$$U^n = \bar{U}^n + e^n$$

$$U^{n+1} = \bar{U}^{n+1} + e^{n+1} =$$

why? $\rightarrow \| \quad \| \quad \|$

$$CU^n = C\bar{U}^n + e^{n+1} \rightarrow e^{n+1} = CU^n - C\bar{U}^n = \underbrace{C(U^n - \bar{U}^n)}_{= Ce^n}$$

$$\therefore \quad e^{n+1} = Ce^n \quad \text{eq } 8.1.7$$

$$k_j = jk_{min} = j\frac{2\pi}{2L} = \qquad j = \cancel{0}, 1, 2, \ldots, N.$$

$$= j\frac{\pi}{L} = j\frac{\pi}{N\Delta x}$$

$$\lambda_{max} = 2L \qquad \lambda_{min} = 2\Delta x$$

$$k_{min} = \frac{2\pi}{\lambda_{max}} = \frac{\pi}{L} \qquad k_{max} = \frac{2\pi}{\lambda_{min}} = \frac{\pi}{\Delta x}.$$

$$\varepsilon_i^n = \sum_{j=-N}^{N} E_j^n e^{Ik_j \cdot i\Delta x} = \sum_{j=-N}^{N} E_j^n e^{Iij\left(\frac{\pi}{N\cancel{\Delta x}}\right)\cancel{\Delta x}} = \sum_{j=-N}^{N} E_j^n e^{Iij\pi/N} \quad \text{eq } 8.1.11$$

$$k_j \Delta x = \frac{j\pi}{N\cancel{\Delta x}} \cdot \cancel{\Delta x} = \frac{j\pi}{N} = \phi \quad \text{eq } 8.1.12.$$

$E_j^n e^{iI\phi}$ eq 8.1.3 $\dfrac{\epsilon_i^{n+1} - \epsilon_i^n}{\Delta t} = -\dfrac{a}{2\Delta x}(\epsilon_{i+1}^n - \epsilon_{i-1}^n)$

$$\frac{E_j^{n+1} e^{iI\phi} - E_j^n e^{iI\phi}}{\Delta t} = -\frac{a}{2\Delta x}\left(E_j^n e^{(i+1)\phi I} - E_j^n e^{(i-1)\phi I}\right)$$

$$\Rightarrow \frac{E^{n+1} - E^n}{\Delta t} = -\frac{a}{2\Delta x}\left(E^n e^{\phi I} - E^n e^{-\phi I}\right) \quad \text{eq } 8.1.13$$

$$E^{n+1} - E^n + \frac{a\Delta}{2\Delta x} E^n (e^{I\phi} - e^{-I\phi}) = 0$$

$$E^{n+1} - E^n + \frac{\sigma}{2} E^n (e^{I\phi} - e^{-I\phi}) = 0 \quad \text{eq } 8.1.13$$

$$G - 1 + \frac{\sigma}{2} 2I \sin\phi = 0 \Rightarrow G = 1 - I\sigma \sin\phi \quad \text{eq } 8.1.17$$

$$|G|^2 = 1 + \sigma^2 \sin^2\phi \quad \text{eq } 8.1.18$$

$$> 1 \quad \forall \phi.$$

eq 7.2.8 $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -\dfrac{a}{\Delta x}(u_i^n - u_{i-1}^n)$ then error satisfies the same eq.

plug in $E_j^n e^{iI\phi}$ & we get

$$\frac{E^{n+1}e^{iI\phi} - E^{n}e^{iI\phi}}{\Delta t} = -\frac{a}{\Delta x}\left(E^{n}e^{iI\phi} - E^{n}e^{(i-1)I\phi}\right)$$

$$(E^{n+1} - E^{n})e^{iI\phi} + \sigma E^{n}\left(e^{iI\phi} - e^{(i-1)I\phi}\right) = 0$$

$$(G-1) + \sigma(1 - e^{-I\phi}) = 0$$

$$G = 1 - \sigma + \sigma e^{-I\phi} \qquad \text{eq } 8.1.19$$

$$= 1 - \sigma + \sigma(\cos\phi - I\sin(\phi))$$

$$= 1 - \sigma + \sigma\cos\phi - I\sigma\sin(\phi) \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$= 1 - \sigma + \sigma(1 - 2\sin^2\phi/2) - I\sigma\sin\phi$$

$$= 1 - \cancel{\sigma} + \cancel{\sigma} - 2\sigma\sin^2\phi/2 - I\sigma\sin\phi$$

$$G = 1 - 2\sigma\sin^2\phi/2 - I\sigma\sin\phi \qquad \text{eq } 8.1.19$$

$$\xi = 1 - 2\sigma\sin^2\phi/2 = 1 - \sigma + \sigma\cos\phi \qquad \text{eq } 8.1.20$$

$$\eta = -\sigma\sin\phi$$

Parametric eqs in complex plane for (ξ, η)

eq 7.2.6 $$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right)$$

Linear Difference eq
error eq satisfies the same eq sub $E_j^n e^{Ii\phi}$ into

$$E_j^{n+1} e^{Ii\phi} - E_j^n e^{Ii\phi} = -\frac{a\Delta t}{2\Delta x}\left(E_j^{n+1} e^{I(i+1)\phi} - E_j^{n+1} e^{I(i-1)\phi}\right)$$

$$\frac{E_j^{n+1}}{E_j^n} - 1 = -\frac{\beta}{2}\left(\frac{E_j^{n+1}}{E_j^n} e^{I\phi} - \frac{E_j^{n+1}}{E_j^n} e^{-I\phi}\right)$$

$$G - 1 = -\frac{\beta}{2}\left(G e^{I\phi} - G e^{-I\phi}\right)$$

$$G\left[1 + \frac{\beta}{\cancel{2}} \cancel{2} I \sin\phi\right] = 1$$

$$G = \frac{1}{1 + I\beta\sin\phi}$$ eq 8.1.22

$$|G|^2 = \frac{1}{1 + \beta^2\sin^2\phi}$$ eq 8.1.23

$$u_{tt} - a^2 u_{xx} = 0$$

let $v = u_t$ + $w = a u_x$

$$v_t = u_{tt} = a^2 u_{xx} = a w_x$$

$$V_x = V_{xt} = \frac{1}{a} w_t$$

$$\Rightarrow w_t = a V_x$$

$$\Rightarrow \quad V_t = a w_x \qquad \text{eq } 8.2.14a$$
$$w_t = a V_x$$

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$U_i^n = v^n e^{Ii\phi}$ eq 8.2.5 is $U_i^{n+1} = C U_i^n + \bar{q}_i$

$$e^{Ii\phi} v^{n+1} = C(E) e^{iI\phi} v^n = G(\phi) v^n e^{iI\phi}$$

Because $C(E)$ is linear, we know that
$$C(E) e^{iI\phi} = G(\phi) e^{iI\phi}$$

$$v^{n+1} = G(\phi) v^n$$

Normal matrix $G^+ G = G G^+$ $G^+ =$ transpose conjugation

Ness for stability $\rho(G) \le 1 + O(\Delta t)$
$\rho(G) \le 1$

sufficient for stability? Yes if from $\rho(G) = \|G\|$

Then $\rho(G) \le 1 + O(\Delta t) \rightarrow \|G\| \le 1 + O(\Delta t)$

$+ \ \rho(G) \le 1 \rightarrow \|G\| \le 1$

for what class of matrices $\ni \|G\| = \rho(G)$ these are the

Normal matrices. $G^+ G = G G^+$

Then

⇒ Amplification matrix G then is (for Euler time discretization see problem 8.1)

$$G = \left[1 - \frac{\Delta t}{2\Delta x} A \left(e^{I\phi} - e^{-I\phi} \right) \right] = \left[1 - \frac{\Delta t}{\Delta x} I \sin\phi A \right]$$

$$= \begin{bmatrix} 1 - \frac{\Delta t}{\Delta x} I \sin\phi\, v_0 & -\frac{\Delta t}{\Delta x} I \sin\phi\, h_0 \\ -\frac{\Delta t}{\Delta x} I \sin\phi\, g & 1 - \frac{\Delta t}{\Delta x} I \sin\phi\, v_0 \end{bmatrix}$$

A nessassary condition is that $\rho(G) < 1$, Finding the eigenvalues

$$\begin{vmatrix} \lambda - 1 + \frac{\Delta t}{\Delta x} I \sin\phi\, v_0 & \frac{\Delta t}{\Delta x} I \sin\phi\, h_0 \\ \frac{\Delta t}{\Delta x} I \sin\phi\, g & \lambda - 1 + \frac{\Delta t}{\Delta x} I \sin\phi\, v_0 \end{vmatrix}$$

$$\left(\lambda - \left(1 - \frac{\Delta t}{\Delta x} I \sin\phi\, v_0 \right) \right)^2 + \left(\frac{\Delta t}{\Delta x} \right)^2 \sin^2\phi\, h_0 g = 0$$

defining: $\sigma_0 \equiv \frac{v_0 \Delta t}{\Delta x}$ + $\sigma \equiv \sqrt{h_0 g} \left(\frac{\Delta t}{\Delta x} \right)$ we get

$$\left(\lambda - \left(1 - I\sigma_0 \sin\phi \right) \right)^2 + \sigma^2 \sin^2\phi = 0 \qquad E\,8.2.2$$

$$\lambda - \left(1 - I\sigma_0 \sin\phi \right) = \pm I \sigma \sin\phi$$

$$\lambda = 1 - I\sigma_0 \sin\phi \pm I\sigma \sin\phi = 1 - I(\sigma_0 \pm \sigma) \sin\phi \qquad E\,8.2.5$$

$$\lambda_\pm = 1 - I(G_0 \pm G)\sin\phi$$

$$\rho(G) = |\lambda_+| = \sqrt{1^2 + (G_0+G)^2 \sin^2\phi} = \sqrt{1 + \left(\frac{\Delta t}{\Delta x}\right)^2 \left(V_0 + \sqrt{gh_0}\right)^2 \sin^2\phi}$$

$$\geq 1$$

unstable.

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w/ $G =$ E8.2.8

$$\begin{vmatrix} \lambda - \cos\phi + G_0 I \sin\phi & -I\frac{\Delta t}{\Delta x} h_0 \sin\phi \\ -I\frac{\Delta t}{\Delta x} g \sin\phi & \lambda - \cos\phi + G_0 I \sin\phi \end{vmatrix}$$

$$= (\lambda - \cos\phi + I G_0 \sin\phi) + \left(\frac{\Delta t}{\Delta x}\right)^2 g h_0 \sin^2\phi = 0$$

$$\lambda_\pm = \cos\phi - I G_0 \sin\phi \pm I\left(\frac{\Delta t}{\Delta x}\right)\sqrt{gh_0}\,\sin\phi$$

$$= \cos\phi - I(G_0 \pm G)\sin\phi \qquad G_0 = v_0 \frac{\Delta t}{\Delta x}$$

$$G = \frac{\Delta t}{\Delta x}\sqrt{gh_0}$$

$$\rho(G)^2 = |\lambda_+|^2 = \cos^2\phi + (G_0 + G)^2 \sin^2\phi$$

Stability if $\sigma_0 + \sigma \le 1$

$$\Rightarrow \frac{\Delta t}{\Delta x}\left(v_0 + \sqrt{g h_0}\right) \le 1$$

$$v_0 + \sqrt{g h_0} \le \frac{\Delta x}{\Delta t}$$

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$$v_i^{n+1} - v_i^n = \sigma\left(w_{i+1}^n - w_i^n\right)$$
$$w_i^{n+1} - w_i^n = \sigma\left(v_i^{n+1} - v_{i-1}^{n+1}\right)$$

$$w_i^{n+1} - 2w_i^n + w_i^{n-1} = w_i^{n+1} - w_i^n - w_i^n + w_i^{n-1} = w_i^{n+1} - w_i^n - \left(w_i^n - w_i^{n-1}\right)$$
$$= \sigma\left(v_i^{n+1} - v_{i-1}^{n+1}\right) - \sigma\left(v_i^n - v_{i-1}^n\right) = \sigma\left(v_i^{n+1} - v_i^n\right) - \sigma\left(v_{i-1}^{n+1} - v_{i-1}^n\right)$$
$$= \sigma^2\left(w_{i+1}^n - w_i^n\right) - \sigma^2\left(w_i^n - w_{i-1}^n\right)$$

$$\Rightarrow \quad w_i^{n+1} - 2w_i^n + w_i^{n-1} = \sigma^2\left(w_{i+1}^n - 2w_i^n + w_{i-1}^n\right) \quad \text{eq E8.2.13}$$

$$G = \begin{pmatrix} 1 & I r e^{I\phi/2} \\ I r e^{-I\phi/2} & 1 - r^2 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - 1 & I r e^{I\phi/2} \\ I r e^{-I\phi/2} & \lambda - 1 + r^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda - 1)(\lambda - 1 + r^2) + r^2 = 0$$

$$(\lambda-1)^2+r^2(\lambda-1)+r^2=0$$

$$\lambda-1=\frac{-r^2\pm\sqrt{r^4-4r^2}}{2}=\frac{-r^2\pm r\sqrt{r^2-4}}{2}$$

$$\lambda=1-\frac{r^2}{2}\pm\frac{r\sqrt{r^2-4}}{2}=\frac{1}{2}(2-r^2)\pm\frac{Ir\sqrt{4-r^2}}{2}$$

Now we are implicitly assuming here that $4-r^2>0$
$4>r^2$

If $r^2<4 \Leftrightarrow 4B^2\sin^2\phi/2<4$

$B^2\sin^2(\phi/2)<1 \Rightarrow |B|<1$

Then $|\lambda_\pm|^2=\frac{1}{4}(4-\cancel{4r^2}+r^4)+\frac{1}{4}r^2(\cancel{4}-r^2)$

$=\frac{1}{4}4=1$ we have conditional stability

If $4-r^2<0$ Then

$(\sqrt{4-r^2})^2\neq 4-r^2$ as this would be negative

If $4-r^2<0 \Rightarrow 4<4\sin^2(\phi/2)B^2 \quad r^2>4$

$1<\sin^2(\phi/2)B^2$

$\Rightarrow |B|>1$

Then $\lambda = \frac{1}{2}(2-r^2) \pm \frac{r\sqrt{r^2-4}}{2}$

$$|\lambda_\pm|^2 = \frac{1}{4}(2-r^2)^2 + \frac{r^2}{4}(r^2-4)$$

The term $\frac{r^2}{4}(r^2-4) > 1$ ∴ Thus $|\lambda_\pm|^2 > 1$

From $\lambda_\pm = \frac{1}{2}(2-r^2) \pm \frac{Ir}{2}\sqrt{4-r^2}$

Since it is the sign of a^2 that changes replace $G = \frac{a\Delta t}{\Delta x}$
w/ $G' = IG = Ia\frac{\Delta t}{\Delta x}$ $\Rightarrow r \Rightarrow r'$

$$\lambda_+ = \frac{1}{2}\left(2+4|G|^2\sin^2(\phi/2)\right) + \frac{I^2}{2}\left(2|G|\sin(\phi/2)\sqrt{4+4|G|^2\sin^2(\phi/2)}\right)$$

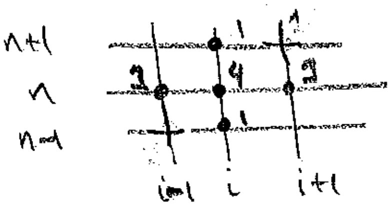
$$\lambda_- = \frac{1}{2}\left(2+4|G|^2\sin^2(\phi/2)\right) - \frac{I^2\left(2|G|\sin(\phi/2)\sqrt{4+4|G|^2\sin^2(\phi/2)}\right)}{2}$$

Then $\lambda_- = 1+2|G|^2\sin^2(\phi/2) + 2|G|\sin(\phi/2)\sqrt{1+|G|^2\sin^2(\phi/2)}$ E 8.2.20

If $|a^2| = 1$ $\quad G^2 = \frac{\Delta t^2}{\Delta x^2} = 1$ w/ equal sized steps

$$w_{i+1}^{n+1} - 2w_i^n + w_{i-1}^{n-1} + w_{i+1}^n - 2w_i^n + w_{i-1}^n = 0$$

$$w_{i+1}^{n+1} + w_{i-1}^{n+1} + w_{i+1}^n + w_{i-1}^n - 4w_i^n = 0$$



Given initial representation of the initial condition numerically discretize to find Fourier representation. Then the numerical "propagation" operator is given by

$V^n = G^n \cdot V^0$ or powers of the matrix G.

$$G = e^{-I\omega\Delta t} \qquad \omega = \xi + I\eta$$

$$= e^{-I(\xi + I\eta)\Delta t} = e^{-I\xi\Delta t} \underbrace{e^{+\eta\Delta t}}_{= |G|}$$

$$G = 1 - 4\beta \sin^2(\phi/2) \qquad \beta = \frac{\alpha \Delta t}{\Delta x^2}$$

$$|G| < 1$$

$$|1 - 4\beta \sin^2(\phi/2)| \le 1$$

$$-1 \le 1 - 4\beta \sin^2(\phi/2) \le 1$$

$$-2 \le -4\beta \sin^2(\phi/2) \le 0$$

$$0 \le \beta \sin^2(\phi/2) \le \frac{1}{2}$$

$$\Rightarrow 0 \le \beta \le 1/2$$

$$\Rightarrow 0 \le \alpha \quad + \quad \frac{\alpha \Delta t}{\Delta x^2} \le \frac{1}{2}$$

$\tilde{v} = \hat{v} e^{-I\tilde{\omega} t} e^{Ikx}$ This is the continuous Fourier decomposition.

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}$$

$$-I\tilde{\omega}\,\hat{v}\,\cancel{e^{-I\hat{\omega} t}}\,\cancel{e^{Ikx}} = \alpha \hat{v}\,\cancel{e^{-I\hat{\omega} t}} (-k^2)\,\cancel{e^{Ikx}}$$

$$-I\hat{\omega} = -\alpha k^2 \Rightarrow \tilde{\omega}(k) = -I\alpha k^2 \qquad \alpha = \frac{\Delta x^2}{\Delta t}\beta$$

$$\hat{\omega}(k) = -I\frac{\Delta x^2}{\Delta t}\beta k^2 = \underbrace{-I\frac{\beta\phi^2}{\Delta t}}_{\text{pure imaginary part of } \omega(k)} \quad \text{eq } 8,3,22 \qquad \phi = k_j \Delta x = \frac{j\pi}{N} \quad \text{discrete } \phi$$

$$\phi \text{ cont} = k\Delta x$$

$$\hat{v} = \hat{v} e^{-I(-I\alpha k^2)} e^{Ikx} = \hat{v} e^{-\alpha k^2} e^{Ikx}$$

Then
$$\varepsilon_D = \frac{|G|}{e^{\hat{\imath}\Delta t}} = \frac{1-4\beta\sin^2(\phi/2)}{e^{-\beta\phi^2/\Delta t \cdot \Delta t}} = \frac{1-4\beta\sin^2(\phi/2)}{e^{-\beta\phi^2}}$$

$$\varepsilon_D = \frac{1-4\beta\left(\frac{\phi}{2} - \frac{\phi^3}{3!\,2^3} + \frac{\phi^5}{5!\,2^5} + O(\phi^7)\right)^2}{1-\beta\phi^2 + \frac{\beta^2\phi^4}{2} - \frac{\beta^3\phi^6}{6} + O(\phi^8)}$$

$$= \frac{1-4\beta\left(\frac{\phi^2}{4} - \frac{2\phi^4}{3!\,2^4} + O(\phi^6)\right)}{1-\beta\phi^2 + \frac{\beta^2\phi^4}{2} + O(\phi^6)} = \frac{1-\beta\phi^2 + \frac{\beta\phi^4}{12} + O(\phi^6)}{1-\beta\phi^2 + \frac{\beta^2\phi^4}{2} + O(\phi^6)}$$

w/ MMA

$$\varepsilon_D = 1 + \left(\frac{B}{12} - \frac{B^2}{2}\right)\phi^4 + O(\phi^6) \qquad B = \frac{\alpha \Delta t}{\Delta x^2}$$

$$= 1 + \left(\frac{\alpha \Delta t}{12 \Delta x^2} - \frac{1}{2}\frac{\alpha^2 \Delta t^2}{\Delta x^4}\right) k^4 \Delta x^4 + O(k^6) \qquad \phi = k \Delta x$$

$$= 1 - \frac{1}{2}\alpha^2 \Delta t^2 k^4 + \frac{\alpha k^4}{12} \Delta x^2 \Delta t + O(k^6)$$

low freq $k \approx 0$ high freq $k \approx \frac{\pi}{N}$

$\phi \approx 0$ $\phi = \pi$

$$\frac{B}{12} - \frac{B^2}{2} = 0 \Rightarrow B = 0 \text{ or } \frac{1}{6} - B = 0$$

Higher order term is

$$\left(-\frac{B}{360} + \frac{B^2}{12} - \frac{B^3}{3}\right)\phi^6 \xrightarrow[B=1/6]{} \frac{1}{3240}\phi^6$$

$$\parallel$$

$$\left(-\frac{\alpha \Delta t}{360 \Delta x^2} + \frac{\alpha^2 \Delta t^2}{12 \Delta x^4} - \frac{\alpha^3 \Delta t^3}{3 \Delta x^6}\right) k^6 \Delta x^6$$

$$= \left(-\frac{\alpha \Delta t \Delta x^4}{360} + \frac{\alpha^2 \Delta t^2 \Delta x^2}{12} - \frac{\alpha^3 \Delta t^3}{3}\right) k^6$$

Think error $O(\Delta t, \Delta x^4)$

$u_t + a u_x = 0$ $\qquad \hat{u} = \hat{v} e^{Ikx} e^{-I\omega t}$ one fourier mode

$-I\omega + aIk = 0$

$\omega = ak$ $\qquad \tilde{u} = \hat{v} e^{Ikx} e^{-Iakt}$

$\hat{\omega} = ka = \tilde{\xi} \qquad \hat{\eta} = 0 \qquad \varepsilon_D = \dfrac{|G|}{e^{\tilde{\eta}\Delta t}} = |G|$

$$\varepsilon_\phi = \frac{\Phi}{\hat{\Phi}} = \frac{\Phi}{ka\Delta t} = \frac{\Phi}{k\sigma}$$

$\varepsilon_\phi > 1 \quad \Rightarrow \quad \Phi > k\sigma \qquad \sin(x - \Phi)$

computed wave crests shifted to right of $k\sigma$ location.

let $\bar{a}$ = computed wave speed

$$\Phi = \bar{a} k \Delta t$$

$$\bar{a} = \frac{\Phi}{k\Delta t} = \frac{\Phi}{k\Delta t a} a = \varepsilon_\phi a$$

$\therefore$ if $\varepsilon_\phi > 1$ $\quad \bar{a} > a$ $\quad$ Also $\quad \varepsilon_\phi = \dfrac{\bar{a}}{a}$

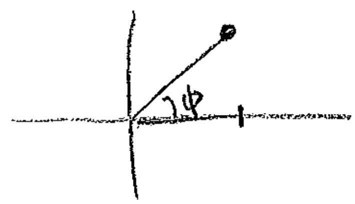
$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{\sigma}{2}(U_{i+1}^n - U_{i-1}^n)$$

let $U_i^n = v^n e^{Iki\Delta x}$

$$v^{n+1} \cancel{e^{Iki\Delta x}} = \frac{1}{2}(e^{Ik\Delta x} + e^{-Ik\Delta x}) v^n \cancel{e^{Iki\Delta x}} - \frac{\sigma}{2}(e^{Ik\Delta x} - e^{-Ik\Delta x}) v^n \cancel{e^{Iki\Delta x}}$$

$$v^{n+1} = \underbrace{\left(\cos(k\Delta x) - \sigma I \sin(k\Delta x)\right)}_{G} v^n$$

$$G = \cos\phi - \sigma I \sin\phi$$



$$|G| < 1 \iff |\sigma| < 1$$

$\{ \phi = k\Delta x \}$

$$|G| = (\cos^2\phi + \sigma^2 \sin^2\phi)^{1/2}$$

$$\Phi = \tan^{-1}\left(\frac{-\sigma\sin\phi}{\cos\phi}\right) = -\tan^{-1}(\sigma\tan\phi)$$

got sign error !! (No the definition of Φ in $\varepsilon_\phi = \frac{\Phi}{\sigma\phi}$ is

$\varepsilon_D = |G|$ How much does the solution dampen at each time step?

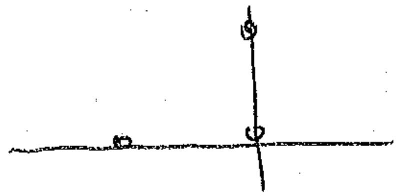
$\left\{ v^{n+1} = G v^n \text{ \& } G = |G| e^{-I\Phi} \right.$ thus minus sign is here

$$\varepsilon_\phi = \frac{\Phi}{\sigma\phi} = \frac{\tan^{-1}(\sigma\tan\phi)}{\sigma\phi}$$

$\sigma = 1 \qquad \varepsilon_D = 1 \qquad \varepsilon_\phi = \dfrac{\tan^{-1}(\tan\phi)}{\sigma\phi} \cong \dfrac{1}{\sigma}$

Explicit upwind $u_t + a u_x = 0$ $u = f(x-at)$

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = -\frac{a}{\Delta x}(u_i^n - u_{i-1}^n)$$



$$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n) \qquad \sigma = \frac{a\Delta t}{\Delta x}$$

let $u_i^n = v^n e^{iI\phi}$ $\phi = k\Delta x$ sum over k to get full

$$v^{n+1}\cancel{e^{iI\phi}} = v^n \cancel{e^{iI\phi}} - \sigma(1 - e^{-I\phi})\, v^n \cancel{e^{iI\phi}}$$

$$v^{n+1} = \underbrace{(1-\sigma+\sigma e^{-I\phi})}\, v^n$$

$$G = 1 - \sigma + \sigma\cos\phi - \sigma I \sin\phi$$

$$= 1 - (1-\cos\phi)\sigma - \sigma I \sin\phi$$

$$|G| = \left((1-\sigma+\sigma\cos\phi)^2 + \sigma^2\sin^2\phi\right)^{1/2} = \left((1-\sigma)^2 + 2(1-\sigma)\sigma\cos\phi + \sigma^2\cos^2\phi + \sigma^2\sin^2\phi\right)^{1/2}$$

$$= \left((1-\sigma)^2 + \sigma^2 + \sigma(1-\sigma)\, 2\cos\phi\right)^{1/2} \quad *$$

Can I get this result easier w/ complex #'s

$$G = 1-\sigma+\sigma e^{-I\phi}$$

$$|G|^2 = G\bar{G} = (1-\sigma+\sigma e^{-I\phi})(1-\sigma+\sigma e^{I\phi})$$

$$|G|^2 = (1-\sigma)^2 + \sigma(1-\sigma)e^{I\phi} + \sigma(1-\sigma)e^{-I\phi} + \sigma^2$$

$$= (1-\sigma)^2 +$$

$$= (1-\sigma)\left[1-\sigma+\sigma e^{I\phi}\right] + \sigma\left[\sigma + (1-\sigma)e^{-I\phi}\right]$$

$\cos\phi$ = in terms of $\sin^2$ $\qquad \sin^2\phi/2 = \dfrac{1-\cos\phi}{2}$

$$\rightarrow \cos\phi = 1 - 2\sin^2(\phi/2)$$

Then From *

$$|G| = \left((1-\sigma)^2 + \sigma^2 + 2\sigma(1-\sigma)(1-2\sin^2(\phi/2))\right)^{1/2}$$

$$= \left((1-\sigma)^2 + \sigma^2 + 2\sigma(1-\sigma) - 4\sigma(1-\sigma)\sin^2(\phi/2)\right)^{1/2}$$

$$= \left(1 - \cancel{2\sigma} + \cancel{\sigma^2} + \cancel{\sigma^2} + \cancel{2\sigma} - \cancel{2\sigma^2} - 4\sigma(1-\sigma)\sin^2(\phi/2)\right)^{1/2}$$

$$= \left(1 - 4\sigma(1-\sigma)\sin^2(\phi/2)\right)^{1/2} \qquad E8,3,6$$

$$\varepsilon_\phi = \tan^{-1}\left[\frac{-\sigma\sin\phi}{1-\sigma+\sigma\cos\phi}\right] \qquad \text{got sign difference}$$

$$\overline{\qquad\qquad}$$

$\sigma\phi$

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\sigma^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$v^{n+1}\cancel{e^{iI\phi}} = v^n\cancel{e^{iI\phi}} - \frac{\sigma}{2}(e^{+I\phi} - e^{-I\phi})v^n\cancel{e^{iI\phi}} + \frac{\sigma^2}{2}(e^{I\phi} - 2 + e^{-I\phi})v^n\cancel{e^{iI\phi}}$$

$$v^{n+1} = \Big[1 - \frac{\sigma}{2}(2I)\sin\phi + \frac{\sigma^2}{2}\underbrace{(e^{I\phi/2} - e^{-I\phi/2})^2}_{(2I\sin(\phi/2))^2}\Big]v^n$$

$$\Rightarrow v^{n+1} = \underbrace{\left[1 - I\sigma\sin\phi - 2\sigma^2\sin^2(\phi/2)\right]}_{G}v^n$$

$$G = 1 - I\sigma\sin\phi - \cancel{2}\frac{\sigma^2}{\cancel{2}}(1-\cos\phi) = 1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi)$$

$$= 1 - \sigma^2 + \sigma^2\cos\phi - I\sigma\sin\phi$$

I.8.3.12

$$= 1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi)$$

If $\sigma^2 < 1/2$ $\quad 1-\sigma^2 > 1/2$

$$|G|^2 = (1 - 2\sigma^2\sin^2(\phi/2))^2 + \sigma^2\sin^2\phi$$

$$= 1 - 4\sigma^2\sin^2(\phi/2) + 4\sigma^4\sin^4(\phi/2) + \sigma^2\sin^2\phi$$

... How manipulate?
What is an st forward
Algebraic way of doing this?

$$|G|^2 = \left(1 - \sigma^2(1-\cos\phi)\right)^2 + \sigma^2\sin^2\phi$$

$$= 1 - 2\sigma^2(1-\cos\phi) + \sigma^4(1-\cos\phi)^2 + \sigma^2\sin^2\phi$$

$$= 1 + 4\sigma^4\sin^4(\phi/2)$$

$\left(1-\cos\phi = 2\sin^2(\phi/2)\right)$

$$= -2\sigma^2 + 2\sigma^2\cos\phi + \sigma^2\sin^2\phi$$

(work only w/ the terms $\underline{}$)

$$= \quad '' \quad '' \quad + \sigma^2 - \sigma^2\cos^2\phi$$

$$= -\sigma^2 + 2\sigma^2\cos\phi - \sigma^2\cos^2\phi$$

$$= -\sigma^2(1 - 2\cos\phi + \cos^2\phi)$$

$$= -\sigma^2(1-\cos\phi)^2 = -\sigma^2 2^2\sin^4(\phi/2)$$

$$= -4\sigma^2\sin^4(\phi/2)$$

Then $|G|^2 = 1 + 4\sigma^4\sin^4(\phi/2) - 4\sigma^2\sin^4(\phi/2)$

$$= 1 + 4\sigma^2(\sigma^2 - 1)\sin^4(\phi/2)$$

$$= 1 - 4\sigma^2(1-\sigma^2)\sin^4(\phi/2) \qquad \text{E8.3.14}$$

$$G = |G|e^{-I\Phi}$$

$$\Phi = -\tan^{-1}\left(\frac{-\sigma\sin\phi}{1-\sigma^2(1-\cos\phi)}\right) = \tan^{-1}\left(\frac{\sigma\sin\phi}{1-\sigma^2(1-\cos\phi)}\right)$$

$$\epsilon_\phi = \frac{\tan^{-1}\left[\dfrac{\sigma \sin\phi}{1 - 2\sigma^2 \sin^2(\phi/2)}\right]}{\sigma\phi} \qquad E8.3.15$$

expand for $\phi \approx 0$ low frequency modes let $Z^n = U^{n-1}$ then 8,3,32 becomes

$$U^{n+1} + b_0 U^n + b_1 U^{n-1} = CU^n + \tilde{Q}$$

$$\Rightarrow U^{n+1} + b_0 U^n + b_1 Z^n = CU^n + \tilde{Q}$$

$$\left.\begin{aligned} U^{n+1} &= -b_1 Z^n - b_0 U^n + CU^n + \tilde{Q} \\ Z^{n+1} &= U^n \end{aligned}\right\} \text{ eq } 8,3,35$$

let $W^n = \begin{pmatrix} U^n \\ Z^n \end{pmatrix}$

$$W^{n+1} = \bar{C} W^n + \bar{Q}$$

$$u_i^{n+1} - u_i^{n-1} = -a\frac{\Delta t}{\Delta x}(u_{i+1}^n - u_{i-1}^n) = -\sigma(u_{i+1}^n - u_{i-1}^n)$$

let $z_i^n = u_i^{n-1}$ then

$$u_i^{n+1} = z_i^n - \sigma(u_{i+1}^n - u_{i-1}^n) \qquad E8,3,17$$

$$z_i^{n+1} = u_i^n$$

let $\bar{w}_i^n = \begin{pmatrix} u_i^n \\ z_i^n \end{pmatrix} \Rightarrow \bar{w}_i^{n+1} = \bar{C}\,\bar{w}_i^n$

$$\bar{C} = ?$$

$$\begin{pmatrix} U_i^{n+1} \\ Z_i^{n+1} \end{pmatrix} = \begin{pmatrix} -\sigma(E-E^{-1}) & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_i^n \\ Z_i^n \end{pmatrix}$$

$$\bar{C} = \begin{pmatrix} -\sigma(E-E^{-1}) & 1 \\ 1 & 0 \end{pmatrix} \quad \text{eq E8.3.19}$$

Then $G = ?$ let $w_i^n = \hat{w}^n e^{Ii\phi}$ w_i^n & $\hat{w}^n$ are vectors

$\phi = k\Delta x$

Be one Fourier mode

Then
$$\hat{w}^{n+1} \cancel{e^{Ii\phi}} = \underbrace{\begin{pmatrix} -\sigma(e^{I\phi} - e^{-I\phi}) & 1 \\ 1 & 0 \end{pmatrix}}_{G \text{ growth matrix}} \hat{w}^n \cancel{e^{Ii\phi}} \quad \text{eq E8.3.20}$$

For stability eigenvalues of G must lie in the unit circle

eig

$$\begin{vmatrix} \lambda + \sigma(2I)\sin\phi & 1 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 2I\sigma\sin\phi\,\lambda - 1 = 0 \Rightarrow \lambda = \frac{-2I\sigma\sin\phi \pm \sqrt{-4\sigma^2\sin^2\phi + 4}}{2}$$

$$\Rightarrow \lambda = -I\sigma\sin\phi \pm \sqrt{1-\sigma^2\sin^2\phi} \quad \text{eq E8.3.21}$$

2nd method applied to 8.3.33

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\frac{a}{2\Delta x}(u_{i+1}^n - u_{i-1}^n)$$

$$\frac{(G v^n - G^{-1} v^n) e^{iI\phi}}{\cancel{2}\Delta t} = -\frac{a}{\cancel{2}\Delta x}(E - E^{-1}) v^n e^{iI\phi}$$

$$= -\frac{a}{\Delta x}(e^{I\phi} - e^{-I\phi}) v^n e^{iI\phi}$$

$$\Rightarrow \frac{G - G^{-1}}{\Delta t} = -\frac{a}{\Delta x} 2I \sin\phi$$

$$G - G^{-1} = -\bar{\sigma}\, 2I \sin\phi$$

$\Rightarrow G^2 - 1 = -\bar{\sigma} 2I \sin\phi\, G \Rightarrow G^2 + 2I\bar{\sigma}\sin\phi\, G - 1 = 0$ same

eq as for the eigenvalues on previous page.

If $\bar{\sigma} > 1$ let $\phi = \frac{\pi}{2}$ $\sin\phi = 1$ &

$$G = -I\bar{\sigma} \pm \sqrt{1 - \bar{\sigma}^2} = -I\bar{\sigma} \pm I\sqrt{\bar{\sigma}^2 - 1}$$

$$= \left(-\bar{\sigma} \pm \sqrt{\bar{\sigma}^2 - 1}\right) I$$

! Then negative sign gives a value of G that is large in magnitude ~~since~~ since $\bar{\sigma}$ is

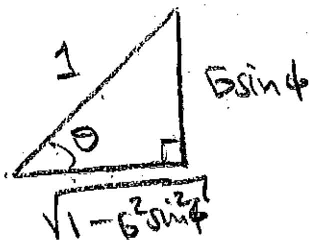
If $\sigma < 1$

$$|\sigma| = \sigma^2 \sin^2\phi + (1 - \sigma^2 \sin^2\phi) = 1 \qquad \text{eq } E8.3.24$$

$$\varepsilon_\phi = \frac{-\tan^{-1}\left[\frac{-\sigma \sin\phi}{\pm\sqrt{1-\sigma^2\sin^2\phi}}\right]}{\sigma\phi} = \frac{\pm\tan^{-1}\left[\frac{\sigma \sin\phi}{\sqrt{1-\sigma^2\sin^2\phi}}\right]}{\sigma\phi}$$

Now

Soh
coa



so $\tan\theta = \dfrac{\sigma\sin\phi}{\sqrt{1-\sigma^2\sin^2\phi}}$

But the $\angle\theta$ is also $\theta = \sin^{-1}\left[\sigma \sin\phi\right]$

$$\therefore \varepsilon_\phi = \pm\frac{\sin^{-1}(\sigma\sin\phi)}{\sigma\phi} \qquad \text{eq } E8.3.25$$

$$U_i^{n+1} - U_i^{n-1} = 2\left(\frac{\alpha \Delta t}{\Delta x^2}\right)\left(U_{i+1}^n - (U_i^{n+1} + U_i^{n-1}) + U_{i-1}^n\right) \qquad E8.3,27a$$

$$U_i^{n+1} - U_i^{n-1} = 2BU_{i+1}^n - 2BU_i^{n+1} - 2BU_i^{n-1} + 2BU_{i-1}^n$$

$$(1+2B)U_i^{n+1} = (1-2B)U_i^{n-1} + 2B(U_{i+1}^n + U_{i-1}^n) \qquad E8.3,27b$$

let $Z_i^n = U_i^{n-1}$ $\qquad Z_i^{n+1} = U_i^n$

$$\begin{pmatrix} (1+2B) & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} U_i^{n+1} \\ Z_i^{n+1} \end{pmatrix} = \begin{pmatrix} 2B(E+E^{-1}) & 1-2B \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_i^n \\ Z_i^n \end{pmatrix}$$

$$\begin{pmatrix} U_i^{n+1} \\ Z_i^{n+1} \end{pmatrix} = \frac{1}{(1+2B)}\begin{pmatrix} 1 & 0 \\ 0 & 1+2B \end{pmatrix}\begin{pmatrix} 2B(E+E^{-1}) & 1-2B \\ 1 & 0 \end{pmatrix}\begin{pmatrix} U_i^n \\ Z_i^n \end{pmatrix}$$

$$= \underline{\frac{1}{(1+2B)}\begin{pmatrix} 2B(E+E^{-1}) & 1-2B \\ 1+2B & 0 \end{pmatrix}}\begin{pmatrix} \; \end{pmatrix}$$

$\bar{c}$ eq E8.3,29

$$G = \begin{pmatrix} \frac{2B\,2\cos\phi}{1+2B} & \frac{1-2B}{1+2B} \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} \lambda - \frac{4B\cos\phi}{1+2B} & -\frac{(1-2B)}{1+2B} \\ -1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 - \frac{4B\cos\phi}{1+2B}\lambda - \frac{(1-2B)}{(1+2B)} = 0$$

$$(1+2B)\lambda^2 - 4B\cos\phi\,\lambda - (1-2B) = 0$$

$$\lambda = \frac{+4B\cos\phi \pm \sqrt{16B^2\cos^2\phi + 4(1+2B)(1-2B)}}{2(1+2B)}$$

$$= \frac{2B\cos\phi \pm \sqrt{4B^2\cos^2\phi + 1 - 4B^2}}{1+2B}$$

Radican $4B^2(\cos^2\phi - 1) = -4B^2\sin^2\phi$

$$\rightarrow \lambda = \frac{2B\cos\phi \pm \sqrt{1-4B^2\sin^2\phi}}{1+2B} \qquad \text{eq FB.3.31}$$

Require $|\lambda| < 1$

$$\Leftrightarrow \quad -1 < \frac{2B\cos\phi \pm \sqrt{1-4B^2\sin^2\phi}}{1+2B} < 1$$

$$-1-2B < 2B\cos\phi \pm \sqrt{1-4B^2\sin^2\phi} < 1+2B$$

$$-1-2B-2B\cos\phi < \pm\sqrt{1-4B^2\sin^2\phi} < 1+2B-2B\cos\phi$$

$$\Leftrightarrow -1-2B(1+\cos\phi) < \pm\sqrt{1-4B^2\sin^2\phi} < 1+2B(1-\cos\phi)$$

Now $1+\cos\phi > 0 \quad \forall \phi$

$+ \quad 1-\cos\phi > 0 \quad \forall\phi$

$\therefore \quad -1-2B(1+\cos\phi) < -1 \quad \forall \phi$

$+ \quad 1+2B(1-\cos\phi) > 1 \quad \forall \phi$

$1-4B^2\sin^2\phi > 0$

$\frac{1}{4\sin^2\phi} > B^2 \quad \forall \phi$

$\Rightarrow B^2 < \frac{1}{4}$

$B < \frac{1}{2}$

$\therefore$ we require

$$-1-2B(1+\cos\phi) < -\sqrt{1-4B^2\sin^2\phi} \quad + \quad +\sqrt{1-4B^2\sin^2\phi} < 1+2B(1-\cos\phi)$$

$$\Rightarrow 1+2B(1+\cos\phi) < \sqrt{1-4B^2\sin^2\phi} \quad + \quad \sqrt{1-4B^2\sin^2\phi} < 1+2B(1-\cos\phi)$$

if $1-4B^2\sin^2\phi > 0 \Rightarrow B^2 < 1/4 \Rightarrow \boxed{B < 1/2}$

$$\cancel{1}+4B(1+\cos\phi)+4B^2(1+\cos\phi)^2 < \cancel{1}-4B^2\sin^2\phi \quad + \quad \cancel{1}-4B^2\sin^2\phi < \cancel{1}+4B(1-\cos\phi)+4B^2(1-\cos\phi)^2$$

$\div 4B$

$$(1+\cos\phi)+B(1+\cos\phi)^2 < -B\sin^2\phi \quad + \quad -B\sin^2\phi < (1-\cos\phi)+B(1-\cos\phi)^2$$

$(1+\cos\phi)+\left((1+\cos\phi)^2+\sin^2\phi\right)B < 0$

$(1+2\cos\phi+1)B < -(1+\cos\phi)$

$2(1+\cos\phi)B < -(1+\cos\phi)$

$B < -1/2$

$-(1-\cos\phi) < B(\underbrace{1-2\cos\phi+\cos^2\phi+\sin^2\phi})$

$< B(2-2\cos\phi)$

$+ \quad B > -1/2$

$\rightarrow\leftarrow$ Thus $B \not< 1/2$ or

$B > 1/2$

$\Rightarrow$ If $B > 1/2$

$$\Rightarrow (1+2B(1+\cos\phi))^2 < 4B^2\sin^2\phi - 1 \quad + \quad (1+2B(1-\cos\phi))^2 > 4B^2\sin^2\phi - 1$$

$$1+4B(1+\cos\phi)+4B^2(1+\cos\phi)^2 < 4B^2\sin^2\phi - 1 \qquad 1+4B(1-\cos\phi)+4B^2(1-\cos\phi)^2 > 4B^2\sin^2\phi - 1$$

$$2+4B(1+\cos\phi)+4B^2\left[(1+\cos\phi)^2-\sin^2\phi\right] < 0 \qquad 2+4B(1-\cos\phi)+4B^2\left[(1-\cos\phi)^2-\sin^2\phi\right] > 0$$

$$1+2B(1+\cos\phi)+2B^2\left[(1+\cos\phi)^2-\sin^2\phi\right] < 0 \qquad 1+2B(1-\cos\phi)+2B^2\left[(1-\cos\phi)^2-\sin^2\phi\right] > 0$$

(1) (2)

(1) gives

$$2\left[(1+\cos\phi)^2-\sin^2\phi\right]B^2+2(1+\cos\phi)B+1 < 0$$

roots are

$$B = \frac{-2(1+\cos\phi) \pm \sqrt{4(1+\cos\phi)^2-4\cdot 2\left[(1+\cos\phi)^2-\sin^2\phi\right]}}{2\left[(1+\cos\phi)^2-\sin^2\phi\right]}$$

$=$

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Lax-Wendroff $\quad |G(\phi)|^2 = 1 - 4\sigma^2(1-\sigma^2)\sin^4(\phi/2)$

$$|G(\pi)|^2 = 1 - 4\sigma^2(1-\sigma^2)$$

$$\hat{V}_G(k) = \frac{d\hat{\omega}}{dk} \qquad \left\{ e^{i(kx - \omega(k)t)} \right\}$$

$$G = e^{\eta \Delta t} e^{-I\Phi} = e^{\eta\Delta t} e^{-I\xi\Delta t} \qquad \text{eq } 8.3.9$$

$$=$$

$$V_G(k) = \frac{d\xi}{dk} = Re\left(\frac{d\omega}{dk}\right)$$

Eq 8.3.6 is $G = e^{-I\omega\Delta t}$ + E8.3.23 is $G = -I\sigma\sin\phi \pm \sqrt{1-\sigma^2\sin^2\phi}$

$$= |G| e^{-I\Phi}$$

$$\Rightarrow \cancel{I}\omega\Delta t = \cancel{I}\sin^{-1}(\sigma\sin\phi)$$

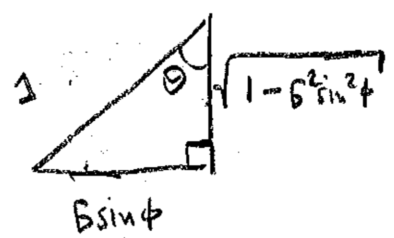
$$= J\exp\{I\Theta\}$$

$$\sigma\sin\phi = \sin\omega\Delta t \qquad \text{eq } 8.3.42$$

$$-\Theta = \tan^{-1}\left[\frac{+\sigma\sin\phi}{\pm\sqrt{1-\sigma^2\sin^2\phi}}\right]$$

$$\phi = (k\Delta x)$$

$$V_G = \frac{d\omega}{dk}$$



$$\Delta x\, \sigma\cos(k\Delta x) = \cos(\omega\Delta t)\Delta t \frac{d\omega}{dk}$$

$$\Rightarrow V_G = \frac{d\omega}{dk} = \frac{\Delta x\, \sigma\cos(\phi)}{\Delta t\cos(\omega\Delta t)}$$

$$\cos\Theta = \sqrt{1-\sigma^2\sin^2\phi}$$

$$\sin\Theta = \sigma\sin\phi$$

$$\sigma = \frac{a\Delta t}{\Delta x} \quad \therefore \quad \frac{\Delta x\,\sigma}{\Delta t} = a$$

$$V_G = \frac{a\cos(\phi)}{\cos(\omega\Delta t)} = \frac{a\cos(\phi)}{\sqrt{1-\underbrace{\sin^2(\omega\Delta t)}}} = \frac{a\cos(\phi)}{\sqrt{1-\beta^2\sin^2(\phi)}} \quad \text{eq } B,3,43$$

$\phi = k\Delta x \qquad k = \frac{2\pi}{\lambda}$

$\frac{2\pi}{\lambda}\Delta x = \frac{\pi}{4}$

$\lambda = 8\Delta x$

$$V_G = \frac{1\cos(\pi/4)}{(1-(.4)^2\sin^2(\pi/4))^{1/2}} = \frac{1/\sqrt{2}}{(1-(4/10)^2(1/2))^{1/2}} = \frac{1/\sqrt{2}}{(1-(2/5)^2(1/2))^{1/2}}$$

$= \qquad\qquad = .737$

let $U^n_{ij} = v^n e^{Ii\phi_x} e^{Ij\phi_y}$ put into 8.4.3

$$v^{n+1} - v^n = \alpha\Delta t\left[\frac{v^n e^{I\phi_x} - 2v^n + v^n e^{-I\phi_x}}{\Delta x^2} + \frac{v^n e^{I\phi_y} - 2v^n + v^n e^{-I\phi_y}}{\Delta y^2}\right]$$

Denote by G the matrix that takes v^n to v^{n+1} or $v^{n+1} = Gv^n$

$$G - 1 = \beta\left[\underbrace{(e^{I\phi_x} + e^{-I\phi_x} - 2)}_{(e^{I\phi_x/2} - e^{-I\phi_x/2})^2} + \left(\frac{\Delta x}{\Delta y}\right)^2(e^{I\phi_y} + e^{-I\phi_y} - 2)\right]$$

$$4(-1)\sin^2(\phi_x/2) \quad + \quad 4(-1)\left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)$$

$$G - 1 = -4\beta\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right) \qquad \text{eq } 8.4.8$$

For stability we have

$$\left|1 - 4\beta\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right)\right| < 1$$

$$\Leftrightarrow \qquad -2 < -4\beta\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right) < 0$$

$$0 < \beta\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right) < \tfrac{1}{2}$$

1st condition gives

since $\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right) > 0$

$$\beta > 0 \Rightarrow \frac{\alpha\Delta t}{\Delta x^2} > 0 \Rightarrow \alpha > 0$$

+ also $\beta\left(\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2)\right) < 1/2$

This will hold if

$$\beta\left(1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right) \le 1/2$$

Because $\sin^2(\phi_x/2) + \left(\frac{\Delta x}{\Delta y}\right)^2 \sin^2(\phi_y/2) \le 1 + \left(\frac{\Delta x}{\Delta y}\right)^2$

$$\to \alpha \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \le \frac{1}{2} \qquad \text{eq } 8.4.11$$

If $\Delta x = \Delta y$ we obtain

$$\frac{2\alpha \Delta t}{\Delta x^2} \le \frac{1}{2} \Rightarrow \Delta t \le \frac{\Delta x^2}{4\alpha} \qquad \text{eq } 8.4.12$$

Given

$$U^{n+1}_{i,j} = \frac{1}{4}\left(U^n_{i,j+1} + U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j-1}\right)$$

$$- \frac{\Delta t}{2\Delta x} A\left(U^n_{i+1,j} - U^n_{i-1,j}\right) - \frac{\Delta t}{2\Delta y} B\left(U^n_{i,j+1} - U^n_{i,j-1}\right)$$

Since this is a linear differential equation, the solution U^n_{ij} can be decomposed in a discrete Fourier series

$$U^n_{ij} = \hat{U}^n e^{i\phi_x I} e^{j\phi_y I}$$

$\phi_x = k_x \Delta x$
$\phi_y = k_y \Delta y$

k_x, k_y discrete wave vectors

$\left\{ k_x = \frac{2\pi}{n_x} \right.$?

Then we get

$$\hat{U}^{n+1} = \frac{1}{4}\left(e^{\phi_y I} + e^{\phi_x I} + e^{-\phi_x I} + e^{-\phi_y I}\right)\hat{U}^n$$

$$- \frac{\Delta t}{2\Delta x} A\left(e^{\phi_x I} - e^{-\phi_x I}\right)\hat{U}^n - \frac{\Delta t}{2\Delta y} B\left(e^{\phi_y I} - e^{-\phi_y I}\right)\hat{U}^n$$

Then G Growth factor for the Fourier coefficient is

$$G = \frac{1}{4}\left(2\cos\phi_y + 2\cos\phi_x\right) I_{n\times n} - \frac{\Delta t}{2\Delta x} A\, 2I \sin(\phi_x) - \frac{\Delta t}{2\Delta y} B\, 2I\sin(\phi_y)$$

$$= \frac{1}{2}(\cos\phi_x + \cos\phi_y) - I\frac{\Delta t}{\Delta x}\sin(\phi_x) A - I\frac{\Delta t}{\Delta y}\sin(\phi_y) B \qquad \text{eq } 8.4.15$$

A Normal matrix G is $\Rightarrow$ $G^* G = G G^*$

Check this for this wy G.

$$G^* = \frac{1}{2}(\cos\phi_x + \cos\phi_y) + I\frac{\Delta t}{\Delta x}\sin(\phi_x) A + I\frac{\Delta t}{\Delta y}\sin(\phi_y) B$$

$$GG^* = \left[\frac{1}{2}(\cos\phi_x + \cos\phi_y) - I\frac{\Delta t}{\Delta x}\sin(\phi_x)A - I\frac{\Delta t}{\Delta y}\sin(\phi_y)B\right] \cdot$$

$$\left[\frac{1}{2}(\cos\phi_x + \cos\phi_y) + I\frac{\Delta t}{\Delta x}\sin(\phi_x)A + I\frac{\Delta t}{\Delta y}\sin(\phi_y)B\right]$$

$$= \frac{1}{4}(\cos\phi_x + \cos\phi_y)^2 + I$$

$G = C_1 - IC_2 - IC_3$ w/ $C_1 = \frac{1}{2}(\cos\phi_x + \cos\phi_y)I$

$$C_2 = -\frac{\Delta t}{\Delta x}\sin(\phi_x)A$$

$$C_3 = -\frac{\Delta t}{\Delta y}\sin(\phi_y)B$$

$$GG^* = (C_1 - IC_2 - IC_3)(C_1 + IC_2 + IC_3)$$

$$= C_1^2 + IC_1C_2 + IC_1C_3 - IC_2C_1 - I^2C_2^2 - I^2C_3^2 - IC_3C_1 - I^2C_3C_2 - I^2C_3^2$$

$$= C_1^2 + IC_1C_2 + IC_1C_3 - IC_2C_3 + C_2^2 + C_3^2 - IC_3C_1 + C_3C_2 + C_3^2$$

Now $C_3C_2 = C_2C_3$ + ~~same for $C_1 + C$~~

$C_1, C_2 + C_3$ all commute

$$= C_1^2 + IC_2C_1 + IC_3C_1$$

$= G^*G$. G is a normal matrix.

$$G = \frac{1}{2}(\cos\phi_x + \cos\phi_y) - \left(\frac{\Delta t}{\Delta x}\sin\phi_x A\right)I - \left(\frac{\Delta t}{\Delta y}\sin\phi_y B\right)I$$

$$\rho(GG^*) = \|G\|_{L^2}^2$$

$$\lambda(G) = \frac{1}{2}(\cos\phi_x + \cos(\phi_y)) - I\left[\frac{\Delta t}{\Delta x}\sin\phi_x \lambda(A) + \frac{\Delta t}{\Delta y}\sin\phi_y \lambda(B)\right]$$

We are told that for the L_2 norm

$$\rho^n(G) = \|G^n\|_{L^2} = \|G\|_{L^2}^n$$

$$\begin{cases} \cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} + \cdots \\ \sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{5!} + \cdots \end{cases}$$

Thus w/ $n = 1$ $\quad \rho(G) = \|G\|_{L^2}$

Assuming 8.4.16

$$\|G\|_{L^2}^2 = \frac{1}{4}(\cos\phi_x + \cos\phi_y)^2 + (\sigma_x\sin\phi_x + \sigma_y\sin\phi_y)^2$$

$$= \frac{1}{4}\left(1 - \frac{\phi_x^2}{2} + \frac{\phi_x^4}{4!} + \cdots + 1 - \frac{\phi_y^2}{2} + \frac{\phi_y^4}{4!} + \cdots\right)^2$$

$$+ \left(\sigma_x\left(\phi_x - \frac{\phi_x^3}{6} + \frac{\phi_x^5}{5!} + \cdots\right) + \sigma_y\left(\phi_y - \frac{\phi_y^3}{6} + \frac{\phi_y^5}{5!} + \cdots\right)\right)^2$$

$$= \frac{1}{4}\left(2 - \frac{1}{2}(\phi_x^2 + \phi_y^2) + \frac{1}{4!}(\phi_x^4 + \phi_y^4) + \cdots\right)^2$$

$$+ \left(\sigma_x\phi_x + \sigma_y\phi_y - \frac{1}{6}(\sigma_x\phi_x^3 + \sigma_y\phi_y^3) + \frac{1}{5!}(\sigma_x\phi_x^5 + \sigma_y\phi_y^5) + \cdots\right)^2$$

$$\|G\|_{L^2}^2 = \frac{1}{4}\left(4 - 2(\phi_x^2 + \phi_y^2) + O(\phi_x^4, \phi_y^4)\ \right)$$

$$+\left(b_x^2\phi_x^2 + 2b_x b_y \phi_x\phi_y + b_y^2\phi_y^2\right) + O(\phi_x^4, \phi_y^4)$$

$$= 1 - \frac{1}{2}(\phi_x^2 + \phi_y^2) + b_x^2\phi_x^2 + 2b_x b_y\phi_x\phi_y + b_y^2\phi_y^2 + O(\phi_x^4, \phi_y^4)$$

$$= 1 - \underbrace{\left[\left(\tfrac{1}{2} - b_x^2\right)\phi_x^2 + \left(\tfrac{1}{2} - b_y^2\right)\phi_y^2 - 2b_x b_y\phi_x\phi_y\right]}_{\text{must be possitive so}} + O(\ ,\ )$$

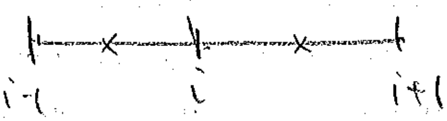
eq 6.4.10

$$\|G\|_{L^2}^2 < 1$$

$f = f(x,y)$

$A = f_{xx}$ $\quad B = f_{xy}$ $\quad C = f_{yy}$

$B^2 - AC$ is one sign



$$U_i^{n+1} = C(x,E)\,U_i^n$$

If

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x^2}\left[\alpha_{i+1/2}(U_{i+1}^n - U_i^n) - \alpha_{i-1/2}(U_i^n - U_{i-1}^n)\right]$$

$$U_i^{n+1} = 1 + \frac{\Delta t}{\Delta x^2}\left[\alpha(x_{i+1/2})(E-1) - \alpha(x_{i-1/2})(1-E^{-1})\right] \quad \text{eq 8.5.8}$$

$$C(x,E) = 1 - \frac{\Delta t}{\Delta x}\,a(x_{i-1/2})(1-E^{-1}) \quad \text{eq 8.5.2}$$

$$(U_i^n)_k = V^n e^{Iik\Delta x}$$

$$G(x,\phi) = 1 + \frac{\Delta t}{\Delta x^2}\left[\alpha(x)(e^{I\phi}-1) - \alpha(x)(1-e^{-I\phi})\right]$$

$$\neq G(x,\phi) = 1 - \frac{\Delta t}{\Delta x}\left(\alpha(x)(1-e^{-I\phi})\right)$$

$G(\phi) = \cos\phi - I\sigma \sin\phi$

$G(\pi) = -1 \qquad |G(\pi)| = 1$

E8.3.6

$$|G| = \left(1 - 4\sigma(1-\sigma)\sin^2(\phi/2)\right)^{1/2} \cong 1 - 2\sigma(1-\sigma)\phi^2/4 + \cdots$$

$$= 1 - \frac{\sigma}{2}(1-\sigma)\phi^2 + \cdots$$

$$|G(\pi)| = (1 - 4\sigma(1-\sigma))^{1/2}$$

$$= (1 - 4\sigma + 4\sigma^2)^{1/2} = \left[(1-2\sigma)^2\right]^{1/2}$$

$$= |1 - 2\sigma| \qquad \text{eq } 8.5.15$$

E8.3.13

$$G = 1 - I\sigma\sin\phi - \sigma^2(1-\cos\phi)$$

$$G \cong 1 - I\sigma\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}\right) - \sigma^2\left(1 - \left(1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} + \cdots\right)\right)$$

$$= 1 - I\sigma\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!}\right) - \sigma^2\left(\frac{\phi^2}{2} - \frac{\phi^4}{4!} + \cdots\right)$$

$$G = 1 - I\sigma\phi - \frac{\sigma^2}{2}\phi^2 + \frac{I\sigma}{6}\phi^3 + O(\phi^4)$$

$$|G| = (GG^*)^{1/2}$$

$$GG^* = 1 - I\sigma\phi - \frac{\sigma^2\phi^2}{2} + I($$

w/ MMA we obtain

$$|G| \cong 1 - \frac{\sigma^2}{2}(1/3 - \sigma^2/4)\phi^4 + O(\phi^6)$$

$$\left(u\frac{\partial u}{\partial x}\right)_i = u_i\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right)$$

$$u_i^n = \sum_{m=-N}^{N} V_m^n e^{Iik_m\Delta x} = \sum_{k_1} V(k_1)e^{Iik_1\Delta x}$$

Then putting in Above gives

$$\sum_{k_1} V(k_1)e^{Iik_1\Delta x}\cdot\left(\frac{1}{2\Delta x}\sum_{k_2} V(k_2)e^{Iik_2\Delta x}e^{Ik_2\Delta x} - \frac{1}{2\Delta x}\sum_{k_2} V(k_2)e^{Iik_2\Delta x}e^{-Ik_2\Delta x}\right)$$

$$= \sum_{k_1} V(k_1)e^{Iik_1\Delta x}\left(\frac{1}{2\Delta x}\sum_{k_2} V(k_2)e^{Iik_2\Delta x}\,2I\sin(k_2\Delta x)\right)$$

$$= \frac{I}{\Delta x} \sum_{k_1} \sum_{k_2} v(k_1) v(k_2) e^{Ii(k_1+k_2)\Delta x} \sin(k_2 \Delta x) \quad \text{eq } 8.5.18$$

$$G_{\text{leap frog}} = -I\sigma \sin\phi \pm \sqrt{1 - \sigma^2 \sin^2 \phi}$$

$$G_{L.F.}(\pi) = 0 \pm \sqrt{1-0} = \pm 1$$

Since $|G(\pi)| = 1$ leap frog scheme is not dissipative in the sense of Kreiss

07-22-01

pg 327 Hirsh Vol I

Eq 8.1.10 is $k_j = \frac{j\pi}{N\Delta x}$ $\quad j = 0, 1, 2, \cdots N$

$k_1 \Delta x =$

Check does $(2\pi k_1 + 2\pi k_2)\Delta x = 2\pi(k_1\Delta x + k_2 \Delta x)$

$= 2\pi\left(\frac{2\pi}{3} + \pi\right) = 2\pi\left(\frac{5\pi}{3}\right) =$

pg 330 Hirsh Vol I

$$b_3 U_{i+1}^{n+1} + b_2 U_i^{n+1} + b_1 U_{i-1}^{n+1} = a_3 U_{i+1}^n + a_2 U_i^n + a_1 U_{i-1}^n$$

$U^n = U$ constant $\forall\ i \ + n$

$$b_3 U^{n+1} + b_2 U^{n+1} + b_1 U^{n+1} = (a_3 + a_2 + a_1) U^n$$

$$(b_3 + b_2 + b_1) U^{n+1} = (a_3 + a_2 + a_1) U^n$$

why equal to 1?

$U^n = U^{n+1}$ if $\quad b_3 + b_2 + b_1 = a_3 + a_2 + a_1 = 1$

Amplification Factor

$$G(b_3 e^{I\phi} + b_2 + b_1 e^{-I\phi}) = a_3 e^{I\phi} + a_2 + a_1 e^{-I\phi}$$

$$G = \frac{a_3 e^{I\phi} + a_2 + a_1 e^{-I\phi}}{b_3 e^{I\phi} + b_2 + b_1 e^{-I\phi}}$$

know $b_2 = 1 - b_1 - b_3$

$a_2 = 1 - a_1 - a_3$ & we get

$$G = \frac{a_3 e^{I\phi} + 1 - a_1 - a_3 + a_1 e^{-I\phi}}{b_3 e^{I\phi} + 1 - b_1 - b_3 + b_1 e^{-I\phi}}$$

$$= \frac{a_3(e^{I\phi} - 1) + a_1(e^{-I\phi} - 1) + 1}{b_3(e^{I\phi} - 1) + b_1(e^{-I\phi} - 1) + 1}$$

$$= \frac{a_3(\cos\phi + I\sin\phi - 1) + a_1(\cos\phi - I\sin\phi - 1) + 1}{b_3(\cos\phi + I\sin\phi - 1) + b_1(\cos\phi - I\sin\phi - 1) + 1}$$

$$= \frac{1 + a_3(\cos\phi - 1) + a_3 I\sin\phi + a_1(\cos\phi - 1) - I a_1 \sin\phi}{''}$$

$$= \frac{1 - (a_3 + a_1)(1 - \cos\phi) + I(a_3 - a_1)\sin\phi}{''}$$

$$|G|^2 = GG^* = \frac{1 - (a_3 + a_1)(1 - \cos\phi) + I(a_3 - a_1)\sin\phi}{1 - (b_3 + b_1)(1 - \cos\phi) + I(b_3 - b_1)\sin\phi} \cdot \frac{1 - (a_3 + a_1)(1 - \cos\phi) - I(a_3 - a_1)\sin\phi}{1 - (b_3 + b_1)(1 - \cos\phi) - I(b_3 - b_1)\sin\phi}$$

The Denominator will be the exactly the same as the numerator & thus we work only w/ the Numerator

$$\left[1-(a_3+a_1)(1-\cos\phi)\right]^2 - I\left[1-(a_3+a_1)(\cancel{1-\cos\phi})\right](a_3-a_1)\sin\phi$$

$$+I\left[1-(a_3+a_1)(1-\cancel{\cos\phi})\right](a_3-a_1)\sin\phi$$

Ah!!

$$+(a_3-a_1)^2\sin^2\phi$$

$(a+Ib)(a-Ib)$

$a^2 - abI + abI + b^2$
$= a^2+b^2$

Now $1-\cos\phi = 2\sin^2(\phi/2)$

$$\Rightarrow \left(1-(a_3+a_1)2\sin^2(\phi/2)\right)^2 + (a_3-a_1)^2\sin^2(\phi/2)\underbrace{\cos^2(\phi/2)}_{=\,1-\sin^2(\phi/2)}4$$

$\sin(\phi) = 2\cos(\phi/2)\sin(\phi/2)$

Thus $\sin^2\phi = 2^2\sin^2(\phi/2)\cos^2(\phi/2)$
$= 4\sin^2(\phi/2)\left(1-\sin^2(\phi/2)\right)$
$= 4\sin^2(\phi/2) - 4\sin^4(\phi/2)$

$$\Rightarrow 1-4(a_1+a_3)\sin^2(\phi/2) + 4(a_1+a_3)^2\sin^4(\phi/2)$$

$$+4(a_3-a_1)^2\sin^2(\phi/2) - 4(a_3-a_1)^2\sin^4(\phi/2)$$

$$= 1 + \left(-4(a_1 + a_3) + 4(a_3^2 - 2a_1a_3 + a_1^2)\right) \sin^2(\phi/2)$$

$$+ \left(4(\cancel{a_1^2} + 2a_1a_3 + \cancel{a_3^2}) - 4(\cancel{a_3^2} - 2a_1a_3 + \cancel{a_1^2})\right) \sin^4(\phi/2)$$

$$B_1 2 a_1a_3$$

$$= 1 + 4\left[(a_3 - a_1)^2 - (a_3 + a_1)\right] \sin^2(\phi/2)$$

$$+ 16 a_1a_3 \sin^4(\phi/2)$$

$$= A_1 B^2 + A_2 B + 1$$

$$f(B) = B_1 B^2 + B_2 B + 1 \geq 0$$

$$\frac{d}{dB}(B_1 B^2 + B_2 B + 1) = 2B_1 B + B_2$$

$$2(16 b_3 b_1)\beta +$$

$$f(B=0) = 1 \qquad f(B=1) = B_1 + B_2 + 1$$

$$= 16 b_1 b_3 + 4\left[(b_3 - b_1)^2 - (b_3 + b_1)\right]$$

$$+1$$

$$= 16 b_1 b_3 + 4\left[b_3^2 - 2b_1b_3 + b_1^2 - b_1 - b_3\right] + 1$$

$$= 4\left[b_3^2 + 2b_1b_3 + b_1^2 - (b_1 + b_3)\right] + 1$$

$$= 4(b_3 + b_1)(b_3 + b_1 - 1) + 1 = 4(b_2 - 1)b_2 + 1$$

$$= 4b_2^2 - 4b_2 + 1 = (1-2b_2)^2 \geq 0$$

$$|b|^2 \leq 1 \Rightarrow \frac{A_1\beta^2 + A_2\beta + 1}{B_1\beta^2 + B_2\beta + 1} \leq 1$$

But $B_1\beta^2 + B_2\beta + 1 > 0 \quad \forall \ \beta \in (0,1)$ Because it is at 0 & at 1? How to check?

Then $A_1\beta^2 + A_2\beta + \cancel{1} \leq B_1\beta^2 + B_2\beta + \cancel{1}$

$$(A_1 - B_1)\beta^2 + (A_2 - B_2)\beta \leq 0 \qquad \beta \neq 0 \quad \beta > 0$$

evaluate at

$(A_1 - B_1)\beta + (A_2 - B_2) \leq 0$ evaluate at $\beta = 0$

$$A_2 - B_2 < 0$$

evaluate at $\beta = 1$ eq B,6,7

$$A_1 - B_1 + A_2 - B_2 \leq 0$$

8.3.16 is

$$U_i^{n+1} = U_i^n + \overset{\beta}{\overset{\shortparallel}{\frac{\alpha\Delta t}{\Delta x^2}}}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) = \beta U_{i+1}^n + (1-2\beta)U_i^n + \beta U_{i-1}^n$$

$b_3 = 0 \quad b_2 = 1 \quad b_1 = 0$

$a_3 = \beta \quad a_2 = 1-2\beta \quad a_1 = \beta$

$\therefore \quad B_1 = 0 \qquad B_2 = 4(\qquad = 0$

$A_1 = 16\beta^2 \qquad A_2 = 4(0^2 - 2\beta) = -8\beta$

$A_2 - B_2 = -8\beta \leq 0 \Rightarrow \beta \geq 0$

$\& \quad A_1 - B_1 + A_2 - B_2 \leq 0 \Rightarrow 16\beta^2 - 8\beta \leq 0$

$8\beta(2\beta - 1) \leq 0 \qquad \beta \leq 1/2$

$$\frac{\partial u}{\partial t} + (a \circ \nabla) u = \nabla \circ (\bar{\bar{\alpha}} \nabla u) \qquad u \in \mathbb{R} \quad a \; \& \; \alpha \text{ constants}$$

$$\frac{1}{\Delta t}\left(u_j^{n+1} - u_j^n\right) + \sum_{m=1}^{M} a_m \frac{\bar{\delta}_m u_j}{\Delta x_m} = \sum_{m=1}^{M} \alpha_m \frac{\delta_m^2 u_j}{\Delta x_m^2}$$

$$\frac{\bar{\delta}_m u_j}{\Delta x_m} = \frac{1}{2\Delta x_j}\left(u_{i,j+1,k} - u_{i,j-1,k}\right)$$

Thus we get

$$u_j^{n+1} = u_j^n - \sum_{m=1}^{M}\left(\sigma_m \bar{\delta}_m u_j - \beta_m \delta_m^2 u_j\right) \qquad \text{eq } 8.6.13$$

$$G = 1 - I\sum_{m=1}^{M} \sigma_m \sin\phi_m - 4\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2) \qquad \text{eq } 8.6.14$$

2nd derivative | 1st de

$$\left(e^{I\phi_j} - 2 + e^{-I\phi_j}\right) \qquad I \sin\phi_j$$

$$= \left(e^{I\phi_j/2} - e^{-I\phi_j/2}\right)^2 = -4\sin^2(\phi_j/2)$$

$$|G|^2 = \left[1 - 4\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2)\right]^2 + \left(\sum_{m=1}^{M} \sigma_m \sin\phi_m\right)^2 \qquad \text{eq } 8.6.15$$

All ϕ_m's $= \pi$ gives

$$\left[1 - 4\sum_{m=1}^{M} B_m\right]^2 \leq 1$$

$$-1 < 1 - 4\sum_{m=1}^{M} B_m \leq 1$$

$$-2 < -4\sum_{m=1}^{M} B_m \leq 0 \Rightarrow 0 \leq \sum_{m=1}^{M} B_m \leq 1/2$$

2nd case

Now $|G|^2 = \left[1-4\sum_{m=1}^{M} B_m \sin^2 \phi_m/2\right]^2 + \left[\sum_{m=1}^{M} \mathrm{б}_m \sin\phi_m\right]^2$

Second case for extreeme comes from $\phi_m \hat{=} 0$

$\left\{ \sin\theta \cong \theta - \frac{\theta^3}{3!} + O(\theta^5) \right.$

$\left\{ \sin^2\theta \cong \theta^2 - \frac{2\theta^4}{3!} + H.O.T \right.$

$$|G|^2 \cong \left[1 - 4\sum_{m=1}^{M} B_m\left(\frac{\phi_m^2}{4} + O(\phi_m^4)\right)\right]^2$$

$$+\left[\sum_{m=1}^{M} \mathrm{б}_m\left(\phi_m - \frac{\phi_m^3}{3!} + O(\phi_m^5)\right)\right]^2$$

$$\cong 1 - 2\cdot 4\sum_{m=1}^{M} B_m\left(\frac{\phi_m^2}{4} + O(\phi_m^4)\right) + O(\phi_m^4)$$

$$+\left[\sum_{m=1}^{M} \mathrm{б}_m \phi_m\right]^2$$

$$= 1 - 2\sum_{m=1}^{M} B_m \phi_m^2 + \left[\sum_{m=1}^{M} \mathrm{б}_m \phi_m\right]^2 + O(\phi_m^4) \quad \text{eq 8.6.18}$$

In vector form w/ $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_m)$

$$|G|^2 = 1 - 2\phi^T \underset{\substack{\uparrow \\ \text{Diagonal} \\ \text{matrix}}}{\overline{\overline{B}}} \phi + \vec{\phi}^T (\mathrm{б}^T \otimes \mathrm{б}) \vec{\phi}$$

I think the notation $\otimes$ means $\Rightarrow ( )( ) \in \mathbb{R}^{M\times M}$.
gives 2nd order tensor (matrix)

$$\vec{b} \otimes \vec{b} = \begin{pmatrix} b_1 b_1 & b_1 b_2 & b_1 b_3 & \cdots \\ b_2 b_1 & b_2 b_2 & & \\ \vdots & & \ddots & \end{pmatrix}$$

$\vec{v} = \begin{pmatrix} \ \end{pmatrix}$

$\vec{v}^T = \begin{pmatrix} \quad \end{pmatrix}$

$$\begin{pmatrix} \ \end{pmatrix} \otimes \begin{pmatrix} \quad \end{pmatrix}$$

Then $\phi^T (\vec{b} \otimes \vec{b}^T) \phi = \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} \quad \end{pmatrix} \begin{pmatrix} \ \end{pmatrix}$

Do a simple $M = 2$ case example Then

$$\vec{b} \otimes \vec{b}^T = \begin{pmatrix} b_1^2 & b_1 b_2 \\ b_2 b_1 & b_2^2 \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} b_1^2 & b_1 b_2 \\ b_2 b_1 & b_2^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} b_1^2 \phi_1 + b_1 b_2 \phi_2 \\ b_1 b_2 \phi_1 + b_2^2 \phi_2 \end{pmatrix}$$

$$= b_1^2 \phi_1^2 + b_1 b_2 \phi_2 \phi_1 + b_1 b_2 \phi_1 \phi_2 + b_2^2 \phi_2^2$$

$$= b_1^2 \phi_1^2 + 2 b_1 b_2 \phi_2 \phi_1 + b_2^2 \phi_2^2 = (b_1 \phi_1 + b_2 \phi_2)^2$$

$$|G| = 1 - \phi^T \underbrace{(2\bar{\bar{B}} - \sigma^T \otimes \sigma)}\phi < 1$$

Has diagonal entries $2B_i - \sigma_i^2 > 0$ or else the matrix

has no chance of being positive definite

$$\sigma_i^2 < 2B_i \qquad B_i = \frac{\alpha_i \Delta t}{\Delta x_i^2}$$

If $B_i = 0$ $\sigma_i^2 < 0$ ⇒

$$2\bar{\bar{B}} - \sigma \otimes \sigma^T = \bar{\bar{r}}\underbrace{(\bar{\bar{I}} - \bar{\bar{r}}^{-1}\sigma\otimes\sigma^T\bar{\bar{r}}^{-1})}\bar{\bar{r}}$$

$$\bar{\bar{A}} = \bar{\bar{I}} - (\bar{\bar{r}}^{-1}\sigma)\otimes(\bar{\bar{r}}^{-1}\sigma)^T$$

Since $\bar{\bar{r}}$ is diagonal & so is $\bar{\bar{r}}^{-1}$

$$= \bar{\bar{I}} - d\otimes d^T \quad \text{eq } 8.6.21$$

⇒ Rank 1 update of identity. defining $d = \bar{\bar{r}}^{-1}\sigma = \begin{pmatrix} \frac{\sigma_1}{\sqrt{2B_1}} \\ \vdots \\ \frac{\sigma_M}{\sqrt{2B_M}} \end{pmatrix}$

$$x^T\bar{\bar{A}}x = x^T x - x^T(d\otimes d^T)x$$

$$\underbrace{(\quad)(\quad)}_{\text{scalar}}\underbrace{(\quad)(\quad)}_{\text{scalar}}$$

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$$x^T \bar{\bar{A}} x = x^T x - (x^T d)^2 \quad \text{eq } 8.6.22$$

For $\bar{\bar{A}}$ to be nonnegative $\Rightarrow$ $x^T \bar{\bar{A}} x > 0$

$$x^T x > (x^T d)^2 \quad \forall \vec{x} \in \mathbb{R}^M$$

let $x = d$ $\quad (d^T d) - (d^T d)^2 > 0$

$$(d^T d)(1 - d^T d) > 0$$

$d^T d \neq 0$ unless $d \equiv 0$ $\rightarrow$ $d^T d > 0$

$\therefore$ $1 > d^T d$ $\qquad$ From def of $\bar{d} =$

$$\sum_{m=1}^{M} \frac{\sigma_m^2}{(2B_m)} < 1$$

$\therefore$ in summary

$$\sum_{m=1}^{M} \frac{\sigma_m^2}{B_m} \sum_{m=1}^{M} B_m \sin^2 \phi_m \leq 2 \sum_{m=1}^{M} B_m \sin^2 \phi_m \quad \text{eq } 8.6.25$$

given

$$|G|^2 = \left[1 - 4\sum_{m=1}^{M} \beta_m \sin^2\left(\frac{\phi_m}{2}\right)\right]^2 + \left[\sum_{m=1}^{M} \beta_m \sin\phi_m\right]^2$$

$$\leq \left[1 - 4\sum_{m=1}^{M} \beta_m \sin^2\left(\frac{\phi_m}{2}\right)\right]^2 + 2\sum_{m=1}^{M} \beta_m \sin^2\phi_m$$

$$= 1 - 8\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2) + 16\left[\sum_{m=1}^{M} \beta_m \sin^2\left(\frac{\phi_m}{2}\right)\right]^2$$

$$+ 8\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2)\cos^2(\phi_m/2)$$

$$= 1 - 8\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2)\left[1 - \cos^2(\phi_m/2)\right] + 16\left[\sum_{m=1}^{M} \beta_m \sin^2\left(\frac{\phi_m}{2}\right)\right]^2$$

$$= 1 - 8\sum_{m=1}^{M} \beta_m \sin^4(\phi_m/2) + 16\left[\sum_{m=1}^{M} \beta_m \sin^2(\phi_m/2)\right]^2 \qquad \text{eq 8.6.26}$$

$$|G|^2 \leq 1 - 8\sum_{m=1}^{M} \beta_m \sin^4(\phi_m/2) + 16\sum_{m=1}^{M} (\sqrt{\beta_m})^2 \sin^4(\phi_m/2) \sum_{m=1}^{M} (\sqrt{\beta_m})^2$$

Schwartz inequality:

$$\sum |a_k b_k| \leq \left(\sum a_k^2\right)^{1/2}\left(\sum b_k^2\right)^{1/2}$$

$$|G|^2 \leq 1 - 8\sum_{m=1}^{M} \beta_m \sin^4(\phi_m/2) + \frac{16}{2}\sum_{m=1}^{M} \beta_m \sin^4(\phi_m/2) \qquad \text{eq 8.6.27}$$

8.4.18 gives

$$U^{n+1}_{i,j} = \frac{1}{4}\left(U^n_{i,j+1} + U^n_{i+1,j} + U^n_{i-1,j} + U^n_{i,j-1}\right) - \frac{\Delta t}{2\Delta x} A\left(U^n_{i+1,j} - U^n_{i-1,j}\right)$$

$$- \frac{\Delta t}{2\Delta y} B\left(U^n_{i,j+1} - U^n_{i,j-1}\right)$$

$\sigma_1 = \frac{\Delta t}{\Delta x} \lambda_{max}(A) \qquad \sigma_2 = \frac{\Delta t}{\Delta y} \lambda_{max}(B)$

$\beta_1 = 1/4 = \beta_2$ Assume

$$0 < \sum_{m=1}^{m=2} (1/4) = \frac{1}{2} \leq \frac{1}{2} \checkmark$$

$$\frac{\left(\frac{\Delta t}{\Delta x}\lambda(A)\right)^2 + \left(\frac{\Delta t}{\Delta y}\lambda(B)\right)^2}{\left(\frac{1}{4}\right)} \leq 2 \qquad \text{eq } 8.4.19$$

$$\frac{\sigma_1^2}{\beta_1} + \frac{\sigma_2^2}{\beta_2} = \frac{\frac{u^2 \Delta t^{\cancel{2}}}{\cancel{\Delta x^2}}}{\frac{\alpha \cancel{\Delta t}}{\cancel{\Delta x^2}}} + \frac{\frac{v^2 \Delta t^{\cancel{2}}}{\cancel{\Delta y^2}}}{\frac{\alpha \cancel{\Delta t}}{\cancel{\Delta y^2}}} = \frac{u^2 \Delta t}{\alpha} + \frac{v^2 \Delta t}{\alpha}$$

$$= \frac{\Delta t}{\alpha}(u^2 + v^2) \leq 2 \qquad E8.6.86$$

Thus $$\Delta t \le \mathrm{Min}\left(\frac{1}{2\alpha}\frac{1}{\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right)},\ \frac{2\alpha}{q^2}\right)$$

$$= \mathrm{Min}\left(\frac{1}{2\alpha}\ \frac{\Delta x^2 \Delta y^2}{\Delta x^2+\Delta y^2},\ \frac{2\alpha}{q^2}\right) \qquad E.8.6.9$$

All $B_m = \beta$

$$\sum_{m=1}^{M} b_m^2 \le 2\beta \quad + \quad 0 < M\beta \le \frac{1}{2}$$

$$\beta > 0 \quad \beta \le \frac{1}{2M}$$

$$\sum_{m=1}^{M} b_m^2 \le 2\beta \le \frac{1}{M} \quad \text{eq } 8.6.28$$

$$R_m = \frac{a_m \Delta x_m}{\alpha_m}$$

8.6.24b $$\sum_{m=1}^{M} \frac{b_m^2}{\beta_m} = \sum_{m=1}^{M} \frac{a_m^2 \Delta t^2}{\cancel{\Delta x_m^2}} \cdot \frac{\cancel{\Delta x_m^2}}{\alpha_m \Delta t} = \sum_{m=1}^{M} \frac{a_m^2 \Delta t}{\alpha_m}$$

$$= \sum \underbrace{\frac{\Delta t\, a_m}{\Delta x_m}}_{b_m} \frac{a_m \Delta x_m}{\alpha_m} = \sum b_m R_m \le 2$$

Or from $$\sum_{m=1}^{M} \frac{a_m^2 \Delta t}{\alpha_m} = \sum_{m=1}^{M} \frac{a_m^2 \Delta x_m^2}{\alpha_m^2} \underbrace{\frac{\alpha_m \Delta t}{\Delta x_m^2}}_{B_m} = \sum_{m=1}^{M} R_m^2 B_m$$

eq 8.6.30

$$\sigma = \frac{a\Delta t}{\Delta x} \qquad \beta = \frac{\alpha \Delta t}{\Delta x^2}$$

$$\frac{U_i^{n+1} - U_i^{n-1}}{2\Delta t} + \frac{a}{2\Delta x}\left(U_{i+1}^n - U_{i-1}^n\right) = \frac{\alpha}{\Delta x^2}\left(U_{i+1}^{n-1} - 2U_i^{n-1} + U_{i-1}^{n-1}\right)$$

$$U_i^{n+1} - U_i^{n-1} = -\sigma\left(U_{i+1}^n - U_{i-1}^n\right) + 2\beta\left(U_{i+1}^{n-1} - 2U_i^{n-1} + U_{i-1}^{n-1}\right)$$

eq EB.6.11

$$G - G^{-1} = -\sigma\left(e^{\phi I} - e^{-\phi I}\right) + 2\beta G^{-1}\left(e^{I\phi_x} - 2 + e^{-I\phi_x}\right)$$

$$G^2 - 1 = -I2\sigma \sin\phi\, G + 2\beta\underbrace{\left(2I\sin(\phi_x/2)\right)^2}_{-4\sin^2(\phi_x/2)}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\therefore \quad G^2 - 1 = -2I\sigma\sin\phi\, G - \frac{8\beta}{2}(1-\cos\phi_x)$$

$$\Rightarrow \quad G^2 + 2I\sigma\sin\phi\, G - 1 + 4\beta(1-\cos\phi) = 0$$

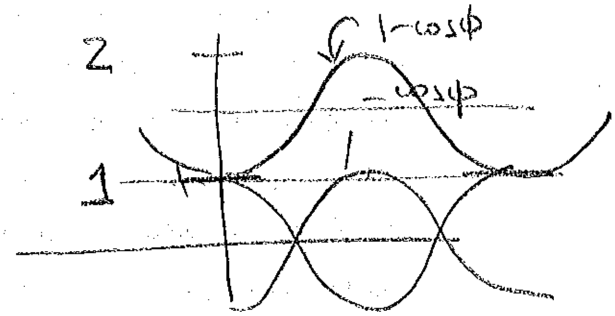
Define $P(\lambda) = \lambda^2 + 2I\sigma\sin\phi\,\lambda - 1 - 4\beta(\cos\phi - 1)$ eq EB.6.12

$\bar{P}(\lambda) = 1 - 2I\sigma\sin\phi\,\lambda - \left(1 + 4\beta(\cos\phi - 1)\right)\lambda^2$ EB.6.13

$|\bar{P}(0)| > |P(0)|$

$1 > |1 + 4B(\cos\phi - 1)| = |1 - 4B(1-\cos\phi)|$

$-1 < 1 - 4B(1-\cos\phi) < 1$



$-2 < -4B(1-\cos\phi) < 0$

$0 < B(1-\cos\phi) < \frac{1}{2}$

$0 < B < \frac{1}{2(1-\cos\phi)} < \frac{1}{2^2} = \frac{1}{4}$

$P_1(\lambda) = \frac{1}{\lambda}\left[\bar{P}(0)P(\lambda) - P(0)\bar{P}(\lambda)\right]$

$= \frac{1}{\lambda}\Big[\lambda^2 + 2I\delta\sin\phi\,\lambda - (1+4B(\cos\phi-1)) + (1+4B(\cos\phi-1))\cdot$

$\quad (1 - 2I\delta\sin\phi\,\lambda - (1+4B(\cos\phi-1))\lambda^2)\Big]$

$= \frac{1}{\lambda}\Big[\lambda^2 + 2I\delta\sin\phi\,\lambda - (1+\cancel{4B(\cos\phi-1)})$

$\quad + (1+\cancel{4B(\cos\phi-1)}) - 2I\delta\sin\phi\,(1+4B(\cos\phi-1))\,\lambda$

$\quad - (1+4B(\cos\phi-1))^2\lambda^2\Big]$

$$P_1(\lambda) = \frac{1}{\lambda}\left[\lambda^2 + 2IG\sin\phi\,\lambda - 2IG\sin\phi\, r\lambda - r^2\lambda^2\right]$$

$$= \frac{1}{\lambda}\left[(1-r^2)\lambda^2 + (1-r)2IG\sin\phi\,\lambda\right]$$

$$= (1-r^2)\lambda + (1-r)2IG\sin\phi \qquad \text{q E8.6.15}$$

$$\bar{P}_1(\lambda) = (1-r^2) - (1-r)2IG\sin\phi\,\lambda$$

$|\bar{P}_1(0)| > |P_1(0)|$ gives

$$|1-r^2| > |2IG(1-r)\sin\phi| = 2G|1-r||\sin\phi|$$

$r = 1-4B(1-\cos\phi)$ told $|r|<1$ $\qquad -1<r<1$

$$|(1+r)\cancel{(1-r)}| > \cancel{|1-r|}\,2G|\sin\phi| \qquad \begin{array}{l} -1<-r<1 \\ 0<1-r<2 \end{array}$$

$$\underset{\Vert}{|1+r|} > 2G|\sin\phi|$$

$$2(1-2B(1-\cos\phi)) > 2G|\sin\phi|$$

$$1-2B(1-\cos\phi) > G|\sin\phi| \qquad \text{E8.6.16}$$

Prob B.1 Idea is discretize the spacial opperator. & obtain an eq like

$$\frac{du_i}{dt} = S u_i + q_i \quad \text{or} \quad \frac{dU}{dt} = SU + q$$ w/ U vector of unknowns.

Then discretize the time operator to obtain $U^{n+1} = CU^n + \bar{Q}$.

Heat diffusion $\quad \frac{du_i}{dt} = \frac{\alpha}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) = S U$

w/ $S = \frac{\alpha}{\Delta x^2}\begin{bmatrix} +1 & -2 & +1 & & & \\ & +1 & -2 & +1 & & \\ & & +1 & -2 & +1 & \\ & & & +1 & -2 & +1 \end{bmatrix}$ tridiagonal matrix w/ +1 on superdiagonal, −2 on diagonal, +1 on subdiagonal.

Discretize the time operator w/ Eulers method

$$\Rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\alpha}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1})$$

$$\Rightarrow u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) = [1 + \beta(E - 2 + E^{-1})]u_i^n$$

Then $C = I + \beta(E - 2 + E^{-1})$ As a matrix

$= \beta E + (1 - 2\beta) + \beta E^{-1}$

$C = \begin{bmatrix} 1-2\beta & \beta & & & \\ \beta & 1-2\beta & \beta & & \\ & \beta & 1-2\beta & \beta & \\ & & \beta & 1-2\beta & \beta \end{bmatrix}$ tridiagonal matrix w/ superdiagonal w/ β, diagonal w/ $1-2\beta$ & subdiagonal w/ β.

let $U_i^n = \underbrace{E^n e^{Ii\phi}}$ $\phi = \frac{j\pi}{N}$ $U_i^n = \sum_{m=-N}^{N} U_m^n e^{I\phi}$

an fourier (discrete) mode (jth mode)

Then putting this into $U_i^{n+1} = U_i^n + B(U_{i+1}^n - 2U_i^n + U_{i-1}^n)$ we get

$$E^{n+1} \cancel{e^{Ii\phi}} = E^n \cancel{e^{Ii\phi}} + B\left(E^n e^{I(i+1)\phi} - 2E^n \cancel{e^{Ii\phi}} + E^n e^{I(i-1)\phi}\right) \div \text{by } E^n e^{iI\phi}$$

$$\frac{E^{n+1}}{E^n} = 1 + B\left(e^{I\phi} - 2 + e^{-I\phi}\right)$$ 1st expression in table.

$$\parallel$$

$$G = 1 + B\left(e^{I\phi/2} - e^{-I\phi/2}\right)^2$$

$$= 1 + B\left(2i \sin \phi/2\right)^2 = 1 - 4B \sin^2 \phi/2$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\therefore \quad G = 1 - 2B(1 - \cos(\phi)) \quad \checkmark$$

Another time discretization
Trapezoidal

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{1}{2}\frac{\alpha}{\Delta x^2}(U_{i+1}^n - 2U_i^n + U_{i-1}^n) + \frac{1}{2}\frac{\alpha}{\Delta x^2}(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1})$$

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{2} S U_i^{n+1} + \frac{\Delta t}{2} S U_i^n$$

$$\left(1 - \frac{\Delta t}{2} S\right) U_i^{n+1} = \left(1 + \frac{\Delta t}{2} S\right) U_i^n$$

In matrix form $\underbrace{\left(1-\frac{\Delta t}{2}\delta\right)}_{B}U^{n+1} = \underbrace{\left(1+\frac{\Delta t}{2}\delta\right)}_{A}U^{n}$

$U^{n+1} = B^{-1}AU^{n}$. The matrices A & B are

$$A = 1 + \frac{\Delta t\,\alpha}{2\,\Delta x^2}\begin{pmatrix} \ddots & & & & \\ +1 & -2 & +1 & & \\ & +1 & -2 & +1 & \\ & & +1 & -2 & +1 \\ & & & & \ddots \end{pmatrix}$$

$$= 1 + \frac{\beta}{2}\begin{pmatrix} \ddots & & & & \\ +1 & -2 & +1 & & \\ & +1 & -2 & +1 & \\ & & +1 & -2 & +1 \\ & & & & \ddots \end{pmatrix}$$

1 = identity matrix

$$= \begin{pmatrix} \ddots & & & & \\ \beta/2 & 1-\beta & \beta/2 & & \\ & \beta/2 & 1-\beta & \beta/2 & \\ & & \beta/2 & 1-\beta & \beta/2 \\ & & & & \ddots \end{pmatrix}$$

$$B = 1 - \frac{\beta}{2}\begin{pmatrix} \ddots & & & & \\ +1 & -2 & +1 & & \\ & +1 & -2 & +1 & \\ & & +1 & -2 & +1 \\ & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \ddots & & & & \\ \beta/2 & 1+\beta & \beta/2 & & \\ & \beta/2 & 1+\beta & \beta/2 & \\ & & \beta/2 & 1+\beta & \beta/2 \\ & & & & \ddots \end{pmatrix}$$

Growth factor: let $U_i^n = E^n e^{i\phi I}$

into $U_i^{n+1} = U_i^n + \frac{\Delta t\,\alpha}{2\,\Delta x^2}(E - 2 + E^{-1})U_i^n + \frac{\Delta t\,\alpha}{2\Delta x^2}(E - 2 + E^{-1})U_i^{n+1}$

$$= U_i^n + \frac{\beta}{2}(E - 2 + E^{-1})U_i^n + \frac{\beta}{2}(E - 2 + E^{-1})U_i^{n+1}$$

$$E^{n+1}e^{i\phi I} = E^n e^{i\phi I} + \frac{\beta}{2}E^n e^{i\phi I}(e^{\phi I} - 2 + e^{-\phi I})$$

$$+ \frac{\beta}{2}E^{n+1}e^{i\phi I}(e^{\phi I} - 2 + e^{-\phi I})$$

$$\Rightarrow \frac{E^{n+1}}{E^n} = 1 + \frac{\beta}{2}(e^{\phi I} - 2 + e^{-\phi I}) + \frac{\beta}{2}(e^{\phi I} - 2 + e^{-\phi I})\frac{E^{n+1}}{E^n}$$

$$\frac{E^{n+1}}{E^n} = \frac{1 + \beta/2(e^{\phi I} - 2 + e^{-\phi I})}{1 - \beta/2(e^{\phi I} - 2 + e^{-\phi I})}$$

$$\| \\ G$$

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Shallow H_2O eqs : (linearized)

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0 \qquad U = \begin{pmatrix} h \\ v \end{pmatrix} \quad A = \begin{pmatrix} v_0 & h_0 \\ g & v_0 \end{pmatrix}$$

$$\frac{dU_i}{dt} = -A\frac{(U_{i+1} - U_{i-1})}{2\Delta x}$$

At grid point i have 2 variables defined h_i & v_i

$U_i \in \mathbb{R}^{2\times 1}$ let $U = \begin{pmatrix} \vdots \\ \bar{U}_{i-1} \\ \bar{U}_i \\ \bar{U}_{i+1} \\ \vdots \end{pmatrix}$ w/ $\bar{U}_{i-1} = \begin{pmatrix} h_{i-1} \\ v_{i-1} \end{pmatrix}$, $\bar{U}_i = \begin{pmatrix} h_i \\ v_i \end{pmatrix}$, $\bar{U}_{i+1} = \begin{pmatrix} h_{i+1} \\ v_{i+1} \end{pmatrix}$

The
$$\frac{dU}{dt} = \underbrace{\begin{bmatrix} 0 & -\frac{A}{2\Delta x} & & & \\ +\frac{A}{2\Delta x} & 0 & -\frac{A}{2\Delta x} & & \\ & +\frac{A}{2\Delta x} & 0 & -\frac{A}{2\Delta x} & \\ & & +\frac{A}{2\Delta x} & 0 & -\frac{A}{2\Delta x} \end{bmatrix}}_{S} \begin{pmatrix} \vdots \\ \bar{U}_{i-1} \\ \bar{U}_i \\ \bar{U}_{i+1} \\ \vdots \end{pmatrix}$$

S is a Block tridiagonal matrix w/

$-\frac{A}{2\Delta x}$ on upper diagonal $\qquad A \in \mathbb{R}^{2\times 2}$

0 on diagonal $\qquad 0 \in \mathbb{R}^{2\times 2}$

$\frac{A}{2\Delta x}$ on lower diagonal

descretize in time. Eulers method.

locally the spacial discretization opperator looks like

$$\frac{dU_i}{dt} = -\frac{A}{2\Delta x}(U_{i+1} - U_{i-1}) \quad \text{becomes}$$

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{A}{2\Delta x}(U_{i+1} - U_{i-1})$$

$$U_i^{n+1} = U_i^n - \frac{A}{2}\frac{\Delta t}{\Delta x}(U_{i+1} - U_{i-1}) \qquad U_i^{n+1} = C U_i^n$$

$$U_i^{n+1} = \left[1 - \frac{A}{2}\frac{\Delta t}{\Delta x}(E - E^{-1})\right] U_i^n \qquad \text{in } 2\times 2 \text{ matrix form}$$

$$\begin{pmatrix} h_i^{n+1} \\ v_i^{n+1} \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\Delta t}{2\Delta x}(E - E^{-1})\begin{pmatrix} V_0 & h_0 \\ g & V_0 \end{pmatrix}\right]\begin{pmatrix} h_i^n \\ V_i^n \end{pmatrix}$$

$$= \begin{bmatrix} 1 - \frac{V_0 \Delta t}{2\Delta x}(E - E^{-1}) & -\frac{h_0 \Delta t}{2\Delta x}(E - E^{-1}) \\ -\frac{g\Delta t}{2\Delta x}(E - E^{-1}) & 1 - \frac{V_0 \Delta t}{2\Delta x}(E - E^{-1}) \end{bmatrix}\begin{pmatrix} h_i^n \\ V_i^n \end{pmatrix}$$

Globally the spacial discretization/temporal discretization opperator looks like:

$$\frac{d\bar{U}}{dt} = \begin{bmatrix} 0 & -\frac{A}{2\Delta x} & & & \\ +\frac{A}{2\Delta x} & 0 & \frac{A}{2\Delta x} & & \\ & +\frac{A}{2\Delta x} & 0 & -\frac{A}{2\Delta x} & \\ & & +\frac{A}{2\Delta x} & 0 & -\frac{A}{2\Delta x} \end{bmatrix}\begin{bmatrix} \vdots \\ \bar{U}_{i-1} \\ \bar{U}_i \\ \bar{U}_{i+1} \\ \vdots \end{bmatrix}$$

$$U^{n+1} - U^n = \begin{bmatrix} 0 & & & & & \\ +\frac{A\Delta t}{2\Delta x} & 0 & -\frac{A\Delta t}{2\Delta x} & & & \\ & +\frac{A\Delta t}{2\Delta x} & 0 & -\frac{A\Delta t}{2\Delta x} & & \\ & & +\frac{A\Delta t}{2\Delta x} & 0 & -\frac{A\Delta t}{2\Delta x} & \\ & & & +\frac{A\Delta t}{2\Delta x} & 0 & -\frac{A\Delta t}{2\Delta x} \end{bmatrix} \begin{bmatrix} \bar{U}_{i-1} \\ \bar{U}_i \\ \bar{U}_{i+1} \end{bmatrix}$$

$$U^{n+1} = \begin{bmatrix} 1 & -\frac{A\Delta t}{2\Delta x} & & & \\ +\frac{A\Delta t}{2\Delta x} & 1 & -\frac{A\Delta t}{2\Delta x} & & \\ & +\frac{A\Delta t}{2\Delta x} & 1 & -\frac{A\Delta t}{2\Delta x} & \\ & & +\frac{A\Delta t}{2\Delta x} & 1 & -\frac{A\Delta t}{2\Delta x} \end{bmatrix} \begin{bmatrix} \vdots \\ \bar{U}_{i-1} \\ \bar{U}_i \\ \bar{U}_{i+1} \\ \vdots \end{bmatrix}$$

locally the spacial / temporal opperator is

$$\bar{U}_i^{n+1} = C\,\bar{U}_i^n \qquad \bar{U} = \begin{pmatrix} h \\ v \end{pmatrix}$$

w/ $$C = \begin{bmatrix} 1 - \frac{V_0 \Delta t}{2\Delta x}(E - E^{-1}) & -\frac{h_0 \Delta t}{2\Delta x}(E - E^{-1}) \\ -\frac{g\Delta t}{2\Delta x}(E - E^{-1}) & 1 - \frac{V_0 \Delta t}{2\Delta x}(E - E^{-1}) \end{bmatrix}$$

sub $\bar{U}_i^n = \bar{V}^n e^{i\phi I}$ $\quad \bar{V} \in \mathbb{R}^{2\times 1}$

gives $$\bar{V}^{n+1} \cancel{e^{i\phi I}} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\Delta t}{2\Delta x} \begin{pmatrix} V_0 & h_0 \\ g & V_0 \end{pmatrix} \left(e^{I\phi} - e^{-I\phi}\right) \right] \bar{V}^n \cancel{e^{i\phi I}}$$

$\Rightarrow$ Amplification matrix G is then

$$G = \left[1 - \frac{\Delta t}{2\Delta x} A \left(e^{I\phi} - e^{-I\phi} \right) \right] = \left[1 - \frac{\Delta t}{\Delta x} I \sin\phi \, A \right]$$

$$= \begin{bmatrix} 1 - \frac{\Delta t}{\Delta x} I \sin\phi \, v_0 & -\frac{\Delta t}{\Delta x} I \sin\phi \, h_0 \\ -\frac{\Delta t}{\Delta x} I \sin\phi \, g & 1 - \frac{\Delta t}{\Delta x} I \sin\phi \, v_0 \end{bmatrix}$$

For the Lax-Friedrich scheme

$$\frac{dU_i}{dt} = -\frac{A}{2\Delta x}(U_{i+1} - U_{i-1})$$

becomes discretizing the time opperator as

$$\frac{dU_i}{dt} = \frac{U_i^{n+1} - \frac{1}{2}(U_{i+1}^n + U_{i-1}^n)}{\Delta t}$$

Then we get

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{A\Delta t}{2\Delta x}(U_{i+1}^n - U_{i-1}^n)$$

$$= \left(\frac{1}{2}I + \frac{A\Delta t}{2\Delta x}\right) U_{i-1}^n + \left(\frac{1}{2}I - \frac{A\Delta t}{2\Delta x}\right) U_{i+1}^n$$

Now C is the matrix $\ni$

$U_i^{n+1} = C U_i^n$ in terms of the increment opperator E^{+1} + E^{-1}

$$C = \frac{1}{2}\left(1 + \frac{A\Delta t}{\Delta x}\right) E^{-1} + \frac{1}{2}\left(1 - \frac{A\Delta t}{\Delta x}\right) E$$

in Matrix form

$$\mathcal{U}^{n+1} = G\,\mathcal{U}^n$$

w/ $\mathcal{U}^n = \begin{pmatrix} \vdots \\ \begin{pmatrix} v \\ h \end{pmatrix}_{i-1} \\ \begin{pmatrix} v \\ h \end{pmatrix}_i \\ \begin{pmatrix} v \\ h \end{pmatrix}_{i+1} \\ \vdots \end{pmatrix}$ all at time level n.

Then

$$G = \begin{pmatrix} \ddots & & & & & \\ \frac{1}{2}\left(1+\frac{A\Delta t}{\Delta x}\right) & O_{2\times 2} & \frac{1}{2}\left(1-\frac{A\Delta t}{\Delta x}\right) & & & \\ & \frac{1}{2}\left(1+\frac{A\Delta t}{\Delta x}\right) & O_{2\times 2} & \frac{1}{2}\left(1-\frac{A\Delta t}{\Delta x}\right) & & \\ & & \frac{1}{2}\left(1+\frac{A\Delta t}{\Delta x}\right) & O_{2\times 2} & \frac{1}{2}\left(1-\frac{A\Delta t}{\Delta x}\right) & \\ & & & & & \ddots \end{pmatrix}$$

G is a block tri-diagonal matrix w/

sub diagonal composed of matrices $\frac{1}{2}\left(\underset{\substack{\uparrow \\ 2\times 2 \text{ identity}}}{1}+\frac{A\Delta t}{\Delta x}\right)$

diagonal composed of $O_{2\times 2}$

+ superdiagonal composed of $\frac{1}{2}\left(1-\frac{A\Delta t}{\Delta x}\right)$

Then the Growth Factor matrix. let $\bar{U}_i^n = \bar{V}^n e^{i\phi I}$ $\quad \bar{V}^n \in \mathbb{R}^{2\times 1}$

Then

$$\bar{V}^{n+1}\cancel{e^{i\phi I}} = \frac{1}{2}\left(I+\frac{A\Delta t}{\Delta x}\right)\bar{V}^n \cancel{e^{i\phi I}} e^{-\phi I} + \frac{1}{2}\left(I-\frac{A\Delta t}{\Delta x}\right)\bar{V}^n \cancel{e^{i\phi I}} e^{+\phi I}$$

$$\bar{V}^{n+1} = \underbrace{\left[\frac{1}{2}\left(I+\frac{A\Delta t}{\Delta x}\right)e^{-\phi I} + \frac{1}{2}\left(I-\frac{A\Delta t}{\Delta x}\right)e^{+\phi I}\right]}_{G \,\in\, \mathbb{C}^{2\times 2}}\bar{V}^n$$

Then $G = \frac{1}{2}\left(e^{\phi I} + e^{-\phi I}\right) + \frac{1}{2}\frac{A\Delta t}{\Delta x}\left(e^{-\phi I} - e^{\phi I}\right)$

$$= \cos\phi\, I - I A \frac{\Delta t}{\Delta x}\sin\phi$$

or in component form w/ $A = \begin{pmatrix} v_0 & h_0 \\ g & v_0 \end{pmatrix}$

$$G = \begin{pmatrix} \cos\phi & 0 \\ 0 & \cos\phi \end{pmatrix} - \frac{I\Delta t}{\Delta x}\sin\phi \begin{pmatrix} v_0 & h_0 \\ g & v_0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi - \frac{v_0\Delta t}{\Delta x} I \sin\phi & -I\frac{\Delta t}{\Delta x} h_0 \sin\phi \\ -I\frac{\Delta t}{\Delta x} g \sin\phi & \cos\phi - I\frac{\Delta t}{\Delta x} v_0 \sin\phi \end{pmatrix}$$

. Wave eq:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \iff \underbrace{\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)}u = 0$$

let $v = u_t$ & $w = a u_x$ $\qquad \left(\frac{\partial^2}{\partial t^2} - a\cancel{\frac{\partial^2}{\partial t \partial x}} + a\cancel{\frac{\partial^2}{\partial x \partial t}} - a^2\frac{\partial^2}{\partial x^2}\right)u = 0$

Then $v_t = u_{tt} = a^2 u_{xx}$

$= a(w_x)$

& $w_t = a u_{xt} = (a u_t)_x = a v_x$

$\therefore$ get $v_t = a w_x$

$w_t = a v_x$

let $U = \begin{pmatrix} v \\ w \end{pmatrix}$

$$U_t = \begin{pmatrix} a w_x \\ a v_x \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} v_x \\ w_x \end{pmatrix} = A U_x$$

$$L = A\frac{\partial}{\partial x}$$

Discretize the spacial derivative 1st w/ central differencing

$$\frac{dU_i}{dt} = \frac{1}{2\Delta x} A\left(U_{i+1} - U_{i-1}\right)$$

$\Rightarrow \quad S = \frac{1}{2\Delta x} A(E - E^{-1})$ is the spacial shift operator

for the vector U_i

$$\vec{U} = \begin{pmatrix} \vdots \\ \vec{U}_{i-1} \\ \vec{U}_i \\ \vec{U}_{i+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ v_{i-1} \\ w_{i-1} \\ v_i \\ w_i \\ v_{i+1} \\ w_{i+1} \\ \vdots \end{pmatrix} \quad \text{Then} \quad \frac{d\vec{U}}{dt} = \begin{pmatrix} \ddots & & & & \\ -\frac{A}{2\Delta x} & 0 & \frac{A}{2\Delta x} & & \\ & -\frac{A}{2\Delta x} & 0 & \frac{A}{2\Delta x} & \\ & & -\frac{A}{2\Delta x} & 0 & \frac{A}{2\Delta x} \\ & & & & \ddots \end{pmatrix}$$

vector of vectors

Discretization of time operator

$$\frac{dv_i}{dt} = \frac{A}{2\Delta x}(v_{i+1} - v_{i-1})$$

|| Backwards Euler

$$\frac{v_i^n - v_i^{n-1}}{\Delta t} = \frac{A}{2\Delta x}(v_{i+1}^n - v_{i-1}^n) \Rightarrow v_i^n = v_i^{n-1} + \frac{A}{2}\frac{\Delta t}{\Delta x}(v_{i+1}^n - v_{i-1}^n)$$

let $n \to n+1$ $$v_i^{n+1} = v_i^n + \frac{A\Delta t}{2\Delta x}(v_{i+1}^{n+1} - v_{i-1}^{n+1}) \quad *$$

Then $\Rightarrow$ $$\frac{A\Delta t}{2\Delta x} v_{i+1}^{n+1} - v_i^{n+1} - \frac{A}{2}\frac{\Delta t}{\Delta x} v_{i-1}^{n+1} = v_i^n$$

$$\begin{pmatrix} \frac{A\Delta t}{2\Delta x} & & & \\ -I & \frac{A\Delta t}{2\Delta x} & & \\ -\frac{A\Delta t}{2\Delta x} & -I & +\frac{A\Delta t}{2\Delta x} & \\ & \frac{A\Delta t}{2\Delta x} & -I & \end{pmatrix} \begin{pmatrix} \vdots \\ v_{i-1}^{n+1} \\ v_i^{n+1} \\ v_{i+1}^{n+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ v_{i-1}^n \\ v_i^n \\ v_{i+1}^n \\ \vdots \end{pmatrix}$$] is this not correct? why not?

eq (*) becomes

$$\underbrace{v_i^{n+1} - \frac{A\Delta t}{2\Delta x}(v_{i+1}^{n+1} - v_{i-1}^{n+1})} = v_i^n$$

$$C^{-1} = 1 - \frac{A\Delta t}{2\Delta x}(E - E^{-1}) = 1 - \frac{A\Delta t}{2\Delta x}E + \frac{A\Delta t}{2\Delta x}E^{-1}$$

C^{-1} in matrix form represents
$$= \begin{pmatrix} I & -\frac{A\Delta t}{2\Delta x} & & \\ \frac{A\Delta t}{2\Delta x} & I & -\frac{A\Delta t}{2\Delta x} & \\ & \frac{A\Delta t}{2\Delta x} & I & -\frac{A\Delta t}{2\Delta x} \end{pmatrix}$$

Amplification factor let $U_i^n = v^n e^{Ii\phi}$

Then $C^{-1}\vec{U}^{n+1} = \vec{U}^n$

$$C^{-1}(v^{n+1}e^{Ii\phi}) = v^n e^{Ii\phi}$$

$$\Rightarrow v^{n+1}\cancel{e^{Ii\phi}} - \frac{A\Delta t}{2\Delta x}\left(v^{n+1}\cancel{e^{Ii\phi}}e^{I\phi} - v^{n+1}\cancel{e^{Ii\phi}}e^{-I\phi}\right) = v^n \cancel{e^{Ii\phi}}$$

$$\underbrace{\left[1 - \frac{A\Delta t}{2\Delta x}(e^{I\phi} - e^{-I\phi})\right]}_{G^{-1}} v^{n+1} = v^n$$

$$v^{n+1} = G v^n$$

$$G^{-1} = 1 - \frac{A\Delta t}{2\Delta x}(2I\sin\phi)$$

$$= 1 - \frac{IA\Delta t}{\Delta x}\sin\phi$$

- Another scheme is the so called forward backwards scheme:

$$\frac{dv_i}{dt} = \frac{a}{\Delta x}(w_{i+1} - w_i)$$

$$\frac{dw_i}{dt} = \frac{a}{\Delta x}(v_i - v_{i-1})$$

$$\frac{dv_i}{dt} = \frac{a}{\Delta x}\begin{bmatrix} 0 & E-1 \\ 1-E^{-1} & 0 \end{bmatrix}\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \frac{1}{\Delta x}\begin{bmatrix} 0 & a(E-1) \\ a(1-E^{-1}) & 0 \end{bmatrix}\begin{bmatrix} v_i \\ w_i \end{bmatrix}$$

$$= \left\{\frac{E}{\Delta x}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} + \frac{1}{\Delta x}\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} + \frac{E^{-1}}{\Delta x}\begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix}\right\}\begin{pmatrix} v_i \\ w_i \end{pmatrix}$$

$$= \left\{\frac{E^{-1}}{\Delta x}\begin{bmatrix} 0 & 0 \\ -a & 0 \end{bmatrix} + \frac{1}{\Delta x}\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} + \frac{E}{\Delta x}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right\}\begin{bmatrix} v_i \\ w_i \end{bmatrix}$$

let $\vec{v} = \begin{pmatrix} \vdots \\ v_{i-1} \\ v_i \\ v_{i+1} \\ \vdots \end{pmatrix}$ vector of vectors one for each node (each node has 2 unknown associated to it)

$$\frac{d\vec{v}}{dt} = \begin{pmatrix} \ddots & & & & \\ & A_0 & A_{+1} & & \\ & A_{-1} & A_0 & A_{+1} & \\ & & A_{-1} & A_0 & A_{+1} \\ & & & & \ddots \end{pmatrix}$$

tridiagonal matrix w/ superdiagonal A_{+1} subdiagonal A_{-1} & diagonal A_0.

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$$v_i^{n+1} - v_i^n = \frac{a\Delta t}{\Delta x}(w_{i+1}^n - w_i^n)$$

$$w_i^{n+1} - w_i^n = \frac{a\Delta t}{\Delta x}(v_i^{n+1} - v_{i-1}^{n+1})$$

The matrix C is defined as the opperator that $v_i^{n+1} = C v_i^n$ or in vector form $U^{n+1} = CU^n$

$U \in \mathbb{R}^2 \quad U = \binom{v}{w}$

$$U_i^{n+1} - U_i^n = \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix} + \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_{i+1}^n \\ w_{i+1}^n \end{pmatrix}$$

$$+ \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_i^{n+1} \\ w_i^{n+1} \end{pmatrix} + \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_{i-1}^{n+1} \\ w_{i-1}^{n+1} \end{pmatrix}$$

$$= \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} U_i^n + \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & +E \\ 0 & 0 \end{pmatrix} U_i^n$$

$$+ \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U_i^{n+1} + \frac{a\Delta t}{\Delta x}\begin{pmatrix} 0 & 0 \\ -E^{-1} & 0 \end{pmatrix} U_i^{n+1}$$

$$= \begin{pmatrix} 0 & \sigma(E-1) \\ 0 & 0 \end{pmatrix} U_i^n + \begin{pmatrix} 0 & 0 \\ \sigma(1-E^{-1}) & 0 \end{pmatrix} U_i^{n+1}$$

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \sigma(1-E^{-1}) & 0 \end{pmatrix}\right] U_i^{n+1} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \sigma(E-1) \\ 0 & 0 \end{pmatrix}\right] U_i^n$$

$$\begin{pmatrix} 1 & 0 \\ -\sigma(1-E^{-1}) & 1 \end{pmatrix} U_i^{n+1} = \begin{pmatrix} 1 & \sigma(E-1) \\ 0 & 1 \end{pmatrix} U_i^n$$

$$U_i^{n+1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ \sigma(1-E^{-1}) & 1 \end{pmatrix}\begin{pmatrix} 1 & \sigma(E-1) \\ 0 & 1 \end{pmatrix} U_i^n$$

$$v_i^{n+1} = \underbrace{\begin{pmatrix} 1 & \sigma(E-1) \\ \sigma(1-E^{-1}) & \sigma^2(E-1)(1-E^{-1})+1 \end{pmatrix}}_{C} v_i^n$$

write C in terms of shift opperators (powers of shift opperators)

$$C = \begin{pmatrix} 1 & \sigma E - \sigma \\ \sigma - E^{-1}\sigma & \sigma^2(E-1-1+E^{-1})+1 \end{pmatrix} = \begin{pmatrix} 1 & -\sigma + \sigma E \\ -\sigma E^{-1} + \sigma & \sigma^2 E^{-1} - 2\sigma^2 + 1 + \sigma^2 E \end{pmatrix}$$

$$C = \underbrace{\begin{pmatrix} 0 & 0 \\ -\sigma & \sigma^2 \end{pmatrix}}_{C_-} E^{-1} + \underbrace{\begin{pmatrix} 1 & -\sigma \\ \sigma & -2\sigma^2+1 \end{pmatrix}}_{C_0} + \underbrace{\begin{pmatrix} 0 & \sigma \\ 0 & \sigma^2 \end{pmatrix}}_{C_+} E$$

Then for update of vector $v^n = \begin{pmatrix} \vdots \\ v_{i-1}^n \\ v_i^n \\ v_i^{n+1} \end{pmatrix}$ we get

$$v^{n+1} = \begin{pmatrix} \ddots & & & & \\ C_- & C_0 & C_+ & & \\ & C_- & C_0 & C_+ & \\ & & C_- & C_0 & C_+ \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ v_{i-1}^n \\ v_i^n \\ v_{i+1}^n \\ \vdots \end{pmatrix}$$

Block tridiagonal matrix w/
C_+ on super-diagonal
C_0 on diagonal
C_- on subdiagonal

- Now G is defined $\ni$ $u_i^n = v^n e^{Ii\phi}$ put into Matrix rep

$U_i^{n+1} = C u_i^n$

$$\Rightarrow v^{n+1} \cancel{e^{Ii\phi}} = \begin{pmatrix} 1 & -G \\ G & 1-2G^2 \end{pmatrix} v^n \cancel{e^{Ii\phi}} + \begin{pmatrix} 0 & 0 \\ -G & G^2 \end{pmatrix} v^n \cancel{e^{Ii\phi}} e^{-I\phi}$$

$$+ \begin{pmatrix} 0 & G \\ 0 & G^2 \end{pmatrix} v^n \cancel{e^{Ii\phi}} e^{+I\phi}$$

$$= \begin{pmatrix} 1 & -G + Ge^{I\phi} \\ G - Ge^{-I\phi} & 1-2G^2 + G^2 e^{-I\phi} + G^2 e^{I\phi} \end{pmatrix} v^n \quad *$$

$$= \begin{pmatrix} 1 & G(e^{I\phi}-1) \\ G(1-e^{-I\phi}) & 1-2G^2 + G^2\, 2I \sin(\phi) \end{pmatrix} v^n$$

$$= \begin{pmatrix} 1 & G e^{I\phi/2} (e^{I\phi/2} - e^{-I\phi/2}) \\ e^{-I\phi/2} G (e^{I\phi/2} - e^{-I\phi/2}) & 1-2G^2 + 2G^2 I \sin(\phi) \end{pmatrix} v^n$$

$$= \begin{pmatrix} 1 & G e^{I\phi/2}\, 2I \sin \phi/2 \\ G e^{-I\phi/2}\, 2I \sin \phi/2 & 1-2G^2 + 2G^2 I \sin(\phi) \end{pmatrix} v^n$$

let $r = 2\sigma \sin\phi/2$ $\qquad \left\{ \sin\phi = 2\sin\phi/2 \cos\phi/2 \right.$

$$G = \begin{pmatrix} 1 & r e^{I\phi/2} \\ r e^{-I\phi/2} & 1 - r^2 \end{pmatrix}$$ see below.

From eq * I get

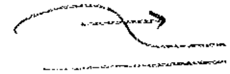
$$1 - 2\sigma^2 + \sigma^2 e^{-I\phi} + \sigma^2 e^{I\phi} = 1 + \sigma^2 (e^{I\phi} + e^{-I\phi} - 2)$$

$$= 1 + \sigma^2 (\underbrace{e^{I\phi/2} - e^{-I\phi/2}})^2$$

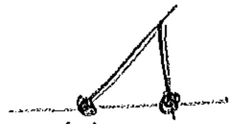
$$= 1 + \sigma^2 \; (2I \sin\phi/2)^2$$

$$= 1 - 4\sigma^2 \sin^2(\phi/2) = 1 - r^2$$

Prob 8.2 Forward space ; Forward time. $U = U(x-at)$



$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + a\frac{(U_{i+1}^n - U_i^n)}{\Delta x} = 0$$



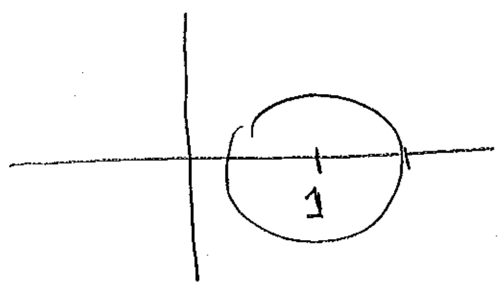
$(U_x)_i = \frac{U_{i+1} - U_i}{\Delta x}$

$U_i^n = \hat{U}^n e^{I\phi_x i}$ Decompose into Fourier decomposition

$$\frac{\hat{U}^{n+1} - \hat{U}^n}{\Delta t} + a\frac{(e^{I\phi_x} - 1)}{\Delta x}\hat{U}^n = 0$$

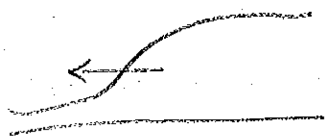
$$\hat{U}^{n+1} = \hat{U}^n - \frac{a\Delta t}{\Delta x}(e^{I\phi_x} - 1)\hat{U}^n = (1 - \sigma(e^{I\phi_x} - 1))\hat{U}^n$$

$$G = (1+\sigma) - \sigma e^{I\phi_x}$$

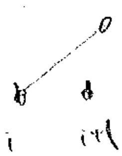


Unconditionally unstable $a > 0$

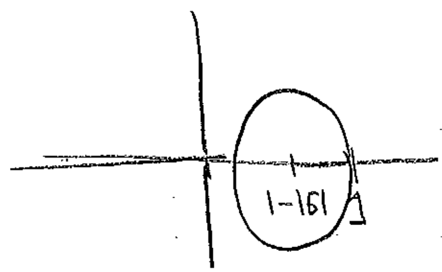
If $a < 0$



Then



$$G = (1 - |\sigma|) + |\sigma| e^{I\phi_x}$$



$|G| < 1$ if $|\sigma| < 1$

conditionally stable $|\sigma| < 1$

To derive the governing PDE $u_t + au_x = 0$

$$\frac{1}{\Delta t}\left(\cancel{u} + \Delta t u_t + \frac{\Delta t^2}{2} u_{tt} + O(\Delta t^3) - \cancel{u}\right) + \frac{a}{\Delta x}\left(\cancel{u} + \Delta x u_x + \frac{\Delta x^2}{2} u_{xx} + \frac{\Delta x^3}{6} u_{xxx} + O(\Delta x^4) - \cancel{u}\right) = 0$$

$$u_t + \frac{\Delta t}{2} u_{tt} + O(\Delta t^2) + a u_x + a\frac{\Delta x}{2} u_{xx} + O(\Delta x^2) = 0$$

Now $u_{tt} + a u_{tx} = 0 \qquad u_t = -a u_x$

$$u_{tt} + a(-a u_x)_x = 0$$

$$u_{tt} - a^2 u_{xx} = 0$$

$$\Rightarrow \quad u_t + \frac{\Delta t}{2} a^2 u_{xx} + a u_x + a\frac{\Delta x}{2} u_{xx} + O(\Delta t^2) + O(\Delta x^2) = 0$$

$$u_t + a u_x = \left(-\frac{\Delta t a^2}{2} - a\frac{\Delta x}{2}\right) u_{xx} + O(\Delta t^2) + O(\Delta x^2)$$

$$= -\frac{a^2}{2}\Delta t\left(1 + \frac{\Delta x}{a\Delta t}\right) u_{xx}$$

If $a > 0$ This coefficient is a negative diffusion giving a exponentially growing solution,

If $a < 0$ & $|\sigma| < 1 \quad \Rightarrow \quad \sigma = \frac{a\Delta t}{\Delta x}$

Then $1 + \frac{\Delta x}{\Delta t a} = 1 - \frac{\Delta x}{|a|\Delta t} > 1$

$\therefore$ coefficient is possitive a diffusion term, but stable.

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Prob 8.3 See notes page 300. + Notes for table 8.1

Prob 8.4

$$v_i^{n+1} - v_i^n = \sigma(w_{i+1}^n - w_i^n)$$

$$w_i^{n+1} - w_i^n = \sigma(v_i^n - v_{i-1}^n)$$

1) Matrix C defined $U^{n+1} = C U^n$

2) Matrix G defined $\hat{U}^{n+1} = G \hat{U}^n$

Define $U_i^n = \begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix}$

$$\begin{pmatrix} v_i^{n+1} \\ w_i^{n+1} \end{pmatrix} = \begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix} + \sigma \begin{pmatrix} 0 & (E-1) \\ 1-E^{-1} & 0 \end{pmatrix} \begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix}$$

$$\begin{pmatrix} v_i^{n+1} \\ w_i^{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \sigma(E-1) \\ \sigma(1-E^{-1}) & 1 \end{pmatrix}}_{\text{matrix } C} \begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\sigma \\ \sigma & 1 \end{pmatrix} \begin{pmatrix} v_i^n \\ w_i^n \end{pmatrix} + \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{i+1}^n \\ w_{i+1}^n \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ -\sigma & 0 \end{pmatrix} \begin{pmatrix} v_{i-1}^n \\ w_{i-1}^n \end{pmatrix} = C_0 U_i^n + C_{+1} U_{i+1}^n + C_{-1} U_{i-1}^n$$

let $U^n = \begin{pmatrix} \vdots \\ U^n_{i-1} \\ U^n_i \\ U^n_{i+1} \\ \vdots \end{pmatrix}$ Then $C = \begin{pmatrix} \ddots & & & & \\ & C_0 & C_{+1} & & \\ & C_{-1} & C_0 & C_{+1} & \\ & & C_{-1} & C_0 & C_{+1} \\ & & & & \ddots \end{pmatrix}$

Defining $U^n_i = \hat{U}^n e^{Ii\phi_x}$ $\qquad \phi_x = k_x \Delta x$

By one Fourier component

we obtain

$$\hat{U}^{n+1}\cancel{e^{Ii\phi_x}} = \begin{pmatrix} 1 & -\sigma \\ \sigma & 1 \end{pmatrix} \hat{U}^n \cancel{e^{Ii\phi_x}} + \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} \hat{U}^n e^{I\phi_x} \cancel{e^{Ii\phi_x}}$$

$$+ \begin{pmatrix} 0 & 0 \\ -\sigma & 0 \end{pmatrix} \hat{U}^n e^{-I\phi_x} \cancel{e^{Ii\phi_x}}$$

$$= \underbrace{\begin{pmatrix} 1 & -\sigma + \sigma e^{I\phi_x} \\ \sigma - \sigma e^{-I\phi_x} & 1 \end{pmatrix}}_{G} \hat{U}^n$$

$$= \underbrace{\begin{pmatrix} 1 & \sigma(e^{I\phi_x} - 1) \\ \sigma(1 - e^{-I\phi_x}) & 1 \end{pmatrix}}_{G} \hat{U}^n = \begin{pmatrix} 1 & \sigma e^{I\phi_x/2}(e^{I\phi_x/2} - e^{-I\phi_x/2}) \\ \sigma e^{-I\phi_x/2}(e^{+I\phi_x/2} - e^{-I\phi_x/2}) & 1 \end{pmatrix} \hat{U}^n$$

$$\rightarrow G = \begin{pmatrix} 1 & 2\sigma I e^{I\phi_x/2} \sin \phi_x/2 \\ \sigma 2 I e^{-I\phi_x/2} \sin \phi_x/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & I r e^{I\phi_x/2} \\ I r e^{-I\phi_x/2} & 1 \end{pmatrix}$$ w/ $r = 2\sigma \sin(\phi_x/2)$

Prob 8.5 8.3.16 is

See MMA file Hirsch Vol 1 2.nb

$$U_i^{n+1} = U_i^n + \frac{\alpha \Delta t}{\Delta x^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^n\right)$$

$$= \beta U_{i-1}^n + (1-2\beta)U_i^n + \beta U_{i+1}^n \quad *$$

Fix Δx. Stability requirements for this scheme require that

$$0 \leq \beta \leq 1/2 \Rightarrow 0 \leq \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Then choose Δt ∋ $\frac{\alpha \Delta t}{\Delta x^2} = \beta$

$$\boxed{\Delta t = \frac{\beta \Delta x^2}{\alpha}}$$

Thus each step of eq * is a time step of size Δt given by

Pseudocode

Numerical controls — physical parameters

Input N (# of nodes of grid), β, α, K (wave # of initial condition)

Compute Δx, $\Delta t = \frac{\beta \Delta x^2}{\alpha}$

compute initial condition w/ given Δx

iterate eq * starting at initial condition,

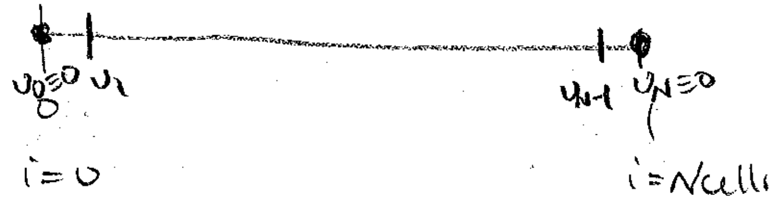
each iterate should be compared against UExact[$t = i\Delta t$]

(N, β) determine $\Delta x, \Delta t$. $\Delta t = \frac{\beta \Delta x^2}{\alpha}$

$\alpha \equiv 1$ since I can not find its value

B.C. for this scheme 07-24-01 2

$\{a, b, c\}$



$$u_i^{n+1} = \beta u_{i-1}^n + (1 - 2\beta) u_i^n + \beta u_{i+1}^n \quad \forall \text{ internal node,}$$

$i = 1, 2, \ldots, nCells - 1$

$i = 1$ $\quad u_0^n \equiv 0$ So stencile is different

$i = nCells - 1$ $\quad u_{ncells}^n \equiv 0$ for those two nodes

vector we compute w/ will have 2 additional elements

$\vec{V}_{comp}$ = runs from 0 to nCells

w/ BC enforce to be $V_0 \equiv 0$ $\quad V_{ncells} \equiv 0$

w/ Rotate + Rotate Right u_1 + $u_{Ncells-1}$ are updated

correctly but u_0 + u_{Ncells} may not be correct.

Thus need to rewrite our update routine at the boundaries

$$\varepsilon_D = \frac{1 - 4\beta \sin^2(\phi/2)}{e^{-\beta\phi^2}}$$

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Prob 8.6 Lax-Friedrichs scheme:

$$* \quad U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{\sigma}{2}(U_{i+1}^n - U_{i-1}^n)$$

Periodic B.C. $u_t + au_x = 0 \qquad u = u(x-at)$

$$\Rightarrow \tilde{U} = \sin \pi k(x-t) \quad \text{Exact sol w/ } a = 1$$

$$\bar{U}_i^n = |G|^n \sin \pi k(x_i - \bar{a}n\Delta t) \qquad \bar{a} = a\varepsilon_\phi = \varepsilon_\phi$$

$$= \frac{\tan^{-1}(\sigma \tan\phi)}{\sigma\phi}$$

$$\bar{U}_i^n = |G|^n \sin[\pi k(x_i - n\Delta t) + n\pi k\Delta t - \pi k\bar{a}n\Delta t]$$

$$= |G|^n \sin[\pi k(x_i - n\Delta t) + \pi k n\Delta t(1-\bar{a})]$$

Lax-Friedrichs scheme is

$$U_i^{n+1} = \frac{1}{2}(1+\sigma)U_{i-1}^n + \frac{1}{2}(1-\sigma)U_{i+1}^n$$

If $\Delta x = .02$

$$\Delta x = \frac{1}{n} = .02 \Rightarrow n = \frac{1}{.02} = 50$$

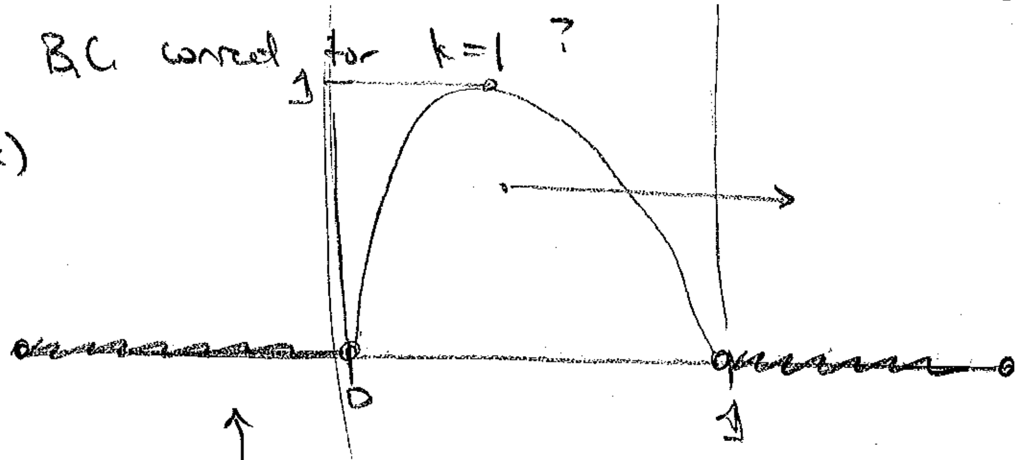
time after n time steps is $0 + \Delta t \cdot n\text{Timesteps}$

$$\hookrightarrow 0 + \frac{\Delta x}{a}\sigma \, n\text{Timesteps}$$

$$\sigma = \frac{\Delta t a}{\Delta x} \Rightarrow \frac{\Delta x}{a}\sigma = \Delta t$$

Are periodic B.C. correct for $k=1$?

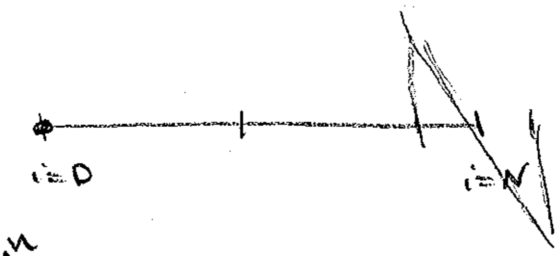
$\sin(\pi x)$



Think conditions outside ↑ shall be 0 (zero)

Lax-Friedrich scheme is

$$U_i^{n+1} = \frac{1}{2}(1+B)U_{i-1}^n + \frac{1}{2}(1-B)U_{i+1}^n$$



$$U_0^{n+1} = \frac{1}{2}\cancel{(1+B)U_{-1}^n}^{\,0} + \frac{1}{2}(1-B)U_1^n = \frac{1}{2}(1-B)U_1^n$$

$$U_N^{n+1} = \frac{1}{2}(1+B)U_{N-1}^n + \frac{1}{2}\cancel{(1-B)U_{N+1}^n}^{\,0} = \frac{1}{2}(1+B)U_{N-1}^n$$

} zero B.C's.

lets get our numerical scheme to more accurately predict or solution at the end points ⇒ Take $\cancel{U_{-1}^n = -U_{+1}^n}$

$+ \cancel{U_{N+1}^n = -}$

Think correct problem is also w/ zero edges

Prob 8.7

upwind scheme is $u_i^{n+1} - u_i^n = -\sigma(u_i^n - u_{i-1}^n)$

$$u_i^{n+1} = (1-\sigma)u_i^n + \sigma u_{i-1}^n \qquad *$$

Prob 8.8 leap frog $u_i^{n+1} - u_i^{n-1} = -\sigma(u_{i+1}^n - u_{i-1}^n)$

$$\Rightarrow u_i^{n+1} = u_i^{n-1} - \sigma(u_{i+1}^n - u_{i-1}^n)$$

$$u_i^{n+1} = u_i^{n-1} - \sigma u_{i+1}^n + \sigma u_{i-1}^n$$

Produce u_i^{n-1} by a backwards step of upwind eq.

To generate can I take $\sigma \Rightarrow -\sigma$ in eq * ?

Thus I know u_i^{n+1} can I compute what u_i^n is?

$$u_i^{n+1} = (1-\sigma)u_i^n + \sigma E^{-1} u_i^n$$

$$= (1+\sigma+\sigma E^{-1})u_i^n$$

$$u_i^n = \frac{1}{(1+\sigma+\sigma E^{-1})} u_i^{n+1} = \frac{1}{1+\sigma} - \frac{\sigma E^{-1}}{(1+\sigma)^2} + \frac{\sigma^2 E^{-2}}{(1+\sigma)^3} - \frac{\sigma^3 E^{-3}}{(1+\sigma)^4} + \cdots$$

one way to invert matrices {Approximations are better than reality}

This is not the correct expansion.

$$U_i^n = \frac{1}{(1+\beta(1+E^{-1}))} U_i^{n+1} = \sum_{k=0}^{\infty} \beta^k (-1)^k (1+E^{-1})^k$$

$$= 1 - \beta(1+E^{-1}) + \beta^2(1+E^{-1})^2 + \cdots$$

* This is the correct expansion

$$= 1 - \beta(1+E^{-1}) + \beta^2(1+2E^{-1}+E^{-2}) + \cdots$$

But many terms are needed to get good agreement. See MMA file

$$= 1 - \beta - \beta E^{-1} + \beta^2 + 2\beta^2 E^{-1} + \beta^2 E^{-2} + \cdots$$

$$= 1 - \beta + \beta^2 + (-\beta + 2\beta^2)E^{-1} + \beta^2 E^{-2} + \cdots$$

{One way is to cheat & use the exact solution to the eq shifted backwards by Δt.}

I'll do 2 time step w/ forward diff scheme & then enter the main loop.

(Prob 8.9) $u_t = \alpha u_{xx}$ Heat conduction

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = \frac{\alpha\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}{\Delta x^2}$$

$$u_i^{n+1} - u_i^{n-1} = 2\underbrace{\left(\frac{\alpha \Delta t}{\Delta x^2}\right)}_{\sigma}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

Following Book turn this two step problem into a one step one

w/ $z_i^n = u_i^{n-1}$

Then define $w_i^n = \begin{pmatrix} u_i^n \\ z_i^n \end{pmatrix}$

$$u_i^{n+1} = z_i^n + 2\sigma\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$z_i^{n+1} = u_i^n$$

$$w_i^{n+1} = \begin{pmatrix} 2\sigma(E - 2 + E^{-1}) & 1 \\ 1 & 0 \end{pmatrix} w_i^n$$

Then Fourier analysis of this eq $w_i^{n+1} = \hat{w}^{n+1} e^{I\phi i}$

$$\hat{w}^{n+1} = \begin{pmatrix} 2\sigma(e^{I\phi} - 2 + e^{-I\phi}) & 1 \\ 1 & 0 \end{pmatrix} \hat{w}^n$$

$$\hat{\omega}^{n+1} = \underbrace{\begin{pmatrix} 2\sigma(-4\sin^2(\phi/2)) & 1 \\ 1 & 0 \end{pmatrix}}_{\text{Amplification matrix.}} \hat{\omega}^n$$

Growth factors follow from its eigenvalue,

$$(e^{I\phi/2} - e^{-I\phi/2})^2 = -4\sin^2\phi/2$$

$$\begin{vmatrix} \lambda + 8\sigma\sin^2(\phi/2) & 1 \\ 1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 8\sigma\sin^2(\phi/2)\lambda - 1 = 0$$

$64/4 = 32/2 = 16$

$$\lambda = \frac{-8\sigma\sin^2(\phi/2) \pm \sqrt{64\sigma^2\sin^4(\phi/2) + 4}}{2}$$

$$= -4\sigma\sin^2(\phi/2) \pm \sqrt{16\sigma^2\sin^4(\phi/2) + 1}$$ all Real.

for negative sign

$$\lambda_- < -4\sigma\sin^2(\phi/2) - 4\sigma\sin^2(\phi/2) < -8\sigma\sin^2(\phi/2)$$

Notice that as long as $\sigma > 0$

$$16\sigma^2 \sin^4(\phi/2) + 1 > 1$$

$$\sqrt{16\sigma^2 \sin^4(\phi/2) + 1} > 1$$

$$-\sqrt{16\sigma^2 \sin^4(\phi/2) + 1} < -1$$

$$-4\sigma \sin^2(\phi/2) - \sqrt{16\sigma^2 \sin^4(\phi/2) + 1} < -1$$

$\therefore$ $|\lambda_-| > 1$ $\rightarrow$ scheme is unstable.

Prob 8.10

$$u_i^{n+1} - u_i^{n-1} = -2\sigma(u_i^n - u_{i-1}^n)$$

Considering an Fourier mode $u_i^n = \hat{u}_k^n e^{I\theta_k i}$ $\theta_k = \frac{\pi}{n}k$

Method 1:

* Since this is a 2 step scheme we must convert this into a single step scheme before we can apply Von Neuman stability Analysis

let $z_i^n = u_i^{n-1}$ Then $z_i^{n+1} = u_i^n$

So eqs are

$$u_i^{n+1} - z_i^n = -2\sigma(u_i^n - u_{i-1}^n)$$
$$z_i^{n+1} = u_i^n$$

$$u_i^{n+1} = z_i^n - 2\sigma(u_i^n - u_{i-1}^n)$$
$$z_i^{n+1} = u_i^n$$

let $V_i^n = \begin{pmatrix} u_i^n \\ z_i^n \end{pmatrix} \Rightarrow V_i^{n+1} = \begin{pmatrix} -2\sigma(1-E^{-1}) & 1 \\ 1 & 0 \end{pmatrix} V_i^n$

Then decomposing a Fourier component of V_i^n

$$V_i^n = \hat{V}_k^n e^{I\theta_k i}$$ $\theta_k = \frac{\pi k}{n}$

$$\hat{V}_1^{n+1} \cancel{e^{I\theta_k i}} = \begin{pmatrix} -2\sigma(1-e^{-I\theta_k}) & 1 \\ 1 & 0 \end{pmatrix} \hat{V}_k^n \cancel{e^{I\theta_k i}}$$

$$\hat{V}_k^{n+1} = \underbrace{\begin{pmatrix} -2\sigma(1-e^{-I\theta_k}) & 1 \\ 1 & 0 \end{pmatrix}}_{G \text{ amplification matrix.}} \hat{V}_k^n$$ ✓

The stability is required by insuring that the eigenvalues all all of magnitude smaller than 1. This gives

$$\begin{vmatrix} \lambda + 2\sigma(1-e^{-I\theta}) & -1 \\ -1 & \lambda \end{vmatrix} = 0$$ ✓

$$\lambda^2 + 2\sigma(1-e^{-I\theta})\lambda - 1 = 0$$ ✓

$$\lambda = \frac{-2\sigma(1-e^{-I\theta}) \pm \sqrt{4\sigma^2(1-e^{-I\theta})^2 + 4}}{2}$$ ✓

$$= -\sigma(1-e^{-I\theta}) \pm \sqrt{\sigma^2(1-e^{-I\theta})^2 + 1}$$

> 1 in magnitude *

Heuristic approach

If $z = z_r + z_I I$

$|z| > \mathrm{Abs}(z_r)$

$|z| > \mathrm{Abs}(z_I)$

* To see this consider that $1 + \sigma^2(1-e^{-I\theta})^2 > 1$

$1 - e^{-I\theta}$ = will not ever be made entirely real !

$$e^{-I\theta/2}\left(e^{I\theta/2} - e^{-I\theta/2}\right)$$

$$= 2I e^{-I\theta/2} \sin(\theta/2) \quad \checkmark$$

Then roots of λ become

$$\lambda = -2I\beta e^{-I\theta/2}\sin(\theta/2) \pm \sqrt{-4e^{-I\theta}\beta^2\sin^2(\theta/2) + 1}$$

$$= -2I\beta e^{-I\theta/2}\sin(\theta/2) \pm \sqrt{1 - (4\beta^2\sin^2(\theta/2))e^{-I\theta}}$$

Need to evaluate the square root.

$$z = 1 - 4\beta^2\sin^2(\theta/2)\cos\theta + 4\beta^2\sin^2(\theta/2)\sin\theta\, I$$

$$\sqrt{z} = \sqrt{|z|}\, e^{I\phi/2} \qquad \phi = \tan^{-1}\left[\frac{y}{x}\right]$$

$$\sqrt{z} = \left(\left(1 - 4\beta^2\sin^2(\theta/2)\cos\theta\right)^2 + \left(4\beta^2\sin^2(\theta/2)\sin\theta\right)^2\right)^{1/2} e^{I\phi/2}$$

$$\text{w/} \quad \phi = \tan^{-1}\left[\frac{4\beta^2\sin^2(\theta/2)\sin\theta}{1 - 4\beta^2\sin^2(\theta/2)\cos\theta}\right]$$

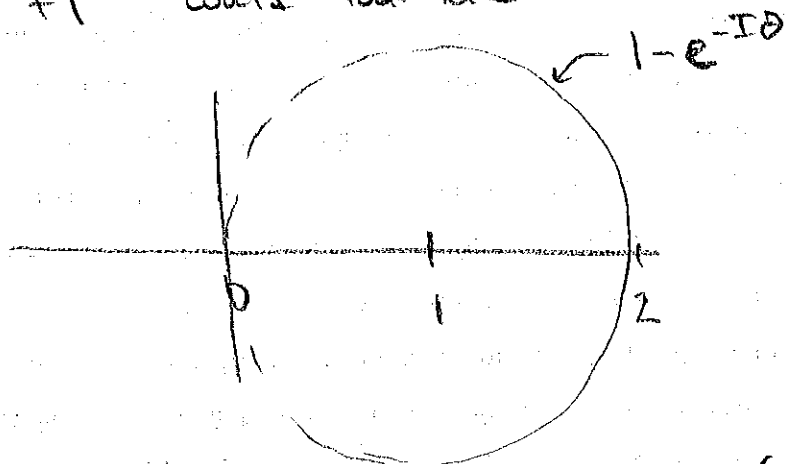
Thus

$$\sqrt{z} = 1 - 8\beta^2\sin^2$$

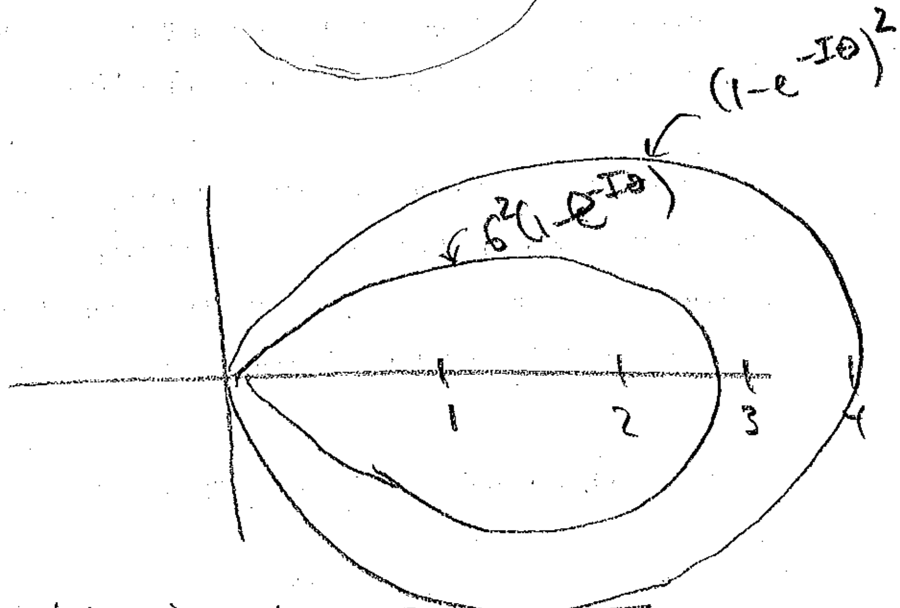
· Stop ... I just want to show that the numerical scheme is unstable $\forall\, \sigma$ I just have to show that for 1 value of $-\pi \le \theta \le \pi$ $\quad |\lambda| > 1$

$$\lambda_\pm = -\sigma(1-e^{-I\theta}) \pm \sqrt{\sigma^2(1-e^{-I\theta})^2+1}$$

plotting $\sigma^2(1-e^{-I\theta})^2+1$ would look like



Then



We see that the largest point is when $\theta = 0$ or π,
trying $\theta = \pi$ we get

$$\lambda_\pm = -\sigma(2) \pm \sqrt{\sigma^2(4)+1} = -2\sigma \pm \sqrt{1+4\sigma^2}$$

$$\lambda_- = -2\sigma - \sqrt{1+4\sigma^2} \qquad |\lambda_-| = 2\sigma + \sqrt{1+4\sigma^2}$$

Since $\sigma > 0$

$$1 + 4\sigma^2 > 1$$

$$2\sigma + \sqrt{1+4\sigma^2} \underset{\neq}{>} 1 \quad \Rightarrow \quad \text{scheme is unstable.}$$

An alternate means of deriving the same expression:

$$U_i^{n+1} - U_i^{n-1} = -2\sigma(U_i^n - U_{i-1}^n)$$

Decompose into Fourier components w/ $U_i^n = \hat{U}_k^n e^{I\phi_k i}$ $\phi_k = \frac{\pi k}{n}$

$\Rightarrow G\hat{U}^n = \hat{U}^{n+1}$

$\therefore \hat{U}^n = G^{-1}\hat{U}^{n+1}$

So the above eq becomes

$$G\hat{U}^n - G^{-1}\hat{U}^n = -2\sigma(1 - e^{-I\phi})\hat{U}^n$$

$$\Rightarrow G - G^{-1} = -2\sigma(1 - e^{-I\phi}) = -4Ie^{-I\phi/2}\sin(\phi/2)\sigma$$

Then

$$G^2 + 4Ie^{-I\phi/2}\sin(\phi/2)G - 1 = 0$$

$$G = \frac{-4Ie^{-I\phi/2}\sin(\phi/2) \pm \sqrt{-16e^{-I\phi}\sin^2(\phi/2) + 4}}{2}$$

$=$

Prob 8.11

7.2.9 is $\dfrac{u_i^{n+1} - u_i^n}{\Delta t} = -\dfrac{a}{\Delta x}(u_i^{n+1} - u_{i-1}^{n+1})$

Decompose into Fourier components $u_i^n = \hat{u}^n e^{I\phi i}$

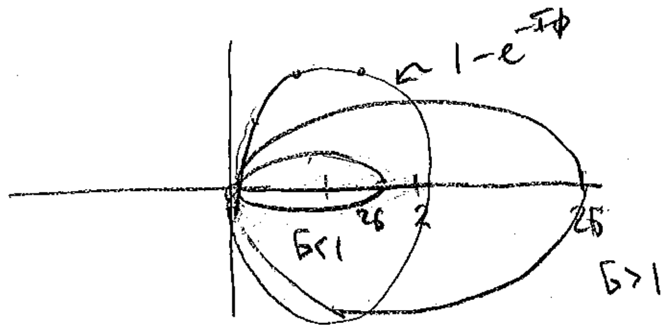
Then Growth factor G is defined by $\hat{u}^{n+1} = G\hat{u}^n$

$$\frac{G\hat{u}^n - \hat{u}^n}{\Delta t} = -\frac{a}{\Delta x}(G\hat{u}^n - G\hat{u}^n e^{-I\phi})$$

$\Rightarrow G - 1 = -\sigma G(1 - e^{-I\phi})$

$\Rightarrow (1 + \sigma(1 - e^{-I\phi}))G = 1$

$$G = \frac{1}{1 + \sigma(1 - e^{-I\phi})}$$



If $\sigma < 1$

$Re(\sigma(1 - e^{-I\phi})) \geq 0$ if $a > 0$

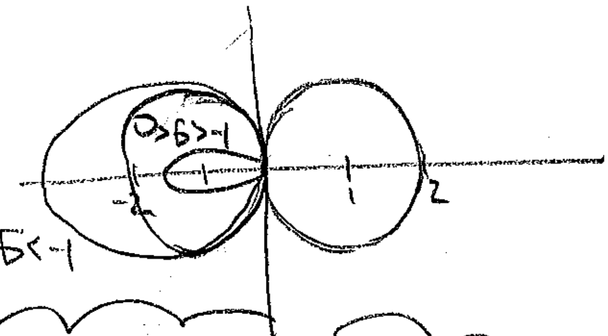
$\therefore Re(1 + \sigma(1 - e^{-I\phi})) \geq 1$ since $|z| \geq |Re(z)|$

$\therefore$

$$|G| = \frac{1}{|1 + \sigma(1 - e^{-I\phi})|} \leq \frac{1}{|Re(1 + \sigma(1 - e^{-I\phi}))|} \leq 1$$

Method is stable if $a > 0$

If $a<0$ $\quad Re(\sigma(1-e^{-I\phi}))$

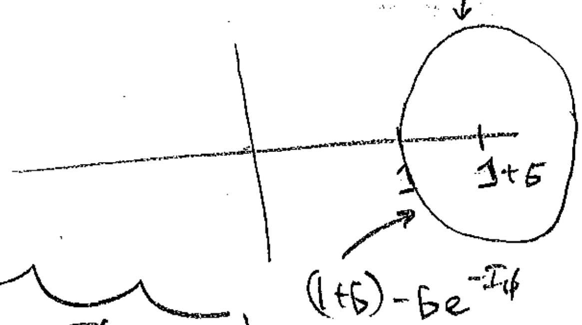


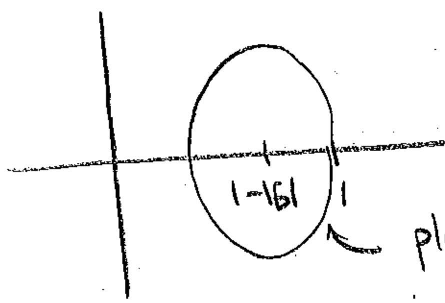
$$G = \frac{1}{1-\sigma(e^{-I\phi}-1)}$$

$$= \frac{1}{1+\sigma-\sigma e^{-I\phi}}$$

$$= \frac{1}{1-|\sigma|+|\sigma|e^{-I\phi}}$$

Another way to see stability when $\sigma>0$ $\quad$ when $\sigma>0$ $\quad G = \frac{1}{(1+\sigma)-\sigma e^{-I\phi}}$





plot of $1-|\sigma|+|\sigma|e^{-I\phi}$

Since $\exists$ points on this circle when

$|z|<1$ The $\frac{1}{|z|}>1$ + scheme is unstable

Prob 8,12

Methods.

a) Firsh order upwind → $u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n)$

b) Lax-Fredrich → $u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n)$

c) Lax-Wendroff → $u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\sigma^2}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$

d) Leap frog → $u_i^{n+1} - u_i^{n-1} = -\sigma(u_{i+1}^n - u_{i-1}^n)$

First order upwind: $u_i^{n+1} = (1-\sigma)u_i^n + \sigma u_{i-1}^n$

Lax-Fredrich: $u_i^{n+1} = \frac{1}{2}(1+\sigma)u_{i-1}^n + \frac{1}{2}(1-\sigma)u_{i+1}^n$

Lax-Wendroff: $u_i^{n+1} = \frac{\sigma}{2}(1+\sigma)u_{i-1}^n + (1-\sigma^2)u_i^n + \frac{\sigma}{2}(-1+\sigma)u_{i+1}^n$

Leap-Frog: $u_i^{n+1} = +\sigma u_{i-1}^n - \sigma u_{i+1}^n + u_i^{n-1}$

Data structure for this should be 2 arrays $\forall$ timestep.

Pseudocode is the following:

Get 2 arrays of data V_{-1} & V_0 from initial condition & a single step method run backwards

Prob 8.12 Here we need the Lax-Wendroff scheme

$$u_i^{n+1} = u_i^n - \frac{b}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{b^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

$$= \left(-\frac{b}{2}+\frac{b^2}{2}\right)u_{i+1}^n + \left(1-b^2\right)u_i^n + \left(\frac{b}{2}+\frac{b^2}{2}\right)u_{i-1}^n$$

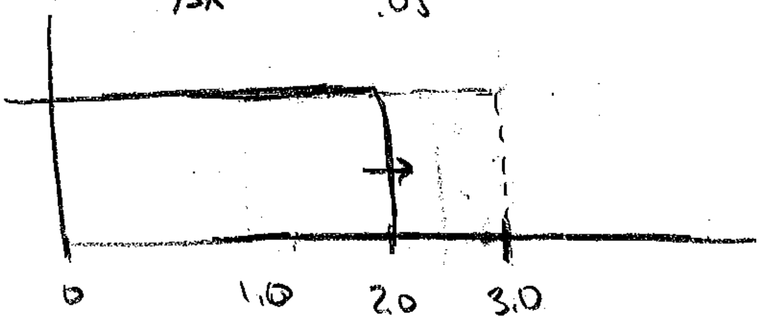
$$= \frac{b}{2}(1+b)u_{i-1}^n + (1-b)(1+b)u_i^n + \frac{b}{2}(-1+b)u_{i+1}^n$$

$b = .8$ nTimesteps $= 80$

$\Delta x = .05$ {From 0 to 4 → n cells = $\frac{4}{ncells} = .05$ → ncells $= \frac{4}{.05} = 80$

$a = 1$

$b = \frac{\Delta t\, a}{\Delta x} = \frac{\Delta t(1)}{.05} = .8$ → $\Delta t = .04$ {From 2 to 4



$3.0 - 50(.04) = 2$ initial conditions location.

Prob 8.14

$$u_t + a u_x + b u_y = 0$$

$$\frac{u^{n+1}_{ij} - u^n_{ij}}{\Delta t} + \frac{a}{\Delta x}\left(u^n_{ij} - u^n_{i-1,j}\right) + \frac{b}{\Delta y}\left(u^n_{ij} - u^n_{i,j-1}\right) = 0$$

Now $\quad u^n_{ij} = \sum_{k_x, k_y} \hat{u}^n e^{I k_x i \Delta x} e^{I k_y j \Delta y}$

& again $\quad v^{n+1} = G v^n$

Then we get

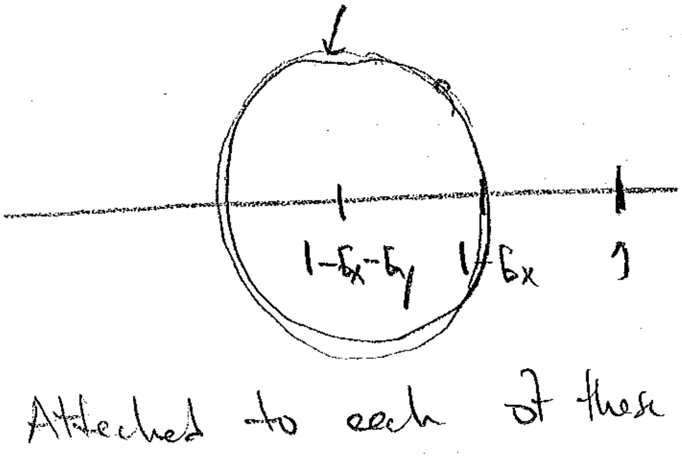
$$G - 1 + \sigma_x\left(1 - e^{-I\phi_x}\right) + \sigma_y\left(1 - e^{-I\phi_y}\right) = 0$$

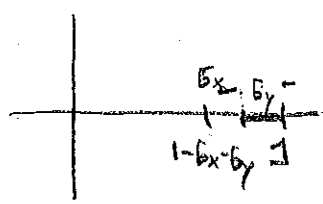
$$G = 1 - \sigma_x\left(1 - e^{-I\phi_x}\right) - \sigma_y\left(1 - e^{-I\phi_y}\right)$$

$$= 1 - \sigma_x - \sigma_y + \sigma_x e^{-I\phi_x} + \sigma_y e^{-I\phi_y}$$

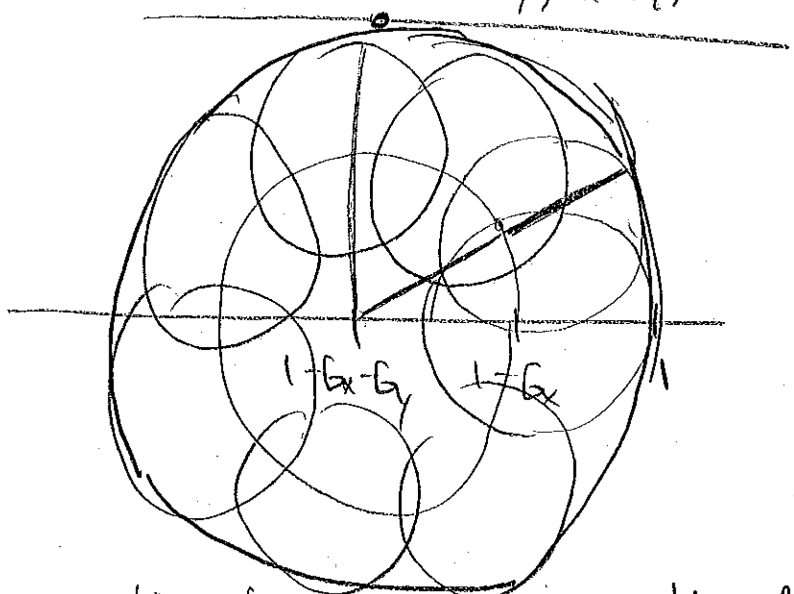
All pts on circ

$1 - \sigma_x - \sigma_y + \sigma_y e^{-I\phi_y}$





Attached to each of these points draw another circl of Radius σ_x i.e



Then question becomes what is the limiting shape of this object? Must be another circle of Radius 6_x+6_y

This should have norm less than 1 for stability !!

$$|6_x+6_y| < 1 \Rightarrow \Delta t\left(\frac{a}{\Delta x}+\frac{b}{\Delta y}\right) < 1 \Rightarrow \begin{matrix} 6_x < 1 \\ \& \ 6_y < 1 \end{matrix}$$

$$G = 1 - 6_x - 6_y + 6_x \cos\phi_x + 6_y \cos\phi_y - (6_x \sin\phi_x + 6_y \sin\phi_y) I$$

$$= 1 - 6_x(\cos\phi_x - 1) - 6_y(\cos\phi_y - 1) - (6_x \sin\phi_x + 6_y \sin\phi_y) I$$

$$|G|^2 = \left(1-(6_x+6_y) + 6_x e^{-I\phi_x} + 6_y e^{-I\phi_y}\right)\left(1-(6_x+6_y) + 6_x e^{I\phi_x} + 6_y e^{I\phi_y}\right)$$

$$= (1-(6_x+6_y))^2 + 26_x(1-(6_x+6_y))\cos\phi_x$$

$$+ 26_y(1-(6_x+6_y))\sin\phi_x$$

$$+$$

Prob 8.15 convection diffusion

$$u_t + a u_x = \alpha u_{xx}$$

Apply leap frog to

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + a\frac{(u_{i+1}^n - u_{i-1}^n)}{2\Delta x} = \frac{\alpha}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

Then to obtain Dufort-Frankel scheme we replace $u_i^n = \frac{u_i^{n+1} + u_i^{n-1}}{2}$

$$\Rightarrow u_i^{n+1} - u_i^{n-1} + \underbrace{\frac{a\Delta t}{\Delta x}}_{\sigma}(u_{i+1}^n - u_{i-1}^n) = 2\underbrace{\frac{\alpha\Delta t}{\Delta x^2}}_{\beta}(u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n)$$

$$u_i^{n+1} - u_i^{n-1} = -\sigma(u_{i+1}^n - u_{i-1}^n) + 2\beta(u_{i+1}^n - u_i^{n+1} - u_i^{n-1} + u_{i-1}^n)$$

Following hint

$$u_i^{n+1} - u_i^{n-1} = -\sigma(u_{i+1}^n - u_{i-1}^n) + 2\beta(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$ ← put all on one time level.

$$+ 2\beta(+2u_i^n - u_i^{n+1} - u_i^{n-1})$$

$$= -\sigma(u_{i+1}^n - u_{i-1}^n) + 2\beta(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$- 2\beta(u_i^{n+1} - 2u_i^n + u_i^{n-1})$$

07-27-01

To test stability let $u_i^n = G^n e^{I\phi i}$ then we get for the growth factor

$$G - G^{-1} = -\sigma(e^{I\phi} - e^{-I\phi}) + 2\beta(e^{I\phi} - 2 + e^{-I\phi})$$

$$-2\beta(G - 2 + G^{-1})$$

$$\Rightarrow G - G^{-1} = -2I\sigma \sin\phi + 2\beta(-4\sin^2\phi/2) - 2\beta(G^{1/2} - G^{-1/2})^2$$

$$\Rightarrow G - G^{-1} = -\sigma 2I \sin\phi + 2\beta \sin^2 \phi/2\,(4(-1)) - 2\beta(G - 2 + G^{-1})$$

$$= -2\sigma \sin\phi I - 8\beta \sin^2(\phi/2) - 2\beta(G - 2 + G^{-1})$$

$$(G - G^{-1}) + 2\beta(G + G^{-1}) = -2\sigma \sin\phi I - 8\beta \sin^2(\phi/2) + 4\beta$$
$$(1+2\beta)G + (-1+2\beta)G^{-1} =$$

$$(1+2\beta)G^2 + (2\sigma \sin\phi I + 8\beta \sin^2(\phi/2) - 4\beta)G + (-1+2\beta) = 0$$

$$2(\underbrace{\sigma \sin\phi I + 4\beta \sin^2(\phi/2) - 2\beta}) G$$

$$\frac{\sigma \sin\phi I + 2\beta(2\sin^2(\phi/2) - 1)}{2}$$

$$\Rightarrow (1+2\beta)G^2 + 2bG + (-1+2\beta) = 0$$

$$G = \frac{-2b \pm \sqrt{4b^2 - 4(1-2\beta)(-1+2\beta)}}{2(1+2\beta)}$$

$$G = \frac{-2b \pm \sqrt{4b^2 + 4(1-2B)(1-2B)}}{2(1+2B)}$$

$$= \frac{-b \pm \sqrt{b^2 + (1-2B)^2}}{1+2B}$$

w/ $b = G \sin\phi I + 2B(2\sin^2(\phi/2) - 1)$

$$(2(1-\cos^2(\phi/2)) - 1)$$

$$1 - 2\cos^2(\phi/2) = -\cos\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= 1 - 2\sin^2\theta$$
$$= -1 + 2\cos^2\theta$$

Thus $b = G\sin\phi I - 2B\cos\phi$

$$= -2B\cos\phi + G\sin\phi I$$

$b^2 =$

Prob 8.17 See Notes pg 308

Prob 8.18 See Notes pg 312

There $\varepsilon_\phi = \pm \dfrac{\sin^{-1}(\sigma \sin\phi)}{\sigma\phi} = 1 + \frac{1}{6}(-1+\sigma^2)\phi^2 + O(\phi^4)$

Prob 8.19 See Notes pg 313

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Prob 8.20

Prob 5.14 gives a scheme of

$$\frac{1}{6}\left[\frac{du_{i-1}}{dt} + 4\frac{du_i}{dt} + \frac{du_{i+1}}{dt}\right] = \frac{1}{2\Delta x}(f_{i+1} - f_{i-1})$$

w/ $f = au$

$$\frac{1}{6}\left[\frac{u^{n+1}_{i-1} - u^{n-1}_{i-1}}{2\Delta t} + \frac{4(u^{n+1}_i - u^{n-1}_i)}{2\Delta t} + \frac{u^{n+1}_{i+1} - u^{n-1}_{i+1}}{2\Delta t}\right]$$

$$= \frac{a}{2\Delta x}(u^n_{i+1} - u^n_{i-1})$$

$$\Rightarrow \quad u^{n+1}_{i-1} - u^{n-1}_{i-1} + 4(u^{n+1}_i - u^{n-1}_i) + u^{n+1}_{i+1} - u^{n-1}_{i+1} =$$

$$6\sigma(u^n_{i+1} - u^n_{i-1})$$

Calc $\quad u^n_i = G^n e^{I\phi i}$

$$Ge^{-I\phi} - G^{-1}e^{-I\phi} + 4(G - G^{-1}) + Ge^{I\phi} - G^{-1}e^{-I\phi}$$

$$+ 6\sigma(e^{I\phi} - e^{-I\phi}) = 0$$

$$\Rightarrow (G - G^{-1})\left[e^{-I\phi} + 4 + e^{I\phi}\right] + 6\sigma 2I\sin\phi = 0$$

$$[4 + 2\cos\phi] \quad + \quad 12\sigma I\sin\phi = 0$$

$$(G-G^{-1})(2+\cos\phi) + 6\sigma I \sin\phi = 0$$

$$G - G^{-1} + \frac{6\sigma I \sin\phi}{2+\cos\phi} = 0$$

$$G^2 + 2\underbrace{\left(\frac{3\sigma \sin\phi}{2+\cos\phi}\right)}_{b} IG - 1 = 0$$

$$G = \frac{-2bI \pm \sqrt{4(-1)b^2+4}}{2} = -bI \pm \sqrt{1-b^2}$$

Then we require $|G| < 1 \iff |G|^2 < 1$

$$\Rightarrow GG^* < 1$$

$$\Rightarrow (-bI \pm \sqrt{1-b^2})(+bI \pm \sqrt{1-b^2}) < 1$$

$$b^2 + (1-b^2) = 1 \qquad \text{So } |G| = 1 \text{ iff}$$

Assuming $\boxed{b^2 < 1}$ $\Rightarrow$ $\frac{36 \sigma^2 \sin^2\phi}{(2+\cos\phi)^2} < 1$

$$\sigma^2 < \frac{(2+\cos\phi)^2}{9\sin^2\phi} > \frac{1}{9} \Rightarrow |\sigma| < \frac{1}{3}$$

Very roughly

What is ~~maximum~~ minimum of $\dfrac{(2+\cos\phi)^2}{9\sin^2\phi}$ on $\phi \in [0, 2\pi]$?

$$\frac{df}{d\phi} = \frac{2(2+\cos\phi)(-\sin\phi)}{9\sin^2\phi} + \frac{(2+\cos\phi)^2(-2)\cos\phi}{9\sin^3\phi} = 0$$

$$= \frac{2(2+\cos\phi)}{9\sin^2\phi}\left[-\sin\phi + -\frac{(2+\cos\phi)\cos\phi}{\sin\phi}\right] = 0$$

$$\Leftrightarrow \quad \sin^2\phi + 2\cos\phi + \cos^2\phi = 0$$

$$1 + 2\cos\phi = 0$$

$$\cos\phi = -\frac{1}{2} \Rightarrow \phi = \frac{4\pi}{3}$$

Check this is a true minimum

$$\sigma^2 < \frac{(2-\frac{1}{2})^2}{9(\frac{\sqrt{3}}{2})^2} = \frac{\cancel{9}/\cancel{4}}{\cancel{9}\,3/\cancel{4}} = \frac{1}{3}$$

$$\Rightarrow \sigma < \frac{1}{\sqrt{3}}$$

The dispersion errors is $\varepsilon_\phi = \dfrac{\Phi}{\tilde{\Phi}} = \dfrac{\Phi}{ka\Delta t} = \dfrac{\Phi}{\sigma\phi}$

$$\Phi = -\tan^{-1}\left(\frac{-b}{\sqrt{1-b^2}}\right) = \tan^{-1}\left(\frac{b}{\sqrt{1-b^2}}\right)$$

$$= \theta = \sin^{-1}(b)$$

Thus $\varepsilon_\phi = \dfrac{\sin^{-1}(b)}{\sigma\phi}$

$$= \frac{\sin^{-1}\left(\frac{3\sigma\sin\phi}{2+\cos\phi}\right)}{\sigma\phi}$$ Dispersion error.

Diffusion error $\varepsilon_D = |G| = 1$

Numerical Group velocity:

setting $G = e^{-I\omega\Delta t} \underset{\text{set}}{=} -Ib \pm \sqrt{1-b^2}$

$$= 1e^{-I\sin^{-1}(b)}$$

$\Rightarrow \quad \omega\Delta t = \sin^{-1}(b)$

$\sin(\omega\Delta t) = b$

$\sin(\omega\Delta t) = \dfrac{3\sigma\sin\phi}{2+\cos\phi}$ $\quad \{\phi = k\Delta x$

$V_G = \dfrac{d\omega}{dk}$

$$\Delta t\cos(\omega\Delta t)\frac{d\omega}{dk} = 3\sigma\left[\frac{\cos\phi\,\Delta x}{2+\cos\phi} - \frac{(-\sin\phi)\cos\phi\,\Delta x}{(2+\cos\phi)^2}\right]$$

$$\cos(\omega\Delta t)\frac{d\omega}{dk} = 3\frac{\sigma\Delta x}{\Delta t}\left[\frac{2+\cos\phi + \sin\phi}{(2+\cos\phi)^2}\right]\cos\phi$$

$\sigma = \dfrac{\Delta t\, a}{\Delta x} \quad \dfrac{\sigma\Delta x}{\Delta t} = a.$

$$\frac{dw}{dt} = \frac{3a\cos\phi}{\cos(\omega\Delta t)}\left[\frac{2+\sin\phi+\cos\phi}{(2+\cos\phi)^2}\right]$$

$$\cos(\omega\Delta t) = \sqrt{1-\sin^2(\omega\Delta t)} = \sqrt{1-b^2}$$

$$\Rightarrow \quad \frac{dw}{dt} = \frac{3a\cos\phi}{\sqrt{1-b^2(\phi)}}\left[\frac{2+\sin\phi+\cos\phi}{(2+\cos\phi)^2}\right]$$