$$a_k x_{k-1} + b_k x_k + c_k x_{k+1} = f_k \quad k = 1, 2, \cdots, N$$

$$\Rightarrow \begin{vmatrix} b_1 & c_1 & & & & & & \\ a_2 & b_2 & c_2 & & & & & \\ 0 & a_3 & b_3 & c_3 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & a_k & b_k & c_k & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & a_{N-2} & b_{N-2} & c_{N-2} & 0 \\ & & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & & a_N & b_N \end{vmatrix} = LU$$

Here we follow the standard procedure of putting the 1's on the diagonal.

$LU =$

$$L = \begin{pmatrix} 1 & & & & & & & \\ \alpha_2 & 1 & & & & & & \\ & \alpha_3 & 1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & \alpha_k & 1 & & & \\ & & & & \ddots & \ddots & & \\ & & & & \alpha_{N-2} & 1 & & \\ & & & & & \alpha_{N-1} & 1 & 0 \\ & & & & & & \alpha_N & 1 \end{pmatrix}$$

$$\begin{pmatrix} B_1 & \gamma_1 & & & & & & \\ & B_2 & \gamma_2 & & & & & \\ & & B_3 & \gamma_3 & & & & \\ & & & \ddots & \ddots & & & \\ & & & & B_k & \gamma_k & & \\ & & & & & \ddots & \ddots & \\ & & & & & & B_{N-1} & \gamma_{N-1} \\ & & & & & & & B_N \end{pmatrix}$$

Then multipling L & U gives the following matrix.

$$\begin{pmatrix} B_1 & \gamma_1 & & & & & \\ \alpha_2 B_1 & \alpha_2 \gamma_1 + B_2 & \gamma_2 & & & & \\ & \alpha_3 B_2 & \alpha_3 \gamma_2 + B_3 & \gamma_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & \alpha_k B_{k-1} & \alpha_k \gamma_{k-1} + B_k & \gamma_k & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \alpha_{N-1} B_{N-2} & \alpha_{N-1} \gamma_{N-2} + B_{N-1} & \gamma_{N-1} \\ & & & & & \alpha_N B_{N-1} & \alpha_N \gamma_{N-1} + B_N \end{pmatrix}$$

Then we get the following equations:

1st row: $\beta_1 = b_1 \qquad \gamma_1 = c_1$

2nd row: $\alpha_2 \beta_1 = a_2 \Rightarrow \alpha_2 = \dfrac{a_2}{\beta_1}$

$$\alpha_2 \gamma_1 + \beta_2 = b_2 \Rightarrow \beta_2 = b_2 - \alpha_2 \gamma_1$$

$$\gamma_2 = c_2$$

3rd row: $\alpha_3 \beta_2 = a_3 \Rightarrow \alpha_3 = \dfrac{a_3}{\beta_2}$

$$\alpha_3 \gamma_2 + \beta_3 = b_3 \Rightarrow \beta_3 = b_3 - \alpha_3 \gamma_2$$

$$\gamma_3 = c_3$$

kth row: $\alpha_k \beta_{k-1} = a_k \Rightarrow \alpha_k = \dfrac{a_k}{\beta_{k-1}}$

$$\alpha_k \gamma_{k-1} + \beta_k = b_k \Rightarrow \beta_k = b_k - \alpha_k \gamma_{k-1}$$

$$\gamma_k = c_k$$

N-1 st row: $\alpha_{N-1} \beta_{N-2} = a_{N-1} \Rightarrow \alpha_{N-1} = \dfrac{a_{N-1}}{\beta_{N-2}}$

$$\alpha_{N-1} \gamma_{N-1} + \beta_{N-1} = b_{N-1} \Rightarrow \beta_{N-1} = b_{N-1} - \alpha_{N-1} \gamma_{N-1}$$

$$\gamma_{N-1} = c_{N-1}$$

<u>Nth row</u>: $\alpha_N \beta_{N-1} = a_N \Rightarrow \alpha_N = \dfrac{a_N}{\beta_{N-1}}$

$$\alpha_N \gamma_{N-1} + \beta_N = b_N \Rightarrow \beta_N = b_N - \alpha_N \gamma_{N-1}$$

Thus in summary given vectors $\vec{a}, \vec{b}, \& \vec{c}$

$$\beta_1 = b_1 \qquad \vec{\gamma} = \vec{c}$$

do $i = 2, N$

$$\alpha_i = \frac{a_i}{\beta_{i-1}}$$

$$\beta_i = b_i - \alpha_i c_{i-1} \xRightarrow{\text{using}} \beta_i = b_i - \frac{a_i}{\beta_{i-1}} c_{i-1}$$ Top part eq A.3

enddo

This computes the vectors needed in the decomposition. To actually solve for x's lets see what we need.

$$\underbrace{\begin{pmatrix} 1 & & & & & \\ \alpha_2 & 1 & & & & \\ & \alpha_3 & 1 & & & \\ & & \alpha_4 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \alpha_N & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \beta_1 & \gamma_1 & & & & \\ & \beta_2 & \gamma_2 & & & \\ & & \beta_3 & \gamma_3 & & \\ & & & \beta_4 & \gamma_4 & \\ & & & & \ddots & \\ & & & & \beta_{N-1} & \gamma_{N-1} \\ & & & & & \beta_N \end{pmatrix}}_{U} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_N \end{pmatrix}$$

To perform back solve let $Ux = y$. Then 1st solve

$$Ly = f$$

$$\Rightarrow \begin{pmatrix} 1 & & & & & \\ \alpha_2 & 1 & & & & \\ & \alpha_3 & 1 & & & \\ & & \alpha_4 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \alpha_N & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_N \end{pmatrix}$$

Then $y_1 = f_1$

do i=2,N

$\{ \alpha_i y_{i-1} + y_i = f_i \} \Rightarrow y_i = f_i - \alpha_i y_{i-1}$

enddo.

Forward substitution Step

Now solve back sol. Step.

$$\begin{pmatrix} B_1 & r_1 & & & & \\ & B_2 & r_2 & & & \\ & & \ddots & \ddots & & \\ & & & B_{N-2} & r_{N-2} & \\ & & & & B_{N-1} & r_{N-1} \\ & & & & & B_N \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_N \end{pmatrix}$$

$x_N = y_N / B_N$

do i=N-1,1,-1

$\{ B_i x_i + r_i x_{i+1} = y_i \Rightarrow x_i = \dfrac{y_i - r_i x_{i+1}}{B_i} = \dfrac{y_i}{B_i} - \dfrac{r_i}{B_i} x_{i+1}$

enddo

lets put this algorithm into Form of A.3.

$a_k x_{k-1} + b_k x_k + c_k x_{k+1} = f_k \qquad k = 1, 2, 3, \cdots, N$

$$\begin{bmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_k & b_k & c_k & \\ & & & & \ddots & \ddots & \ddots \\ & & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & & a_N & b_N \end{bmatrix} = LU.$$

Now standard decomposition has 1s on the diagonal of L. I think the Hirsh Book uses 1's on the diagonal of U. (Yes that is correct). This will give the Hirsh algo

$$LU = \begin{bmatrix} \beta_1 & & & & \\ \alpha_2 & \beta_2 & & & \\ & \alpha_3 & \beta_3 & & \\ & & \ddots & \ddots & \\ & & \alpha_k & \beta_k & \\ & & & \ddots & \ddots \\ & & & \alpha_N & \beta_N \end{bmatrix} \begin{bmatrix} 1 & \gamma_1 & & & \\ & 1 & \gamma_2 & & \\ & & 1 & \gamma_3 & \\ & & & \ddots & \ddots \\ & & & 1 & \gamma_k \\ & & & & \ddots & \ddots \\ & & & & 1 & \gamma_{N-1} \\ & & & & & 1 \end{bmatrix}$$

Multiplying L & U we get

$$LU = \begin{bmatrix} B_1 & B_1\gamma_1 & & & & & \\ \alpha_2 & \alpha_2\gamma_1 + B_2 & B_2\gamma_2 & & & & \\ & \alpha_3 & \alpha_3\gamma_2 + B_3 & B_3\gamma_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \alpha_k & \alpha_k\gamma_{k-1} + B_k & B_k\gamma_k & \\ & & & & \ddots & \ddots & \ddots \\ & & & & \alpha_{N-1} & \alpha_{N-1}\gamma_{N-2} + B_{N-1} & B_{N-1}\gamma_{N-1} \\ & & & & & \alpha_N & \alpha_N\gamma_{N-1} + B_N \end{bmatrix}$$

Then equating coefficients we see that $\bar{a} = \bar{\alpha}$.

1st row $\quad B_1 = b_1 \;;\; c_1 = B_1\gamma_1 \Rightarrow \gamma_1 = \dfrac{c_1}{B_1}$

2nd row $\quad \alpha_2\gamma_1 + B_2 = b_2 \Rightarrow B_2 = b_2 - \alpha_2\gamma_1 \;;\; c_2 = B_2\gamma_2 \Rightarrow \gamma_2 = \dfrac{c_2}{B_2}$

⋮

kth row: $\quad \alpha_1 = a_k \;;\; \alpha_k\gamma_{k-1} + B_k = b_k \Rightarrow B_k = b_k - \alpha_k\gamma_{k-1} \;;\; B_k\gamma_k = c_k \Rightarrow \gamma_k = \dfrac{c_k}{B_k}$

⋮

N-1 st row: $\quad \alpha_{N-1} = a_{N-1} \;;\; B_{N-1} = b_{N-1} - \alpha_{N-1}\gamma_{N-2} \;;\; \gamma_{N-1} = \dfrac{c_{N-1}}{B_{N-1}}$

Nth row: $\quad \alpha_N = a_N \;;\; B_N = b_N - \alpha_N\gamma_{N-1}$

Thus routine to calculate $\bar{\alpha}, \bar{B}, + \bar{r}$ is

$$\{ \bar{\alpha} = \bar{a} \}$$

$B_1 = b_1$

do $i = 2, N$

$$r_{i-1} = \frac{c_{i-1}}{B_{i-1}}$$

$$B_i = b_i - \alpha_i r_{i-1} \overset{\text{using } *}{\Rightarrow} B_i = b_i - a_i \frac{c_{i-1}}{B_{i-1}} \qquad \text{1st eq in A,3}$$

enddo

Then Forward/Backwards substitutions since $LUx = f$

and $Ux = y$ then $Ly = f$

$$\begin{bmatrix} B_1 & & & & & \\ \alpha_2 & B_2 & & & & \\ & \alpha_3 & B_3 & & & \\ & & \ddots & \ddots & & \\ & & & \alpha_k & B_k & \\ & & & & \ddots & \ddots \\ & & & & \alpha_N & B_N \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{bmatrix}$$

Then $y_1 = f_1 / B_1$

do $i = 2, N$

$$\alpha_i y_{i-1} + B_i y_i = f_i \Rightarrow y_i = \frac{f_i - \alpha_i y_{i-1}}{B_i} \overset{\text{using } \bar{\alpha} = \bar{a}}{=} \frac{-a_i y_{i-1} + f_i}{B_i} \qquad \text{2nd eq in A,3}$$

enddo

Then Backwards solve is solving $Ux = y$

$$\begin{bmatrix} 1 & r_1 & & & & \\ & 1 & r_2 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & r_k & \\ & & & & \ddots & \ddots \\ & & & & 1 & r_{N-1} \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \\ \vdots \\ y_{N-1} \\ y_N \end{bmatrix}$$

$\Rightarrow \quad x_N = y_N$

do $i = N-1, 1, -1$

$x_i + r_i x_{i+1} = y_i \Rightarrow x_i = y_i - r_i x_{i+1} \Rightarrow x_i = y_i - x_{i+1} \dfrac{c_i}{\beta_i}$ eq 7.1.4

enddo

# of operations:

From 1.3 $\quad \beta_k = \dfrac{\overset{1}{b_k \beta_{k-1}} - \overset{2}{a_k c_{k-1}}}{\beta_{k-1}} \Big) 3$ is 3 operations

$r_k = \left(\dfrac{-a_k r_{k-1} + f_k}{\beta_k} \right) \Big) 2$ is 2 operations

How get # of operations?

$x_k =$

eq 10,2,12 is

$$\frac{dv}{dt} = \frac{\alpha}{\Delta x^2}\underbrace{\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ 1 & & & & 1 & -2 \end{pmatrix}}_{1 \ \cdots \ x} v$$

Periodic matrix of size $N+1$ called $B_p(\bar{a},\bar{b},\bar{c})$

decompose solution X to $B_p X = F$ as

$$X = X^{(1)} + x_{N+1} X^{(2)} \quad \text{w/ } X^{(1)} \text{ the solution of } [B_p]X^{(1)} = [F]$$

$[B_p]$ obtained from B_p by removing the last row & column of B_p

x_1 ($i=1$) ———— x_{N+1} ($i=N+1$) $\Leftarrow$ Note: The unknowns are ordered from 1, to N+1

Periodic conditions imply that $U_1 = U_{N+1}$ $\Leftarrow$ periodicity conditions ?

deleting last row & column results in matrix called $B^{(N)}(\bar{a},\bar{b},\bar{c})$

Then $$B_p\left(X^{(1)} + x_{N+1}X^{(2)}\right) = F = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N \\ F_{N+1} \end{pmatrix}$$

$$B_p X^{(1)}$$

$$\begin{pmatrix} b_1 & & & & \\ a_2 & b_2 & & & \\ & & b_3 & & \\ & & & \ddots & \\ & & & & b_N \end{pmatrix}$$

True system to solve:

$$\left(\begin{array}{cccccc|c} b_1 & c_1 & & & & & a_1 \\ a_2 & b_2 & c_2 & & & & 0 \\ & a_3 & b_3 & c_3 & & & \vdots \\ & & & \ddots & & & \vdots \\ & & & & & c_{N-1} & 0 \\ & & & & a_N & b_N & c_N \\ \hline c_{N+1} & 0 & & & 0 & a_{N+1} & b_{N+1} \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ x_{N+1} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \\ F_{N+1} \end{pmatrix}$$

let $X = X^{(1)} + x_{N+1} X^{(2)} = \begin{pmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_N^1 \\ 0 \end{pmatrix} + x_{N+1} \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_N^2 \\ 1 \end{pmatrix}$ ← Note 1 in last location of vector $X^{(2)}$.

$$\Rightarrow \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & \ddots & & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ \vdots \\ x_N^1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ \vdots \\ F_N \end{pmatrix} *$$

so that when one substitutes $X = X^{(1)} + x_{N+1} X^{(2)}$ into full set of equations one gets

$$\Rightarrow \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & \ddots & & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{pmatrix} \begin{pmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_{N-1}^1 \\ x_N^1 \end{pmatrix} + x_{N+1} \begin{pmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & \ddots & & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ x_2^3 \\ \vdots \\ x_{N-1}^2 \\ x_N^2 \end{pmatrix}$$

$$+ x_{N+1} \begin{pmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_N \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{pmatrix}$$

& Last $N+1$'st eq becomes:

$$c_{N+1}x_1^1 + a_{N+1}x_N^1 + x_{N+1}\left(c_{N+1}x_1^2 + a_{N+1}x_N^2 + b_{N+1}\right) = F_{N+1} \quad ** \quad \checkmark$$

Because of * 1st N eq.s become

$$\cancel{x_{N+1}}\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & & \ddots & & \\ & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & a_N & b_N \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ \vdots \\ x_{N-1}^2 \\ x_N^2 \end{bmatrix} = -\cancel{x_{N+1}}\begin{bmatrix} a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ c_N \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & & \ddots & & \\ & & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & & a_N & b_N \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \\ \vdots \\ x_{N-1}^2 \\ x_N^2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -c_N \end{bmatrix} \quad \text{eq A.8 \& A.9}$$

Now we know $X^{(1)}$ & $X^{(2)}$ x_{N+1} is determined from eq **

or

$$x_{N+1} = \frac{F_{N+1} - c_{N+1}x_1^1 - a_{N+1}x_N^1}{b_{N+1} + c_{N+1}x_1^2 + a_{N+1}x_N^2} \quad \text{eq A.10}$$

Algorithm would be 1) Solve tridag system w/ $F_1, F_2, \dots, F_N$ as R.H.S & w/o $N+1$'s row & column. produces $x_1^1, x_2^1, \dots, x_N^1$

2) Solve tridag system w/ $(-a_1, 0, 0, \dots, 0, -c_N)$ as R.H.S produces $x_1^2, x_2^2, \dots, x_N^2$'s

3) compute x_{N+1} 4) produce X from eq A.6

$$
\begin{bmatrix}
b(N1) & c(N1) & & & & & a(N1) \\
a(N1+1) & b(N1+1) & c(N1+1) & & & & 0 \\
& a(N1+2) & b(N1+2) & c(N1+2) & & & \\
& & & \ddots & & & \\
& & & a(N) & b(N) & c(N) & \\
c(N+1) & & & & a(N+1) & b(N+1) &
\end{bmatrix}
\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}
$$

$$
\approx
\begin{bmatrix}
a(N1) & b(N1) & c(N1) & & & \\
& a(N1+1) & b(N1+1) & c(N1+1) & & \\
& & a(N1+2) & b(N1+2) & c(N1+2) & \\
& & & & \ddots &
\end{bmatrix}
$$

```
loop    do k = N1+1, N
              cc(k-1) = cc(k-1)·BB(k-1)
              BB(k)   = BB(k) - AA(k) cc(k-1)
```