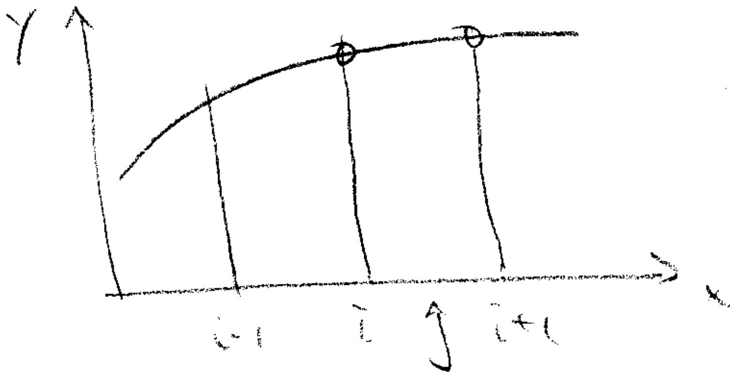
PS 168 Hirsch Vol I

Forward difference

$$(U_x)_i = \frac{U_{i+1} - U_i}{\Delta x} + O(\Delta x)$$

$=$



$$x_{i+1/2} = \frac{x_i + x_{i+1}}{2}$$

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$$(U_x)_i = \frac{a U_i + b U_{i+1} + c U_{i+2}}{\Delta x} + O(\Delta x^2)$$

$$U_{i+2} = U_i + 2\Delta x (U_x)_i + \frac{4\Delta x^2}{2}(U_{xx})_i + \frac{8\Delta x^3}{3!}(U_{xxx})_i + O(\Delta x^4)$$

$$U_{i+1} = U_i + \Delta x (U_x)_i + \frac{\Delta x^2}{2}(U_{xx})_i + \frac{\Delta x^3}{3!}(U_{xxx})_i + O(\Delta x^4)$$

Mult + Add

$$(c+b+a)U_i + (2c+b)\Delta x (U_x)_i + (2c + b/2)\Delta x^2 (U_{xx})_i + O(\Delta x^3)$$

$$= (U_x)_i + O(\Delta x$$

$$a + b + c = 0$$

$$2c + b = 1 \qquad 2c - 4c = 1 \Rightarrow -2c = 1 = c = -1/2$$

$$2c + b/2 = 0 = 4c + b = 0 = b = -4c = b = 2$$

$$a + \underbrace{2 + -1/2}_{3/2} = 0$$

$$a = -3/2$$

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$$(u_{xx})_i = \frac{(u_x)_{i+1} - (u_x)_i}{\Delta x}$$

$$= \frac{(u_{i+1} - u_i)/\Delta x^{+} - (u_i - u_{i-1})/\Delta x}{\Delta x}$$

$$= \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^{?})$$

$$\cancel{u_i} + \Delta x \cancel{(u_x)_i} + \frac{\Delta x^2}{2}(u_{xx})_i + \Delta x^3/\cancel{3!}\cancel{(u_{xxx})_i} + \Delta x^4/4!\,(u_{xxxx})_i + O(\Delta x^5)$$

$$+ \; -2\cancel{u_i}$$

$$+ \cancel{u_i} - \cancel{\Delta x}(u_x)_i + \frac{\Delta x^2}{2}(u_{xx})_i - \frac{\Delta x^3}{3!}\cancel{(u_{xxx})_i} + \frac{\Delta x^4}{4!}(u_{xxxx})_i + O(\Delta x^5)$$

$$\div \Delta x^2$$

$$\frac{\Delta x^2 (u_{xx})_i}{\Delta x^2} + \frac{2\Delta x^4}{4!\,\Delta x^2}(u_{xxxx})_i + O\frac{(\Delta x^6)}{\Delta x^2}$$

$$= (u_{xx})_i + \frac{\Delta x^2}{2 \cdot 6}(u_{xxxx})_i + \cdots$$

$$(u_{xx})_i = \frac{au_i + bu_{i-1} + cu_{i-2}}{\Delta x^2} +$$

Using expanded forms given on pg 170

$$a + b + c = 0$$

$$2c + b = 0 \rightarrow b = -2c$$

$$(4c + b) = 2$$

$$2c = 2 \Rightarrow c = 1 \Rightarrow b = -2$$

$$a - 2 + 1 = 0$$

$$a - 1 = 0 \Rightarrow a = 1$$

$$\therefore (u_{xx})_i = \frac{u_i - 2u_{i-1} + u_{i-2}}{\Delta x^2} + \frac{?}{\Delta x^2}$$

coefficient of $O(\Delta x^3)$ terms

$1 \Delta x \cdot u_{xxx}$

$$-\frac{8}{6}(1) - \frac{1}{6}(-2) = \frac{-6}{6} = -1$$

$$(u_{xx})_i = \frac{u_i - 2u_{i-1} + u_{i-2}}{\Delta x^2} + \Delta x \cdot u_{xxx} + \cdots$$ 1st order

$$(u_{xx})_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$ 2nd order

$$(u_{xx})_{i-1} = \frac{u_i - 2u_{i-1} + u_{i-2}}{\Delta x^2} + O(\Delta x^2)$$ 2nd order at

$x = x_{i-1}$

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$$\delta^+ u_i = u_{i+1} - u_i = E u_i - u_i = (E-1) u_i$$

$$\delta^- u_i = u_i - u_{i-1} = u_i - E^{-1} u_i = (1 - E^{-1}) u_i$$

$$\delta^- =$$

$$\delta^- u_i = \cancel{(1-E^{-1})} u_i = u_i - u_{i-1}$$

$$= E^{-1}(u_{i+1} - u_i)$$

$$= E^{-1}(\delta^+ u_i)$$

$$\delta^+ \delta^- u_i = \delta^+ (u_i - u_{i-1}) = (u_{i+1} - u_i) - (u_i - u_{i-1})$$

$$= u_{i+1} - 2u_i + u_{i-1} = (u_{i+1} - u_i) - (u_i - u_{i-1})$$

$$= \delta^- (u_{i+1} - u_i)$$

$$= \delta^- \delta^+ u_i \Rightarrow \delta^+ \delta^- = \delta^- \delta^+$$

$$(\delta^+ - \delta^-) u_i = u_{i+1} - u_i - (u_i - u_{i-1}) = \text{Above} \checkmark$$

$$\delta^2 u_i = \delta(u_{i+1/2} - u_{i-1/2}) = (u_{i+1} - u_i) - (u_i - u_{i-1}) = \text{Above}$$

$$\delta u_i = u_{i+1/2} - u_{i-1/2} = \left(E^{1/2} - E^{-1/2}\right) u_i$$

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$\delta = \cdots$$

$$\delta^{+2} = \delta^+ \delta^+ = (E-1)^2 = E^2 - 2E + 1$$

$$\delta^{+3} = (E-1)^3 = E^3 - 3E^2 + 3E - 1$$

$$E = e^{\Delta x D}$$

For exponential fn

$$E e^{ax} = e^{\Delta x D} e^{ax} = \sum_{k=0}^{\infty} \frac{\Delta x^k D^k}{k!} e^{ax}$$

$$= e^{ax} e^{a \Delta x} = e^{ax} \underbrace{\sum_{k=0}^{\infty} \frac{\Delta x^k a^k D^k}{k!}}_{e^{a\Delta x}}$$

equality ✓

If $u(x)$ = polynomial of degree n

$$E P_n(x) = P_n(x+\Delta x) \stackrel{?}{=} \sum_{k=0}^{\infty} \frac{\Delta x^k D^k}{k!} P_n(x)$$

$$RHS = \Delta^k P_n(x) = 0 \text{ if } k \geq n+1$$

$$= \sum_{k=0}^{n} \frac{\Delta x^k}{k!} \frac{\partial^k}{\partial x^k} P(x) = \text{taylor exp. of } P(x+\Delta x) \text{ eq.}$$

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$$\Delta x D = \ln E$$
$$= \ln(\delta^+ + 1) = \ln(1+\delta^+) = \sum_{k=0}^{\infty} \frac{(-1)^k (\delta^+)^{k+1}}{k+1}$$

$$= \delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} - \frac{\delta^{+4}}{4} + \cdots$$

keeping ~~terms~~ only the 1st term

$$D = (U_x)_i = \frac{\delta^+ U}{\Delta x} = \frac{U_{i+1} - U_i}{\Delta x}$$

w/ Truncation error of

$$U_{i+1} = U_i + \Delta x (U_x)_i + \frac{\Delta x^2}{2} (U_{xx})_i + O(\Delta x^3)$$

$$\therefore \frac{U_{i+1} - U_i}{\Delta x} = (U_x)_i + \frac{\Delta x}{2} (U_{xx})_i + O(\Delta x^2)$$

If 1st 2 terms are kept we get

$$\Delta x D = \delta^+ - \frac{\delta^{+2}}{2}$$ applying to U_x gives

$$\Rightarrow (U_x)_i = \frac{1}{\Delta x}\left[(U_{i+1} - U_i) - \frac{1}{2}\delta^+ (U_{i+1} - U_i)\right]$$
$$= \frac{1}{\Delta x}\left[U_{i+1} - U_i - \frac{1}{2}(U_{i+2} - U_{i+1} - U_{i+1} + U_i)\right]$$

$$\therefore\ (U_x)_i = \frac{1}{\Delta x}\left[-\frac{1}{2}(U_{i+2}) + 2U_{i+1} - \frac{3}{2}U_i\right]$$

$$= \frac{-3U_i + 4U_{i+1} - U_{i+2}}{2\Delta x} + O(\Delta x^2)$$

To find actual truncation error term, taylor expand each on

$$U_{i+1} = U_i + \Delta x(U_x)_i + \frac{\Delta x^2}{2}(U_{xx})_i + \frac{\Delta x^3}{3!}(U_{xxx})_i + O(\Delta x^4)$$

$$U_{i+2} = U_i + 2\Delta x(U_x)_i + 4\frac{\Delta x^2}{2}(U_{xx})_i + \frac{8\Delta x^3}{3!}(U_{xxx})_i + O(\Delta x^4)$$

& combine as above

$$(-3+4-1)U_i + (4-2)(U_x)_i\,\Delta x + \left(\frac{4}{2}-\frac{4}{2}\right)\Delta x^2(U_{xx})_i +$$

$$\left(\frac{4}{3!}-\frac{8}{3!}\right)\Delta x^3(U_{xxx})_i + O(\Delta x^4)$$

$$= 2(U_x)_i\,\Delta x + \frac{1}{6}(-4)\,\Delta x^3(U_{xxx})_i + O(\Delta x^4)$$

$$\therefore\ \frac{-3U_i + 4U_{i+1} - U_{i+2}}{2\Delta x} = (U_x)_i - \frac{1}{3}\Delta x^2(U_{xxx})_i + O(\Delta x^3)$$

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$\delta^+ = \Delta x \, U_x + O(\Delta x^2)$ from 1st order Formula & ~~discus~~ discussion on bottom of pg 173

$$\delta^{+n} = O(\Delta x^n)$$

$$D = \frac{\ln(1+\delta^+)}{\Delta x} = \frac{1}{\Delta x} \sum_{k=0}^{\infty} \frac{(-1)^k \, \delta^{+k+1}}{k+1}$$

$\dfrac{\delta^{+k+1}}{\Delta x} = O(\Delta x^k)$ Thus keeping only k terms

give accuracy for D that is $O(\Delta x^{k-1})$.

keeping 1 term (δ^+) give $D = \dfrac{\delta^+}{\Delta x} + O(\Delta x^1)$

$$D = \frac{\delta^+ - \delta^+/2}{\Delta x} + O(\Delta x^2)$$ If I keep

1 term get $O(\Delta x)$
2 terms get $O(\Delta x^2)$
3 " " $O(\Delta x^3)$
⋮

] don't follow book?

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$$\delta^- = 1 - E^{-1} \Rightarrow \sout{\text{multiply by } E \quad E\delta^- = E - 1}$$

$$\sout{\Rightarrow E = E\delta^- + 1}$$

$$\Delta x D = \ln E$$

$$= \ln(\cancel{E}\delta^- + 1)$$

$$E^{-1} = \cdot\cdot$$

$$E^{-1}U(x) = U(x - \Delta x) = U(x) - \Delta x U_x + \frac{\Delta x^2}{2} U_{xx} - \frac{\Delta x^3}{3!} U_{xxx} + \cdots$$

$$= \left(1 - \Delta x D + \frac{\Delta x^2 D^2}{2} - \frac{(\Delta x D)^3}{3!} + \cdots\right) U$$

$$= e^{-\Delta x D} U$$

$\therefore$ symbolically $E^{-1} = e^{-\Delta x D}$

$$\Rightarrow \boxed{\Delta x D = -\ln E^{-1}}$$

$\therefore$

$\Delta x D = -\ln(1 - \delta^-)$ from Above

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k (-1)^{k+1} \delta^{-\,k+1}}{k+1}$$

$$= + \sum_{k=0}^{\infty} \frac{\delta^{-k+1}}{k+1} = + \left(\delta^- + \frac{\delta^{-2}}{2} + \frac{\delta^{-3}}{3} + \frac{\delta^{-4}}{4} + \cdots \right)$$

2nd order accuracy ⇒ take 2 terms

$$D = \frac{+1}{\Delta x}\left(\delta^- + \delta^{-2}/2 \right)$$

$$Du = +\frac{1}{\Delta x}\left(u_i - u_{i-1} + \frac{1}{2}\delta^-(u_i - u_{i-1}) \right)$$

$$= \frac{+1}{\Delta x}\left(u_i - u_{i-1} + \frac{1}{2}(u_i - u_{i-1} - u_{i-1} + u_{i-2}) \right)$$

$$= +\frac{1}{2\Delta x}\left(2u_i - 2u_{i-1} + u_i - 2u_{i-1} + u_{i-2} \right)$$

$$= \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} + O(\Delta x^2)$$

To get actual $O(\Delta x^2)$ term use of taylor expands

$$u_{i-2} = u_i - 2\Delta x (u_x)_i + \frac{4\Delta x^2}{2!}(u_{xx})_i - \frac{8\Delta x^3}{3!}(u_{xxx})_i + O(\Delta x^4)$$

$$u_{i-1} = u_i - \Delta x (u_x)_i + \frac{\Delta x^2 (u_{xx})_i}{2} - \frac{\Delta x^3}{3!}(u_{xxx})_i + O(\Delta x^4)$$

$\therefore$

$$(3-4+1)u_i + (4-2)\Delta x(u_x)_i + (2-2)\frac{\Delta x^2 (u_{xx})_i}{2} + \left(\frac{4}{3} - \frac{1}{6}\right)\Delta x^3 (u_{xxx})_i$$
$$+ O(\Delta x^4)$$

$$= \quad 2\Delta x (u_x)_i - \frac{1}{6}(9)\Delta x^3 (u_{xxx})_i + O(\Delta x^4)$$

$$\therefore (u_x)_i = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} - \frac{3}{2^2}\Delta x^2 (u_{xxx})_i + O(\Delta x^3)$$

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$$\delta U_i = U_{i+1/2} - U_{i-1/2} = (E^{1/2} - E^{-1/2}) U_i$$

$$\Rightarrow \delta =$$

Note: powers to E are all gotta from eq 4.2.16

$$\boxed{E^p = e^{p\Delta x D}}$$

$$\therefore \quad E^{-1} = e^{-\Delta x D}$$

$$\& \quad E^{\pm 1/2} = e^{\pm \frac{\Delta x}{2} D}$$

$$\delta = e^{\frac{\Delta x}{2} D} - e^{-\frac{\Delta x}{2} D} = 2\sinh\left(\frac{\Delta x}{2} D\right)$$

$$\Delta x D = 2\sinh^{-1}(\delta/2) = ?$$

From Gradstein & Ryzik

$$\operatorname{Arsh}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{2^{2k}(k!)^2 (2k+1)} x^{2k+1} = x F\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; -x^2\right)$$

$$= x - \frac{2x^3}{4 \cdot 3} + \frac{4!\, x^5}{2^4 2^2 5} - \frac{6!}{2^6 3!^2 7} x^7 + O(x^9)$$

$k = 0, \quad 1 \quad 2 \quad 3$

$$= x - \frac{x^3}{2\cdot 3} + \frac{1\cdot 3}{2^3\, 5}x^5 - \frac{\cancel{6}\cdot 5\cdot \cancel{4}}{2^{\cancel{6}4}\,\cancel{3!}\cdot 7}x^7 + O(x^9)$$

$$= x - \frac{x^3}{2\cdot 3} + \frac{1\cdot 3x^5}{2\cdot 4\cdot 5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + O(x^9)$$

$\therefore$

$$\Delta x D = 2\left[\left(\frac{\delta}{2}\right) - \frac{1}{2\cdot 3}\left(\frac{\delta}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\left(\frac{\delta}{2}\right)^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{\delta}{2}\right)^7 + O(\delta^9)\right]$$

$$= \delta - \frac{1}{3\cdot 8}\delta^3 + \frac{3}{4\cdot 5\cdot 2^5}\delta^5 - \frac{\overset{5}{\cancel{15}}}{4\cdot \underset{2}{\cancel{6}}\cdot 7\cdot 2^7}\delta^7 + O(\delta^9)$$

$$= \delta - \frac{1}{24}\delta^3 + \frac{3}{640}\delta^5 - \frac{5\delta^7}{\underbrace{7168}_{=2^{10}\cdot 7}} + O(\delta^9)$$

---

keeping only the 1st term gives:

$$(u_x)_i = \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} + O(\Delta x^2)$$

$$u_{i+1/2} = u_i + \frac{\Delta x}{2}(u_x)_i + \frac{\Delta x^2}{4}\frac{(u_{xx})_i}{2!} + \frac{\Delta x^3}{8}\frac{(u_{xxx})_i}{3!} + O(\Delta x^4)$$

$$u_{i-1/2} = u_i - \frac{\Delta x}{2}(u_x)_i + \frac{\Delta x^2}{4}\frac{(u_{xx})_i}{2!} - \frac{\Delta x^3}{8}\frac{(u_{xxx})_i}{3!} + O(\Delta x^4)$$

$$\therefore\ u_{i+1/2} - u_{i-1/2} = \Delta x (u_x)_i + 2\frac{\Delta x^3}{8}\frac{(u_{xxx})_i}{3!} + O(\Delta x^5)$$

$$\therefore\ (u_x)_i = \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} - \frac{\Delta x^3 (u_{xxx})_i}{\underbrace{4\cdot 6}_{24}}$$

keeping 2 terms gives:

$$\Delta x D = \delta - \delta^3/24 \quad \therefore$$

$$\Delta x D u(x) = u_{i+1/2} - u_{i-1/2} - \frac{1}{2}\delta^2 (u_{i+1/2} - u_{i-1/2})$$

$$= u_{i+1/2} - u_{i-1/2} - \frac{1}{2}\delta\Big(u_{i+1} - u_i - (u_i - u_{i-1})\Big)$$

$$= u_{i+1/2} - u_{i-1/2} - \frac{\delta}{2}(u_{i+1} - 2u_i + u_{i-1})$$

$$= u_{i+1/2} - u_{i-1/2} - \frac{1}{2}(u_{i+3/2} - 2u_{i+1/2} + u_{i-1/2} - u_{i+1/2} + 2u_{i-1/2} - u_{i-3/2})$$

$$= u_{i+1/2} - u_{i-1/2} - \frac{1}{2}(u_{i+3/2} - 3u_{i+1/2} + 3u_{i-1/2} - u_{i-3/2})$$

Doh.

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta^3 = (E^{1/2} - E^{-1/2})^3 = E^{3/2} - 3E^{2/2}E^{-1/2} + 3E^{1/2}E^{-2/2} - E^{-3/2}$$

$$= E^{3/2} - 3E^{1/2} + 3E^{-1/2} - E^{-3/2}$$

Thus

$$\Delta x (u_x)_i = \delta u - \frac{1}{24}\delta^3 u$$

$$= u_{i+1/2} - u_{i-1/2} - \frac{1}{24}\left(u_{i+3/2} - 3u_{i+1/2} + 3u_{i-1/2} - u_{i-3/2}\right)$$

$$= \frac{1}{24}\left(-u_{i+3/2} + (24+3)u_{i+1/2} + (-24-3)u_{i-1/2} + u_{i-3/2}\right)$$

$$= \frac{1}{24}\left(-u_{i+3/2} + 27u_{i+1/2} - 27u_{i-1/2} + u_{i-3/2}\right)$$

$$\therefore (u_x)_i = \frac{-u_{i+3/2} + 27u_{i+1/2} - 27u_{i-1/2} + u_{i-3/2}}{24\Delta x} + O(\Delta x^?) \quad *$$

To figure out the truncation error expand each term in term ...

$$U_{i+3/2} = U_i + \frac{3}{2}\Delta x (U_x)_i + \left(\frac{3}{2}\right)^2 \frac{\Delta x^2}{2!}(U_{xx})_i + \left(\frac{3}{2}\right)^3 \frac{\Delta x^3}{3!}(U_{xxx})_i + \left(\frac{3}{2}\right)^4 \frac{\Delta x^4}{}$$

Doing many calculations of this type I feel we will get For the taylor expanded top of equation *

$$\left(-1 + \cancelto{0}{27 - 27} + 1\right)U_i + \Delta x (U_x)_i\left(-\left(\frac{3}{2}\right) + 27\left(\frac{1}{2}\right) - 27\left(\frac{-1}{2}\right) - \frac{3}{2}\right)$$

$$+ \frac{\Delta x^2}{2!}(U_{xx})_i\left(-\left(\frac{3}{2}\right)^2 + \cancelto{0}{27\left(\frac{1}{2}\right)^2 - 27\left(\frac{-1}{2}\right)^2} + \left(\frac{-3}{2}\right)^2\right)$$

$$+ \frac{\Delta x^3}{3!}(U_{xxx})_i\left(-\left(\frac{3}{2}\right)^3 + 27\left(\frac{1}{2}\right)^3 - 27\left(\frac{-1}{2}\right)^3 + \left(\frac{-3}{2}\right)^3\right)$$

$$+ \frac{\Delta x^4}{4!}(U_{xxxx})_i\left(-\left(\frac{3}{2}\right)^4 + \cancelto{0}{27\left(\frac{1}{2}\right)^4 - 27\left(\frac{-1}{2}\right)^4} + \left(\frac{-3}{2}\right)^4\right)$$ ← Notice every even partial vanish

{ This can be seen by looking at the coefficients of U_{i+p} If they Add to Zero w/o any sign changes every even term will vanish }

$$+ \frac{\Delta x^5}{5!}(U_{xxxxx})_i\left(-\left(\frac{3}{2}\right)^5 + 27\left(\frac{1}{2}\right)^5 - 27\left(-\frac{1}{2}\right)^5 + \left(-\frac{3}{2}\right)^5\right) + O(\Delta x^7)$$

$O(\Delta x^6)$ vanishes

$$\Rightarrow \Delta x (u_x)_i \left(-3+27\right)$$

$$+ \frac{\Delta x^3}{3!} \frac{(u_{xxx})_i}{2^3} \cancelto{0}{\left(-3^3 + 27 + 27 - \underset{\underset{27}{\parallel}}{3^3}\right)}$$

$$+ \frac{\Delta x^5}{5!} \frac{(u_{xxxxx})_i}{2^5} \left(-(3^5) + 27 + 27 - 3^5\right) + O(\Delta x^7)$$

$$= \Delta x (u_x)_i (24) + \frac{\Delta x^5}{5!\,2^5} (-432)(u_{xxxxx})_i + O(\Delta x^7)$$

$$= 24 \Delta x (u_x)_i - \frac{27}{5!\,2} \Delta x^5 (u_{xxxxx})_i + O(\Delta x^7)$$

$$= 24 \Delta x (u_x)_i - \frac{3^{\cancel{3}2}}{5 \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 1 \cdot 2} \Delta x^5 (u_{xxxxx})_i + O(\Delta x^7)$$

$$24 \Delta x (u_x)_i - \frac{9}{2 \cdot 400} \Delta x^5 (u_{xxxxx})_i + O(\Delta x^7)$$

In summary:

$$(u_x)_i = \frac{1}{24\Delta x}\left(-u_{i+3/2} + 27u_{i+1/2} - 27u_{i-1/2} + u_{i-3/2}\right)$$

$$+ \frac{\cancel{9}\,3}{2\cdot 8\,\cancel{24}(400)}\Delta x^4\left(\frac{\partial^5 u}{\partial x^5}\right)_i + O(\Delta x^6)$$

$$= \frac{\left(-u_{i+3/2} + 27u_{i+1/2} - 27u_{i-1/2} + u_{i-3/2}\right)}{24\Delta x}$$

$$+ \frac{3}{6400}\Delta x^4\left(\frac{\partial^5 u}{\partial x^5}\right)_i + O(\Delta x^6)$$

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$$\Delta x D = \sinh^{-1}(\bar{\delta})$$

$$= \left[\bar{\delta} - \frac{1}{2\cdot 3}\bar{\delta}^3 + \frac{3}{2\cdot 4\cdot 5}\bar{\delta}^5 + O(\bar{\delta}^7) \right]$$

keeping

1st term $\Rightarrow$

$$\Delta x D = \bar{\delta}$$

$$\Delta x D u = \frac{1}{2}(E - E^{-1})u =$$

$$= \frac{1}{2}\Big(\cancel{u_i} + \Delta x (u_x)_i + \frac{\Delta x^2}{\cancel{2}}\cancel{(u_{xx})_i} + \frac{\Delta x^3}{3!}(u_{xxx})_i +$$

$$- \cancel{u_i} + \Delta x (u_x)_i - \frac{\Delta x^2}{\cancel{2}}\cancel{(u_{xx})_i} + \frac{\Delta x^3}{3!}(u_{xxx})_i + \dots \Big)$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2}\Delta x (u_x)_i + \cancel{2}\frac{\Delta x^3}{3!}(u_{xxx})_i + O(\Delta x^5) \right)$$

$$\Rightarrow \quad (u_x)_i = \frac{\bar{\delta} u}{\Delta x} - \frac{\Delta x^2}{6}(u_{xxx})_i + O(\Delta x^4)$$

keeping 2 terms

$$\Delta x D = \bar{\delta} - \frac{\bar{\delta}^3}{6}$$

$$\delta^3 = \frac{1}{2^3}(E - E^{-1})^3 = \frac{1}{8}(E^3 - 3E^2E^{-1} + 3EE^{-2} - E^{-3})$$

$$= \frac{1}{8}(E^3 - 3E + 3E^{-1} - E^{-3})$$

$\therefore$ stencil for δ^3 depends on

$i+3,\ i+1,\ i-1,\ i-3$

$$\mu^2 = \frac{1}{4}(E^{1/2} + E^{-1/2})^2 = \frac{1}{4}(E^1 + 2 + E^{-1})$$

$$\cancel{= \frac{1}{4}(}$$

or $$\mu^2 = \frac{1}{4}(E^{1/2} + E^{-1/2})^2 = \frac{1}{4}(\delta + E^{-1/2} + E^{-1/2})^2 \quad \{\text{As } \delta = E^{1/2} - E^{-1/2}$$

$$= \frac{1}{4}(\delta + 2E^{-1/2})^2 = \frac{\delta^2}{4} + \frac{1}{4}(4)\delta E^{-1/2} + \frac{4}{4}E^{-1}$$

$$= \frac{\delta^2}{4} + \underbrace{\delta E^{-1/2}}_{\parallel} + E^{-1}$$

$$(E^{1/2} - E^{-1/2})E^{-1/2} = 1 - E^{-1}$$

$$\therefore \mu^2 = \frac{\delta^2}{4} + 1$$

Note : From 4.228

$$1 = u\left(1+\frac{s^2}{4}\right)^{-1/2} = u\Big(1$$

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty}\binom{-1/2}{k}(x^k) = \binom{-1/2}{0} + \binom{-1/2}{1}x + \binom{-1/2}{2}x^2 + \binom{-1/2}{3}x^3 + O(x^4)$$

$$=$$

__Now__:

term #: 0 1 ... $\overset{n-1}{k-(n-1)}$

$$\binom{k}{n} = \frac{k!}{n!(k-n)!} = \frac{k(k-1)(k-2)\cdots\overbrace{(k-n+1)}}{n!}$$

top terms
0, 1, 2, ..., ~~n~~ n-1
⇒ __n__ terms
Sam as ~~bottom~~.

$\therefore \binom{-1/2}{0} = 1$ As Pattern $k(k-1)\cdots(k-0+1)$ is never executed

$$\binom{-1/2}{1} = \frac{(-1/2)}{1!} = \frac{-1}{2}$$

$$\binom{-1/2}{2} = \frac{(-1/2)(-1/2-1)}{2!} = \frac{(-1/2)(-3/2)}{2} = \frac{3}{8}$$

$$\binom{-1/2}{3} = \frac{(-1/2)(-1/2-1)(-1/2-2)}{3!} = \frac{(-1)(1/2)(3/2)(5/2)}{6} = \frac{-15}{48}$$

$$\therefore (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{15}{48}x^3 + O(x^4)$$

$\therefore$

$$1 = \mu\left(1+\frac{\delta^2}{4}\right)^{-1/2} = \mu\left(1-\frac{1}{2}\left(\frac{\delta^2}{4}\right)+\frac{3}{8}\left(\frac{\delta^2}{4}\right)^2-\frac{15}{48}\left(\frac{\delta^2}{4}\right)^3+O(\delta^8)\right)$$

$$= \mu\left(1-\frac{\delta^2}{8}+\frac{3\delta^4}{128}-\frac{5}{1024}\delta^6+O(\delta^8)\right)$$

Thus $\overset{\text{remembering}}{\wedge}$ $\Delta x D = \delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168}$ from 4.2.3

$$\therefore \quad 1 \cdot \Delta x D = \mu\left(\delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + O(\delta^9)\right)\left(1-\frac{\delta^2}{8}+\frac{3\delta^4}{128}-\frac{5\delta^6}{1024}+O(\delta^8)\right)$$

$$= \mu\left(\delta + \left(-\frac{1}{8}-\frac{1}{24}\right)\delta^3 + \left(\frac{3}{128}+\frac{1}{24\cdot 8}+\frac{3}{640}\right)\delta^5 + O(\delta^7)\right)$$

$$= \mu\left(\delta - \frac{\delta^3}{6} + \frac{1}{30}\delta^5 + O(\delta^7)\right)$$

$$= \mu\left(\delta - \frac{\delta^3}{3!} + \frac{4}{5\cdot 4\cdot 3\cdot 2\cdot 1}\delta^5 + O(\delta^7)\right)$$

But $\mu = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)$ & $\delta = \frac{1}{2}\left(E - E^{-1}\right)$

Thus

$$= \mu\delta\left(1 - \frac{\delta^2}{3!} + \frac{4\,\delta^4}{5!} + O(\delta^6)\right)$$

$$+\ \mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$$

$$= \frac{1}{2}(E - E^{-1}) = \bar{\delta}$$

Check $\quad 1 = \mu\left(1 + \frac{\delta^2}{4}\right)^{-1/2}$ multiply

$$\Delta x D = \sinh^{-1}(\delta/2)$$

$$\Delta x D = \mu\left(1 + \frac{\delta^2}{4}\right)^{-1/2} \sinh^{-1}(\delta/2) = \text{From MMA}$$

$$= \mu\left(\delta - \frac{\delta^3}{6} + \frac{\delta^5}{30} - \frac{\delta^7}{140} + O(\delta^8)\right)$$

$$= \underbrace{\mu\delta}\left(1 - \frac{\delta^2}{6} + \frac{\delta^4}{30} - \frac{\delta^6}{140} + O(\delta^7)\right)$$

$$\bar{\delta}\left(1 - \frac{\delta^2}{6} + \frac{\delta^4}{30} - \frac{\delta^6}{140} + O(\delta^8)\right)$$

Bob's coeff is $\frac{4 \cdot 9}{7!} = \frac{1}{140}$ same ✓

Taking one term (or $\bar{\delta}$)

$$\Delta x D \approx \bar{\delta}$$

$$\Delta x D U \approx \bar{\delta} U = \frac{1}{2}(E - E^{-1})U$$

<u>Now</u>:

$$\frac{1}{2}(E-E^{-1})U = \frac{1}{2}\Big(\cancel{U_i} + \Delta x (U_x)_i + \cancel{\frac{\Delta x^2}{2}(U_{xx})_i} + \frac{\Delta x^3}{6}(U_{xxx})_i + O(\Delta x^4)$$

$$- \cancel{U_i} + \Delta x (U_x)_i - \cancel{\frac{\Delta x^2}{2}(U_{xx})_i} + \frac{\Delta x^3}{6}(U_{xxx})_i \cancel{+} - O(\Delta x^4) \Big)$$

$$= \Delta x (U_x)_i + \frac{\Delta x^3}{6}(U_{xxx})_i + O(\Delta x^5)$$

$$\therefore (U_x)_i = \frac{U_{i+1} - U_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6}(U_{xxx})_i + O(\Delta x^4)$$

Taking two terms

$$\Delta x D \approx \bar{\delta}\left(1 - \frac{\delta^2}{6}\right)$$

$$= \frac{1}{2}(E - E^{-1})\left(1 - \frac{1}{6}\left(E^{1/2} - E^{-1/2}\right)^2\right)$$

$$= \frac{1}{2}(E - E^{-1})\left(1 - \frac{1}{6}(E - 2 + E^{-1})\right)$$

$$= \frac{1}{2}\Big[\cancel{E - E^{-1}} - \frac{1}{6}$$

$$= \frac{1}{2}(E - E^{-1})\left(-\frac{E}{6} - \frac{E^{-1}}{6} + \frac{4}{3}\right)$$

$$= \frac{1}{2}\left(-\frac{E^2}{6} - \cancel{\frac{1}{6}} + \frac{4}{3}E + \cancel{\frac{1}{6}} + \frac{E^{-2}}{6} - \frac{4}{3}E^{-1}\right)$$

$$= \frac{1}{2}\left(-\frac{E^2}{6} + \frac{4}{3}E - \frac{4}{3}E^{-1} + \frac{E^{-2}}{6}\right)$$

$$= \frac{-E^2 + 8E - 8E^{-1} + E^{-2}}{12}$$

Now expand $-E^2 + 8E - 8E^{-1} + E^{-2}$ operating on a fn $u(x)$ to obtain the truncation Error for this scheme

$$\Rightarrow$$

$$\Rightarrow -u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}$$

$$= (-1 + \overset{0}{\cancel{8 - 8}} + 1)u_i + (-2 + 8 + 8 - 2)(u_x)_i \Delta x$$

$$+ \left(-1 \cdot 2^2 + \overset{0}{\cancel{8(1^2) - 8(-1)^2}} + 1(-2)^2\right)\Delta x^2 \frac{(u_{xx})_i}{2!}$$

$$+ \left(-1(2)^3 + 8(1^3) - 8(-1)^3 + 1(-2)^3\right)\Delta x^3 \frac{(u_{xxx})_i}{6!}$$

$\Rightarrow + 0 \cdot \Delta x^4$

← All even ordered terms vanish if the coefficients sum to 0

$$+ \left(-1(2)^5 + 8\cdot 1^5 - 8(-1)^5 + 1(-2)^5\right) \frac{\Delta x^5}{5!}\left(\frac{\partial^5 u}{\partial x^5}\right)_i + O(\Delta x^7)$$

$$= 12(u_x)_i \Delta x - \frac{48}{5!}\left(\frac{\partial^5 u}{\partial x^5}\right)_i \Delta x^5 + O(\Delta x^7)$$

$$\frac{48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{2}{5}$$

Thus
$$(u_x)_i = \frac{1}{12\Delta x}\left(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}\right)$$

$$+ \frac{2}{12\cdot 5}\Delta x^4 \left(\frac{\partial^5 u}{\partial x^5}\right)_i + O(\Delta x^6)$$

$$= \frac{1}{12\Delta x}\left(-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}\right)$$

$$+ \frac{\Delta x^4}{30}\left(\frac{\partial^5 u}{\partial x^5}\right)_i + O(\Delta x^5)$$

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eq 4.2.18

$$D^n u_i = \frac{1}{\Delta x^n}\left(\ln(1+\delta^+)\right)^n u_i$$

$$= \frac{1}{\Delta x^n}\left[\sum_{k=0}^{\infty}\frac{(-1)^k \delta^{+\,k+1}}{k+1}\right]^n u_i$$

$$= \frac{1}{\Delta x^n}\left[\delta^+ - \frac{\delta^{+2}}{2} + \frac{\delta^{+3}}{3} - \frac{\delta^{+4}}{4} + \frac{\delta^{+5}}{5} + O(\delta^{+6})\right]^n u_i$$

$$= \frac{1}{\Delta x^n}(\delta^+)^n\left[1 - \frac{\delta^+}{2} + \frac{\delta^{+2}}{3} - \frac{\delta^{+3}}{4} + \frac{\delta^{+4}}{5} + O(\delta^{+5})\right]^n$$

Now $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ Thus the above becomes:

$$= \frac{1}{\Delta x^n}(\delta^+)^n \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left(\frac{\delta^+}{2} - \frac{\delta^{+2}}{3} + \frac{\delta^{+3}}{4} - \frac{\delta^{+4}}{5} + O(\delta^{+5})\right)^k$$

$$= \frac{1}{\Delta x^n}(\delta^+)^n\Big[\ 1$$

$$- \binom{n}{1}\left(\frac{\delta^+}{2} - \frac{\delta^{+2}}{3} + \frac{\delta^{+3}}{4} - \frac{\delta^{+4}}{5} + O(\delta^{+5})\right)$$

$$+ \binom{n}{2}\left(\frac{\delta^+}{2} - \frac{\delta^{+2}}{3} + \frac{\delta^{+3}}{4} - \frac{\delta^{+4}}{5} + O(\delta^{+5})\right)^2$$

$$-\binom{n}{3}\left(\frac{\delta^{+}}{2}+O(\delta^{+2})\right)^3 + O(\delta^{+4})\Big]$$

$$= \frac{\delta^{+n}}{\Delta x^n}\Big[1 - n\frac{\delta^{+}}{2} + n\frac{\delta^{+2}}{3} - n\frac{\delta^{+3}}{4} + O(\delta^{+4})$$

$$+ \frac{n(n-1)}{2}\Big(\frac{\delta^{+2}}{4} - \frac{\delta^{+3}}{3}$$

$$(x_1 + x_2 + x_3 + x_4 + \cdots)^n = x_1^n + \underbrace{n x_1 x_2 + n x_1 x_3}_{} + \cdots$$

nth power of 1st term

n · product of 1st term w/ every other term

$+$

$$\frac{n(n-1)}{2} x_1 x_2 + n(n$$

$$\| $$

$$\binom{n}{2}$$

... what is the general term ...

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k$$

w/ $c_0 = a_0^n$

$$c_m = \frac{1}{m a_0}\sum_{k=1}^{m}(kn - m + k)\, a_k c_{m-k} \qquad m \geq 1$$

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Thus consider using this w/

$$\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}(\delta^+)^{k+1}\right]^n = \delta^{+n}\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\delta^{+k}\right]^n$$

Now it has the form $\sum a_k x^k$

$$a_0 = 1$$
$$a_1 = \frac{-1}{2}$$
$$a_2 = \frac{1}{3}$$
$$a_3 = \frac{-1}{4}$$

$$\cancel{= (\delta^+)^n\left[1^n + \frac{1}{1(1)}\sum_{k=1}^{1}(kn-1+k)a_k c_0\right]}$$

$$c_0 = a_0^1 = 1$$

$$c_1 = \frac{1}{1(1)}\sum_{k=1}^{1}(kn-1+k)a_k c_{1-k} = (n-1+1)a_1 c_0 = n\left(\frac{-1}{2}\right) = \frac{-n}{2}$$

$$c_2 = \frac{1}{2(1)}\sum_{k=1}^{2}(kn-2+k)a_k c_{2-k}$$

$$= \frac{1}{2}\left[(n-2+1)\left(\frac{-1}{2}\right)\left(\frac{-n}{2}\right) + (2n-2+2)\left(\frac{1}{3}\right)1\right]$$

$$= \frac{1}{2}\left[\frac{n(n-1)}{4} + \frac{2n}{3}\right] = \frac{1}{2}\left[\cancel{\text{[illegible]}}\right.$$

$$= \frac{1}{2}\left[\frac{3n^2-3n}{12} + \frac{8n}{12}\right] = \frac{1}{24}(3n^2+5n) = \frac{n(3n+5)}{24}$$

$$c_3 = \frac{1}{3(1)} \sum_{k=1}^{3} (kn-3+k)\, a_k c_{3-k}$$

$$= \frac{1}{3}\left[(n-3+1)a_1c_2 + (2n-3+2)a_2c_1 + (3n-3+3)a_3c_0\right]$$

$$= \frac{1}{3}\left[(n-2)\left(\frac{-1}{2}\right)\frac{n(3n+5)}{24} + (2n-1)\left(\frac{1}{3}\right)\left(\frac{-n}{2}\right) + 3n\left(\frac{-1}{4}\right)\right]$$

$$= \frac{n}{3\cdot 24}\left[\frac{-(n-2)(3n+5)}{2} - (2n-1)\cdot 4 - 3\cdot 6\right]$$

$$= \frac{n}{2\cdot 3\cdot 24}\left[\cancel{4n} - (3n^2+5n-6n-10) - 8(2n-1) - 3\cdot 2\cdot 6\right]$$

$$= \frac{-n}{2\cdot 3\cdot 24}\left[3n^2+5n-6n-10+16n-8+36\right]$$

$$= \frac{-n}{2\cdot 3\cdot 24}\left[3n^2+15n+18\right]$$

$$= \frac{-n}{48}(n+2)(n+3)$$

$$\therefore \left.\frac{\partial^n u}{\partial x^n}\right|_i = \frac{\delta^{+n}}{\Delta x^n}\left\{1 - \frac{n}{2}\delta^{+} + \frac{n(3n+5)}{24}\delta^{+2} - \frac{n(n+2)(n+3)}{48}\delta^{+3} + O(\delta^{+4})\right\} u_i$$

δ

$$\Rightarrow \left.\frac{\partial^n u}{\partial x^n}\right|_i = \frac{1}{\Delta x^n}\Big[\delta^{+n} - \frac{n}{2}\delta^{+n+1} + \frac{n(3n+5)}{24}\delta^{+n+2}$$

$$- \frac{n(n+2)(n+3)}{48}\delta^{+(n+3)} + O(\delta^{+(n+4)})\Big] u_i$$

Backwards diff operator: from fundamental operator identity

$$D = \cancel{\frac{1}{\Delta x}} \ln E \qquad \text{or eq } 4.2.16$$

$$E^{-1} = e^{-\Delta x D}$$

$$\overbrace{\delta^- = 1 - E^{-1}}$$

$$\Rightarrow -\Delta x D = \ln E^{-1}$$

$$E^{-1} = 1 - \delta^-$$

$$D = \frac{-1}{\Delta x}\ln E^{-1}$$

$$\therefore \quad D = \frac{-1}{\Delta x}\ln(1-\delta^-)^n$$

Book does not have $(-1)^n$ in 1st eq in 4.2.34.

$$\therefore \quad D^n = \frac{(-1)^n}{\Delta x^n}\left[\sum_{k=0}^{\infty}\frac{(-1)^k(-1)^{k+1}}{k+1}\delta^{-k+1}\right]^n$$

~~Book does not have $(-1)^n$~~

$$= \frac{(-1)^n(-1)^n}{\Delta x^n}\left[\sum_{k=0}^{\infty}\frac{(\delta^-)^{k+1}}{k+1}\right]^n = \frac{(\delta^-)^n}{\Delta x^n}\left[\sum_{k=0}^{\infty}\frac{(\delta^-)^k}{k+1}\right]^n$$

$a_0 = 1 \quad a_1 = \frac{1}{2}, \quad a_2 = \frac{1}{3}, \quad a_4 = \frac{1}{4}.$

Remember:

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_k x^k$$

$$c_0 = a_0^n$$

$$c_m = \frac{1}{m a_0}\sum_{k=1}^{m}(kn-m+k)a_k c_{m-k} \qquad m \geq 1$$

$$c_0 = 1$$

$$c_1 = \frac{1}{1}\sum_{k=1}^{1}(kn-1+k)a_k c_{1-k} = n\left(\frac{1}{2}\right)(1) = \frac{n}{2}$$

$$c_2 = \frac{1}{2}\sum_{k=1}^{2}(kn-2+k)a_k c_{2-k} = \cancel{\sum_{k=1}^{2}}$$

$$= \frac{1}{2}\left[(n-2+1)a_1 c_1 + (2n-2+2)a_2 c_0\right]$$

$$= \frac{1}{2}\left[(n-1)\left(\frac{1}{2}\right)\left(\frac{n}{2}\right) + 2n\left(\frac{1}{3}\right)1\right]$$

$$= \cancel{\frac{n}{2\cdot 6}\left[3n-3+4\right]} = \frac{n}{2\cdot 12}\left[3n-3+8\right] = \frac{n(3n+5)}{24}.$$

$$c_3 = \frac{1}{3}\sum_{k=1}^{3}(kn-3+k)a_k c_{3-k}$$

$$= \frac{1}{3}\left[(n-3+1)a_1c_2 + (2n-3+2)a_2c_1 + (3n-3+3)a_3c_0\right]$$

$$= \frac{1}{3}\left[(n-2)\left(\frac{1}{2}\right)\frac{n(3n+5)}{24} + (2n-1)\left(\frac{1}{3}\right)\frac{n}{2} + 3n\left(\frac{1}{4}\right)\right]$$

$$= \frac{n}{3}\,\frac{1}{2\cdot 24}\left[(n-2)(3n+5) + 8(2n-1) + 12\cdot 3\right]$$

$$= \frac{n}{2\cdot 3\cdot 24}\left[3n^2 + 5n - 6n - 10 + 16n - 8 + 36\right]$$

. . .

Note: This calculation is just the previous one w/ $\delta^+ \to -\delta^-$. Thus the answer is the old calculation w/ $\delta^+ \to -\delta^-$

gives 4.2.34

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$$D^n u_i = \left(\frac{2\sinh^{-1}\delta/2}{\Delta x} \right)^n u_i$$

$$= \frac{1}{\Delta x^n}\left(2\sinh^{-1}\delta/2 \right)^n u_i$$

$$= \frac{1}{\Delta x^n}\left[\delta - \frac{\delta^3}{24} + \frac{3\delta^5}{640} - \frac{5\delta^7}{7168} + \cdots \right]^n u_i$$ from eq 4.2.23

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$$\left(\sum_{k=0}^{\infty} a_k x^k \right)^n = \sum_{k=0}^{\infty} c_k x^k$$

$$= \frac{\delta^n}{\Delta x^n}\left[1 - \frac{\delta^2}{24} + \frac{3\delta^4}{640} - \frac{5\delta^6}{7168} + \cdots \right]^n u_i$$

w/ $c_0 = a_0^n$

$$c_m = \frac{1}{m a_0} \sum_{k=1}^{m} (kn - m + k) a_k c_{m-k} \qquad m \geq 1$$

$a_0 = 1$; $a_1 = 0$; $a_2 = \frac{-1}{24}$; $a_3 = 0$; $a_4 = \frac{3}{640}$; $a_5 = 0$, $a_6 = \frac{-5}{7168}$

Then $c_0 = 1$

$$c_1 = \frac{1}{1(1)}\left[\sum_{k=1}^{1} (kn - 1 + k) a_k c_{1-k} \right] = n \overset{a_1 = 0}{\cancel{a_1}} c_0 = 0$$

$$c_2 = \frac{1}{2(1)} \sum_{k=1}^{2} (kn - 2 + k) a_k c_{2-k} = \frac{1}{2}\left[(n-2+1) a_1 c_1 + (2n-2+2) a_2 c_0 \right]$$

$$c_2 = \frac{1}{2}\left[\cancelto{0}{(n-1)} + 2n\left(\frac{-1}{24}\right)1\right] = -\frac{n}{24}$$

$$c_3 = \frac{1}{3(1)}\sum_{k=1}^{3}(kn-3+k)\,a_k c_{3-k}$$

$$= \frac{1}{3}\left[\cancelto{a_1=0}{(n-2)a_1}c_2 + \cancelto{c_1=0}{(2n-1)}a_2 c_1 + \cancelto{a_3=0}{(3n)}a_3 c_0\right] = 0$$

$$c_4 = \frac{1}{4(1)}\sum_{k=1}^{4}(kn-4+k)\,a_k c_{4-k}$$

$$= \frac{1}{4}\left[\cancelto{a_1=0}{(n-3)}a_1 c_3 + (2n-2)a_2 c_2 + \cancelto{a_3=0}{(3n-1)}a_3 c_1 + (4n)\,a_4 c_0\right]$$

$$= \frac{1}{4}\left[2(n-1)\left(\frac{-1}{24}\right)\left(\frac{-n}{24}\right) + 4n\left(\frac{3}{640}\right)\right]$$

$$= \frac{1}{2}n\left[\frac{(n-1)}{\underbrace{24^2}_{(2^3\cdot 3)^2}} + \frac{6}{\underbrace{640}_{2^{5}\cdot 5}}\right] = \frac{n}{2}\,\frac{1}{2^6}\left[\frac{n-1}{9} + \frac{\cancelto{3}{6}}{\cancel{2}5}\right]$$

$$2^6\cdot 3^2 \qquad 2^7\,5$$

$$= \frac{n}{2^7}\left[\frac{n-1}{9} + \frac{3}{5}\right]$$

$$= \frac{n}{2^7}\left[\frac{5n-5+27}{45}\right]$$

$2^4 = 16$

$2^5 = 32$

$2^6 = 64$

$$= \frac{n}{64}\left[\frac{5n+22}{90}\right]$$

~~I — g#~~

$$\cancel{c_4 = \frac{n}{2^7}\left[\frac{10n+17}{90}\right]}$$ ~~diff than book~~

$$c_5 = \frac{1}{5(1)}\sum_{k=1}^{5}(kn-5+k)\,a_k\,c_{5-k}$$

$$= \frac{1}{5}\Big[\overset{a_1=0}{\cancel{(n-4)a_1}}\,c_4 + \overset{c_3=0}{\cancel{(2n-3)a_2}}\,c_3 + \overset{a_3=0}{\cancel{(3n-2)a_3}}\,c_2$$

$$+\,(4n-1)\overset{c_1=0}{\cancel{a_4 c_1}} + (5n)\overset{a_5=0}{\cancel{a_5}}\,c_0\Big]$$

$$= \frac{\cancel{1}}{5} \qquad 0$$

$$c_6 = \frac{1}{6}\sum_{k=1}^{6}(kn-6+k)\,a_k\,c_{6-k}$$

$$= \frac{1}{6}\Big[\overset{a_1=0}{\cancel{(n-5)a_1}}\,c_5 + (2n-4)a_2c_4 + \overset{a_3=0}{\cancel{(3n-3)a_3}}\,c_3$$

$$+\,(4n-2)a_4c_2 + \overset{a_5=0}{\cancel{(5n-1)a_5}}\,c_1 + (6n)a_6c_0\Big]$$

$$= \frac{1}{6}\Big[2(n-2)\left(\frac{-1}{24}\right)\frac{n}{64}\left(\frac{5n+22}{90}\right) + 2(2n-1)\left(\frac{3}{640}\right)\left(\frac{-n}{24}\right)$$

$$+\,6n\left(\frac{-5}{7168}\right)\Big]$$

$$C_6 = \frac{n}{6(24)(64)}\left[-\frac{2(n-2)(5n+22)}{90} - \frac{2(2n-1)(3)}{10} - \frac{\cancel{6}^{3}\cdot 5\cdot 3}{\cancel{14}\,7}\right]$$

$$= \frac{-n}{6\cdot 24\cdot 64}\,\frac{1}{90\cdot 7}\Big[+14(n-2)(5n+22) + 9\cdot 7\cdot 2(2n-1)(3)$$

$$+\,9\cdot 5\cdot 90\Big]$$

$$= \frac{-n}{2\cdot 3\cdot 2\cdot 3\cdot 2^2\, 8^2}\,\frac{1}{2\cdot 7\cdot 5\cdot 3^2}\Big[14(5n^2+12n-44) + 3\cdot 18\cdot 7(2n-1)$$

$$+\,45\;90\Big]$$

$$= \frac{-n}{2^5 3^4\, 5\cdot 7\cdot 8^2}\left[70n^2 + 168n - 616 + 756n - 378 + 4050\right]$$

$$= \frac{-n}{2^5 3^4\, 5\cdot 7\, 2^6}\Big[70n^2 + 924n$$

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n even All powers of δ are even

$$\delta^n = \frac{1}{2^n}\left(E^{1/2} - E^{-1/2}\right)^n$$

$$\delta^2 = \frac{1}{4}\left(E - E^{-1}\right)$$

$$\delta^4 = \delta^2\delta^2 = \frac{1}{16}\left(E^2 - 1\cdot(2) + E^{-2}\right)$$

$$\vdots$$

δ^n n even integer mesh pts only.

$$\mu^2 = 1 + \delta^2/4 \quad 4.2.28$$

$$\mu = + \sqrt{1 + \delta^2/4}$$

$$\therefore \quad \frac{\mu}{\sqrt{1+\delta^2/4}} = 1$$

Multiply eq 4.2.35 by This

$$D^n u_i = \frac{\mu}{(1+\delta^2/4)^{1/2}} \left(\frac{2}{\Delta x} \sinh^{-1}\left(\frac{\delta}{2}\right)\right)^n u_i$$

$$= \mu (1 + \delta^2/4)^{-1/2} \frac{\delta^n}{\Delta x^n} \left\{ 1 - \frac{n\delta^2}{24} + \frac{n}{64}\left(\frac{22+5n}{90}\right)\delta^4 + O(\delta^6) \right\}$$

using eq 4.2.35

$$= \mu \frac{\delta^n}{\Delta x^n} \underbrace{\left(1 - \frac{\delta^2}{8} + \frac{3\delta^4}{128} - \frac{5\delta^6}{1024} + \cdots\right)}_{\text{using eq 4.2.29}} \left(1 - \frac{n\delta^2}{24} + \frac{n}{64}\left(\frac{22+5n}{90}\right)\delta^4 + O(\delta^6)\right)$$

$$= \mu \frac{\delta^n}{\Delta x^n} \left(1 - \left(\frac{n}{24} + \frac{1}{8}\right)\delta^2 + \left(\frac{n}{64}\left(\frac{22+5n}{90}\right) + \frac{3}{128} + \frac{n}{8 \cdot 24}\right)\delta^4 + O(\delta^6) \right)$$

multiply 8 + 24

$$= \mu \frac{\delta^n}{\Delta x^n} \left(1 - \frac{(n+3)}{24}\delta^2 + \frac{5n^2 + 52n + 135}{5760}\delta^4 + O(\delta^6) \right)$$

$$\mu = \frac{1}{2}\left(v_{i+1/2} + v_{i-1/2}\right) = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) v_i$$

Then n even δ^n's all generate differences at integer mesh pts. n odd

$$\delta^n = \frac{1}{2^n}\left(E^{1/2} - E^{-1/2}\right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} E^{k/2} E^{-(n-k)/2}$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} E^{-n/2} E^{k}$$

n even δ^n involves ~~st~~ only integer mesh pts but
n odd δ^n " " half-integer mesh pts

$$\therefore \quad \mu\delta^n = \frac{1}{2^{n+1}}\Bigg(\underbrace{\sum_{k=0}^{n} \binom{n}{k} E^k E^{\frac{1-n}{2}} + \sum_{k=0}^{n} \binom{n}{k} E^k E^{\frac{-1-n}{2}}}_{\text{Both involve only integer mesh pts.}} \Bigg)$$

$$(u_{xx})_i = \frac{1}{\Delta x^2}\left[\delta^{+2} - \delta^{+3} + \frac{11}{12}\delta^{+4} - \frac{4(5)}{24}\delta^{+5} + \mathcal{O}(\delta^{+6})\right]u_i$$

$$= \frac{1}{\Delta x^2}\left[\delta^{+2} - \delta^{+3} + \frac{11}{12}\delta^{+4} - \frac{5}{6}\delta^{+5} + \mathcal{O}(\delta^{+6})\right]u_i \qquad 4.2.37$$

$(u_{xx})_i$ = Same w/ All positives

$$(u_{xx})_i = \frac{1}{\Delta x^2}\delta^2\left[1 - \frac{\delta^2}{12} + \frac{1}{\cancel{3}2}\left(\frac{\cancel{3}2}{90}\right)\delta^4 - \underbrace{\frac{2}{45}\left(\frac{5}{7}+\frac{1}{5}\right)}\delta^6 + \mathcal{O}(\delta^8)\right]u_i$$

$$= \frac{\delta^2}{\Delta x^2}\left[1 - \frac{\delta^2}{12} + \frac{\delta^4}{90} - \frac{\delta^6}{560} + \mathcal{O}(\delta^8)\right] \qquad \frac{2}{45}\frac{(25+7)}{35}$$

$$= \frac{1}{\Delta x^2}\left[\delta^2 - \frac{\delta^4}{12} + \frac{\delta^6}{90} - \frac{\delta^8}{560} + \mathcal{O}(\delta^{10})\right] \qquad \underline{4.2.39}$$

$$(u_{xx})_i = \frac{\mu\delta^2}{\Delta x^2}\left[1 - \frac{5}{24}\delta^2 + \frac{259}{5760}\delta^4 + \mathcal{O}(\delta^6)\right]$$

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$$\delta^+ = E - 1$$

$$(U_{xx})_i = \frac{1}{\Delta x^2}\delta^{+2} U_i = \frac{1}{\Delta x^2}(E-1)^2 U_i$$

$$= \frac{1}{\Delta x^2}(E^2 - 2E + 1) U_i = \frac{1}{\Delta x^2}(U_{i+2} - 2U_{i+1} + U_i)$$

taylor expand ..

Note: we can get the order of accuracy quickly by

Noting $\delta^+ = E - 1 = O(\Delta x)$

$\therefore$ ~~so~~ dropping the $\frac{\delta^{+3}}{\Delta x^2}$ term introduces an order

of $O(\frac{\Delta x^3}{\Delta x^2}) = O(\Delta x)$

But Notice in the central difference droppin

δ^2

$\therefore$ $\delta U_i = O(\Delta x)$ Thus in cent. difference

dropping $\frac{\delta^4}{\Delta x^2}$ introduces error $O(\frac{\delta^4}{\Delta x^2}) = O(\frac{\Delta x^4}{\Delta x^2})$

$= O(\Delta x^2)$ & we have 2nd order accuracy

Back diff

$$\delta^- = 1 - E^{-1}$$

$$(U_{xx})_i = \frac{1}{\Delta x^2}\left(\delta^{-2}\right) U_i$$

$$= \frac{1}{\Delta x^2}\left(1 - 2E^{-1} + E^{-2}\right) U_i$$

$$= \frac{1}{\Delta x^2}\left(U_i - 2U_{i-1} + U_{i-2}\right)$$

Central difference:

$$(U_{xx})_i = \frac{1}{\Delta x^2}\,\delta^2 U_i = \frac{1}{\Delta x^2}\left(U_{i+1} - 2U_i + U_{i-1}\right)$$

Cent diff: half integer mesh pts

$$(U_{xx})_i = \mu \frac{\delta^2}{\Delta x^2} U_i = \mu \frac{1}{\Delta x^2}\left(U_{i+1} - 2U_i + U_{i-1}\right) \quad \left\{ \mu = \text{Avg op} \quad \frac{(U_{i+1/2} + U_{i-1/2})}{2} \right.$$

$$= \frac{1}{2\Delta x^2}\left\{ U_{i+3/2} - 2U_{i+1/2} + U_{i-1/2} + U_{i+1/2} - 2U_{i-1/2} + U_{i-3/2} \right)$$

$$= \frac{1}{2\Delta x^2}\left\{ U_{i+3/2} - U_{i+1/2} - U_{i-1/2} + U_{i-3/2} \right]$$

Forward difference Pg 178 Hirsch Vol I

$$(u_{xx})_i = \frac{1}{\Delta x^2}\left(\delta^{+2} - \delta^{+3}\right) u_i \qquad \delta^+ = E - 1$$

$$= \frac{1}{\Delta x^2}\left(E^2 - 2E + 1 - (E-1)^3\right) u_i$$

$$= \frac{+1}{\Delta x^2}\left(2 - 5E + 4E^2 - E^3\right) u_i$$

$$= \frac{1}{\Delta x^2}\left(2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}\right)$$

Backward difference

$$(u_{xx})_i = \frac{1}{\Delta x^2}\left(\delta^{-2} + \delta^{-3}\right) u_i \qquad \delta^- = 1 - E^{-1}$$

$$= \frac{1}{\Delta x^2}\left(2 - 5E^{-1} + 4E^{-2} - E^{-3}\right) u_i$$

$$= \frac{1}{\Delta x^2}\left(2u_i - 5u_{i-1} + 4u_{i-2} - u_{i-3}\right)$$

Central difference:

$$(u_{xx})_i = \frac{1}{\Delta x^2}\left(\delta^2 - \frac{\delta^4}{12}\right) u_i \qquad \delta = \frac{E^{1/2} - E^{-1/2}}{2}$$

$$= \frac{1}{12\Delta x^2}\left(-30u_i - u_{i-2} + 16u_{i-1} + 16u_{i+1} - u_{i+2}\right)$$

$$= \frac{1}{12\Delta x^2}\left(-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}\right)$$

Central diff : half-integ mesh pts

w/ $\delta = E^{1/2} - E^{-1/2}$

$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$

$$(u_{xx})_i = \frac{\mu}{\Delta x^2}\left(\delta^2 - \frac{5\delta^4}{24}\right)$$

$$= \frac{1}{48\Delta x^2}\Big(-5u_{i-5/2} + 39u_{i-3/2} - 34u_{i-1/2} - 34u_{i+1/2}$$

$$+ 39u_{i+3/2} - 5u_{i+5/2}\Big)$$

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial}{\partial x}\cdot\right)u_i = \frac{\delta^+\left(k_{i-1/2}\,\delta^-\right)}{\Delta x^2}u_i$$

$$\delta^- = 1 - E^{-1}$$

$$\delta^+ = E - 1$$

$$= \frac{\delta^+\left(k_{i-1/2}\left(u_i - u_{i-1}\right)\right)}{\Delta x^2} = \underbrace{\frac{k_{i+1/2}\left(u_{i+1} - u_i\right) - k_{i-1/2}\left(u_i - u_{i-1}\right)}{\Delta x^2}}_{\text{eq } 4.1.2.50}$$

= expanding this in a taylor series

$$= \underbrace{\left(k'(x)u'(x) + k(x)u''(x)\right)}_{\frac{d}{dx}\left(k(x)u'(x)\right)} + O(\Delta x^2)$$

---

inverting the forward & backward differences:

$$\frac{\partial}{\partial x}\left(k(x)u_x\right) \cong \frac{1}{\Delta x^2}\,\delta^-\left(k_{i+1/2}\,\delta^+\right)u_i$$

$$= \frac{1}{\Delta x^2}\,\delta^-\left(k_{i+1/2}\left(u_{i+1} - u_i\right)\right)$$

$$= \frac{1}{\Delta x^2}\left(k_{i+1/2}\left(u_{i+1} - u_i\right) - k_{i-1/2}\left(u_i - u_{i-1}\right)\right)$$

Central difference:

$$(u_{xxx})_i = \frac{1}{\Delta x^3}(\delta)^3 u_i \qquad \delta = E^{1/2} - E^{-1/2}$$

$$= \frac{1}{\Delta x^3}\left(E^{3/2} - 3E^{1}E^{-1/2} + 3E^{1/2}E^{-1} + E^{-3/2}\right)u_i$$

$$= \frac{1}{\Delta x^3}\left(E^{3/2} - 3E^{1/2} + 3E^{-1/2} + E^{-3/2}\right)u_i$$

$$= \frac{1}{\Delta x^3}\left(u_{i+3/2} - 3u_{i+1/2} + 3u_{i-1/2} + u_{i-3/2}\right) + O(\Delta x^2)$$

4th order approximate: 1/2 integer mesh pts

$$(u_{xxx})_i \approx \frac{1}{\Delta x^3}\delta^3\left|1 - \frac{3}{24}\delta^2\right|u_i \qquad \delta = E^{1/2} - E^{-1/2}$$

$$= \frac{1}{\Delta x^3}\left(E^{3/2} - 3E^{1/2} + 3E^{-1/2} + E^{-3/2}\right)\Big[\underbrace{1 - \frac{1}{8}\left(E - 2 + E^{-1}\right)}_{\frac{1}{4} - \frac{E}{8} - \frac{E^{-1}}{8}}\Big]u_i$$

$$= \frac{1}{\Delta x^3}\Big[$$

$3/2 \quad 1/2 \quad -1/2 \quad -3/2$

$1 \ (-3) \ 0 \ 3) \ 1$

$E^{1} \quad E^{0} \quad E^{-1}$

$-\frac{1}{8} \quad \frac{9}{8} \quad -\frac{1}{8}$

$E^{5/2}$	$E^{3/2}$	$E^{1/2}$	E^{0}	$E^{-1/2}$	$E^{-3/2}$	$E^{-5/2}$
	$-\frac{1}{8}$	$\frac{3}{8}$	0	0	$-\frac{3}{8}$	$\frac{1}{8}$
0	$\frac{9}{8}$	$-\frac{27}{8}$	0	$\frac{27}{8}$	$\frac{9}{8}$	$\boxed{0}$
$-\frac{1}{8}$	$\frac{3}{8}$	$-\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\boxed{}$

$\Rightarrow$ Summing columns gives

$-\frac{1}{8} \quad \frac{11}{8} \quad -\frac{27}{8} \quad 0 \quad \frac{26}{8} \quad \frac{6}{8} \quad -\frac{1}{8}$ off s

---

$\frac{n}{2}$ odd Forma —

under addition group

$\frac{3}{2}$ $\frac{1}{2}$

1 -3

$$(U_{xxx})_i = \frac{1}{\Delta x^3}(\delta^{+3})U_i = \frac{1}{\Delta x^3}(E-1)^3 U_i$$

$$= \frac{1}{\Delta x^3}(E^3 - 3E^2 + 3E - 1)U_i \qquad \text{eq } 4.2.52$$

$$= \ldots$$

$$(U_{xxx})_i = \frac{1}{\Delta x^3}\left(\delta^{+3} - \frac{3}{2}\delta^{+4}\right)$$

$$= \frac{1}{2\Delta x^3}\left(-3U_{i+4} + 14U_{i+3} - 24U_{i+2} + 18U_{i+1} - 5U_i\right)$$

Backward difference: taking only 1 term from eq 4.2.34

$$(U_{xxx})_i = \frac{1}{\Delta x^3}\left[(\delta^-)^3\right]U_i \qquad \delta^- = 1 - E^{-1}$$

$$= \frac{1}{\Delta x^3}\left[(1-E^{-1})^3\right]U_i$$

$$= \frac{1}{\Delta x^3}\left[1 - 3E^{-1} + 3E^{-2} - E^{-3}\right]U_i$$

$$= \frac{1}{\Delta x^3}(U_i - 3U_{i-1} + 3U_{i-2} - U_{i-3}) + O(\Delta)$$ Do remainder

w/ 2nd order accuracy:

$$(U_{xxx})_i = \frac{1}{\Delta x^3}\left[(\delta^-)^3 + \frac{3}{2}(\delta^-)^4\right] + O(\Delta x^2)$$

$$\Rightarrow (u_{xxx})_i = \frac{1}{\Delta x^3}\left[1-3E^{-1}+3E^{-2}-E^{-3}+\frac{3}{2}(1-E^{-1})^4\right]+O(\Delta x^2)$$

$(a+b)^n$ pascals Δ :

$$\begin{array}{lccccccccc}
n=0 & & & & & 1 & & & & \\
n=1 & & & & 1 & & 1 & & & \\
n=2 & & & 1 & & 2 & & 1 & & \\
n=3 & & 1 & & 3 & & 3 & & 1 & \\
n=4 & 1 & & 4 & & 6 & & 4 & & 1 \\
 & a^4 & & a^3b & & a^2b^2 & & ab^3 & & b^4
\end{array}$$

$\therefore$

$$(1-E^{-1})^4 = 1-4E^{-1}+6E^{-2}-4E^{-3}+E^{-4}$$

$\therefore$

$$(u_{xxx})_i = \frac{1}{\Delta x^3}\Big[1-3E^{-1}+3E^{-2}-E^{-3}$$
$$\frac{3}{2}-6E^{-1}+9E^{-2}-6E^{-3}+\frac{3}{2}E^{-4}\Big]u_i$$

$$= \frac{1}{\Delta x^3}\left[\frac{5}{2}-9E^{-1}+12E^{-2}-7E^{-3}+\frac{3}{2}E^{-4}\right]u_i + O(\Delta x^2)$$

$$= \frac{1}{2\Delta x^3}\left[5u_i-18u_{i-1}+24u_{i-2}-14u_{i-3}+3u_{i-4}\right]u_i + O(\Delta x^2)$$

2nd order accurate third derivative

$$(u_{xxx})_i = \frac{\delta^3}{\Delta x^3}\left(1-\frac{\delta^2}{8}\right)u_i \qquad \delta = E^{1/2} - E^{-1/2}$$

$$\delta^3 = E^{3/2} - 3E^{1/2} + 3E^{-1/2} - E^{-3/2}$$

$$\delta^2 = E - 2 + E^{-1}$$

$$(u_{xxx})_i = \frac{1}{\Delta x^3}\left(\delta^3 - \frac{\delta^5}{8}\right)u_i$$

$$\delta^5 = E^{5/2} - 3E^{3/2} + 3E^{1/2} - E^{-1/2}$$
$$-2E^{3/2} + 6E^{1/2} - 6E^{-1/2} + 2E^{-3/2}$$
$$E^{1/2} - 3E^{-1/2} + 3E^{-3/2} - E^{-5/2}$$
$$+$$
$$= E^{5/2} - 5E^{3/2} + 10E^{1/2} - 10E^{-1/2} + 5E^{-3/2} - E^{-5/2}$$

Then $\frac{1}{8}$ to sum $\delta^3 - \frac{\delta^5}{8}$

$$= \frac{1}{8}\left(8\delta^3 - \delta^5\right)$$

Then $8\delta^3 = 8E^{3/2} - 24E^{1/2} + 24E^{-1/2} - 24E^{-3/2}$

$-\delta^5 = 5E^{3/2} - 10E^{1/2} + 10E^{-1/2} - 5E^{-3/2} + E^{-5/2} - E^{+5/2}$

Adding

$$-E^{5/2} + 13E^{3/2} - 34E^{1/2} + 34E^{-1/2} - 29E^{-3/2} + E^{-5/2}$$

$\therefore$

$$(u_{xxx})_i = \frac{1}{8\Delta x^3}\left(-u_{i+5/2} + 13u_{i+3/2} - 34u_{i+1/2} + 34u_{i-1/2} - 29u_{i-3/2}\right.$$

$$\left. + u_{i-5/2}\right)$$

Central difference integer mesh: eq 4.2.36

1st order accurate:

$$(u_{xxx})_i = \mu \frac{\delta^3}{\Delta x^3} u_i$$

μ = Avg operator

$= \frac{1}{2}(E^{1/2} + E^{-1/2})$

$\delta = E^{1/2} - E^{-1/2}$

$$= \frac{1}{2\Delta x^3}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})(E - 2 + E^{-1})u_i$$

$$= \frac{1}{2\Delta x^3}(E - E^{-1})(E - 2 + E^{-1})u_i$$

$$= \frac{1}{2\Delta x^3}(E^2 - 2E + 1 - 1 + 2E^{-1} - E^{-2})u_i$$

$$(u_{xxx})_i = \frac{1}{2\Delta x^3}\left(E^2 - 2E + 2E^{-1} - E^{-2}\right)u_i$$

$$= \frac{1}{2\Delta x^3}\left(u_{i+2} - 2u_i + 2u_{i-1} - u_{i-2}\right)$$

<u>2nd order accuracy:</u>

$$(u_{xxx})_i = \frac{\mu}{\Delta x^3}\,\delta^3\left[1 - \frac{1}{4}\delta^2\right]u_i$$

$$= \frac{1}{2\Delta x^3}\left(E^2 - 2E + 2E^{-1} - E^{-2}\right)\frac{\left(4 - (E - 2 + E^{-1})\right)}{4}u_i$$

$$= \frac{-1}{8\Delta x^3}\left(E^2 - 2E + 2E^{-1} - E^{-2}\right)\left(E - 6 + E^{-1}\right)u_i$$

$$= \frac{-1}{8\Delta x^3}\left(E^3 - 2E^2 + \cancel{2} - E^{-1} - 6E^2 + 12E - 12E^{-1} + 6E^{-2}\right.$$

$$\left. E^1 - \cancel{2} + 2E^{-2} - E^{-3}\right)u_i$$

$$= \frac{-1}{8\Delta x^3}\left(E^3 - 8E^2 + 13E - 13E^{-1} + 8E^{-2} - E^{-3}\right)u_i$$

$$= \frac{1}{8\Delta x^3}\left(-u_{i+3} + 8u_{i+2} - 13u_{i+1} + 13u_{i-1} - 8u_{i-2} + u_{i-3}\right)$$

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<u>Forward difference</u>: 1st order

$$(u_{xxxx})_i = \frac{1}{\Delta x^4}\,\delta^{+4} u_i \qquad \delta^{+} = (E-1)$$

Pascals Δ

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$$(u_{xxxx})_i = \frac{1}{\Delta x^4}\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right)u_i$$

$$= \frac{1}{\Delta x^4}\left(u_{i+4} - 4u_{i+3} + 6u_{i+2} - 4u_{i+1} + u_i\right)$$

<u>Backwards difference</u>: 1st order

$$(u_{xxxx})_i = \frac{1}{\Delta x^4}\,\delta^{-4} u_i \qquad \delta^{-} = 1 - E^{-1}$$

Again from pascals Δ

$$\delta^{-4} = 1 - 4E^{-1} + 6E^{-2} - 4E^{-3} + E^{-4}$$

$$\therefore (u_{xxxx})_i = \frac{1}{\Delta x^4}\left(u_i - 4u_{i-1} + 6u_{i-2} - 4u_{i-3} + u_{i-4}\right)$$

Central difference 2nd order accurate

$$(u_{xxxx})_i = \frac{1}{\Delta x^4}\delta^4 \qquad \delta = E^{1/2} - E^{-1/2}$$

$$\delta^4 = E^{4/2} - 4E^{3/2}E^{-1/2} + 6E^{2/2}E^{-2/2} - 4E^{1/2}E^{-3/2} + E^{-2}$$
$$= E^2 - 4E + 6 - 4E^{-1} + E^{-2}$$

$$(u_{xxxx})_i = \frac{1}{\Delta x^4}\left(u_{i+2} - 4u_{i+1} + 6 - 4u_{i-1} + u_{i-2}\right)$$

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$$(4.3.2) \Rightarrow \qquad \left(1 + \frac{\delta^2}{6}\right) D = \frac{1}{\Delta x}\mu\delta + O(\Delta x^4)$$

$$\mu = \tfrac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

$$\delta = E^{1/2} - E^{-1/2}$$

$$\delta\mu = \frac{1}{2}\left(E + 1 - 1 - E^{-1}\right)$$

$$= \tfrac{1}{2}\left(E - E^{-1}\right)$$

$\therefore$ Above eq 4.3.2 becomes

$$\left(1 + \frac{\delta^2}{6}\right) D = \left(1 + \frac{E - 2 + E^{-1}}{6}\right) D$$

$$= \left(\frac{E + 4 + E^{-1}}{6}\right) D$$

operating on $U(x,t)$

$$\frac{1}{6}\left[(U_x)_{i+1} + 4(U_x) + (U_x)_{i-1}\right] = \frac{U_{i+1} - U_{i-1}}{2\Delta} + O(\Delta x^4)$$

$$\left(1 + \frac{\delta^+}{2}\right)\Delta x\, D = \delta^+ + O(\Delta x^3)$$

$$\delta^+ = E - 1$$

$$\frac{1}{2}(2+\delta^+)\Delta x D = \frac{1}{2}(2+E-1)\Delta x D$$

$$= \frac{1}{2}(E+1)\Delta x D = \delta^+ + \mathcal{O}(\Delta x^3)$$

operate on U_i each side

$$\Rightarrow \frac{1}{2}\left[(U_x)_{i+1} + (U_x)_i\right] = \frac{1}{\Delta x}(U_{i+1} - U_i) + \mathcal{O}(\Delta x^2)$$

4th order approximate

$$\frac{1}{6}\left[(U_x)_{i+1} + 4(U_x)_i + (U_x)_{i-1}\right] = \frac{U_{i+1} - U_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^4)$$

$$\begin{pmatrix} \ddots & & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & 1 & 4 & 1 \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ (U_x)_{i-1} \\ (U_x)_i \\ (U_x)_{i+1} \\ \vdots \end{pmatrix} = \frac{3}{\Delta x} \begin{pmatrix} \vdots \\ U_i - U_{i-2} \\ U_{i+1} - U_{i-1} \\ U_{i+2} - U_i \\ \vdots \end{pmatrix}$$

eq (4.3.5)

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \end{pmatrix} \begin{pmatrix} (U_x)_{i-1} \\ (U_x)_i \\ (U_x)_{i+1} \end{pmatrix} = \frac{2}{\Delta x} \begin{pmatrix} U_i - U_{i-1} \\ U_{i+1} - U_i \\ U_{i+2} - U_{i+1} \end{pmatrix}$$

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$$\delta = E^{1/2} - E^{-1/2}$$

$$\left(1 + \frac{\delta^2}{12}\right)(u_{xx})_i = \frac{1}{\Delta x^2}\delta^2 u_i + O(\Delta x^4)$$

$$\Rightarrow \left(1 + \frac{1}{12}\left(E^1 - 2 + E^{-1}\right)\right)(u_{xx})_i = \frac{1}{\Delta x^2}(u_{i+1} - 2u_i + u_{i-1}) + O(\Delta x^4)$$

$$\Rightarrow \frac{1}{12}\left(10 + E + E^{-1}\right)(u_{xx})_i = \frac{1}{\Delta x^2} \quad " \quad \ldots$$

$$\delta^+ = E - 1$$

$$1 + \delta^{+2} = (2 + E^2 - 2E)$$

$\therefore$ eq 4.3.8 $\Rightarrow$

$$(u_{xx})_{i+2} - 2(u_{xx})_{i+1} + 2(u_{xx})_i = \frac{1}{\Delta x^2}(u_{i+1} - u_i) + O(\Delta x^2)$$