,. \qquad (116)
$$

We can continue this process to show that the product on the right-hand-side evaluates to one. Towards this end note that Equations 115 and 116 are specific examples of the representation

$$
q_1 q_2 q_3 = \prod_{n\geq 1}(1+q^{2^p n})\prod_{n\geq 1}(1-q^{2^p(2n-1)})\,. \qquad (117)
$$

for $p = 1$ and $p = 2$ respectively. Assuming this form holds more generally we can "continue this process" by splitting the factors in the first product above into "even" and "odd" values for n say $n \to 2n$ and $n \to 2n-1$. This gives

$$
q_1 q_2 q_3 = \prod_{n\geq 1}(1+q^{2^p(2n)})\prod_{n\geq 1}(1+q^{2^p(2n-1)})\prod_{n\geq 1}(1-q^{2^p(2n-1)})\,.
$$

Multiplying the two "right-most" products together term by term we get

$$
q_1 q_2 q_3 = \prod_{n\geq 1}(1+q^{2^{p+1} n})\prod_{n\geq 1}(1-q^{2^{p+1}(2n-1)})\,,
$$

which is Equation 117 evaluated at $p \to p+1$. As Equation 117 must then hold as $p \to \infty$ (and since $|q| < 1$) we must have

$$q_1 q_2 q_3 = 1\,.$$

To show the final result we "imagine" that we multiply by $\prod_{n\geq 1}(1-q^{2n-1})$ and consider P defined as

$$P = \prod_{n\geq 1}(1+q^n)\prod_{n\geq 1}(1-q^{2n-1})\,.$$

Then following the same ideas from the previous part of this problem we have

$$\begin{aligned}
P &= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1+q^{2n-1})\prod_{n\geq 1}(1-q^{2n-1}) \\
&= \prod_{n\geq 1}(1+q^{2n})\prod_{n\geq 1}(1-q^{2(2n-1)}) \\
&= \prod_{n\geq 1}(1+q^{2^p n})\prod_{n\geq 1}(1-q^{2^p(2n-1)}) \quad \text{for} \quad p = 1\,.
\end{aligned}$$

Where in the last expression we have generalized the expression before it. We can write P as

$$\begin{aligned}
P &= \prod_{n\geq 1}(1+q^{2^{p+1}n})\prod_{n\geq 1}(1+q^{2^p(2n-1)})\prod_{n\geq 1}(1-q^{2^p(2n-1)}) \\
&= \prod_{n\geq 1}(1+q^{2^{p+1}n})\prod_{n\geq 1}(1-q^{2^{p+1}(2n-1)})\,.
\end{aligned}$$

We can continue this procedure indefinitely. If we let $p \to \infty$ we see that we must conclude

$$\prod_{n\geq 1}(1+q^n)\prod_{n\geq 1}(1-q^{2n-1}) = 1\,,$$

which is equivalent to the desired expression.

Exercise 10

Consider the product

$$P = (1-x^2)\prod_{n\geq 1}(1+x^{2^n})\,.$$

Based on this functional form we define P_N as

$$P_N = (1-x^2)\prod_{n=1}^{N}(1+x^{2^n})\,.$$

Then we have that

$$P_1 = (1-x^2)(1+x^2) = 1-x^4\,,$$

and

$$P_2 = (1 - x^2)\prod_{n=1}^{2}(1 + x^{2^n}) = P_1(1 + x^4) = (1 - x^4)(1 + x^4) = 1 - x^8 \,.$$

We claim that $\lim_{N\to\infty} P_N = 1$. From the above it looks like the functional form for P_N is given by

$$P_N = 1 - x^{2^{N+1}} \,.$$

We have shown this expression is true for $N = 1$ and $N = 2$. Assuming this is true up to N then for $N + 1$ we have

$$P_{N+1} = P_N(1 + x^{2^{N+1}}) = (1 - x^{2^{N+1}})(1 + x^{2^{N+1}}) = 1 - x^{2^{N+2}} \,,$$

showing that our expression is true for $N + 1$ also. Then as $|x| < 1$ we have $x^n \to 0$ as $n \to \infty$ and thus $P_N \to 1$ as $N \to \infty$ showing the desired expression.

Exercise 11

We can get a hint at how to evaluate this by considering the expression for $n = 1$. This is

$$2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \,.$$

This looks very much like the identity $2\sin(v)\cos(v) = \sin(2v)$ and using that we see that the above is $\sin(x)$. Now in

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \,.$$

If we keep replacing the $\sin(v)$ with $\sin(v) = 2\sin\left(\frac{v}{2}\right)\cos\left(\frac{v}{2}\right)$ we get the given expression. Doing this we have

$$\begin{aligned}
\sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\
&= 2^2\sin\left(\frac{x}{4}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) \\
&= 2^3\sin\left(\frac{x}{8}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) \\
&\vdots \\
&= 2^n\sin\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) \,.
\end{aligned}$$

From this we can write

$$\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) = \frac{\sin(x)}{2^n\sin\left(\frac{x}{2^n}\right)} \,.$$

Notice that we can write

$$\frac{1}{2^n\sin\left(\frac{x}{2^n}\right)} = \frac{1}{x}\left(\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)}\right)$$

Notice that as $n \to \infty$ we have $\frac{x}{2^n} \to 0$ so that $\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)} \to 1$ thus the product we seek is given by

$$\cos\left(\frac{x}{2^n}\right)\cos\left(\frac{x}{2^{n-1}}\right)\cdots\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{2}\right) = \frac{\sin(x)}{x},$$

as we were to show.

Exercise 13

The first two statements are simple algebra

$$\frac{1}{t} - \frac{1}{t+1} = \frac{t+1}{t(t+1)} - \frac{t}{t(t+1)} = \frac{1}{t(t+1)},$$

and

$$\begin{aligned}\frac{1}{t} - \frac{1}{t+1} - \frac{1}{(t+1)(t+2)} &= \frac{1}{t(t+1)} - \frac{1}{(t+1)(t+2)} \\ &= \frac{t+2}{t(t+1)(t+2)} - \frac{t}{t(t+1)(t+2)} = \frac{2}{t(t+1)(t+2)}.\end{aligned}$$

These are $n = 1$ and $n = 2$ of the general expression

$$\frac{1}{t} - \frac{1}{t+1} - \sum_{r=1}^{n-1} \frac{r!}{(t+1)(t+2)\cdots(t+r+1)} = \frac{n!}{t(t+1)(t+2)\cdots(t+n)}. \tag{118}$$

If we assume that the above is true for $n \le N$ then the right-hand-side of Equation 118 for $n \to n+1$ is

$$\begin{aligned}\frac{1}{t} - \frac{1}{t+1} - \sum_{r=1}^{n} \frac{r!}{(t+1)(t+2)\cdots(t+r+1)} &= \frac{n!}{t(t+1)(t+2)\cdots(t+n)} - \frac{n!}{(t+1)(t+2)\cdots(t+n+1)} \\ &= \frac{n!}{(t+1)(t+2)\cdots(t+n)}\left(\frac{1}{t} - \frac{t}{t+n+1}\right) \\ &= \frac{n!}{(t+1)(t+2)\cdots(t+n)}\left(\frac{t+n+1-t}{t(t+n+1)}\right) \\ &= \frac{(n+1)!}{t(t+1)(t+2)\cdots(t+n)(t+n+1)},\end{aligned} \tag{119}$$

showing that Equation 118 is true for $N + 1$ also.

Exercise 14

Consider the product on the right-hand-side of Equation 119 which is

$$\frac{1}{t}\prod_{n\ge 1}\frac{n}{t+n} = \frac{1}{t}\prod_{n\ge 1}\left(1 - \frac{t}{t+n}\right).$$

Now as $\sum \frac{t}{t+n}$ diverges from this section of the book we know that the above product converges to zero. Taking the limit of $n \to \infty$ in Equation 119 we get

$$\frac{1}{t} - \frac{1}{t+1} - \sum_{r=1}^{\infty} \frac{r!}{(t+1)(t+2)\cdots(t+r+1)} = 0\,,$$

which is the desired expression.

Exercise 15

We are asked about the product

$$\prod_{n\geq 0} \frac{a+n}{b+n}\,.$$

If we write

$$\frac{a+n}{b+n} = \frac{b+n+a-b}{b+n} = 1 + \frac{a-b}{b+n} = 1 - \left(\frac{b-a}{b+n}\right)\,,$$

or product is

$$\prod_{n\geq 0} \left(1 - \left(\frac{b-a}{b+n}\right)\right)\,.$$

As $\sum \frac{b-a}{b+n}$ diverges this product must converge to zero.

Chapter 19 (Theorems on Limits: Cesàro Sums)

A comment about Theorem 62

While this theorem maybe somewhat difficult to remember it is made easier by the fact that it is of the very same form as L'Hospital's rule which is learned in calculus. For example if we seek to evaluate the limit of

$$\frac{a_n}{b_n},$$

we can do that by evaluating the limit of

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n},$$

which is a form that looks very much like a "derivative over a derivative" and is what we would consider to evaluate certain limits of the form

$$\frac{f(x)}{g(x)}.$$

Because of this I find it easier to recall this theorem as "a discrete L'Hospital's rule".

Examples XXIV

Example 1

Recall that $S_n^{(r)}$ is defined as

$$S_n^{(r)} = s_n + rs_{n-1} + \frac{r(r+1)}{2!}s_{n-2} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_0.$$

Now from this we have that

$$S_{n+1}^{(r)} = s_{n+1} + rs_n + \frac{r(r+1)}{2!}s_{n-1} + \cdots + \frac{r(r+1)\cdots(r+n-1)}{n!}s_1 + \frac{r(r+1)\cdots(r+n)}{(n+1)!}s_0.$$

From these two we can compute the difference $S_{n+1}^{(r)} - S_n^{(r)}$ as

$$
\begin{aligned}
S_{n+1}^{(r)} - S_n^{(r)} &= s_{n+1} + (r-1)s_n + \left(\frac{r(r+1)}{2} - r\right) s_{n-1} \\
&+ \left(\frac{r(r+1)(r+2)}{3!} - \frac{r(r+1)}{2!}\right) s_{n-2} + \cdots + \\
&+ \left(\frac{r(r+1)\cdots(r+n-1)}{n!} - \frac{r(r+1)\cdots(r+n-2)}{(n-1)!}\right) s_1 \\
&+ \left(\frac{r(r+1)\cdots(r+n)}{(n+1)!} - \frac{r(r+1)\cdots(r+n-1)}{n!}\right) s_0 \\
&= s_{n+1} + (r-1)s_n + \frac{r}{2}\left[r+1-2\right] s_{n-1} + \frac{r(r+1)}{3!}\left[(r+2)-3\right] s_{n-2} + \cdots \\
&+ \frac{r(r+1)\cdots(r+n-2)}{n!}\left[r+n-1-n\right] s_1 \\
&+ \frac{r(r+1)\cdots(r+n-2)(r+n-1)}{(n+1)!}\left[r+n-(n+1)\right] s_0 \\
&= s_{n+1} + (r-1)s_n + \frac{(r-1)r}{2} s_{n-1} + \frac{(r-1)r(r+1)}{3!} s_{n-2} + \cdots \\
&+ \frac{(r-1)r(r+1)\cdots(r+n-2)}{n!} s_1 + \frac{(r-1)r(r+1)\cdots(r+n-2)(r+n-1)}{(n+1)!} s_0 = S_{n+1}^{(r-1)} .
\end{aligned}
$$

Next for $A_n^{(r)}$ we recall its definition

$$
A_n^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)}{n!} . \tag{120}
$$

From this definition we have

$$
A_{n+1}^{(r)} = \frac{(r+1)(r+2)\cdots(r+n)(r+n+1)}{(n+1)!} .
$$

From these we compute the needed difference from

$$
\begin{aligned}
A_{n+1}^{(r)} - A_n^{(r)} &= \frac{(r+1)(r+2)\cdots(r+n)}{n!}\left[\frac{r+n+1}{n+1} - 1\right] \\
&= \frac{(r+1)(r+2)\cdots(r+n)}{n!}\left[\frac{r}{n+1}\right] \\
&= \frac{r(r+1)(r+2)\cdots(r+n)}{(n+1)!} = A_{n+1}^{(r-1)} .
\end{aligned}
$$

We are also told that

$$
\frac{S_{n+1}^{(r-1)}}{A_{n+1}^{(r-1)}} \to l .
$$

If I use the two earlier derivations I can write this as

$$
\frac{S_{n+1}^{(r)} - S_n^{(r)}}{A_{n+1}^{(r)} - A_n^{(r)}} \to l . \tag{121}
$$

Now to show the desired result we will use Theorem 62 (the discrete L'Hospital's Rule) with $a_n \equiv S_n^{(r)}$ and $b_n = A_n^{(r)}$. Equation 121 gives one of the conditions needed for Theorem 62.

The other is that $b_n = A_n^{(r)}$ is a sequence of positive numbers that increases steadily to positive infinity. From the definition of b_n given by Equation 120, we see that b_n is a sequence of positive numbers. To show that b_n increases to positive infinity write it as

$$
\begin{aligned}
b_n = A_n^{(r)} &= \frac{(r+1)(r+2)\cdots(r+n-1)(r+n)}{n!} \\
&= \left[\left(\frac{r+1}{1}\right)\left(\frac{r+2}{2}\right)\cdots\left(\frac{r+n-1}{n-1}\right)\right]\left(\frac{r+n}{n}\right) = A_{n-1}^{(r)} \times \left(\frac{r+n}{n}\right) \\
&= b_{n-1} \times \left(1 + \frac{r}{n}\right) > b_{n-1} \,.
\end{aligned}
$$

An application of Theorem 62 then shows that

$$
\frac{S_n^{(r)}}{A_n^{(r)}} \to l \,,
$$

also.

Example 2

Part (i): For this limit to use the "discrete L'Hospital's rule" we let

$$
a_n \equiv s_n + 2s_{n-1} + \cdots + ns_1 \,,
$$

and $b_n \equiv n^2$. Note that $b_n > 0$ and $b_n \to \infty$ as $n \to \infty$.

For a_n our first difference is given by

$$
\begin{aligned}
a_{n+1} - a_n &= s_{n+1} + 2s_n + 3s_{n-1} + \cdots + ns_2 + (n+1)s_1 \\
&\quad - s_n - 2s_{n-1} - \cdots - (n-1)s_2 - ns_1 \\
&= s_{n+1} + s_n + s_{n-1} + \cdots + s_2 + s_1 \,,
\end{aligned}
$$

and for b_n our first difference is given by

$$
b_{n+1} - b_n = (n+1)^2 - n^2 = 2n + 1 \,.
$$

The ratio needed for the discrete L'Hospital's rule look like

$$
\frac{2(a_{n+1} - a_n)}{b_{n+1} - b_n} = \frac{2(s_{n+1} + s_n + \cdots + s_2 + s_1)}{2n+1} = \frac{s_{n+1} + s_n + \cdots + s_2 + s_1}{n + \frac{1}{2}}
$$

As $s_n \to s$ the above ratio converges to s also (another application of the discrete L'Hospital's rule will also prove that statement). Thus by the discrete L'Hospital's rule for sequences we can conclude that

$$
\frac{2a_n}{b_n} \to s \,,
$$

as $n \to \infty$.

Part (ii): For this limit to use the "discrete L'Hospital's rule" we let

$$a_n \equiv \sum_{k=1}^{n} s_k \sum_{l=k}^{n} \frac{1}{l}\,.$$

and $b_n = n$. Note that $b_n > 0$ and $b_n \to \infty$ as $n \to \infty$. For this definition of a_n we have

$$a_{n+1} = \sum_{k=1}^{n+1} s_k \left(\sum_{l=k}^{n+1} \frac{1}{l} \right),$$

and thus the first difference is given by

$$\begin{aligned}
a_{n+1} - a_n &= \sum_{k=1}^{n+1} s_k \left(\sum_{l=k}^{n+1} \frac{1}{l} \right) - \sum_{k=1}^{n} s_k \left(\sum_{l=k}^{n} \frac{1}{l} \right) \\
&= s_{n+1} \left(\sum_{l=n+1}^{n+1} \frac{1}{l} \right) + \sum_{k=1}^{n} s_k \left(\sum_{l=k}^{n+1} \frac{1}{l} - \sum_{l=k}^{n} \frac{1}{l} \right) \\
&= \frac{s_{n+1}}{n+1} + \sum_{k=1}^{n} \frac{s_k}{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} s_k\,.
\end{aligned}$$

The first difference of b_n is $b_{n+1} - b_n = 1$. Thus we need to consider the limit as $n \to \infty$ of

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{1}{n+1} \sum_{k=1}^{n+1} s_k\,.$$

as discussed in the book the above limit tends to s as $n \to \infty$ (another application of the discrete L'Hospital's rule will also prove this). Thus by the discrete L'Hospital's rule for sequences we can conclude that

$$\frac{a_n}{b_n} \to s\,,$$

as $n \to \infty$.

Example 3

From the description of $P(n)$ I believe we can write it as

$$P(n) = \sum_{1 \le i \le n; 1 \le j \le n}^{n} i^p j^p\,.$$

We can modify how we write this if we note that

$$\begin{aligned}
(1^p + 2^p + 3^p + \cdots n^p)(1^p + 2^p + 3^p + \cdots n^p) &= 1^p \cdot 1^p + 1^p \cdot 2^p + 1^p \cdot 3^p + \cdots + 1^p \cdot n^p \\
&+ 2^p \cdot 1^p + 2^p \cdot 2^p + 2^p \cdot 3^p + \cdots + 2^p \cdot n^p \\
&\cdots \\
&+ n^p \cdot 1^p + n^p \cdot 2^p + \cdots + n^p \cdot n^p\,.
\end{aligned}$$

Thus based on this we have

$$P(n) = \left(\sum_{k=1}^{n} k^p\right)^2 = (1^p + 2^p + 3^p + \cdots + (n-1)^p + n^p)^2 \,.$$

To use the discrete L'Hospital's rule for sequences we will take

$$\begin{aligned} a_n &\equiv P(n) \quad \text{and} \\ b_n &\equiv (n+1)^{2p+2} \,. \end{aligned}$$

Note that $b_n > 0$ and $b_n \to \infty$ as $n \to \infty$.

For the sequence $P(n)$ we have

$$\begin{aligned} P(n+1) - P(n) &= \left(\sum_{k=1}^{n+1} k^p\right)^2 - \left(\sum_{k=1}^{n} k^p\right)^2 \\ &= \left(\sum_{k=1}^{n+1} k^p - \sum_{k=1}^{n} k^p\right)\left(\sum_{k=1}^{n+1} k^p + \sum_{k=1}^{n} k^p\right) \\ &= (n+1)^p \left((n+1)^p + 2\sum_{k=1}^{n} k^p\right) \,. \end{aligned}$$

Now to evaluate $\sum_{k=1}^{n} k^p$ in the above expression we will use the fact that for large n we have

$$\sum_{k=1}^{n} k^p \sim \frac{(n+1)^{p+1}}{p+1} \,.$$

This means that

$$\begin{aligned} P(n+1) - P(n) &\sim (n+1)^p \left[(n+1)^p + 2\frac{(n+1)^{p+1}}{p+1}\right] \\ &= (n+1)^{2p} + \frac{2}{p+1}(n+1)^{2p+1} \,. \end{aligned}$$

For large n this tends to

$$P(n+1) - P(n) \to \frac{2}{p+1}(n+1)^{2p+1} \,.$$

For b_n it can be shown that

$$b_{n+1} - b_n = \Delta(n+1)^{2p+2} = (2p+2)(n+1)^{2p+1} \,,$$

for large n. This is related to a similar relationship for derivatives i.e. $\frac{d}{dx}x^r = rx^{r-1}$.

Using these we have

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \to \frac{\frac{2}{p+1}(n+1)^{2p+1}}{(2p+2)(n+1)^{2p+1}} = \frac{1}{(p+1)^2} \,.$$

Note that the answer above is different than the one in the book by a factor of $\frac{1}{2}$. If anyone sees anything wrong with what I have done please contact me.

Example 4

The sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ of this series takes the form

$$s_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} .$$

To compute the $(C, 1)$ sum we need to compute

$$\begin{aligned} t_n &= \frac{1}{n} \sum_{k=1}^{n} s_k = \frac{1}{n} (\text{number of odd numbers between 1 and } n) \\ &= \frac{1}{n} \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n+1}{2} & n \text{ odd} \end{cases} = \begin{cases} \frac{1}{2} & n \text{ even} \\ \frac{1}{2} + \frac{1}{2n} & n \text{ odd} \end{cases} . \end{aligned}$$

The limit of this sequence is $\frac{1}{2}$ as $n \to \infty$.

Example 5

In this example the book gives a proof of this statement when $a = b = 0$.

This statement can also be proved when $a = 0$ and $b \neq 0$ and we will give a sketch of this proof here. As a_n and b_n are convergence sequences, they are bounded so there exists an A and B such that $|a_n| < A$ and $|b_n| < B$ for all n. As a_n converges to zero i.e. $a_n \to 0$ if we are given a value of $\epsilon > 0$ (and $k > 0$) we can find a N such that

$$|a_n| < k\epsilon ,$$

for all $n \geq N$. Define the expression we seek to take the limit of as e_n so that

$$e_n \equiv \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) = \frac{1}{n} \sum_{j=1}^{n} a_j b_{n+1-j} .$$

Note that

$$|e_n| \leq \frac{1}{n} \sum_{j=1}^{n} |a_j b_{n+1-j}| < \frac{B}{n} \sum_{j=1}^{n} |a_j| .$$

Taking $n > N$ and breaking the above sum up into two parts we have

$$\begin{aligned} |e_n| &\leq \frac{B}{N} \left(\sum_{j=1}^{N} |a_j| + \sum_{j=N+1}^{n} |a_j| \right) \\ &< \frac{B}{n} \left(AN + k\epsilon(n - N) \right) \\ &= \frac{(A - k\epsilon)BN}{n} + Bk\epsilon . \end{aligned}$$

By making n large we can make the first term as small as we like and the second term can be made as small as we like by selecting a specific value for k. Thus we can make the right-hand-side of this expression as small as we like (when n is large enough). This is the statement that $|e_n| \to 0$ as $n \to \infty$ and thus $e_n \to 0$.

Note that the arguments used above, to prove that $\lim_{n\to\infty} e_n = 0$ when $a = 0$ and $b \neq 0$ are symmetric and the above proof works to prove this statement when $a \neq 0$ and $b = 0$ as well.

Next we will consider the case where both $a \neq 0$ and $b \neq 0$. To do that, lets write the expression for e_n as

$$\begin{aligned} e_n &= \frac{1}{n}(a_1 b_n + a_2 b_{n-1} + \cdots a_n b_1 - nab) \\ &= \frac{1}{n}((a_1 b_n - ab) + (a_2 b_{n-1} - ab) + \cdots + (a_n b_1 - ab)) \\ &= \frac{1}{n}\left(\sum_{j=1}^{n}(a_j b_{n+1-j} - ab)\right) . \end{aligned}$$

Note that each term in the above sum is trivially bounded as

$$|a_j b_{n+1-j} - ab| \leq |a_j b_{n+1-j}| + |ab| \leq AB + |ab| \equiv K .$$

Here K is an upper bound on each of these terms.

Now as in the notes with this example if $n > 2N$ then one of the subscripts j or $n+1-j$ must be larger than N when $n > 2N$. Thus lets consider the value of e_n when $n > 2N$ when broken down into two sums as

$$e_n = \frac{1}{n}\left(\sum_{j=1}^{2N}(a_j b_{n+1-j} - ab) + \sum_{j=2N+1}^{n}(a_j b_{n+1-j} - ab)\right) ,$$

so that we can bound $|e_n|$ as

$$|e_n| \leq \frac{1}{n}(2NK) + \frac{1}{n}\sum_{j=2N+1}^{n}|a_j b_{n+1-j} - ab| .$$

At this point we have not specified what the value N should be. Note that because a_n and b_n converge to a and b respectively if we are given a value of $\epsilon > 0$ we can find values N_a and N_b such that

$$|a_n - a| < k_a\epsilon \quad \text{and} \quad |b_n - b| < k_b\epsilon ,$$

when $n \geq N_a$ and $n \geq N_b$ for any $k_a > 0$ and $k_b > 0$. To use this we will break the difference above into two different differences that we can bound. We do that with

$$\begin{aligned} |a_j b_{n+1-j} - ab| &= |a_j b_{n+1-j} - a_j b + a_j b - ab| \\ &= |a_j(b_{n+1-j} - b) + b(a_j - a)| \\ &\leq A|b_{n+1-j} - b| + B|a_j - a| \\ &= Ak_a\epsilon + Bk_b\epsilon = (Ak_a + Bk_b)\epsilon . \end{aligned}$$

This is valid if $n > 2N$ where $N > \max(N_a, N_b)$. If we use this we can find an upper bound on $|e_n|$ as

$$
\begin{aligned}
|e_n| &\leq \frac{2NK}{n} + \frac{1}{n}(n - 2N)(Ak_a + Bk_b)\epsilon \\
&= \frac{2NK}{n} + \left(\frac{n - 2N}{n}\right)(Ak_a + Bk_b)\epsilon \\
&= 2N(K - (Ak_a + Bk_b)\epsilon)\frac{1}{n} + (Ak_a + Bk_b)\epsilon \,.
\end{aligned}
$$

By making n large we can make the first term as small as we like and the second term can be made as small as we like by selecting specific values for k_a and k_b. Thus we can make the right-hand-side of this expression as small as we like (when n is large enough). This is the statement that $|e_n| \to 0$ as $n \to \infty$. Thus $e_n \to 0$.

Example 6

Define

$$
\begin{aligned}
a_n &\equiv \sum_{k=1}^{n} d_k \\
b_n &\equiv \sum_{k=1}^{n} c_k \,,
\end{aligned}
$$

Note that $b_n > 0$ and b_n tends to infinity as n does. The first difference of these two sequences is given by

$$
\begin{aligned}
a_{n+1} - a_n &= d_{n+1} \\
b_{n+1} - b_n &= c_{n+1} \,.
\end{aligned}
$$

Thus the ratio of the differences equals

$$
\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{d_{n+1}}{c_{n+1}} \,.
$$

This tends to s as $n \to \infty$. By the discrete L'Hospital's rule for sequences the limit of $\frac{a_n}{b_n}$ is also s.

Chapter 20 (Fourier Series)

Notes on The Proof of Theorem 65

Recalling Equation 33 if we start the sum at $r = 1$ and "expand" the right-hand-side we get

$$1 + \sum_{r=1}^{n} \cos(r\theta) = \frac{\sin\left(\theta\left(n + \frac{1}{2}\right)\right)}{2\sin(\theta/2)} + \frac{1}{2}\,,$$

which simplifies to a result used in this proof.

As a second lemma we will derive the sine sum needed in the evaluation of σ_n which is

$$\sum_{k=1}^{n} \sin\left(\left(k - \frac{1}{2}\right)x\right) = \sum_{k=1}^{n} \sin\left(\frac{2k-1}{2}x\right)\,. \tag{122}$$

To evaluate this sum note that because Euler's identity of

$$e^{ix} = \cos(x) + i\sin(x)\,,$$

we note that sums of $\sin(x)$ can be expressed as the **imaginary** part of sums of terms e^{ix}. Thus we look to evaluate

$$\sum_{k=1}^{n} e^{i\left(k-\frac{1}{2}\right)x} = e^{-\frac{x}{2}i} \sum_{k=1}^{n} e^{ikx}\,.$$

For the sum above we have

$$\begin{aligned}
\sum_{k=1}^{n} e^{ikx} &= \frac{e^{ix} - e^{i(n+1)x}}{1 - e^{ix}} \\
&= \frac{(e^{ix} - e^{i(n+1)x})(1 + e^{-ix})}{(1 - e^{ix})(1 + e^{-ix})} = \frac{e^{ix} - e^{i(n+1)x} + 1 - e^{inx}}{1 - e^{ix} + e^{-ix} - 1} \\
&= \frac{e^{ix} - e^{i(n+1)x} + 1 - e^{inx}}{-(e^{ix} - e^{-ix})} = \frac{e^{ix} - e^{i(n+1)x} + 1 - e^{inx}}{-2i\sin(x)}\,.
\end{aligned}$$

If we multiply this by $e^{-\frac{x}{2}i}$ we get

$$\frac{i}{2\sin(x)}\left(e^{\frac{x}{2}i} - e^{i\left(n+\frac{1}{2}\right)x} + e^{-\frac{x}{2}i} - e^{i\left(n-\frac{1}{2}\right)x}\right)\,.$$

The imaginary part of this is

$$\frac{1}{2\sin(x)}\left(\cos\left(\frac{x}{2}\right) - \cos\left(\left(n + \frac{1}{2}\right)x\right) + \cos\left(\frac{x}{2}\right) - \cos\left(\left(n - \frac{1}{2}\right)x\right)\right)\,.$$

Lets expand the $\cos\left(\left(n \pm \frac{1}{2}\right)x\right)$ terms to get

$$\frac{1}{2\sin(x)}\left(2\cos\left(\frac{x}{2}\right) - \cos(nx)\cos\left(\frac{x}{2}\right) + \sin(nx)\sin\left(\frac{x}{2}\right) - \cos(nx)\cos\left(\frac{x}{2}\right) - \sin(nx)\sin\left(\frac{x}{2}\right)\right)\,,$$

or

$$\frac{\cos\left(\frac{x}{2}\right)}{\sin(x)}\left(1-\cos(nx)\right) .$$

Using $\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ the above is

$$\frac{1}{2\sin\left(\frac{x}{2}\right)}(1-\cos(nx)) .$$

Using

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= \cos^2(x) - 1 + \cos^2(x) = 2\cos^2(x) - 1 .\end{aligned}$$

This means that

$$1-\cos(2x) = 2 - 2\cos^2(x) = 2(1-\cos^2(x)) = 2\sin^2(x) ,$$

so the above can be written as

$$\frac{\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)} ,$$

which is the argument of the integrand in the expression for σ_n.

Examples XXV

Exercise 1

If we look at the example in this chapter under "intervals other than $(-\pi, \pi)$ we compute the Fourier coefficients of x under the interval $(0, 2\pi)$ where one derives

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} , \tag{123}$$

which is a transformation of the desired result.

Exercise 2

Warning: I was not able to finish this problem. If anyone has any insight as for ways to proceed please let me know.

Using the results of Section 5.2 with $a = -\alpha$ and $b = 2\pi - \alpha$ we have $b - a = 2\pi$. Then the formula for a_k is given by

$$\begin{aligned}\pi a_k &= \frac{2\pi}{2\pi}\int_{-\alpha}^{2\pi-\alpha} f(\theta)\cos\left(\frac{\pi k(2\theta+\alpha-2\pi+\alpha)}{2\pi}\right)d\theta = \int_{-\alpha}^{2\pi-\alpha} f(\theta)\cos\left(\frac{k}{2}(2\theta-2\pi+2\alpha)\right)d\theta \\ &= \int_{-\alpha}^{2\pi-\alpha} f(\theta)\cos(k\theta-k\pi+k\alpha)d\theta = \cos(\pi k)\int_{-\alpha}^{2\pi-\alpha} f(\theta)\cos(k(\theta+\alpha))d\theta .\end{aligned}$$

Using this we have

$$\frac{\pi a_k}{\cos(\pi k)} = \frac{1}{2}(\pi - \alpha)\int_{-\alpha}^{\alpha} \theta \cos(k(\theta+\alpha))d\theta + \frac{1}{2}\alpha \int_{\alpha}^{2\pi-\alpha} (\pi - \theta)\cos(k(\theta+\alpha))d\theta\,.$$

In both of these we will let $v = \theta + \alpha$ to get

$$\frac{\pi a_k}{\cos(\pi k)} = \frac{1}{2}(\pi - \alpha)\int_{0}^{2\alpha} (v-\alpha) \cos(kv)dv + \frac{1}{2}\alpha \int_{2\alpha}^{2\pi} (\pi - (v-\alpha))\cos(kv)dv\,.$$

Integrating this I was not able to get an expression that looked like that given in the book.

Exercise 3

We extend x evenly so that our function $f(x)$ would be defined as

$$f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

This means that $b_n = 0$ for $n \geq 1$ and that

$$\pi a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)dx = 2\int_0^{\pi} x\cos(nx)dx\,.$$

for $n \geq 0$. If $n = 0$ this is

$$\pi a_0 = 2\int_0^{\pi} x dx = x^2\Big|_0^{\pi} = \pi^2 \quad \text{so} \quad a_0 = \pi\,.$$

If $n \geq 1$ then using integration by parts this is

$$\begin{aligned} \pi a_n &= 2\left(\frac{x\sin(nx)}{n}\bigg|_0^{\pi} - 2\int_0^{\pi} \frac{\sin(nx)}{n} dx \right. \\ &= 0 + \frac{2}{n}\left(\frac{\cos(nx)}{n}\bigg|_0^{\pi} = \frac{2}{n^2}((-1)^n - 1)\,. \right. \end{aligned}$$

This means that when n is even $a_n = 0$ and when n is odd we have

$$a_n = -\frac{4}{\pi n^2}\,.$$

This means that our Fourier series for $f(x)$ is given by

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1,3,5,\ldots} a_n \cos(nx) \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2} \\ &= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos(x) + \frac{1}{3^2}\cos(3x) + \frac{1}{5^2}\cos(5x) + \cdots\right)\,. \end{aligned}$$

Exercise 4

For an even function over $(-\pi, \pi)$ the Fourier cosign series takes the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,,$$

with (when $f(x) = x^2$)

$$\pi a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)dx = 2\int_0^{\pi} x^2 \cos(nx)dx\,.$$

Integrating once (when $n \neq 0$) we get

$$\begin{aligned}\frac{\pi}{2} a_n &= \left.\frac{x^2 \sin(nx)}{n}\right|_0^{\pi} - 2\int_0^{\pi} \frac{x\sin(nx)}{n} dx \\ &= -\frac{2}{n}\int_0^{\pi} x\sin(nx)dx\,.\end{aligned}$$

Thus integrating a second time

$$\begin{aligned}-\frac{\pi n}{4} a_n &= -\left.\frac{x\cos(nx)}{n}\right|_0^{\pi} + \int_0^{\pi} \frac{\cos(nx)}{n} dx \\ &= -\frac{\pi}{n}\cos(n\pi) + \frac{1}{n}\left(\left.\frac{\sin(nx)}{n}\right|_0^{\pi}\right. = -\frac{\pi}{n}\cos(n\pi)\,.\end{aligned}$$

This means that

$$a_n = \frac{4}{n^2}(-1)^n\,.$$

For $n = 0$ we have

$$\pi a_0 = 2\int_0^{\pi} x^2 dx = \frac{2}{3}\left. x^3\right|_0^{\pi} = \frac{2}{3}\pi^3 \quad \text{so} \quad a_0 = \frac{2\pi^2}{3}\,.$$

Using all of these we find

$$x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos(nx)\,. \tag{124}$$

Exercise 5

Warning: I was not able to finish this problem. If anyone has any insight as for ways to proceed please let me know.

We consider the odd extension of this function from $(0, \pi)$ to $(-\pi, 0)$ then $a_n = 0$ and

$$\pi b_n = 2\int_0^{\pi} f(x)\sin(nx)dx\,.$$

Given the definition of $f(x)$ we can evaluate the integral above. We find

$$\begin{aligned}
\frac{\pi}{2} b_n &= \int_0^{\pi/3} \frac{\pi}{3} \sin(nx)dx + \int_{\pi/3}^{2\pi/3} 0 \sin(nx)dx - \int_{2\pi/3}^{\pi} \frac{\pi}{3} \sin(nx)dx \\
&= -\frac{\pi}{3} \left(\frac{\cos(nx)}{n} \right|_0^{\pi/3} + \frac{\pi}{3} \left(\frac{\cos(nx)}{n} \right|_{2\pi/3}^{\pi} \\
&= -\frac{\pi}{3n} \left(\cos\left(\frac{n\pi}{3}\right) - 1 \right) + \frac{\pi}{3n} \left((-1)^n - \cos\left(\frac{2\pi}{3} n\right) \right) ,
\end{aligned}$$

which seems different from the expression given in the book.

Exercise 6

We have computed the Fourier expansion of x in Equation 123 and the Fourier expansion of x^2 in Equation 124 adding these two we get the Fourier expansion of $x + x^2$.

Exercise 7

We have computed the Fourier expansion of x^2 in Equation 124. We can subtract this expression from π^2 to get the Fourier expansion of $\pi^2 - x^2$.

Miscellaneous Examples

Exercise 1

To study convergence of the infinite series we will use d'Alembert's test. We first need to compute the ratio $\frac{|n^k z^n|}{|(n+1)^k z^{n+1}|}$ where we find

$$\frac{|n^k z^n|}{|(n+1)^k z^{n+1}|} = \frac{1}{|z|}\frac{n^k}{(n+1)^k}\,.$$

Thus the limit of this fraction as $n \to \infty$ is the value $\frac{1}{|x|}$. By d'Alembert's test our sum will converge if $\frac{1}{|x|} > 1$ which happens if $|x| < 1$. This test also tells us that the sum will diverge if $|x| > 1$. If $x = \pm 1$ then the terms of the series don't limit to zero as $n \to \infty$ and thus the sum also diverges.

Next we define

$$F_k(z) \equiv \sum_{n=1}^{\infty} n^k z^n\,.$$

As discussed above $F_k(z)$ is absolutely convergent on $|z| < 1$. Thus we can take the derivative of $F_k(x)$ term-by-term and find

$$F_k'(z) = \sum_{n=1}^{\infty} n^{k+1} z^{n-1}\,,$$

so multiplying both sides by z we get

$$zF_k'(z) = \sum_{n=1}^{\infty} n^{k+1} z^n\,. \tag{125}$$

The right-hand-side of this is equal to the definition of $F_{k+1}(z)$.

If we take $k = 1$ we find that

$$F_1(z) = \sum_{n=1}^{\infty} nz^n\,.$$

This sum can be evaluated explicitly by taking the derivative of the expression $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ or we can "look it up" to find that

$$F_1(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}\,.$$

Lets write this as

$$-\left(\frac{-z}{(1-z)^2}\right) = -\left(\frac{1-z-1}{(1-z)^2}\right) = -\left(\frac{1}{1-z} - \frac{1}{(1-z)^2}\right) = \frac{1}{(1-z)^2} - \frac{1}{1-z}\,,$$

which is of the desired summation form for $F_1(z)$ of

$$\sum_{r=0}^{1} \frac{(-1)^r A_r}{(1-z)^{2-r}}\,,$$

with $A_0 = 1$ and $A_1 = 1$ (both positive constants). To use induction we now assume that $F_k(z)$ is of this form for all $k \geq K$ that is

$$F_k(z) = \sum_{r=0}^{k} \frac{(-1)^r A_r}{(1-z)^{k-r+1}}\,,$$

for all $1 \leq k \leq K$ and consider $F_{K+1}(z)$ via Equation 125 (for notational simplicity we take $K \to k$). We find

$$\begin{aligned}
F_{k+1}(z) &= z\frac{d}{dz}\sum_{r=0}^{k} \frac{(-1)^r A_r}{(1-z)^{k-r+1}} \\
&= z\sum_{r=0}^{k} \frac{(-1)^r(-1)(k-r+1)A_r(-1)}{(1-z)^{k-r+2}} = z\sum_{r=0}^{k} \frac{(-1)^r(k-r+1)A_r}{(1-z)^{k-r+2}} \\
&= (-z)\sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k-r+2}} = (1-z-1)\sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k-r+2}} \\
&= \sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k-r+1}} - \sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k-r+2}}\,.
\end{aligned}$$

We "increment" k in the denominator of fractions in the first sum above to write $F_{k+1}(z)$ as

$$F_{k+1}(z) = \sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k+1-(r+1)+1}} - \sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k+1-r+1}}\,.$$

We next adjust the summation index in the first sum up by one to get

$$F_{k+1}(z) = \sum_{r=1}^{k+1} \frac{(-1)^{r}(k-r+2)A_{r-1}}{(1-z)^{k+1-r+1}} - \sum_{r=0}^{k} \frac{(-1)^{r+1}(k-r+1)A_r}{(1-z)^{k+1-r+1}}\,.$$

Release the first term in the first sum to get

$$F_{k+1}(z) = \frac{(k+1)A_0}{(1-z)^{k+1+1}} + \sum_{r=1}^{k+1} \frac{(-1)^r}{(1-z)^{k+1-r+1}}[(k-r+2)A_{r-1} + (k-r+1)A_r]\,.$$

If we introduce

$$\begin{aligned}
\tilde{A}_0 &= (k+1)A_0 \\
\tilde{A}_r &= (k-r+2)A_{r-1} + (k-r+1)A_r \quad \text{for} \quad 1 \leq r \leq k+1\,,
\end{aligned}$$

Notice that the above sum is equal to

$$F_{k+1}(z) = \sum_{r=0}^{k+1} \frac{(-1)^r \tilde{A}_r}{(1-z)^{k+1-r+1}}\,,$$

with $\tilde{A}_r > 0$ and proving the summation expression for $F_k(z)$.

Exercise 3

Assume that $\lim_{n\to\infty} a_n = L \neq 0$. Then comparing the terms $\frac{a_n}{1+n^2 a_n}$ to the convergent series $\frac{1}{n^2}$ we have

$$\lim_{n\to\infty} \left(\frac{\frac{1}{n^2}}{\frac{a_n}{1+n^2 a_n}} \right) = \lim_{n\to\infty} \frac{1}{n^2} \left(\frac{1+n^2 a_n}{a_n} \right)$$
$$= \lim_{n\to\infty} \frac{\frac{1}{n^2} + a_n}{a_n} = \frac{0+L}{L} = 1\,,$$

and thus by the comparison test in limiting form we have that the series with terms $\frac{a_n}{1+n^2 a_n}$ converges also. If $\lim_{n\to\infty} a_n = \infty$ we can write the above limit as

$$\lim_{n\to\infty} \frac{\frac{1}{n^2 a_n} + 1}{1} = 1\,,$$

and again have shown convergence of $\sum \frac{a_n}{1+n^2 a_n}$.

If $\sum a_n$ is divergent then and $\lim_{n\to\infty} a_n = L \neq 0$ then

$$\lim_{n\to\infty} \frac{a_n}{1+a_n} = \frac{L}{1+L} \neq 0\,,$$

and the series $\sum \frac{a_n}{1+a_n}$ diverges.

Warning: I was not able to show the desired result when $\lim_{n\to\infty} a_n = 0$. If anyone sees how to do this please contact me.

Exercise 6

Part of this is worked in Exercise 10 in Examples XVI (uniform convergence) on Page 94.

Exercise 7

Notice that both of these sums are absolutely convergence by the comparison test with a series with terms $\frac{1}{n^3}$ and thus the sums can be evaluated in any order.

Part (i): Note that

$$\frac{1}{9n^3 - n} = \frac{1}{n(9n^2-1)} = \frac{1}{n(3n-1)(3n+1)}\,.$$

From "partial fractions" we can write this as

$$\frac{1}{n(3n-1)(3n+1)} = \frac{A}{n} + \frac{B}{3n-1} + \frac{C}{3n+1}\,,$$

with $A = -1$, $B = \frac{3}{2}$, and $C = \frac{3}{2}$ so that

$$\frac{1}{n(3n-1)(3n+1)} = -\frac{1}{n} + \frac{3}{2(3n-1)} + \frac{3}{2(3n+1)}\,.$$

Based on this lets define the partial sum of n terms s_n as

$$s_n = \sum_{k=1}^{n} a_k = -\sum_{k=1}^{n} \frac{1}{k} + \frac{3}{2}\sum_{k=1}^{n} \frac{1}{3k-1} + \frac{3}{2}\sum_{k=1}^{n} \frac{1}{3k+1}\,.$$

We will now work each of these individual sums into harmonic sums of the form

$$\sum_{k=1}^{n} \frac{1}{k} = \log(n) + \gamma_n\,, \tag{126}$$

and then evaluate them by taking the limit as $n \to \infty$. To do this we will write s_n as

$$\begin{aligned}
s_n &= -\sum_{k=1}^{n} \frac{1}{k} + \frac{3}{2}\left[\sum_{k=1}^{n} \frac{1}{3k-1} + \sum_{k=1}^{n} \frac{1}{3k} + \sum_{k=1}^{n} \frac{1}{3k+1}\right] - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} \\
&= -\sum_{k=1}^{n} \frac{1}{k} + \frac{3}{2}\sum_{k=2}^{3n+1} \frac{1}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k} = -\frac{3}{2}\sum_{k=1}^{n} \frac{1}{k} + \frac{3}{2}\left[\sum_{k=1}^{3k+1} \frac{1}{k} - 1\right].
\end{aligned}$$

Changing these sums using Equation 126 we get

$$\begin{aligned}
s_n &= -\frac{3}{2}(\log(n) + \gamma_n) + \frac{3}{2}(\log(3n+1) + \gamma_{3n+1} - 1) \\
&= \frac{3}{2}\left(\log\left(\frac{3n+1}{n}\right)\right) - \frac{3}{2}\gamma_n + \frac{3}{2}\gamma_{3n+1} - \frac{3}{2}\,.
\end{aligned}$$

Taking the limit $n \to \infty$ gives

$$s_n \to \frac{3}{2}\ln(3) - \frac{3}{2} = \frac{3}{2}(\ln(3) - 1)\,.$$

In the `R` code `misc_examples_viii_xxv_exercise_7.R` we numerically evaluate the given summation and "verify" graphically that the summation limits to the above number.

Part (ii): Note that using "partial fractions" we can write this as

$$\frac{1}{(n+1)(2n+1)} = \frac{A}{n+1} + \frac{B}{2n+1}\,,$$

with $A = -1$, and $B = 2$ so that

$$\frac{1}{(n+1)(2n+1)} = -\frac{1}{n+1} + \frac{2}{2n+1}\,.$$

Evaluating the sequence of partial sums s_n we find

$$\begin{aligned} s_n &= -\sum_{k=1}^{n} \frac{1}{k+1} + 2\sum_{k=1}^{n} \frac{1}{2k+1} \\ &= -\sum_{k=1}^{n} \frac{1}{k+1} + 2\left[\sum_{k=1}^{n} \frac{1}{2k+1} + \sum_{k=1}^{n} \frac{1}{2k}\right] - 2\sum_{k=1}^{n} \frac{1}{2k} \\ &= -\sum_{k=2}^{n+1} \frac{1}{k} + 2\left[\sum_{k=1}^{2n+1} \frac{1}{k} - 1\right] - \sum_{k=1}^{n} \frac{1}{k} \\ &= -1 + 2\sum_{k=1}^{2n+1} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{k} - \frac{1}{n+1}\,. \end{aligned}$$

Changing these sums using Equation 126 we get

$$\begin{aligned} s_n &= -1 + 2(\ln(2n+1) + \gamma_{2n+1}) - 2(\ln(n) + \gamma_n) - \frac{1}{n+1} \\ &= -1 + 2\ln\left(\frac{2n+1}{n}\right) + 2\gamma_{2n+1} - 2\gamma_n - \frac{1}{n+1}\,. \end{aligned}$$

Taking the limit $n \to \infty$ gives

$$s_n \to 2\ln(2) - 1\,.$$

In the R code `misc_examples_viii_xxv_exercise_7.R` we numerically evaluate the given summation and "verify" graphically that the summation limits to the above number.

Exercise 9

Now if $|a_n\phi(n)|$ converges then $|n^k a_n|$ converges by Theorem 9 i.e. if $\frac{u_n}{v_n} \to L > 0$ then u_n and v_n either both converge or both diverge. Then if we take $u_n \equiv n^k a_k$ and assume that $u_n \geq u_{n+1}$ i.e. that u_n is monotonically decreasing we can use Pringsheim's theorem directly to show that

$$nu_n = n^{k+1} a_n \to 0\,.$$

Exercise 10

As we are told that $\sum a_n x^n$ is absolutely convergent when $|x| \leq R$ we know that

$$\lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \lim_{n\to\infty} \frac{|a_n x^n|}{|a_{n+1} x^{n+1}|} = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1} x|} = l\,,$$

with $l \geq 1$. If this was not true then our series would diverge. We thus have

$$\lim_{n\to\infty} \frac{|a_n x^n|}{n!} \times \frac{(n+1)!}{|a_{n+1} x^{n+1}|} = \lim_{n\to\infty} \frac{|a_n|(n+1)}{|a_{n+1}||x|} = \infty\,,$$

for all x. As this is larger than one this series converges for all x.

Exercise 11

From the given expression it looks like the terms of this series $\sum u_n$ can be written as

$$u_n = \frac{(2n)!}{(2 \cdot 4 \cdots 2n)^2} \left(\sum_{k=1}^{n} \frac{1}{k} \right) x^{2n} = \frac{(2n)!}{(2^n n!)^2} \left(\sum_{k=1}^{n} \frac{1}{k} \right) x^{2n} .$$

To evaluate the radius of convergence we will use d'Alembert's test. We consider

$$\begin{aligned}
\frac{|u_n|}{|u_{n+1}|} &= \frac{(2n)! \left(\sum_{k=1}^{n} \frac{1}{k} \right) |x|^{2n}}{(2^n n!)^2} \times \frac{(2^{n+1}(n+1)!)^2}{(2(n+1))! \left(\sum_{k=1}^{n+1} \frac{1}{k} \right) |x|^{2(n+1)}} \\
&= \left(\frac{(2n)!}{(2n+2)(2n+1)(2n!)} \right) \left(\frac{\sum_{k=1}^{n} \frac{1}{k}}{\sum_{k=1}^{n+1} \frac{1}{k}} \right) \left(\frac{2^{2n+2}(n+1)^2(n!)^2}{2^{2n}(n!)^2} \right) \left(\frac{1}{|x|^2} \right) \\
&= \left(\frac{1}{(2n+2)(2n+1)} \right) \left(\frac{\sum_{k=1}^{n} \frac{1}{k}}{\sum_{k=1}^{n+1} \frac{1}{k}} \right) \left(\frac{2^2(n+1)^2}{1} \right) \left(\frac{1}{|x|^2} \right) \\
&= \frac{4}{|x|^2} \left(\frac{n+1}{2n+2} \right) \left(\frac{n+1}{2n+1} \right) \left(\frac{\ln(n) + \gamma_n}{\ln(n+1) + \gamma_{n+1}} \right) .
\end{aligned}$$

This tends to

$$\frac{1}{|x|^2} ,$$

as $n \to \infty$. For convergence we need this limit to be larger than one or $|x| < 1$.

Exercise 12

From the observation that

$$(1 - re^{i\theta})(1 - re^{-i\theta}) = 1 - r(e^{i\theta} + e^{-i\theta}) + r^2 = 1 - 2r\cos(\theta) + r^2 ,$$

using partial fractions we can write

$$\frac{1}{1 - 2r\cos(\theta) + r^2} = \frac{1}{(1 - re^{i\theta})(1 - re^{-i\theta})} = \frac{A}{1 - re^{i\theta}} + \frac{B}{1 - re^{-i\theta}} .$$

Cross multiplying and setting the real and imaginary parts equal gives $A = B$ and

$$A = \frac{1}{2(1 - r\cos(\theta))} .$$

This means that we have shown that

$$\frac{1 - r\cos(\theta)}{1 - 2r\cos(\theta) + r^2} = \frac{1}{2} \left(\frac{1}{1 - re^{i\theta}} \right) + \frac{1}{2} \left(\frac{1}{1 - re^{-i\theta}} \right) .$$

Using geometric sums for the two expressions on the right-hand-side of the above gives

$$\begin{aligned}
\frac{1 - r\cos(\theta)}{1 - 2r\cos(\theta) + r^2} &= \frac{1}{2}\sum_{k=0}^{\infty} r^k e^{ik\theta} + \frac{1}{2}\sum_{k=0}^{\infty} r^k e^{-ik\theta} = \frac{1}{2}\sum_{k=0}^{\infty} r^k (e^{ik\theta} + e^{-ik\theta}) \\
&= \sum_{k=0}^{\infty} r^k \cos(k\theta) \\
&= 1 + r\cos(\theta) + r^2\cos(2\theta) + r^3\cos(3\theta) + \cdots
\end{aligned}$$

which is the desired expression.

References

[1] G. Almkvist and B. Berndt. Gauss, landen, ramanujan, the arithmetic-geometric mean, ellipses, and the ladies diary. *Am. Math. Monthly*, 95(7):585–608, Aug. 1988.

[2] S. Mahajan. *Street-fighting mathematics: the art of educated guessing and opportunistic problem solving.* The MIT Press, Cambridge, 2010.