Solution Manual for: Introduction To Stochastic Models by Roe Goodman.

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Introduction

This book represents an excellent choice for a student interested in learning about probability models. Similar to the book [3], but somewhat more elementary, this book is very well written and explains the most common applications of probability. The problems are quite enjoyable. This is an excellent choice for someone looking to extend their probability knowledge. These notes were written to help clarify my understanding of the material. It is hoped that others find these notes helpful. Please write me if you find any errors.

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Chapter 1: Sample Spaces

Exercise Solutions

Exercise 1 (sample spaces)

Part (a): The sample space for this experiment are pairs of integers (i, j) where the value of i is the result of the first die and j is the result of the second die. When we toss to die we get for the sample space

$$
\begin{array}{c}
(1,1)\,,(1,2)\,,(1,3)\,,(1,4)\,,(1,5)\,,(1,6) \\
(2,1)\,,(2,2)\,,(2,3)\,,(2,4)\,,(2,5)\,,(2,6) \\
(3,1)\,,(3,2)\,,(3,3)\,,(3,4)\,,(3,5)\,,(3,6) \\
(4,1)\,,(4,2)\,,(4,3)\,,(4,4)\,,(4,5)\,,(4,6) \\
(5,1)\,,(5,2)\,,(5,3)\,,(5,4)\,,(5,5)\,,(5,6) \\
(6,1)\,,(6,2)\,,(6,3)\,,(6,4)\,,(6,5)\,,(6,6)\,.
\end{array}
$$

Thus there are 36 possible outcomes in the sample space.

Part (b): The outcomes in the event E are given by

$$
\begin{array}{rcl}
E & = & (1,2)\,,(1,4)\,,(1,6) \\
 & & (2,1)\,,(2,2)\,,(2,3)\,,(2,4)\,,(2,5)\,,(2,6) \\
 & & (3,2)\,,(3,4)\,,(3,6) \\
 & & (4,1)\,,(4,2)\,,(4,3)\,,(4,4)\,,(4,5)\,,(4,6) \\
 & & (5,2)\,,(5,4)\,,(5,6) \\
 & & (6,1)\,,(6,2)\,,(6,3)\,,(6,4)\,,(6,5)\,,(6,6)\,.
\end{array}
$$

The outcomes in event F are given by

$$
\begin{array}{rcl}
F & = & (1,1)\,,(1,2)\,,(1,3)\,,(1,4)\,,(1,5)\,,(1,6) \\
 & & (2,1)\,,(2,3)\,,(2,5) \\
 & & (3,1)\,,(3,2)\,,(3,3)\,,(3,4)\,,(3,5)\,,(3,6) \\
 & & (4,1)\,,(4,3)\,,(4,5) \\
 & & (5,1)\,,(5,2)\,,(5,3)\,,(5,4)\,,(5,5)\,,(5,6) \\
 & & (6,1)\,,(6,3)\,,(6,5)\,.
\end{array}
$$

The outcomes in the event $E \cap F$ are

$$
\begin{array}{rcl}
E \cap F & = & (1,2)\,,(1,4)\,,(1,6) \\
 & & (2,1)\,,(2,3)\,,(2,5) \\
 & & (3,2)\,,(3,4)\,,(3,6) \\
 & & (4,1)\,,(4,3)\,,(4,5) \\
 & & (5,2)\,,(5,4)\,,(5,6) \\
 & & (6,1)\,,(6,3)\,,(6,5)\,.
\end{array}
$$

These are the outcomes that have at one even and one odd number on at least one roll. The event $E \cup F$ are all die rolls that have an even number or an odd number on at least one roll. This is all die rolls and thus is the entire sample space. The event E^c is given by

$$\begin{aligned} E^c &= (1,1), (1,3), (1,5) \\ &\quad (3,1), (3,3), (3,5) \\ &\quad (5,1), (5,3), (5,5), \end{aligned}$$

and is the event that no even number shows on either die roll. The event $E^c \cap F = E^c$ since E^c is a subset of the event F. The event $E^c \cup F = F$ again because E^c is a subset of F.

Exercise 2 (three events)

Part (a): This is the event $E \cup F \cup G$.

Part (b): This is the event $(E \cup F \cup G)^c = E^c \cap F^c \cap G^c$ by deMorgon's law.

Part (c): This is the event $(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G)$.

Part (d): This is the event

$$(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G) \cup (E^c \cap F^c \cap G^c).$$

Exercise 3 (proving deMorgon's law)

We want to prove

$$(A \cup B)^c = A^c \cap B^c. \tag{1}$$

We can prove this by showing that an element of the set on the left-hand-side is an element of the right-hand-side and vice versa. If x is an element of the left-hand-side then it is not in the set $A \cup B$. Thus it is not in A or in B. Thus it is in $A^c \cap B^c$. Similar arguments work to show the opposite direction.

Exercise 4 (indicator functions)

Part (a): The function $I_E I_F$ is one if and only if when an even from E and F has occurred. This is the definition of $E \cap F$.

Part (b): Two mutually exclusive events E and F means that if the event E occurs the event F cannot occur and vice versa. The function $I_E + I_F$ has the value of 1 if event E or F occurs. This is the definition of the set $E \cup F$. Since the events E and F cannot simultaneously occur both I_E and I_F cannot be one at the same time. Thus $I_E + I_F$ is the indicator function for $E \cup F$.

Part (c): The function $1 - I_E$ is one when the event E does not occur and is zero when the event E occurs. This is the definition of I_{E^c}.

Part (d): The indicator function of $E \cup F$ is 1 minus the indicator function for $(E \cup F)^c$ or

$$I_{E \cup F} = 1 - I_{(E \cup F)^c} \,.$$

By deMorgon's law we have

$$(E \cup F)^c = E^c \cap F^c \,,$$

thus the indicator function for the event $(E \cup F)^c$ is the product of the indicator functions for E^c and F^c thus we have

$$I_{E \cup F} = 1 - I_{E^c} I_{F^c} = 1 - (1 - I_E)(1 - I_F) \,.$$

We can multiply the product above to get

$$I_{E \cup F} = I_E + I_F - I_E I_F \,.$$

Chapter 2: Probability

Notes on the Text

Notes on the proof of the general inclusion-exclusion formula

By considering the union of the $n+1$ events as the union of n events with a single additional event E_{n+1} and then using the *two* set inclusion-exclusion formula we get

$$P\left(\cup_{i=1}^{n+1} E_i\right) = P\left(\cup_{i=1}^{n} E_i\right) + P(E_{n+1}) - P\left(\cup_{i=1}^{n} (E_i \cap E_{n+1})\right) . \tag{2}$$

The induction hypothesis applied to the two terms that have the union of n events. We find that the first term in the right-hand-side in Equation 2 is given by

$$\begin{aligned}
P\left(\cup_{i=1}^{n} E_i\right) &= \sum_{i=1}^{n} P(E_i) & (3)\\
&- \sum_{i_1<i_2}^{n} P(E_{i_1} \cap E_{i_2}) & (4)\\
&+ \sum_{i_1<i_2<i_3}^{n} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) & (5)\\
&- \sum_{i_1<i_2<i_3<i_4}^{n} P(E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap E_{i_4}) + & (6)\\
&\vdots\\
&+ (-1)^{n-1} P(E_1 \cap E_2 \cap \cdots \cap E_n) . & (7)
\end{aligned}$$

The third term in Equation 2 is given by

$$\begin{aligned}
P\left(\cup_{i=1}^{n} (E_i \cap E_{n+1})\right) &= \sum_{i_1}^{n} P(E_{i_1} \cap E_{n+1}) & (8)\\
&- \sum_{i_1<i_2}^{n} P(E_{i_1} \cap E_{i_2} \cap E_{n+1}) & (9)\\
&+ \sum_{i_1<i_2<i_3}^{n} P(E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap E_{n+1}) & (10)\\
&\vdots\\
&+ (-1)^{n-2} \sum_{i_1<i_2<\cdots<i_{n-1}}^{n} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}} \cap E_{n+1}) & (11)\\
&+ (-1)^{n-1} P(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1}) . & (12)
\end{aligned}$$

We then add together the right-hand-side of these two expressions (as specified via Equation 2) in a specific way that will proves the general induction step. We first add part 3 with

$P(E_{n+1})$ to get $\sum_{i=1}^{n+1} P(E_i)$, then add parts 4 and the negative of 8 to get

$$-\sum_{i_1<i_2}^{n+1} P(E_{i_1} \cap E_{i_2})\,.$$

Now add parts 5 and the negative of 9 to get

$$\sum_{i_1<i_2<i_3}^{n+1} P(E_{i_1} \cap E_{i_2} \cap E_{i_3})\,.$$

We keep going in this way until we get to the end, where we add parts 7 and the negative of 11 to get all intersections with n events. That is

$$\begin{aligned}
&(-1)^{n-1}P(E_1 \cap E_2 \cap \cdots \cap E_n) + (-1)^{n-1} \sum_{i_1<i_2<\cdots<i_{n-1}}^{n} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}} \cap E_{n+1}) \\
= \; & (-1)^{n-1} \sum_{i_1<i_2<\cdots<i_n}^{n+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}} \cap E_{i_n})\,.
\end{aligned}$$

We then take the negative last part as

$$(-1)^{n+1}P(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1})\,.$$

When we add all of these pieces together we get

$$\begin{aligned}
P\left(\cup_{i=1}^{n+1} E_i\right) &= \sum_{i=1}^{n+1} P(E_i) \\
&- \sum_{i_1<i_2}^{n+1} P(E_{i_1} \cap E_{i_2}) \\
&+ \sum_{i_1<i_2<i_3}^{n+1} P(E_{i_1} \cap E_{i_2} \cap E_{i_3}) \\
&\vdots \\
&- (-1)^n \sum_{i_1<i_2<\cdots<i_{n-1}}^{n+1} P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_{n-1}} \cap E_{i_n}) \\
&+ (-1)^{n+1}P(E_1 \cap E_2 \cap \cdots \cap E_n \cap E_{n+1})\,.
\end{aligned}$$

which shows the induction step.

Exercise Solutions

Exercise 1 (some hands of cards)

Part (a):

$$P(\text{Two Aces}) = \frac{\binom{4}{2}\binom{52-4}{3}}{\binom{52}{5}} = 0.03993\,.$$

Part (b):

$$P(\text{Two Aces and Three Kings}) = \frac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}} = 9.235 \cdot 10^{-6}\,.$$

Exercise 2

Part (a):

$$P(E) = \frac{\binom{4}{2}\binom{13}{2}\binom{13}{2}}{\binom{52}{4}} = 0.13484\,.$$

Here $\binom{4}{2}$ are the ways we can choose two suits to use for the suits and $\binom{13}{2}$ selects the cards to use in each of these suits.

Part (b):

$$P(E) = \frac{\binom{13}{2}\binom{52-13}{2}}{\binom{52}{4}} = 0.21349\,.$$

Exercise 3

Part (a): Since each ball is replaced on each draw we can get any of the numbers between 1 and n on each draw. Thus the sample space is ordered n-tuples where each number is in the range between 1 and n. This set has n^n elements in it.

Part (b): To have each ball drawn once we can do this in $n!$ ways, thus our probability is

$$\frac{n!}{n^n}\,.$$

Part (c): Since $n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ we can simplify the above probability as

$$\frac{n!}{n^n} \sim \frac{\sqrt{2\pi n}}{e^n}\,.$$

Exercise 5

Part (a):

$$P(E) = \frac{\binom{5}{2}\binom{15}{2}}{\binom{20}{4}} = 0.2167\,.$$

Part (b): The probability to get a single red ball in this case is $p = \frac{5}{20} = \frac{1}{4}$. To get two red balls (only) from the four draws will happen with probability

$$\binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 = 0.2109\,.$$

Exercise 6

Let E_r, E_b, and E_w be the event that the three drawn balls are red, blue, and white respectively. Then the even we want to compute the probability of $E = E_r \cup E_b \cup E_w$. Since each of these events is mutually exclusive we can compute $P(E)$ from

$$\begin{aligned}
P(E) &= P(E_r) + P(E_b) + P(E_w) \\
&= \frac{\binom{4}{3}}{\binom{15}{3}} + \frac{\binom{5}{3}}{\binom{15}{3}} + \frac{\binom{6}{3}}{\binom{15}{3}} = 0.0747\,.
\end{aligned}$$

Exercise 7

Part (a): Let G_1 be the event that the first drawn ball is green and G_2 be the event that the second drawn ball is green. Then the event we want to calculate the probability of is

G_1G_2. To compute this we have

$$P(G_1G_2) = P(G_2|G_1)P(G_1) = \left(\frac{1}{5}\right)\left(\frac{2}{6}\right) = \frac{1}{15} = 0.0666\,.$$

Part (b): To have no green balls at the end of our experiment means we must have picked a green ball twice in our three draws. This is the event E given by

$$E = G_1G_2G_3^c \cup G_1^cG_2G_3 \cup G_1G_2^cG_3\,.$$

Here G_3 is the event we draw a green ball on our third draw and G_1 and G_2 were defined earlier. Each of these events in the union is mutually exclusive and we can evaluate them by conditioning on the sequence of events. Thus we have

$$\begin{aligned}
P(E) &= P(G_1G_2G_3^c) + P(G_1^cG_2G_3) + P(G_1G_2^cG_3) \\
&= P(G_3^c|G_2G_1)P(G_2|G_1)P(G_1) + P(G_3|G_1^cG_2)P(G_2|G_1^c)P(G_1^c) + P(G_3|G_1G_2^c)P(G_2^c|G_1)P(G_1) \\
&= 1\left(\frac{1}{5}\right)\left(\frac{2}{6}\right) + \frac{1}{5}\left(\frac{2}{6}\right)\left(\frac{4}{6}\right) + \frac{1}{5}\left(\frac{4}{5}\right)\left(\frac{2}{6}\right) = 0.16444\,.
\end{aligned}$$

Exercise 8 (picking colored balls)

Part (a): We have

$$P(E_R) = \frac{\binom{2}{2}\binom{8-2}{2}}{\binom{8}{4}} = 0.21429$$

$$P(E_R \cap E_Y) = \frac{\binom{2}{2}\binom{2}{2}}{\binom{8}{4}} = 0.014286\,.$$

Part (b): The probability of the event E of interest is $1 - P(A)$ where A is the event that there is a ball of every different color. Thus we compute

$$P(E) = 1 - \frac{\binom{2}{1}\binom{2}{1}\binom{2}{1}\binom{2}{1}}{\binom{8}{4}} = 0.77143\,.$$

Exercise 9 (the probability of unions of sets)

Part (a): If A, B, and C are mutually exclusive then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.1 + 0.2 + 0.3 = 0.6\,.$$

Part (b): If A, B, and C are independent then the probability of intersecting events is easy to compute. For example, $P(A \cap B) = P(A)P(B)$ and we compute using the inclusion-exclusion identity that

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B)P(C) + P(A)P(B)P(C) \\ &= 0.6 - 0.02 - 0.03 - 0.06 + 0.006 = 0.496\,. \end{aligned}$$

Part (c): In this case we are given the values of the needed intersections so again using the inclusion-exclusion identity we have

$$P(A \cup B \cup C) = 0.6 - 0.04 - 0.05 - 0.08 + 0.01 = 0.44\,.$$

Exercise 10 (right-handed and blue eyed people)

We are told that $P(A) = 0.9$, $P(B) = 0.6$, $P(C) = 0.4$ and $P(B|C) = 0.7$.

Part (a): The event we want is $B \cap C$. We can compute this from what we know. We have

$$P(B \cap C) = P(B|C)P(C) = 0.7(0.4) = 0.28\,.$$

Part (b): The event we want is $A \cap B \cap C$. We have (using independence)

$$P(A \cap B \cap C) = P(B \cap C)P(A) = 0.28(0.9) = 0.252\,.$$

Part (c): The event we want is $A \cup B \cup C$. We have

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \\ &= P(A) + P(B) + P(C) - P(A)P(B) - P(A)P(C) - P(B \cap C) + P(A \cap B \cap C) \\ &= 0.9 + 0.6 + 0.4 - 0.9(0.6) - 0.9(0.4) - 0.28 + 0.252 = 0.972\,. \end{aligned}$$

Exercise 11

Part (a): We will use Bayes' rule

$$P(F|E) = \frac{P(E|F)P(F)}{P(E)}\,.$$

Now $P(E|F) = \frac{5}{8}$ and $P(F) = \frac{4}{9}$. We can evaluate $P(E)$ as

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c) = \frac{5}{8}\left(\frac{4}{9}\right) + \frac{4}{8}\left(\frac{5}{9}\right) = \frac{5}{9}\,.$$

Using this we find $P(F|E) = \frac{\frac{5}{8}\left(\frac{4}{9}\right)}{\frac{5}{9}} = \frac{1}{2}$.

Part (b): Since $P(F|E) = \frac{1}{2} \neq P(F) = \frac{4}{9}$ these two events are *not* independent.

Exercise 12 (the chain rule of probability)

Since $P(E \cap F) = P(E|F)P(F)$ via the definition of conditional probability. We can apply this relationship twice to $E \cap F \cap G$ to get

$$P(E \cap F \cap G) = P(E|F \cap G)P(F \cap G) = P(E|F \cap G)P(F|G)P(G) .$$

Exercise 13 (the mixed up mail problem)

From the book the probability of a complete mismatch is

$$1 - P(A_c) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^N}{N!} .$$

When $N = 2$ we get

$$1 - P(A_c) = \frac{1}{2} .$$

When $N = 4$ we get

$$1 - P(A_c) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = \frac{3}{8} = 0.375 .$$

When $N = 6$ we get

$$1 - P(A_c) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} = 0.3680556 .$$

As N increases $1 - P(A_c)$ limits to $1 - e^{-1}$. Since the sum is an alternating series the error in stopping the summation at the term N is smaller than the last neglected term. That is

$$|P(A_c) - e^{-1}| \leq \frac{1}{(N+1)!} .$$

We then need to pick a value of N to have this smaller than 10^{-3}. Putting various values of N into the above formula we find for $N = 6$ gives

$$|P(A_c) - e^{-1}| \leq \frac{1}{7!} = 2 \cdot 10^{-4} ,$$

and thus the summation accurate to three decimals.

Exercise 14 (3 die in a box)

Let A be the event that the fair die is thrown, B the even the die that always returns a 6 is thrown, and C the event that the die that only returns 1 or 6 is thrown. Let E be the event that a 6 shows when the chosen die is thrown. We want to calculate $P(A|E)$. From Bayes' rule we have

$$
\begin{aligned}
P(A|E) &= \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C)} \\
&= \frac{\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{6}\right)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{1}{10},
\end{aligned}
$$

when we evaluate.

Exercise 15 (coins in a box)

Let E be the event the box picked has at least one dime, then the box picked needs to be the box B or C. Let A, B, C be the events that we initially draw from the boxes A, B, and C respectively. Let Q be the event that the coin drawn in a quarter. With these definitions we want to compute $P(B \cup C|Q)$. Since B and C are mutually independent we can compute them with by adding. Thus

$$
P(B \cup C|Q) = P(B|Q) + P(C|Q).
$$

Each of the events on the right-hand-side can be computed using Bayes' rule as

$$
P(B \cup C|Q) = \frac{P(Q|B)P(B)}{P(Q)} + \frac{P(Q|C)P(C)}{P(Q)}.
$$

We first compute $P(Q)$ using

$$
\begin{aligned}
P(Q) &= P(Q|A)P(A) + P(Q|B)P(B) + P(Q|C)P(C) \\
&= 1\left(\frac{1}{3}\right) + \frac{1}{3}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{11}{18}.
\end{aligned}
$$

Thus we find

$$
P(B \cup C|Q) = \frac{\frac{1}{3}\left(\frac{1}{3}\right)}{\frac{11}{18}} + \frac{\frac{1}{2}\left(\frac{1}{3}\right)}{\frac{11}{18}} = \frac{5}{11} = 0.4545.
$$

Exercise 16 (two cards from a deck)

Part (a): Let A be the event at least one card in the hand is an ace. Let B be the event that both cards in the hand are aces. Then since $B \subset A$ we have

$$
P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\frac{\binom{4}{2}}{\binom{52}{2}}}{\frac{\binom{4}{2}}{\binom{52}{2}} + \frac{\binom{4}{1}\binom{52-4}{1}}{\binom{52}{2}}} = \frac{6}{198} = 0.0303.
$$

Part (a): Let A be the event one card is the ace of spades and the other card is unknown (arbitrary). Let B be the event that one card is the ace of spades and the other card is an ace also. Again since $B \subset A$ we have

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\frac{\binom{3}{1}}{\binom{52}{2}}}{\frac{\binom{51}{1}}{\binom{52}{2}}} = \frac{3}{51} = 0.050882\,.$$

Exercise 17 (a stopping bus)

Part (a): Each passenger has a $1/3$ chance of of getting off at each stop (assuming that the passenger must get off at one of the stops). The probability that k people get off at the first stop is then

$$\binom{n}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k}.$$

Part (b): Let E be the event that the day is Sunday. Let O be the event that no one gets off at the first stop. We want to compute $P(E|O)$. By Bayes' rule we have

$$\begin{aligned} P(E|O) &= \frac{P(O|E)P(E)}{P(O)} = \frac{P(O|E)P(E)}{P(O|E)P(E) + P(O|E^c)P(E^c)} \\ &= \frac{\binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{2-0}\left(\frac{1}{7}\right)}{\binom{2}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{2-0}\left(\frac{1}{7}\right) + \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^{4-0}\left(\frac{6}{7}\right)} = 0.027273\,. \end{aligned}$$

Exercise 18 (fatal diseases)

Let D be the even we have the disease, and T be the even that our test comes back positive. Then from the problem we have that $P(D) = 10^{-5}$, $P(T|D) = 0.9$, and $P(T^c|D^c) = 0.99$.

Part (a): We want to compute $P(T)$. We have

$$P(T) = P(T|D)P(D) + P(T|D^c)P(D^c) = 0.9(10^{-5}) + (1 - 0.99)(1 - 10^{-5}) = 0.01\,.$$

Part (b): We want to compute $P(D|T)$. We have

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)} = \frac{0.9(10^{-5})}{0.01} = 9 \cdot 10^{-4}\,.$$

Exercise 19 (more fatal diseases)

In this case we are to assume that $P(T^c|D^c) = 1 - \delta$ then as in Exercise 18 we get

$$P(D|T) = \frac{0.9(10^{-5})}{0.9(10^{-5}) + \delta(1 - 10^{-5})} \,.$$

We want to have $P(D|T) \geq \frac{1}{2}$. This means that we have to have

$$9 \cdot 10^{-6} \geq \frac{9}{2} \cdot 10^{-6} + \frac{\delta}{2}(1 - 10^{-5}) \,,$$

or solving for δ we get

$$\delta \leq \frac{9 \cdot 10^{-6}}{1 - 10^{-5}} = 9 \cdot 10^{-6} \,.$$

Note that this is different than the answer in the back of the book. If anyone sees anything wrong with what I have done (or agrees with me) please contact me.

Exercise 20 (answering correctly by guessing)

Here $f(p)$ is the probability a student marked a correct answer by guessing. From the stated example, this is the expression for $P(H_2|E)$ or

$$f(p) \equiv P(H_2|E) = \frac{1-p}{mp + 1 - p} \,. \tag{13}$$

From this we calculate

$$f'(p) = -\frac{1}{mp + 1 - p} - \frac{(1-p)(m-1)}{(mp + 1 - p)^2} = \frac{2p - m}{(mp + 1 - p)^2} \,,$$

when we simplify. Now since $p \leq 1$ we have $2p \leq 2$ and so $2p - m \leq 2 - m$. This last expression (or $2 - m$) is less than 0 since $m \geq 2$ (we must have at least 2 answers to a given question). Thus $f'(p) < 0$ and $f(p)$ is a strictly monotone decreasing function as we were to show.

Exercise 21 (independent events?)

We have

$$P(E) = \frac{\binom{4}{2}}{\binom{5}{3}} = \frac{3}{5} \,,$$

since $\left(\begin{array}{c}4\\2\end{array}\right)$ is the number of ways to draw a set of three numbers (from the digits $1-5$) of which the digit 1 is one of the numbers. By similar logic we have $P(F) = P(E)$. The event $E \cap F$ is given by

$$P(E \cap F) = \frac{\left(\begin{array}{c}3\\1\end{array}\right)}{\left(\begin{array}{c}5\\3\end{array}\right)} = \frac{3}{10}.$$

To be independent we must have $P(E)P(F) = P(E \cap F)$. The left-hand-side of this expression is $\left(\frac{3}{5}\right)^2$ which is not equal to $P(E \cap F)$. Thus the two events are *not* independent.

Exercise 22 (independent events?)

There are 9 primes between 1 and 30 which are $2, 3, 5, 7, 11, 13, 17, 19, 23$. Thus

$$P(X \text{ is prime}) = \frac{9}{30} = \frac{3}{10}.$$

At the same time we compute

$$P(16 \leq X \leq 30) = \frac{15}{30} = \frac{1}{2},$$

and

$$P((X \text{ is prime}) \cap (16 \leq X \leq 30)) = \frac{3}{30} = \frac{1}{10},$$

since there are only three primes in the range $16-30$. The product of the probability of the two events X is prime and $16 \leq X \leq 30$ is $\frac{3}{10} \cdot \frac{1}{2} = \frac{3}{20}$. Since this is not equal to the probability of the intersection of these two events we conclude that the two events are *not* independent.

Exercise 23 (Al and Bob flip)

Part (a): On each round three things can happen: Al can win the game A, Bob can win the game B, or the game can continue C. Lets compute the probability of each of these events. We find

$$\begin{aligned}
P(A) &\equiv P(\text{Al wins}) \\
&= P(\text{Al gets 2 heads and Bob gets 1 or 0 heads}) + P(\text{Al gets 1 head and Bob gets 0 heads}) \\
&= P(A_2 \cap B_1) + P(A_2 \cap B_0) + P(A_1 \cap B_0) \\
&= P(A_2)P(B_1) + P(A_2)P(B_0) + P(A_1)P(B_0) \\
&= \frac{1}{4}\left[\left(\begin{array}{c}3\\1\end{array}\right)\left(\frac{1}{2}\right)^3\right] + \frac{1}{4}\left[\left(\begin{array}{c}3\\0\end{array}\right)\left(\frac{1}{2}\right)^3\right] + \frac{1}{2}\left[\left(\begin{array}{c}3\\0\end{array}\right)\left(\frac{1}{2}\right)^3\right] = \frac{3}{16} = 0.1875.
\end{aligned}$$

Part (b): Here we find

$$
\begin{aligned}
P(C) &= P(A_0 \cap B_0) + P(A_1 \cap B_1) + P(A_2 \cap B_2) \\
&= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2) \\
&= \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + \left(\begin{array}{c} 2 \\ 1 \end{array}\right) \left(\frac{1}{2}\right)^2 \left(\begin{array}{c} 3 \\ 1 \end{array}\right) \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 \left(\begin{array}{c} 3 \\ 2 \end{array}\right) \left(\frac{1}{2}\right)^3 \\
&= \frac{5}{16} = 0.3125\,.
\end{aligned}
$$

From these two event we can compute $P(B)$. We find

$$
P(B) = 1 - P(A) - P(C) = 1 - \frac{3}{16} - \frac{5}{16} = \frac{1}{2}\,.
$$

Part (c): This must be a sequence of CCA and thus has a probability of

$$
\frac{5}{16} \cdot \frac{5}{16} \cdot \frac{3}{16} = 0.0183\,.
$$

Part (d): This must be one of the sequences A, CA, CCA, $CCCA$, $CCCCA$ etc. Thus the probability this happens is given by

$$
\begin{aligned}
P(\text{event}) &= \frac{3}{16} + \frac{3}{16} \cdot \frac{5}{16} + \frac{3}{16} \cdot \left(\frac{5}{16}\right)^2 + \frac{3}{16} \cdot \left(\frac{5}{16}\right)^3 + \cdots \\
&= \frac{3}{16} \sum_{k=0}^{\infty} \left(\frac{5}{16}\right)^k = \frac{3}{16}\left(\frac{1}{\frac{11}{16}}\right) = \frac{3}{11} = 0.2727\,.
\end{aligned}
$$

Another way to solve this problem is to recognize that Al wins if the event A happens before the event B. From the book this happens with the probability

$$
P(\text{even}) = \frac{P(A)}{P(A) + P(B)} = \frac{\frac{3}{16}}{\frac{3}{16} + \frac{1}{2}} = \frac{3}{11}\,,
$$

the same answer.

Exercise 24 (rolling a 6 last)

Part (a): $P(E) = \frac{2}{6} = \frac{1}{3}$ and $P(F) = \frac{1}{6}$. From the book the probability that the event E happens before the event F is

$$
\frac{P(E)}{P(E) + P(F)} = \frac{\frac{1}{3}}{\frac{3}{6}} = \frac{2}{3}\,.
$$

Part (b): Let $E^{(1)}$ the the event that E happens on the first roll, $F^{(1)}$ the even that F happens on the first roll, and $G^{(1)}$ that neither E or F happens on the first roll. Let W be the event in question. Then conditioning on the first event we have

$$
\begin{aligned}
P(W) &= P(W|E^{(1)})P(E^{(1)}) + P(W|F^{(1)})P(F^{(1)}) + P(W|G^{(1)})P(G^{(1)}) \\
&= 1\left(\frac{1}{3}\right) + P(W|F^{(1)})\left(\frac{1}{6}\right) + P(W)\left(1 - \frac{1}{3} - \frac{1}{6}\right) .
\end{aligned}
$$

We need to evaluate $P(W|F^{(1)})$. In one method we can evaluate this by conditioning on the outcome of the second event. Thus we have

$$
\begin{aligned}
P(W|F^{(1)}) &= P(W|F^{(1)}, E^{(2)})P(E^{(2)}) + P(W|F^{(1)}, F^{(2)})P(F^{(2)}) + P(W|F^{(1)}, G^{(2)})P(G^{(2)}) \\
&= \frac{1}{3} + 0 + P(W|F^{(1)}, G^{(2)})\left(1 - \frac{1}{3} - \frac{1}{6}\right) .
\end{aligned}
$$

Since $P(W|F^{(1)}, G^{(2)}) = P(W|F^{(1)})$ we can solve for $P(W|F^{(1)})$ to find

$$
P(W|F^{(1)}) = \frac{2}{3} .
$$

As another way to evaluate this probability is to note that it is the probability that we get one event E before one event F which was computed in the first part of this problem and we have $P(W|F^{(1)}) = \frac{2}{3}$. Using this result we have that

$$
P(W) = \frac{1}{3} + \frac{1}{6} \cdot \frac{2}{3} + P(W)\left(\frac{1}{2}\right) ,
$$

which solving for $P(W)$ gives $P(W) = \frac{8}{9} = 0.88888$.

Exercise 25 (team A and B)

From the problem statement we have $p = P(A) = 0.6$ and $q = P(B) = 0.4$.

Part (a): This is like the problem of the points where we want the probability we will win $k = 3$ times before our opponent wins $n = 3$ times. Thus in this case $n = k$ where $k = 3$. Then from the book with $N = k + n - 1 = 2k - 1$ we have

$$
P(E_{k,k}) = \sum_{i=k}^{2k-1} \binom{2k-1}{i} p^i q^{2k-1-i} . \tag{14}
$$

With $k = 3$ and the numbers for this problem we have

$$
P(E_{3,3}) = \sum_{i=3}^{5} \binom{5}{i} 0.6^i 0.4^{2k-1-i} = 0.68255 .
$$

Part (b): If each team has won one game for A to win we need A to win 2 games before B wins two games. Again we have the problem of the points where $n = k = 2$. Using Equation 14 we get

$$
P(E_{2,2}) = \sum_{i=2}^{3} \binom{3}{i} 0.6^i 0.4^{3-i} = 0.64799 .
$$

Part (c): If each team has won two games for A to win we need A to win 1 more game before B wins 1 more game. This happens with probability $P(A) = 0.6$. Another way to get this same answer is to again say that this is the problem of the points where $n = k = 1$. Using Equation 14 we get

$$P(E_{1,1}) = \sum_{i=1}^{1} \left(\begin{array}{c} 1 \\ i \end{array} \right) 0.6^i 0.4^{1-i} = 0.6 \,,$$

the same answer.

Exercise 26 (the problem of the points)

Part (a): We must win k times before our opponent wins n times. Let $e_{k,n}$ be this probability and condition on whether we win or loose the first game. In words this recursion is easier to understand. We have

$$\begin{aligned} P(\text{We win } k \text{ before our opponent wins } n) &= pP(\text{We win } k-1 \text{ before our opponent wins } n) \\ &+ qP(\text{We win } k \text{ before our opponent wins } n-1) \,. \end{aligned}$$

In symbols this is

$$e_{k,n} = pe_{k-1,n} + qe_{k,n-1} \,. \tag{15}$$

Part (b): We want to solve the above recursion relationship with the given boundary conditions $e_{0,n} = 1$ and $e_{k,0} = 0$. Let $k = 1$ in Equation 19 to get

$$e_{1,n} = pe_{0,n} + qe_{1,n-1} = p + qe_{1,n-1} \,. \tag{16}$$

To solve for $e_{1,n}$ in the above note that $e_{1,1} = p$, and we will let $n = 2$ and $n = 3$ and then derive a general expression from the pattern we see. For $n = 2$ we have

$$e_{1,2} = p + qe_{1,1} = p(1+q) \,.$$

Let $n = 3$ in Equation 16 to get

$$e_{1,3} = p + qe_{1,2} = p + qp(1+q) = p(1+q+q^2) \,.$$

In general, the solution to $e_{1,n}$ looks like

$$e_{1,n} = p(1 + q + q^2 + \cdots q^{n-1}) = p\frac{1-q^n}{1-q} \quad \text{for} \quad n \geq 1 \,. \tag{17}$$

If we let $k = 2$ in Equation 19 we get a linear difference equation for $e_{2,n}$. We could solve this difference equations using techniques like in [1] but since we are only asked to compute $e_{2,3}$ we will just do it by iteration. Using Equation 19 repeatedly we have

$$\begin{aligned} e_{2,3} &= pe_{1,3} + qe_{2,2} \\ &= p(pe_{0,3} + qe_{1,2}) + q(pe_{1,2} + qe_{2,1}) \\ &= p(p + q(pe_{0,2} + qe_{1,1})) + q(p(pe_{0,2} + qe_{1,1}) + q(pe_{1,1} + qe_{2,0})) \\ &= p^2 + p^2q + pq^2e_{1,1} + p^2q + pq^2e_{1,1} + pq^2e_{1,1} \\ &= p^2 + 2p^2q + 3pq^2(pe_{0,1} + qe_{1,0}) \\ &= p^2 + 2p^2q + 3p^2q^2 \,. \end{aligned}$$

If we let $q = 1 - p$ two write the above only in terms of p to get

$$p^2 + 2p^2q + 3p^2q^2 = p^2 + 2p^2(1-p) + 3p^2(1-2p+p^2) = 6p^2 - 8p^3 + 3p^4\,.$$

Recall that eq 2.32 from the book is

$$P(E_{k,n}) = \sum_{i=k}^{N} \left(\begin{array}{c} N \\ i \end{array} \right) p^i q^{N-i}\,, \tag{18}$$

with $N = k+n-1$. Lets check our result for $e_{2,3}$ against this expression. Since $e_{2,3} \equiv P(E_{2,3})$ we have $k = 2$, $n = 3$ and $N = 2+3-1 = 4$ so $P(E_{2,3})$ via Equation 18 is given by

$$\begin{aligned} P(E_{2,3}) &= \sum_{i=2}^{4} \left(\begin{array}{c} 4 \\ i \end{array} \right) p^i q^{4-i} \\ &= \left(\begin{array}{c} 4 \\ 2 \end{array} \right) p^2q^2 + \left(\begin{array}{c} 4 \\ 3 \end{array} \right) p^3q^1 + \left(\begin{array}{c} 4 \\ 4 \end{array} \right) p^4 \\ &= 6p^2q^2 + 4p^3q + p^4\,. \end{aligned}$$

If we let $q = 1 - p$ in the above we get

$$P(E_{2,3}) = 6p^2(1-2p+p^2) + 4p^3(1-p) + p^4 = 6p^2 - 8p^3 + 3p^4\,,$$

the same as before.

Exercise 27 (gamblers ruin)

Part (a): Let $f_{a,k}$ be this probability that Ann goes broke when she starts with \$$a$ dollars (Bob starts with \$$N - a$ dollars) playing at most k games and condition on whether Ann wins or looses the first game. In words this recursion is easier to understand. We have

$$\begin{aligned} P(\text{Ann goes broke with \$}a\text{ in }k\text{ games}) &= pP(\text{Ann goes broke with \$}(a+1)\text{ in }k-1\text{ games}) \\ &+ qP(\text{Ann goes broke with \$}(a-1)\text{ in }k-1\text{ games})\,. \end{aligned}$$

In symbols this is

$$f_{a,k} = pf_{a+1,k-1} + qf_{a-1,k-1}\,. \tag{19}$$

Part (b): We have

$$f_{2,3} = pf_{3,2} + qf_{1,2} = p(0) + q(pf_{2,1} + qf_{0,1}) = qp(0) + q^2 = q^2\,.$$

Part (c): Since $f_{2,3}$ is the probability that Ann will go broke with \$2 and Bob has \$3. In this case Ann will go broke if she looses twice, which happens with the probability q^2 the same as above.

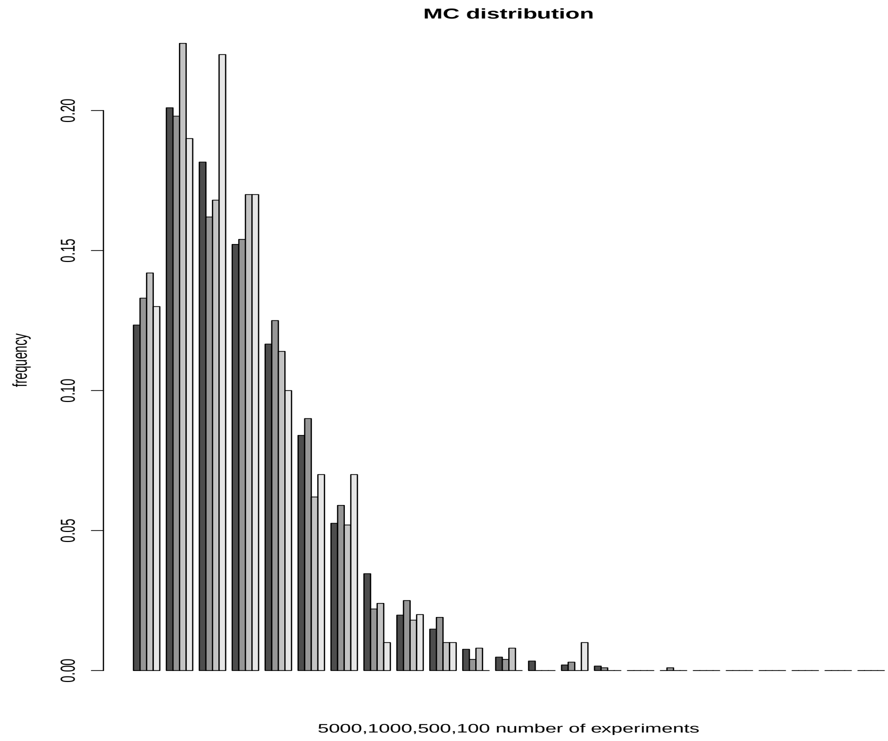

Figure 1: The waiting game (for three total flips).

Exercises 28-32 (simulations)

Please see the `R` files `chap_2_ex_28.R`—`chap_2_ex_32.R`, where we perform these simulations.

For exercise 29 in Figure 1 we plot the relative frequency of each experimental outcomes for the waiting game as stated.

When we run `chap_2_ex_30.R` we get

```
[1] "nSims=    10 fraction with three or more of the same BDs=  0.000000"
[1] "nSims=    50 fraction with three or more of the same BDs=  0.040000"
[1] "nSims=   100 fraction with three or more of the same BDs=  0.010000"
[1] "nSims=   500 fraction with three or more of the same BDs=  0.022000"
[1] "nSims=  1000 fraction with three or more of the same BDs=  0.009000"
[1] "nSims=  5000 fraction with three or more of the same BDs=  0.014200"
[1] "nSims= 10000 fraction with three or more of the same BDs=  0.015700"
```

For exercise 31 in Figure 2 we plot the relative frequency of each experimental outcomes for the waiting game as stated.

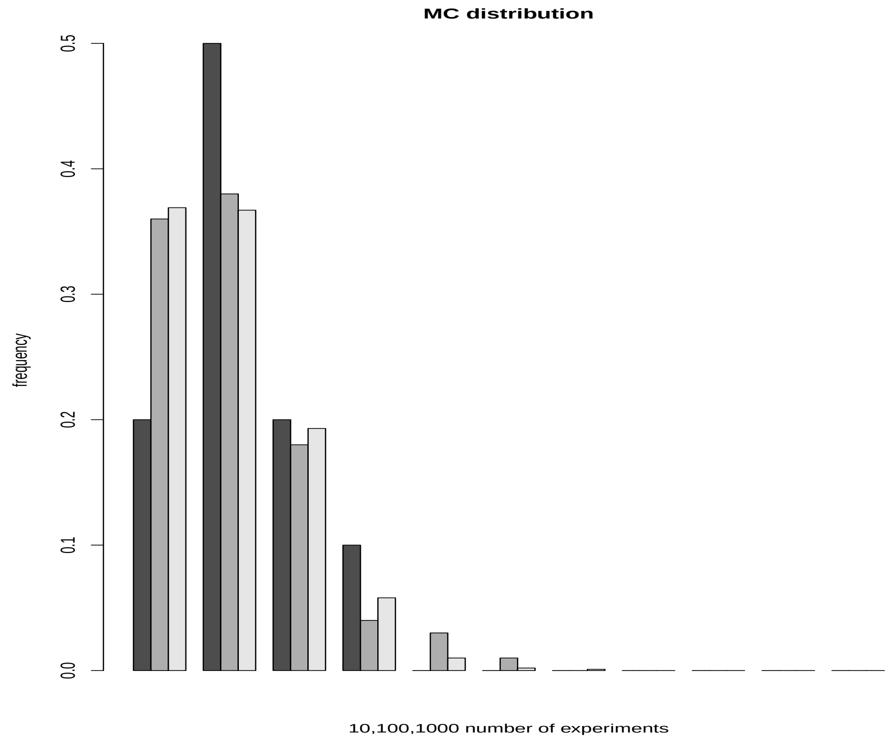

Figure 2: The frequency of the various outcomes in the mixed-up mail problem for 10, 100, 1000 random simulations.

For exercise 32 we are trying to determine the area of the ellipse given by

$$x^2 + \kappa^2 y^2 = 1 \quad \text{or} \quad x^2 + \frac{y^2}{\left(\frac{1}{\kappa^2}\right)} = 1\,.$$

From this second form of the equation we can see that the "domain" of the area of this ellipse is $-1 \leq x \leq +1$ (when $y \approx 0$) and $-\frac{1}{\kappa} \leq y \leq +\frac{1}{\kappa}$ (when $x \approx 0$), thus has κ gets smaller (closer to 0) the ranges of valid y expand greatly. Thus our ellipse gets "long an skinny". In that case, one would expect that a great number of random draws would need to be performed to estimate the true area $\frac{\pi}{\kappa}$ accurately. To do this we simulate uniform random variables x and y according to the above distributions and then compute whether or not

$$x^2 + \kappa^2 y^2 \leq 1\,.$$

If this inequality is true then we have a point in the object and we increment a variable N_{obj}. We do this procedure N_{box} times. We expect that if we do this procedure enough times that the fraction of times the point falls in the object is related to its area via

$$\frac{N_{\text{obj}}}{N_{\text{box}}} \sim \frac{A_{\text{obj}}}{A_{\text{box}}}\,.$$

Solving for A_{obj} and using what we know for A_{box} we would get

$$A_{\text{obj}} = \left(\frac{N_{\text{obj}}}{N_{\text{box}}}\right) A_{\text{box}} = \left(\frac{N_{\text{obj}}}{N_{\text{box}}}\right)\left(\frac{4}{\kappa}\right)\,.$$

Using this information we can implement the R code `chap_2_ex_31.R`. When we run that code we get

	1000	5000	10000	50000	1e+06
1	0.019605551	0.004326676	0.006495860	0.0041738873	0.0001377179
0.9	0.001780197	0.004326676	0.003440085	0.0029770421	0.0003503489
0.75	0.032597271	0.011970790	0.004963296	0.0011991836	0.0006185864
0.5	0.017059071	0.002034845	0.009164986	0.0002569863	0.0002497630
0.25	0.011966113	0.001525549	0.003440085	0.0025696054	0.0005523779
0.1	0.024957834	0.018336988	0.001525549	0.0007106757	0.0002340680

This is a matrix showing the relative error in the Monte-Carlo approximation to the area (for different values of κ in each row) and then the number of random draws used to estimate the approximation (with more samples as we move to the right). In general, for smaller κ the area is harder to compute using this method.

Figure 3: A schematic of the integration region used to prove $E[X] = \int_0^\infty P\{X > a\}da$.

Chapter 3: Distributions and Expectations of RVs

Notes on the Text

Notes on proving $E[X] = \int_0^\infty P\{X > a\}da$

Note that we can write the integral $\int_0^\infty P\{X > a\}da$ in terms of the density function for X as

$$\int_0^\infty P\{X > a\}da = \int_{a=0}^\infty \left(\int_{x=a}^\infty f(x)dx \right) da\,.$$

Next we want to change the order of integration in the integral on the right-hand-side of this expression. In Figure 3 we represent the current order of integration as the *horizontal* gray30 stripe. What is meant by this statement is that the double integral on the right-hand-side above can be viewed as specifying a value for a on the A-axis and then integrating horizontally along the X-axis over the domain $[a, +\infty)$. To change the order of integration we need to instead consider the differential of area given by the *vertical* gray60 stripe in Figure 3. In this ordering, we first specify a value for x on the X-axis and then integrating vertically along the A-axis over the domain $[0, a]$. Symbolically this procedure is given by

$$\int_{x=0}^\infty \int_{a=0}^x f(x)dadx\,.$$

This can be written as

$$\int_{x=0}^{\infty} \left(\int_{a=0}^{x} da \right) f(x) dx = \int_{x=0}^{\infty} x f(x) dx = E[X] \, ,$$

as we were to show.

Notes on Example 3.17 a Cauchy random variable

When θ is a uniform random variable over the domain $[-\frac{\pi}{2}, +\frac{\pi}{2}]$ and X is defined as $X = \tan(\theta)$ then we can derive the distribution function for X following the book

$$P\{X \leq b\} = P\{\tan(\theta) \leq b\} = P\{-\frac{\pi}{2} < \theta \leq \arctan(b)\} \, .$$

Then since θ is a uniform random variable the above probability is equal to the length of the interval in θ divided by the length of the total possible range of θ. This is as in the books Example 3.7 where $P\{X \in I\} = \frac{\text{Length}(I)}{\beta - \alpha}$. Thus

$$P\{-\frac{\pi}{2} < \theta \leq \arctan(b)\} = \frac{\arctan(b) - (-\frac{\pi}{2})}{\pi} = \frac{\arctan(b) + \frac{\pi}{2}}{\pi} \, .$$

I believe that the book has a sign error in that it states $-\frac{\pi}{2}$ rather than $+\frac{\pi}{2}$. The density function is then the derivative of the distribution function or

$$f_X(b) = \frac{d}{db} F_X(b) = \frac{1}{\pi(1 + b^2)} \, ,$$

which is the Cauchy distribution.

Notes on Example 3.21 the modified geometric RV

We evaluate

$$\begin{aligned} P\{X \geq n\} &= p \sum_{k=n}^{\infty} q^k \\ &= p \left(\sum_{k=0}^{\infty} q^k - \sum_{k=0}^{n-1} q^k \right) \\ &= p \left(\frac{1}{1-q} - \frac{1-q^n}{1-q} \right) = p \left(\frac{q^n}{1-q} \right) \, , \end{aligned}$$

the expression given in the book.

Exercise Solutions

Exercise 1 (a random variable)

We find

$$\mu_X = 0.3(0) + 0.2(1) + 0.4(3) + 0.1(6) = 2\,.$$

and

$$E[X^2] = 0.3(0^2) + 0.2(1^2) + 0.4(3^2) + 0.1(6^2) = 7.4\,.$$

Then

$$\sigma_X^2 = E[X^2] - \mu_X^2 = 7.4 - 2^2 = 3.4\,,$$

so $\sigma_X = \sqrt{3.4} = 1.84$. The distribution function for this random variable is given by

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ 0.3 & 0 < x \le 1 \\ 0.5 & 1 < x \le 3 \\ 0.9 & 3 < x \le 6 \\ 1.0 & 6 < x \end{cases}$$

Exercise 2 (flipping fair coins)

Part (a):

$$\left(\begin{array}{c} 5 \\ 2 \end{array}\right)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{2^4} = 0.3125\,.$$

Part (b):

$$\left(\begin{array}{c} 4 \\ 1 \end{array}\right)\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3\frac{1}{2} = \frac{4}{2^5} = 0.125\,.$$

Exercise 3 (10 balls in an urn)

The range of the variable X is $1 \le X \le 8$. We now compute the probability density of this random variable. To begin we have

$$P_X\{X = 8\} = \frac{\left(\begin{array}{c} 1 \\ 1 \end{array}\right)\left(\begin{array}{c} 2 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 10 \\ 3 \end{array}\right)} = 0.0083\,.$$

since we have to exactly pick the set $8, 9, 10$ in order to get the smallest number to be 8. Next we have

$$P_X\{X = 7\} = \frac{\binom{1}{1}\binom{3}{2}}{\binom{10}{3}} = 0.025\,,$$

since in this case we have to pick the number 7 in the set which we can do in $\binom{1}{1}$ ways. After we pick this 7 we then need to pick two other numbers from the set $8, 9, 10$ which we can do in $\binom{3}{2}$ ways. Next we have

$$P_X\{X = 6\} = \frac{\binom{1}{1}\binom{4}{2}}{\binom{10}{3}} = 0.05\,,$$

since in this case we have to pick the number 6 in the set which we can do in $\binom{1}{1}$ ways. After we pick this 6 we then need to pick two other numbers from the set $7, 8, 9, 10$ which we can do in $\binom{4}{2}$ ways. The remaining probabilities are computed using the same logic. We find

$$P_X\{X = 5\} = \frac{\binom{1}{1}\binom{5}{2}}{\binom{10}{3}} = 0.0833$$

$$P_X\{X = 4\} = \frac{\binom{1}{1}\binom{6}{2}}{\binom{10}{3}} = 0.125$$

$$P_X\{X = 3\} = \frac{\binom{1}{1}\binom{7}{2}}{\binom{10}{3}} = 0.175$$

$$P_X\{X = 2\} = \frac{\binom{1}{1}\binom{8}{2}}{\binom{10}{3}} = 0.233$$

$$P_X\{X = 1\} = \frac{\binom{1}{1}\binom{9}{2}}{\binom{10}{3}} = 0.3\,.$$

These calculations are done in the `R` file `prob_3.R`.

Exercise 4 (ending with white balls)

Let X be the random variable that indicates the number of white balls in urn I after the two draws and exchanges take place. The range of the random variable X is $0 \leq X \leq 2$. We can compute the probability of each value of X by conditioning on what color balls we draw at each stage. Let D_1 and D_2 be the two draws which can be of the colors W for white and R for red. Then we have

$$\begin{aligned}
P\{X=0\} &= P\{X=0|D_1=W,D_2=W\}P(D_1=W,D_2=W) \\
&+ P\{X=0|D_1=W,D_2=R\}P(D_1=W,D_2=R) \\
&+ P\{X=0|D_1=R,D_2=W\}P(D_1=R,D_2=W) \\
&+ P\{X=0|D_1=R,D_2=R\}P(D_1=R,D_2=R) \\
&= 0 + 1\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + 0 + 0 = \frac{1}{12}.
\end{aligned}$$

Next we compute $P\{X=1\}$, where we find

$$\begin{aligned}
P\{X=1\} &= P\{X=1|D_1=W,D_2=W\}P(D_1=W,D_2=W) \\
&+ P\{X=1|D_1=W,D_2=R\}P(D_1=W,D_2=R) \\
&+ P\{X=1|D_1=R,D_2=W\}P(D_1=R,D_2=W) \\
&+ P\{X=1|D_1=R,D_2=R\}P(D_1=R,D_2=R) \\
&= 1\left(\frac{1}{3}\right)\left(\frac{3}{4}\right) + 0 + 0 + 1\left(\frac{2}{3}\right)\left(\frac{2}{4}\right) = \frac{7}{12}.
\end{aligned}$$

Next we compute $P\{X=2\}$, where we find

$$\begin{aligned}
P\{X=2\} &= P\{X=2|D_1=W,D_2=W\}P(D_1=W,D_2=W) \\
&+ P\{X=2|D_1=W,D_2=R\}P(D_1=W,D_2=R) \\
&+ P\{X=2|D_1=R,D_2=W\}P(D_1=R,D_2=W) \\
&+ P\{X=2|D_1=R,D_2=R\}P(D_1=R,D_2=R) \\
&= 0 + 0 + 1\left(\frac{2}{3}\right)\left(\frac{2}{4}\right) = \frac{1}{3}.
\end{aligned}$$

Note that these three numbers add to 1 as they must.

Exercise 5 (playing a game with dice)

Part (a): Let X the the random variable representing the total amount won. If we first roll a $1, 2, 3, 4, 5$ we get 0. If we first roll a 6 and then roll one of $1, 2, 3, 4, 5$ on our second roll we get 10. If we roll two 6s then we get $10 + 30 = 40$. Thus the probabilities for each of these events is given by

$$
\begin{aligned}
P\{X = 0\} &= \frac{5}{6} \\
P\{X = 10\} &= \frac{1}{6}\left(\frac{5}{6}\right) = \frac{5}{36} \\
P\{X = 40\} &= \left(\frac{1}{6}\right)^2 = \frac{1}{36}\,.
\end{aligned}
$$

Part (b): The fair value for this game is its expectation. We find

$$
E[X] = 0\left(\frac{5}{6}\right) + 10\left(\frac{5}{36}\right) + 40\left(\frac{1}{36}\right) = 2.5\,.
$$

Exercise 8 (flipping heads)

Part (a): $\left(\begin{array}{c} 6 \\ 3 \end{array}\right) p^3 q^3$.

Part (b): $p^3 q^3$.

Part (c): In this case we need two heads in the first four flips and we don't care what the outcome of the last flip is. Thus we have $\left(\left(\begin{array}{c} 4 \\ 2 \end{array}\right) p^2 q^2\right) p$.

Exercise 9 (a continuous random variable)

Part (a): We must have $\int_0^1 f(x)dx = 1$ or

$$
\begin{aligned}
c\int_0^1 x(1-x)dx &= c\int_0^1 (x - x^2)dx \\
&= c\left(\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{c}{6}\,.
\end{aligned}
$$

Thus $c = 6$.

Part (a): We compute

$$
\begin{aligned}
\mu_X = \int_0^1 x(6x(1-x))dx &= 6\int_0^1 (x^2 - x^3)dx \\
= 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right|_0^1 &= \frac{1}{2}\,,
\end{aligned}
$$

when we simplify. We compute

$$
\begin{aligned}
E[X^2] &= \int_0^1 x^2(6x(1-x))dx = 6\int_0^1 (x^3 - x^4)dx \\
&= 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right|_0^1 = \frac{3}{10},
\end{aligned}
$$

when we simplify. With these two values we can thus compute

$$
\sigma_X^2 = E[X^2] - \mu_X^2 = \frac{3}{10} = \frac{1}{4} = \frac{1}{20},
$$

so $\sigma_X = \sqrt{0.05}$.

Exercise 10 (the law of the unconscious statistician)

$$
E[X^n] = \int_0^1 x^n(1)dx = \left.\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}.
$$

$$
\begin{aligned}
\text{Var}(X^n) &= E[X^{2n}] - E[X^n]^2 \\
&= \int_0^1 x^{2n}(1)dx - \frac{1}{(n+1)^2} = \left.\frac{x^{2n+1}}{2n+1}\right|_0^1 - \frac{1}{(n+1)^2} \\
&= \frac{x^{2n+1}}{2n+1} - \frac{1}{(n+1)^2} = \frac{n^2}{(2n+1)(n+1)^2},
\end{aligned}
$$

when we simplify.

Exercise 11 (battery life)

The density and distribution function for an exponential random variable is given by $f_X(x) = \lambda e^{-\lambda x}$ and $F_X(x) = 1 - e^{-\lambda x}$.

Part (a): To have us fail in the first year is the event $\{X \leq 1\}$ which has probability

$$
F_X(1) = 1 - e^{-1/3} = 0.28346\,.
$$

Part (b): This is

$$
P\{1 < X < 2\} = F_X(2) - F_X(1) = 1 - e^{-2/3} - (1 - e^{-1/3}) = 0.2031\,.
$$

Part (c): Because of the memoryless property of the exponential distribution the fact that the batter is still working after one year does not matter in the calculation of the requested probability. Thus this probability is the same as that in Part (a) of this problem or 0.28346.

Exercise 12 (scaling random variables)

We have

$$P\{Y \le a\} = P\{X \le \frac{a}{c}\} = \int_{-\infty}^{a/c} f_X(x)dx\,.$$

Let $y = cx$ so that $dy = cdx$ and we get

$$\int_{-\infty}^{a} f_X\left(\frac{y}{c}\right)\frac{dy}{c} = \frac{1}{c}\int_{-\infty}^{a} f_X\left(\frac{y}{c}\right)dy\,.$$

Thus

$$f_Y(a) = \frac{d}{da}P\{Y \le a\} = \frac{1}{c}f_X\left(\frac{y}{c}\right)\,,$$

as we were to show.

Exercise 13 (scaling a gamma RV)

A gamma RV with parameters (λ, n) has a density function given by

$$f_X(x) = \frac{1}{\Gamma(n)}(\lambda x)^{n-1}\lambda e^{-\lambda x}\,.$$

If $Y = cX$ then via a previous exercise we have

$$f_Y(y) = \frac{1}{c}f_X\left(\frac{y}{c}\right) = \frac{1}{c\Gamma(n)}\left(\frac{\lambda}{c}y\right)^{n-1}\lambda e^{-\lambda/cy} = \frac{1}{\Gamma(n)}\left(\frac{\lambda}{c}y\right)^{n-1}\left(\frac{\lambda}{c}\right)e^{-\lambda/cy}\,,$$

which is a gamma RV with parameters $(\lambda/c, n)$.

Exercise 14

Part (a): We find

$$\begin{aligned}
P\{Y \le 4\} &= P\{X^2 \le 4\} = P\{-2 \le X \le +2\} \\
&= P\{0 \le X \le 2\} \quad \text{since } f(x) \text{ is zero when } x < 0 \\
&= \int_0^2 2xe^{-x^2}dx = -\left.e^{-x^2}\right|_0^2 = -(e^{-4} - 1) = 1 - e^{-4} = 0.9816\,.
\end{aligned}$$

Part (b): We find

$$\begin{aligned}
F_Y(a) &= P\{Y \le a\} = P\{X^2 \le a\} = P\{-a^{1/2} \le X \le a^{1/2}\} \\
&= P\{0 \le X \le a^{1/2}\} = \int_0^{a^{1/2}} 2xe^{-x^2}dx = -\left.e^{-x^2}\right|_0^{a^{1/2}} = 1 - e^{-a}\,.
\end{aligned}$$

Thus $f_Y(a) = e^{-a}$.

Exercise 15 (a discrete RV)

From the discussion in the book the discrete density function for this RV is given by

$$P\{X = 0\} = 0.3\,, \quad P\{X = 1\} = 0.2\,, \quad P\{X = 3\} = 0.4\,, \quad P\{X = 6\} = 0.1\,,$$

which is the same density as exercise 1.

Exercise 16 (a geometric RV is like an exponential RV)

When X is a geometric RV with parameter p then $p_X(k) = q^{k-1}p$ for $k = 1, 2, 3, \cdots$. Let n be a large positive integer and let $Y \equiv \frac{X}{n}$ then we have

$$\begin{aligned} F_Y(a) &= P\{Y \le a\} = P\{X \le na\} \\ &= \sum_{k=1}^{\lfloor na \rfloor} q^{k-1}p = p \sum_{k=0}^{\lfloor na \rfloor - 1} q^k = p\left(\frac{1 - q^{\lfloor na \rfloor}}{1-q}\right) = 1 - q^{\lfloor na \rfloor} = 1 - (1-p)^{\lfloor na \rfloor}\,. \end{aligned}$$

Let $p = \frac{\lambda}{n}$ i.e. define λ with this expression. Then the above is

$$1 - \left(1 - \frac{\lambda}{n}\right)^{\lfloor na \rfloor}\,.$$

The limit of this expression as $n \to \infty$ is

$$\left(1 - \frac{\lambda}{n}\right)^{\lfloor na \rfloor} \to (e^{-\lambda})^a = e^{-\lambda a}\,.$$

Thus

$$P\{Y \le a\} \approx 1 - e^{-\lambda a} \quad \text{for} \quad a > 0\,,$$

as we were to show.

Exercise 18 (simulating a negative binomial random variable)

In exercise 29 on Page 20 denoted "the waiting game" we implemented code to simulate random draws from a negative binomial distribution. We can use that to compare the relative frequencies obtained via simulation with the exact probability density function for a negative binomial random variable. Recall that a negative binomial random variable S can be thought of as the number of trials to obtain $r \ge 1$ successes and has a probability mass function given by $p_S(k) = \left(\begin{array}{c} k-1 \\ r-1 \end{array}\right) p^r q^{k-r}$.

Chapter 4: Joint Distributions of Random Variables

Notes on the Text

Notes on Example 4.3

Some more steps in this calculation give

$$E[\pi R^2] = \int_0^{2\pi} \int_0^1 (\pi r^2) \frac{1}{\pi} dA = \int_0^{2\pi} \int_0^1 r^3 dr d\theta \,.$$

Notes on Example 4.7

Recall that when θ is measured in radian that the area of a sector of a circle of radius r is given by

$$A_{\text{Sector}} = \left(\frac{\theta}{2\pi} \right) \pi r^2 = \frac{\theta}{2} r^2 \,.$$

Then we find that

$$\frac{\frac{1}{2}\theta r^2}{\pi} = \frac{\theta r^2}{2\pi} \,.$$

Notes on Example 4.15 ($\phi_X(t)$ for a Gamma distribution)

Following the book we have

$$\phi_X(t) = \frac{\lambda^r}{\Gamma(r)} \int_0^\infty e^{-(\lambda - t)x} x^{r-1} dx \,.$$

Let $v = (\lambda - t)x$ so that $dv = (\lambda - t)dx$ and the above becomes

$$\begin{aligned} \phi_X(t) &= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty e^{-v} \frac{v^{r-1}}{(\lambda - t)^{r-1}} \frac{dv}{(\lambda - t)} \\ &= \frac{\lambda^r}{\Gamma(r)} \cdot \frac{1}{(\lambda - t)^r} \int_0^\infty v^{r-1} e^{-v} dv = \frac{\lambda^r}{(\lambda - t)^r} \,. \end{aligned}$$

Note that we must have $\lambda - t > 0$ so that the integral limit when $x = +\infty$ corresponds to $v = +\infty$. This means that we must have $t < \lambda$.

Notes on Example 4.24

Note that $\frac{|\bar{X}_n - \mu|}{\mu}$ is the relative error in $\bar{X}_n$'s approximation to μ. If we want this approximation to be with in a 1% relative error this means that we want to bound

$$P\left\{\frac{|\bar{X}_n - \mu|}{\mu} \geq 0.01\right\}. \tag{20}$$

Chebyshev's inequality applied to the sample mean is

$$P\{|\bar{X}_n - \mu| \geq \delta\} \leq \frac{\sigma^2}{n\delta^2}.$$

To match the desired relative error bound above we take $\delta = 0.01\mu$ to get

$$P\{|\bar{X}_n - \mu| \geq 0.01\mu\} \leq \frac{\sigma^2}{n(0.01\mu)^2}.$$

Since we are told that $\sigma = 0.1\mu$ the right-hand-side of the above becomes

$$\frac{\sigma^2}{n(0.01\mu)^2} = \frac{0.1^2}{n(0.01)^2} = \frac{100}{n},$$

when we simplify. Thus if we take $n \geq 1000$ then we will have the right-hand-side less than $\frac{1}{10}$.

Notes on Example 4.25

Recall from section 3.5 in the book that when $Y = \alpha X + \beta$ we have

$$P\{Y \leq c\} = F_X\left(\frac{c - \beta}{\alpha}\right),$$

so the density function for Y is given by

$$f_Y(c) = \frac{d}{dc}P\{Y \leq c\} = \frac{d}{dc}F_X\left(\frac{c-\beta}{\alpha}\right) = \frac{1}{\alpha}f_X\left(\frac{c-\beta}{\alpha}\right).$$

In this case where $\bar{X}_n = \frac{1}{n}S_n$ and S_n has a Cauchy density

$$f_n(s) = \frac{n}{\pi}\left(\frac{1}{n^2 + s^2}\right),$$

we then have $\alpha = \frac{1}{n}$ and $\beta = 0$ thus

$$f_{X_n}(x) = nf_n\left(\frac{x}{\frac{1}{n}}\right) = nf_n(nx) = \frac{n^2}{\pi^2}\left(\frac{1}{n^2 + n^2x^2}\right),$$

as stated in the book.

Notes on the Central Limit Theorem

For Z_n given as in the book we can show

$$\begin{aligned} Z_n &\equiv \frac{\bar{X}_n - \mu}{\left(\frac{\sigma}{\sqrt{n}}\right)} = \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) = \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \frac{1}{n}\sum_{i=1}^{n}\mu\right) \\ &= \frac{1}{\sqrt{n}\sigma}\sum_{i=1}^{n}(X_i - \mu) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i - \mu}{\sigma}. \end{aligned} \tag{21}$$

Notes on Example 4.28

Now $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + X_3 + \cdots + X_n)$ so

$$\text{Var}(\bar{X}_n) = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}(X_1) = \frac{1}{n}\text{Var}(X_1) = \frac{1}{n}\left(\frac{\mu}{10}\right)^2.$$

Thus to introduce the standardized RV for $\bar{X}_n$ into Equation 20 we would have

$$P\left\{|Z_n| \geq \frac{0.01\mu}{\frac{1}{\sqrt{n}}\left(\frac{\mu}{10}\right)}\right\} \leq 0.1\,,$$

or

$$P\{|Z_n| \geq \frac{\sqrt{n}}{10}\} \leq 0.1\,, \tag{22}$$

the equation in the book.

Notes on the Proof of the Central Limit Theorem

There were a few steps in this proof that I found it hard to follow without writing down a few of the steps. I was able to follow the arguments that showed that

$$\begin{aligned} \phi_n(t) &= 1 + \frac{t^2}{2n} + \left(\frac{t}{\sqrt{n}}\right)^3 r\left(\frac{t}{\sqrt{n}}\right) \\ &= 1 + \frac{t^2}{2n}\left(1 + \frac{2t}{\sqrt{n}}r\left(\frac{t}{\sqrt{n}}\right)\right) \\ &\equiv 1 + \frac{t^2}{2n}\left(1 + \epsilon(t,n)\right)\,, \end{aligned}$$

where we have defined $\epsilon(t,n) = \frac{2t}{\sqrt{n}}r\left(\frac{t}{\sqrt{n}}\right)$. Note that as $n \to +\infty$ we have that $r\left(\frac{t}{\sqrt{n}}\right) \to r(0)$ a finite value and thus $\epsilon(t,n) \to 0$ as $n \to +\infty$.

$$\log(\phi_n(t)) = n\log\left(1 + \frac{t^2}{2n}(1 + \epsilon(t,n))\right)$$

For large n with $\log(1+u) \approx u$ as $u \to 0$ we have that $\log(\phi_n(t))$ goes to

$$n\left(\frac{t^2}{2n}\right) = \frac{t^2}{2}\,.$$

Therefore

$$\lim_{n\to\infty} \phi_n(t) = e^{t^2/2}\,,$$

as claimed.

Exercise Solutions

Exercise 1 (examples with a discrete joint distribution)

Part (a): The marginal distributions $f_X(x)$, is defined as $f_X(x) = \sum_y f_{X,Y}(x,y)$ (a similar expression holds for $f_Y(y)$) so we find

$$\begin{aligned}
f_X(0) &= 0.1 + 0.1 + 0.3 = 0.5 \\
f_X(1) &= 0 + 0.2 + 0 = 0.2 \\
f_X(2) &= 0.1 + 0.2 + 0 = 0.3\,.
\end{aligned}$$

and

$$\begin{aligned}
f_Y(0) &= 0.1 + 0 + 0.1 = 0.2 \\
f_Y(1) &= 0.1 + 0.2 + 0.2 = 0.5 \\
f_Y(2) &= 0.3 + 0 + 0 = 0.3\,.
\end{aligned}$$

Part (b): The expectations of X and Y are given by

$$\begin{aligned}
E[X] &= \sum_x x f_X(x) = 0(0.5) + 1(0.2) + 2(0.3) = 0.8 \\
E[Y] &= \sum_y y f_X(y) = 0(0.2) + 1(0.5) + 2(0.3) = 1.1\,.
\end{aligned}$$

Part (c): Now the variables X and Y can take on values from the set $\{0, 1, 2\}$, so that the random variable $Z = X - Y$ can take on values between the "endpoints" $0 - 2 = -2$ and $2 - 0 = 2$. That is values from the set $\{-2, -1, 0, +1, +2\}$. The probability of each of these points is given by

$$\begin{aligned}
f_Z(-2) &= f_{X,Y}(0,2) = 0.3 \\
f_Z(-1) &= f_{X,Y}(0,1) + f_{X,Y}(1,2) = 0.1 + 0.0 = 0.1 \\
f_Z(0) &= f_{X,Y}(0,0) + f_{X,Y}(1,1) + f_{X,Y}(2,2) = 0.1 + 0.2 + 0 = 0.3 \\
f_Z(+1) &= f_{X,Y}(0,1) + f_{X,Y}(2,1) = 0 + 0.2 = 0.2 \\
f_Z(+2) &= f_{X,Y}(2,0) = 0.1\,.
\end{aligned}$$

Part (d): The expectation of Z computed directly is given by

$$E[Z] = -2(0.3) + (-1)(0.1) + 0(0.3) + 1(0.2) + 2(0.1) = -0.6 - 0.1 + 0.2 + 0.2 = -0.3\,.$$

While using linearity we have the expectation of Z given by

$$E[Z] = E[X] - E[Y] = 0.8 - 1.1 = -0.3\,,$$

the same result.

Exercise 2 (a continuous joint density)

Part (a): We find the marginal distribution $f(x)$ given by

$$\begin{aligned}
f(x) &= \int f(x,y)dy = \int_0^\infty x^2 y e^{-xy} dy \\
&= x^2 \int_0^\infty y e^{-xy} dy = x^2 \left[\left. \frac{y e^{-xy}}{(-x)} \right|_0^\infty + \frac{1}{x} \int_0^\infty e^{-xy} dy \right] \\
&= \frac{x^2}{x} \int_0^\infty e^{-xy} dy = x \frac{1}{(-x)} e^{-xy} \Big|_0^\infty = -1(0-1) = 1\,,
\end{aligned}$$

a uniform distribution. The marginal distribution for Y in the same way is given by

$$\begin{aligned}
f(y) &= \int f(x,y)dx = \int_1^2 x^2 y e^{-xy} dx \\
&= y \int_1^2 x^2 e^{-xy} dx = y \int_y^{2y} \frac{v^2}{y^2} e^{-v} \frac{dv}{y} = \frac{1}{y^2} \int_y^{2y} v^2 e^{-v} dv\,,
\end{aligned}$$

where we have used the substitution $v = xy$ (so that $dv = ydx$). Integrating this last expression we get

$$f(y) = \frac{e^{-2y}}{y^2}(-2 - 4y - 4y^2) + \frac{e^{-y}}{y^2}(2 + 2y + y^2))\,,$$

for the density of Y.

Part (b): We find our two expectations given by

$$\begin{aligned}
E[X] &= \int_1^2 x f(x) dx = \frac{3}{2} \quad \text{and} \\
E[Y] &= \int_0^\infty y f(y) dx = \ln(4)\,.
\end{aligned}$$

Part (c): Using the definition of the covariance we can derive $\mathrm{Cov}(X,Y) = E[XY] -$

$E[X]E[Y]$. To use this we first need to compute $E[XY]$. We find

$$
\begin{aligned}
E[XY] &= \int xyf(x,y)dxdy = \int x^3y^2e^{-xy}dxdy \\
&= \int_{x=1}^{2} x^3 \int_{y=0}^{\infty} y^2e^{-xy}dydx \\
&= \int_{x=1}^{2} x^3 \left(\frac{2}{x^3}\right) dx = \int_{x=1}^{2} 2dx = 2\,.
\end{aligned}
$$

Using Part (b) above we then see that

$$
\text{Cov}(X,Y) = 2 - \frac{3}{2}(2\ln(2)) = 2 - 3\ln(2)\,.
$$

The algebra for this problem is worked in the Mathematica file `chap_4_prob_2.nb`.

Exercise 3 (the distribution function for the maximum of n uniform RVs)

Part (a): We find

$$
\begin{aligned}
P\{M < x\} &= P\{x_1 < x\,, x_2 < x\,, x_3 < x\,, \cdots, x_n < x\} \\
&= \prod_{i=1}^{n} P\{x_i < x\} = \prod_{i=1}^{n} x = x^n\,,
\end{aligned}
$$

since $\max(x_1, x_2, \cdots, x_n) < x$ if and only if all the individual x_i are less than or equal to x.

Part (b): We next find that the density function for M is given by

$$
f_M(x) = \frac{d}{dx}P\{M \le x\} = nx^{n-1}\,,
$$

so that we obtain

$$
E[M] = \int_0^1 xnx^{n-1}dx = n\int_0^1 x^n dx = \left.\frac{nx^{n+1}}{n+1}\right|_0^1 = \frac{n}{n+1}\,,
$$

for the expected value of the maximum of n independent uniform random variables.

Exercise 4 (the probability we land in a circle of radius a)

Part (a): We have

$$\begin{aligned}
P\{R \leq a\} &= \int_{\Omega=\{R\leq a\}} f(x,y)dxdy \\
&= \int_{\Omega=\{\sqrt{x^2+y^2}\leq a\}} f(x,y)dxdy \\
&= \int_{\Omega=\{\sqrt{x^2+y^2}\leq a\}} \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\right) dxdy \\
&= \frac{1}{2\pi}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} e^{-r^2/2} r dr d\theta = \int_{r=0} a e^{-r^2/2} r dr\,.
\end{aligned}$$

To evaluate this let $v = \frac{r^2}{2}$ so that $dv = rdr$ to get the above integral is equal to

$$\int_0^{a^2/2} e^{-v} dv = 1 - e^{-a^2/2}\,.$$

Part (b): We then find the density function for R given by

$$f_R(a) = \frac{d}{da} P\{R \leq a\} = -e^{-a^2/2}(-a) = a e^{-a^2/2}\,.$$

Exercise 5 (an unbalanced coin)

Let X_1 be the number of flips needed to get the first head, X_2 the number of additional flips needed to get the second head X_2, and X_3 the number of flips needed to get the third head. Then N is given by $X_1 + X_2 + X_3$ and

$$E[N] = E[X_1] + E[X_2] + E[X_3]\,.$$

Each X_i is a geometric RV with $p = \frac{1}{4}$, thus $E[X_i] = \frac{1}{p} = 4$. Thus $E[N] = 3E[X_1] = 12$. As each X_i is independent we have

$$\text{Var}(N) = \sum_{i=1}^{3} \text{Var}(X_i) = \sum_{i=1}^{3}\left(\frac{q}{p^2}\right) = \frac{3\left(\frac{3}{4}\right)}{\left(\frac{1}{4}\right)^2} = 36\,.$$

Exercise 6

Part (a): We have

$$\begin{aligned}
E[I_A] = P(A) &= \frac{|\{2,4,6,8\}|}{9} = \frac{4}{9} \\
E[I_B] = P(B) &= \frac{|\{3,6,9\}|}{9} = \frac{3}{9} = \frac{1}{3}\,.
\end{aligned}$$

Note that $E[I_A^2] = E[I_A]$ and the same for $E[I_B^2]$. Thus

$$\mathrm{Var}(I_A) = E[I_A^2] - E[I_A]^2 = \frac{4}{9} - \frac{16}{81} = \frac{20}{81}\,,$$

and

$$\mathrm{Var}(I_B) = E[I_B^2] - E[I_B]^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}\,.$$

Now we have

$$\begin{aligned}\mathrm{Cov}(I_A, I_B) &= E[I_A I_B] - E[I_A]E[I_B] = P(A \cap B) - P(A)P(B)\\ &= \frac{|\{6\}|}{9} - \frac{1}{3}\cdot\frac{4}{9} = \frac{1}{9} - \frac{4}{27} = -\frac{1}{27}\,.\end{aligned}$$

Part (b): Since $\mathrm{Cov}(I_A, I_B) = -\frac{1}{27}$ when we repeat this experiment n times we would find

$$\mathrm{Cov}(X, Y) = n\mathrm{Cov}(I_A, I_B) = -\frac{n}{27}\,.$$

Exercise 7 (counting birthdays)

Following the hint in the book the number of distinct birthdays X can be computed from X_i using $X = \sum_{i=1}^{365} X_i$. Thus the expectation of X is given by

$$E[X] = \sum_{i=1}^{365} E[X_i] = 365E[X_1]\,.$$

Now

$$\begin{aligned}E[X_i] &= P\{X_i = 1\} = P\{\text{At least one of the } N \text{ people has day } i \text{ as their birthday}\}\\ &= 1 - P\{\text{None of the } N \text{ people has day } i \text{ as their birthday}\}\\ &= 1 - \left(1 - \frac{1}{365}\right)^N\,.\end{aligned}$$

Using this we have

$$E[X] = 365\left(1 - \left(1 - \frac{1}{365}\right)^N\right)\,.$$

If we let $p = \frac{1}{365}$ and $q = 1 - p$ we get

$$E[X] = 365(1 - q^N) = \frac{1 - q^N}{p}\,.$$

Note: I was not sure how to compute $\mathrm{Var}(X)$. If anyone has any idea on how to compute this please contact me.

Exercise 8 (Sam and Sarah shopping)

We are told that Sam's shopping time is $T_{\text{Sam}} \sim U[10, 20]$ and Sarah's shopping time is $T_{\text{Sarah}} \sim U[15, 25]$. Then we want to evaluate $P\{T_{\text{Sam}} \leq T_{\text{Sarah}}\}$. We find

$$
\begin{aligned}
P\{T_{\text{Sam}} \leq T_{\text{Sarah}}\} &= \iint_{T_{\text{Sam}} \leq T_{\text{Sarah}}} p(t_{\text{Sam}}, t_{\text{Sarah}}) dt_{\text{Sam}} dt_{\text{Sarah}} \\
&= \int_{t_{\text{Sam}}=10}^{20} \int_{t_{\text{Sarah}}=\max(15, t_{\text{Sam}})}^{25} \frac{1}{10} \cdot \frac{1}{10} dt_{\text{Sarah}} dt_{\text{Sam}} \\
&= \int_{t_{\text{Sam}}=10}^{15} \int_{t_{\text{Sarah}}=15}^{25} \frac{1}{100} dt_{\text{Sarah}} dt_{\text{Sam}} + \int_{t_{\text{Sam}}=15}^{20} \int_{t_{\text{Sarah}}=t_{\text{Sam}}}^{25} \frac{1}{100} dt_{\text{Sarah}} dt_{\text{Sam}} \\
&= \frac{1}{100}(10)(5) + \int_{t_{\text{Sam}}=15}^{20} \frac{1}{100}(25 - t_{\text{Sam}}) dt_{\text{Sam}} \\
&= \frac{1}{2} - \frac{1}{100} \left. \frac{(25 - t_{\text{Sam}})^2}{2} \right|_{15}^{20} = \frac{7}{8}.
\end{aligned}
$$

Exercise 9 (some probabilities)

Part (a): We have

$$
\begin{aligned}
P\{Y \leq X\} &= \iint_{Y \leq X} 1 p_{X,Y}(x, y) dx dy = \int_{x=0}^{1} \int_{y=0}^{1} 1(2y) dy dx \\
&= \int_{x=0}^{1} y^2 \Big|_0^x dx = \int_{x=0}^{1} x^2 dx = \left. \frac{x^3}{3} \right|_0^1 = \frac{1}{3}.
\end{aligned}
$$

Part (b): Let $Z = X + Y$ then as X and Y are mutually independent we have

$$
f_Z(z) = \int f_X(z - y) f_Y(y) dy = \int f_Y(z - x) f_X(x) dx .
$$

Using the second expression above and the domain of X we have

$$
f_Z(z) = \int_{x=0}^{1} f_Y(z - x) dx .
$$

Let $v = z - x$ so that $dv = -dx$ and $f_Z(z)$ becomes

$$
f_Z(z) = \int_{z}^{z-1} f_Y(v)(-dv) = \int_{z-1}^{z} f_Y(v) dv .
$$

Now since $0 < X < 1$ and $0 < Y < 1$ we must have that $0 < Z < 2$ as the domain of the RV Z. Note that if $0 < z < 1$ then $z - 1 < 0$ and the integrand $f_Y(v)$ in the above integral is zero on this domain of v. Thus

$$
f_Z(z) = \int_0^z f_Y(v) dv = \int_0^z 2v dv = v^2 \Big|_0^z = z^2 \quad \text{for} \quad 0 < z < 1 .
$$

If $1 < z < 2$ by similar logic then the density function $f_Z(z)$ becomes

$$f_Z(z) = \int_{z-1}^{1} f_Y(v)dv = \int_{z-1}^{1} 2vdv = v^2\Big|_{z-1}^{1} = 1 - (z-1)^2 = 2z - z^2 \quad \text{for} \quad 1 < z < 2\,.$$

Exercise 10

Part (a): By the geometrical interpretation of probability (i.e. the area of the given triangle) we have $f_{X,Y}(x, y) = \frac{1}{2}(4)(1) = 2$ for (x, y) in the region given i.e. $1 \le x \le 4$ and $0 \le y \le 1 - \frac{1}{4}x$.

Part (b): We have

$$f_X(x) = \int f_{X,Y}(x, y)dy = \int_0^{1-\frac{1}{4}x} 2dy = 2\left(1 - \frac{1}{4}x\right)\,,$$

for $1 \le x \le 4$. Next we have

$$f_Y(y) = \int f_{X,Y}(x, y)dx = \int_0^{4(1-x)} 2dx = 8(1 - y)\,,$$

for $0 \le y \le 1$.

Part (c): As $f_X(x)f_Y(y) \neq f_{X,Y}(x, y)$ these random variables are not independent.

Exercise 11

Part (a): Because X and Y are independent we have that

$$\begin{aligned}
P\{X + Y \ge 1\} &= \iint_{X+Y\ge 1} f_X(x)f_Y(y)dxdy \\
&= \int_{x=0}^{1}\int_{y=1-x}^{\infty} e^{-y}dydx = \int_{x=0}^{1} -\left(e^{-y}\Big|_{1-x}^{\infty}\right. \\
&= \int_{x=0}^{1} -(0 - e^{x-1})dx = \int_{x=0}^{1} e^{x-1}dx = e^{x-1}\Big|_0^1 = 1 - e^{-1} = 0.6321\,.
\end{aligned}$$

Part (b): Again because X and Y are independent we have that

$$\begin{aligned}
E[Y^2e^X] &= \iint y^2 e^x f_X(x)f_Y(y)dxdy \\
&= \int_0^1\int_0^{\infty} y^2e^xe^{-y}dydx = \int_0^{\infty} y^2e^{-y}dy \int_0^1 e^x dx \\
&= (e-1)\int_0^{\infty} y^2e^{-y}dy = 2(e-1) = 3.43656\,.
\end{aligned}$$

Note this result is different than in the back of the book. If anyone sees anything wrong with what I have done (or agrees with me) please let me know.

Exercise 12 (hitting a circular target)

Part (a): When the density of hits is uniform over the target the average point received for a hit is proportional to the fraction of the area each region occupies. The total target has an area given by $\pi 3^3 = 9\pi$. The area of the one point region is

$$A_1 = 9\pi - 4\pi = 5\pi\,.$$

The area of the four point region is given by

$$A_4 = 4\pi - \pi = 3\pi\,.$$

The area of the ten point region is given by $A_{10} = \pi$. Thus the probabilities of the one, four, and ten point region are given by

$$\frac{5}{9}\,,\quad \frac{1}{3}\,,\quad \frac{1}{9}\,.$$

Thus the expected point value is given by

$$E[P] = \frac{5}{9}(1) + \frac{1}{3}(4) + \frac{1}{9}(10) = 3\,.$$

Part (b): Lets first normalize the given density. We need to find c such that

$$c\iint (9 - x^2 - y^2)dxdy = 1\,,$$

or

$$c\iint (9 - r^2)rdrd\theta = 1 \quad \text{or} \quad 2\pi c\int_0^3 (9r - r^3)dr = 1\,,$$

or when we perform the radial integration we get

$$\frac{81\pi}{2}c = 1\,.$$

Thus $c = \frac{2}{81\pi}$. We now can compute the average points

$$E[P] = \iint_{A_{10}} 10c(9 - x^2 - y^2)dxdy + \iint_{A_4} 4c(9 - x^2 - y^2)dxdy + \iint_{A_1} 1c(9 - x^2 - y^2)dxdy\,.$$

The angular integrations all integrate to 2π and thus we get

$$\frac{E[P]}{2\pi c} = 10\int_0^1 (9 - r^2)rdr + 4\int_1^2 (9 - r^2)rdr + \int_2^3 (9 - r^2)rdr = \frac{351}{4}\,.$$

Solving for $E[P]$ we get $E[P] = \frac{13}{3} = 4.3333$.

Exercise 13 (the distribution of the sum of three uniform random variables)

Note in this problem I will find the density function for $\frac{1}{3}(X_1 + X_2 + X_3)$ which is a variant of what the book asked. Modifying this to exactly match the question in the book should be relatively easy.

If X is a uniform random variable over $(-1, +1)$ then it has a p.d.f. given by

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} ,$$

while the random variable $Y = \frac{X}{3}$ is another uniform random variable with a p.d.f. given by

$$p_Y(y) = \begin{cases} \frac{3}{2} & -\frac{1}{3} \leq x \leq \frac{1}{3} \\ 0 & \text{otherwise} \end{cases} .$$

Since the three random variables $X/3$, $Y/3$, and $Z/3$ are independent the characteristic function of the sum of them is the product of the characteristic function of each one of them. For a uniform random variable over the domain (α, β) on can show that the characteristic function $\phi(t)$ is given by

$$\phi(t) = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} e^{itx} dx = \frac{e^{it\beta} - e^{it\alpha}}{it(\beta - \alpha)} ,$$

note this is a slightly different than the normal definition of the Fourier transform [2], which has e^{-itx} as the exponential argument. Thus for each of the random variables $X/3$, $Y/3$, and $Z/3$ the characteristic function since $\beta = \frac{1}{3}$ and $\alpha = -\frac{1}{3}$ looks like

$$\phi(t) = \frac{3(e^{it(1/3)} - e^{-it(1/3)})}{2it} .$$

Thus the sum of two uniform random variables like $X/3$ and $Y/3$ has a characteristic function given by

$$\phi^2(t) = -\frac{9}{4t^2}(e^{it(2/3)} - 2 + e^{-it(2/3)}) ,$$

and adding in a third random variable say $Z/3$ to the sum of the previous two will give a characteristic function that looks like

$$\phi^3(t) = -\frac{27}{8i}\left(\frac{e^{it}}{t^3} - \frac{3e^{it(1/3)}}{t^3} + \frac{3e^{-it(1/3)}}{t^3} - \frac{e^{-it}}{t^3}\right) .$$

Given the characteristic function of a random variable to compute its probability density function from it we need to evaluate the *inverse Fourier transform* of this function. That is we need to evaluate

$$p_W(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)^3 e^{-itw} dt .$$

Note that this later integral is equivalent to $\frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)^3 e^{+itw} dt$ (the standard definition of the inverse Fourier transform) since $\phi(t)^3$ is an even function. To evaluate this integral then it will be helpful to convert the complex exponentials in $\phi(t)^3$ into trigonometric functions by writing $\phi(t)^3$ as

$$\phi(t)^3 = \frac{27}{4}\left(\frac{3\sin\left(\frac{t}{3}\right)}{t^3} - \frac{\sin(t)}{t^3}\right). \tag{23}$$

Thus to solve this problem we need to be able to compute the inverse Fourier transform of two expressions like

$$\frac{\sin(\alpha t)}{t^3}.$$

To do that we will write it as a product with two factors as

$$\frac{\sin(\alpha t)}{t^3} = \frac{\sin(\alpha t)}{t} \cdot \frac{1}{t^2}.$$

This is helpful since we (might) now recognize as the *product* of two functions each of which we know the Fourier transform of. For example one can show [2] that if we define the step function $h_1(w)$ as

$$h_1(w) \equiv \left\{ \begin{array}{cc} \frac{1}{2} & |w| < \alpha \\ 0 & |w| > \alpha \end{array} \right.,$$

then the Fourier transform of this step function $h_1(w)$ is the first function in the product above or $\frac{\sin(\alpha t)}{t}$. Notationally, we can write this as

$$\mathcal{F}\left[\left\{ \begin{array}{cc} \frac{1}{2} & |w| < \alpha \\ 0 & |w| > \alpha \end{array} \right.\right] = \frac{\sin(\alpha t)}{t}.$$

In the same way if we define the ramp function $h_2(w)$ as

$$h_2(w) = -w\,u(w),$$

where $u(w)$ is the unit step function

$$u(w) = \left\{ \begin{array}{cc} 0 & w < 0 \\ 1 & w > 0 \end{array} \right.,$$

then the Fourier transform of $h_2(w)$ is given by $\frac{1}{t^2}$. Notationally in this case we then have

$$\mathcal{F}\left[-wu(w)\right] = \frac{1}{t^2}.$$

Since the inverse of a function that is the product of two functions for which we know the individual inverse Fourier transform of is the *convolution* integral of the two inverse Fourier transforms we have that

$$\mathcal{F}^{-1}\left[\frac{\sin(\alpha t)}{t^3}\right] = \int_{-\infty}^{\infty} h_1(x)h_2(w-x)dx,$$

the other ordering of the integrands

$$\int_{-\infty}^{\infty} h_1(w-x)h_2(x)dx,$$

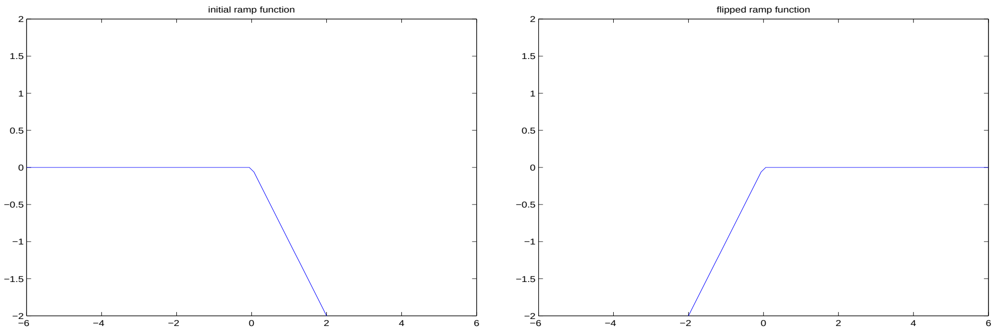

Figure 4: **Left:** The initial function $h_2(x)$ (a ramp function). **Right:** The ramp function flipped or $h_2(-x)$.

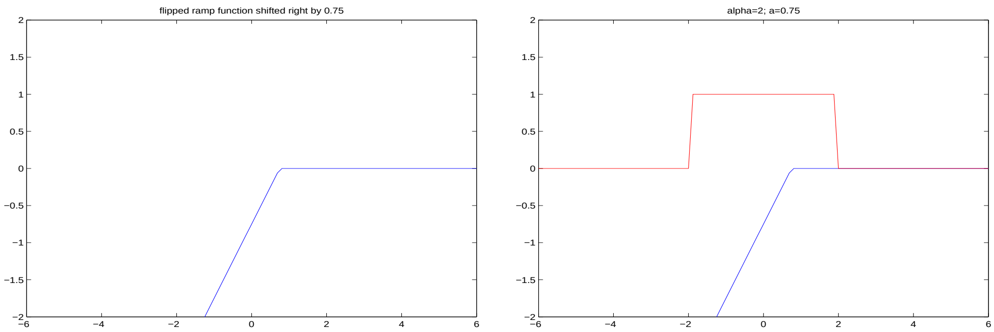

Figure 5: **Left:** The function $h_2(x)$, flipped and shifted by $w = 3/4$ to the right or $h_2(-(x - w))$. **Right:** The flipped and shifted function plotted together with $h_1(x)$ allowing visualizations of function overlap as w is varied.

can be shown to be an equivalent representation. To evaluate the above convolution integral and finally obtain the p.d.f for the sum of three uniform random variables we might as well select a formulation that is simple to evaluate. I'll pick the *first* formulation since it is easy to flip and shift to the ramp function $h_2(\cdot)$ distribution to produce $h_2(w - x)$. Now since $h_2(x)$ looks like the plot given in Figure 4 (left) we see that $h_2(-x)$ then looks like Figure 4 (right). Inserting a right shift by the value w we have $h_2(-(x - w)) = h_2(w - x)$, and this function looks like that shown in Figure 5 (left). The shifted factor $h_2(w - x)$ and our step function $h_1(x)$ are plotted together in Figure 5 (right). These considerations give a functional form for the p.d.f of $g_\alpha(w)$ given by

$$
\begin{aligned}
g_\alpha(w) &= \begin{cases} 0 & w < -\alpha \\ \int_{-\alpha}^{w} \frac{1}{2}(x - w)dx & -\alpha < w < +\alpha \\ \int_{-\alpha}^{+\alpha} \frac{1}{2}(x - w)dx & w > \alpha \end{cases} \\
&= \begin{cases} 0 & w < -\alpha \\ -\frac{1}{4}(\alpha + w)^2 & -\alpha < w < +\alpha \\ -\alpha w & w > \alpha \end{cases} ,
\end{aligned}
$$

when we evaluate each of the integrals. Using this and Equation 23 we see that

$$
\begin{aligned}
\mathcal{F}^{-1}[\phi^3(t)] &= \frac{27}{4}(3g_{1/3}(w) - g_1(w)) \\
&= -\frac{81}{4}\left\{\begin{array}{cc} 0 & w < -\frac{1}{3} \\ \frac{1}{4}(\frac{1}{3}+w)^2 & -\frac{1}{3} < w < +\frac{1}{3} \\ \frac{1}{3}w & w > \frac{1}{3} \end{array}\right. + \frac{27}{4}\left\{\begin{array}{cc} 0 & w < -1 \\ \frac{1}{4}(1+w)^2 & -1 < w < +1 \\ w & w > 1 \end{array}\right. \\
&= \left\{\begin{array}{cc} 0 & w < -1 \\ \frac{27}{16}(1+w)^2 & -1 < w < -\frac{1}{3} \\ -\frac{9}{8}(-1+3w^2) & -\frac{1}{3} < w < +\frac{1}{3} \\ \frac{27}{16}(-1+w)^2 & \frac{1}{3} < w < 1 \\ 0 & w > 1 \end{array}\right. ,
\end{aligned}
$$

which is equivalent to what we were to show.

Exercise 14 (some moment generating functions)

Part (a): We find

$$\phi_X(t) = E[e^{Xt}] = \frac{1}{3}e^{-t} + \frac{1}{3} + \frac{1}{3}e^t \,.$$

Part (a): We find

$$E[X] = \frac{1}{3}(-1) + \frac{1}{3}(0) + \frac{1}{3}(+1) = 0 \,,$$

and

$$E[X^2] = \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(1) = \frac{2}{3} \,.$$

To use the moment-generating function we use

$$E[X^n] = \left.\frac{d^n}{dt^n}\phi_X(t)\right|_{t=0} \tag{24}$$

For the moment generating function calculated above using this we get

$$E[X] = \left.\frac{d}{dt}\phi_X(t)\right|_{t=0} = \left(-\frac{1}{3}e^{-t} + \frac{1}{3}e^t\right|_{t=0} = 0 \,,$$

and

$$E[X^2] = \left.\frac{d^2}{dt^2}\phi_X(t)\right|_{t=0} = \left(\frac{1}{3}e^{-t} + \frac{1}{3}e^t\right|_{t=0} = \frac{2}{3} \,,$$

the same as before.

Exercise 15 (more moment generating functions)

We need

$$\frac{d}{dt}\phi_X(t) = \frac{2e^t}{3-t} + \frac{2e^t+1}{(3-t)^2} = \frac{(8-2t)e^t+1}{(3-t)^2} \,,$$

and

$$\frac{d^2}{dt^2}\phi_X(t) = -\frac{2(1+e^t(17-8t+t^2))}{(-3+t)^3}\,.$$

Thus

$$\begin{aligned} E[X] &= \left.\frac{d}{dt}\phi_X(t)\right|_{t=0} = 1 \\ E[X^2] &= \left.\frac{d^2}{dt^2}\phi_X(t)\right|_{t=0} = \frac{4}{3}\,. \end{aligned}$$

With these two we compute

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = \frac{1}{3}\,.$$

Exercise 16 (estimating probabilities)

Part (a): Recall that Chebyshev's inequality for the sample mean $\bar{X}_n$ is given by

$$P\{|\bar{X}_n - \mu| \geq \delta\} \leq \frac{\sigma^2}{n\delta^2}\,,$$

if $\delta = 1$, $n = 100$, $\mu = 50$, and $\sigma = 5$ we get

$$P\{|\bar{X}_n - 50| \geq 1\} \leq \frac{25}{n} = \frac{1}{4}\,.$$

This is equivalent to

$$P\{\bar{X}_n \leq 49 \quad \text{or} \quad \bar{X}_n \geq 51\} \leq \frac{1}{4}\,.$$

or

$$1 - P\{\bar{X}_n \leq 49 \quad \text{or} \quad \bar{X}_n \geq 51\} \geq \frac{3}{4}\,.$$

or

$$P\{49 \leq \bar{X}_n \leq 51\} \geq \frac{3}{4}\,.$$

This gives the value $c = \frac{3}{4}$.

Part (b): We have

$$\begin{aligned} P\{49 \leq \bar{X}_n \leq 51\} &= P\left\{\frac{49-50}{\frac{5}{\sqrt{n}}} \leq \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \leq \frac{51-50}{\frac{5}{\sqrt{n}}}\right\} \\ &= \Phi\left(\frac{51-50}{5/10}\right) - \Phi\left(\frac{49-50}{5/10}\right) = 0.954\,. \end{aligned}$$

Exercise 17

Since for the random variable X we have $\mu = np = 100(1/2) = 50$ and $\sigma = \sqrt{npq} = \sqrt{50(1/2)} = 5$ when we convert to a standard normal RV we find

$$\begin{aligned} P\{X > 55\} &= P\{X \geq 45.5\} \quad \text{since } X \text{ is integer values i.e. the "continuity correction"} \\ &= 1 - P\{X < 45.5\} = 1 - P\left\{\frac{X - 50}{5} < \frac{45.5 - 50}{5}\right\} \\ &= 1 - \Phi\left(\frac{45.5 - 50}{5}\right) = 0.816 \,. \end{aligned}$$

Exercise 18

We can calculate that the probability of rolling a sum of a 7 using two dice is given by $\frac{1}{6}$. Then X is a binomial RV with parameters $n = 100$ and $p = \frac{1}{6}$. Thus

$$\begin{aligned} \mu &= np = 83.33 \\ \sigma &= \sqrt{npq} = 8.33 \,. \end{aligned}$$

Exercise 19

Part (a): We have that

$$\begin{aligned} E[X_1] &= -a\left(\frac{1}{8}\right) + 0\left(\frac{3}{4}\right) + a\left(\frac{1}{8}\right) = 0 \\ E[X_1^2] &= a^2\left(\frac{1}{8}\right) + 0^2\left(\frac{3}{4}\right) + a^2\left(\frac{1}{8}\right) = \frac{1}{4}a^2 \,. \end{aligned}$$

Thus

$$\text{Var}(X_1) = E[X_1^2] = \frac{1}{4}a^2 = 1 \quad \Rightarrow \quad a = 2 \,.$$

With this value of a we have

$$P\{|X_1| \geq 2\} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \,,$$

and Chebyshev's inequality is sharp, i.e. it holds with equality.

Part (b): We can see that $E[X_2] = 0$ and can compute

$$E[X_2^2] = \int_{-b}^{b} x^2\left(\frac{1}{2b}\right) dx = \frac{1}{2b}\left(\frac{x^3}{2}\bigg|_{-b}^{b}\right) = \frac{1}{6b}(2b^3) = \frac{b^2}{3} \,.$$

For $\text{Var}(X_2) = 1$ we must have $b = \sqrt{3} = 1.73$. Then

$$P\{|X_2| \geq 2\} = 0 \,,$$

since X_2 can only be as large as $\sqrt{3} < 2$.

Part (c): We can see that $E[X_3] = 0$ and can compute

$$E[X_3^2] = 0.005c^2 + 0.005c^2 = 0.01c^2 \,.$$

To have $\text{Var}(X_3) = 1$ we need $\frac{c^2}{100} = 1$ or $c = 10$. With this we find

$$P\{|X_3| \geq 2\} = 0.01 \,.$$

Part (d): We can see that $E[X_3] = 0$ and can compute directly

$$P\{|X_4| \geq 2\} = 2\Phi(-2) = 2(1 - \Phi(2)) = 2(1 - 0.97725) = 0.0455 \,.$$

Exercise 20

Part (a): We can compute using integration by parts that

$$\begin{aligned}
P\{Z > t\} &= \int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_t^\infty x^{-1} x e^{-\frac{x^2}{2}} dx \quad \text{using} \quad \int u dv = uv - \int v du \\
&= \frac{1}{\sqrt{2\pi}} x^{-1}(-e^{-\frac{x^2}{2}})\Big|_t^\infty - \frac{1}{\sqrt{2\pi}} \int_t^\infty (-x^{-2})(-e^{-\frac{x^2}{2}}) dx \\
&= \frac{1}{\sqrt{2\pi}} \left(0 + \frac{1}{t} e^{-\frac{t^2}{2}}\right) - \frac{1}{\sqrt{2\pi}} \int_t^\infty x^{-2} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi} t} e^{-\frac{t^2}{2}} - \frac{1}{\sqrt{2\pi}} \int_t^\infty x^{-2} e^{-\frac{x^2}{2}} dx \,,
\end{aligned}$$

which is the desired expression.

Part (b): From the above expression we see that

$$R(t) \equiv (2\pi)^{-1/2} \int_t^\infty x^{-2} e^{-\frac{x^2}{2}} dx \,.$$

Now $R(t) > 0$ since the function integrated is nonnegative. Next note that on the region of integration the following chain of inequalities hold true

$$x > t \quad \text{so} \quad \frac{1}{x} < \frac{1}{t} \quad \text{so} \quad \frac{1}{x^2} < \frac{1}{t^2} \,.$$

In addition, from $x > t$ we have that $1 < \frac{x}{t}$. Thus

$$\frac{1}{x^2} < \frac{1}{t^2}(1) < \frac{x}{t^3} \,.$$

When we up this inequality into the expression for $R(t)$ we get

$$
\begin{aligned}
R(t) &= \frac{1}{\sqrt{2\pi}} \int_t^\infty x^{-2} e^{-\frac{x^2}{2}} dx < \frac{1}{\sqrt{2\pi} t^3} \int_t^\infty x e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi} t^3} \left(-e^{-\frac{x^2}{2}} \right|_t^\infty = \frac{1}{\sqrt{2\pi} t^3} e^{-\frac{t^2}{2}} \, .
\end{aligned}
$$

Part (c): We find

$$
P\{Z > 3.5\} \approx (2\pi)^{-1/2} (3.5)^{-1} e^{-\frac{3.5^2}{2}} = 2.493 \cdot 10^{-4} \, .
$$

Part (d): We find

$$
P\{Z > 6\} \approx (2\pi)^{-1/2} 6^{-1} e^{-\frac{6^2}{2}} = 1.01 \cdot 10^{-9} \, .
$$

The error correction to this value is

$$
0 < R(t) < (2\pi)^{-1/2} 6^{-3} e^{-\frac{6^2}{2}} = 2.8 \cdot 10^{-11} \, .
$$

Chapter 5: Conditional Expectation

Notes on the Text

Notes on Example 5.2 (the expected value for second roll)

The expected value for the second roll is

$$\frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6}(21) = \frac{7}{2} = 3.5\,.$$

Notes on Example 5.7

For

$$f_{X,Y}(x,y) = \frac{1}{10y} \quad 0 < x < y < 10\,,$$

to compute $f_X(x)$ we "integrate out" the variable y from the joint density function $f_{X,Y}(x,y)$. We find

$$f_X(x) = \int f_{X,Y}(x,y)dy = \int_x^{10} \frac{dy}{10y}\,.$$

Notes on Example 5.8 (using iterated expectations)

To evaluate

$$E[X] = \int_0^{10} -\frac{x}{10}\ln\left(\frac{x}{10}\right)dx = -\frac{1}{10}\int_0^{10} x\log\left(\frac{x}{10}\right)dx\,,$$

we let $v = \frac{x}{10}$ so $dv = \frac{dx}{10}$ get

$$-\int_0^1 10v\ln(v)dv\,.$$

To integrate this we will use

$$\int x\ln(x)dx = \frac{x^2}{4}(2\ln(x)-1)+c\,. \tag{25}$$

To verify the above integral identity we start by evaluating $\int \ln(x)dx$, where we use the integration by parts formula $\int udv = uv - \int vdu$ with $u = \ln(x)$ and $dv = dx$ to get

$$\int \ln(x)dx = x\ln(x) - \int x\left(\frac{1}{x}\right)dx = x\ln(x) - x + c\,. \tag{26}$$

Now in the integral of interest given by Equation 25 we will again use integration by parts with $u = x$ and $dv = \ln(x)dx$ and result in Equation 26 to get

$$\begin{aligned}\int x\ln(x)dx &= x(x\ln(x) - x) - \int 1(x\ln(x) - x)dx \\ &= x^2\ln(x) - x^2 - \int x\ln(x)dx + \frac{x^2}{2} + c\,.\end{aligned}$$

Solving for $\int x\ln(x)dx$ in the previous expression we find

$$\int x\ln(x)dx = \frac{1}{2}\left(x^2\ln(x) - \frac{x^2}{2}\right) + c = \frac{x^2}{4}(2\ln(x) - 1) + c\,, \tag{27}$$

which is the expression we were trying to prove. Using this integral we find

$$E[X] = -10\left(\frac{v^2}{4}(2\ln(v) - 1)\right|_0^1 = -10\left(\frac{1}{4}(-1) - 0\right) = \frac{5}{2}\,,$$

the same as before.

Exercise Solutions

Exercise 1

Part (a): X and Y alone are geometric random variables with probability of success $p = \frac{1}{6}$. From the table in the back of the book we have that the expectation of a geometric RV is given by $E[X] = E[Y] = \frac{1}{p} = 6$.

Part (b): To compute $E[X|Y = 1]$ we note that since the event $\{Y = 1\}$ is true means that the first roll must be a five. Once this five has been rolled we will then have to try to roll a six, which happens on average after $\frac{1}{p} = 6$ rolls. Thus

$$E[X|Y = 1] = 1 + 6 = 7\,.$$

Note that X and Y are not independent since $E[X|Y = 1] \neq E[X]$.

Exercise 2

We are told that $P\{C = c\} = \frac{4^c e^{-4}}{c!}$ and $P\{T = t\} = \frac{2^t e^{-2}}{t!}$ for $c \geq 0$ and $t \geq 0$.

Part (a): Since C and T are independent we have

$$P\{C = 4, T = 0\} = P\{C = 4\}P\{T = 0\} = \left(\frac{4^4 e^{-4}}{4!}\right)\left(\frac{2^0 e^{-2}}{0!}\right) = 0.02644\,.$$

Part (b): Using $P(A|B) = \frac{P(A\cap B)}{P(B)}$ we compute

$$\begin{aligned}
P\{C=4|C+T=4\} &= \frac{P\{C=4, C+T=4\}}{P\{C+T=4\}} \\
&= \frac{P\{C=4, T=0\}}{P\{C=4,T=0\} + P\{C=3,T=1\} + P\{C=2,T=2\} + P\{C=1,T=3\} + P\{C=4,T=0\}} \\
&= 0.19753\,.
\end{aligned}$$

We find

$$\begin{aligned}
E[C|C+T=4] &= \sum_{c=0}^{4} cP\{C=c|C+T=4\} \\
&= \sum_{c=0}^{4} c\frac{P\{C=c, T=4-c\}}{P\{C+T=4\}} \\
&= \frac{1}{P\{C+T=4\}} \sum_{c=0}^{4} cP\{C=c, T=4-c\} = 2.667\,.
\end{aligned}$$

From the book as X and Y are Poisson RV then $p_{X|Z}(k|n)$ is a binomial RV with parameters $p = \frac{\lambda}{\lambda+\mu}$ and n. In this problem $p = \frac{4}{4+2} = \frac{2}{3}$ and $n = 4$. Thus

$$P\{C=4\} = \left(\begin{array}{c} 4 \\ 4 \end{array}\right) p^4 q^0 = p^4 = \left(\frac{2}{3}\right)^4 .$$

The expectation of C conditioned on $C+T=4$ is then $np = 4\left(\frac{2}{3}\right) = 2.6667$.

Exercise 3

Part (a): We are told that $p_Y(y) = e^{-y}$ and $p_{X|Y}(x|y) = \frac{1}{y}e^{-\frac{1}{y}x}$ since in this case $E[X|Y=y] = y$ as expected. Now

$$E[X] = E[E[X|Y]] = E[Y] = +1\,.$$

Part (b): We have

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = \frac{1}{y}e^{-\frac{1}{y}x}e^{-y} = \frac{1}{y}e^{-\left(\frac{x}{y}+y\right)}\,.$$

Exercise 4

Part (a): We are told $Y \sim e^{-y}$ and $X|Y=y \sim U[y, 3y]$ so that

$$E[X|Y=y] = \frac{3y+y}{2} = 2y\,,$$

since the conditional distribution is uniform. Using the "law of iterated expectation" or $E[X] = E[E[X|Y]]$ we have

$$E[X] = E[2Y] = 2E[Y] = 2\,.$$

Part (b): We find

$$\begin{aligned}E[X^2|Y=y] &= \int_{x=y}^{3y} x^2 \left(\frac{1}{2y}\right) dx = \frac{1}{2y}\left(\frac{x^3}{3}\right|_y^{3y} \\ &= \frac{1}{6y}(27y^3 - y^3) = \frac{1}{6}26y^2 = \frac{13}{3}y^2\,.\end{aligned}$$

Since $E[X^2] = E[E[X^2|Y]]$ we have

$$E[X^2] = \frac{13}{3}E[Y^2] = \frac{13}{3}\left(\text{Var}(Y) + E[Y]^2\right) = \frac{13}{3}(1+1^2) = \frac{26}{3}\,.$$

Then

$$\text{Var}(X) = \frac{26}{3} - 4 = \frac{26}{3} - \frac{12}{3} = \frac{14}{3} = 4.6667\,.$$

The back of the book has the value 8.6667 which I think is a typo. If anyone agrees or disagrees with me please contact me.

Exercise 5

Part (a): We are told $Y \sim U[0,1]$ and $X|Y=y \sim U[Y,1]$. We want to compute $E[X]$. Note that $E[X|Y] = \frac{1+Y}{2}$ thus

$$E[X] = E[E[X|Y]] = \frac{1}{2} + \frac{1}{2}E[Y] = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}\,.$$

Part (b): We need to compute $E[Y]$ and we find $E[Y] = \frac{1}{2}$. As we are told the two boys should split the money according to the fractions

$$\frac{3/4}{3/4+1/2} = \frac{3}{5} \quad \text{and} \quad \frac{1/2}{3/4+1/2} = \frac{2}{5}\,,$$

for Tom and Huck respectively. Thus Tom should get $\frac{3}{5}(10) = 6$ and Huck should get $\frac{2}{5}(10) = 4$.

Exercise 6

Part (a):

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)dx = \int_0^y \frac{e^{-y}}{y}dx = e^{-y} \quad \text{for} \quad 0 < y < +\infty\,.$$

Part (b): We have

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y} \quad \text{for} \quad 0 < x < y\,,$$

or a uniform density.

Part (c): We have

$$E[X^2|Y] = \int x^2 f_{X|Y}(x|y)dx = \int_0^y x^2 \left(\frac{1}{y}\right) dx = \frac{1}{y^2}\left(\frac{x^3}{3}\right|_0^y = \frac{1}{3y}y^3 = \frac{y^2}{3}\,.$$

Exercise 7

We are told that $P\{N = k\} = \left(\begin{array}{c} 10 \\ k \end{array}\right) 0.7^k 0.3^{10-k}$ for $0 \le k \le 10$ is the distribution of users of computers in one class period. Then for each ith student that has used the computer we will require X_i storage where $X_i \sim U[0, 10]$. Thus the total storage used/required is

$$S = \sum_{i=1}^{N} X_i\,,$$

with $N \sim \text{binomial}(10, 0.7)$. To evaluate $E[S]$ we will use the "law of iterated expectation" or $E[S] = E[E[S|N]]$ we find

$$E[S] = E[E[S|N]] = E[N(5)] = 5E[N] = 5(10)(0.7) = 35\,.$$

In the same way we have

$$E[S^2] = E[E[S^2|N]] = E\left[E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \middle| N\right]\right]\,.$$

Note that the sum squared can be written as

$$\left(\sum_{i=1}^{N} X_i\right)^2 = \sum_{i=1}^{N} X_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} X_i X_j\,.$$

To use this expression to evaluate the inner expectation $E[\cdot|N]$ we recall

$$E[X_i^2] = \text{Var}(X_i) + E[X_i]^2 = \frac{10^2}{12} + 5^2 = 33.33\,,$$

and $E[X_i X_j] = E[X_i]E[X_j] = 5^2 = 25$ by independence. With these two facts we find that

$$\begin{aligned}
E\left[\left(\sum_{i=1}^{N} X_i\right)^2 \middle| N\right] &= N(33.33) + 2\left(\begin{array}{c} N \\ 2 \end{array}\right) 25 \\
&= 33.33N + \frac{2N(N-1)}{2}(25) \\
&= 33.33N + 25N(N-1) = 25N^2 + 8.33N\,.
\end{aligned}$$

Now to compute the expectation of this we recall that

$$E[N^2] = \text{Var}(N) + E[N]^2 = npq + (np)^2 = 10(0.7)(0.3) + (10^2)(0.7)^2 = 51.1\,,$$

and then find

$$E[S^2] = 25(51.1) + 8.33(10)(0.7) = 1335.81\,,$$

thus we get for the variance

$$\text{Var}(S) = E[S^2] - E[S]^2 = 1335.81 - 35^2 = 110.81\,.$$

Exercise 8

Part (a): We are told that $Y \sim U[0,2]$ and $X|Y \sim ye^{-yx}$ and thus in this case then

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = ye^{-yx}\left(\frac{1}{2}\right)\,.$$

Part (b): We find

$$\begin{aligned}
P\{X > 5|Y = y\} &= \int_{x=5}^{\infty} ye^{-yx}dx = \left.\frac{ye^{-yx}}{(-y)}\right|_5^{\infty} \\
&= -\left(e^{-yx}\Big|_{x=5}^{\infty}\right. = -(e^{-\infty} - e^{-5y}) = e^{-5y} \quad \text{for} \quad 0 < y < 2\,.
\end{aligned}$$

Part (c): We find

$$\begin{aligned}
P\{X > 5\} &= \int_{y=0}^{2} P\{X > 5|Y = y\}\left(\frac{1}{2}\right)dy \\
&= \frac{1}{2}\int_{y=0}^{2} e^{-5y}dy = \frac{1}{2}\left(-\frac{1}{5}e^{-5y}\Big|_0^2\right. = -\frac{1}{10}(e^{-10} - 1) = \frac{1}{10}(1 - e^{-10})\,.
\end{aligned}$$

Exercise 9

Part (a): We want to evaluate $\text{Cov}(X,Y)$. From the discussion in the book $E[X] = 0$, $E[Y] = \beta$, $\text{Var}(X) = 1$, and $\text{Var}(Y) = \beta$. Using these facts when needed we find

$$\begin{aligned}
\text{Cov}(X,Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[(X - 0)(Y - \beta)] = E[X(Y - \beta)] \\
&= E[XY] - \beta E[X] = E[XY] \\
&= E[E[XY|X]] = E[XE[Y|X]] \\
&= E[X(\alpha X + \beta)] = \alpha E[X^2] + \beta E[X] = \alpha\,.
\end{aligned}$$

Part (b): With the above value for $\text{Cov}(X,Y)$ we compute

$$\rho = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} = \frac{\alpha}{\sqrt{\alpha^2 + \sigma^2}}\,.$$

Exercise 10

Part (a): We have

$$
\begin{aligned}
f_{X|Y}(X|Y=y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\frac{1}{2\pi\sigma}\exp\left\{-\frac{(y-\alpha x-\beta)^2}{2\sigma^2}-\frac{x^2}{2}\right\}}{\frac{1}{\sqrt{2\pi}\sqrt{\alpha^2+\sigma^2}}\exp\left\{-\frac{(y-\beta)^2}{2(\alpha^2+\sigma^2)}\right\}} \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\frac{\sigma}{\alpha^2+\sigma^2}}}\exp\left\{-\frac{\left(x-\frac{\alpha}{\alpha^2+\sigma^2}(y-\beta)\right)^2}{2\left(\frac{\sigma^2}{\alpha^2+\sigma^2}\right)}\right\},
\end{aligned}
$$

with some algebra. Thus $X|Y=y$ is a normal random variable with mean $\frac{\alpha}{\alpha^2+\sigma^2}(y-\beta)$ and a variance $\frac{\sigma^2}{\alpha^2+\sigma^2}$.

Chapter 6: Markov Chains

Notes On The Text

Examples of Markov Chains in Standard Form (Examples 6.24-6.25)

For the waiting game we have a one-step transition matrix of $P = \begin{bmatrix} q & p \\ 0 & 1 \end{bmatrix}$, from which we see that state 0 is transient and state 1 is absorbing. In standard form, writing the absorbing states first P becomes

$$P = \begin{bmatrix} 1 & 0 \\ p & q \end{bmatrix}.$$

If we partition into blocks as $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ we see that $R = p$ and $Q = q$ as claimed in the book.

For the gamblers ruin problem with a total fortune of $N = 3$ we have a one-step transition matrix in the natural state ordering $(0, 1, 2, 3)$ of

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so we see that the states 0 and 3 are absorbing the and states 1 and 2 are transient. Writing this in standard form where we write the absorbing states before the transient states (specifically in the order given by $0, 3, 1, 2$) we have a one-step transition matrix given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & p & 0 \\ q & 0 & 0 & p \\ 0 & p & q & 0 \end{bmatrix}.$$

When we write this in a partitioned form as $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ we see that $R = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$. We can then invert the matrix $I - Q = \begin{bmatrix} 1 & -p \\ -q & 1 \end{bmatrix}$ to get the result in the book.

Notes on the proof that $|E_i[(I_j(s)-a_j)(I_j(t)-a_j)]| \le C(r^{t-s}+r^t)$

With the steady-state decomposition of $p_{ij}(t)$ given by $p_{ij}(t) = a_j + e_{ij}(t)$ such that $|e_{ij}(t)| \le br^t$ we have from the definition of $m(s,t)$ that

$$\begin{aligned}
m(s,t) &= p_{jj}(t-s)p_{ij}(s) - a_j p_{ij}(t) - a_j p_{ij}(s) + a_j^2 \\
&= (a_j + e_{jj}(t-s))(a_j + e_{ij}(s)) - a_j(a_j + e_{ij}(t)) - a_j(a_j + e_{ij}(s)) + a_j^2 \\
&= a_j^2 + a_j e_{ij}(s) + a_j e_{jj}(t-s) + e_{jj}(t-s)e_{ij}(s) \\
&- a_j^2 - a_j e_{ij}(t) - a_j^2 - a_j e_{ij}(s) + a_j^2 \\
&= a_j(e_{jj}(t-s) - e_{ij}(t)) + e_{jj}(t-s)e_{ij}(s)\,,
\end{aligned}$$

which is the books result. Using the fact that our error term is geometrically bounded as $|e_{ij}(t)| \le br^t$ and $|a_j| \le 1$ using the triangle inequality we see that

$$\begin{aligned}
|m(s,t)| &\le |e_{jj}(t-s)| + |e_{ij}(t)| + |e_{jj}(t-s)||e_{ij}(s)| \\
&\le br^{t-s} + br^t + b^2 r^t \\
&= br^{t-s} + (b+b^2)r^t \\
&\le (b+b^2)r^{t-s} + (b+b^2)r^t = (b+b^2)(r^{t-s}+r^t)\,,
\end{aligned}$$

the inequality stated in Lemma 6.8.

In the section following Lemmas 6.8 entitled "the completion proof of Theorem 6.8" the argument about replacing t with infinity is a bit difficult to follow. A better argument is by writing out the summation as

$$\sum_{s=1}^{t} r^{t-s} = r^{t-1} + r^{t-2} + \cdots r + 1 = \sum_{k=0}^{t-1} r^k \le \sum_{k=0}^{\infty} r^k\,.$$

Examples computing the mean recurrence and sojourn times (Example 6.32)

For the vending machine model our Markov chain has a one-step transition probability matrix P given by

$$P = \left[\begin{array}{cc} 1-\delta & \delta \\ \gamma & 1-\gamma \end{array}\right].$$

Now to evaluate $E_0[T_1]$ recall that this is the expected number of steps to get to state 1 given that we start in state 0. Since there are only two states in this Markov chain this number of steps must equal the number of steps taken where when we don't change state i.e. the sojourn time in state 0 or $E_0[T_0] = E[S_0]$. This we know equals

$$\frac{1}{1-p_{00}} = \frac{1}{1-(1-\delta)} = \frac{1}{\delta}\,.$$

In the same way

$$E_1[T_0] = E[S_1] = \frac{1}{1-p_{11}} = \frac{1}{\gamma}\,.$$

An alternative way to calculate these expressions is to solve $(I-P)M = U - D$, for M. To do this we first compute $I-P$ to get

$$I - P = \begin{bmatrix} \delta & -\delta \\ -\gamma & \gamma \end{bmatrix},$$

and next compute $U-D$ to get

$$\begin{aligned} U - D &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} R_{00} & 0 \\ 0 & R_{00} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1+\delta/\gamma & 0 \\ 0 & 1+\gamma/\delta \end{bmatrix} \\ &= \begin{bmatrix} -\delta/\gamma & 1 \\ 1 & -\gamma/\delta \end{bmatrix}. \end{aligned}$$

Then the matrix equation $(I-P)M = U - D$ becomes

$$\begin{bmatrix} \delta & -\delta \\ -\gamma & \gamma \end{bmatrix} \begin{bmatrix} 0 & R_{01} \\ R_{10} & 0 \end{bmatrix} = \begin{bmatrix} -\delta/\gamma & 1 \\ 1 & -\gamma/\delta \end{bmatrix}.$$

Multiplying the matrices together we have

$$\begin{bmatrix} -\delta R_{10} & \delta R_{01} \\ \gamma R_{10} & -\gamma R_{01} \end{bmatrix} = \begin{bmatrix} -\delta/\gamma & 1 \\ 1 & -\gamma/\delta \end{bmatrix},$$

from which we see that a solution is $R_{10} = 1/\gamma$ and $R_{01} = 1/\delta$ as given in the book.

The inventory model has a one-step transition probability matrix of

$$P = \begin{bmatrix} 0 & p & q \\ p & q & 0 \\ 0 & p & q \end{bmatrix},$$

with a steady-state distribution $\pi = \begin{bmatrix} \frac{p}{2} & \frac{1}{2} & \frac{q}{2} \end{bmatrix}$. We compute $I-P$ and find

$$I - P = \begin{bmatrix} 1 & -p & -q \\ -p & 1-q & 0 \\ 0 & -p & 1-q \end{bmatrix} = \begin{bmatrix} 1 & -p & -q \\ -p & p & 0 \\ 0 & -p & p \end{bmatrix},$$

and

$$\begin{aligned} U - D &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{p} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2}{q} \end{bmatrix} \\ &= \begin{bmatrix} 1-\frac{2}{p} & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1-\frac{2}{q} \end{bmatrix}, \end{aligned}$$

So that the matrix system we need to solve to find the mean recurrence and sojourn times is $(I-P)M = U - D$ becomes

$$\begin{bmatrix} 1 & -p & -q \\ -p & p & 0 \\ 0 & -p & p \end{bmatrix} \begin{bmatrix} 0 & T_{01} & T_{02} \\ T_{10} & 0 & T_{12} \\ T_{20} & T_{21} & 0 \end{bmatrix} = \begin{bmatrix} 1-\frac{2}{p} & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1-\frac{2}{q} \end{bmatrix}.$$

Multiplying the two matrices on the left hand side we obtain

$$\begin{bmatrix} -pT_{10} - qT_{20} & T_{01} - qT_{21} & T_{02} - pT_{12} \\ pT_{10} & -pT_{01} & -pT_{02} + pT_{12} \\ -pT_{10} + pT_{20} & pT_{21} & -pT_{12} \end{bmatrix} = \begin{bmatrix} 1 - \frac{2}{p} & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 - \frac{2}{q} \end{bmatrix} .$$

The equation for the $(2,1)$st component gives $pT_{10} = 1$ or $T_{10} = \frac{1}{p}$. The equation for the $(2,2)$ component gives $-pT_{01} = -1$ or $T_{01} = \frac{1}{p}$. The equation for the $(3,2)$ component gives $pT_{21} = 1$ or $T_{21} = \frac{1}{p}$. The equation for the $(3,3)$ component gives

$$T_{12} = -\frac{1}{p} + \frac{2}{pq} = \frac{p+1}{pq} .$$

The equation for the $(1,3)$ component using what we found for T_{12} above

$$T_{02} = pT_{12} + 1 = \frac{p+1}{q} + 1 = \frac{2}{q} .$$

Finally, the equation for the $(3,1)$ component gives

$$-pT_{10} + pT_{20} = 1 ,$$

or

$$T_{20} = \frac{1}{p} + T_{10} = \frac{2}{p} .$$

In summary then we have

$$R = \begin{bmatrix} \frac{2}{p} & \frac{1}{p} & \frac{2}{q} \\ \frac{1}{p} & 2 & \frac{p+1}{pq} \\ \frac{2}{p} & \frac{1}{p} & \frac{2}{q} \end{bmatrix} .$$

Exercise Solutions

Exercise 1 (the number of flips to get to state n)

In this experiment T_n is the count of the number of trials until we get n heads. This is the definition of a negative-binomial random variable and T_n is distributed as such.

Exercise 2 (vending machine breakdowns)

Part (a): For the machine to be in good working order on all days between Monday and Thursday means that we must transition from state 0 to state 0 three times. The probability this happens is given by

$$(1 - \delta)^3 = 0.8^3 = 0.512 .$$

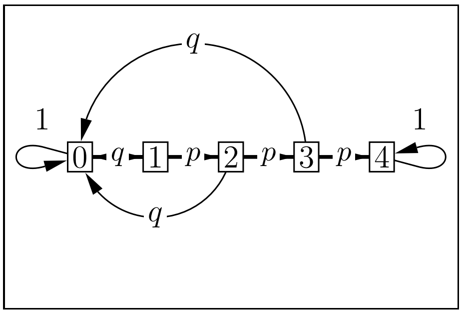

Figure 6: The transition diagram for Exercise 4.

Part (b): For the machine to be in working order on Thursday can be computed by summing the probabilities over all possible paths that start with a working machine on Monday and end with a working machine on Thursday. By enumeration, we have four possible transitions that start with the state of our vending machine working on Monday and end with it working on Thursday. The four transitions are

$$\begin{array}{lll} 0 \to 0 \to 0 \to 0 & \text{with probability} & (1-\delta)^3 \\ 0 \to 1 \to 0 \to 0 & \text{with probability} & \delta\gamma(1-\delta) \\ 0 \to 0 \to 1 \to 0 & \text{with probability} & \delta(1-\gamma)\gamma \\ 0 \to 1 \to 1 \to 0 & \text{with probability} & (1-\delta)\delta\gamma \end{array}$$

Adding up these probabilities we find that the probability requested is given by 0.8180. These to calculations are done in the MATLAB file `chap_6_prob_2.m`.

Exercise 3 (successive runs)

Given that you have two successive wins, one more will allow you to win the game. This event happens with probability p. If you loose (which happens with probability $1-p$) your number of successive wins is set back to zero and you have four remaining rounds in which to win the game. You can do this in two ways, either winning the first three games directly or loosing the first game and winning the remaining three. The former event has probability of p^3 while the later event has probability qp^3. Thus the total probability one wins this game is given by

$$p + q(p^3 + qp^3) = 0.5591\,.$$

This simple calculation is done in the MATLAB script `chap_6_prob_3.m`.

Exercise 4 (more successive runs)

Part (a): Let our states for this Markov chain be denoted $0, 1, 2, 3, 4$ with our system in state i if we have won i consecutive games. A transition diagram for this system is given as in Figure 6.

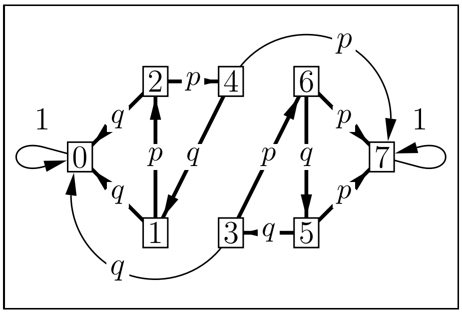

Figure 7: The transition diagram for Exercise 5.

Part (b): Given that we have won three consecutive rounds we can win our entire game and have a total of four consecutive wins if we win the next game, which happens with probability p. If we don't win the next game we transition to the state 0 and have four more attempts to get to state 4. We must flip heads in each of the four remaining rounds to end up in the state 4 at the end of these trials. This will happen with probability p^4. Thus the total probability we win is given by

$$p + q(p^4) = 0.5002\,.$$

This simple calculation is done in the MATLAB script `chap_6_prob_4.m`.

Exercise 5 (bold play to 7 dollars)

Part (a): Let i denote the amount of money that the player currently has at the end of the given timestep. We will assume that the rules for this game are the same as that for the bold play example from the book. That is, we bet as much as possible at any given round as long as the winnings would put us under (or equal too) our target of seven dollars. In this case the transition diagram for this Markov chain looks like that in Figure 7.

Part (b): We desire to calculate the probability we will arrive in state 7 in six or fewer rounds of play. To solve this problem we enumerate all possible paths from 1 to 7 that have less than six legs. In this case there are only two possible paths. Their probabilities are

$$\begin{array}{rll} 1 \to 2 \to 4 \to 7 & \text{with probability} & p^3 \\ 1 \to 2 \to 4 \to 1 \to 2 \to 4 \to 7 & \text{with probability} & (p^2q)p^3 = p^5 q\,. \end{array}$$

Thus the probability we reach our goal in six or fewer plays is given by

$$p^3 + p^5 q\,.$$

Part (c): To calculate the probability we get to 7 in an infinite number of plays note from Figure 7 that we can never get to the states $3, 5$ or 6 from the state 1 under the bold play policy rules. While we can get to state 7 under and infinite number of plays if we cycle through the states $1 \to 2 \to 4 \to 1$ before jumping on a path to state 7 from state 4. Thus the probability we can get to state 7 under an infinite number of plays is

$$p^3 + p^2q(p^3) + (p^2q)^2(p^3) + (p^2q)^3(p^3) + \cdots\,,$$

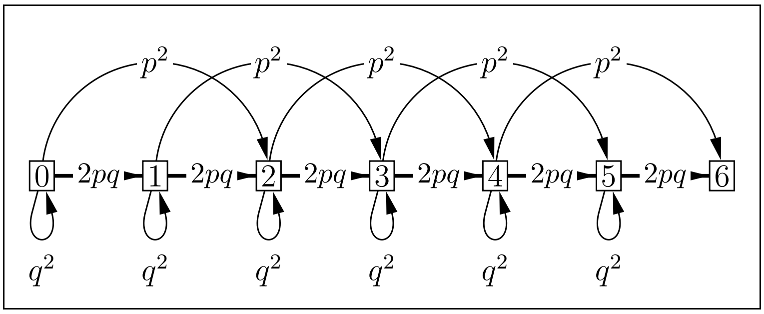

Figure 8: The transition diagram for Exercise 6. That the pattern continues indefinitely to the right.

where each factor $(p^2q)^k$ represents the probability we cycle k times before getting on a path that takes us directly (in three steps) to state 7. When we sum this infinite series we find it equals

$$p^3 \sum_{k=0}^{\infty} (p^2q)^k = \frac{p^3}{1 - p^2q}.$$

Exercise 6 (repeated play with two fair coins)

On any given toss we can obtain 0, 1, or 2 heads with probabilities q^2, $2pq$, and p^2 respectively. A transition diagram for this process is given in Figure 8. The transition matrix for this process p_{ij} is given as

$$P = \begin{bmatrix} q^2 & 2pq & p^2 & & & \\ 0 & q^2 & 2pq & p^2 & & \\ 0 & 0 & q^2 & 2pq & p^2 & \\ & & & \ddots & & \end{bmatrix},$$

which we recognized as a matrix with q^2 on its diagonal, $2pq$ on its upper diagonal and p^2 on its upper-upper diagonal.

Exercise 7 (some example of Markov chains)

Part (a): We desire to calculate the joint probability of X and Y i.e. $P(X, Y)$ when X and Y can take two values: 0 for working and 1 for not working. Given that the state of the machine on Monday is working we can compute $P(X, Y)$ as $P(Y|X)P(X)$ to find

$$\begin{aligned} P(X = 0, Y = 0) &= P(Y = 0|X = 0)P(X = 0) = (1 - \delta)(1 - \delta) \\ P(X = 0, Y = 1) &= P(Y = 1|X = 0)P(X = 0) = \delta(1 - \delta) \\ P(X = 1, Y = 0) &= P(Y = 0|X = 1)P(X = 1) = \gamma\delta \\ P(X = 1, Y = 1) &= P(Y = 1|X = 1)P(X = 1) = (1 - \gamma)\delta. \end{aligned}$$

One can check that this is indeed a valid probability mass function by verifying that $\sum_{X,Y} P(X,Y) = 1$.

Part (b): The marginal mass functions $P(X)$ and $P(Y)$ are given by $P(X) = \sum_y P(X, Y = y)$ and $P(Y) = \sum_x P(X = x, Y)$. Using these definitions we find

$$\begin{aligned}
P(X = 0) &= \sum_y P(X = 0, Y = y) = 1 - 2\delta + \delta^2 + \delta - \delta^2 = 1 - \delta \\
P(X = 1) &= \sum_y P(X = 1, Y = y) = \delta \\
P(Y = 0) &= (1 - \delta)^2 + \gamma\delta \\
P(Y = 1) &= \delta - \delta^2 + \delta - \gamma\delta = 2\delta - \delta^2 - \gamma\delta\,.
\end{aligned}$$

To be independent would require that $P(X,Y) = P(X)P(Y)$. That this is not true can be seen by taking $X = 0$ and $Y = 0$. We then see that

$$P(X = 0, Y = 0) = (1 - \delta)^2\,,$$

while from above

$$P(X = 0)P(Y = 0) = (1 - \delta)\left((1 - \delta)^2 + \gamma\delta\right)\,,$$

which are not equal, showing X and Y are not independent. Nor are they identically distributed since in general $P(X = 0) = 1 - \delta \neq P(Y = 0) = (1 - \delta)^2 + \gamma\delta$.

Exercise 8 (the vending machine example continued)

Part (a): To show that $\{X(n)\}$ is not a Markov chain it suffices to show that

$$P\{X(2) = 0 | X(1) = 0, X(0) = 1\} \neq P\{X(2) = 0 | X(1) = 0\}\,.$$

Here I have chosen to start the chain at time 0. The left hand side of the above from the problem statement is $q_{01} = \frac{1}{2}$, while the right hand side can be obtained by marginalizing out $X(0)$ or

$$\begin{aligned}
P\{X(2) = 0 | X(1) = 0\} &= P\{X(2) = 0 | X(1) = 0, X(0) = 0\} P\{X(0) = 0\} \\
&+ P\{X(2) = 0 | X(1) = 0, X(0) = 1\} P\{X(0) = 1\} \\
&= q_{00} P\{X(0) = 0\} + q_{01} P\{X(0) = 1\} \\
&= \frac{3}{4} P\{X(0) = 0\} + \frac{1}{2}(1 - P\{X(0) = 0\}) \\
&= \frac{1}{2} + \frac{1}{4} P\{X(0) = 0\} \neq q_{01} = \frac{1}{2}\,,
\end{aligned}$$

unless $P\{X(0) = 1\} = 0$ which is a very special initial condition for our system to start in. For example, taking $P\{X(0) = 1\} = 1$ (meaning that with certainty our system starts in the working state) the above right hand side would not equal q_{01} and $\{X(n)\}$ is not a Markov chain.

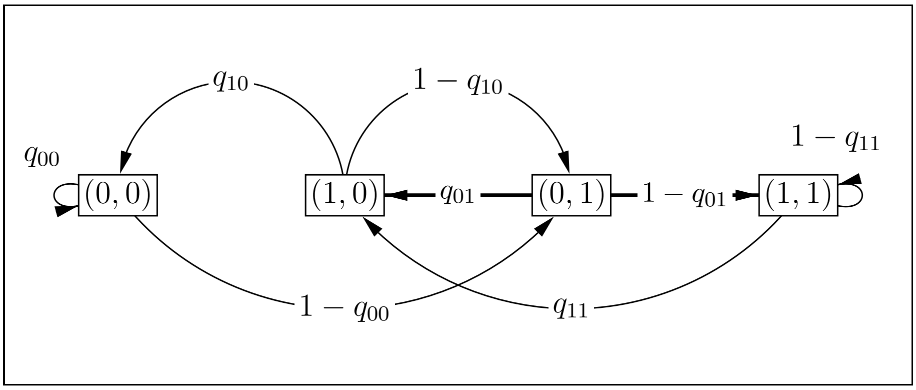

Figure 9: The transition diagram for Exercise 8.

Part (b): Our enlarged state space consists of the ordered pairs $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ and has a transition diagram given by Figure 9. The fact that this is a Markov chain can be seen by the given definition of the transitions probabilities associated with the process $\{X(n)\}$. For example since $P\{X(n+1)=0|X(n-1)=j,X(n)=k\}$ is equivalent to

$$P\{(X(n+1),X(n))=(0,k)|(X(n),X(n-1))=(k,j)\}=q_{jk}\,,$$

and

$$P\{(X(n+1),X(n))=(1,k)|(X(n),X(n-1))=(k,j)\}=1-q_{jk}\,.$$

The state defined by the ordered pair $(X(n+1),X(n))$ depends only on the previous value of this vector.

Part (c): If our machine is working on Monday and Tuesday in terms of the enlarged space we are in the state $(0,0)$. The question as to whether our machine will be working on Thursday means that on Thursday it will be in the state $(0,0)$ or $(0,1)$ after two transitions. These two transitions are from the state indexed by (Monday,Tuesday), to (Tuesday,Wednesday), to (Wednesday,Thursday). Thus to find our probability we sum all possible paths in our enlarged state space that move from the current state of $(0,0)$ to a final state of either $(0,0)$ or $(1,0)$ in two steps. We have only two such paths given by

$$\begin{array}{lll}(0,0)\to(0,0)\to(0,0) & \text{with probability} & q_{00}^2 \\ (0,0)\to(0,1)\to(1,0) & \text{with probability} & (1-q_{00})q_{01}\,.\end{array}$$

Thus the total probability is

$$q_{00}^2+(1-q_{00})q_{01}=\left(\frac{3}{4}\right)^2+\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)=\frac{11}{16}\,.$$

Exercise 9 (bold play v.s. timid play)

Part (a): For bold play we calculated in the book the probability of winning in six or fewer rounds to be

$$B_6(p)=p^3+p^3q\,,$$

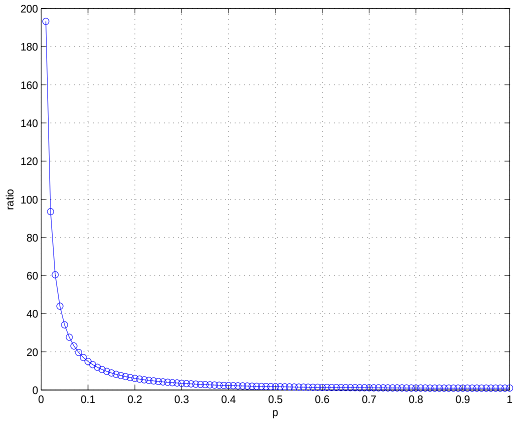

Figure 10: A plot of the ratio $B_6(p)/T_6(p)$ as a function of the probability of winning each trail p.

while for timid play we found this same thing given by

$$T_6(p) = p^4(1 + 3pq)\,.$$

So we see that the desired ratio

$$\frac{B_6(p)}{T_6(p)} = \frac{1+q}{p(1+3pq)}\,.$$

For value of p between 0 and 1 this ratio is plotted in Figure 10

Part (b): In Figure 10, we see that this ratio is always larger than zero. The simple calculations for and the plot for this problem are performed in the MATLAB script `chap_6_prob_9.m`.

Exercise 10 (the genetic chain example)

From the genetic chain example we calculated transition probabilities p_{ij} given by

$$p_{ij} = \frac{\left(\begin{array}{c} 2i \\ j \end{array}\right)\left(\begin{array}{c} 2N-2i \\ N-j \end{array}\right)}{\left(\begin{array}{c} 2N \\ N \end{array}\right)}\,.$$

for $0 \le i \le N$ and $\max(0, N-2(N-i)) \le j \le \min(2i, N)$ where the states i and j represent the number of normal genes. The lower limit of j is obtained by recognizing that if we take as many bad genes as possible we will obtain the smallest number of good genes.

When $N = 3$ we compute $\left(\begin{array}{c} 2N \\ N \end{array}\right) = \left(\begin{array}{c} 6 \\ 3 \end{array}\right) = 20$ and we can compute each p_{ij} in tern to

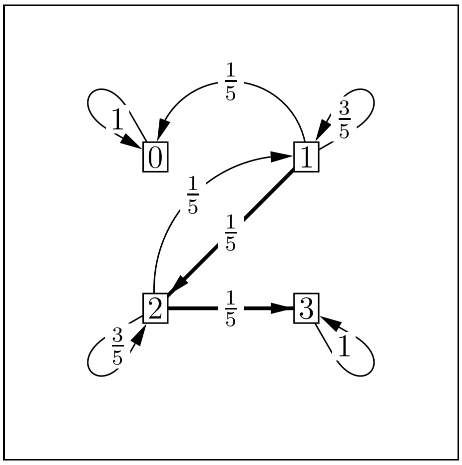

Figure 11: The transition diagram for Exercise 10.

get the one step transition probability matrix given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and a transition diagram given by Figure 11. The simple calculations for this problem are performed in the MATLAB script `chap_6_prob_10.m`.

Exercise 11 (the inventory model example)

With an inventory cap of $S = 2$ and a restocking value of $s = -1$ our possible states are $2, 1, 0, -1$ and our daily demand is for 0 or 1 item. Then with p the probability we have a demand for an item (and q the complement probability) our transition diagram looks like that in Figure 12.

This Markov chain has a transition diagram given by (with the states ordered as 2, 1, 0, and -1) as

$$P = \begin{bmatrix} q & p & 0 & 0 \\ 0 & q & p & 0 \\ 0 & 0 & q & p \\ q & p & 0 & 0 \end{bmatrix}.$$

Exercise 12 (simulating the inventory example)

Using the starting value of $x_0 = 3$, an inventory cap $S = 3$, a restocking threshold of $s = 0$, a stochastic demand $D(n)$ we can simulate our inventory example using the following recursion

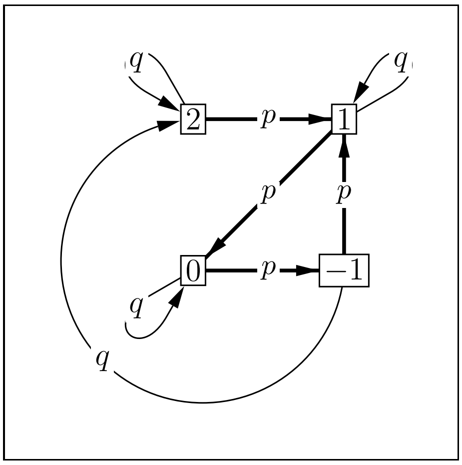

Figure 12: The transition diagram for Exercise 11.

formula

$$
\begin{aligned}
X(n+1) &= \begin{cases} X(n) - D(n+1) & \text{if} \quad X(n) > s \\ S - D(n+1) & \text{if} \quad X(n) \le s \end{cases} \\
&= \begin{cases} X(n) - D(n+1) & \text{if} \quad X(n) > 0 \\ 3 - D(n+1) & \text{if} \quad X(n) \le 0 \end{cases} ,
\end{aligned}
$$

for $n \ge 0$. The stochastic demand $D(n)$ can be simulated using the random numbers supplied for this problem to obtain the following five demands:

$$1\,, \quad 1\,, \quad 1\,, \quad 2\,, \quad 2\,.$$

Using these demands and the above recurrence relation we can compute the value of the next state for each of the five days. The logic to do this is coded up in the MATLAB script `chap_6_prob_12.m`. When this script is run we obtain the sequence of states $X(n)$ for $0 \le n \le 5$ given by:

$$3\,, \quad 2\,, \quad 1\,, \quad 0\,, \quad 1\,, \quad -1\,.$$

Exercise 13 (example stochastic matrices)

Recall that a Markov chain is irreducible if every state communicates with every other state. Two states communicate if $i \to j$ and also $j \to i$.

The Matrix P_1: For P_1 we have a transition diagram given by Figure 13. We have two classes of states $\{1, 3\}$ and $\{2\}$ both of which are ergodic. The class consisting of $\{1, 3\}$ is periodic with period 2, while the class $\{2\}$ is not periodic. The standard form for the transition matrix consists of of grouping states by classes such that the ergodic classes come before the transient classes. Thus for P_1 we have

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{bmatrix} .$$

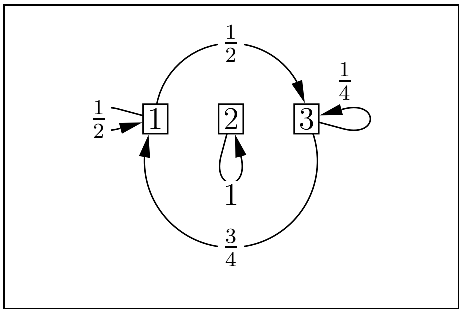

Figure 13: The transition diagram for Exercise 13 under transition matrix P_1.

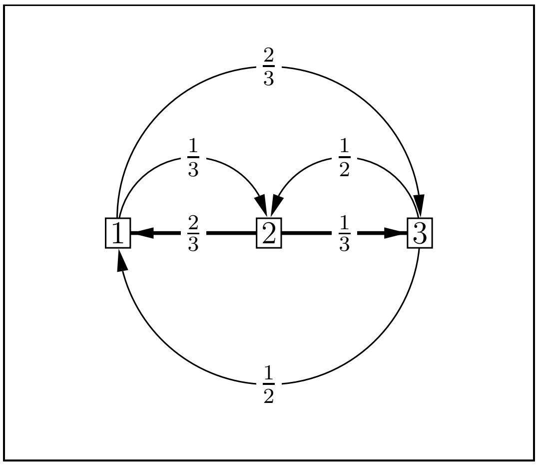

Figure 14: The transition diagram for Exercise 13 under transition matrix P_2.

The Matrix P_2: For P_2 we have a transition diagram given by Figure 14. The irreducible Markov chains in this example are P_2 only. For P_2 our matrix is irreducible and the only class is $\{1, 2, 3\}$ which is ergodic and periodic with period 2. For P_2 since it is irreducible the form given is already in standard form.

The Matrix P_3: For our matrix P_3 our the classes of states are given by $\{1, 2, 3\}$, $\{4\}$, and $\{5\}$. The class $\{1, 2, 3\}$ and $\{4\}$ are transient, while the state/class $\{5\}$ is absorbing. The standard form for the transition matrix consists of of grouping states by classes such that the ergodic classes come before the transient classes. For P_3 we will list our states as $5, 4, 1, 2, 3$ to get

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{8} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} .$$

The Matrix P_4: For the matrix P_4 the classes of states are given by $\{1, 5\}$ and $\{2, 3, 4\}$. Then both classes are ergodic. The class $\{1, 5\}$ is periodic with period 1 while the class $\{2, 3, 4\}$ is periodic with period 2. The standard form for the transition matrix consists of of grouping states by classes such that the ergodic classes come before the transient classes.

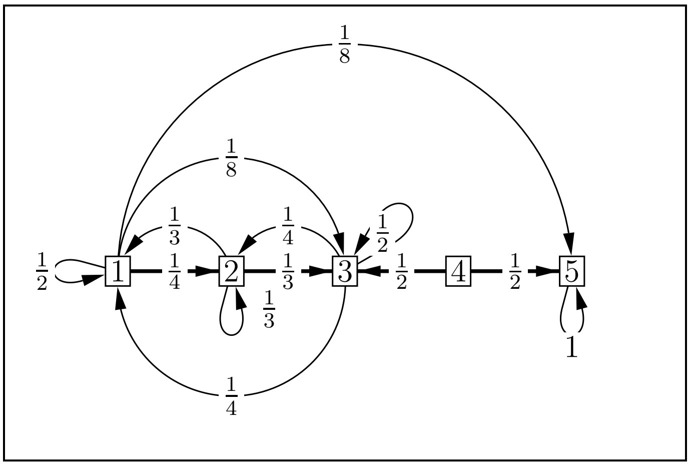

Figure 15: The transition diagram for the transition matrix P_3.

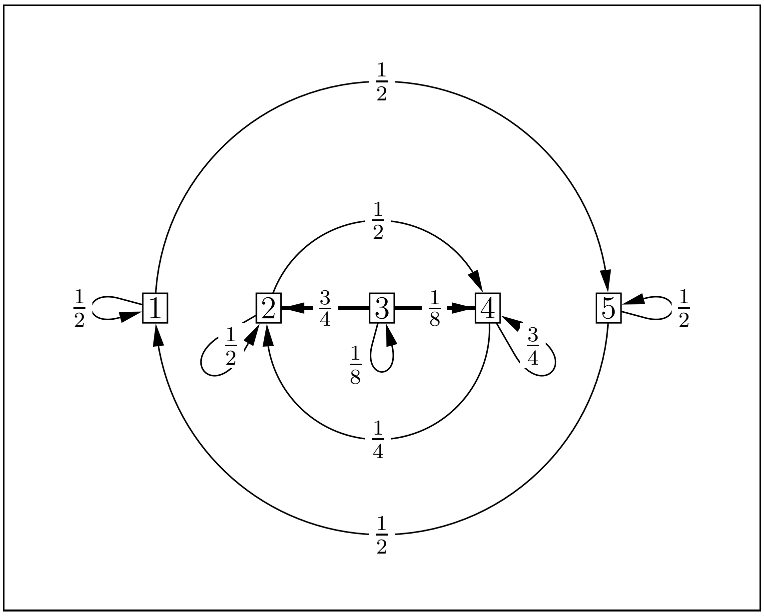

Figure 16: The transition diagram for the transition matrix P_4.

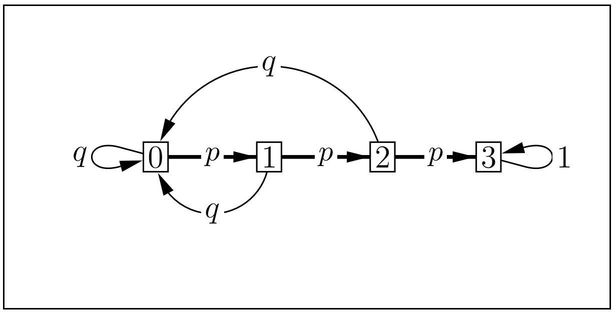

Figure 17: The transition diagram for Exercise 14.

For P_4 we will list our states in the order $1, 5, 2, 3, 4$ to get

$$P_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} & \frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \end{bmatrix} .$$

Exercise 14 (flipping a fair coin)

We begin by defining a Markov chain where the state represents the number of consecutive heads. Thus for this problem the possible states are $0, 1, 2, 3$ and the transition diagram for this process looks like that in Figure 17 (where p is the probability we obtain a head on a single flip of the coin).

This Markov chain has a one-step transition probability matrix given by (with the order of the states $0, 1, 2, 3$)

$$P = \begin{bmatrix} q & p & 0 & 0 \\ q & 0 & p & 0 \\ q & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Now the state 3 is absorbing and the states $\{0, 1, 2\}$ are transient. So to write the transition probability in standard form we reorder the states such that the absorbing states are first. That is, we take the ordering of the states to be $3, 0, 1, 2$ to get

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q & p & 0 \\ 0 & q & 0 & p \\ p & q & 0 & 0 \end{bmatrix} .$$

It is this matrix that will answer questions regarding the expected number of times a given state will be visited and the expected number of total steps taken until an absorbing state is reached. To determine these answers we need to partition the above matrix into the block matrix $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$, from which we see that $R = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$, and $Q = \begin{bmatrix} q & p & 0 \\ q & 0 & p \\ q & 0 & 0 \end{bmatrix}$.

Part (a): The number of expected tails we will obtain before we get three heads is the expected number of times we will visit state 0, since each time we visit that state will have had to have flipped a tail to land there. In general the expected number of times we visit a state j starting at a state i is given by the "(i,j) component" of the matrix $N = (I-Q)^{-1}$. Computing this inverse we find

$$
\begin{aligned}
(I-Q)^{-1} &= \begin{bmatrix} 1-q & -p & 0 \\ -q & 1 & -p \\ -q & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} p & -p & 0 \\ -(1-p) & 1 & -p \\ -(1-p) & 0 & 1 \end{bmatrix}^{-1} \\
&= \frac{1}{p^3} \begin{bmatrix} 1 & p & p^2 \\ 1-p^2 & p & p^2 \\ 1-p & p(1-p) & p^2 \end{bmatrix}.
\end{aligned}
$$

This matrix inverse is done in the Mathematica file `chap_6_prob_14.nb`. The expected of times we visit state 0 starting from state 0 is the $(1,1)$ component of the matrix $(I-Q)^{-1}$. We find

$$
E_0[V_0] = \frac{1}{p^3} = 8\,,
$$

when we assume a fair coin $p = 1/2$.

Part (b): The number of flips we expect to make would be the expected number total number of flips taken starting in state 0 to get to the state 3. This is given by the sum of the elements in the first row of $(I-Q)^{-1}$ or

$$
\frac{1}{p^3}(1+p+p^2) = 8(1+\frac{1}{2}+\frac{1}{4}) = 14\,.
$$

Part (c): If the first flip comes up heads we have now moved to state 1 and we desire to evaluate $E_1[V_0]$ or the expected number of visits to state 0 given that we start in state 1. From the matrix $(I-Q)^{-1}$ this it the $(2,1)$ element or

$$
\frac{1-p^2}{p^3} = \frac{1}{p^3} - \frac{1}{p} = 8 - 2 = 6\,.
$$

which is smaller than the result from Part (a) as expected since starting in state 1 we are one step closer to our goal of being in state 3.

Part (d): As in Part (a) this is $E_1[W]$ or the sum of the second row of the $(I-Q)^{-1}$ matrix. We find

$$
\frac{1}{p^3}(1-p^2+p+p^2) = 8(1+\frac{1}{2}) = 12\,,
$$

smaller than in Part (b) as expected.

Exercise 15 (standard form for the genetic chain)

Part (a): In Exercise 10 (above) we computed the one-step transition matrix and the transition diagram for this Markov chain. This chain is not irreducible since states $i = 0$ and $i = 3$ are absorbing. In addition, the states $\{1, 2\}$ form a transient class. Because the class $\{1, 2\}$ is transient eventually the descendants end in states $i = 0$ or $i = 3$ i.e. will have all normal or abnormal genes.

Part (b): In standard form we write the ergodic classes (in this case these are absorbing) before the transient classes. For this example we choose to list the states in the order $0, 3, 1, 2$ and find our one step transition probability in that case given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{bmatrix}.$$

Block partitioning this matrix as $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ we find $R = \frac{1}{5}I$ and $Q = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ so that $N = (I - Q)^{-1} = \frac{5}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Exercise 16 (the probability we end with various gene configurations)

Part (a): We desire to calculate the probability we end in state $i = 3$ (all normal subgenes) given that we start in state $i = 2$ (we have two normal subgenes). The solution to this is an appropriate element of the matrix NR. Computing the matrix NR we find

$$NR = (I - Q)^{-1}R = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The rows of NR are indexed by the initial states $1, 2$ and the columns of NR are indexed by the final states $0, 3$ we see that the probability we desire is the component $(2, 2)$ (one based) of the matrix NR or the value $\frac{2}{3}$.

Part (b): The expected number of generations before we end in any absorbing state given we start in state i is given by summing the appropriate row in the $(I - Q)^{-1}$ matrix. Since we are told we start in the state $i = 2$ we want to sum the second row of N. When we do that we find

$$E_2[W] = \frac{5}{3}(1 + 2) = 5.$$

Exercise 17 (standard form for bold play)

Part (a): The single step transition diagram for bold play is shown in Figure 6.6 in the book. There we see that the states 0 and 5 are absorbing and the states $1, 2, 3,$ and 4 are transient. Thus if we order our states as $0, 5, 1, 2, 3, 4$ we have a standard form one-step transition matrix P of the form

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 & 0 \\ q & 0 & 0 & 0 & 0 & p \\ 0 & p & q & 0 & 0 & 0 \\ 0 & p & 0 & 0 & q & 0 \end{bmatrix} .$$

For P in standard form we look for a block decomposition given by $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$, from which we see that for the matrix P above we have $R = \begin{bmatrix} q & 0 \\ q & 0 \\ 0 & p \\ 0 & p \end{bmatrix}$, and $Q = \begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{bmatrix}$.

Part (b): Using the above Q we find

$$Q^2 = \begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{bmatrix} \begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & p^2 \\ 0 & 0 & pq & 0 \\ 0 & pq & 0 & 0 \\ q^2 & 0 & 0 & 0 \end{bmatrix} ,$$

and that

$$Q^3 = QQ^2 = \begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & p^2 \\ 0 & 0 & pq & 0 \\ 0 & pq & 0 & 0 \\ q^2 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & p^2q & 0 \\ pq^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & qp^2 \\ 0 & q^2p & 0 & 0 \end{bmatrix} ,$$

and finally

$$Q^4 = QQ^3 = \begin{bmatrix} 0 & p & 0 & 0 \\ 0 & 0 & 0 & p \\ q & 0 & 0 & 0 \\ 0 & 0 & q & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & p^2q & 0 \\ pq^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & qp^2 \\ 0 & q^2p & 0 & 0 \end{bmatrix} = \begin{bmatrix} p^2q^2 & 0 & 0 & 0 \\ 0 & p^2q^2 & 0 & 0 \\ 0 & 0 & q^2p^2 & 0 \\ 0 & 0 & 0 & p^2q^2 \end{bmatrix} = p^2q^2 I .$$

One way to compute the expression $(I - Q)^{-1}$ is to use its power series as

$$(I - Q)^{-1} = I + Q - Q^2 + Q^3 - Q^4 + \cdots .$$

To do this we need to be able to compute powers of the matrix Q. If we write every integer n as $n = 4m + p$ with $0 \le p \le 3$ we see that

$$Q^{4m+p} = (Q^4)^m Q^p = (p^2q^2I)^m Q^p = p^{2m} q^{2m} Q^p .$$

So the inverse $(I - Q)^{-1}$ can be written as four terms

$$(I - Q)^{-1} = \sum_{m=0}^{\infty} Q^{4m} + Q \sum_{m=0}^{\infty} Q^{4m} + Q^2 \sum_{m=0}^{\infty} Q^{4m} + Q^3 \sum_{m=0}^{\infty} Q^{4m} .$$

The first summation above represents terms all with $p = 0$ (all powers of Q's are multiples of 4) the second summation above represents all terms with $p = 1$ (all powers of Q can be expressed as $4m + 1$) etc. We then see that

$$
\begin{aligned}
(I - Q)^{-1} &= (I + Q + Q^2 + Q^3) \sum_{m=0}^{\infty} Q^{4m} \\
&= (I + Q + Q^2 + Q^3) \sum_{m=0}^{\infty} (p^2 q^2)^m \\
&= (I + Q + Q^2 + Q^3) \left(\frac{1}{1 - p^2 q^2} \right) \\
&= \frac{1}{1 - p^2 q^2} \left[\begin{array}{cccc} 1 & p & p^2 q & p^2 \\ pq^2 & 1 & pq & p \\ q & pq & 1 & qp^2 \\ q^2 & q^2 p & q & 1 \end{array} \right] .
\end{aligned}
$$

Part (c): The expected length of the game is the number of steps taken until an absorbing state is reached. Since we start in state 1, to compute this we sum the first row of the $(I - Q)^{-1}$ matrix to find

$$
\frac{1 + p + p^2 q + p^2}{1 - p^2 q^2} . \tag{28}
$$

Part (d): The probability of getting to state 5 starting in state 1 is given by the $(1, 2)$th element of the $(I - Q)^{-1} R$ matrix. Computing this matrix we find

$$
\begin{aligned}
(I - Q)^{-1} R &= \frac{1}{1 - p^2 q^2} \left[\begin{array}{cccc} 1 & p & p^2 q & p^2 \\ pq^2 & 1 & pq & p \\ q & pq & 1 & qp^2 \\ q^2 & q^2 p & q & 1 \end{array} \right] \left[\begin{array}{cc} q & 0 \\ q & 0 \\ 0 & p \\ 0 & p \end{array} \right] \\
&= \frac{1}{1 - p^2 q^2} \left[\begin{array}{cc} q + qp & p^3 q + p^3 \\ pq^3 + q & p^2 q + p^2 \\ q^2 + pq^2 & p + qp^3 \\ q^3 + q^3 p & pq + p \end{array} \right] .
\end{aligned}
$$

So the $(1, 2)$th element of the above matrix is given by

$$
\frac{p^3 (1 + q)}{1 - p^2 q^2} . \tag{29}
$$

This is the same result as obtained using path analysis in Example 6.8 from the book.

Part (e): When we evaluate Part (c) and (d) for the various probabilities of winning p considered in Table 6.3 from the book

```
0.2000    0.3000    0.4000    0.5000    0.6000    0.7000
0.0149    0.0490    0.1133    0.2143    0.3533    0.5227
1.3140    1.5507    1.8319    2.1429    2.4579    2.7397
```

These agree quite well with the simulation results in Table 6.3. They are calculated in the MATLAB script `chap_6_prob_17.m`.

Exercise 18 (standard form for the gamblers ruin problem)

Part (a): For the gamblers ruin example we have a transition diagram given in Figure 6.3 of the book. From there we see that the states 0 and N are absorbing while all others are transient. When $N = 5$ writing the states in the order $0, 5, 1, 2, 3, 4$, we have a standard form transition matrix P given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & p & 0 & 0 \\ 0 & 0 & q & 0 & p & 0 \\ 0 & 0 & 0 & q & 0 & p \\ 0 & p & 0 & 0 & q & 0 \end{bmatrix}.$$

Partitioning this as $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ we have $R = \begin{bmatrix} q & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & p \end{bmatrix}$ and $Q = \begin{bmatrix} 0 & p & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & q & 0 \end{bmatrix}$.

Part (b): When $p = 3/10$ and $p = 7/10$ we can explicitly compute $(I - Q)^{-1}$. This is done in the Mathematica file `chap_6_prob_18.nb`.

Part (c): The win probability, starting in any given state, can be determined by the elements of $(I - Q)^{-1}R$. This matrix is of dimension 4×2. The state 1 corresponds to the first row of this matrix. The final state 5 corresponds to the the second column of this matrix. We calculate the win probabilities when $p = 3/10$ and $p = 7/10$ to be

$$0.0195 \quad \text{and} \quad 0.5798\,.$$

The average game duration starting in the state 1 is given by the sum of the first row of the matrix $(I - Q)^{-1}$. We find that under $p = 3/10$ and $p = 7/10$ these sums are given by

$$2.255 \quad \text{and} \quad 4.747\,.$$

In the gamblers ruin simulation results we find when $p = 3/10$ a win frequency of 0.017 and an expected game length of 2.21, while when $p = 7/10$ we find a win frequency of 0.582 and an expected game length of 4.81. Both agree quite well with the analytic results presented above.

Exercise 19 (winning with bold or timid play)

Part (a): The probability we win under bold play in the gamblers ruin problem is calculated in Exercise 17 Equation 29. For timid play following the same procedure as in Exercise 17, if we draw a Markov chain for this strategy we can follow the same methods there and compute $T(p)$, the probability we win under timid play. However, if we recognize that under the timid play policy the game is *equivalent* to the gamblers ruin game where we in fact calculated $T(p)$ for $p = 3/10$ and $p = 7/10$ in Exercise 18. Using both these results we calculate

$$\frac{B(0.3)}{T(0.3)} = \frac{0.049}{0.0195} = 2.512 \quad \text{and} \quad \frac{B(0.7)}{T(0.7)} = \frac{0.5227}{0.5798} = 0.901\,.$$

Part (b): The simulations in Examples 6.2 and 6.5 in the book give ratios of these probabilities as

$$\frac{\hat{B}(0.3)}{\hat{T}(0.3)} = \frac{0.042}{0.017} = 2.4706 \quad \text{and} \quad \frac{\hat{B}(0.7)}{\hat{T}(0.7)} = \frac{0.483}{0.582} = 0.8299\,,$$

which are quite close to the results calculated above.

Part (c): In Exercise 9 we looked for the probability we win in six or fewer rounds while in the above we allow an infinite number of rounds. We found earlier that $B_6(p) \geq T_6(p)$ for all p, while the calculation above found that when $p = 0.7$, $B(p) < T(p)$. There is no contradiction since in the latter expression we are considering an infinite number of plays. Thus with a small probability of winning each round p bold play is to be preferred while with large p timid play is the better policy.

Exercise 20 (maximal profit from our vending machine)

The vending machine repair model has a one-step transition probability matrix P of

$$P = \begin{bmatrix} 1-\delta & \delta \\ \gamma & 1-\gamma \end{bmatrix},$$

which since this Markov chain is regular has a unique stationary vector of

$$\begin{bmatrix} \frac{\gamma}{\delta+\gamma} & \frac{\delta}{\delta+\gamma} \end{bmatrix}.$$

This stationary distribution determines the long term time average of the probability we are in the given state, so our expected average profit under the description given here is

$$\text{Profit}[\gamma, \delta] = 200\left(\frac{\gamma}{\delta+\gamma}\right) - \left(\frac{10}{1-\gamma}\right)\left(\frac{\delta}{\delta+\gamma}\right).$$

It is this expression we desire to maximize with respect to γ. Taking the γ derivative and setting the result equal to zero we find

$$\frac{\text{Profit}[\gamma, \delta]}{d\gamma} = \frac{200}{\delta+\gamma} - \frac{200\gamma}{(\delta+\gamma)^2} - \frac{10}{(1-\gamma)^2}\left(\frac{\delta}{\delta+\gamma}\right) + \frac{10\delta}{(1-\gamma)(\delta+\gamma)^2} = 0\,.$$

Taking a breakdown rate of $\delta = 0.2$ we can use the MATLAB function `fsolve` to find the solution to the above equation. We find a solution given by $\gamma = 0.8$ and gives an optimal profit of 20. This calculation is done in the MATLAB script `chap_6_prob_20.m`.

Exercise 21 (the machine maintenance model)

Part (a): The one-step transition matrix for this example is given by

$$P = \left[\begin{array}{ccc} 1-\delta & \delta(1-\epsilon) & \delta\epsilon \\ 0 & 1-\phi & \phi \\ \gamma\rho & (1-\gamma)\rho & 1-\rho \end{array}\right] = \left[\begin{array}{ccc} \frac{4}{5} & \frac{3}{50} & \frac{7}{50} \\ 0 & \frac{2}{5} & \frac{3}{5} \\ \frac{8}{25} & \frac{2}{25} & \frac{3}{5} \end{array}\right],$$

when we put in the given numbers.

Part (b): The stationary distribution α is the unique row vector, α, such that $\alpha P = \alpha$ and has a sum of components equal to one. Writing the steady state conditions as

$$\begin{aligned} \alpha(P - I) &= 0 \\ \alpha \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right] &= 1\,. \end{aligned}$$

Taking the transpose of each equation gives

$$\begin{aligned} (P^T - I)\alpha^T &= 0 \\ \left[\begin{array}{ccc} 1 & 1 & 1 \end{array}\right] \alpha^T &= 1\,. \end{aligned}$$

If we denote α as the row vector given by (a, b, c) the system above becomes in terms of the components a, b, c becomes the following

$$\left[\begin{array}{ccc} -\delta & 0 & \gamma\rho \\ \delta(1-\epsilon) & -\phi & (1-\gamma)\rho \\ \delta\epsilon & \phi & -\rho \\ 1 & 1 & 1 \end{array}\right] \left[\begin{array}{c} a \\ b \\ c \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}\right].$$

Solving this system using Gaussian elimination or something equivalent gives

$$\left[\begin{array}{c} a \\ b \\ c \end{array}\right] = \frac{1}{217} \left[\begin{array}{c} 120 \\ 22 \\ 75 \end{array}\right] = \left[\begin{array}{c} 0.5530 \\ 0.1014 \\ 0.3456 \end{array}\right].$$

These calculations are done in the MATLAB file `chap_6_prob_21_N_22.m`.

Part (c): Recalling what the given state definitions mean for this problem we have that the ratio of the number of days the machine is in good working order relative to that of when its it poor working order to be

$$\frac{a}{b} = \frac{0.5530}{0.1014} = 5.454\,,$$

while the number of days the machine is in good working order relative to the number of days the machine is broken to be

$$\frac{a}{c} = \frac{0.5530}{0.3456} = 1.600\,.$$

Exercise 22 (additional information on the machine maintenance chain)

Part (a): The expected number of days between breakdowns is the expected amount of time our system spends in states 0 (working and in good condition) or 1 (working in poor condition) before returning to state 2 (out of order). This is the mean recurrence time for state 2 and is given by

$$\frac{1}{c} = \frac{1}{0.3456} = 2.89 \,.$$

Part (b-d): This is the expected number of steps that will be taken (starting in state 0) until we reach state 2 and is given by the $(1,3)$ component of the R matrix i.e. the value R_{02}. So to answer questions like this we need to compute the entries of the matrix R by solving the matrix equation $(I-P)M = U - D$, for the matrix M which is related to R by $M = R - D$. We begin by computing $I - P$ to find

$$I - P = \begin{bmatrix} 1/5 & -3/50 & -7/50 \\ 0 & 3/5 & -3/5 \\ -8/25 & -2/25 & 2/5 \end{bmatrix} .$$

We next compute the matrix $U - D$ is given by

$$U - D = \begin{bmatrix} -97/120 & 1 & 1 \\ 1 & -195/22 & 1 \\ 1 & 1 & -142/75 \end{bmatrix} .$$

Recall that U is a matrix of all ones and D is a diagonal matrix with the mean recurrence time for state i for the element D_{ii}.

To find the matrix M we must explicitly look for a matrix that has *zero* diagonal elements. Thus we must solve

$$\begin{bmatrix} 1/5 & -3/50 & -7/50 \\ 0 & 3/5 & -3/5 \\ -8/25 & -2/25 & 2/5 \end{bmatrix} \begin{bmatrix} 0 & T_{01} & T_{02} \\ T_{10} & 0 & T_{12} \\ T_{20} & T_{21} & 0 \end{bmatrix} = \begin{bmatrix} -97/120 & 1 & 1 \\ 1 & -195/22 & 1 \\ 1 & 1 & -142/75 \end{bmatrix} .$$

Multiplying the two matrices on the left hand side we obtain

$$\begin{bmatrix} -\frac{3}{50}T_{10} - \frac{7}{50}T_{20} & \frac{1}{5}T_{01} - \frac{7}{50}T_{21} & \frac{1}{5}T_{02} - \frac{3}{50}T_{12} \\ \frac{3}{5}T_{10} - \frac{3}{5}T_{20} & -\frac{3}{5}T_{21} & \frac{3}{5}T_{12} \\ -\frac{2}{25}T_{10} + \frac{2}{5}T_{20} & -\frac{8}{25}T_{01} + \frac{2}{5}T_{21} & -\frac{8}{25}T_{02} - \frac{2}{25}T_{12} \end{bmatrix} = \begin{bmatrix} -97/120 & 1 & 1 \\ 1 & -195/22 & 1 \\ 1 & 1 & -142/75 \end{bmatrix} .$$

Solving the easy equations first we see that by equating the $(2,2)$ component on both sides gives $T_{21} = \frac{5}{3}\left(\frac{195}{22}\right) = 14.77$. The equation from the $(2,3)$ component gives $T_{12} = \frac{5}{3}(1) = 1.666$. The $(1,2)$ component then gives

$$T_{01} = 5\left(1 + \frac{7}{50}T_{21}\right) = 15.3409 \,.$$

The $(1,3)$ component gives

$$T_{02} = 5\left(1 + \frac{3}{50}T_{12}\right) = 5.50 \,.$$

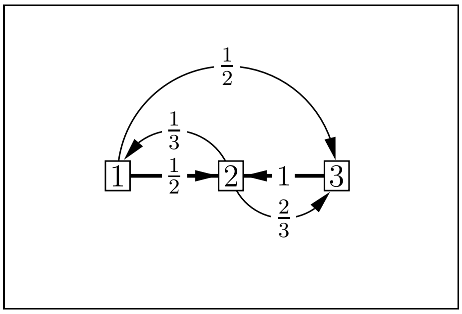

Figure 18: The transition diagram for Exercise 23.

Finally, the $(1,1)$ and $(2,1)$ equations must be considered together since they are more coupled to find T_{10} and T_{20}. Solving this system we find

$$T_{10} = 5.2083 \quad \text{and} \quad T_{20} = 3.5417\,,$$

and the matrix M becomes

$$M = \begin{bmatrix} 0 & 15.3409 & 5.5000 \\ 5.2083 & 0 & 1.6667 \\ 3.5417 & 14.7727 & 0 \end{bmatrix}.$$

so that $R = M + D$ then becomes

$$R = \begin{bmatrix} 1.8083 & 15.3409 & 5.5000 \\ 5.2083 & 9.8636 & 1.6667 \\ 3.5417 & 14.7727 & 2.8933 \end{bmatrix}.$$

Using the R matrix we see that if we are in good working order (in state 0) the number of expected days until we enter state 2 (broken down) is $R_{02} = 5.5$. If we are in poor working condition (in state 1) the number of days until we enter state 0 (working again) is given by $R_{10} = 5.208$. Finally, if we are not working (in state 2) the expected number of days until we are working again is given by $R_{20} = 3.54$.

These calculations are performed in the MATLAB script `chap_6_prob_21_N_22.m`.

Exercise 23 (a given Markov chain)

Part (a): The transition diagram for the given P can be seen in Figure 18. For a Markov chain to be regular means it is irreducible and aperiodic. This chain is aperiodic because every path is not a multiple of an integer period and this chain is irreducible since every state communicates with every other state.

Part (b): The stationary distribution is the unique row vector $\pi = (a, b, c)$ that satisfies

$\pi P = \pi$ and satisfies $a + b + c = 1$. Considering $\pi P = \pi$ we have in equation form

$$\begin{aligned}
\left(\frac{1}{3}\right) b &= a \\
\left(\frac{1}{2}\right) a + c &= b \\
\left(\frac{1}{2}\right) a + \left(\frac{2}{3}\right) b &= c\,.
\end{aligned}$$

The first equation gives $b = 3a$. The second equation gives

$$c = b - \frac{1}{2}a = \frac{5}{2}a\,.$$

The third equation becomes

$$b = \frac{3}{2}\left(c - \frac{1}{2}a\right) = 3a\,.$$

Then the constraint that $a + b + c = 1$ in terms of the variable a becomes

$$a + 3a + \frac{5}{2}a = 1 \Rightarrow a = \frac{2}{13}\,.$$

Then b and c are given by

$$\begin{aligned}
b &= 3\left(\frac{2}{13}\right) = \frac{6}{13} \\
c &= \frac{5}{2}\left(\frac{2}{13}\right) = \frac{5}{13}\,.
\end{aligned}$$

Thus our stationary distribution is

$$\pi = \left[\begin{array}{ccc} \frac{2}{13} & \frac{6}{13} & \frac{5}{13} \end{array}\right].$$

Part (c): The recurrence for state i is the number of steps taken before we return to state i. I would guess that state 2 has the shortest mean recurrence time since with probability one we end up there if we land in state 3. That is many paths seem to lead to state 2. I would expect state 1 to have the longest mean recurrence time since since we can only get there from state 2 and only then with a probability of 1/3. Thus few paths go to state 1. From the stationary distribution found above we calculate the mean recurrence times to be

$$\left[\begin{array}{ccc} R_{11} & R_{22} & R_{33} \end{array}\right] = \left[\begin{array}{ccc} 1/a & 1/b & 1/c \end{array}\right] = \left[\begin{array}{ccc} 13/2 & 13/6 & 13/5 \end{array}\right] = \left[\begin{array}{ccc} 6.5 & 2.16 & 2.6 \end{array}\right],$$

which verifies our hypothesis on which states have the longest/shortest recurrence times.

Exercise 24 (where to place our goal)

I would expect that the state with the largest mean recurrence time would take the longest time to visit and if the goal where placed there result in the smallest expect payoff. From Exercise 23 above I would expect that that placing our goal at state 3 which has a mean recurrence time of 2.6 v.s. state 2 which has a mean recurrence time of 2.16, would yield the highest long run profit.

To check this we compute the mean first entrance times i.e. the elements of the matrix $R_{ij} = E_i[T_j]$, by solving for the matrix M (which is related to R) in the system $(I-P)M = U-D$. In the MATLAB script `chap_6_prob_24.m` we compute

$$I - P = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/3 & 1 & -2/3 \\ 0 & -1 & 1 \end{bmatrix},$$

and

$$U - D = \begin{bmatrix} -11/2 & 1 & 1 \\ 1 & -7/6 & 1 \\ 1 & 1 & -8/5 \end{bmatrix}.$$

So that the equation $(I-P)M = U-D$ then becomes

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/3 & 1 & -2/3 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & T_{23} \\ T_{31} & T_{32} & 0 \end{bmatrix} = \begin{bmatrix} -11/2 & 1 & 1 \\ 1 & -7/6 & 1 \\ 1 & 1 & -8/5 \end{bmatrix}.$$

Multiplying the two matrices on the left hand side we obtain

$$\begin{bmatrix} -\frac{1}{2}T_{21} - \frac{1}{2}T_{31} & T_{12} - \frac{1}{2}T_{32} & T_{13} - \frac{1}{2}T_{23} \\ T_{21} - \frac{2}{3}T_{31} & -\frac{1}{3}T_{12} - \frac{2}{3}T_{32} & -\frac{1}{3}T_{13} + T_{23} \\ -T_{21} + T_{31} & T_{32} & -T_{23} \end{bmatrix} = \begin{bmatrix} -11/2 & 1 & 1 \\ 1 & -7/6 & 1 \\ 1 & 1 & -8/5 \end{bmatrix}.$$

From this we see that the equation for the $(3,2)$ component gives $T_{32} = 1$. The equation for the $(3,3)$ component gives $T_{23} = \frac{8}{5}$. The $(2,2)$ equation gives

$$T_{12} = -3\left(-\frac{7}{6} + \frac{2}{3}T_{23}\right) = \frac{3}{10}.$$

The $(2,3)$ equation is

$$T_{13} = -3\left(1 - T_{23}\right) = \frac{9}{5}.$$

We need to now solve for T_{21} and T_{31} together since they are more tightly coupled than the previously considered components of T. Using the equations given by the $(1,1)$ and the $(2,1)$ components we find

$$T_{21} = 5 \quad \text{and} \quad T_{31} = 6.$$

Using all of these results we find that our matrix M is given by

$$M = \begin{bmatrix} 0 & 3/10 & 9/5 \\ 5 & 0 & 8/5 \\ 6 & 1 & 0 \end{bmatrix},$$

and the matrix R is given by

$$R = \begin{bmatrix} 13/2 & 3/10 & 9/5 \\ 5 & 13/6 & 8/5 \\ 6 & 1 & 13/5 \end{bmatrix}.$$

From which we see starting in state 1 on average it takes 3/10 to first get to state 2 and 9/5 amount of time to get to state 3. A goal at state 3 would yield a better long term profit.

Exercise 25 (the weak form of the law of large numbers)

Part (a): For the repeated independent trials chain the one-step probability matrix P does not depend on on the current state i and its (i, j)th element P_{ij} is equal to π_j. Thus the matrix P has constant rows, where each row is equal to the steady-state transition probability π.

To be regular a Markov chain must be aperiodic and irreducible. This Markov chain is obviously aperiodic since we can get from any state i to any other state j in one jump and it is irreducible since every state communicates with every other. This is assuming $\pi_j \neq 0$ for all j.

The stationary distribution is the unique row vector α such that $\alpha P = \alpha$ and has components that sum to one. If we hypothesis that $\alpha = \pi$, we can verify that this is true by considering the jth component element of the product αP. We find

$$(\alpha P)_j = \sum_{k=1}^{n} \alpha_k P_{kj} = \sum_{k=1}^{n} \pi_k \pi_j = \pi_j \sum_{k=1}^{n} \pi_k = \pi_j \,.$$

So we see that $\pi P = \pi$, showing that the stationary distribution α is the same as the state transition distribution π.

Part (b): At a heuristic level the weak form of the law of large numbers states that if we perform an experiment n times and count the number of times that a given outcome occurs, n_o the ratio of this number to n should approach the *probability* of this outcome as n goes to infinity. Theorem 6.8 is of the same flavor as this law in that the expression

$$A_j(n) = \frac{1}{n} \sum_{t=1}^{n} I_j(t) \,,$$

counts the number of times we visit state j and Theorem 6.8 states that this limit as $n \to \infty$ this is equal to the jth component of the stationary distribution π_j i.e. the long term proportion of time we find our chain in the state j.

Exercise 26 (simulating the genetic chain)

To do this problem we will draw samples from the Markov chain for the genetic chain using the MATLAB function, `mc_sample.m`, found in Kevin Murphy's Bayes' Net toolbox. This function generates samples from a Markov chain given an initial probability distribution π_0 over the states and a one-step probability transition matrix P.

Part (a): To explore the probability that a decedent will have all normal subgenes we will generate M Markov chains of N timesteps each and then use the weak form of the law of large numbers to estimate the probability we end in the third state by computing

$$\hat{P}_2\{\text{Game ends at } 3\} = \frac{\#\{X(N) = 3\}}{M} \,.$$

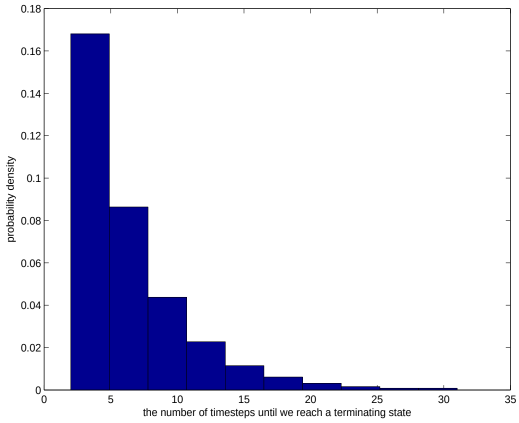

Figure 19: The probability density function of the number of timesteps until we reach an absorbing state for Exercise 26.

Here the notation "#" means to count the number of times the marker is found to be in state 3 on the last timestep.

In exercise 16 starting in state $i = 2$ we found that the exact probability our decedent gene will eventually have all normal subgenes (end in state $i = 3$) is given by

$$P_2\{ \text{ Game ends at 3 } \} = \frac{2}{3} .$$

The approximate value obtained via. Monte Carlo simulation with 10000 chains is 0.66818 and is quite close to the exact value.

Part (b): To compute the distribution of the number of generations before a descendant has either all normal ($X(N) = 3$) or all abnormal ($X(N) = 0$) subgenes we perform the following. For each sample we will compute the first index i^* such that $X(i^*) = 3$ or $X(i^*) = 0$. This index is one larger than the number of timesteps we must take to get to a terminal state. The average of these values of $i^* - 1$ gives the desired statistic to compare with the result in Exercise 16. The individual samples provide information on the distribution of timesteps.

From Exercise 16 the expected number of generations before a descendant has either all normal or all abnormal subgenes is given by

$$E_2[W] = \sum_{j \in T} n_{2j} = 5 .$$

The approximate value we obtain when we average all the Monte Carlo chains is 4.9893. The standard deviation of these samples is given by 4.428. The probability density of the number of timesteps until we reach an absorbing state is shown in Figure 19.

These calculations are done in the MATLAB file `chap_6_prob_26.m`.

Chapter 7: The Poisson Process

Exercise Solutions

Exercise 1 (some electronic components)

Part (a): An exponential distribution with mean 1000 hours is given by

$$p_T(t) = \frac{1}{1000} e^{-\frac{1}{1000}t} \quad \text{for} \quad t \geq 0\,,$$

and is zero otherwise. Then the probability that a component will fail during the time period $(900, 1000)$ is given by

$$P\{900 < T < 1000\} = \int_{900}^{1000} p_T(t)dt = F_T(1000) - F_T(900)\,,$$

where $F_T(\cdot)$ is the cumulative distribution function for an exponential random variable and is given by

$$F_T(t) = P\{T < t\} = 1 - e^{-\frac{1}{1000}t}\,.$$

Using this we find that

$$P\{900 < T < 1000\} = (1 - e^{-1}) - (1 - e^{-0.9}) = e^{-0.9} - e^{-1} = 0.0386\,.$$

Part (b): By the memoryless property of the exponential random variable the fact that the component is functioning after 900 hours of service makes no difference to the probability that the component will fail at any given time t^*. Mathematically this is expressed as

$$P\{T < 1000 | T > 900\} = P\{T < 100\} = F_T(100) = 1 - e^{-0.1} = 0.0951\,.$$

Exercise 2 (a serially connected machine)

If we introduce A and B as exponential random variables denoting the lifetime of the components A and B then the lifetime of the two component machine with A and B in serial is given by the random variable C where $C = \min(A, B)$. From the discussion in the book if both A and B are exponentially distributed with failure rates α and β, then the random variable C is exponentially distributed with a failure rate given by $\alpha + \beta$. Since we are told that $\alpha = 0.01$ for this problem this expression is given by $0.01 + \beta$. To have a probability that our serial machine lasts at least 50 hours requires that

$$P\{C \geq 50\} = 0.5\,,$$

or

$$1 - P\{C \leq 50\} = 0.5\,,$$

or finally

$$F_C(50) = 0.5\,,$$

where $F_C(\cdot)$ is the cumulative distribution function for the random variable C. Since this is given by

$$1 - e^{-50(0.01+\beta)} = 0.5\,,$$

we can solve for β in the above. When we do this we find that

$$\beta = \frac{\ln(2)}{50} - 0.01 = 0.003862\,.$$

Which gives a mean of $1/\beta = 258.87$ hours. A value of β *smaller* than this will increase the chances that the machine is operating after 50 hours.

Exercise 3 (serial lifetime of two exponential components)

The lifetime of he communications satellite will be distributed as the random variable representing the minimum of the random variables representing the lifetimes of the two components A and B. Since A and B are exponentially distributed with failure rates $\lambda = 2$ (per year) and $\mu = 1$ (per year) the random variable representing the minimum of these two variables is an exponential random variable with failure rate given by $\lambda + \mu = 3$ (per year).

Part (a): The probability that the satellite will fail within the first year is given by the cumulative distribution function of the minimum of A and B. Defining $C = \min(A, B)$ we have that

$$F_C(c) = 1 - e^{-(\lambda+\mu)c}\,,$$

so that our desired probability is

$$F_C(1) = 1 - e^{-3} = 0.9502\,.$$

Part (b): Let I_A be an indicator random variable denoting if system component A failed first i.e. $I_A = 1$ if $A \le B$ and $I_A = 0$ if $A > B$. Then we want to compute $P\{C \le 1, I_A = 1\}$. From Lemma 7.1 in the book the random variable C and I_A are independent so this probability becomes

$$P\{C \le 1, I_A = 1\} = P\{C \le 1\}P\{I_A = 1\} = \left(1 - e^{-3}\right)\left(\frac{2}{1+2}\right) = 0.63348\,.$$

where we have used the result from Lemma 7.1 where

$$P\{I_A = 1\} = E[I_A] = \frac{\mu}{\mu + \lambda} = \frac{2}{3}\,.$$

Exercise 4 (a memoryless discrete random variable)

Part (a): A geometric random variable with parameter p is defined as one that has $P\{X = k\} = q^{k-1}p$, for $k = 1, 2, \cdots$. From which we see that

$$\begin{aligned} P\{X > k\} &= 1 - P\{X \leq k\} \\ &= 1 - \sum_{i=1}^{k} P\{X = i\} \\ &= 1 - \sum_{i=1}^{k} pq^{i-1} \\ &= 1 - p\sum_{i=0}^{k-1} q^i \\ &= 1 - p\left(\frac{q^i}{q-1}\right|_0^k \\ &= 1 - (1 - q^k) = q^k \,. \end{aligned}$$

Part (b): Now consider the requirement that the random variable X has no memory i.e.

$$P\{X > k + m | X > m\} = P\{X > k\} \,.$$

which using the definition of conditional probabilities is equivalent to

$$\frac{P\{X > k + m\}}{P\{X > m\}} = P\{X > k\} \,,$$

Or multiplying across by the denominator we have

$$P\{X > k + m\} = P\{X > k\}P\{X > m\} \,.$$

Defining the reliability function $R(k) = P\{X > k\}$ for $k \geq 0$, we see that $R(\cdot)$ satisfies

$$R(k + m) = R(k)R(m) \quad \text{with} \quad R(0) = 1 \,.$$

Following the steps found in this chapter of the book the unique solution to this functional equation is given by $R(mk) = R(m)^k$, where m is an integer. Taking $m = 1$ for convenience since our function R is defined only at the integers, we have $R(k) = R(1)^k$. Defining $q = R(1)$ and remembering the definition of the reliability function R our solution is then

$$P\{X > k\} = q^k \,.$$

Thus the random variable X must be a geometric random variable as claimed. Note that this is slightly different than the problem formulation suggested in the book in that all greater than *or equal* signs are replaced with strictly greater than signs this is more consistent with the continuous case.

Exercise 5 (set descriptions of a Poisson process)

Part (a): For this part of the problem we will evaluate the expression

$$\max\{k : S_k \le t\}\,.$$

From the discussion in the book the following two sets are equivalent

$$\{S_n \le t\} \quad \text{and} \quad \{N(t) \ge n\}\,.$$

Now if we take k to be the largest integer such that $S_k \le t$ while $S_{k+1} > t$, we see from the equivalent two sets above, this means that $N(t) \ge k$ while $N(t) < k+1$. This information together with the fact that $N(t)$ is integer valued imply that $N(t)$ must equal k, i.e. $N(t) = k$. These results imply that

$$\max\{k : S_k \le t\} = N(t)\,,$$

as we were to show.

Part (b): For this part of the problem we will evaluate the expression

$$\min\{t : N(t) \ge n\}\,.$$

As in Part (a) of this problem the following two sets are equivalent

$$\{S_n \le t\} \quad \text{and} \quad \{N(t) \ge n\}\,.$$

Now if t is the *smallest* value of t such that $N(t) \ge n$, then this means that

$$N(t-\epsilon) < n \quad \text{and} \quad N(t) \ge n \quad \forall \epsilon > 0\,.$$

From the equivalent two sets above this means that

$$S_n > t - \epsilon \quad \forall \epsilon > 0 \quad \text{and} \quad S_n \le t\,,$$

or the statement equivalent to this is $S_n = t$. This implies that $\min\{t : N(t) \ge n\} = S_n$.

Exercise 6 (a telephone switchboard)

The information that on average that one call comes every ten minutes is equivalent to stating that the rate of the Poisson process is given by solving $\lambda(10) = 1$, which gives that $\lambda = \frac{1}{10}$. The units of which are reciprocal minutes.

Part (a): For this part of the problem we desire to calculate

$$P\{N(10) = 0, N(15) - N(10) = 1\}\,.$$

By the independent increments property of a Poisson process is given by

$$P\{N(10) = 0\}P\{N(15) - N(10) = 1\}\,,$$

and by the shift-invariant property of a Poisson process (meaning that only the amount of time elapsed between 15 and 10 seconds is what matters) is equal to

$$P\{N(10) = 0\}P\{N(5) = 1\} = e^{-1}\left(\frac{1}{2}e^{-1/2}\right) = 0.1115\,.$$

Part (b): The fourth call will happen at a time given by $T_4 = \sum_{i=1}^{4} X_i$ where X_i are independent exponential random variables all with failure rate $\lambda = \frac{1}{10} = 0.1$. With this the expected time of the fourth call will be

$$E[T_4] = \sum_{i=1}^{4} E[X_i] = \sum_{i=1}^{4} \frac{1}{\lambda} = \sum_{i=1}^{4} 10 = 40\,.$$

Part (c): We desire to compute $P\{N(20) - N(10) \geq 2\}$, which by the shift-invariant property of the Poisson process is equal to $P\{N(20-10) \geq 2\} = P\{N(10) \geq 2\}$. Now since all probabilities must add to one we have

$$1 = P\{N(10) = 0\} + P\{N(10) = 1\} + P\{N(10) \geq 2\}\,,$$

so we find that

$$\begin{aligned} P\{N(10) \geq 2\} &= 1 - P\{N(10) = 0\} - P\{N(10) = 1\} \\ &= 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 0.2642\,. \end{aligned}$$

Part (d): Given that n calls have happened by time t their location within the interval $(0, t)$ are given by n independent independent uniform (over $(0, t)$) random variables. Thus the probability this only call occurred during the last $1/3$ of the time interval is given by $1/3$.

Part (e): By the independent increments property of the Poisson process, the fact that one call arrived in the time interval $0 < t < 15$ has no affect on what will happen in the next five minutes. Thus

$$\begin{aligned} P\{N(20) - N(15) \geq 1 | N(15) - N(0) = 1\} &= P\{N(20) - N(15) \geq 1\} \\ &= P\{N(5) \geq 1\} \\ &= 1 - P\{N(5) = 0\} \\ &= 1 - e^{-0.1(5)} = 1 - e^{-0.5} = 0.3934\,. \end{aligned}$$

Exercise 7 (play till we get struck by lightning)

Sam will be able to play his game if no lightning strikes during the time interval $(0, s)$. This will happen with probability

$$P\{N(s) = 0\} = e^{-\lambda s}\,.$$

Since for this problem $\lambda = 3$ (per hour) this expression then becomes

$$P\{N(s) = 0\} = e^{-3s} .$$

Since the times s provided are given in minutes they must be converted to fractions of an hour. For the s's given we have in fractions of an hour the following

$$2/60 \, , \quad 10/60 \, , \quad 20/60$$

or

$$1/30 \, , \quad 1/6 \, , \quad 1/3 \, .$$

This gives probabilities of

$$e^{-1/10} = 0.905 \, , \quad e^{-1/2} = 0.606 \, , \quad e^{-1} = 0.3678 \, .$$

Exercise 8 (surviving two lighting strikes)

In this case Sam can survive at most one lighting strike. Thus the probability that he can finish his game is now given by

$$P\{N(s) = 0\} + P\{N(s) = 1\} = e^{-\lambda s} + \lambda s e^{-\lambda s} = e^{-3s}(1 + 3s) \, .$$

Where we have used the value of $\lambda = 3$. Using the values of s given in Problem 7 above (in terms of fractions of an hour) we find these three probabilities to be

$$\begin{aligned} e^{-1/10}\left(1 + \frac{1}{10}\right) &= 0.995 \\ e^{-1/2}\left(1 + \frac{1}{2}\right) &= 0.909 \\ e^{-1}\left(1 + 1\right) &= 2e^{-1} = 0.735 \end{aligned}$$

Exercise 9 (the covariance of a Poisson process)

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Then we desire to calculate $\text{Cov}(N(t), N(s))$ which we do by manipulating the expression we desire such that we introduce "increment variables". We begin by assuming that $t \geq s$. We find

$$\begin{aligned} \text{Cov}(N(t), N(s)) &= \text{Cov}(N(t) - N(s) + N(s), N(s)) \\ &= \text{Cov}(N(t) - N(s), N(s)) + \text{Cov}(N(s), N(s)) \, . \end{aligned}$$

But by the independent increments property of the Poisson process, the random variables $N(t) - N(s)$ and $N(s)$ are independent, so

$$\text{Cov}(N(t) - N(s), N(s)) = 0 \, .$$

In addition we have from the definition of covariance that $\text{Cov}(N(s), N(s)) = \text{Var}(N(s))$, which for a Poisson process is given by λs. Combining these results we find that

$$\text{Cov}(N(t), N(s)) = \lambda s \,.$$

In the case when $s \geq t$ then all of the above manipulations still hold but with t and s switched. Considering this we find that

$$\text{Cov}(N(t), N(s)) = \lambda \text{min}(s, t) \,.$$

Exercise 10 (the conditional distribution of $N(s)$ given $N(t) = n$)

We are told that $\{N(t), t \geq 0\}$ is a Poisson distribution (with rate λ) and we are asked to find

$$P\{N(s) = m | N(t) = n\} \,.$$

We are assuming here that both $t \geq s$ and $n \geq m$. We will do this by using the definitions of conditional probability and properties of the Poisson process. The desired probability above is equal to (introducing "increment variables")

$$P\{N(t) - N(s) = n - m, N(s) = m | N(t) = n\} \,.$$

Which by the definition of conditional probability is equal to

$$\frac{P\{N(t) - N(s) = n - m, N(s) = m\}}{P\{N(t) = n\}} \,.$$

By the independent increments properties of the Poisson process the above is equal to

$$\frac{P\{N(t) - N(s) = n - m\} P\{N(s) = m\}}{P\{N(t) = n\}} \,.$$

Finally using the stationary increments property of a Poisson process on the first term in the numerator of the above fraction, we can simplify the increment variable in the above to give

$$\frac{P\{N(t - s) = n - m\} P\{N(s) = m\}}{P\{N(t) = n\}} \,.$$

Since we can compute each of these probabilities for a Poisson process we find that the above equals

$$\begin{aligned}
P\{N(s) = m | N(t) = n\} &= \frac{\left(\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-m}}{(n-m)!}\right)\left(\frac{e^{-\lambda s}(\lambda s)^m}{m!}\right)}{\left(\frac{e^{-\lambda t}(\lambda t)^n}{n!}\right)} \\
&= \frac{n!}{m!(n-m)!} \frac{(\lambda(t-s))^{n-m}(\lambda s)^m}{(\lambda t)^n} \\
&= \left(\begin{array}{c} n \\ m \end{array}\right) \frac{(t-s)^{n-m} s^m}{t^n} \\
&= \left(\begin{array}{c} n \\ m \end{array}\right) \frac{(t-s)^{n-m} s^m}{t^{n-m}\, t^m} \\
&= \left(\begin{array}{c} n \\ m \end{array}\right) \left(1 - \frac{s}{t}\right)^{n-m} \left(\frac{s}{t}\right)^m \,.
\end{aligned}$$

Which we recognize as a binomial random variables with parameters n and $p = \frac{s}{t}$ as claimed.

Exercise 11 (waiting for the bus)

Part (a): Assuming that the number of people that arrive to wait for a bus is a Poisson random variable, the probability that there are n people waiting at the bus stop at time t is given by the standard expression for a Poisson process i.e.

$$P\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!},$$

which has an expected number of people at time t is given by λt. This is the desired expression for $E[N|T = t]$.

Part (b): The probability distribution for $E[N|T]$ is obtained from that from the distribution of the random variable T. This latter distribution is uniform, so the distribution of the random variable $E[N|T]$ will also be uniform (since it is just a multiple of T). If we denote the random variable $E[N|T]$ as X we have

$$P_X(x) = \begin{cases} 1/\lambda & 0 < x < \lambda \\ 0 & \text{otherwise} \end{cases},$$

or a uniform distribution over the range $(0, \lambda)$.

Part (c): The expectation of N can be computed by conditioning on the random variable T. We have

$$E[N] = E[E[N|T]] = E[\lambda T] = \lambda E[T] = \frac{\lambda}{2}.$$

Since we know T to be a uniform random variable over $(0, 1)$.

Part (d): To compute the probability that there be no people at the bus stop when the bus arrives, we can compute this probability by conditioning on the time when the bus arrives. If the bus arrives at time t, then the probability that no people are there is given by (using the expression from Part (a) above)

$$P\{N(t) = 0|T = t\} = e^{-\lambda t}.$$

Thus the total probability desired is given by integrating this expression against the distribution function for T (which is uniform over $(0, 1)$). We find

$$\begin{aligned} P\{N = 0\} &= \int_0^1 P\{N(t) = 0|T = t\}dt \\ &= \int_0^1 e^{-\lambda t}dt = \left.\frac{e^{-\lambda t}}{(-\lambda)}\right|_0^1 = \frac{1 - e^{-\lambda}}{\lambda}. \end{aligned}$$

Chapter 8: Continuous-Time Stochastic Processes

Notes On The Text

Proving that $N(t) \equiv N_0(\tau(t))$ is a nonhomogenous Poisson process (page 219)

With the definition of τ given by

$$\tau = \int_0^t \lambda(s) ds \,,$$

we define the stochastic process $N(t)$ by $N(t) \equiv N_0(\tau(t))$. Lets now compute the expression $P\{N(t+h) - N(t) = 1\}$. We find that

$$\begin{aligned} P\{N(t+h) - N(t) = 1\} &= P\{N_0(\tau(t+h)) - N_0(\tau(t)) = 1\} \\ &\approx P\{N_0(\tau(t) + \lambda(t)h + o(h)) - N_0(\tau(t)) = 1\} \\ &\approx 1(\lambda(t)h + o(h)) + o(h) \\ &= \lambda(t)h + o(h) \,. \end{aligned}$$

which follows from the third property of Poisson processes (here applied to the Poisson process $N_0(t)$ which has a failure rate of $\lambda = 1$). The expression $P\{N(t+h) - N(t) > 1\}$ can be evaluated in the same way and shown to be $o(h)$.

Chapter 8: Continuous-Time Stochastic Processes

Exercise Solutions

Exercise 1 (the sum of two Poisson processes)

We will first prove that the sum of two Poisson random variables is a Poisson random variable. Let X and Y be Poisson random variables with parameters λ_1 and λ_2 respectively. We can evaluate the distribution of $X+Y$ by computing the characteristic function of $X+Y$. Since X and Y are independent Poisson random variables the characteristic functions of $X+Y$ is given by

$$\begin{aligned}
\phi_{X+Y}(u) &= \phi_X(u)\phi_Y(u) \\
&= e^{\lambda_1(e^{iu}-1)}e^{\lambda_2(e^{iu}-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^{iu}-1)}\,.
\end{aligned}$$

From the direct connection between characteristic functions to and probability density functions we see that the random variable $X+Y$ is a Poisson random variable with parameter $\lambda_1+\lambda_2$, the sum of the Poisson parameters of the random variables X and Y.

Now for the problem at hand, since $N_1(t)$ and $N_2(t)$ are both Poisson random variables with parameters $\lambda_1 t$ and $\lambda_2 t$ respectively, from the above discussion the random variable $N(t)$ defined by $N_1(t)+N_2(t)$ is a Poisson random variable with parameter $\lambda_1 t+\lambda_2 t$ and thus has a probability of the event $N(t)=j$ given by

$$P\{N(t)=j\} = \frac{e^{-(\lambda_1 t+\lambda_2 t)}(\lambda_1 t+\lambda_2 t)^j}{j!} = \frac{e^{-(\lambda_1+\lambda_2)t}((\lambda_1+\lambda_2)t)^j}{j!}\,,$$

showing that $N(t)$ is a Poisson process with rate $\lambda_1+\lambda_2$.

Exercise 2 (Mark and Twain proofreading)

Part (a): We are told from the problem statement that $X(t)$ is a Poisson process representing with rate 10 (per hour) the number of mistakes *Mark* finds after searching for t time (in hours) and $Y(t)$ is a Poisson process with rate 15 (per hour) representing the number of mistakes that *Twain* finds after searching for a time t (in hours). The by Problem 1 the total number of mistakes found together is given by $X(t)+Y(t)$ is a Poisson process with rate given by $10+15=25$ (per hour). In one hour, the probability that we find twenty errors is given by

$$P\{X(1)+Y(1)=20\} = \frac{e^{-25}25^{20}}{20!} = 0.0519\,.$$

Using the **Matlab** or **Octave** command `poisspdf`.

Part (b): We are told that $X(t) + Y(t) = 20$ and we want to compute

$$P\{X(t) = k | X(t) + Y(t) = 20\}\,,$$

which we can do from the definition of conditional probability. We find that

$$\begin{aligned}
P\{X(t) = k | X(t) + Y(t) = 20\} &= \frac{P\{X(t) = k, X(t) + Y(t) = 20\}}{P\{X(t) + Y(t) = 20\}} \\
&= \frac{P\{X(t) = k, Y(t) = 20 - k\}}{P\{X(t) + Y(t) = 20\}} \\
&= \frac{P\{X(t) = k\}P\{Y(t) = 20 - k\}}{P\{X(t) + Y(t) = 20\}} \\
&= \left(\frac{e^{-10t}\,(10t)^k}{k!}\right)\left(\frac{e^{-15t}\,(15t)^{(20-k)}}{(20-k)!}\right)\left(\frac{20!}{e^{-25t}\,(25t)^{20}}\right) \\
&= \frac{20!}{k!(20-k)!}\frac{(10t)^k\,(15t)^{(20-k)}}{(25t)^{20}} \\
&= \binom{20}{k}\left(\frac{10}{25}\right)^k\left(\frac{15}{25}\right)^{20-k}\,,
\end{aligned}$$

or a binomial distribution with parameter $p = \frac{2}{5}$ and $n = 20$. Which has an expectation given by $np = 20\left(\frac{2}{5}\right) = 8$.

Exercise 3 (defective segments of tape)

WARNING: For some reason I get a different answer than the back of the book for this problem. I anyone finds anything wrong with my logic below please let me know.

Part (a): In Example 8.2 from this book we have a probability of a defect begin present in a length of tape Δt given by $p = \lambda\Delta t = (0.001)1 = 0.001$ and we have $\frac{100}{1} = 100$ intervals of length Δt. Letting N be the number of defects found in the entire length of tape, for this part of the problem we want to calculate

$$\begin{aligned}
P\{N \le 2\} &= P\{N = 0\} + P\{N = 1\} + P\{N = 2\} \\
&= \binom{100}{0}p^0(1-p)^{100} + \binom{100}{1}p^1(1-p)^{99} + \binom{100}{2}p^2(1-p)^{98} \\
&= 0.99985\,.
\end{aligned}$$

Part (b): Using the Poisson approximation to the binomial we can approximate each binomial distribution with a Poisson distribution with a parameter $\lambda = pn = 0.001 \cdot 100 = 0.1$ and the probability above becomes

$$\begin{aligned}
P\{N \le 2\} &= P\{N = 0\} + P\{N = 1\} + P\{N = 2\} \\
&= e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2}e^{-\lambda} \\
&= 0.99716\,.
\end{aligned}$$

See the **Matlab** or **Octave** function `chap_8_ex_3.m` for these calculations.

Exercise 4 (a discrete time counting process)

The discrete process $B(n)$ counts the number of times that the random events occurring at time $t = 1, 2, \cdots, n$ occur. We assume that our discrete random process is shift-invariant meaning that the probability of k events occurring in an interval of length j, and beginning at index i, i.e. $P\{N(i+j-1)-N(i) = k\}$ is the same for any starting position i. Mathematically this is represented as

$$P\{N(i+j-1) - N(i) = k\} = P\{N(j) = k\} \quad \text{for} \quad 0 \leq k \leq j\,,$$

We can (and will) define $N(0)$ to be zero for consistency. Also we assume that our process has independent increments meaning that if we are given a sequence of integers

$$0 \leq i_1 < i_2 < \cdots < i_n \leq n$$

the random variable X_k denoting the number of events that occur in the kth interval

$$X_k = B(i_{k+1}) - B(i_k) \quad \text{for} \quad 0 \leq X_k \leq i_{k+1} - i_k + 1\,,$$

are independent. This problem asks us to determine the distribution of the random variables $B(n)$. That is we desire to compute the value of $P\{B(n) = k\}$. This do by introducing independent increments by recognizing that our desired probability can be written as follows,

$$P\{B(n) = k\} = P\left\{\sum_{l=0}^{n-1}(B(l+1) - B(l)) = k\right\}\,,$$

here we have defined $B(0) = 0$. Note that each increment random variable, defined as $B(l+1) - B(l)$ is independent by the independent increments property and from the shift invariant each of these random variables is characterized by only one number. Specifically, by the shift invariant property we see that for all l's we have

$$P\{B(l+1) - B(l) = i\} = P\{B(1) = i\}\,,$$

and since $B(1)$ can take only only two values (either zero or one) we have the constraint that $0 \leq i \leq 1$. Defining $p \equiv P\{B(1) = 1\}$ and $X_l = P\{B(l+1) - B(l)\}$ we see that our problem of evaluating $P\{B(n) = k\}$ is equivalent to that of evaluating

$$P\left\{\sum_{l=0}^{n-1} X_l = k\right\}\,,$$

where the random variables X_i are independent Bernoulli trials each with a probability of success given by p. Thus the random variable $B(n)$ is a binomial random variable with parameters (n, p) as claimed.

Exercise 5 (passing vehicles)

Part (a): From the discussion in the book the number of trucks on the road counted after some initial time is a Poisson process with rate $p\lambda = 0.2(1) = 0.2$ (per minute). Let $X(t)$ be this random process. Then we are asked to compute

$$\begin{aligned} P\{X(t) > 2\} &= 1 - P\{X(t) = 0\} - P\{X(t) = 1\} \\ &= 1 - e^{-\lambda pt} - e^{-\lambda pt} \\ &= 1 - 2e^{-\lambda pt} \,. \end{aligned}$$

When $t = 5$ (minutes) the above becomes

$$P\{X(t) > 2\} = 1 - 2e^{-0.2(5)} = 0.2642 \,.$$

Part (b): For an arbitrary t, in this part of the problem we are asked to compute $E[N(t)|X(t) = 2]$, where $N(t)$ is the Poisson process representing the total number of vehicles that pass after some time t (we can let $t = 5$ to evaluate the specific expressions). Now to evaluate the above expectation we introduce a variable we know to be independent of $X(t)$, namely $N(t) - X(t)$. This unexpected independence property of $X(t)$ and $N(t) - X(t)$ is discussed in the book. We find that

$$\begin{aligned} E[N(t)|X(t) = 2] &= E[N(t) - X(t) + X(t)|X(t) = 2] \\ &= E[N(t) - X(t)|X(t) = 2] + E[X(t)|X(t) = 2] \\ &= E[N(t) - X(t)] + 2 \,. \end{aligned}$$

Where the last step is possible because the random variables $N(t) - X(t)$ and $X(t)$ are independent. Now the random variable $N(t) - X(t)$ is a Poisson process with rate $(1-p)\lambda = 0.8(1) = 0.8$ and thus has an expectation given by $(1 - p)\lambda t = 0.8t$. When $t = 5$ we find that the above expression becomes

$$E[N(t)|X(t) = 2] = (0.8) \cdot 5 + 2 = 6 \,.$$

Part (c): If ten vehicles have passed the probability that two are vans is given by evaluating a binomial distribution. We find

$$\binom{10}{2} (0.2)^2 (1 - 0.2)^8 = 0.3019 \,.$$

Exercise 6 (print jobs)

Part (a): Let $X(t)$ be the random variable denoting a count of the number of jobs that go to the printer. Then $X(t)$ is a Poisson process with rate $p\lambda = \frac{1}{5}(3) = \frac{3}{5}$ (per minute). Then the probability that $X(t) = 0$ is given by

$$P\{X(5) = 0\} = e^{-\frac{3}{5}(5)} = e^{-3} = 0.04978 \,.$$

Part (b): For an arbitrary t, in this part of the problem we are asked to compute $E[N(t)|X(t) = 4]$, where $N(t)$ is the Poisson process representing the total number of print jobs that arrive a the computer center after some time t (we can let $t = 5$ to evaluate the specific expression desired for this part of the problem). Now to evaluate the above expectation we introduce a variable we know to be independent of $X(t)$, namely $N(t) - X(t)$. This unexpected independence property of $X(t)$ and $N(t) - X(t)$ is discussed in the book. We find that

$$\begin{aligned}
E[N(t)|X(t) = 4] &= E[N(t) - X(t) + X(t)|X(t) = 4] \\
&= E[N(t) - X(t)|X(t) = 4] + E[X(t)|X(t) = 4] \\
&= E[N(t) - X(t)] + 4\,.
\end{aligned}$$

Where the last step is possible because the random variables $N(t) - X(t)$ and $X(t)$ are independent. Now the random variable $N(t) - X(t)$ is a Poisson process with rate $(1-p)\lambda = (1 - \frac{1}{5})(3) = \frac{12}{5}$ (per minute). The expectation of this random variable at time $t = 5$ is given by $(1 - p)\lambda t = \frac{12}{5} \cdot 5 = 12$. Thus when $t = 5$ we find that the above expression becomes

$$E[N(5)|X(5) = 4] = 12 + 4 = 16\,.$$

Problem 7 (A will take every other one)

Part (a): If we consider the sequence of inter arrival times $T_1, T_2, \cdots, T_i$, etc, for the original Poisson process we know that the random variables T_i are exponential distributed with rate λ. Now if we denote the interarrival times for the worker A as $\hat{T}_i$, since worker A takes every other arriving event we see that the interarrival times are related to the interarrival times for the *original* Poisson T_i process as

$$\begin{aligned}
\hat{T}_1 &= T_1 \\
\hat{T}_2 &= T_2 + T_3 \\
\hat{T}_3 &= T_4 + T_5 \\
&\vdots \\
\hat{T}_i &= T_i + T_{i+1}\,.
\end{aligned}$$

Now since both T_i and T_{i+1} are exponential random variables the distribution of the random variable $\hat{T}_i$ is given by a gamma random variable with $n = 2$. That is the density function of $\hat{T}_i$ for $i \geq 2$ is given by

$$p_{\hat{T}_i}(t) = \frac{(\lambda t)^{2-1}\lambda e^{-\lambda t}}{\Gamma(2)} = \frac{(\lambda t)^1 \lambda e^{-\lambda t}}{1!} = \lambda^2 t e^{-\lambda t}\,.$$

Part (b): Since the inter arrival times are *not* exponentially distributed $N_A(t)$ is not a Poisson process.

Problem 8 (the expectation and variance of a compound Poisson process)

We can represent the amount of money paid out by our insurance company after t time in weeks as

$$X(t) = \sum_{i=1}^{N(t)} Y_i \,,$$

with Y_i exponentially distributed with mean 2000 and $N(t)$ a Poisson process with rate $\lambda = 5$ (per week) and time measured in weeks $t = 4$ (weeks). This type of process is defined as a compound Poisson process and is discussed in the textbook. There it is shown that

$$E[X(4)] = \mu E[N(t)] = \mu(\lambda t) \,,$$

where μ is the mean of the random variables Y_i. In this problem we have $\mu = 2000$, $\lambda = 5$ (per week), and $t = 4$ (weeks) giving

$$E[X(4)] = 2000 \cdot 5 \cdot 4 = 40000 \,.$$

It is also shown that

$$\text{Var}[X(t)] = \lambda t \sigma^2 + \lambda t \mu^2 \,,$$

where σ is the standard deviation of the random variables Y_i. For exponential random variables we have that

$$\sigma^2 = \mu^2 = 4 \cdot 10^6 \,.$$

So that we see that

$$\text{Var}(X(t)) = 5(4)(4 \cdot 10^6) + 5(4)(4 \cdot 10^6) = 1.6 \cdot 10^8 \,.$$

Problem 9 (events from one Poisson process before an event from another)

Part (a): We know that the times between the arrival events (the interarrival times) in any Poisson process are exponentially distributed. By the memoryless property of the exponential distribution the absolute value of time does not matter since the arrival of events from t *onward* follow the same exponential distribution. From this discussion, starting at a time t the probability a man enters before a women after this time is the same as the probability that the next interarrival time for the men's Poisson process is less than the next interarrival time for the womens's Poisson process. If we let M be an exponential random variable with rate α representing the next interarrival time of the men and W be an exponential random variable with rate β representing the next interarrival time of the women. Then the probability we desire to compute is $P\{M < W\}$. From the discussion in the book we know that

$$P\{M < W\} = \frac{\alpha}{\alpha + \beta} \,.$$

While the complementary result $P\{M > W\}$ (i.e. that a woman enters first) is given by

$$P\{M > W\} = 1 - \frac{\alpha}{\alpha + \beta} = \frac{\beta}{\alpha + \beta}, \,.$$

This result is also shown in [3]. After either a man or a woman arrives by the memoryless property of the exponential distribution we are back to the same situation as earlier. That is the probability that a man enters the store next is given by $\frac{\alpha}{\alpha+\beta}$ and that a women enters the store next is given by $\frac{\beta}{\alpha+\beta}$. This argument continues to hold true as more people enter the store.

Now if we consider the arrival of the first women after time t a *success*, the desired probability distribution for X, the number of men who enter the store *before* the first women, is equivalent to the probability distribution of the number of failure in a sequence of independent Bernoulli trials (each having a probability of success given by $\frac{\beta}{\alpha+\beta}$). This is a modified geometric distribution and we have a probability mass function given by

$$P\{X = k\} = \left(\frac{\beta}{\alpha+\beta}\right)^k \frac{\beta}{\alpha+\beta} \quad \text{for} \quad k = 0, 1, \cdots, \infty .$$

Part (b): From the discussion in Part (a) of this problem and again considering the arrival of a women to be a "success" we are looking for the probability distribution for S, the number of Bernoulli trials required to obtain r successes. This prescription describes a negative binomial random variable and we see that

$$P\{S = k\} = \left(\begin{array}{c} k-1 \\ r-1 \end{array}\right) \left(\frac{\alpha}{\alpha+\beta}\right)^{k-r} \left(\frac{\beta}{\alpha+\beta}\right)^r \quad \text{for} \quad k = r, r+1, \cdots, \infty .$$

Which is the desired result.

Problem 10 (a nonhomogenous Poisson process)

For a nonhomogenous Poisson process with intensity function $\lambda(t)$ the number of events that occur between the times $t+s$ and s are given by a Poisson process with a mean

$$\int_0^{t+s} \lambda(\tau) d\tau - \int_0^s \lambda(\tau) d\tau = \int_s^t \lambda(\tau) d\tau .$$

That is, the probability we have k events between the times s and $t+s$ is given by

$$P\{N(t+s) - N(s) = k\} = \frac{1}{k!} \left(e^{-\int_s^t \lambda(\tau) d\tau} \left(\int_s^t \lambda(\tau) d\tau \right)^k \right) .$$

Part (a): In this case, using the numbers and formula for $\lambda(\tau)$ from the book, and remembering that $t = 0$ corresponds to 10 A.M. so that 1 P.M corresponds to $t = 3$ we find that

$$\begin{aligned}
\int_s^t \lambda(\tau) d\tau &= \int_{3.5}^{3.75} 20(4-\tau) d\tau \\
&= \left(\frac{-20(4-\tau)^2}{2} \right|_{3.5}^{3.75} \\
&= 10(0.5^2) - 10(0.25^2) = 1.875 .
\end{aligned}$$

With this expression we can evaluate the desired probability. We find

$$
\begin{aligned}
P\{N(1.75) - N(1.5) \geq 2\} &= 1 - P\{N(1.75) - N(1.5) = 0\} \\
&- P\{N(1.75) - N(1.5) = 1\} \\
&= 1 - e^{-1.875} - 1.875e^{-1.875} = 0.5591\,.
\end{aligned}
$$

See the Matlab or Octave file `chap_8_prob_10.m` for these calculations.

Part (b): In this case, we find that

$$
\begin{aligned}
\int_s^t \lambda(\tau)d\tau &= \int_{3.75}^{40} 20(4-\tau)d\tau \\
&= \left(\frac{-20(4-\tau)^2}{2} \right|_{3.75}^{4} \\
&= 10(0.25^2) = 0.625\,.
\end{aligned}
$$

With this expression we can evaluate the desired probability. We find

$$
\begin{aligned}
P\{N(1.75) - N(1.5) \geq 2\} &= 1 - P\{N(1.75) - N(1.5) = 0\} \\
&- P\{N(1.75) - N(1.5) = 1\} \\
&= 1 - e^{-0.625} - 0.625e^{-0.625} = 0.1302\,.
\end{aligned}
$$

Again see the file `chap_8_prob_10.m` for these calculations.

Problem 12 (the expected number of events in nonhomogenous Poisson process)

From the discussion in the book for a nonhomogenous Poisson process with an intensity function $\lambda(s)$ the average number of events that occur in an interval of time say t to $t+r$ is given by

$$
\int_t^{t+r} \lambda(s)ds\,.
$$

From the given intensity function we can explicitly evaluate this integral giving

$$
\int_t^{t+r} \lambda(s)ds = \int_t^{t+r} 3s^2 ds = (t+r)^3 - t^3\,.
$$

Part (a): For the interval of time $(0,1]$ we have $t = 0$ and $r = 1$ so the average number of events in this interval is given by $1^3 - 0^3 = 1$.

Part (b): For the interval of time $(1,2]$ we have $t = 1$ and $r = 1$ so the average number of events in this interval is given by $2^3 - 1^3 = 7$, quite a few more events than in Part (a) showing the relevance of nonhomogenous Poisson process to modeling situations where the number of events increases (or decreases) in time.

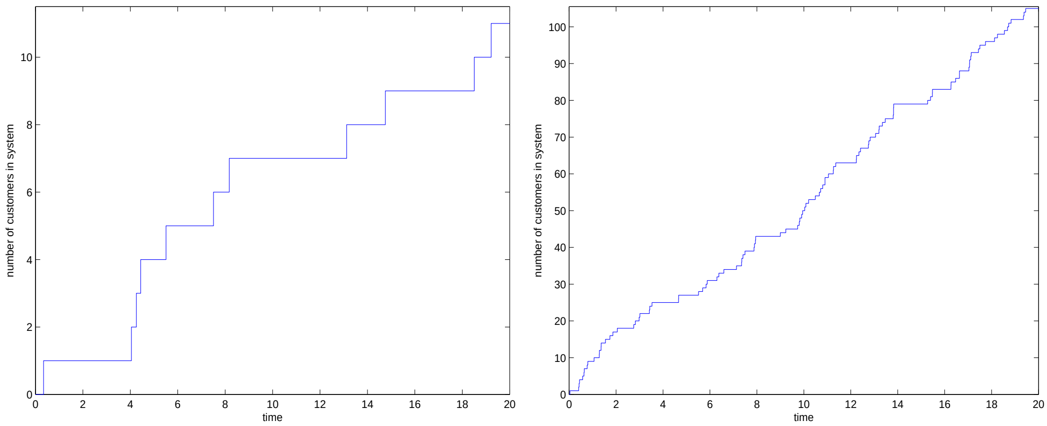

Figure 20: Examples of long service times relative to the arrival rate. **Left:** Here the mean arrival time is $1/\lambda = 2$ and the service times are drawn from a uniform distribution with limits $u_l = 20$ and $u_r = 40$. **Right:** Here the mean time between arrivals is taken to be $1/\lambda = 0.2$ shorter than before. Note that significantly more customers build up. All number are in the same time units.

Part (c): For the first event to happen *after* time t means that our nonhomogenous Poisson process must have no events up until time t. Thus the event $\{T > t\}$ is equivalent to the event $\{N(t) = 0\}$ where $N(\cdot)$ is our nonhomogenous Poisson process. We then have

$$\begin{aligned} P\{T > t\} &= P\{N(t) = 0\} \\ &= \frac{\left(\int_0^t \lambda(s)ds\right)^0}{0!} e^{-\int_0^t \lambda(s)ds} \\ &= e^{-\int_0^t \lambda(s)ds} = e^{-t^3}, \end{aligned}$$

as claimed. To compute the density function for T recall that the cumulative distribution function for T is given by $P\{T < t\} = 1 - P\{T > t\} = 1 - e^{-t^3}$, so taking the derivative of the cumulative density function to get the distribution function gives

$$f_T(t) = 3t^2 e^{-t^3}.$$

Problem 13 (simulating a general single-server queuing system)

Rather than perform this task by hand it seems more instructive to program a computer to do so. In the Matlab file `chap_8_prob_13.m` one can find a function that will simulate a single-server queuing system for arbitrary input λ (actually the input parameter is `lmean` or the average number of events that occur in one time unit) and bounds on the uniform random variable representing the service times given by `ul` and `ur`. The driver script `chap_8_prob_13_Script.m` demonstrates a variety of behaviors obtained by varying these inputs.

A key fact in simulating a single-server queue is that given a standard uniform random variable U (drawn uniformly between $[0, 1]$) a second uniform random variable X drawn

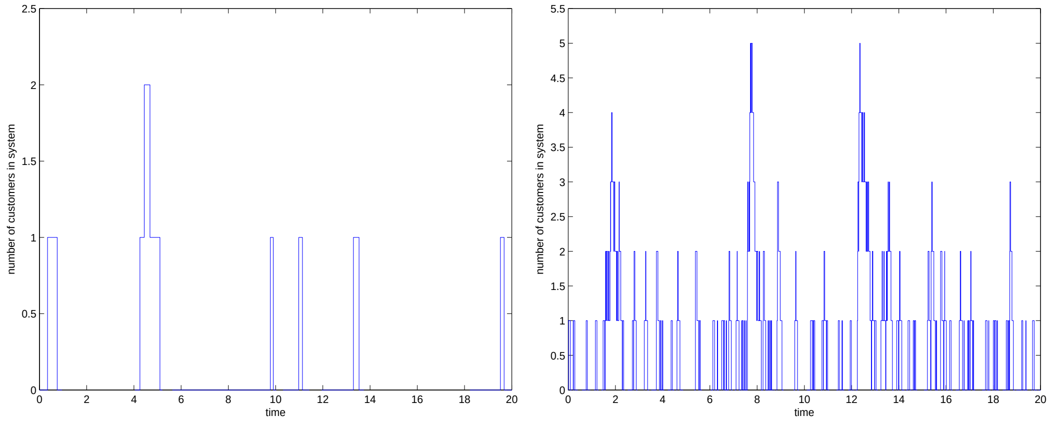

Figure 21: Examples of very short service times relative to the arrival rate. **Left:** Here the mean arrival time is given by $1/\lambda = 2$ and the service times are drawn from a uniform distribution with limits $u_l = 0.1$ and $u_r = 0.5$. **Right:** Here the mean arrival time is given by $1/\lambda = 0.1$ and the service times are drawn from a uniform distribution with limits $u_l = 0.05$ and $u_r = 0.075$. All times are in the same units.

from between u_l and u_r can be obtained as

$$X = u_l + (u_r - u_l)U \,,$$

and an exponential random variable with rate λ (mean $1/\lambda$) is given by

$$Y = -\frac{1}{\lambda}\log(U) \,.$$

Running the above MATLAB script produces various sample paths for several parameter settings for λ , u_l, and u_r. For example, consider the case where there is a relatively long service time relative to the arrival rate λ so that customers build up. Two such examples are shown in Figure 20 (left) and (right).

An alternative case is where we have very short service times, relative to the arrival rate. Two examples of sample paths according to this process are given in Figure 21 (left) and (right).

Problem 14 (simulating a M/M/1 queuing system)

As in Problem 13 of this chapter rather than perform the requested task by hand it seems more instructive to program a computer to do so. In the Matlab file `chap_8_prob_14.m` one can find a function that will simulate a M/M/1 queuing system for arbitrary arrival rate λ and service rate μ. Actually the input parameters are `lmean`, or the average number of events that occur in one second ($= 1/\lambda$) and `smean`, or the average number of customers that can be serviced occur in one second ($= 1/\mu$). A driver script called `chap_8_prob_14_Script.m` demonstrates a variety of behaviors obtained by varying these two inputs.

Note: I'll have to say that programming the *logic* required to implement a general M/M/1 queue is not as straightforward as one might think. One finds it is relatively easy to *explain* how to simulate a M/M/1 queue by just thinking about how the events arrive and are processed, but somehow it is more difficult to program this logic into a computer!

Chapter 9: Birth and Death Processes

Notes On The Text

The derivation of Kolmogorov's forward equation

The Chapman-Kolmogorov equation is

$$P_{ij}(t+h) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(h) \,. \tag{30}$$

But for a small amount of time h we have

$$\begin{aligned} P_{i,i+1}(h) &= \lambda_i h + o(h) \\ P_{i,i-1}(h) &= \mu_i h + o(h) \\ P_{i,i}(h) &= 1 - \lambda_i h - \mu_i h + o(h) \\ \sum_{j \neq i, i \pm 1} P_{ij}(h) &= o(h) \,. \end{aligned}$$

Using these results we can expand the summation in the Chapman-Kolmogorov equation above by explicitly considering the term with $k = j-1$ (representing a birth in state $j-1$ into the state j), the term with $k = j+1$ (representing a death in state $j+1$ into the state j), the term with $k = j$ (representing no transition), and all other terms with $k \neq j, j \pm 1$ (representing transitions to these further away states). When we do this we get

$$\begin{aligned} P_{ij}(t+h) &= P_{i,j-1}(t) P_{j-1,j}(h) + P_{i,j}(t) P_{j,j}(h) + P_{i,j+1}(t) P_{j+1,j}(h) + \sum_{k \neq j, j \pm 1} P_{ik}(t) P_{kj}(h) \\ &= P_{i,j-1}(t)(\lambda_{j-1} h + o(h)) + P_{i,j}(t)(1 - \lambda_j h - \mu_j h + o(h)) + P_{i,j+1}(t)(\mu_{j+1} h + o(h)) + o(h) \end{aligned}$$

Converting this into the expression for the forward difference in the definition of the derivative we obtain

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lambda_{j-1} P_{i,j-1}(t) - \lambda_j P_{i,j}(t) - \mu_j P_{i,j}(t) + \mu_{j+1} P_{i,j+1}(t) + \frac{o(h)}{h}$$

On taking $h \to 0$ we obtain

$$\frac{dP_{ij}(t)}{dt} = \lambda_{j-1} P_{i,j-1}(t) - (\lambda_j + \mu_j) P_{i,j}(t) - \mu_{j+1} P_{i,j+1}(t) \,,$$

which is Kolmogorov's forward equation.

Exercise Solutions

Exercise 1 (the logistic process)

Part (a): The described continuous-time stochastic process $\{X(t) : t \geq 0\}$ can be modeled as a birth-death process with a birth rate given by $\lambda_n = \alpha(N - n)$ and death rate given by

$\mu_n = \beta n$. The transition from state n to state $n+1$ happens with a probability of $\frac{\lambda_n}{\lambda_n+\mu_n}$, while the transition from state n to state $n-1$ happens with a probability of $\frac{\mu_n}{\lambda_n+\mu_n}$.

Part (b): For a birth-death processes, the sojourn time B, or the time spent in state n before transitioning to either state $n-1$ or $n+1$ is given by an exponential random variable with a rate of $\lambda_n + \mu_n$. Because of this the random variable B has a distribution function, F, given by

$$F(b) = 1 - e^{-(\lambda_n+\mu_n)b}.$$

So we have the desired probability given by

$$P\{B \geq 1 | X(0) = n\} = 1 - F(1) = e^{-(\lambda_n+\mu_n)}.$$

For the given expressions for λ_n and μ_n we find

$$\lambda_n + \mu_n = \alpha(N-n) + \beta n = \frac{1}{5}(5-n) + \frac{1}{4}n = \frac{25-n}{20}.$$

So that when $n = 2$ we compute $P\{B \geq 1 | X(0) = 2\} = 0.31663$, while when $n = 4$ we compute $P\{B \geq 1 | X(0) = 2\} = 0.34993$.

Exercise 2 (a linear growth model-with immigration)

If we have the possibility that our population can increase its size due to immigration (at a rate θ) our birth-rate rate in the standard linear growth model would be modified to

$$\lambda_n = \begin{cases} n\lambda + \theta & n < N \\ n\lambda & n \geq N \end{cases},$$

with a death rate $\mu_n = n\mu$, as before.

Exercise 3 (a machine repair model-part 1)

In this machine-repair models we will take the stochastic process $X(t)$ to represent the number of *broken* machines at time t. If we assume that we have M total machines ($M = 4$) and s total servicemen ($s = 2$), then this machine repair model can be considered a birth-death process with a birth rate given by

$$\lambda_n = (M-n)\lambda \quad \text{for} \quad n \leq M,$$

and a death-rate given by

$$\mu_n = \begin{cases} n\mu & n \leq s \\ s\mu & n > s \end{cases}.$$

With the parameters of this problem $M = 4$, $s = 2$, $\lambda = 1$, and $\mu = 2$ the specifications of the above equation become

$$\begin{aligned} \lambda_n &= 4 - n \quad \text{for} \quad 0 \leq n \leq 4 \\ \mu_n &= \begin{cases} 2n & 0 \leq n \leq 2 \\ 4 & 2 < n \leq 4 \end{cases}. \end{aligned}$$

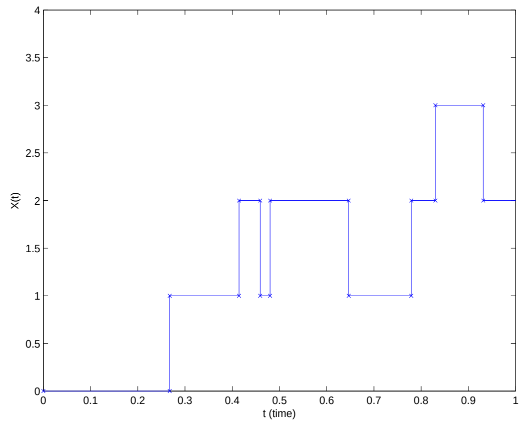

Figure 22: **Left:** An example trajectory generated according to the given machine repair model.

We can simulate a birth-death process with these rates using the method discussed in this chapter. To do this we remember that the sojourn times are generated using an exponential random variable who's rate is given by $\lambda_n + \mu_n$ and we have a transition from the state n to the state $n+1$ (representing one more broken machine) with a probability of $\frac{\lambda_n}{\lambda_n+\mu_n}$ and a transition from the state n to the state $n-1$ (representing machine repaired and put back in service) with a probability $\frac{\mu_n}{\lambda_n+\mu_n}$. Obviously, from the state $n = 0$ (no machines broken) we can only go to the state 1. We can *numerically* enforce this boundary condition by setting

$$\mu_n = \begin{cases} 2n + \epsilon & 0 \le n \le 2 \\ 4 & 2 < n \le 4 \end{cases} ,$$

where ϵ is small number specified such that an exponential random variable with a rate, ϵ, could (numerically) be sampled from but would have almost zero chance of resulting in an event. A simulation as requested for this problem is done in the MATLAB script `prob_9_prob_3.m`. When this script is run it produces an example sample path from the process $X(t)$. One such example trajectory is shown in Figure 22.

Exercise 4 (a machine repair model-part 2)

We are told that at 11:00 A.M. our random variable X (the number of machines broken) is given by $X = 2$. Then by the memoryless property of the exponential random variables that make up the "births" and "deaths" associated with this machine repair model, the fact that we have two broken machines makes no difference on T, the random variable representing the sojourn time from this state. When $X = 2$ the sojourn time is given by an exponential RV with rate $\lambda_2 + \mu_2 = 2 + 4 = 6$. If $F(t)$ is the cumulative distribution function for such a RV we have

$$P\{T \ge 0.25\} = 1 - F(0.25) = e^{-6(0.25)} = 0.2231 .$$

Part (b):

Warning: I could not get the same result as in the back of the book for this part of the problem. If anyone sees an error in the logic presented here please let me know.

We are asked to compute the probability that the two broken machines will be repaired before another one breaks. Since we assume that each machine is currently being worked on by a repairman the repair time, R_i, for *each* machine $i = 1, 2$ is an exponential random variable with a rate of 2. While the time till the next breakdown, B_i, is an exponential random variable with a rate 1. Now since there are two independent machines that could breakdown, the time to *any* breakdown, B, is an exponential RV with a rate 2. Thus we want to evaluate the probability of the combined event $R_1 \leq B$ *and* $R_2 \leq B$. By the independence of these two component events this joint probability is equal to the product of the two individual probabilities. From previous work on the probability that one exponential random variable will be less than, or happen *before* another one, we see that this join event has a probability of

$$\left(\frac{2}{2+2}\right)\left(\frac{2}{2+2}\right) = \frac{4}{16} = 0.25\,.$$

Exercise 5 (matrix representation of birth-death processes)

Given the matrix A, we can compute the ith component of $q'(t)$ by performing the desired multiplication $q(t)A$. This multiplication (in terms of the components of A) is given by

$$q_i'(s) = \sum_{j=0}^{N} q_j(s)A_{ji}\,.$$

If $i = 0$ this becomes

$$\begin{aligned} q_0'(s) &= q_0(s)A_{00} + q_1(s)A_{10} \\ &= -\lambda_0 q_0(s) + \mu_1 q_1(s)\,. \end{aligned}$$

If $i \neq 0, N$ this becomes

$$\begin{aligned} q_i'(s) &= q_{i-1}(s)A_{i-1,i} + q_i(s)A_{i,i} + q_{i+1}(s)A_{i+1,i} \\ &= q_{i-1}(s)\lambda_{i-1} - (\lambda_i + \mu_i)q_i(s) + q_{i+1}(s)\mu_{i+1}\,. \end{aligned}$$

Finally, if $i = N$ this becomes

$$q_N'(s) = \lambda_{N-1}q_{N-1}(s) - \mu_N q_N(s)\,.$$

Exercise 6 (a pure birth process-part 1)

Part (a): From the definition of expectation and $q_i(t)$ we have

$$E[X(t)] = \sum_{i=0}^{\infty} iP\{X(t) = i\} = \sum_{i=0}^{\infty} iq_i(t)\,,$$

which is the expected value of $q_i(t)$. For a pure birth process $q_i(t)$ is a negative-binomial distribution with parameters $r = n$ and $p = e^{-\lambda t}$. Thus its expected value is $\frac{r}{p}$. With the parameters above we obtain an expected value of

$$\frac{n}{e^{-\lambda t}} = ne^{\lambda t}\,.$$

Part (b): We are told that $X(0)$ is only known in terms of a probability distribution and we desire to calculate, $E[X(t)]$, the expectation of our Yule process (a pure birth process). Following the hint of conditioning on $X(0)$ we have

$$E[X(t)] = \sum_{n=0}^{\infty} E[X(t)|X(0) = n]P\{X(0) = n\}\,.$$

From the previous part of this problem we know that a Yule-process has $E[X(t)|X(0) = n] = ne^{\lambda t}$, so the above becomes

$$E[X(t)] = \sum_{n=0}^{\infty} ne^{\lambda t}P\{X(0) = n\} = e^{\lambda t}E[X(0)]\,.$$

Part (c): If we define $\phi(t) = E[X(t)]$, then from the previous part of this problem $\phi(t) = e^{\lambda t}E[X(0)]$, so taking the time derivative we find

$$\phi'(t) = \lambda e^{\lambda t}E[X(0)] = \lambda\phi(t)\,,$$

as expected.

Exercise 7 (a pure birth process-part 2)

Part (a): From the discussions in the book, if the population is of size N at time t an expressions for $X(t)$ can be written as

$$X(t) = Y_1(t) + Y_2(t) + \cdots + Y_N(t)\,,$$

where $Y_i(t)$ are geometric random variables (RV) representing the number of trials that the ith population member needs to perform to get their first success. That is, they are geometric RVs with parameter of success $e^{-\lambda t}$.

Part (b): To compute $\mathrm{Var}(X(t))$ we may use the conditional variance formula by conditioning on N, the initial size of the population. The conditional variance formula is given by

$$\mathrm{Var}[X] = E[\mathrm{Var(X|N)}] + \mathrm{Var}(E[X|N])\,.$$

Now in the first term $\mathrm{Var}(X|N)$ is the variance of the sum of N (fixed) independent geometric random variables and so has a variance given by $N\left(\frac{1-p}{p^2}\right)$. The expectation of this is then $E[N]\left(\frac{1-p}{p^2}\right)$. In the second term the expression $E[X|N]$ is the expectation of the sum of N

independent geometric RV and so has an expectation of $\frac{N}{p}$. The variance of this expression is $\frac{1}{p^2}\text{Var}(N)$. Thus

$$\text{Var}(X(t)) = \left(\frac{1-p}{p^2}\right) E[N] + \frac{1}{p^2}\text{Var}(N)\,.$$

Since in the Yule-process $Y_i(t)$ is a geometric RV with a parameter p equal to $e^{-\lambda t}$, the above expression becomes

$$\text{Var}(X(t)) = \frac{(1-e^{-\lambda t})E[N] + \text{Var}(N)}{e^{-2\lambda t}} = e^{\lambda t}(e^{\lambda t}-1)E[N] + e^{2\lambda t}\text{Var}(N)\,.$$

Exercise 8 (the Kolmogorov backward equations)

The Chapman-Kolmogorov equation to consider is

$$P_{ij}(h+t) = \sum_{k=0}^{\infty} P_{ik}(h)P_{kj}(t)\,.$$

For a small amount of time, h, we have order expansions for the nearest-neighbor probabilities given by

$$\begin{aligned}
P_{i,i+1}(h) &= \lambda_i h + o(h) \\
P_{i,i-1}(h) &= \mu_i h + o(h) \\
P_{ii}(h) &= 1 - \lambda_i h - \mu_i h + o(h) \\
\sum_{j\neq i,i\pm 1} P_{ij}(h) &= o(h)\,.
\end{aligned}$$

Using these results we can expand the summation in the Chapman-Kolmogorov equation above by explicitly considering the terms $P_{ik}(h)$ for $k=i$ and $k=i\pm 1$, to get

$$\begin{aligned}
P_{ij}(h+t) &= P_{i,i-1}(h)P_{i-1,j}(t) + P_{ii}(t)P_{ij}(t) + P_{i,i+1}(h)P_{i+1,j}(t) + \sum_{k\neq i,i\pm 1} P_{ik}(h)P_{kj}(t) \\
&= (\mu_i h + o(h))P_{i-1,j}(t) + (1-\lambda_i h - \mu_i h + o(h))P_{ij}(t) + (\lambda_i h + o(h))P_{i+1,j}(t) + o(h) \\
&= P_{i+1,j}(t)\lambda_i h + P_{i-1,j}(t)\mu_i h + P_{ij}(t)(1-\lambda_i-\mu_i)h + o(h)\,.
\end{aligned}$$

Part (b): Subtracting $P_{ij}(t)$ and dividing by h to obtain the expression for the forward difference in the definition of the derivative we obtain

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i+\mu_i)P_{ij}(t) + \frac{o(h)}{h}\,.$$

On taking the limit $h \to 0$ we obtain

$$\frac{dP_{ij}(t)}{dt} = \lambda_i P_{i+1,j}(t) - (\lambda_i+\mu_i)P_{ij}(t) - \mu_i P_{i-1,j}(t)\,,$$

which is Kolmogorov's backward equation.

Part (c): Given a matrix $P(t)$ defined as

$$P(t) = \begin{bmatrix} P_{0,0} & \cdots & P_{0,j} & \cdots & P_{0,N} \\ P_{1,0} & & \vdots & & P_{1,N} \\ & & P_{i-1,j} & & \\ & P_{i,j-1} & P_{ij} & P_{i,j+1} & \\ & & P_{i+1,j} & & \\ P_{N-1,0} & & \vdots & & P_{N-1,N} \\ P_{N,0} & & & & P_{N,N} \end{bmatrix},$$

the product $AP(t)$ where A is given in problem 5 an (i,j)th element $0 \le i \le N$ and $0 \le j \le N$ given by

$$\mu_i P_{i-1,j}(t) - (\mu_i + \lambda_i) P_{ii}(t) + \lambda_i P_{i+1,j}(t)\,.$$

Setting this equal to $\frac{dP_{ij}(t)}{dt}$ is Kolmogorov's backwards equation. While the product $P(t)A$ has an (i,j)th component given by

$$\lambda_{j-1} P_{i,j-1}(t) - (\mu_j + \lambda_j) P_{ij}(t) + \mu_{j+1} P_{i,j+1}(t)\,,$$

which when set equal to $\frac{dP_{ij}(t)}{dt}$ is equal to Kolmogorov's backwards equation.

Chapter 10: Steady-State Probabilities

Exercise Solutions

Exercise 1 (the balance equations)

Part (a): The balance equations for p_j the steady-state probabilities are given by

$$0 = \lambda_{j-1} p_{j-1} - (\lambda_j + \mu_j) p_j + \mu_{j+1} p_{j+1} , \tag{31}$$

for $j = 1, 2, \cdots, N$. At the left endpoint, $j = 1$, we define $\lambda_{-1} = 0$ and $\mu_0 = 0$, while at the right endpoint, $j = N$, we define $\lambda_N = 0$, and $\mu_{N+1} = 0$. With the definition of the row vector p, given in this problem and the definition of the matrix A, one can see that the component equations in the product $pA = 0$ are equivalent to the above equation.

Part (b): Recall that the time dependent probabilities $q_j(t)$, defined as $q_j(t) = P\{X(t) = j\}$, satisfy the dynamic equation

$$\frac{dq_j(t)}{dt} = \lambda_{j-1} q_{j-1}(t) - (\lambda_j + \mu_j) q_j(t) + \mu_{j+1} q_{j+1}(t) .$$

From this we see that if we define the *row* vector $q(t)$ to have components, $q_j(t)$, the above equation can be written in matrix form as $\frac{dq(t)}{dt} = q(t)A$. If initially we have $q(0) = p$ where p is the steady-state row vector such that $pA = 0$, evaluating $\frac{dq}{dt}$ at $t = 0$ gives

$$\frac{dq(0)}{dt} = q(0)A = pA = 0 .$$

Thus initially, our differential equation indicates a *zero* change from its initial value $q(0)$ i.e. it stays constant at $q(t) = p$ for all time.

Exercise 2 (an example $M/M/1$ queuing system)

The exact solutions to the time dependent probabilities $q_j(t) = P\{X(t) = j\}$ for a $M/M/1$ queue where the two possible states for j are $j = 0, 1$ is derived in this chapter. They were found to be

$$q_0(t) = \frac{\mu}{\lambda + \mu} + C_0 e^{-(\lambda+\mu)t} \quad \text{and} \quad q_1(t) = \frac{\lambda}{\lambda + \mu} - C_0 e^{-(\lambda+\mu)t} ,$$

with C_0 a constant determined by the initial probability distribution for $X(0)$.

Part (a): If $X(0) = 0$, we have $q_0(0) = 1$ while $q_1(0) = 0$, since when $X(0) = 0$ the probability we have *no* customers at $t = 0$ is 1 (since that is what we are told), while the probability we have one customer is 0 (since this is the opposite of what we are told). Using the second condition, $q_1(0) = 0$, to evaluate C_0, we find that

$$C_0 = \frac{\lambda}{\lambda + \mu},$$

so that the functions $q_0(t)$ and $q_1(t)$ become

$$q_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \quad \text{and} \quad q_1(t) = \frac{\lambda}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}).$$

From these we find that $P\{X(1) = 0\} = q_0(1)$ is given by

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)} = 0.8426.$$

and $P\{X(10) = 0\} = q_0(10)$ is given by

$$\frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-10(\lambda+\mu)} = 0.602695.$$

Part (b): If we are told that $X(0) = 1$ we have $q_0(0) = 0$ while $q_1(0) = 1$. Using the first expression, $q_0(0) = 0$, we find that

$$C_0 = -\frac{\mu}{\lambda+\mu}.$$

So that the functions $q_0(t)$ and $q_1(t)$ then become

$$q_0(t) = \frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)t}) \quad \text{and} \quad q_1(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}$$

Using these we find that $P\{X(1) = 0\} = q_0(1)$ is given by

$$\frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)}) = \frac{0.3}{0.5}(1 - e^{-0.5}) = 0.23608.$$

and $P\{X(10) = 0\} = q_0(10)$ is given by

$$\frac{\mu}{\lambda+\mu}(1 - e^{-(\lambda+\mu)10}) = \frac{0.3}{0.5}(1 - e^{-5}) = 0.59595.$$

These simple calculations are done in the MATLAB file `chap_10_calculations.m`.

Part (c): In both case above, the limiting probabilities should be

$$\begin{aligned} p_0 &\equiv \lim_{t\to\infty} P\{X(t) = 0\} = \frac{\mu}{\lambda+\mu} = \frac{0.3}{0.5} = 0.59999 \\ p_1 &\equiv \lim_{t\to\infty} P\{X(t) = 1\} = \frac{\lambda}{\lambda+\mu} = \frac{0.2}{0.5} = 0.40000, \end{aligned}$$

which are exact. The probabilities $P\{X(10) = 0\}$ calculated in the two parts above (since $t = 10$ is a relatively large time) are very close to the steady-state probability p_0.

Exercise 3 (the approach to steady-state)

The steady-state probability for the state with no customers is given by $p_0 = \frac{\mu}{\mu+\lambda}$, while the time dependent expression for this same event $\{X(t) = 0\}$ in this case (by considering Problem 2 Part (a)) is given by

$$q_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} .$$

For this to be within 10% of the steady-state probability, p_0, we desire

$$\left| \frac{q_0(t) - \frac{\mu}{\lambda+\mu}}{\frac{\mu}{\lambda+\mu}} \right| = \frac{\lambda}{\mu} e^{-(\lambda+\mu)t} \le 0.1 .$$

Since we know that $q_0(t) > \frac{\mu}{\lambda+\mu}$, the absolute values in the first expression are not strictly needed. On solving for t in the above we find

$$t \ge \left(\frac{1}{\lambda+\mu} \right) \log(\frac{10\lambda}{\mu}) = 3.794240 ,$$

when we put in the values of $\lambda = 0.2$ and $\mu = 0.3$. This simple calculation is done in the MATLAB file `chap_10_calculations.m`.

Exercise 4 (the average approach to steady-state)

In the same way as in Problem 3 we have

$$P\{X(t) = 0\} = q_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} ,$$

and the steady-state time-average will still be $\frac{\mu}{\lambda+\mu}$. Since we can explicitly compute the time average of $q_0(t)$ for any T as

$$\begin{aligned} I(T) &\equiv \frac{1}{T} \int_0^T P\{X(t) = 0\} dt = \frac{\mu}{\lambda+\mu} + \frac{1}{T} \left(\frac{\lambda}{\mu+\lambda} \right) \left(-\frac{1}{\lambda+\mu} e^{-(\lambda+\mu)t} \right|_0^T \\ &= \frac{\mu}{\lambda+\mu} + \frac{1}{T} \left(\frac{\lambda}{(\mu+\lambda)^2} \right) (1 - e^{-(\lambda+\mu)T}) . \end{aligned}$$

To be within 10% of the steady-state probability would require a value of T such that the integral above, $I(T)$, satisfies

$$\left| \frac{I(T) - \frac{\mu}{\lambda+\mu}}{\frac{\mu}{\lambda+\mu}} \right| \le 0.1 .$$

The absolute value in the above expression is not strictly needed since $I(T) > \frac{\mu}{\lambda+\mu}$. The above simplifies to

$$\frac{1}{T} \left(\frac{\lambda}{\mu} \right) \left(\frac{1}{\mu+\lambda} \right) (1 - e^{-(\lambda+\mu)T}) \le 0.1 .$$

To find a value of T that satisfies this expression we can take the approximation that $e^{-(\lambda+\mu)T}) \approx 0$, when T is relatively large, and solve the resulting inequality for T. When we do this we find

$$T \geq \frac{10\lambda}{\mu(\mu+\lambda)} = 13.333333\,.$$

This simple calculation is done in the MATLAB file `chap_10_calculations.m`.

Exercise 5 (an $M/M/1$ queue with capacity $N = 2$)

From Example 10.2 when $N = 2$, $\lambda_0 = \lambda_1 = \lambda > 0$, and $\mu_0 = \mu_1 = \mu > 0$ the matrix A is given by

$$A = \left[\begin{array}{ccc} -\lambda_0 & \lambda_0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 \\ 0 & \mu_2 & -\mu_2 \end{array}\right] = \left[\begin{array}{ccc} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{array}\right]\,.$$

Part (a): The characteristic polynomial for this matrix is given by $M(x) = \det(xI - A)$, which in this case is

$$\left|\begin{array}{ccc} x+\lambda & -\lambda & 0 \\ -\mu & x+\lambda+\mu & -\lambda \\ 0 & -\mu & x+\mu \end{array}\right| = 0\,.$$

Expanding this determinant about the first row gives

$$(x+\lambda)((x+\lambda+\mu)(x+\mu) - \mu\lambda) + \lambda(-\mu(x+\mu)) = 0\,.$$

Expanding this and simplifying we finally obtain the equation

$$x^3 + 2(\mu+\lambda)x^2 + (\mu^2 + \mu\lambda + \lambda^2)x = 0\,.$$

From the above, $x = 0$ is one solution, while the remaining quadratic has its solutions given by

$$\begin{aligned} x &= \frac{-2(\mu+\lambda) \pm \sqrt{4(\mu+\lambda)^2 - 4(\mu^2+\mu\lambda+\lambda^2)}}{2} \\ &= -(\mu+\lambda) \pm \sqrt{\mu\lambda}\,, \end{aligned}$$

the desired roots. Considering the root $-\mu - \lambda + \sqrt{\mu\lambda}$, we know from the "inequality of arithmetic and geometric means" given by

$$\sqrt{\mu\lambda} \leq \frac{1}{2}(\mu+\lambda)\,,$$

that the root we are considering has an upper bound of

$$-\mu - \lambda + \sqrt{\mu\lambda} \leq -\mu - \lambda + \frac{1}{2}(\mu+\lambda) = -\frac{1}{2}(\mu+\lambda) < 0\,,$$

showing that the second root is explicitly negative as requested.

Exercise 6 (the linear growth model with immigration)

The linear growth model with immigration and no population limit had birth and death rates given by

$$
\begin{aligned}
\lambda_n &= \begin{cases} n\lambda + \theta & n < N \\ n\lambda & n \geq N \end{cases} \quad n\lambda + \theta \quad \text{for} n \geq 0 \\
\mu_n &= n\mu \quad \text{for} \quad n \geq 1 \,,
\end{aligned}
$$

so that the ratios

$$
\rho_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \,, \tag{32}
$$

become

$$
\rho_n = \frac{\theta(\lambda + \theta)(2\lambda + \theta) \cdots ((n-1)\lambda + \theta)}{\mu(2\mu)(3\mu) \cdots (n\mu)} \quad \text{for} \quad n < N \,.
$$

and

$$
\rho_n = \frac{\theta(\lambda + \theta)(2\lambda + \theta) \cdots ((N-1)\lambda + \theta)(N\lambda) \cdots n\lambda}{\mu(2\mu)(3\mu) \cdots (n\mu)} \quad \text{for} \quad n \geq N \,.
$$

We now want show that $\sum_{n=1}^{\infty} \rho_n$ converges. As suggested in the book we can apply the ratio test. The ratio test states that if

$$
\lim_{n \to \infty} \frac{\rho_{n+1}}{\rho_n} < 1 \,,
$$

the above series converges. For the expression for ρ_n given here we have the ratio of ρ_{n+1}/ρ_n of

$$
\begin{aligned}
\frac{\rho_{n+1}}{\rho_n} &= \left(\frac{\theta(\lambda + \theta)(2\lambda + \theta) \cdots ((N-1)\lambda + \theta)(N\lambda) \cdots (n+1)\lambda}{\mu(2\mu)(3\mu) \cdots (n\mu)((n+1)\mu)} \right) \left(\frac{\mu(2\mu)(3\mu) \cdots (n\mu)}{\theta(\lambda + \theta)(2\lambda + \theta) \cdots ((N-1)\lambda + \theta} \right. \\
&= \frac{n\lambda + \theta}{(n+1)\mu} \,.
\end{aligned}
$$

when $n \geq N$. This expression has a limit as $n \to \infty$ of

$$
\lim_{n \to \infty} \frac{\rho_{n+1}}{\rho_n} = \frac{\lambda}{\mu} \,,
$$

which will be less than one if $\lambda < \mu$.

Exercise 7 (the backwards recursion formula for p_j)

Part (a): Writing our steady-state probability balance equation in the "backwards" form

$$
\lambda_{j-1} p_{j-1} = (\lambda_j + \mu_j) p_j - \mu_{j+1} p_{j+1} \quad \text{for} \quad j = N, N-1, \cdots
$$

with the convention that $\mu_{N+1} = 0$ and $\lambda_N = 0$. Taking $j = N$ the above requires

$$
\lambda_{N-1} p_{N-1} = (0 + \mu_N) p_N - 0 = \mu_N p_N \,.
$$

Second, taking $j = N-1$ we obtain

$$\begin{aligned}
\lambda_{N-2} p_{N-2} &= (\lambda_{N-1} + \mu_{N-1}) p_{N-1} - \mu_N p_N \\
&= (\lambda_{N-1} + \mu_{N-1}) p_{N-1} - \lambda_{N-1} p_{N-1} = \mu_{N-1} p_{N-1} \,,
\end{aligned}$$

where we have used the relationship $\mu_N p_N = \lambda_{N-1} p_{N-1}$ found when we took $j = N$. Continuing in this way we obtain

$$\lambda_{N-k-1} p_{N-k-1} = \mu_{N-k} p_{N-k} \quad \text{for} \quad k = 0, 1, 2, \cdots N-1 \,.$$

Solving for p_{N-k-1} in terms of p_{N-k} using the above we have

$$p_{N-k-1} = \left(\frac{\mu_{N-k}}{\lambda_{N-k-1}} \right) p_{N-k} \quad \text{for} \quad k = 0, 1, 2, \cdots N-1 \,.$$

Iterating this expression by taking $k = 0, 1, 2, \cdots$ we find

$$\begin{aligned}
p_{N-1} &= \left(\frac{\mu_N}{\lambda_{N-1}} \right) p_N \\
p_{N-2} &= \left(\frac{\mu_{N-1}}{\lambda_{N-2}} \right) p_{N-1} = \left(\frac{\mu_{N-1}}{\lambda_{N-2}} \right) \left(\frac{\mu_N}{\lambda_{N-1}} \right) p_N \\
p_{N-3} &= \left(\frac{\mu_{N-2}}{\lambda_{N-3}} \right) p_{N-2} = \left(\frac{\mu_{N-2}}{\lambda_{N-3}} \right) \left(\frac{\mu_{N-1}}{\lambda_{N-2}} \right) \left(\frac{\mu_N}{\lambda_{N-1}} \right) p_N \\
&\vdots \\
p_{N-k} &= \left(\frac{\mu_N}{\lambda_{N-1}} \right) \left(\frac{\mu_{N-1}}{\lambda_{N-2}} \right) \cdots \left(\frac{\mu_{N-k+1}}{\lambda_{N-k}} \right) p_N \quad \text{for} \quad k = 1, 2, \cdots, N-1 \,.
\end{aligned}$$

Converting the index $N-k$ into j we find

$$p_j = \frac{\mu_N \mu_{N-1} \cdots \mu_{j+2} \mu_{j+1}}{\lambda_{N-1} \lambda_{N-2} \cdots \lambda_{j+1} \lambda_j} p_N \quad \text{for} \quad j = 0, 1, \cdots, N-1 \,, \tag{33}$$

as our desired expression.

Part (b): To find the normilization expression for p_N we require that $\sum_{j=0}^{N} p_j = 1$ or separating out the term p_N equivalently

$$p_N + \sum_{j=0}^{N-1} p_j = 1 \,.$$

But since we have an expression for p_j in terms of p_N where $j = 0, \cdots N-1$ we find that the above becomes

$$p_N + p_N \sum_{j=0}^{N-1} \left(\frac{\mu_N \mu_{N-1} \cdots \mu_{j+2} \mu_{j+1}}{\lambda_{N-1} \lambda_{N-2} \cdots \lambda_{j+1} \lambda_j} \right) = 1 \,.$$

Solving for p_N in the above we obtain

$$p_N = \left(1 + \sum_{j=0}^{N-1} \left(\frac{\mu_N \mu_{N-1} \cdots \mu_{j+2} \mu_{j+1}}{\lambda_{N-1} \lambda_{N-2} \cdots \lambda_{j+1} \lambda_j} \right) \right)^{-1} \,,$$

as the required normalization condition.

Exercise 8 (terminating end conditions)

We are told that

$$(\lambda_0, \mu_0) = (0, 0)\,, \quad (\lambda_1, \mu_1) = (1, 1)\,, \quad (\lambda_2, \mu_2) = (0, 0)\,.$$

Part (a): The balance equation for every possible state $j = 0, 1, 2$ are given by Equation 31. When we take $j = 0$ we have

$$0 = 0 - 0 + 1p_1 \Rightarrow p_1 = 0\,.$$

When $j = 1$ we have

$$0 = 0 - (\lambda_1 + \mu_1)p_1 + 0 \Rightarrow p_1 = 0\,.$$

When $j = 2$ we have

$$0 = \lambda_1 p_1 - 0 + 0 \Rightarrow p_1 = 0\,.$$

Part (b): From the above any three values for p_0, p_1, and p_2 that sum to one and have $p_1 = 0$ will satisfy the balance equations. One such system is $(p, 0, 1-p)$ showing an infinite number of solutions is possible, and the steady-state probabilities are not unique.

Exercise 9 (the birth-death ratios for the machine-repair model)

In steady-state from the balance equations, one obtains a solution for p_n of

$$p_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} p_0 = \rho_n p_0\,.$$

Now for the machine-repair model we have a birth-rate given by

$$\lambda_n = (M - n)\lambda \quad \text{for} \quad n \le M\,,$$

and a death-rate given by

$$\mu_n = \left\{ \begin{array}{ll} n\mu & n \le s \\ s\mu & n > s \end{array} \right.\,.$$

We begin by computing p_n for $n \le s < M$ and find

$$\begin{aligned} \rho_n &= \frac{(M\lambda)((M-1)\lambda)((M-2)\lambda)\cdots(M-(n-1))\lambda}{\mu(2\mu)(3\mu)\cdots(n\mu)} \\ &= \frac{M(M-1)(M-2)\cdots(M-(n-1))\lambda^n}{n!\mu^n} \\ &= \left(\frac{M(M-1)(M-2)\cdots(M-(n-1))}{n!} \right) \left(\frac{\lambda}{\mu} \right)^n \quad \text{for} \quad n \le s\,. \end{aligned}$$

In the cases where $s < n \le M$ we have

$$\rho_n = \frac{(M\lambda)((M-1)\lambda)((M-2)\lambda)\cdots(M-(n-1))\lambda}{\mu(2\mu)(3\mu)\cdots(s\mu)\cdot(s\mu)\cdots(s\mu)}\,,$$

Where in the denominator after the product $\mu(2\mu)(3\mu)\cdots(n\mu)$ we have $n-s$ terms of the type $s\mu$. Simplifying this expression some we obtain

$$\begin{aligned}
\rho_n &= \frac{M(M-1)(M-2)\cdots(M-(n-1))\lambda^n}{s!\mu^s s^{n-s}\mu^{n-s}} \\
&= \frac{M(M-1)(M-2)\cdots(M-(n-1))}{s!s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n \quad s < n \leq M\,,
\end{aligned}$$

which are the two desired results.

Exercise 10 (a gasoline station)

This problem can be modeled as an $M/M/1$ queuing problem where the "customers" are the arriving cars which we are told come at a rate of 20 cars per hour. Customers pull in if there are two or *fewer* cars already. This means that there are *three* locations where cars can be serviced so $N=3$. Finally, the vehicles depart after they are serviced which happens at a *rate* of $\mu = \frac{1}{(5/60)} = 12$ cars per hour. With these values we have $\rho = \frac{\lambda}{\mu} = \frac{20}{12} = \frac{5}{3}$, and our steady-state probabilities p_n are given by

$$p_n = \rho^n p_0 \quad \text{for} \quad n = 0, 1, \cdots, N\,, \tag{34}$$

with p_0 given by

$$p_0 = \left(1 + \sum_{n=1}^{N} \rho^n\right)^{-1} = \left(\frac{1-\rho^{N+1}}{1-\rho}\right)^{-1}. \tag{35}$$

When $\rho = 5/3$ and $N=3$ we find that $p_0 = 0.0993$ so that

$$p_0 = 0.0993\,, \quad p_1 = 0.1654\,, \quad p_2 = 0.2757\,, \quad p_3 = 0.4596\,.$$

Part (a): If we define $X(t)$ to be our time dependent random variable denoting the number of cars in the station at time t we see that the valid values for $X(t)$ are 0, 1, 2, 3. In steady-state, the single attendant will be busy if $X(t)=1$ or $X(t)=2$, or $X(t)=3$. Thus in steady-state the proportion of time that the attendant is busy is $p_1+p_2+p_3$. From the above this equals 0.900735.

Part (b): In steady-state, the proportion of time customers enter the station is the proportion of time that $X(t) \neq 3$ and is given by $1-p_3 = 0.540441$.

Part (c): If the service time is twice as fast the rate μ now becomes 2μ. With this the proportion of time customers enter the station $1-p_3$ now becomes 0.813711. The average *number* of customers that enter the station in Part (b) above is given by $\lambda(1-p_3) \approx 10.808$, where as the average number of customers that enter the station *now* under the new faster service is $\lambda(1-p_3) \approx 16.274$. The gain in customers is around 5.46 more. The calculations for this problem are performed in the MATLAB script `chap_10_prob_10.m`. This script calls the MATLAB function `mm1_queue_cap_N.m` which computes the steady-state probability distribution of a $M/M/1$ queue with capacity N i.e. it implements Equations 34 and 35.

Exercise 11 (a printing shop)

This problem can be formulated as a machine-repair model with $M = 4$ machines (printing presses) and $s = 2$ servicemen (repairmen). The breakdown rate for the machines is $\lambda = 1/10$ machines per hour, while the repair rate is $1/8$ machines per hour. Thus $\rho = \frac{\lambda}{\mu} = \frac{4}{5}$, while the ρ_n is given by Equations 10.35 and 10.36 in the book. These equations are implemented in the MATLAB function `machine_repair_model.m`. Thus function is called from the MATLAB script `chap_10_prob_11.m` to produce the steady-state probability distribution p_n corresponding to the described machine repair model.

Part (a): The average number of machines not in use is the average of the number of machines broken and can be computed via $\sum_{n=0}^{M} np_n$ with p_n the steady-state probabilities. When the above script is run we find that this average is given by 2.02670. We also compute the coefficient of loss for the machines and find it to be 0.11201.

Part (b): If we define $X(t)$ to be our time dependent random variable denoting the number of broken printers we see that both repairmen will be busy if $X(t) = 2$, $X(t) = 3$, or $X(t) = 4$, so the proportion of time both repairmen are busy is given by $p_2 + p_3 + p_4$. When we run the above script we find that this number is 0.6596. We also compute the coefficient of loss for the repairmen and find 0.21068.

Exercise 12 (dividing the work up)

If we assume that each repairman has exclusive responsibility for two presses then the machine-repair model in Problem 11 becomes *two* machine-repair models with $M = 2$, $s = 1$, and λ and μ as before. In this case for each repairman we compute

$$p_0 = 0.2577\,, \quad p_1 = 0.4124\,, \quad p_2 = 0.3299\,,$$

so that the proportion of time each repairman is busy is $p_1 + p_2 = 0.74226$ with a coefficient of loss for the repairmen of $C_{\text{Repairman}} = 0.25773$. Thus the repairmen are busier but the coefficient of loss for the repairmen is slightly larger. This problem is implemented in the MATLAB script `chap_10_prob_12.m` which again calls the MATLAB function `machine_repair_model.m`.

Exercise 13 (finding space for all customers)

We are told that $\rho = \lambda/8$ and $N = 2$ so that the normalizing value for p_0 in a $M/M/1$ queue is given by Equation 35 is

$$p_0 = \left(\frac{1-\rho^3}{1-\rho}\right)^{-1}.$$

With this computed we have steady-state probabilities $p_n = \rho^n p_0$ for $n = 0, 1, 2, \cdots$.

Part (a): To have a 50% chance of joining the queue we require that $p_0 + p_1 \geq 0.5$ or

$$\frac{1-\rho}{1-\rho^3}(1+\rho) \geq 0.5\,,$$

because in that case at least 50% of the time there are at most one person in line. Since we can factor $1-\rho^3$ as $1-\rho^3 = (1-\rho)(1+\rho+\rho^2)$ we see that the above gives

$$1+\rho \geq \frac{1}{2} + \frac{1}{2}\rho + \frac{\rho^2}{2}\,.$$

or

$$\rho^2 - \rho - 1 \leq 0\,. \tag{36}$$

Solving this quadratic for ρ we find that two values of ρ make this quadratic equal to zero. These values of ρ are given by

$$\rho = -0.6180\,, \quad \rho = 1.6180\,.$$

Since a test value of ρ say $\rho = 0$ between these two values makes the inequality in Equation 36, true, the region *between* these two roots is the valid one above. Then any arrival rate λ such that ρ is less than or equal to 1.6180 will result in the chance that the average customer will have greater than a 50% chance of being served. The value of λ corresponding to this upper limit is $\lambda = 8(1.6180) = 12.9443$.

Part (b): To get served immediately we must have no customers already in line. That will happen with probability $p_0 = 0.1910$. This problem is implemented in the MATLAB script `chap_10_prob_13.m`

Exercise 14 (bottling plants)

As the problem statement is not exactly a machine-repair model, to calculate the steady-state probabilities, we have to explicitly solve the balance equations. To do this we begin by first computing ρ_n from Equation 32, repeated here for convenience

$$\rho_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n} \quad n = 1, 2, 3, \cdots$$

From the description given above we have that $\lambda_n = 2(3-n)$, for $n \geq 3$ and

$$\begin{aligned} \mu_n &= 3 \quad \text{for} \quad n < 3 \\ \mu_n &= 4 \quad \text{for} \quad n = 3\,. \end{aligned}$$

So that computing ρ_n we find

$$\begin{aligned} \rho_1 &= \frac{\lambda_0}{\mu_1} = \frac{6}{3} = 2 \\ \rho_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = \frac{6\cdot 4}{3\cdot 3} = \frac{8}{3} \\ \rho_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = \frac{6\cdot 4\cdot 2}{3\cdot 3\cdot 4} = \frac{4}{3}\,. \end{aligned}$$

Then the normalization required is to compute p_0 given by

$$p_0 = \left(1 + \sum_{n=1}^{3} \rho_n\right)^{-1} = \left(1 + 2 + \frac{8}{3} + \frac{4}{3}\right)^{-1} = 0.142857\,.$$

So that $p_n = \rho_n p_0$ gives

$$p_0 = 0.1429\,, \quad p_1 = 0.2857\,, \quad p_2 = 0.3810\,, \quad p_3 = 0.1905\,,$$

as the steady-state probabilities of our system. This problem is worked in the MATLAB script `prob_10_prob_14.m`.

Part (b): The average number of machines that are broken is given by $\sum_{n=0}^{3} n p_n$. Computing this sum using the steady-state probablites found above we obtain 1.619048. The average rate at which machines breakdown in steady-state is given by

$$\sum_{n=0}^{3} \lambda_n p_n = \sum_{n=0}^{2} \lambda_n p_n = 2.761905\,.$$

Exercise 15 (a discouraged arrival model)

We are told that $\mu = 1$ is the departure rate from our queue and the a-priori arrival rate is λ customers per minute. In addition, if we have n customers already in line then the probability the entering customr joins the queue is given by $\frac{1}{n+1}$.

Part (a): Given that we are in state n we move to state $n+1$ with a rate of $\lambda\left(\frac{1}{n}\right)$. This is because arrivals according to a Poisson process at a rate λ which are then "filtered" by a Bernoulli process (i.e. allowed to be counted with a probability p) is a Poisson process with a rate λp. Thus our birth-rate and death-rates are given by

$$\begin{aligned} \lambda_n &= \lambda\left(\frac{1}{n+1}\right) \quad \text{for} \quad n \geq 0 \\ \mu_n &= 1\,. \end{aligned}$$

Part (b): The average number of customers in the system is given by $\sum_{n=0}^{\infty} p_n$, where p_n are the steady-state solutions to the balance equations. To solve them we first compute the ratios ρ_n given by

$$\rho_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} = \lambda^n\,(1)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\cdots\left(\frac{1}{n}\right) = \frac{\lambda^n}{n!}\,.$$

The normilization constant p_0 is next given by

$$p_0 = \left(1 + \sum_{n=1}^{\infty} \rho_n\right)^{-1} = \left(1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!}\right)^{-1} = e^{-\lambda}\,.$$

Then the steady-state probabilities are given by

$$p_n = \rho_n p_0 = e^{-\lambda} \frac{\lambda^n}{n!} .$$

With these expressions the average number of customers in the system is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} n p_n &= \sum_{n=0}^{\infty} n \frac{\lambda^n e^{-\lambda}}{n!} = \sum_{n=1}^{\infty} n \frac{\lambda^n e^{-\lambda}}{n!} = e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^{n+1}}{n!} = \lambda e^{-\lambda} e^{\lambda} = \lambda .
\end{aligned}$$

Chapter 11: General Queuing Systems

Notes On The Text

The derivation of the average arrival rate λ_a for a $M/M/1$ queue

The average arrival rate (Formula 11.4) is given by $\lambda_a = \sum_{n=0}^{\infty} \lambda_n p_n$. For a $M/M/1$ queue with capacity N, the arrival rate as a function of n is given by $\lambda_n = \lambda$ for $n < N$ and $\lambda_n = 0$ for $n \geq N$, since when $n = N$ no customers can arrive as there is no space for them to occupy. With these, the above becomes

$$\lambda_a = \sum_{n=0}^{N-1} \lambda p_n = \frac{\lambda(1-\rho)}{1-\rho^{N+1}} \sum_{n=0}^{N-1} \rho^n \,,$$

Using the result that $p_n = \frac{\rho^n(1-\rho)}{1-\rho^{N+1}}$ for a steady-state $M/M/1$ queue with capacity N. Using the summation identity $\sum_{n=0}^{N-1} \rho^n = \frac{1-\rho^N}{1-\rho}$ we see that λ_a is equal to

$$\lambda_a = \frac{\lambda(1-\rho^N)}{1-\rho^{N+1}} \,.$$

Now the probability our entire system is occupied is given by $p_N = \frac{\rho^N(1-\rho)}{1-\rho^{N+1}}$ giving that the probability complement of p_N is given by

$$1 - p_N = \frac{1-\rho^{N+1} - \rho^N(1-\rho)}{1-\rho^{N+1}} = \frac{1-\rho^N}{1-\rho^{N+1}} \,.$$

Thus using this we see the expression for λ_a above becomes

$$\lambda_a = \lambda(1-p_N) \,,$$

as claimed. This can be seen more intuitively if we notice that people will be added to the queue as long as the queue is not full i.e. has less than N people in the queue. The probability there is space for at least one more person is the probability complement of p_N. Thus people arrive at a rate λ but are then filtered with a Bernoulli process that only accepts new arrivals events with a probability $1-p_N$ so the total arrival rate from these two combined processes is $\lambda(1-p_N)$, the same as before.

The derivation of the steady-state probabilities p_n for $M/M/s$ queues

To normalize the steady-state probabilities p_n's for a $M/M/s$ queue requires computing

$$\begin{aligned}
\frac{1}{p_0} &= 1 + \sum_{n=1}^{\infty} \rho_n \\
&= 1 + \sum_{n=1}^{s-1} \frac{\alpha^s}{s!} + \sum_{n=s}^{\infty} \frac{\alpha^s}{s!} \rho^{n-s} \\
&= e_{s-1}(\alpha) + \frac{\alpha^s}{s!} \sum_{n=s}^{\infty} \rho^{n-s} ,
\end{aligned}$$

where we have put in the previously derived expressions for ρ_n for a $M/M/s$ queue and also introduced the definition of $e_m(x)$, the truncated exponential series. Continuing the evaluation of the second summation above we find

$$\frac{1}{p_0} = e_{s-1}(\alpha) + \frac{\alpha^s}{s!} \sum_{n=0}^{\infty} \rho^n = e_{s-1}(\alpha) + \frac{\alpha^s}{s!} \left(\frac{1}{1-\rho} \right) ,$$

as the normalization condition required for p_n. As a special case of the above if we take $s = 1$ we find that

$$p_0 = \left(e_0(\alpha) + \frac{\alpha}{1-\rho} \right)^{-1} = \left(1 + \frac{\alpha}{1-\rho} \right)^{-1} .$$

Since $\alpha = \frac{\lambda}{\mu}$ and when $s = 1$ we have $\rho = \frac{\lambda}{s\mu} = \frac{\lambda}{\mu} = \alpha$, we see that the above is equal to

$$p_0 = \left(\frac{1-\rho+\alpha}{1-\rho} \right)^{-1} = 1 - \rho ,$$

as claimed in the book, and which was derived when we considered the $M/M/1$ queue in an earlier chapter.

The algebraic manipulations required to derive $M/M/s$ the queue waiting times $P\{T_Q > t\}$

Given the expression derived in the book for $P\{T_Q > t\}$ as

$$P\{T_Q > t\} = p_s e^{-\mu s t} \sum_{n=s}^{\infty} \left\{ \sum_{k=0}^{n-s} \frac{(\mu s t)^k}{k!} \right\} \rho^{n-s} ,$$

we begin by noticing that we can start the outer summation at $n = 0$ by shifting the n index in the summation up by s i.e. $P\{T_Q > t\}$ becomes

$$P\{T_Q > t\} = p_s e^{-\mu s t} \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \frac{(\mu s t)^k}{k!} \right\} \rho^n . \tag{37}$$

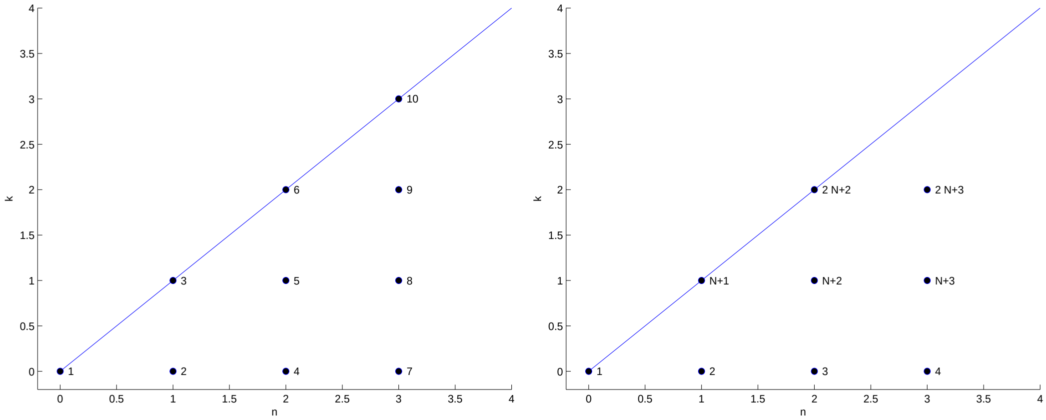

Figure 23: **Left:** The order of the (n, k) terms visited in the summation 37, when we sum in n first and then k second. **Right:** The order of the (n, k) terms visited in the summation 38, when we sum in k first and then in n second.

Next we interchange the order of summation by noting that in the above summation we are summing in the order given in Figure 23 (left), that is we sum by first setting the value of n and summing in k. Observe that the sum presented above can also be done by summing along k first and then n second as shown in Figure 23 (right). In that figure, "N" represents the "infinity" required when one sums fully in n before incrementing k. Changing the given summation in this way we have $P\{T_Q > t\}$ given as

$$\begin{aligned} P\{T_Q > t\} &= p_s e^{-\mu s t} \sum_{k=0}^{\infty} \left\{ \sum_{n=k}^{\infty} \frac{(\mu s t)^k}{k!} \right\} \rho^n \\ &= p_s e^{-\mu s t} \sum_{k=0}^{\infty} \frac{(\mu s t)^k}{k!} \left(\sum_{n=k}^{\infty} \rho^n \right) . \end{aligned} \tag{38}$$

Since this inner sum can be transformed as

$$\sum_{n=k}^{\infty} \rho^n = \sum_{n=0}^{\infty} \rho^{n+k} = \rho^k \sum_{n=0}^{\infty} \rho^n = \frac{\rho^k}{1-\rho} ,$$

the above becomes

$$P\{T_Q > t\} = \frac{p_s e^{-\mu s t}}{1-\rho} \sum_{k=0}^{\infty} \frac{(\mu s t \rho)^k}{k!} .$$

As the expression in parenthesis becomes $\mu s t \rho = \mu s t \left(\frac{\lambda}{s\mu} \right) = \lambda t$, we have

$$P\{T_Q > t\} = \frac{p_s e^{-\mu s t}}{1-\rho} e^{\lambda t} = \frac{p_s e^{-(\mu s - \lambda) t}}{1-\rho} .$$

If we take $t = 0$ then the probability we will have to wait in this $M/M/s$ queue *at all* is given by

$$P\{T_Q > 0\} = \frac{p_s}{1-\rho} \equiv C(s, \alpha) ,$$

where we have defined the function $C(s, \alpha)$, known as Erlang's loss function.

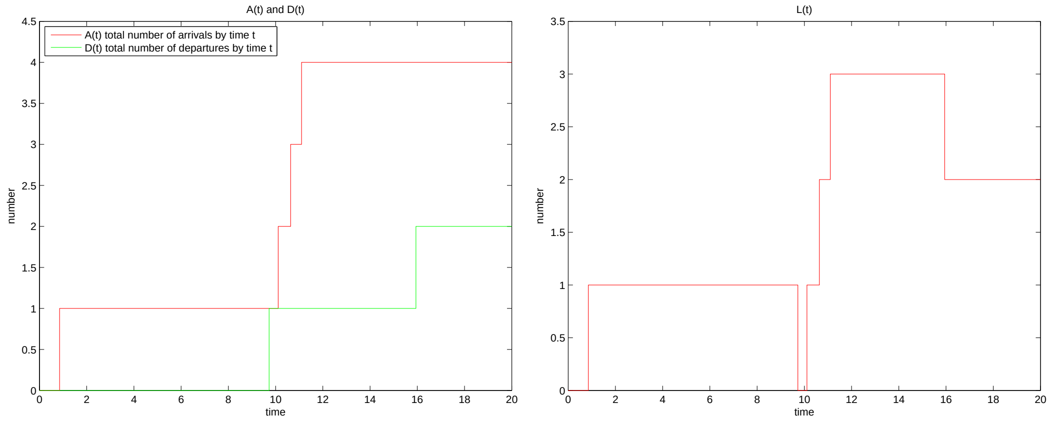

Figure 24: **Left:** A plot of $A(t)$ and $D(t)$ for the simulation in Exercise 1. **Right:** A plot of $L(t) = A(t) - D(t)$ for the simulation in Exercise 1.

Exercise Solutions

Note: Most of the results below match exactly the exercise solutions found in the back of the book. However, in some of the problems below I could not match these results exactly and couldn't find any errors in my proposed reasoning. If anyone see any errors with what I have done below please let me know and I'll correct them as quickly as possible.

Exercise 1 (the functions $A(t)$ and $D(t)$

Part (a): Given $A(t)$ the number of arrivals by time t and $D(t)$ the number of departures by time t, the number of customers present at time t, denoted $L(t)$, is given by $L(t) = A(t) - D(t)$. Simulating both $A(t)$ and $D(t)$ one can immediately find $L(t)$ by subtraction. See Figure 24 (left) for a plot of $A(t)$ and $D(t)$ and Figure 24 (right) for a plot of $L(t)$.

Part (b): The requested expressions are defined in terms of the functions above $A(t)$, $D(t)$, and $L(t)$ as

$$
\begin{aligned}
L(20) &= \frac{1}{20}\int_0^{20} L(t)dt \\
W(20) &= \frac{S(20)}{A(20)} = \frac{1}{A(20)}\int_0^{20} L(t)dt \\
\lambda_a(20) &= \frac{A(20)}{20}\,.
\end{aligned}
$$

Here $S(20)$ is the total time spent by all customers during the time $(0, 20]$. Note that we observe before computing these expressions from our simulation that the expression $\lambda_a(20)W(20)$ equals

$$
\frac{A(20)}{20}\cdot\frac{1}{A(20)}\int_0^{20} L(t)dt = \frac{1}{20}\int_0^{20} L(t)dt = L(20)\,,
$$

as requested. A simulation for this problem is implemented in the MATLAB script `chap_11_prob_1.m`. When we run this simulation with the parameters from Chapter 8 of $u_l = 5$ and $u_r = 10$, we find $L(20) = 1.36$, $W(20) = 6.805$, and $\lambda_a(20) = 0.2$.

These numbers are slightly different than those found in the book.

Exercise 2 (deriving L for a $M/M/1$ queue with capacity N)

From the discussion in the text, to determine the average number of customers in our system L, we needed to calculate the sum $\sum_{n=0}^{N} n\rho^n$. Following the book we have

$$
\begin{aligned}
\sum_{n=0}^{N} n\rho^n &= \rho\frac{d}{d\rho}\sum_{n=0}^{N}\rho^n = \rho\frac{d}{d\rho}\left\{\frac{1-\rho^{N+1}}{1-\rho}\right\} \\
&= \rho\left[\frac{1-\rho^{N+1}}{(1-\rho)^2} - \frac{(N+1)\rho^N}{1-\rho}\right] \\
&= \frac{\rho}{(1-\rho)^2}\left[1-\rho^{N+1}-(N+1)\rho^N(1-\rho)\right] \\
&= \frac{\rho}{(1-\rho)^2}\left[1-(N+1)\rho^N + N\rho^{N+1}\right] .
\end{aligned}
$$

Since the average number of customers in our entire system L (consisting of the queue plus the person being served) needs to be multiplied by $\frac{1-\rho}{1-\rho^{N+1}}$ to give

$$
L = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\sum_{n=0}^{\infty} n\rho^n .
$$

Using the above summation we see get

$$
L = \frac{\rho}{(1-\rho)(1-\rho^{N+1})}\left[1-(N+1)\rho^N + N\rho^{N+1}\right] , \tag{39}
$$

the requested expression.

Exercise 3 (the average number of customers being served in a $M/M/1$ queue)

Since L is defined as the expected number of customers in the entire system and L_Q is the same thing but for the queue only (not considering the person at the server) we see that the average number of customers being serviced is $L - L_Q$ which for the $M/M/1$ queue with capacity N is given by

$$
L - L_Q = \frac{\lambda_a}{\mu} = \frac{\lambda}{\mu}(1-p_N) = \rho(1-p_N) ,
$$

following the discussion in the book. Since we know an analytic expression for p_N under this queuing system given by

$$
p_N = \frac{\rho^N(1-\rho)}{1-\rho^{N+1}} ,
$$

we can calculate $1 - p_N = \frac{1-\rho^N}{1-\rho^{N+1}}$, to find the utilization of the server given by

$$L - L_Q = \frac{\rho(1-\rho^N)}{1-\rho^{N+1}}\,.$$

If ρ is very small we see that we have

$$L - L_Q = \frac{\rho(1-\rho^N)}{1-\rho^{N+1}} \sim \rho\,,$$

and when ρ is very large we have

$$L - L_Q = \frac{\rho(1-\rho^N)}{1-\rho^{N+1}} = \frac{\rho^{N+1}(\rho^{-N}-1)}{\rho^{N+1}(\rho^{-(N+1)}-1)} \sim 1 - \rho^{-N}\,,$$

the required expressions.

Exercise 4 (library copy machines)

This description fits the model of a $M/M/1$ queue with infinite capacity (but we expect that on average a much smaller capacity will be utilized). Since the mean service time is 3 minutes (1/20 of an hour) the service *rate* is $\mu = \left(\frac{1}{20}\right)^{-1} = 20$ customers an hour.

Part (a): If we assume Poisson arrivals at a rate of λ and desire to have an average of $L = 3$ customers in the system, then from the results on $M/M/1$ queues of this type we have that in steady-state L is related to the arrival rate λ and the service rate μ by

$$L = \frac{\lambda}{\mu - \lambda}\,.$$

Solving this for the arrival rate λ we find that

$$\lambda = \frac{L\mu}{(1+L)} = \frac{3(20)}{4} = 15\,,$$

arriving customers per hour. Note that this is the maximum rate, any arrival rate less than this will produce smaller values of L.

Part (b): With the rates given above the utilization factor for this system is $\rho = \frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$. The average waiting time in the queue is then

$$W_Q = \frac{\rho}{(1-\rho)\mu} = \frac{(3/4)}{(1/4)(20)} = \frac{3}{20} = 0.15\,,$$

of an hour or 9 minutes.

Part (c): If $L = 6$ then the values of λ, ρ, and W_Q all change. In this case they become $\lambda = 17.142857$, $\rho = 0.857143$, and $W_Q = 0.3$ hours or 18 minutes. The simple calculations for this problem can be found in the MATLAB script `chap_11_prob_4.m`.

Exercise 5 (barber shop queuing)

Part (a): We can consider this a $M/M/1$ queue with $N = 2$ space for people to wait for service and we are told that our pure arrival rate is $\lambda = 6$, while our departure rate is independent of state given by $\mu = 3$ (both are in units of people per hour). If there is no one present, $n = 0$, and $\lambda_0 = 6$. In the other cases by recalling the result that a Poisson process with rate λ that is "filtered" by a Bernoulli process with probability p is another Poisson process with rate $p\lambda$, when $n > 0$ we have

$$\lambda_1 = 0.5(6) = 3 \quad \text{and} \quad \lambda_2 = 0.5(6) = 3\,.$$

Here n is the "state" and represents the number of people in the barbershop $0 \leq n \leq N+s = 3$.

Part (b): We will solve the steady-state balance equations exactly since we have exact knowledge of the transition rates λ_n and μ_n for all n. Following the results found earlier the unique limiting normalized probability distribution $\{p_n\}$ is given by

$$p_n = \rho_n p_0 = \left(\frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\right) p_0\,.$$

with

$$p_0 = \left(1 + \sum_{n=1}^{\infty}\rho_n\right)^{-1}.$$

Then we compute

$$\begin{aligned}
\rho_1 &= \frac{\lambda_0}{\mu_1} = \frac{6}{3} = 2 \\
\rho_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = 2\left(\frac{3}{3}\right) = 2 \\
\rho_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = 2\,.
\end{aligned}$$

so that $p_0 = \left(1 + \sum_{n=1}^{3} 2\right)^{-1} = \frac{1}{7}$. Thus we find

$$p_0 = \frac{1}{7} \quad p_1 = \frac{2}{7} \quad p_2 = \frac{2}{7} \quad p_3 = \frac{2}{7}\,.$$

Part (c): The average number of customers in the shop at any moment L is given by

$$L = \sum_{n=0}^{3} n p_n = \frac{2}{7} + \frac{4}{7} + \frac{6}{7} = \frac{12}{7} = 1.7143\,.$$

Part (d): The barber will receive \$10 for every customer served. The service rate μ_n is a function of the state n, which in steady-state occurs with probability p_n. The average service rate μ_a is

$$\mu_a = \sum_{n=1}^{3}\mu_n p_n = 3\left(\frac{2}{7}\right) + 3\left(\frac{2}{7}\right) + 3\left(\frac{2}{7}\right) = \frac{18}{7} = 2.5714\,.$$

Then the steady-state long term profit is given by $P = 10\mu_a = 25.7143$ per hour.

Part (e): We are asked the average waiting time W. Using Little's law we know $L = \lambda_a W$, so $W = \frac{L}{\lambda_a}$. Calculating λ_a as

$$\lambda_a = \sum_{n=0}^{2} \lambda_n p_n = 6\left(\frac{1}{7}\right) + 3\left(\frac{2}{7}\right) + 3\left(\frac{2}{7}\right) = \frac{18}{7} = 2.5714\,,$$

and using L calculated above we get that $W = 0.667$ of an hour or 40 minuets.

Exercise 6 (some averages for the gas station queue)

From the earlier problem the average number of cars at the station is denoted L. Since this example is a $M/M/1$ queue with capacity $N = 3$ (the total number of spots in the system) and a utilization factor of $\rho = \frac{\lambda}{\mu} = \frac{20}{12} = \frac{5}{3}$, we see that L is given by Equation 39. Using the values above we find that $L = 2.095588$.

To answer the question as to how long each car spends at the station is to determine W, which can be obtained by using Little's law as $W = \frac{L}{\lambda_a}$, if we have λ_a, or using the results from this chapter we have that the expression for W for a $M/M/1$ queue is given by

$$W = \frac{L}{(1 - p_N)\lambda} = \left(\frac{1 - \rho^{N+1}}{1 - \rho^N}\right)\left(\frac{L}{\lambda}\right) = 0.1938\,,$$

hours or 11.63 minutes.

These two expressions, L and W, for a $M/M/1$ queue are calculated in the MATLAB functions `L_MM1_queue.m` and `W_MM1_queue.m`. These are called in the MATLAB script `chap_11_prob_6.m`.

Exercise 7 (the steady-state average birth rate equals the average death rate)

We desire to show that the average birth rate $\lambda_a = \sum_{n=0}^{\infty} \lambda_n p_n$ equals the average death rate $\mu_n = \sum_{n=0}^{\infty} \mu_n p_n$. Since we know that the balance equations have a solution p_n given by

$$p_n = \rho_n p_0 = \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}\right) p_0\,,$$

we can write the expression for λ_a as follows

$$
\begin{aligned}
\lambda_a &= \sum_{n=0}^{\infty} \lambda_n p_n = \sum_{n=0}^{\infty} \lambda_n \rho_n p_0 \\
&= p_0 \sum_{n=0}^{\infty} \lambda_n \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right) \\
&= p_0 \sum_{n=0}^{\infty} \mu_{n+1} \left(\frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1} \lambda_n}{\mu_1 \mu_2 \cdots \mu_n \mu_{n+1}} \right) \\
&= \sum_{n=0}^{\infty} \mu_{n+1} \rho_{n+1} p_0 = \sum_{n=0}^{\infty} \mu_{n+1} p_{n+1} = \sum_{n=1}^{\infty} \mu_n p_n \,,
\end{aligned}
$$

or the desired expression.

Exercise 8 (the car-wash-and-vacuum facility)

Part (a): The equations governing the steady-state probabilities for each of the states 0, 1, and 2 can be found by using the heuristic that the rate leaving a state must equal the rate of entering and give the following

$$
\begin{aligned}
\lambda p_0 &= 12 p_2 \\
4 p_1 &= \lambda p_0 \\
12 p_2 &= 4 p_1 \,.
\end{aligned}
$$

As in the book we find that p_0 in terms of λ is given by

$$
p_0 = \left(1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} \right)^{-1} = \left(1 + \frac{\lambda}{4} + \frac{\lambda}{12} \right)^{-1} = \left(1 + \frac{\lambda}{3} \right)^{-1} . \tag{40}
$$

so that the other probabilities are given (in terms of λ) solving the steady-state equations above as

$$
\begin{aligned}
p_1 &= \frac{\lambda}{4} \left(1 + \frac{\lambda}{3} \right)^{-1} \\
p_2 &= \frac{1}{3} p_1 = \frac{\lambda}{12} \left(1 + \frac{\lambda}{3} \right)^{-1} .
\end{aligned}
$$

Part (b): The average gross return to the customer who owns the facility would be $D\lambda_a$ or

$$
D\lambda(1 - p_1 - p_2) = D\lambda p_0 = D\lambda \left(1 + \frac{\lambda}{4} + \frac{\lambda}{12} \right)^{-1} ,
$$

since if the system is in state 1 (someone is washing) or if the system is in state 2 (someone is vacuuming) the service station is blocked and the arrival rate λ needs to be reduced appropriately. That is it is only open for customers if it is in state 0.

Part (c): The probability an arriving customer will find the system free is given by Equation 40, which can be simplified as

$$p_0 = \frac{3}{\lambda}\left(1 + \frac{3}{\lambda}\right)^{-1} .$$

If λ is large this becomes $p_0 \approx \frac{3}{\lambda}$, while the amount the business earns is $D\lambda p_0 \approx 3D$ per hour.

Exercise 9 (the enlarged car-wash-and-vacuum facility)

Part (a): This is an example of a two stage queue where the first stage is the the washing station and the second stage is the vacuuming station. We are told that the processing rate of the washing station is $\mu_1 = 4$ cars/hour and that the processing rate of the vacuuming station is $\mu_2 = 12$ cars/hour, while the car arrival rate λ is unknown (as of yet). Then the steady-state equations for this of two station queue become

$$\begin{aligned}
12p_{01} &= \lambda p_{00} && (41)\\
4p_{10} + 12p_{b1} &= (12 + \lambda)p_{10} && (42)\\
\lambda p_{00} + 12p_{11} &= 4p_{10} && (43)\\
\lambda p_{01} &= 16p_{11} && (44)\\
4p_{11} &= 12p_{b1} && (45)\\
p_{00} + p_{10} + p_{01} + p_{11} + p_{b1} &= 1 , && (46)
\end{aligned}$$

which are obtained by balancing the rate of leaving a state with the rate of entering a state. We can solve this system of equations by writing every probability in terms of p_{00} and then using the normalization Equation 46 to derive an equation involving λ only. This latter equation can then be solved for λ. Using Equation 41 we have p_{01} in terms of p_{00} given by

$$p_{01} = \frac{\lambda}{12} p_{00} .$$

Using Equation 44 to express p_{11} in terms of p_{00} we have

$$p_{11} = \frac{\lambda}{16} p_{01} = \frac{\lambda^2}{12(16)} p_{00} .$$

Using Equation 45 to express p_{b1} in terms of p_{00} we have

$$p_{b1} = \frac{1}{3} p_{11} = \frac{\lambda^2}{3(12)(16)} p_{00} .$$

Using Equation 42 to express p_{10} in terms of p_{00} we have

$$p_{10} = \frac{12}{8+\lambda} p_{b1} = \frac{\lambda^2}{3(16)(8+\lambda)} p_{00} .$$

Using all of these expressions in Equation 46 we find

$$p_{00} = \left(1 + \frac{\lambda^2}{3(16)(8+\lambda)} + \frac{\lambda}{12} + \frac{\lambda^2}{12(16)} + \frac{\lambda^2}{3(12)(16)}\right)^{-1} . \tag{47}$$

With this expression we can compute all probabilities above as functions of λ.

Part (b): If each customer pays \$D to use the facility then the average return is given by $\lambda(p_{00} + p_{01})D$, since a customer can only enter the facility if the washing station is free i.e. the system is in the state $0,0$ or $0,1$. From the above this is

$$\lambda\left(p_{00} + \frac{\lambda}{12}p_{00}\right)D = \lambda\left(1 + \frac{\lambda}{12}\right)p_{00}D\,,$$

with p_{00} a function of λ given by Equation 47.

Part (c): If λ is large, an arriving customer will be able to enter the facility a proportion of time given by

$$p_{00} + p_{01} = p_{00} + \frac{\lambda}{12}p_{00} = \left(1 + \frac{\lambda}{12}\right)p_{00} = \frac{1 + \frac{\lambda}{12}}{1 + \frac{\lambda}{12} + \frac{\lambda^2}{3(16)(8+\lambda)} + \frac{\lambda^2}{9(16)}} .$$

If λ is large, the above becomes

$$p_{00} + p_{01} \approx \frac{\lambda/12}{\lambda^2/9(16)} = \frac{12}{\lambda} .$$

This is different than the book's result, but I don't see an error in my logic.

Using this result we calculate that the business earns

$$\lambda D(p_{00} + p_{01}) = 12D\,,$$

dollars per hour, a significant increase over the $3D$ earned by the single station queue in Exercise 8.

Exercise 10 (verifying Jackson's solution for $p_{m,n}$)

We want to show that Jackson's solution $p_{m,n} = (1-\rho_1)\rho_1^m(1-\rho_2)\rho_2^n$, with $\rho_1 = \frac{\lambda}{\mu_1}$ and $\rho_2 = \frac{\lambda}{\mu_2}$ for the steady-state probabilities of two queues linked in series satisfies the steady-state balance equation given by

$$\lambda p_{m-1,n} + \mu_1 p_{m+1,n-1} + \mu_2 p_{m,n+1} = (\lambda + \mu_1 + \mu_2)p_{m,n}\,.$$

we have the left hand side LHS of the balance equation given by

$$\begin{aligned}
\text{LHS} &= \lambda(1-\rho_1)\rho_1^{m-1}(1-\rho_2)\rho_2^n + \mu_1(1-\rho_1)\rho_1^{m+1}(1-\rho_2)\rho_2^{n-1} + \mu_2(1-\rho_1)\rho_1^m(1-\rho_2)\rho_2^{n+1} \\
&= (1-\rho_1)(1-\rho_2)\left(\mu_1\rho_1^m\rho_2^n + \lambda\rho_1^m\left(\frac{\lambda}{\mu_2}\right)^{-1}\rho_2^n + \frac{\mu_2\lambda}{\mu_2}\rho_1^m\rho_2^n\right) \\
&= (1-\rho_1)(1-\rho_2)\rho_1^m\rho_2^n(\mu_1 + \mu_2 + \lambda) = (\mu_1 + \mu_2 + \lambda)p_{m,n}\,,
\end{aligned}$$

as claimed.

Exercise 11 (the motor vehicle licensing agency)

From the example in the book we assume that the arrival rate is $\lambda = 7$ people per hour, the rate of the processing clerk is $\mu_1 = 8$ applications per hour, and the rate of the cashier is $\mu_2 = 21$ customers per hour. Thus the two utilization factors ρ_1 and ρ_2 are given by $\rho_1 = \frac{\lambda}{\mu_1} = \frac{7}{8}$ and $\rho_2 = \frac{\lambda}{\mu_2} = \frac{7}{21} = \frac{1}{3}$.

Part (a): This is the probability we are in the state $p_{0,0}$ under steady-state operation can be calculated using Jackson's theorem from

$$p_{m,n} = (1-\rho_1)\rho_1^m(1-\rho_2)\rho_2^n \quad \text{for} \quad m \geq 0 \quad \text{and} \quad n \geq 0\,.$$

Thus we have

$$p_{0,0} = (1-\rho_1)(1-\rho_2) = \frac{1}{12} = 0.08333\,.$$

Part (b): This is the probability $p_{1,0}$ and in steady-state is given by

$$p_{1,0} = (1-\rho_1)\rho_1(1-\rho_2) = \left(1-\frac{7}{8}\right)\left(\frac{7}{8}\right)\left(1-\frac{1}{3}\right) = 0.0729\,.$$

This is different than the answer in the back of the book, but I can't find an error in my assumptions.

Part (c): The average time a given customer spends in the system is given by

$$W = \frac{L}{\lambda} = \frac{1}{\lambda}\left(\frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2}\right) = \frac{1/\mu_1}{1-\lambda/\mu_1} + \frac{1/\mu_2}{1-\lambda/\mu_2}\,.$$

If this expression is to be at most 1/2 hour we require that the minimum acceptable value of μ_1 satisfy

$$\frac{1/\mu_1}{1-\lambda/\mu_1} + \frac{1/\mu_2}{1-\lambda/\mu_2} = \frac{1}{2}\,.$$

With the values of λ and μ_2 given above we find that the minimum value for μ_1 must satisfy

$$\frac{1/\mu_1}{1-\lambda/\mu_1} = 0.428571\,,$$

on solving for μ_1 we find that $\mu_1 = 9.333333$ customers per hour.

These simple calculations are done in the MATLAB script `chap_11_prob_11.m`.

Exercise 12 (now waiting to vacuum)

Part (a): The states for this system can still be taken as in the example for the extended car wash facility in the book but with the removal of the state $(b, 1)$. This is because a

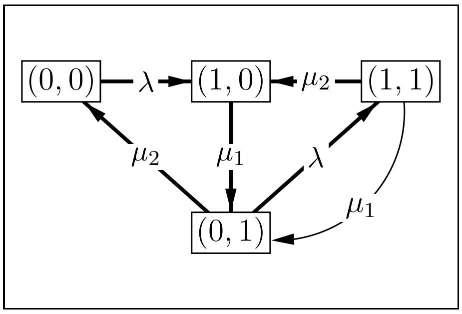

Figure 25: The steady-state transition diagram for Exercise 12. Note that the link from state $(1,1)$ to $(0,1)$ represents a car leaving if the vacuum facility is occupied after this customer finishes their car wash.

customer who finishes washing their car will leave if the vacuum station is occupied. Thus the state transition diagram becomes that seen in the Figure 25.

Expressing the heuristic that the rate entering equals the rate leaving gives the equations for the steady-state probabilities shown in Table 1. Coupled with these we also have the normalization condition

$$p_{00} + p_{10} + p_{01} + p_{11} = 1\,.$$

With the numbers given here $\mu_1 = 4$, $\mu_2 = 2$, and $\lambda = 3$ cars per hour the above equations become

$$\begin{aligned} 2p_{01} &= 3p_{00} \\ 3p_{00} + 2p_{11} &= 4p_{10} \\ 4p_{11} + 4p_{10} &= 2p_{01} + 3p_{11} \\ 3p_{01} &= 2p_{11} + 4p_{11} \end{aligned}$$

We can solve this system for every probability in terms of p_{00} and then using the normalization condition to solve for p_{00}. We find that $p_{00} = 0.228$, and that the others are given by

$$\begin{aligned} p_{10} &= 0.257 \\ p_{01} &= 0.342 \\ p_{11} &= 0.171\,. \end{aligned} \tag{48}$$

Part (b): The average rate at which customers *enter* the facility is

$$\lambda_a = \lambda p_{00} + \lambda p_{01} = 1.71\,,$$

cars per hour.

Part (c): The average number of customers in the facility is given by

$$L = 0p_{00} + 1p_{10} + 1p_{01} + 2p_{11} = 0.257 + 0.342 + 2(0.1714) = 0.94\,.$$

state	rate entering	=	rate leaving
$(0,0)$	$\mu_2 p_{01}$	$=$	λp_{00}
$(1,0)$	$\lambda p_{00} + \mu_2 p_{11}$	$=$	$\mu_1 p_{10}$
$(0,1)$	$\mu_1 p_{11} + \mu_1 p_{10}$	$=$	$\mu_2 p_{01} + \lambda p_{01}$
$(1,1)$	λp_{01}	$=$	$\mu_2 p_{11} + \mu_1 p_{11}$

Table 1: The steady-state equations for Exercise 12

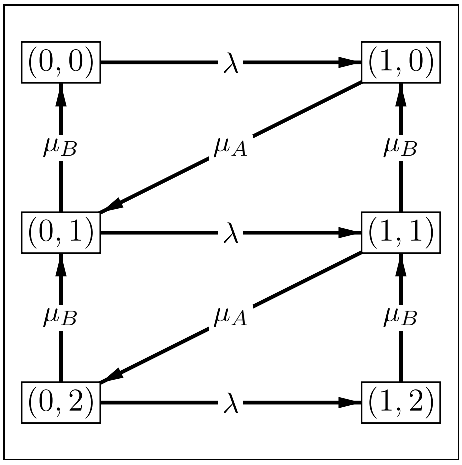

Figure 26: The steady-state transition diagram for Exercise 13.

Part (d): Using Little's law, the average amount of time an entering customer spends in the facility is

$$W = \frac{L}{\lambda_a} = \frac{0.94}{1.71} = 0.549\,,$$

hours or 32.9 minutes.

Exercise 13 (one space at station B)

As suggested, let the state of the system be denoted by (m, n) if there are m customers at station A and n customers at station B. From the given description, the ranges of these two variables are $m = 0, 1$ and $n = 0, 1, 2$. We draw a steady-state transition diagram for these states in Figure 26. Using the idea that in steady-state the rate of entry must equal the rate of departure we can write down the steady-state equations for $p_{m,n}$. This is done in Table 2.

Part (b): Using the numbers given of $\lambda = 3$, $\mu_A = 4$, and $\mu_B = 2$ customers per hour we

state	rate entering	=	rate leaving
$(0,0)$	$\mu_B p_{01}$	$=$	λp_{00}
$(0,1)$	$\mu_A p_{10} + \mu_B p_{02}$	$=$	$\mu_B p_{01} + \lambda p_{01}$
$(1,0)$	$\mu_B p_{11} + \lambda p_{00}$	$=$	$\mu_A p_{10}$
$(1,1)$	$\lambda p_{01} + \mu_B p_{12}$	$=$	$\mu_B p_{11} + \mu_A p_{11}$
$(0,2)$	$\mu_A p_{11}$	$=$	$\mu_B p_{02} + \lambda p_{02}$
$(1,2)$	λp_{02}	$=$	$\mu_B p_{12}$

Table 2: The steady-state equations for Exercise 13

find our steady-state equations become

$$\begin{aligned}
2p_{01} &= 3p_{00} \\
4p_{10} + 2p_{02} &= 2p_{01} + 3p_{01} = 5p_{01} \\
2p_{11} + 3p_{00} &= 4p_{10} \\
3p_{01} + 2p_{12} &= 2p_{11} + 4p_{11} = 6p_{11} \\
4p_{11} &= 2p_{02} + 3p_{02} = 5p_{02} \\
3p_{02} &= 2p_{12}\,.
\end{aligned}$$

with the normalization condition that

$$p_{00} + p_{01} + p_{02} + p_{10} + p_{11} + p_{12} = 1\,.$$

As a matrix, this system is given by

$$\begin{bmatrix} -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & -5 & 2 & 4 & 0 & 0 \\ 3 & 0 & 0 & -4 & 2 & 0 \\ 0 & 3 & 0 & 0 & -6 & 2 \\ 0 & 0 & -5 & 0 & 4 & 0 \\ 0 & 0 & 3 & 0 & 0 & -2 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} p_{00} \\ p_{01} \\ p_{02} \\ p_{10} \\ p_{11} \\ p_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Solving this with MATLAB we find

$$p_{00} = 0.1311\,, p_{01} = 0.1967\,, p_{02} = 0.1311\,, p_{10} = 0.1803\,, p_{11} = 0.1639\,, p_{12} = 0.1967\,.$$

The average number of customers in the system is the computed as

$$L = 0p_{00} + 1p_{01} + 2p_{02} + 1p_{10} + 2p_{11} + 3p_{12} = 1.5574\,.$$

This problem is worked in the MATLAB script `chap_11_prob_13.m`.

Exercise 14 (the waiting time of J. Doe)

Part (a): From the expression in the book the average queue time W_Q for this type of queue is given by

$$W_Q = W - \frac{1}{\mu} = \frac{\rho}{(1-\rho)\mu} = 0.75\,,$$

of an hour or 45 minutes and

$$L_Q = \lambda W_Q = \frac{\rho^2}{1-\rho} = 2.25\,,$$

customers.

Part (b): Since now we *know* that there are two customers ahead of J. Doe, the time he will have to wait is the sum of two exponentially distributed service times (the time required to service both customers). The distribution of the sum of n identically distributed exponential random variables is a Gamma random variable. If we denote T the random denoting how long J. Doe has to wait for service and $F_T(t;\lambda,r)$ the cumulative distribution function for a Gamma random variable with parameters $(\lambda = 4, r = 2)$ we have

$$P\{T > t\} = 1 - F_T(t;\lambda,r) = 1 - 0.593994 = 0.406006\,,$$

when we take $t = 1/2$. This problem is worked in the MATLAB script `chap_11_prob_14.m`.

Exercise 15 (analytic expressions for the statistics in a $M/M/2$ queue))

If we define E to be the event that a customer must queue, for a $M/M/s$ queue we have an Erlang's loss function $C(s,\alpha)$ given by here

$$\begin{aligned} P\{E\} &= C(s,\alpha) = \frac{\alpha^s}{s!(1-\rho)} p_0 \\ &= \frac{\alpha^s}{s!(1-\rho)} \left(e_{s-1}(\alpha) + \frac{\alpha^s}{s!(1-\rho)} \right)^{-1}, \end{aligned}$$

where α is called the traffic intensity defined by $\alpha = \frac{\lambda}{\mu}$, ρ is called the utilization factor, defined by $\rho = \frac{\lambda}{s\mu} = \frac{\alpha}{s}$, and $e_s(\alpha)$ is the truncated exponential series defined by $e_s(x) = \sum_{n=0}^{s} \frac{x^n}{n!}$.

Part (a): When $s = 2$ we have $e_{2-1}(\alpha) = e_1(\alpha) = 1 + \alpha$ so the above becomes

$$\begin{aligned} P\{E\} &= C(2,\alpha) = \frac{\alpha^2}{2(1-\rho)} \left(1 + \alpha + \frac{\alpha^2}{2(1-\rho)} \right)^{-1} \\ &= \frac{\alpha^2}{2+\alpha}\,. \end{aligned}$$

Part (b): Again from the book we have that for a $M/M/s$ queues that W_Q the average wait time in the queue only is given by

$$W_Q = \frac{1}{s\mu - \lambda} C(s,\alpha)\,.$$

When $s = 2$ and using the above this becomes

$$\begin{aligned} W_Q &= \frac{1}{2\mu - \lambda} \left(\frac{\alpha^2}{2+\alpha} \right) = \frac{1}{\mu} \left(\frac{1}{2 - \lambda/\mu} \right) \left(\frac{\alpha^2}{2+\alpha} \right) \\ &= \frac{\alpha^2}{\mu(2-\alpha)(2+\alpha)}\,, \end{aligned}$$

as expected.

Part (c): Since the expression for W_Q for a $M/M/2$ queue was computed in Part (b) the steady-state total waiting time W is (since the average waiting time once we get to the server is $\frac{1}{\mu}$)

$$
\begin{aligned}
W &= W_Q + \frac{1}{\mu} = \frac{1}{\mu}\left(\frac{\alpha^2}{(2-\alpha)(2+\alpha)}\right) + \frac{1}{\mu} \\
&= \frac{1}{\mu}\left(\frac{4}{4-\alpha^2}\right) = \frac{1}{\mu}\left(\frac{1}{1-\alpha^2/4}\right).
\end{aligned}
$$

Since $\rho = \frac{\lambda}{s\mu} = \frac{\alpha}{s} = \frac{\alpha}{2}$, when we have $s = 2$ servers, the above becomes

$$
W = \frac{1}{\mu}\left(\frac{1}{1-\rho^2}\right).
$$

Part (d): The probability that the system is empty is given by

$$
\begin{aligned}
p_0 &= \left(e_1(\alpha) + \frac{\alpha^2}{2(1-\rho)}\right)^{-1} \\
&= \left(1 + \alpha + \frac{\alpha^2}{2(1-\rho)}\right)^{-1} = \left(\frac{(1+2\rho)(1-\rho)+2\rho^2}{1-\rho}\right)^{-1} \\
&= \left(\frac{1+\rho}{1-\rho}\right)^{-1} = \frac{1-\rho}{1+\rho}.
\end{aligned}
$$

Exercise 16 (evaluating statistics for the $M/M/2$ queue)

Since with the given numbers we have

$$
\begin{aligned}
\rho &= \frac{\lambda}{s\mu} = \frac{18}{2(10)} = \frac{9}{10} \quad \text{and} \\
\alpha &= \frac{\lambda}{\mu} = \frac{18}{10} = \frac{9}{5},
\end{aligned}
$$

we have that the requested queue statistics are given by

Part (a): $p_0 = \frac{1-\rho}{1+\rho} = 0.052632$.

Part (b): The probability the customer must queue is given by $C(2, \alpha) = 0.852632$.

Part (c): We have \$$W_Q = 0.500000$ hours or 30 minuets.

Part (d): We have $W_Q = 0.426316$ of an hour or 25.57 minutes.

Part (e): We have $L_Q = W_Q\lambda = 7.673684$

Part (f): We have $W = \frac{1}{\mu} + W_Q = 0.526316$ of an hour or 31.57 minutes.

Part (g): We have $L = W\lambda = 9.473684$.

Part (h): If we define F to be the event that the customers queue time is greater than t we have

$$\begin{aligned} P\{F\} &= P\{T_Q > 0\}e^{-(s\mu-\lambda)t} \\ &= C(2,\alpha)e^{-(2\mu-\lambda)(1)} = 0.115391\,, \end{aligned}$$

where we have used the fact that $t = 1$ hour.

All of these simple calculations are done in the MATLAB script `chap_11_prob_16.m`.

Exercise 17 (comparing two servers against one server twice as fast)

Part (a): As suggested, we will compute the ratio $\$W_Q^{(1)}/W_Q^{(2)}$. Using the formula in the book that $\$W_Q = \frac{1}{s\mu-\lambda}$ we find

$$\frac{\$W_Q^{(1)}}{\$W_Q^{(2)}} = \frac{\left(\frac{1}{(1)2\mu-\lambda}\right)}{\left(\frac{1}{(2)\mu-\lambda}\right)} = 1\,,$$

Thus $\$W_Q^{(1)} = \$W_Q^{(2)}$ i.e. the expected waiting time (given that a customer has to wait) is the same if we have two serves or one server who works twice as fast.

Part (b): We have

$$\begin{aligned} W_Q^{(1)} &= \frac{L}{\lambda} - \frac{1}{2\mu} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} - \frac{1}{2\mu} \\ W_Q^{(2)} &= \frac{1}{\mu}\frac{\alpha_2^2}{(2-\alpha_2)(2+\alpha_2)} = \frac{1}{\mu}\frac{\alpha_1^2}{(1-\alpha_1)(1+\alpha_1)} = \frac{1}{\mu}\frac{\rho^2}{(1-\rho)(1+\rho)}\,. \end{aligned}$$

Working on $W_Q^{(1)}$ with some algebra we can write it as $\frac{1}{\mu}\left(\frac{1+\rho}{2(1-\rho)}\right)$, so that

$$\frac{W_Q^{(1)}}{W_Q^{(2)}} = \frac{1+\rho}{2(1-\rho)}\left(\frac{(1-\rho)(1+\rho)}{\rho^2}\right) = \frac{1}{2}\left(\frac{1}{\rho}+1\right)^2\,.$$

Since we are told that $\rho = \frac{\lambda}{\mu} < \frac{1}{2}$ we know that $\frac{1}{\rho} > 2$ and the above gives

$$\frac{W_Q^{(1)}}{W_Q^{(2)}} > \frac{1}{2}(2+1)^2 = \frac{9}{2}\,.$$

Thus we see that $W_Q^{(1)} > W_Q^{(2)}$, and the queue wait under the one server model is more than in the two server model.

Part (c): We have

$$
\begin{aligned}
W^{(1)} &= \frac{L}{\lambda} = \frac{1}{2\mu - \lambda} = \frac{1}{\mu}\left(\frac{1}{2-\rho}\right) \\
W^{(2)} &= \frac{1}{\mu}\left(\frac{1}{1-\rho^2}\right).
\end{aligned}
$$

So that the ratio is given by

$$
\frac{W^{(1)}}{W^{(2)}} = \frac{1-\rho^2}{2-\rho}.
$$

It is not clear what the magnitude of the expression on the right hand side is. Since we know that $\rho < \frac{1}{2}$ we can take $\rho = \frac{1}{3}$ and evaluate it. When we do this we find the right hand side of the above given by $\frac{8}{15} < 1$. So in this case $W^{(1)} < W^{(2)}$. Lets see if this holds in general. That is we try to show

$$
\frac{1-\rho^2}{2-\rho} < 1. \tag{49}
$$

This expression is equivalent to $\rho^2 - \rho + 1 > 0$. Considering the two roots of this quadratic equation we find

$$
\rho = \frac{1 \pm \sqrt{1-4(1)}}{2} = \frac{1 \pm i\sqrt{3}}{2},
$$

since these are complex this quadratic does not change signs over the real numbers. Thus evaluating at any point will determine the sign of this expression. Since we have *already* done this for $\rho = 1/3$ and found that $\rho^2 - \rho + 1 > 0$ we see that this quadratic is always positive (equivalently Equation 49 is true) and we can conclude that

$$
\frac{W^{(1)}}{W^{(2)}} < 1 \quad \text{or} \quad W^{(1)} < W^{(2)}.
$$

Thus in summary, given that you have to wait the two queues are identical, the wait in the two queue is longer than in the one server case, while the total system time is smaller in the one server case.

Exercise 18 (the library copy machines)

For the two machine situation we have that the average waiting time at the reading room copier is

$$
W_{1Q} = W_1 - \frac{1}{\mu} = \frac{1}{5} - \frac{1}{20} = 0.15,
$$

hours or 9 minutes and $L_{1Q} = W_{1Q}\lambda_1 = 2.25$. For the reserve room copier we have

$$
W_{2Q} = W_2 - \frac{1}{\mu} = \frac{1}{15} - \frac{1}{20} = 0.016667,
$$

hours or 1 minute and $L_{2Q} = W_{2Q}\lambda_2 = 0.083$, for the individual waiting times for the two machines. When they are placed at a central location we have $\lambda_1 + \lambda_2 = 20$ and $\rho = \frac{\lambda}{2\mu} = \frac{1}{2}$. So that

$$
W_Q = W - \frac{1}{\mu} = \frac{1}{15} - \frac{1}{20} = 0.0166,
$$

of an hour or 1 minute. While $L_Q = 0.3333$.

Thus from this analysis, the combined system is better in that it has a smaller W_Q and a reduced L_Q. The simple calculations for this problem are performed in the the MATLAB script `chap_11_prob_18.m`.

Exercise 19 (the reciprocal of Erlang's C function

Erlang's C function is given by

$$C(s, \alpha) = \frac{\alpha^s}{s!(1-\rho)} p_0 = \frac{\alpha^s}{s!(1-\rho)} \left(e_{s-1}(\alpha) + \frac{\alpha^s}{s!(1-\rho)} \right)^{-1},$$

so the reciprocal of this expression is

$$\begin{aligned} C(s, \alpha)^{-1} &= \frac{s!(1-\rho)}{\alpha^s} \left(e_{s-1}(\alpha) + \frac{\alpha^s}{s!(1-\rho)} \right) \\ &= 1 + \frac{s!(1-\rho)}{\alpha^s} e_{s-1}(\alpha). \end{aligned}$$

Now to relate ρ to α recall that $\alpha = \frac{\lambda}{\mu}$ and $\rho = \frac{\lambda}{s\mu} = \frac{\alpha}{s}$, so using this we have

$$\begin{aligned} C(s, \alpha)^{-1} &= 1 + \frac{s!(1-\alpha/s)}{\alpha^s} e_{s-1}(\alpha) \\ &= 1 + \frac{(s-1)!(s-\alpha)}{\alpha^s} e_{s-1}(\alpha), \end{aligned}$$

as requested.

Exercise 20 (limits of the traffic intensity α)

From Exercise 19 if $0 < \alpha < s$ and $\alpha \to s^-$ we see that $C(s, \alpha)^{-1} \to 1^-$ so $C(s, \alpha) \to 1^+$, the result we were to show.

Exercise 21 (the cost of operating a dock)

Part (a): The cost to operate the dock depends on the the cost to unload the ships and the cost to have them wait in the "queue" of the harbor. We therefore get a total cost of

$$E = 4800\mu(1) + 100L,$$

which fits the optimization models introduced earlier in this chapter. Since this is a single server queue this has an optimum $\mu_{\min}$ given by

$$\mu_{\min} = \lambda \left\{ 1 + \left(\frac{c}{\lambda b} \right)^{1/2} \right\} = 0.125,$$

ships per hour.

Part (b): The dock will be free p_0 percent of the time, which in this case is given by $p_0 = 1 - \rho = 1 - \frac{\lambda}{\mu_{\min}} = \frac{1}{3}$.

Part (c): The probability that the ship will spend more than 24 hours anchored waiting for the dock to be free can be computed by conditioning on the fact that the ship has to wait. Thus the desired probability is

$$\begin{aligned} P\{T_Q > t\} &= P\{T_Q > t | T_Q > 0\} P\{T_Q > 0\} \\ &= \frac{p_s e^{-(s\mu-\lambda)t}}{1-\rho} . \end{aligned}$$

Since $s = 1$ this becomes

$$P\{T_Q > t\} = \frac{p_1 e^{-(s\mu-\lambda)t}}{1-\rho} = \frac{\rho(1-\rho) e^{-(s\mu-\lambda)t}}{1-\rho} = \rho e^{-(\mu-\lambda)t} = 0.24525 \,,$$

when we put in the above numbers ($t = 24$). These simple calculations are done in the MATLAB file `chap_11_prob_21.m`.

Exercise 22 (Erlang's loss function)

Part (a): For Erlang's loss system the average number of orders taken is given by

$$\lambda_a = (1 - B(s, \alpha))\lambda \,.$$

Here Erlang's loss formula, $B(s, \alpha)$, is defined as

$$B(s, \alpha) = \frac{\alpha^s}{s! e_s(\alpha)} \,,$$

and in this problem we have a traffic intensity α, of $\alpha = \frac{\lambda}{\mu} = 2$. So with one operator $s = 1$ and we find $B(1, 2) = \frac{2}{3}$. Using the formula for λ_a above we find in this case that $\lambda_a = \frac{20}{3} = 6.6667$ per hour. While with two operator case $s = 2$ and we have $B(2, 2) = \frac{2}{5}$ so that $\lambda_a = 12$ per hour.

Part (b): The total profit rate for s serves is given by

$$E(s) = 15\lambda_a - 10bs \,.$$

The for one and two serves we have

$$\begin{aligned} E(1) &= 15\left(\frac{20}{3}\right) - 10b = 100 - 10b \\ E(2) &= 15\,(12) - 20b = 180 - 10b \,. \end{aligned}$$

So the two operator arraignment gives a larger profit rate if

$$E(2) > E(1) \Rightarrow b < \frac{15(12) - 100}{10} = 8 \,.$$

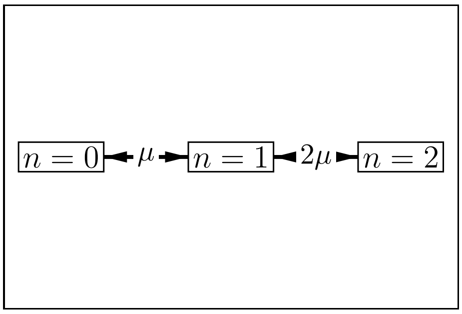

Figure 27: The steady-state transition diagram for Exercise 23.

state	rate entering	=	rate leaving
$n = 0$	μp_1	$=$	λp_0
$n = 1$	$\lambda p_0 + \mu_2 p_1$	$=$	$\mu p_1 + \lambda p_1$
$n = 2$	λp_1	$=$	$2\mu p_2$

Table 3: The steady-state equations for Exercise 23

Exercise 23 (Erlang's loss system)

Part (a): The states for this system can be taken to be $n = 0$, $n = 1$, and $n = 2$ representing $0, 1$ or 2 customers in the system at the given time. Then in steady-state the transition diagram looks like that shown in Figure 27.

The steady-state equations are then given by Table 3. Solving these with $\lambda = 3$ we find $p_1 = \frac{3}{\mu} p_0$, and $p_2 = \frac{3}{2\mu} p_1 = \frac{9}{2\mu^2} p_0$, so the normalization condition $p_0 + p_1 + p_2 = 1$ gives

$$p_0 = \frac{1}{1 + \frac{3}{\mu} + \frac{9}{2\mu^2}} \,,$$

and the other probabilities follow. Note that these agree with the general probabilities from Erlang's loss system given in the book.

Part (b): A caller will get a busy signal only if the system is in the state $n = 2$ which happens a proportion of the time given by

$$p_2 = \frac{9/(2\mu^2)}{1 + \frac{3}{\mu} + \frac{9}{2\mu^2}} \,.$$

Management wants this too be less than $1/4$. This imposes a requirement on μ of

$$\frac{9}{(2\mu^2)} < \frac{1}{4}\left(1 + \frac{3}{\mu} + \frac{9}{2\mu^2}\right) \,,$$

or equivalently this becomes a quadratic in $1/\mu$ of

$$\frac{27}{9}\frac{1}{\mu^2} - \frac{3}{4}\left(\frac{1}{\mu}\right) - \frac{1}{4} < 0 \,.$$

Solving the above quadratic (set equal to zero) for $1/\mu$ we find the only positive zero given by $\mu^* = 0.4065$, so $1/\mu^* = 2.46862$. Thus we require $\mu > \mu^* = 2.46862$, for managements requirement to be met. This problem is worked in the MATLAB script `chap_11_prob_23.m`

Exercise 24 (a large number of servers s)

Under Erlang's loss system our steady-state probabilities are given by

$$p_n = \begin{cases} \left(\frac{\alpha^n}{n!}\right) p_0 & n \leq s \\ \rho^{n-s} p_s & n > s \end{cases},$$

with p_0 given by

$$p_0 = \left(e_{s-1}(\alpha) + \frac{\alpha^s}{s!(1-\rho)} \right)^{-1}.$$

Part (a): If s is very large relative to n then we can assume that we are in the $n \leq s$ case for p_n above and in addition find that p_0 is approximately $p_0 \approx e^{-\alpha}$ which we get by recognizing that $\rho = \frac{\lambda}{s\mu} \approx 0$ so $e_{s-1}(\alpha) + \frac{\alpha^s}{s!(1-\rho)} \approx e_s(\alpha) \approx e^\alpha$, when s is large. Combining these two expressions we find

$$p_n \approx \frac{\alpha^n}{n!} e^{-\alpha},$$

or a Poisson distribution as claimed.

Part (b): Erlang's loss formula is defined as $B(s,\alpha) = \frac{\alpha^s}{s! e_s(\alpha)}$, which for large s by using Sterling's formula on the $s!$ in the denominator becomes

$$B(s,\alpha) = \frac{\alpha^s}{(2\pi)^{1/2} s^{s+1/2} e^{-s} e^{\alpha}} = (2\pi s)^{-1/2} \left(\frac{e\alpha}{s}\right)^s e^{-\alpha},$$

as requested.

Part (c): Since $B(s,\alpha)$ identifies the proportion of potential customers who are lost we desire to find the number of servers s such that $B(s,\alpha) \leq 0.05$, specifically when $\alpha = 10$. Following the hint when $s = \alpha$ the value of $B(s,\alpha) \approx (2\pi s)^{-1/2} = (2\pi(10))^{-1/2} \approx 0.126157$, which is larger than the target of 0.05. Note that this approximation to $B(s,\alpha)$ is monotonically decreasing as a function of s since

$$(e\alpha)^s < s^s,$$

for s large. Thus since $B(s,s) = 0.126157$, when $s = 10$ is too large we need to increase s to find the required value for the number of servers. In the MATLAB script `chap_11_prob_24.m` we loop over s find this number to be $s = 15$.

Exercise 25 (the optimum server rate)

Assuming a loss model where each server costs $b\mu$ dollars per hour and customer time in the system costs c dollars per hour we find a loss function, $E(\alpha)$, given by

$$E(\alpha) = \frac{b\lambda s}{\alpha} + c\alpha + \left(\frac{c\alpha}{s-\alpha}C(s,\alpha)\right).$$

For a single server queue this becomes

$$E(\alpha) = \frac{b\lambda}{\alpha} + \frac{c\alpha}{1-\alpha},$$

and has a minimum at $\alpha_{\text{min}} = \frac{z}{1+z}$ where $z = \left(\frac{b\lambda}{c}\right)^{1/2}$. To find the optimal service rate μ_{min} given a fixed arrival rate λ we let

$$\alpha_{\text{min}} = \frac{\lambda}{\mu_{\text{min}}} = \frac{z}{1+z},$$

and solving for μ_{min}. We find

$$\mu_{\text{min}} = \lambda(1+z^{-1}) = \lambda\left\{1+\left(\frac{c}{b\lambda}\right)^{1/2}\right\},$$

as requested.

Exercise 26 (simulation of the enlarged car-wash-facility)

Example 11.9 is a simulation of Example 11.2, the enlarged car-wash-facility without a blocking state. This later model has an analytic expression for all of the required parts which is computed in Exercise 9. Here we are told that the processing rates for the two stations are $\mu_1 = \frac{1}{15}$ cars-per-minute and $\mu_2 = \frac{1}{30}$ cars-per-minute (2 cars-per-hour), while the default arrival rate is $\lambda = 3$ cars-per-hour or $\frac{1}{20}$ cars per minute.

Part (a): The average customer time of the first facility (the wash facility) is $1/\mu_1 = 15$ minutes, while the average time found in the simulation was 16.39 minutes. The average length at the second facility (the vacuum facility) is $1/\mu_2 = 30$ minutes, while the simulation found 35.24 minutes.

Part (b): The proportion of time the was facility is busy in steady-state is $p_{10}+p_{11} = 0.4280$, while the simulation found this to be 0.4167.

Part (c): The proportion of time the vacuum station is busy in steady-state is $p_{01}+p_{11} = 0.5130$, while the simulation found the average utilization of this station to be 0.4992.

Part (d): The arrival rate at the wash facility is $\lambda_{\text{aw}} = (p_{00}+p_{01})\lambda = 1.71$ cars-per-hour, while the simulation gave 1.525 cars-per-hour.

Part (e): The arrival rate at the vacuum facility, using Little's law, is

$$\lambda_{\text{av}} = \frac{L_v}{W_v} = \frac{1p_{11} + 1p_{01}}{1/\mu_2} = 1.026\,,$$

cars-per-hour. The simulation gave $\lambda_{\text{av}} = 0.85$ cars-per-hour.

Part (f): The proportion of wash customers who also use the vacuum facility is $\frac{p_{11}}{p_{10}} = 0.6653$, while the simulation gave $\frac{34}{61} = 0.557$.

This problem is worked in the MATLAB script `chap_11_prob_26.m`.

Exercise 27 (simulating dock usages)

Recall that this example is a $M/M/1$ queuing system with $\mu = 3$ ships-per-day, $\lambda = 2$ ships-per-day. Using these we have $\rho = \frac{\lambda}{\mu} = \frac{2}{3} = 0.667$.

Part (a): The expected total number of arrivals in time $T = 100$ hours ($\frac{100}{24}$ days) is given by $T\lambda = \left(\frac{100}{24}\right) 2 = 8.333333$ ships, while the simulation had 12 arrivals.

Part (b): The steady-state queue length is given by

$$L_Q = \lambda W_Q = \frac{\rho^2}{1-\rho} = 1.333333\,,$$

ships while the simulation had this 1.65 ships.

Part (c): The average queue time in this situation is given by W_Q

$$W_Q = W - \frac{1}{\mu} = \frac{\rho}{(1-\rho)\mu} = 1\,,$$

days or 24 hours. In simulation this was computed at 17.55 hours.

Part (d): The steady-state probability that we find the dock free is given by $p_0 = 1 - \rho = 0.33333$, while in simulation we approximate this with $\frac{73}{226} = 0.323009$.

Part (e): The expected average unloading time is $1/\mu = 0.333333$ days or 8 hours. In the simulation we find this to be 7.35 hours.

Part (f): The steady-state proportion of the time the dock is busy is given by $p_1 = 1 - p_0 = \rho = 0.6667$, while in simulation we find its average utilization to be 0.6796.

This problem is worked in the MATLAB script `chap_11_prob_27.m`.

Exercise 28 (the probability of a long wait at the dock)

See Exercise 21 (c), where we find the exact steady-state probability of this event to be 0.24525 or 24.5%.

Chapter 12: Renewal Processes

Exercise Solutions

Exercise 1 (the lifetime of light bulbs)

Part (a): We have

$$P\{X > 2\} = 1 - P\{X \le 2\} = 1 - F_X(2)\,,$$

with $F_X(\cdot)$ the cumulative density function for the random variable X. From the discussion in the book, given the failure rate function $h(t)$, the cumulative density function $F_X(\cdot)$ is given by

$$1 - F_X(x) = \exp\left\{-\int_0^x h(t)dt\right\} = \exp\{-\frac{x^2}{2}\}\,.$$

So $P\{X > 2\} = \exp(-2) = 0.1353$.

Part (b): We have a density function given by

$$f_X(x) = \frac{dF_X(x)}{dx} = xe^{-x^2/2}\,.$$

Part (c): We find the expectation of X given by

$$E[X] = \int_0^\infty x^2 e^{-x^2/2}dx\,.$$

To evaluate this let $v = x^2/2$ so that $dv = xdx$ or $dx = \frac{dv}{\sqrt{2v}}$. The above then becomes

$$\begin{aligned} E[X] &= \int_0^\infty 2ve^{-v}\left(\frac{dv}{\sqrt{2v}}\right) = \sqrt{2}\int_0^\infty v^{1/2}e^{-v}dv \\ &= \Gamma(1/2)\sqrt{2}\int_0^\infty v\left(\frac{v^{-1/2}e^{-v}}{\Gamma(1/2)}\right)dv\,. \end{aligned}$$

Note that the expression $\frac{v^{-1/2}e^{-v}}{\Gamma(1/2)}$ is the density function for a Gamma random variable V with parameters $\lambda = 1$ and $r = 1/2$. Since the above is the expectation for this Gamma random variable we see that it is equal to

$$\Gamma(1/2)\sqrt{2}\left(\frac{1/2}{1}\right) = \sqrt{\frac{\pi}{2}}\,.$$

To compute $\text{Var}(X)$ we use the fact that $\text{Var}(X) = E[X^2] - E[X]^2$. Computing $E[X^2]$ we have

$$\begin{aligned} E[X^2] &= \int_0^\infty x^3 e^{-x^2/2}dx = \int_0^\infty (\sqrt{2v})^3 e^{-v}\left(\frac{dv}{\sqrt{2v}}\right) = \int_0^\infty 2ve^{-v}dv \\ &= 2\int_0^\infty ve^{-v}dv\,. \end{aligned}$$

This last expression we recognize as two times the value of $E[V]$ if V is distributed as an exponential random variable with $\lambda = 1$. In that case $E[V] = \frac{1}{\lambda} = 1$. Thus $E[X^2] = 2$ and our variance is then

$$\text{Var}(X) = 2 - \frac{\pi}{2}\,.$$

Exercise 2 (Weibull failure rates)

Part (a): If X fits a Weibull distribution with $\alpha = 2$ then its failure rate function $h(x)$ is given by $h(x) = 2\lambda x$ and so $H(t) = \lambda t^2$ so that $R(t) = e^{-\lambda t^2}$. Considering the requested first difference we find

$$\begin{aligned}
\log(R(t+\Delta t)) - \log(R(t)) &= -\lambda(t+\Delta t)^2 + \lambda t^2 \\
&= -\lambda(t^2 + 2t\Delta t + \Delta t^2) + \lambda t^2 \\
&= -2\lambda t\Delta t - \lambda \Delta t^2\,,
\end{aligned}$$

which is linear in t.

Part (b): We are told that $P\{X < 100|X > 90\} = 0.15$, so $P\{X > 100|X > 90\} = 1 - 0.15 = 0.85$. Since the probability our component has a lifetime greater than t, is given by $1 - F(t) \equiv R(t)$ we see that

$$P\{X > 100|X > 90\} = \frac{P\{X > 100\}}{P\{X > 90\}} = \frac{R(100)}{R(90)} = e^{-\lambda(100^2 - 90^2)} = 0.85\,.$$

Solving for λ in the above we find $\lambda = 8.5536 \cdot 10^{-5}$, so $h(t) = 2\lambda t = 1.717 \cdot 10^{-4} t$.

Exercise 3 (electrical overloads)

Part (a): The probability the lifetime X is greater than t can be related to the given Poisson process. Now $P\{X > t\} = P\{N(t) < 2\}$, since our system functions as long as we have had less than two arrivals. This later expression becomes

$$P\{N(t) < 2\} = P\{N(t) = 0\} + P\{N(t) = 1\}\,,$$

and since $P\{N(t) = n\} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ the above becomes

$$P\{N(t) < 2\} = e^{-\lambda t} + (\lambda t)e^{-\lambda t} = (1 + \lambda t)e^{-\lambda t}\,,$$

thus

$$P\{X < t\} = 1 - P\{X > t\} = 1 - (1 + \lambda t)e^{-\lambda t}\,.$$

Thus our distribution function for our components lifetime X is given by

$$\begin{aligned}
f_X(t) &= \frac{dP\{X < t\}}{dt} \\
&= \lambda(1 + \lambda t)e^{-\lambda t} - \lambda e^{-\lambda t} = (\lambda + \lambda^2 t - \lambda)e^{-\lambda t} \\
&= \lambda^2 t e^{-\lambda t} = \lambda e^{-\lambda}\left(\frac{\lambda t}{\Gamma(2)}\right)\,.
\end{aligned}$$

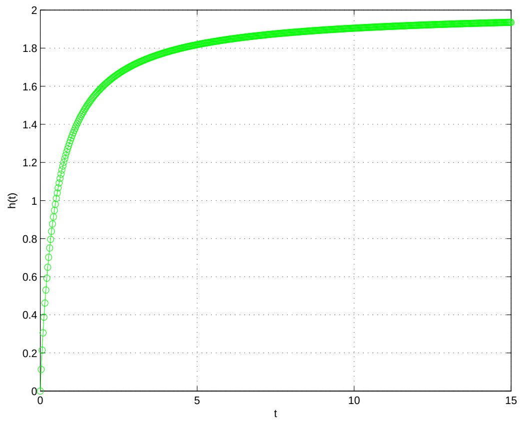

Figure 28: A plot of the failure rate function, $h(t)$, for Exercise 3. We have taken $\lambda = 2$ and notice how $h(t) \to \lambda$ as $t \to \infty$.

which is the Gamma distribution with parameters $(2, \lambda)$. Note that $\Gamma(2) = (2-1)! = 1$.

Part (b): The failure rate function $h(t)$ can be obtained from

$$h(t) = -\frac{d}{dt}\log(1 - F(t))\,.$$

We find in this case that

$$1 - F(t) = (1 + \lambda t)e^{-\lambda t}\,.$$

so that

$$h(t) = \frac{F'(t)}{1 - F(t)} = \frac{\lambda^2 t e^{-\lambda t}}{(1 + \lambda t)e^{-\lambda t}} = \frac{\lambda^2 t}{1 + \lambda t}\,.$$

To visualize this function let $\lambda = 2$ and we get the plot shown in Figure 28.

For large t we see that

$$h(t) = \frac{\lambda^2 t}{1 + \lambda t} \to \frac{\lambda^2}{\lambda} = \lambda\,,$$

so the failure rate is approximately constant for large times.

Exercise 4 (a modified Weibull distribution)

The reliability function, $R(t)$, is defined as $R(t) = 1 - F(t)$ where $F(t)$ is the cumulative distribution function. In terms of the failure rate function $h(t)$ we have

$$R(t) = 1 - F(t) = \exp\left\{-\int_0^t h(x)dx\right\} = \exp\left\{-\beta t + \frac{\lambda}{\alpha}t^\alpha\right\}\,,$$

for the cumulative reliability function.

Exercise 5 (the do-it-yourself car wash as a $M/G/1$ queue)

Example 11.1 was the example of the do-it-yourself car wash facility where there is only one spot for a customer and two activities to perform, washing and vacuuming. Once the initial spot is occupied, the facility will not be able to accept new customer until a time $X_1 + X_2$ (the sum of the time to wash and vacuum) has passed. After $X_1 + X_2$ we must wait an additional time Y for another customer to arrive. Here Y is an exponential random variable with rate λ. Thus, from the time the wash facility gets occupied we have to wait on average

$$E[X_1 + X_2 + Y] = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\lambda},$$

for the next customer.

Exercise 6 (an integral equation for the mean-value function $M(t)$)

Part (a): Since $N(t) = 0$ *before* the first renewal (which we know happens at the time x) we must have $N(t) \equiv 0$ when $t < x$. Thus $E[N(t)|X_1 = x] = 0$, when $t < x$. To compute $E[N(t)|X_1 = x]$ for $t > x$ we have

$$\begin{aligned}
E[N(t)|X_1 = x] &= E[1 + N(t - x)|X_1 = x] \\
&= 1 + E[N(t - x)|X_1 = x] \\
&= 1 + E[N(t - x)],
\end{aligned}$$

using the fact that all renewals that come after the first one are independent of when exactly that renewal took place. This later equation becomes

$$E[N(t)|X_1 = x] = 1 + M(t - x) \quad \text{when} \quad x \le t,$$

as we were to show.

Part (b): By conditioning on the time when the first renewal takes place we have

$$\begin{aligned}
M(t) &= E[N(t)] = \int_0^\infty E[N(t)|X_1 = x]f(x)dx \\
&= \int_0^t (1 + M(t - x))f(x)dx \\
&= \int_0^t f(x)dx + \int_0^t M(t - x)f(x)dx \\
&= F(t) + \int_0^t M(t - x)f(x)dx,
\end{aligned}$$

with $f(x)$ the density function and $F(x)$ the cumulative density function.

Part (c): Taking the derivative of the above integral equation (with $m(t) \equiv M'(t)$) gives

$$\begin{aligned}
m(t) &= f(t) + M(t - t)f(t) + \int_0^t m(t - x)f(x)dx \\
&= f(t) + \int_0^t m(t - x)f(x)dx, \qquad (50)
\end{aligned}$$

since $M(0) = 0$.

Part (d): If $m(t) = \lambda$ (a constant) then Equation 50 above becomes

$$\lambda = f(t) + \lambda \int_0^t f(x)dx\,. \tag{51}$$

If we differentiate this with respect to t we obtain

$$0 = f'(t) + \lambda f(t) \quad \text{or} \quad f'(t) = -\lambda f(t)\,.$$

Thus $f(t) = Ce^{-\lambda t}$. Putting this back into Equation 51 we find

$$\lambda = Ce^{-\lambda t} + \frac{\lambda C}{(-\lambda)} \left(e^{-\lambda x}\right|_0^t = Ce^{-\lambda t} - C(e^{-\lambda t} - 1) = C\,.$$

Thus $f(t) = \lambda e^{-\lambda t}$, or the density function for an exponential random variable.

Exercise 7 (the cumulative distribution for the cycle length in a $M/G/1$ queue))

Define the random variable Y to be the length of the cycle (total time of the busy period plus idle period). Then we want to compute $F(t) \equiv P\{Y < t\}$. Now Y can be decomposed as $Y = X + V$. Here X the random variable denoting the length of the service time with a distribution given by $G(t)$. While V is the random variable denoting the idle time which, because we assume that the arrivals are exponential, has a distribution given by an exponential random variable with rate λ. Then by conditioning on the length of the idle time V we have

$$\begin{aligned} F(t) &= P\{Y < t\} = P\{X + V < t\} \\ &= \int P\{X + v < t | V = v\} f_V(v) dv\,. \end{aligned}$$

Now the above becomes

$$F(t) = \int P\{X < t - v\} f_V(v) dv\,.$$

Here X is a non-negative random variable so the *largest* value V can take is t and our limits of the above integral become

$$F(t) = \int_0^t P\{X < t - v\} f_V(v) dv\,.$$

Using the known distribution and density functions for X and V respectively, we find

$$F(t) = \int_0^t G(t - v) \lambda e^{-\lambda v} dv\,, \tag{52}$$

the requested expression.

Exercise 8 (the distribution function for cycle times in a $M/M/1$ queue)

Part (a): From Exercise 7, the distribution of the length of the busy period plus the idle time i.e. the so called cycle time is given by Equation 52 above. If the service times are exponentially distributed with a rate μ, then the distribution function $G(t)$ takes the form $G(t) = 1 - e^{-\mu t}$, and Equation 52 becomes

$$\begin{aligned} F(t) &= \lambda \int_0^t (1 - e^{-\mu(t-x)}) e^{-\lambda x} dx \\ &= \lambda \left[\left. \frac{e^{-\lambda t}}{-\lambda} \right|_0^t - e^{-\mu t} \int_0^t e^{-(\lambda-\mu)x} dx \right] \\ &= \lambda \left[\frac{1 - e^{-\lambda t}}{\lambda} - e^{-\mu t} \left(\frac{e^{-(\lambda-\mu)t} - 1}{-(\lambda - \mu)} \right) \right] \\ &= \frac{1}{\lambda - \mu} \left(\lambda - \mu + \mu e^{-\lambda t} - \lambda e^{-\mu t} \right) . \end{aligned}$$

Thus the *density* function is given by the derivative of the above or

$$\begin{aligned} f(t) &= F'(t) = \frac{1}{\lambda - \mu} (-\mu \lambda e^{-\lambda t} + \lambda \mu e^{-\mu t}) \\ &= \left(\frac{\mu \lambda}{\mu - \lambda} \right) (e^{-\lambda t} - e^{-\mu t}) , \end{aligned}$$

as we were to show.

Part (b): If we attempt t take the limit $\mu \to \lambda$ we notice that we have an indeterminate limit of the type $0/0$ so we need to apply L'Hospital's rule to evaluate it. We find

$$\begin{aligned} \lim_{\mu \to \lambda} f(t) &= \lim_{\mu \to \lambda} \left(\frac{\mu \lambda (e^{-\lambda t} - e^{-\mu t})}{\mu - \lambda} \right) \\ &= \lim_{\mu \to \lambda} \frac{\lambda (e^{-\lambda t} - e^{-\mu t}) + \lambda \mu (t e^{-\mu t})}{1} = \lambda^2 t e^{-\lambda t} , \end{aligned}$$

which we obtain by taking the derivative with respect to μ on the "top and bottom". We notice that this last expression is a Gamma distribution with parameters $(2, \lambda)$. That we know this must be true follows from the fact that when $\mu = \lambda$ the length of a cycle is the sum of two exponential random variables with the same rate. Sums of this type are distributed as Gamma random variables.

Exercise 9 (bounds on the distribution function for $S_n = X_1 + X_2 + \cdots X_n$)

Part (a): Chebyshev's inequality for an arbitrary random variable X with mean μ and standard deviation σ is the statement that for all $\delta > 0$ we have

$$P\{|X - \mu| \geq \delta\} \leq \left(\frac{\sigma}{\delta} \right)^2 .$$

This above inequality implies (by considering the possible absolute values) that

$$P\{X - \mu \geq \delta\} \leq \left(\frac{\sigma}{\delta}\right)^2 \quad \text{and} \quad P\{X - \mu \leq -\delta\} \leq \left(\frac{\sigma}{\delta}\right)^2 .$$

For this problem we will consider our random variable to be the time of the n-th renewal or $S_n = \sum_{i=1}^n X_i$, so that $E[S_n] = n\tau$ and $\text{Var}(S_n) = n\sigma^2$. Using the second of these expressions we have that

$$P\{S_n - n\tau \leq -\delta\} \leq \frac{n\sigma^2}{\delta^2} ,$$

or

$$P\{S_n \leq -\delta + n\tau\} \leq \frac{n\sigma^2}{\delta^2} ,$$

by moving the tern $n\tau$ to the other side of the inequality. Now define $t = -\delta + n\tau$ so that $\delta = n\tau - t$ and the above becomes

$$P\{S_n \leq t\} \leq \frac{n\sigma^2}{(t - n\tau)^2} ,$$

since $P\{S_n \leq t\}$ *is* $F_n(t)$ we have the desired expression. Note also that as $\delta > 0$ this requires $n > t/\tau$.

Exercise 10 (more bounds on the distribution function for $S_n = X_1 + X_2 + \cdots X_n$)

Part (a): Consider the expression $P\{S_{k+m} \leq t | \sum_{i=1}^m X_i = x\}$. Then as before this equals

$$P\{S_k \leq t - x\} = F_k(t - x) ,$$

since if I tell you the value of $\sum_{i=1}^m X_i$ is x, then

$$S_{k+m} = \sum_{i=1}^m X_i + \sum_{i=m+1}^{m+k} X_i = x + \sum_{i=m+1}^{m+k} X_i ,$$

so that the probability $P\{S_{k+m} \leq t\}$ is equal to the probability $P\{x + \sum_{i=m+1}^{m+k} X_i \leq t\}$. Equivalently, that the sum of the k elements $X_{m+1}, X_{m+2}, \cdots , X_{m+k}$, are less than $t - x$. This is equivalent to $F_k(t - x)$. Since we know an expression for $P\{S_{k+m} \leq t | S_m = x\}$, we can compute $P\{S_{k+m} \leq t\}$ alone, by conditioning on the value of the random variable S_m. That is

$$\begin{aligned} F_{k+m}(t) &= P\{S_{k+m} \leq t\} = \int_0^t P\{S_{k+m} \leq t | S_m = x\} P\{S_m = x\} dx \\ &= \int_0^t F_k(t - x) f_m(x) dx , \end{aligned}$$

as we were to show.

Part (b): Now as all distribution functions are non-decreasing we know that $F_k(t - x) \leq F_k(t)$ and we have from Part (a) that

$$F_{k+m}(t) \leq \int_0^t F_k(t) f_m(x) dx = F_k(t) F_m(t) ,$$

as we were to show.

Part (c): Given the inequality derived in Exercise 9 from Chebyshev's inequality for a fixed t we have

$$F_r(t) \leq \frac{r\sigma^2}{(r\tau - t)^2} \to \frac{\sigma^2}{r\tau^2} \quad \text{as} \quad r \to +\infty \,.$$

Thus when t is fixed we can find an r large enough such that $F_r(t) \leq 1$ i.e. any integer r such that $r \gg O(\frac{\tau^2}{\sigma^2})$. Then using this value of r let any integer n be decomposed as $n = rq + k$ with the remainder term k such that $0 \leq k < r$ so that

$$F_n(t) = F_{rq+k}(t) \leq F_k(t) F_{rq}(t) \,,$$

using the result from Part (b).

Exercise 11 (stopping rules?)

Recall that a random variable N is a a stopping rule for the sequence of random variables $X_1, X_2, \cdots$ if for every integer $n \geq 0$ the event $\{N = n\}$ is independent of all the random variables that follow $X_{n+1}, X_{n+2}, \cdots$. Intuitively this says that our determination of when to stop at element n, can be made by only considering the samples X_i up to and including X_n. Thus once we have seen enough "events", we don't need to see any more to make our decision.

Part (a): Since the event $\{N = n\}$ only depends on the values of X_i for $i \leq n$ and not on any for $i > n$ this *is* a random stopping rule.

Part (b): Since the only two value for N' are $N' = 2$ and $N' = 3$, from which the choice of which to take is made after seeing the value of X_2 so this *is* a random stopping rule.

Part (c): Since in this case we prescribe $N'' = 1$ or a "stop" if $X_2 \neq 1$ this is *not* a random stopping rule. This is because to have the rule stop us at 1 i.e. $N'' = 1$ we cannot use any information from the sequence *after* X_1.

Exercise 12 (calculating $E[\sum_{j=1}^N X_j]$)

Wald's' identity is that if N is a random stopping rule for the sequence $\{X_j\}$ the elements of which have expectations given by $E[X_i] = \tau$ then

$$E[\sum_{j=1}^{N} X_j] = \tau \cdot E[N] \,.$$

Thus to use this theorem we need to be able to calculate both τ (the expectation of the renewal time i.e. $E[X_1]$) and $E[N]$, the expected number of renewals. Since X_i is Bernoulli the expected value of X_i is $E[X_i] = p$. The expected value of N depends on the specified random stopping rule and will be calculated in each specific case below.

Part (a): Using Wald's identity and the definition of N in this case we stop when we have received 5 successes. A random variable of this type is distributed as a negative-binomial with parameters $(5, p)$ and has an expectation given by $\frac{5}{p}$. Thus we find

$$E[\sum_{j=1}^{N} X_j] = p\left(\frac{5}{p}\right) = 5\,,$$

as we would expect. To calculate this directly we can condition the sum on the value taken for N. We have

$$E[\sum_{j=1}^{N} X_j] = E[\sum_{j=1}^{5} X_j | N = 5]P\{N = 5\} + E[\sum_{j=1}^{6} X_j | N = 6]P\{N = 6\} + \cdots\,.$$

Note we start our sum at $N = 5$ since sums with fewer terms than five are not possible under this stopping rule. This later expression becomes

$$5pP\{N = 5\} + 6pP\{N = 6\} + \cdot = p\left(\sum_{n=5}^{\infty} nP\{N = n\}\right)\,.$$

Now since $P\{N = n\}$ is given by a negative-binomial random with parameters $(5, p)$ and we recognize the above summation as the expected value of n when n is given by a negative-binomial random variable. Finally we have

$$E[\sum_{j=1}^{N} X_j] = p\left(\frac{5}{p}\right) = 5\,,$$

as before.

Part (b): In this case to use Wald's identity we need to calculate $E[N']$. We find

$$E[N'] = 2p + 3(1 - p) = 3 - p\,.$$

Thus

$$E[\sum_{j=1}^{N} X_j] = (3 - p)p\,.$$

To calculate this directly again by conditioning on the value of N' this can be computed as

$$\begin{aligned} E[\sum_{j=1}^{N'} X_j] &= E[\sum_{j=1}^{2} X_j | N' = 2]P\{N' = 2\} + E[\sum_{j=1}^{3} X_j | N' = 3]P\{N' = 3\} \\ &= 2p(p) + 3p(1 - p) = 3 - p\,, \end{aligned}$$

the same as before.

Part (c): We cannot use Wald's identity as before since the rule N'' is not a random stopping rule, but we maybe able to compute the expectation directly by conditioning

$$\begin{aligned} E[\sum_{j=1}^{N''} X_j] &= E[\sum_{j=1}^{2} X_j | N'' = 2] P\{N'' = 2\} + E[\sum_{j=1}^{1} X_j | N'' = 1] P\{N'' = 1\} \\ &= 2p(p) + p(1-p) = p^2 + p\,. \end{aligned}$$

Incidentally this *is* what Wald's identity would give also.

Exercise 13 (betting on Bernoulli outcomes)

Part (a): This is *not* a random stopping rule since you don't know when your first loss will happen. One can't use any future outcomes from the sequence to determine when to stop.

Part (b): Let W be your expected winnings. Then

$$E[W] = \sum_{i=0}^{\infty} w_i p_i = 0q + 1pq + 2p^2 q + \cdots\,,$$

which is the expectation of $K - 1$, where K is a geometric random variable with parameter p. Thus $E[W] = \frac{1}{p} - 1 = \frac{1-p}{p} = \frac{q}{p}$ dollars.

Wald's identity would be the statement that $E[\sum_{i=1}^{N} X_i] = E[N] \cdot E[X_i]$, with X_i the amount one wins on trial i. Thus

$$E[X_i] = 1p - 1q = 2p - 1\,,$$

and N the random variable denoting the number of trials one bets on. Thus

$$E[N] = 0q + 1pq + 2p^2 q + \cdots = \frac{q}{p}\,,$$

since this is the same sum as before. Thus $E[N]E[X_i] = \frac{q}{p}(2p-1)$ and is *not* equal to what we had before, showing that if N is not a random stopping rule one cannot use Wald's identity.

Exercise 14 (examples of the Renewal-Reward theorem)

Part (a): The elementary renewal theorem is the statement that over an interval of length t, the number of renewals is approximately t/τ so that the long term renewal rate is $1/\tau$ i.e. that

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = \frac{1}{\tau}\,.$$

Viewing Example 11.1 as a $M/G/1$ queue in the renewal framework we have that

$$\lambda_a = \lim_{t \to \infty} \frac{E[N(t)]}{t}\,,$$

with $\tau = E[X_i]$ the average time between renewals. This was calculated in Exercise 5 to be $\tau = \frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\lambda}$ so

$$\lambda_a = \frac{1}{\tau} = \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\lambda}} = \frac{\lambda}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}} .$$

Part (b): Defining $Y(t)$ to be a renewal process i.e. $Y(t) = \sum_{i=1}^{N(t)} R_i$, then for large t the expected total reward given by the Renewal-Reward theorem is $\frac{rt}{\tau}$. That is

$$\lim_{t\to\infty} \frac{E[Y(t)]}{t} = \frac{r}{\tau} .$$

So in the car-wash-example of Example 11.1 we have that $Y(t)$ is the total profit made up to and including time t. If we assume that each customer pays one dollar per hour of service time then $\lim_{t\to\infty} \frac{E[Y(t)]}{t}$ is the long term occupation percentage. From the renewal reward theorem this equals $\frac{r}{\tau}$, with $r = E[R_i]$. This is

$$r = E[R_i] = 1\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) ,$$

so the long term occupational percentage then becomes

$$\lim_{t\to\infty} \frac{E[Y(t)]}{t} = \frac{\frac{1}{\mu_1} + \frac{1}{\mu_2}}{\frac{1}{\mu_1} + \frac{1}{\mu_2} + \frac{1}{\lambda}} = \frac{\lambda\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right)}{1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}} .$$

Part (c): Now assuming an exponential model as in Example 11.1 we have (by defining $p_0 \equiv \left(1 + \frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2}\right)^{-1}$, that $\lambda_a = \lambda p_0$ which agrees with Part (a) and that

$$L = p_1 + p_2 = \frac{\lambda}{\mu_1} p_0 + \frac{\lambda}{\mu_2} p_0 ,$$

which agrees with Part (b).

Exercise 15 (a $M/G/1$ queue)

Part (a): Given that the customer places an order, the average service time $E[S]$ is given by

$$E[S] = \sum_s sp(s) = 2(0.5) + 4(0.4) + 6(0.1) = 3.2 ,$$

minutes.

Part (b): If we view this as a $M/G/1$ queue then the renewal time X is the time to first service the customer plus the time to wait until the next customer arrives. Thus

$$E[X] = \frac{1}{\lambda} + E[S] = 6 + 3.2 = 9.2 ,$$

minutes. Here we have used the fact that the average waiting time for the exponential distribution of the arrivals is $\frac{1}{\lambda}$.

Part (c): If we use the elementary renewal theorem we have

$$\lambda_a = \lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\tau} = \frac{1}{\left(\frac{1}{\lambda} + E[S]\right)} = \frac{1}{9.2} = 0.108\,,$$

customers per minute.

Exercise 16 (exponentially distributed flashers)

We are told to assume that $g_Y(y) = \lambda e^{-\lambda y}$, i.e. that the lifetime Y of our flasher is exponentially distributed. In that case our distribution function is $G(T) = 1 - e^{-\lambda T}$ and $L(T)$ the expected length of the replacement cycle is given by

$$L(T) = \int_0^T R(y)dy = \int_0^T (1 - G(y))dy = \int_0^T e^{-\lambda y}dy = \left.\frac{e^{-\lambda y}}{(-\lambda)}\right|_0^T = \frac{1}{\lambda}(1 - e^{-\lambda T})\,.$$

Then $C(T)$ our long term cost as described in the age-replacement model becomes

$$C(T) = \frac{C_1 + C_2 G(T)}{L(T)} = \frac{C_1 + C_2(1 - e^{-\lambda T})}{\frac{1}{\lambda}(1 - e^{-\lambda T})} = \lambda C_1 (1 - e^{-\lambda T})^{-1} + \lambda C_2\,.$$

To show that this function is a monotonically decreasing function of T we take the derivative of this expression. We find

$$C'(T) = -\lambda C_1 (1 - e^{-\lambda T})^{-2}(\lambda e^{-\lambda T}) = -\frac{\lambda^2 C_1 e^{-\lambda T}}{(1 - e^{-\lambda T})^2} < 0\,,$$

for all T.

Exercise 17 (uniformly distributed flashers)

Part (a): In this case our density and distribution functions for the flashers lifetime, Y, is given by

$$g(y) = \begin{cases} 1 & 1 < y < 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad G(y) = \begin{cases} 0 & y < 1 \\ y - 1 & 1 < y < 2 \\ 1 & y > 2 \end{cases}\,.$$

Next we compute the expected length of the replacement cycle, $L(T)$, when $1 < T < 2$ as

$$\begin{aligned} L(T) &= \int_0^T R(y)dy = \int_0^T (1 - G(y))dy = T - \int_0^T G(y)dy \\ &= T - \int_1^T (y - 1)dy = T - \left(\frac{1}{2}T^2 - T + \frac{1}{2}\right) \\ &= -\frac{1}{2}T^2 + 2T - \frac{1}{2}\,. \end{aligned}$$

Note that this expression is valid at the two boundary cases $T = 1$ and $T = 2$ also. For T in the range $1 < T < 2$ the average long term cost assuming a scheduled replacement cost of C_1 and an additional cost of C_2 for random failure is

$$C(T) = \frac{C_1 + C_2 G(T)}{L(T)} = \frac{C_1 + C_2(T-1)}{-\frac{1}{2}T^2 + 2T - \frac{1}{2}} .$$

Part (b): The value of T that minimizes this cost $C(T)$ is found by solving $C'(T) = 0$ which is

$$C'(T) = \frac{C_2}{-\frac{1}{2}T^2 + 2T - \frac{1}{2}} - \frac{(C_1 + C_2(T-1))(-T+2)}{\left(-\frac{1}{2}T^2 + 2T - \frac{1}{2}\right)^2} = 0 .$$

or

$$C_2 \left(-\frac{1}{2}T^2 + 2T - \frac{1}{2} \right) - (C_1 + C_2(T-1))(-T+2) = 0 .$$

when $C_1 = 200$ and $C_2 = 100$ we can solve this using the quadratic equation. Using Mathematica we find the positive root given by 1.449 (in units of years).

Part (c): The average cost under this policy is given by $C(1.449)$ and is given by 181.65 dollars per year.

Part (d): If we replace the flashers when they randomly fail we find $C(2) = 200$, so the policy in Part (c) is slightly better.

Some simple calculations for this problem can be found in the Mathematica file `chap_12_prob_17.nb`.

Exercise 18 (adding a salvage value)

Part (a): Assuming we routinely replace our flasher after T amount of time and that a flasher that has not expired when replaced has a salvage value of $10(2-T)$. Given this, we can calculate a new expected cost per cycle depending on whether or not our flasher has actually expired when we replace it. Defining Y to be the random variable denoting the lifetime of the given flasher we find

$$\begin{aligned} E[\text{Cost of a cycle}] &= (C_1 - 10(2-T))P\{Y > T\} + (C_1 + C_2)P\{Y \leq T\} \\ &= (C_1 - 10(2-T))(1 - G(T)) + (C_1 + C_2)G(T) . \end{aligned}$$

Here we have subtracted the salvage value of the flasher from the routine maintenance trip cost C_1 when the flasher is still working at the time of replacement i.e. when $P\{Y > T\}$. Then the expected long term cost rate (defined as $C(T)$) in this case is given by

$$\begin{aligned} C(T) &\equiv E[\text{Long term cost rate}] = \frac{E[\text{Cost of a cycle}]}{E[\text{length of a cycle}]} \\ &= \frac{(C_1 - 10(2-T))(1 - G(T)) + (C_1 + C_2)G(T)}{L(T)} , \end{aligned}$$

with $G(T)$ and $L(T)$ as in Exercise 17. In that case the above becomes

$$C(T) = \frac{(C_1 - 10(2-T))(2-T) + (C_1 + C_2)(T-1)}{-\frac{1}{2}T^2 + 2T - \frac{1}{2}}.$$

Part (b): To minimize $C(T)$ we solve the equation $C'(T) = 0$ for T. Taking this derivative and solving the resulting equation for T is done in the Mathematica file `chap_12_prob_18.nb`. When we do this we find that our solution is given by $T = 1.3712$ years. This is a *smaller* time than found in Exercise 17. Which makes sense because we obtain some benefit when the flasher is still working when we replace it.

Part (c): The average cost under this policy is given by $C(1.3712)$ and is given by 179.04 dollars per year. This again is a smaller cost than what we found in Exercise 17 as would be expected.

Exercise 19 (taking ticket reservations)

Part (a): If we define X_i to be the length of time required for the operator to service the ith call, then the total time the operator spends servicing calls up to time t is given by $Y(t) = \sum_{i=1}^{N(t)} X_i$, where $N(t)$ is a Poisson process with rate λ. The long time *proportion* of time the operator is busy is given by $\lim_{t\to\infty} \frac{E[Y(t)]}{t}$, which can be calculated with the Renewal-Reward theorem since

$$\lim_{t\to\infty} \frac{E[Y(t)]}{t} = \frac{r}{\tau} = \frac{E[X_i]}{1/\lambda}.$$

In the case here, $E[X_i] = 20$ seconds, since X_i is given by a uniform random variable distributed between $[15, 25]$ seconds. Since the incoming rate is $\lambda = 2$ calls per minute the expression above becomes

$$\frac{E[X_i]}{1/\lambda} = \frac{20}{60/2} = \frac{2}{3}.$$

Part (b): If under steady-state conditions $2/3$ of the time the operator is busy then $1 - (2/3) = 1/3$ of the time the operator is free and all received calls will not be put on hold.

Exercise 21 (bus trip costs)

Using the variables from the dispatch shuttle bus example in the book we see that $a = 10$ and $c = 20$, while the largest N can be is 10 since that is the capacity of a bus. Then to find a range of values for the average interarrival time, τ, requires $N_{\text{min}} > 10$ or

$$\left(\frac{2a}{\tau c}\right)^{1/2} > 10.$$

When we put in a and c from above and solve for τ we find $\tau < 0.01$ hours or $\tau < 3/5$ minutes. **Note that this is different than the result given in the back of the book**.

Exercise 22 (the optimal number of customers to wait for)

For the average long-term cost rate given in the dispatching shuttle bus example we found

$$F(N) = \frac{a}{N\tau} + \frac{c(N-1)}{2},$$

so that the derivative of this expression is given by

$$F'(N) = -\frac{a}{N^2\tau} + \frac{c}{2}.$$

Setting this equal to zero and solving for N we find

$$\frac{a}{N^2\tau} = \frac{c}{2} \Rightarrow N = +\sqrt{\frac{2a}{\tau c}}.$$

Exercise 23 (a shuttle bus system)

Part (a): Since a bus will only depart once it is filled and that there is space for ten passengers each of which arrives after the previous one at a time given by an exponential random variable Y_i (with rate λ) each bus will leave after a time $\sum_{i=1}^{10} Y_i$. Sums of this type are given by Gamma random variables with parameters $(10, \lambda)$.

Part (b): Relating the numbers given to the shuttle bus example we have $c = 10$, $a = 20$, and $N = 10$ so the average long term cost rate is given by

$$\frac{a}{N\tau} + \frac{c(N-1)}{2} = 2\lambda + 45,$$

when $\tau = \frac{1}{\lambda}$. So to operate for 1000 hours would require a cost of

$$2000\lambda + 45000,$$

dollars.

Exercise 24 (the expected light bulb lifetime)

Following the discussion in the section entitled lifetime sampling and the inspection paradox, the expected value for A the random variable denoting the age of our component is given as

$$E[A] = \frac{\tau}{2} + \frac{\sigma^2}{2\tau},$$

with $\tau = E[X]$ the expected lifetime and $\sigma^2 = \mathrm{Var}(X)$ the variance of the lifetime. From the light bulbs described in problem 1 we have calculated both of these and found $\tau = \sqrt{\frac{\pi}{2}} = 1.2533$ and $\mathrm{Var}(X) = 2 - \frac{\pi}{2} = 0.429$ so that $E[A] = 0.7979$.

Exercise 25 (the car batteries of Mr. Jones)

We can model this problem as in the section "Lifetime Sampling and the Inspection Paradox". In this case we want to evaluate $P\{A \leq 1\}$. As shown there this is given by

$$P\{A \leq c\} = \frac{1}{\tau} \int_0^c R(x) dx \,.$$

Now $\tau = E[X]$ and $R(x)$ is the reliability function for the random variable X the lifetime of a battery.

Part (a): In this case $\tau = E[X] = 1$ and $R(x) = 1 - F(x) = 1 - \frac{1}{2}x$, so that

$$P\{A \leq c\} = \frac{1}{1} \int_0^c \left(1 - \frac{1}{2}x\right) dx = c - \left.\frac{x^2}{4}\right|_0^c = c - \frac{c^2}{4} \,.$$

When $c = 1$ this gives $P\{A \leq 1\} = 1 - \frac{1}{4} = \frac{3}{4}$.

Part (b): In this case $\tau = 1$ while $R(x) = 1 - F(x) = 1 - (1 - e^{-x}) = e^{-x}$, so

$$P\{A \leq c\} = 1 \int_0^c e^{-x} dx = 1 - e^{-c} \,.$$

Thus $P\{A \leq 1\} = 1 - e^{-1} = 0.632$.

Exercise 26 (how long will this battery last?)

The expected remaining lifetime of the battery currently in use is computed in the section on "Lifetime Sampling and the Inspection Paradox". There we find that the expected remaining lifetime, $E[Z]$, is given by

$$E[Z] = \frac{\tau}{2} + \frac{\sigma^2}{2\tau} \,.$$

When X is distributed as a uniform random variable over $(0, 2)$ we have $\tau = 1$ and $\sigma^2 = \frac{(2-0)^2}{12} = \frac{1}{3}$ so

$$E[Z] = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \,,$$

of a year. The expected total lifetime of the batter in current use is

$$\tau + \frac{\sigma^2}{\tau} = 1 + \frac{1}{3} = \frac{4}{3} \,,$$

of a year.

Exercise 27 (a numerical illustration of the inspection paradox)

Part (a): We compute the direct average of the X_i's to be $\frac{1}{5}\sum_{i=1}^5 X_i = 2.6580$.

Part (b): We can approximate the integral $\int_0^T A(t)dt$ by recognizing that it represents the area under the $A(t)$ graph. The $A(t)$ function is a sawtooth graph where each full sawtooth's is a triangle with a base and height of equal length. Thus each full triangle has an area of $A_i = \frac{1}{2}X_i^2$. The *last* (fifth) sawtooth is not complete in this example and has an area given by (see Figure 12.4 in the book) of

$$U_4 \equiv \frac{1}{2}(T - S_4)^2 = \frac{1}{2}\left(T - \sum_{i=1}^{4} X_i\right)^2 = 1.3613\,.$$

Thus we find

$$\frac{1}{T}\int_0^T A(t)dt = \frac{1}{11}\left(\sum_{i=1}^{4}\frac{1}{2}X_i^2 + U_4\right) = 1.2697\,.$$

Part (c): In the same way as in Part (b) counting each sawtooth of the function $Z(t)$ as having an area given by the triangle it subtends we find

$$\frac{1}{T}\int_0^T Z(t)dt = \frac{1}{11}\left(\sum_{i=1}^{4}\frac{1}{2}X_i^2 + \frac{1}{2}(T - S_4)(X_5 + X_5 - T + S_4)\right) = 1.6132\,.$$

Here the last uncompleted "triangle" (corresponding to the lifetime of the fifth component) is actually a trapezoid and has an area given by the appropriate formula. In this case the trapezoid is denoted by the following four points (going clockwise) $(S_4, 0)$, (S_4, X_4), $(T, X_5 - (T - S_4))$, and $(T, 0)$. Thus if we take the height of this trapezoid to be the line between the points $(S_4, 0)$ and $(T, 0)$, the first "base" of this trapezoid to be the line between the points $(S_4, 0)$ and (S_4, X_5), and the second base of the trapezoid to be the distance between the points $(T, X_5 - T + S_4)$ and $(T, 0)$ we see that our trapezoid has an area given by

$$V_4 \equiv \frac{1}{2}(T - S_4)(X_5 + (X_5 - T + S_4)) = \frac{1}{2}(T - S_4)(X_5 + S_5 - T) = 5.1398\,.$$

Exercise 28 (a analytical illustration of the inspection paradox)

Part (a-b): See Problem 27 where we do this calculation for a specific number of arrivals. The calculation is the same for an arbitrary number of them.

Part (c): From the Renewal-Reward theorem if we take rewards given by $R_n = X_n^2$ then for large t the expected total reward up to time t is approximately $\frac{tr}{\tau}$ that is

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{N(t)} X_i^2 = \frac{E[X_i^2]}{E[X_i]}\,.$$

Thus since as discussed in the hint $U_N = (T - S_n)^2 \le (S_{N+1} - S_N)^2 =\le X_{N+1}^2$ we see that

$$\frac{1}{T}\int_0^T A(t)dt = \frac{1}{2T}\sum_{i=1}^{N(T)} X_i^2 + \frac{U_N}{2T} \le \frac{1}{2T}\sum_{i=1}^{N(T)+1} X_i^2 \,,$$

so taking the limit as T goes to infinity in the above expression we find

$$\lim_{T\to\infty} E\left[\frac{1}{T}\int_0^T A(t)dt\right] \le \lim_{T\to\infty}\frac{1}{2T}E\left[\sum_{i=1}^{N(T)+1} X_i^2\right] = \frac{1}{2}\left(\frac{E[X_i^2]}{E[X_i]}\right) \,.$$

as desired to be shown.

Exercise 29 (W_Q as a function of ρ for various $M/G/1$ queues)

These specifications follow from the section on waiting times for the $M/G/1$ queue where the book calculates

$$W_Q = \frac{\rho}{1-\rho}\left(\frac{\tau}{2} + \frac{\sigma^2}{2\tau}\right) \,. \tag{53}$$

Part (a): For this distribution of X we have $\tau = \rho$ and $\sigma^2 = \rho^2$ so that W_Q as a function of ρ becomes

$$W_Q = \frac{\rho}{1-\rho}\left(\frac{\rho}{2} + \frac{\rho^2}{2\rho}\right) = \frac{\rho^2}{1-\rho} \,.$$

Part (b): For this distribution of X we have $\tau = \rho$ and $\sigma^2 = \frac{(2\rho-0)^2}{12} = \frac{\rho^2}{3}$ so that W_Q as a function of ρ becomes

$$W_Q = \frac{\rho}{1-\rho}\left(\frac{\rho}{2} + \frac{\rho^2}{3(2\rho)}\right) = \frac{2}{3}\left(\frac{\rho^2}{1-\rho}\right) \,.$$

Part (c): For this distribution of X we have $\tau = n/\lambda = 2/(2/\rho) = \rho$ and $\sigma^2 = \frac{n}{\lambda^2} = \frac{2}{(2/\rho)^2} = \frac{\rho^2}{2}$ so that W_Q as a function of ρ becomes

$$W_Q = \frac{\rho}{1-\rho}\left(\frac{\rho}{2} + \frac{\rho^2}{2(2\rho)}\right) = \frac{3}{4}\left(\frac{\rho^2}{1-\rho}\right) \,.$$

Exercise 30 (the expression for L_Q, W, and L for a $M/G/1$ queue)

Part (a): Equation 53 gives the expression for W_Q for a $M/G/1$ queue so that from Little's law we have

$$L_Q = \lambda W_Q = \frac{\lambda\rho}{1-\rho}\left(\frac{\tau}{2} + \frac{\sigma^2}{2\tau}\right) \,. \tag{54}$$

Since $W = W_Q + \frac{1}{\mu} = W_Q + \tau$ again using Equation 53 we have

$$W = \frac{\rho}{1-\rho}\left(\frac{\tau}{2} + \frac{\sigma^2}{2\tau}\right) + \tau\,. \tag{55}$$

Using Little's formula we find that $L = \lambda W$, since $\lambda\tau = \rho$ this is

$$L = \frac{\lambda\rho}{1-\rho}\left(\frac{\tau}{2} + \frac{\sigma^2}{2\tau}\right) + \rho\,. \tag{56}$$

Part (b): Under the assumption of exponential service times $\tau = \frac{1}{\mu}$ and $\sigma^2 = \tau^2 = \frac{1}{\mu^2}$ so each of the above expression simplifies. We first find that the subexpression $\left(\frac{\tau}{2} + \frac{\sigma^2}{2\tau}\right)$, which appears in every expression above as

$$\frac{\tau}{2} + \frac{\sigma^2}{2\tau} = \frac{\tau}{2} + \frac{\tau}{2} = \tau = \frac{1}{\mu}\,.$$

Using this, the requested expressions become

$$\begin{aligned}
L_Q &= \frac{\lambda\rho}{1-\rho}\left(\frac{1}{\mu}\right) = \frac{\rho^2}{1-\rho} \\
W &= \frac{\rho}{1-\rho}\left(\frac{1}{\mu}\right) + \frac{1}{\mu} = \frac{1}{(1-\rho)\mu} \\
L &= \frac{\lambda\rho}{1-\rho}\left(\frac{1}{\mu}\right) + \rho = \frac{\rho}{1-\rho}\,,
\end{aligned}$$

all of which agree with the results from a $M/M/1$ queue derived earlier.

Exercise 31 (statistics of particular $M/G/1$ queues)

Part (a): Following the discussion in the section entitled "Regeneration Points" we define B the length of a busy period and calculated $E[B] = \frac{1}{\mu(1-\rho)} = \frac{\tau}{1-\rho}$. With the given uniform distribution for the servers we find this becomes

$$E[B] = \frac{(\epsilon/2)}{1-\lambda(\epsilon/2)} = \frac{(\epsilon/2)}{1-\epsilon} = \frac{\epsilon}{2(1-\epsilon)}\,,$$

when $\lambda = 2$.

Part (b): From Equation 24 since for X distributed as a uniform random variable over $(0,\epsilon)$ that $\tau = \frac{\epsilon}{2}$ and $\sigma^2 = \frac{(\epsilon-0)^2}{12} = \frac{\epsilon^2}{12}$, $\rho = \frac{\lambda}{\mu} = \lambda\tau = \frac{\lambda\epsilon}{2}$ so

$$W_Q = \frac{(\lambda\epsilon/2)}{(1-(\lambda\epsilon/2))}\left(\frac{\epsilon}{4} + \frac{\epsilon^2}{12(2(\epsilon/2))}\right) = \frac{\lambda\epsilon^2}{3(2-\lambda\epsilon)}\,,$$

with $\lambda = 2$ this becomes

$$W_Q = \frac{\epsilon^2}{3(1-\epsilon)}\,.$$

Part (c): From Little's law or Equation 54 in Exercise 30 above we find that

$$L_Q = \lambda W_Q = \frac{2\epsilon^2}{3(1-\epsilon)} \,.$$

References

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