# Solutions To Selected Exercises In: Finite-difference Equations and Simulations by Francis Begnaud Hildebrand

John L. Weatherwax*

February 10, 2008

## Relations among difference operators (Page 7)

All of these relations among difference operators can be proved by expansion. For instance to prove that $\delta^2 = \Delta^2$ we expand the operator $\delta^2$ we get

$$\begin{aligned}\delta^2 y_k &= \delta(y_{k+1/2} - y_{k-1/2}) && (1)\\ &= (y_{k+1} - y_k) - (y_k - y_{k-1}) && (2)\\ &= (y_{k+1} - 2y_k + y_{k-1}) = \Delta^2 y && (3)\end{aligned}$$

## Formulation of difference equations example (Page 10-11)

$$\bar{T}\{(y_{k+1} - y_k) - (y_k - y_{k-1})\} + hf_k = 0 \tag{4}$$

If we know $y_k$ for $k = 0, 1, \ldots, N+1$

$$\frac{\Delta y_k}{h} = \tan\phi_k \tag{5}$$

$$\sqrt{\tan\phi^2 + 1} = ((\sec\phi)^2)^{1/2} = \sec\phi \tag{6}$$

$T_k = \bar{T}\sec\phi$

$$T_k = \bar{T}\sqrt{1 + \left(\frac{\Delta y_k}{h}\right)^2} \tag{7}$$

*wax@alum.mit.edu

which is Eq. 35.

$$\frac{T_k}{h}(y_{k+1} - y_k) - \frac{T_{k-1}}{h}(y_k - y_{k-1}) + f_k = 0 \tag{8}$$
$$T_k(y_{k+1} - y_k) - T_{k-1}(y_k - y_{k-1}) + hf_k = 0 \tag{9}$$
$$T_k \Delta y_k - T_{k-1}\Delta y_{k-1} + hf_k = 0 \tag{10}$$
$$\Delta(T_{k-1}\Delta y_{k-1}) + hf_k = 0 \tag{11}$$

$f_k = p_k \Delta s_k$

# Page 13

$$I_{k+1} - 2\cos\phi I_k + I_{k-1} = 0 \tag{12}$$

Substituting the canonical solution $I_k = \lambda^k$ into the above difference equation gives a characteristic equation of

$$\lambda^2 - 2\cos\phi\lambda + 1 = 0 \tag{13}$$

This quadratic equation has a solution of

$$\lambda = \frac{2\cos\phi \pm \sqrt{4\cos\phi^2 - 4}}{2} \tag{14}$$
$$= \cos\phi \pm i\sqrt{1 - \cos\phi^2} \tag{15}$$
$$= \cos\phi \pm i|\sin\phi| \tag{16}$$

If $0 \leq \phi \leq \pi$, then $\sin\phi > 0$ and the above becomes

$$\lambda = \cos\phi \pm i\sin\phi \equiv e^{i\phi}, e^{-i\phi} \tag{17}$$

Therefore the solution to $I_k$ is

$$I_k = Ae^{ik\phi} + Be^{-ik\phi} \tag{18}$$

with two arbitrary constants $A$ and $B$ to be determined by the initial conditions. The initial conditions for $I_0$ and $I_1$ give

$$I_0 = \int_0^\pi 0\, d\theta = 0 \tag{19}$$
$$I_1 = \int_0^\pi 1\, d\theta = \pi \tag{20}$$

Thus the system to be solved to obtain $A$ and $B$ is

$$A + B = 0 \tag{21}$$
$$Ae^{i\phi} + Be^{-i\phi} = \pi \tag{22}$$

Solving the first equation for $A$ and substituting it into the second gives

$$-Be^{i\phi} + Be^{-i\phi} = \pi \tag{23}$$

converting to trigonometric functions gives

$$B\left[-\cos(\phi) - i\sin(\phi) + \cos(\phi) - i\sin(\phi)\right] = \pi \tag{24}$$

which simplifies to

$$-B\left[2i\sin(\phi)\right] = \pi \tag{25}$$

and finally gives

$$B = \frac{\pi i}{2\sin(\phi)} \tag{26}$$

Thus the total solution to $I_k$ is then

$$\begin{aligned} I_k &= \frac{\pi i}{2\sin(\phi)}\left[-e^{ik\phi} + e^{-ik\phi}\right] & (27) \\ &= \frac{\pi \sin(k\phi)}{\sin(\phi)} & (28) \end{aligned}$$

# Chapter 1

## Problem 1

**Part (a):** From the given difference equation we have

$$y_{k+1} = 2y_k\cos(\alpha) - y_{k-1} \qquad \text{for} \qquad k = 1, 2, 3, \ldots \tag{29}$$

Thus with the initial conditions given $y_0 = 0$ and $y_1 = 1$ we obtain via iteration that

$$\begin{aligned} y_2 &= 2\cos(\alpha) & (30) \\ y_3 &= 2\cos(\alpha)(2\cos(\alpha)) - 1 = 4\cos(\alpha)^2 - 1 & (31) \\ y_4 &= 2\cos(\alpha)(4\cos(\alpha)^2 - 1) - 2\cos(\alpha) = 8\cos(\alpha)^3 - 4\cos(\alpha) & (32) \end{aligned}$$

**Part (b):** To show that

$$y_{k+1} = \frac{\sin(k\alpha)}{\sin(\alpha)}$$

satisfies the difference equation we compute

$$\begin{aligned} y_{k+1} &= \frac{\sin((k+1)\alpha)}{\sin(\alpha)} & (33) \\ &= \frac{\sin(k\alpha)\cos(\alpha) + \cos(k\alpha)\sin(\alpha)}{\sin(\alpha)} & (34) \\ &= \frac{\sin(k\alpha)\cos(\alpha)}{\sin(\alpha)} + \cos(k\alpha) & (35) \\ &= y_k\cos(\alpha) + \cos(k\alpha) & (36) \end{aligned}$$

In a similarly way

$$y_{k-1} = \frac{\sin(k\alpha)\cos(\alpha)}{\sin(\alpha)} - \cos(k\alpha) = y_k \cos(\alpha) - \cos(k\alpha) \tag{37}$$

Therefore, the left hand side of the given difference equation

$$y_{k+1} - 2\cos(\alpha) y_k + y_{k-1}$$

becomes

$$y_k \cos(\alpha) + \cos(k\alpha) - 2\cos(\alpha) y_k + y_k \cos(\alpha) - \cos(k\alpha) = 0 \tag{38}$$

From the explicit expression for $y_k$ we calculate

$$\begin{aligned}
y_0 &= 0 & (39)\\
y_1 &= 1 & (40)\\
y_2 &= \frac{\sin(2\alpha)}{\sin(\alpha)} = \frac{2\sin(\alpha)\cos(\alpha)}{\sin(\alpha)} = 2\cos(\alpha) & (41)\\
y_3 &= \frac{\sin(3\alpha)}{\sin(\alpha)} = \frac{\sin(2\alpha+\alpha)}{\sin(\alpha)} = \frac{\sin(2\alpha)\cos(\alpha) + \sin(\alpha)\cos(2\alpha)}{\sin(\alpha)} & (42)\\
&= \frac{2\sin(\alpha)\cos(\alpha)^2 + \sin(\alpha)(2\cos(\alpha)^2 - 1)}{\sin(\alpha)} & (43)\\
&= 2\cos(\alpha)^2 + 2\cos(\alpha)^2 - 1 = 4\cos(\alpha)^2 - 1 & (44)\\
y_4 &= \frac{\sin(4\alpha)}{\sin(\alpha)} = \frac{2\sin(2\alpha)\cos(2\alpha)}{\sin(\alpha)} = \frac{4\sin(\alpha)\cos(\alpha)(2\cos(\alpha)^2 - 1)}{\sin(\alpha)} & (45)\\
&= 8\cos(\alpha)^3 - 4\cos(\alpha) & (46)
\end{aligned}$$

Where in the above we have used the trigonometric identity

$$\cos(2\alpha) = 2\cos(\alpha)^2 - 1$$

## Problem 2

**Part (a):** Given the difference equation

$$y_{k+1} - \lambda y_k + y_{k-1} = 0 \qquad \text{for} \qquad k = 1, 2, 3 \tag{47}$$

and boundary conditions of $y_0 = 0$ and $y_4 = 0$ we begin by evaluating the difference equation at $k = 1, 2, 3$ to produce a linear system of equations for the unknown $y_k$'s. We obtain

$$\begin{aligned}
y_2 - \lambda y_1 &= 0 \qquad \text{when} \qquad k = 1 & (48)\\
y_3 - \lambda y_2 + y_1 &= 0 \qquad \text{when} \qquad k = 2 & (49)\\
-\lambda y_3 + y_2 &= 0 \qquad \text{when} \qquad k = 3 & (50)
\end{aligned}$$

or in matrix format

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{51}$$

To guarantee a nontrivial solution to this system we must have the determinant of the coefficient matrix vanish. Taking this determinant we obtain

$$2\lambda - \lambda^3 = 0 \tag{52}$$

which has solutions $\lambda = 0$ and $\lambda = \pm\sqrt{2}$.

**Part (b):** Given the expression

$$\lambda_n = 2\cos(\frac{n\pi}{4})$$

we can immediately check that for $n = 1, 2, 3$ the same values as in part (a) are obtained. Evaluating this expression at each value of $n$ we obtain the requested correspondence.

$$\begin{aligned} \lambda_1 &= \sqrt{2} && (53) \\ \lambda_2 &= 0 && (54) \\ \lambda_3 &= -\sqrt{2} && (55) \\ & && (56) \end{aligned}$$

To check the proposed solutions

$$y_k^{(n)} = \sin(\frac{n\pi k}{4}) \quad \text{for} \quad n = 1, 2, 3$$

We compute

$$y_{k+1}^{(n)} = \sin(\frac{n\pi k}{4})\cos(\frac{n\pi}{4}) + \cos(\frac{n\pi k}{4})\sin(\frac{n\pi}{4}) \tag{57}$$

$$y_{k-1}^{(n)} = \sin(\frac{n\pi k}{4})\cos(\frac{n\pi}{4}) - \cos(\frac{n\pi k}{4})\sin(\frac{n\pi}{4}) \tag{58}$$

Adding the two equations above we get

$$y_{k+1}^{(n)} + y_{k-1}^{(n)} = 2\cos(\frac{n\pi}{4})\sin(\frac{n\pi k}{4}) \tag{59}$$

Which we recognize as $+\lambda y_k^{(n)}$ showing that the proposed $y_k^{(n)}$ are indeed nontrivial solutions.

## Problem 4

**Part (a):** Show

$$2^s = 1 + \frac{s}{1!} + \frac{s(s-1)}{2!} + \ldots + \frac{s(s-1)(s-2)\ldots(s-r+1)}{r!} \tag{60}$$

Consider an application of Newton's formula for polynomial interpolation for the function $f(s) = 2^s$.

$$2^s = 2^0 + \frac{s}{1!}\Delta(2^s)|_{s=0} + \frac{s(s-1)}{2!}\Delta^2 2^s|_{s=0} + \frac{s(s-1)(s-2)}{3!}\Delta^3 2^s|_{s=0} + \ldots \tag{61}$$

Now

$$\Delta 2^s|_{s=0} = 2^{s+1} - 2^s|_{s=0} = 2 - 1 = 1 \tag{62}$$

and

$$\Delta^2 2^s|_{s=0} = \Delta(2^{s+1} - 2^s)|_{s=0} = 2^{s+2} - 2^{s+1} - 2^{s+1} + 2^s|_{s=0} = 4 - 4 + 1 = 1 \tag{63}$$

We can further investigate the forward difference of this function with the following difference table

| s | $2^s$ | $\Delta 2^s$ | $\Delta^2 2^s$ | $\Delta^3 2^s$ | $\Delta^4 2^s$ | $\Delta^5 2^s$ |
|---|---|---|---|---|---|---|
| 0 | 1 | | | | | |
| | | 1 | | | | |
| 1 | 2 | | 1 | | | |
| | | 2 | | 1 | | |
| 2 | 4 | | 2 | | 1 | |
| | | 4 | | 2 | | 1 |
| 3 | 8 | | 4 | | 2 | |
| | | 8 | | 4 | | 2 |
| 4 | 16 | | 8 | | 4 | |
| | | 16 | | 8 | | |
| 5 | 32 | | 16 | | | |
| | | 32 | | | | |
| 6 | 64 | | | | | |

Based on the above, I claim that $\Delta^k 2^s|_{s=0} = 1$ for every $k$. We can prove this by induction. Above we have shown that it is true for $k = 1$. The induction hypothsis is that we assume it is true up to and including $K$. The proof will be complete if we can show that the hypothsis holds if for $k = K + 1$. We have (to be evaluated at $s = 0$)

$$\Delta^{K+1} 2^s = \Delta^K \Delta 2^s = \Delta^K 2^{s+1} - \Delta^K 2^s = 2\Delta^K 2^s - \Delta^K 2^s = 2 - 1 = 1 \tag{64}$$

therefore Newtons formula for polynomial interpolation in this case becomes

$$2^s = 1 + \frac{s}{1} + \frac{s(s-1)}{2!} + \frac{s(s-1)(s-2)}{3!} + \dots \frac{s(s-1)(s-2)\dots(s-r+1)}{r!} + \dots \tag{65}$$

Which is what we were trying to show

**Part (b):** To obtain the polynomial $p(s)$ of degree three that agrees with $f(s) = 2^s$ for $s = 0, 1, 2, 3$ we take the first four terms of the Newton polynomial. This is

$$2^s \approx 1 + s + \frac{s(s-1)}{2} + \frac{s(s-1)(s-2)}{6} \tag{66}$$

Expanding out each term we obtain

$$2^s \approx 1 + \frac{5s}{6} + \frac{s^3}{6} \tag{67}$$

We can check that this expression aggrees with $2^s$ for the four $s$ values specified above.

$$f(0) = 1 \tag{68}$$
$$f(1) = 2 \tag{69}$$
$$f(2) = 4 \tag{70}$$
$$f(3) = 8 \tag{71}$$
$$f(4) = 15 \neq 16 \tag{72}$$

## Problem 5

**Part (a):** To expand $\frac{1}{s+1}$ in terms of a polynomial in $s$ that agrees with this function for the integers we must compute

$$\Delta^k \frac{1}{s+1} \tag{73}$$

for all integers $k$. Based on the following difference table

| s | $\frac{1}{s+1}$ | $\Delta \frac{1}{s+1}$ | $\Delta^2 \frac{1}{s+1}$ | $\Delta^3 \frac{1}{s+1}$ | $\Delta^4 \frac{1}{s+1}$ |
|---|---|---|---|---|---|
| 0 | 1 | | | | |
| | | $-\frac{1}{2}$ | | | |
| 1 | $\frac{1}{2}$ | | $\frac{1}{3}$ | | |
| | | $-\frac{1}{6}$ | | $-\frac{1}{4}$ | |
| 2 | $\frac{1}{3}$ | | $\frac{1}{12}$ | | $\frac{1}{5}$ |
| | | $-\frac{1}{12}$ | | $-\frac{1}{20}$ | |
| 3 | $\frac{1}{4}$ | | $\frac{1}{20}$ | | |
| | | $-\frac{1}{20}$ | | | |
| 4 | $\frac{1}{5}$ | | | | |

we hypothosize that

$$\Delta^k \frac{1}{s+1}|_{s=0} = \frac{(-1)^k}{k+1} \qquad \text{for} \qquad k = 0, 1, 2, \ldots \tag{74}$$

We can prove this by recognizing that

$$\frac{1}{s+1} = s^{(-1)} \tag{75}$$

where $s^{(-1)}$ is the factorial function. Which has difference properties [?] (Page 20) just like differentials

$$\Delta s^{(-1)} = (-1)s^{(-2)} \tag{76}$$
$$\Delta^2 s^{(-1)} = (-1)(-2)s^{(-3)} \tag{77}$$
$$\Delta^3 s^{(-1)} = (-1)(-2)(-3)s^{(-4)} \tag{78}$$
$$\vdots \tag{79}$$
$$\Delta^k s^{(-1)} = (-1)^k k! s^{(-k-1)} \tag{80}$$
$$\tag{81}$$

Expanding $s^{(-k-1)}$ we get

$$s^{(-k-1)} = \frac{(-1)^k k!}{(s+1)(s+2)(s+3)\ldots(s+k)(s+k+1)} \tag{82}$$

From this expression we can see that evaluating the above expression at $s = 0$ gives

$$\Delta^k \left( \frac{1}{s+1} \right) = \frac{(-1)^k k!}{(k+1)!} = \frac{(-1)^k}{k+1} \tag{83}$$

When this is put into Newton's forward-difference interpolation forumla we obtain WWX HERE!!!

## Problem 12

Given the expression for the Bessel function of order $k$

$$J_k(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(\theta) - k\theta) d\theta \tag{84}$$

We desire to express $J_k(x)$ as a solution to a difference operator $\mathcal{L}$

$$\mathcal{L} J_k = A J_{k+1} + B J_k + C J_{k-1} = 0 \tag{85}$$

so from the integral expression for the Bessel function we have as the result of our $\mathcal{L}$ operator

$$\frac{1}{\pi} \int_0^\pi A \cos(x \sin(\theta) - (k+1)\theta) + B \cos(x \sin(\theta) - k\theta) + C \cos(x \sin(\theta) - (k-1)\theta) d\theta \tag{86}$$

We begin by expanding each cos term in the above so that it has arguments $x \sin(\theta) - k\theta$ only. To do this we use the following two trigonometric identities

$$\cos(x \sin(\theta) - (k+1)\theta) = \cos(x \sin(\theta) - k\theta) \cos(\theta) + \sin(x \sin(\theta) - k\theta) \sin(\theta) \tag{87}$$

$$\cos(x \sin(\theta) - (k-1)\theta) = \cos(x \sin(\theta) - k\theta) \cos(\theta) - \sin(x \sin(\theta) - k\theta) \sin(\theta) \tag{88}$$

When we sum up the contributions from each part we obtain the following

$$= \frac{1}{\pi} \int_0^\pi (A \cos(\theta) + B + C \cos(\theta)) \cos(x \sin(\theta) - k\theta) d\theta \tag{89}$$

$$+ \frac{1}{\pi} \int_0^\pi (A - C) \sin(\theta) \sin(x \sin(\theta) - k\theta) d\theta \tag{90}$$

If we allow $A = C$ we can elliminate the second integral and the argument of the first integral becomes

$$2A \cos(\theta) + B \tag{91}$$

with this assignment the first integral becomes

$$\frac{1}{\pi} \int_0^\pi (2A \cos(\theta) + B) \cos(x \sin(\theta) - k\theta) d\theta \tag{92}$$

and we see that if we define

$$
\begin{aligned}
u &= x\sin(\theta) - k\theta \qquad \text{then} && (93)\\
du &= (x\cos(\theta) - k)d\theta && (94)
\end{aligned}
$$

with this definition we can make the expression in the integral above $(2A\cos(\theta) + B)$ equal to the differential of $u$ by the assignments $2A = x$ and $B = -k$. This makes the integral expression above equal to

$$
\frac{1}{\pi}\int_0^{\pi} du\cos(u) = \frac{1}{\pi}\sin(u)|_0^{\pi} = 0 \tag{95}
$$

Finally with these substitutions for $A$,$B$, and $C$ we have that the linear difference equation that $J_k(x)$ satisfies is

$$
\frac{x}{2}J_{k+1}(x) - kJ_k(x) + \frac{x}{2}J_{k-1}(x) = 0 \tag{96}
$$

which is the expression desired.

## Problem 25

**Part (a):** To sum the following expression

$$
1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots + n(n+1)(n+2) \tag{97}
$$

We recognized that we can express this sum in closed form by recalling the following definitions of the factorial function [?]

$$
t^{(r)} = \begin{cases} t(t-1)(t-2)(t-3)\ldots(t-r+1) & r = +1, +2, +3, \ldots \\ 1 & r = 0 \\ \frac{1}{(t+1)(t+2)(t+3)\ldots(t-r)} & r = -1, -2, -3, \ldots \end{cases} \tag{98}
$$

From this we can see that the expression we are suming is given by

$$
(t+2)^{(3)} = t(t+1)(t+2) \tag{99}
$$

Thus by the fundamental theorm of summation [?] we can then write

$$
\begin{aligned}
\sum_{k=1}^{n}(k+2)^{(3)} &= \sum_{k=-1}^{n-2} k^{(3)} && (100)\\
&= \left.\frac{k^{(4)}}{4}\right|_{-1}^{n-1} && (101)\\
&= \left.\frac{1}{4}k(k-1)(k-2)(k-3)\right|_{-1}^{n-1} && (102)\\
&= \frac{1}{4}\left((n-1)(n-2)(n-3)(n-4) - (-1)(-2)(-3)(-4)\right) && (103)\\
&= \frac{1}{4}\left(\prod_{k=1}^{4}(n-k) - 4!\right) && (104)
\end{aligned}
$$

**Part (b):** As in part (a) we recognize that the individual terms in the summation is equivalent to the factorial function $n^{(-3)}$. Again with the fundamental theorm of summation we obtain

$$\sum n^{(-3)} = \frac{n^{(-2)}}{(-2)} \tag{105}$$

So with the given limits we obtain

$$\sum_{n=1}^{N} n^{(-3)} = \left.\frac{n^{(-2)}}{(-2)}\right|_{1}^{N+1} \tag{106}$$

Since $n^{(-2)} = \frac{1}{(n+1)(n+2)}$ the above is

$$\begin{aligned} \sum_{n=1}^{N} n^{(-3)} &= \frac{-1}{2}\left(\frac{1}{(N+2)(N+3)} - \frac{1}{2\cdot 3}\right) &(107)\\ &= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{(N+2)(N+3)}\right) &(108)\end{aligned}$$

**Part (c):** This summation can be evaluated by partial fractions. For example we have

$$\frac{1}{(2n+1)(2n+3)} = \frac{1}{2}\left(\frac{1}{2n+1}\right) - \frac{1}{2}\left(\frac{1}{2n+3}\right) = \frac{1}{2}\left(\frac{1}{2n+1}\right) - \frac{1}{2}\left(\frac{1}{2(n+1)+1}\right) \tag{109}$$

Thus the sum to to be evaluated is

$$\sum_{n=0}^{N} \frac{1}{(2n+1)(2n+3)} = \sum_{n=0}^{N} \Delta\left((-1)\frac{1}{2}\frac{1}{2n+1}\right) \tag{110}$$

or

$$\sum_{n=0}^{N} \frac{1}{(2n+1)(2n+3)} = \left.\frac{-1}{2}\frac{1}{2n+1}\right|_{0}^{N+1} = \frac{-1}{2}\left(\frac{1}{2N+3} - 1\right) = \frac{1}{2}\left(1 - \frac{1}{2N+3}\right) \tag{111}$$

**Part (d):** To sum

$$1\cdot 3 + 2\cdot 4 + 3\cdot 5 + \ldots + n(n+2)$$

we regcognize this sum as being equivalent to

$$\sum_{k=1}^{n}(k^2+2k) = 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} k^2 = 2\frac{1}{2}n(n+1) + \frac{1}{6}n(n+1)(1+2n) \tag{112}$$

or upon simplifying we obtain

$$\sum_{k=1}^{n}(k^2+2k) = \frac{n}{3}(n+1)(n+2) \tag{113}$$