

$$\begin{pmatrix} B_1 & r_1 & & & & & & & & \\ & B_2 & r_2 & & & & & & & \\ & & B_3 & r_3 & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & B_k & r_k & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & B_{N-1} & r_{N-1} & & \\ & & & & & & & B_N & & \end{pmatrix}$$

Then multiplying $L + U$ gives the following matrix.

$$\begin{pmatrix} B_1 & r_1 & & & & & & & & \\ \alpha_2 B_1 & \alpha_2 r_1 + B_2 & r_2 & & & & & & & \\ & \alpha_3 B_2 & \alpha_3 r_2 + B_3 & r_3 & & & & & & \\ & & & \ddots & \ddots & & & & & \\ & & & & \alpha_k B_{k-1} & \alpha_k r_{k-1} + B_k & r_k & & & \\ & & & & & \vdots & \vdots & & & \\ & & & & & & \alpha_{N-1} B_{N-2} & \alpha_{N-1} r_{N-2} + B_{N-1} & r_{N-1} & \\ & & & & & & & \alpha_N B_{N-1} & \alpha_N r_{N-1} + B_N & \end{pmatrix}$$

Then we get the following equations:

$$1^{\text{st}} \text{ row: } \beta_1 = b_1 \quad \gamma_1 = c_1$$

$$2^{\text{nd}} \text{ row: } \alpha_2 \beta_1 = a_2 \Rightarrow \alpha_2 = \frac{a_2}{\beta_1}$$

$$\alpha_2 \gamma_1 + \beta_2 = b_2 \Rightarrow \beta_2 = b_2 - \alpha_2 \gamma_1$$

$$\gamma_2 = c_2$$

$$3^{\text{rd}} \text{ row: } \alpha_3 \beta_2 = a_3 \Rightarrow \alpha_3 = \frac{a_3}{\beta_2}$$

$$\alpha_3 \gamma_2 + \beta_3 = b_3 \Rightarrow \beta_3 = b_3 - \alpha_3 \gamma_2$$

$$\gamma_3 = c_3$$

$$k^{\text{th}} \text{ row: } \alpha_k \beta_{k-1} = a_k \Rightarrow \alpha_k = \frac{a_k}{\beta_{k-1}}$$

$$\alpha_k \gamma_{k-1} + \beta_k = b_k \Rightarrow \beta_k = b_k - \alpha_k \gamma_{k-1}$$

$$\gamma_k = c_k$$

$$N-1^{\text{st}} \text{ row: } \alpha_{N-1} \beta_{N-2} = a_{N-1} \Rightarrow \alpha_{N-1} = \frac{a_{N-1}}{\beta_{N-2}}$$

$$\alpha_{N-1} \gamma_{N-1} + \beta_{N-1} = b_{N-1} \Rightarrow \beta_{N-1} = b_{N-1} - \alpha_{N-1} \gamma_{N-1}$$

$$\gamma_{N-1} = c_{N-1}$$

Nth row:

$$\alpha_N \beta_{N-1} = a_N \Rightarrow \alpha_N = \frac{a_N}{\beta_{N-1}}$$

$$\alpha_N \gamma_{N-1} + \beta_N = b_N \Rightarrow \beta_N = b_N - \alpha_N \gamma_{N-1}$$

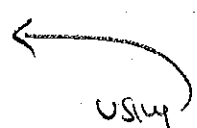
Thus in summary gives vectors $\vec{a}, \vec{b}, \vec{c}$.

$$B_i = b_i$$

$$\vec{Y} = \vec{c}$$

$$do\ i = 2, N$$

$$\alpha_i = \frac{a_i}{B_{i-1}}$$

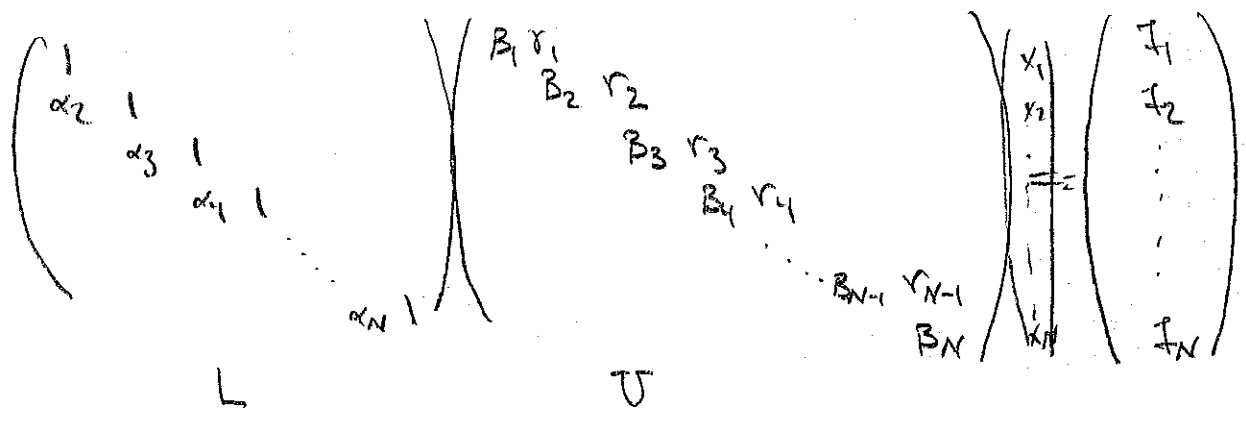


$$B_i = b_i - \alpha_i c_{i-1} \Rightarrow B_i = b_i - \frac{a_i c_{i-1}}{B_{i-1}}$$

Top part eq A.3

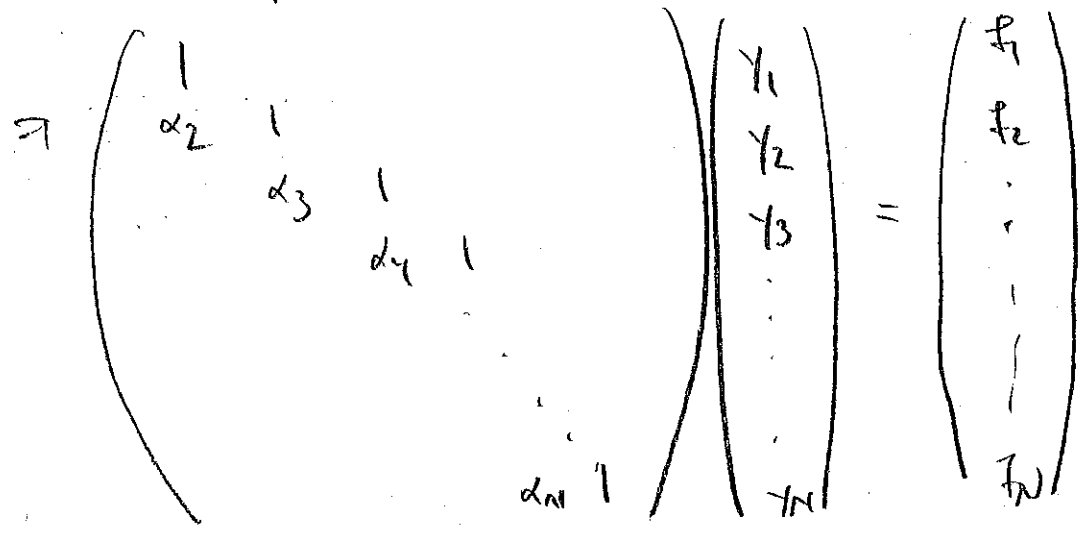
enddo

This computes the vectors needed in the decomposition. To actually solve for x 's lets see what we need.



To perform back solve let $Ux = y$. Then 1st solve

$$Ly = z$$



Then

$$y_1 = f_1$$

do $i=2, N$

$$\{ \alpha_i y_{i-1} + y_i = f_i \} \Rightarrow y_i = f_i - \alpha_i y_{i-1}$$

enddo.

} Forward substitution
step

Now solve back sub. step.

$$\begin{pmatrix} B_1 & r_1 \\ & B_2 & r_2 \\ & & \ddots \\ & & & B_{N-2} & r_{N-2} \\ & & & & B_{N-1} & r_{N-1} \\ & & & & & B_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

$$x_N = y_N / B_N$$

do $i=N-1, 1, -1$

$$\{ B_i x_i + r_i x_{i+1} = y_i \} \Rightarrow x_i = \frac{y_i - r_i x_{i+1}}{B_i} = \frac{y_i}{B_i} - \frac{r_i}{B_i} x_{i+1}$$

enddo

lets put this algorithm into form of A.3.

$$a_k x_{k-1} + b_k x_k + c_k x_{k+1} = f_k \quad k=1, 2, 3, \dots, N$$

$$\begin{bmatrix}
 b_1 & c_1 & & & & & \\
 a_2 & b_2 & c_2 & & & & \\
 & a_3 & b_3 & c_3 & & & \\
 & & & & \ddots & & \\
 & & & & a_k & b_k & c_k \\
 & & & & & & & \ddots & & \\
 & & & & & & & & a_{N-1} & b_{N-1} & c_{N-1} \\
 & & & & & & & & & a_N & b_N
 \end{bmatrix} = LU$$

Now standard decomposition has 1s on the diagonal of L. I think the Hirsch Book uses 1's on the diagonal of U. (Yes that is correct). This will give the Hirsch algo

$$LU =
 \begin{bmatrix}
 B_1 & & & & & & \\
 \alpha_2 & B_2 & & & & & \\
 \alpha_3 & & B_3 & & & & \\
 & & & \ddots & & & \\
 \alpha_k & & & & B_k & & \\
 & & & & & \ddots & \\
 \alpha_N & & & & & & & B_N
 \end{bmatrix}
 \begin{bmatrix}
 1 & r_1 & & & & & \\
 & 1 & r_2 & & & & \\
 & & 1 & r_3 & & & \\
 & & & & \ddots & & \\
 & & & & & 1 & r_k \\
 & & & & & & & \ddots & & \\
 & & & & & & & & 1 & r_{N-1} \\
 & & & & & & & & & 1
 \end{bmatrix}$$

Multiplying $L+U$ we get

$$LD = \begin{bmatrix}
 B_1 & B_1 Y_1 & & & & \\
 \alpha_2 & \alpha_2 Y_1 + B_2 & B_2 Y_2 & & & \\
 & \alpha_3 & \alpha_3 Y_2 + B_3 & B_3 Y_3 & & \\
 & & & \alpha_k & \alpha_k Y_{k-1} + B_k & B_k Y_k \\
 & & & & & \alpha_{N-1} & \alpha_{N-1} Y_{N-2} + B_{N-1} & B_{N-1} Y_{N-1} \\
 & & & & & & \alpha_N & \alpha_N Y_{N-1} + B_N
 \end{bmatrix}$$

Then equating coefficients we see that $\bar{a} = \bar{\alpha}$.

1st row $B_1 = b_1$; $a_1 = B_1 Y_1 \Rightarrow Y_1 = \frac{a_1}{B_1}$

2nd row $\alpha_2 Y_1 + B_2 = b_2 \Rightarrow B_2 = b_2 - \alpha_2 Y_1$; $a_2 = B_2 Y_2 \Rightarrow Y_2 = \frac{a_2}{B_2}$

⋮

kth row: $\alpha_k = a_k$; $\alpha_k Y_{k-1} + B_k = b_k \Rightarrow B_k = b_k - \alpha_k Y_{k-1}$; $B_k Y_k = a_k \Rightarrow Y_k = \frac{a_k}{B_k}$

⋮

$(N-1)$ st row: $\alpha_{N-1} = a_{N-1}$; $B_{N-1} = b_{N-1} - \alpha_{N-1} Y_{N-2}$; $Y_{N-1} = \frac{a_{N-1}}{B_{N-1}}$

N th row: $\alpha_N = a_N$; $B_N = b_N - \alpha_N Y_{N-1}$

Thus routine to calculate $\bar{\alpha}$, \bar{B} , & \bar{r} is

$$B_1 = b_1$$

$$\bar{\alpha} = \bar{a}$$

do $i = 2, N$

$$r_{i-1} = \frac{c_{i-1}}{B_{i-1}}$$

$$B_i = b_i - \alpha_i r_{i-1}$$

using \bar{a}

$$\Rightarrow B_i = b_i - \alpha_i \frac{c_{i-1}}{B_{i-1}}$$

1st eq in A.3

enddo

Then Forward/Backwards substitution since $LUx = f$

and $Ux = y$ then $Ly = f$

$$\begin{bmatrix} B_1 \\ \alpha_2 & B_2 \\ \alpha_3 & & B_3 \\ & \alpha_k & & B_k \\ & & & \alpha_N & B_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}$$

Then

$$y_1 = f_1 / B_1$$

do $i = 2, N$

$$\alpha_i y_{i-1} + B_i y_i = f_i \Rightarrow y_i = \frac{f_i - \alpha_i y_{i-1}}{B_i} = \frac{-\alpha_i y_{i-1} + f_i}{B_i}$$

enddo

2nd eq in A.3

Then Backwards solve is solving $Ux = y$

$$\begin{bmatrix} 1 & r_1 & & & & \\ & 1 & r_2 & & & \\ & & & \ddots & & \\ & & & & 1 & r_k \\ & & & & & \ddots \\ & & & & & & 1 & r_{N-1} \\ & & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$$

$$\Rightarrow x_N = y_N$$

do $i = N-1, 1, -1$

$$x_i + r_i x_{i+1} = y_i \Rightarrow x_i = y_i - r_i x_{i+1} \Rightarrow x_i = y_i - x_{i+1} \frac{c_i}{B_i}$$

eq 7.4

enddo

of operations:

$$\text{From A.3 } B_k = \frac{b_k B_{k-1} - a_k c_{k-1}}{B_{k-1}} \Bigg) 3 \text{ is } 3 \text{ operations}$$

$$r_k = \left(\frac{-a_k v_{k+1} + i_k}{B_k} \right) \Bigg) 2 \text{ is } 2 \text{ operations}$$

$$x_k =$$

How get # of operations?

eq (10.2.12) is

$$\frac{dy}{dt} = \frac{\alpha}{\Delta x^2} \left(\begin{array}{cccc} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & & \ddots \\ & & & & & -2 & 1 \\ & & & & & 1 & -2 \end{array} \right) y$$

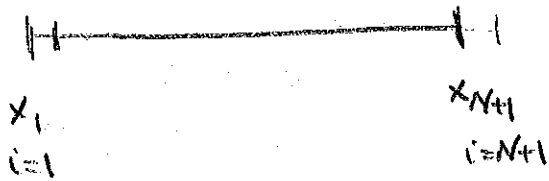
}
}
}
}

Periodic matrix of size $N+1$ called $B_p(\bar{a}, \bar{b}, \bar{c})$

decompose solution X to $B_p X = F$ as

$$X = X^{(1)} + x_{N+1} X^{(2)} \quad \text{w/ } X^{(1)} \text{ the solution of } [B_p] X^{(1)} = [F]$$

$[B_p]$ obtained from B_p by removing the last row & column of B_p



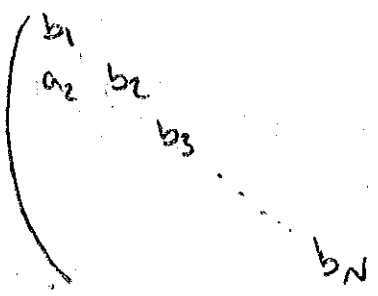
← Note: The unknowns are ordered from 1, to $N+1$

Periodic conditions imply that $u_1 = u_{N+1}$ ← periodically conditions?

deleting last row & column results in matrix called $B^{(N)}(\bar{a}, \bar{b}, \bar{c})$

Then $B_p (X^{(1)} + x_{N+1} X^{(2)}) = F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N \\ F_{N+1} \end{pmatrix}$

$B_p X^{(1)}$



∴ last N+1'st eq becomes:

$$c_{N+1}x_1' + a_{N+1}x_N' + x_{N+1}(c_{N+1}x_1^2 + a_{N+1}x_N^2 + b_{N+1}) = F_{N+1} \quad ** \checkmark$$

Because of * 1st N eq.s become

$$x_{N+1} \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & & \ddots & & \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ \vdots \\ x_{N-1}^2 \\ x_N^2 \end{bmatrix} = -x_{N+1} \begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_N \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & & \ddots & & \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ \vdots \\ x_{N-1}^2 \\ x_N^2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ -c_N \end{bmatrix} \quad \text{eq A, B \& A, 9}$$

Now we know $\Sigma^{(1)} + \Sigma^{(2)}$ x_{N+1} is obtained from eq **

or

$$x_{N+1} = \frac{F_{N+1} - c_{N+1}x_1' - a_{N+1}x_N'}{b_{N+1} + c_{N+1}x_1^2 + a_{N+1}x_N^2} \quad \text{eq A.10}$$

- Algorithm would be 1) solve tri-diag system w/ F_1, F_2, \dots, F_N as R.H.S. & w/o N+1's row & column. produces x_1', x_2', \dots, x_N'
- 2) solve tri-diag system w/ $(-a_1, 0, 0, \dots, 0, -c_N)$ as R.H.S. produces $x_1^2, x_2^2, \dots, x_N^2$'s
- 3) compute x_{N+1} 4) produce Σ from eq A.6

$$\begin{bmatrix} b(N) & c(N) & & & & & a(N) \\ a(N+1) & b(N+1) & c(N+1) & & & & 0 \\ & a(N+2) & b(N+2) & c(N+2) & & & \\ & & & & \ddots & & \\ & & & & & a(N) & b(N) & c(N) \\ (N+1) & & & & & a(N+1) & b(N+1) & \end{bmatrix}$$

$$\approx \begin{bmatrix} a(N) & b(N) & c(N) & & & \\ & a(N+1) & b(N+1) & c(N+1) & & \\ & & a(N+2) & b(N+2) & c(N+2) & \\ & & & & \ddots & \\ & & & & & \end{bmatrix}$$

loop do $k = N+1, N$

$$c(k-1) = c(k-1) \cdot BB(k-1)$$

$$BB(k) = BB(k) - AA(k) \cdot c(k-1)$$