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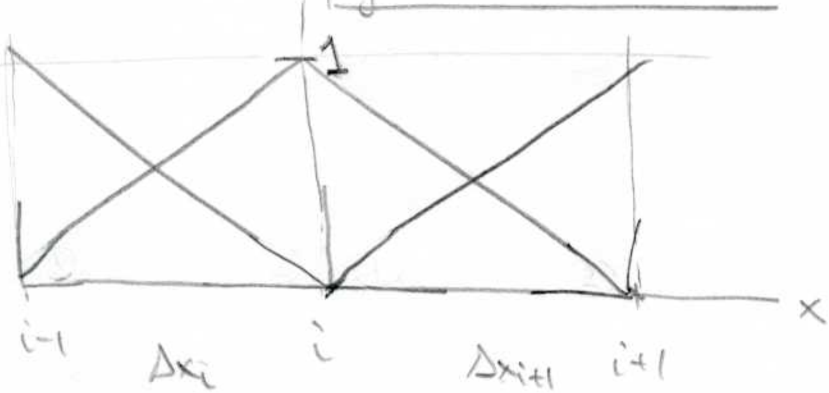

$$N_i^{(1)} = \frac{x - x_{i-1}}{\Delta x_i} \qquad N_{i-1}^{(1)} = \frac{-(x - x_i)}{\Delta x_i} = \frac{x_i - x}{\Delta x_i} \qquad \text{eqs 5.2.1}$$

↑ means 1 at global node i in element 1

In element 2:

$$N_i^{(2)}(x) = \frac{-(x - x_{i+1})}{\Delta x_{i+1}} = \frac{x_{i+1} - x}{\Delta x_{i+1}} \qquad \text{eqs 5.2.2}$$

$$N_{i+1}^{(2)}(x) = \frac{x - x_i}{\Delta x_{i+1}}$$

local $\xi_i(x) = \dfrac{x - x_{i-1}}{\Delta x_i}$

The $\underset{\uparrow}{N_1}(\xi) = 1 - \xi_i(x) = \dfrac{\Delta x_i - x + x_{i-1}}{\Delta x_i} = \dfrac{x_i - \cancel{x_{i-1}} - x - \cancel{x_{i-1}}}{\Delta x_i}$

1 corresponds to left local node

$$= \frac{x_i - x}{\Delta x_i}$$

$+ \quad N_2(\xi) = \xi = \dfrac{x - x_{i-1}}{\Delta x_i}$ 2 corresponds to right local node.

— — — — —

$$\tilde{U}(x) = U_{i-1} N_{i-1}^{(1)}(x) + U_i N_i^{(1)}(x)$$

$$= U_{i-1}\left[\frac{x_i - x}{\Delta x_i}\right] + U_i\left[\frac{x - x_{i-1}}{\Delta x_i}\right]$$

$$= -U_{i-1}\left[\frac{x - x_i - x_{i-1} + x_{i-1}}{\Delta x_i}\right] + U_i\left[\frac{x - x_{i-1}}{\Delta x_i}\right]$$

$$= -U_{i-1}\left[\frac{(x - x_{i-1})}{\Delta x_i} + -\frac{(x_i - x_{i-1})}{\Delta x_i}\right] + \frac{U_i}{\Delta x_i}(x - x_{i-1})$$

$$= U_{i-1} - U_{i-1}\frac{(x - x_{i-1})}{\Delta x_i} + \frac{U_i}{\Delta x_i}(x - x_{i-1})$$

$$= U_{i-1} + \frac{(x - x_{i-1})}{\Delta x_i}(U_i - U_{i-1}) \qquad x_{i-1} < x < x_i \quad \text{eq } 5.2.9$$

On elt 2

$$\tilde{U}(x) = U_i N_i^{(2)}(x) + U_{i+1} N_{i+1}^{(2)}(x)$$

$$= U_i\left(\frac{x_{i+1} - x}{\Delta x_{i+1}}\right) + U_{i+1}\left(\frac{x - x_i}{\Delta x_{i+1}}\right) \qquad x_i < x < x_{i+1}$$

$$= U_i\left(\frac{x_{i+1} - x_i}{\Delta x_{i+1}} + \frac{x_i - x}{\Delta x_{i+1}}\right) + U_{i+1}\left(\frac{x - x_i}{\Delta x_{i+1}}\right)$$

$\Delta x_{i+1} = x_{i+1} - x_i$

$$\Rightarrow \tilde{U}(x) = U_i\left(1 + \frac{x_i - x}{\Delta x_{i+1}}\right) + U_{i+1}\left(\frac{x - x_i}{\Delta x_{i+1}}\right)$$

$$= U_i + \frac{(x - x_i)}{\Delta x_{i+1}}\left(U_{i+1} - U_i\right) \quad \text{eq } 5.2.10$$

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$$(U_x)_i = \left(\frac{\partial U}{\partial x}\right)_i^{(1)} = \frac{U_i - U_{i-1}}{\Delta x_i}$$

In elt 2 $\quad (U_x)_i = \left(\frac{\partial U}{\partial x}\right)_i^{(2)} = \frac{U_{i+1} - U_i}{\Delta x_{i+1}}$

At node i $\underbrace{\text{slopes from left}}_{\text{positive}} \neq \underbrace{\text{slopes from right}}_{\text{negative}}$

See picture to Left.

Define

$$(U_x)_i = \frac{1}{\Delta x_i + \Delta x_{i+1}}\left[\Delta x_{i+1}\left(\frac{\partial U}{\partial x}\right)_i^{(1)} + \Delta x_i\left(\frac{\partial U}{\partial x}\right)_i^{(2)}\right]$$

$$= \frac{1}{\Delta x_i + \Delta x_{i+1}}\left[\Delta x_{i+1}\frac{(U_i - U_{i-1})}{\Delta x_i} + \Delta x_i\frac{(U_{i+1} - U_i)}{\Delta x_{i+1}}\right] = \text{eq } 4.5.4$$

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eq 5.2.3 → $\xi = \frac{x - x_{i-1}}{\Delta x_i} \Rightarrow x = \Delta x_i \xi + x_{i-1}$

$$x = (x_i - x_{i-1})\xi + x_{i-1}$$

$$= x_i \xi + x_{i-1}(1-\xi) = \sum_i x_i N_i(\xi)$$

eq 5.2.16 sum over local nodes.

+ eq 5.2.5 are same

$$N_1 = -\tfrac{1}{2}\xi(1-\xi) \qquad N_2 = 1-\xi^2 \qquad N_3 = \tfrac{1}{2}\xi(1+\xi)$$

$$N_1 + N_2 + N_3 = \frac{\xi^2}{2} + 1 - \xi^2 + \frac{\xi^2}{2} = 1.$$

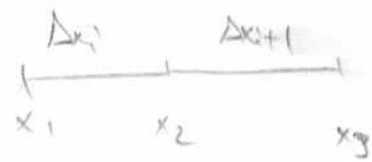
$$\frac{\partial N_j}{\partial x} = \frac{\partial N_j}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} = \frac{\partial N_j}{\partial \xi}\frac{1}{\partial x/\partial \xi} = \frac{\partial N_j}{\partial \xi}\frac{1}{\sum_k x_k(\partial N_k/\partial x)}$$

$$\frac{\partial x}{\partial \xi} = \sum_{j=1}^{3} x_j \frac{\partial N_j}{\partial \xi} = x_1\left(-\tfrac{1}{2}(1-\xi) + \tfrac{1}{2}\xi\right) + x_2(-2\xi) + x_3\left(\tfrac{1}{2}(1+\xi) + \tfrac{1}{2}\xi\right)$$

$$\frac{\partial x(\xi)}{\partial \xi} = x_1\left(-\frac{1}{2}\right) + \frac{1}{2}x_3 + \xi\left(\frac{x_1}{2} + \frac{x_1}{2} - 2x_2 + \frac{x_3}{2} + \frac{x_3}{2}\right)$$

$$= \frac{1}{2}(x_3 - x_1) + \xi(x_1 - 2x_2 + x_3)$$

$$= \xi(-\Delta x_i + \Delta x_{i+1}) + \frac{1}{2}(\Delta x_{i+1} + \Delta x_i)$$



For $j = i-1$

$$\frac{\partial N_{i-1}}{\partial x} = \frac{-\frac{1}{2}(1-\xi) + \frac{1}{2}\xi}{\xi(\Delta x_{i+1} - \Delta x_i) + \frac{1}{2}(\Delta x_{i+1} + \Delta x_i)} = \frac{\xi - \frac{1}{2}}{\xi(\quad) + \frac{1}{2}(\quad)}$$

$$= \frac{2\xi - 1}{2\xi(\Delta x_{i+1} - \Delta x_i) + (\Delta x_{i+1} + \Delta x_i)} \qquad (1) \qquad \text{eq } 5.2.21$$

$$\left(\frac{\partial N_{i-1}}{\partial x}\right)_{i-1} = \frac{-3}{-2(\Delta x_{i+1} - \Delta x_i) + \Delta x_{i+1} + \Delta x_i} = \frac{-3}{-\Delta x_{i+1} + 3\Delta x_i}$$

evaluate at $\xi = -1$

derivative of Global basis fns w.r.t global x values

$$\left(\frac{\partial N_{i-1}}{\partial x}\right)_i = \frac{-1}{\Delta x_{i+1} + \Delta x_i} \qquad \left(\frac{\partial N_{i-1}}{\partial x}\right)_{i+1} = \frac{1}{3\Delta x_{i+1} - \Delta x_i} \qquad \text{eqs } 5.2.22$$

$\xi = 0$ from (1)

$\xi = +1$ into (1)

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$$U_x = \sum_{j} u_j \frac{\partial N_j}{\partial x} = \sum_{\text{"}i\text{"}=i-1,i,i+1} u_i \frac{\partial N_i}{\partial x} = u_{i-1}\frac{\partial N_{i-1}}{\partial x} + u_i \frac{\partial N_i}{\partial x} + u_{i+1}\frac{\partial N_{i+1}}{\partial x}$$

Sum of j over all nodes of structure

Sum over local (only 1) element containing the pt x U_x is to be evaluated at.

If in 2 node element eq 5.1.4 is satisfied $N_I^{(e)}(\vec{x}_J) = \delta_{IJ}$ + Also $\forall$ N

* w/ 2 unknowns per node U + U_x requires 2 "N" fns for each node in a 2 node element $\Rightarrow$ 4 unknowns

Thus each N fn must satisfy

$$N(\xi_1) = 1$$
$$N'(\xi_1) = 0$$
$$N(\xi_2) = 0$$
$$N'(\xi_2) = 0 \quad \text{etc}$$

Then to satisfy 4 conditions w/ each basis fn require a 3rd order polynomial

Following the notation in the book let H_1^0 + w/ $\xi_1 = 0$ $\xi_2 = 1$

$$H_1^0(\xi_1) = 1 \qquad H_1^{0'}(\xi_1) = 0 \qquad H_1^0(\xi_2) = 0 \qquad H_1^{0'}(\xi_2) = 0$$

$$H_1^0 = a + b\xi + c\xi^2 + d\xi^3$$

$$H_1^0(0) = a = 1 \qquad H_1^{0'}(0) = b = 0$$

Then $H_1^0(1) = 1 + c + d = 0 \Rightarrow$

& $H_1^{0\prime}(1) = 2c + 3d = 0 \Rightarrow c = -\frac{3}{2}d$

$1 - \frac{3}{2}d + d = 0 \Rightarrow 1 - \frac{d}{2} = 0 \Rightarrow d = 2$, then

$c = -3$

$\therefore H_1^0(\xi) = 1 - 3\xi^2 + 2\xi^3$

For $H_1^1(\xi) \Rightarrow$

$H_1^1(\xi_1) = 0, \quad H_1^{1\prime}(\xi_1) = 1, \quad H_1^1(\xi_2) = 0, \quad H_1^{1\prime}(\xi_2) = 0$

Then $a = 0 \qquad b = 1 \qquad 1 + c + d = 0 \qquad 1 + 2c + 3d = 0$

$\Rightarrow d = -(1+c)$ put in ↑

$\Rightarrow 1 + 2c - 3 - 3c = 0$

$-2 - c = 0 \quad c = -2$

$d = 1$

$H_1^1(\xi) = \xi - 2\xi^2 + \xi^3 = \xi(1 - 2\xi + \xi^2)$

$= \xi(1-\xi)^2$

For $H_2^0(\xi) \Rightarrow$

$H_2^0(\xi_1) = 0, \quad H_2^{0\prime}(\xi_1) = 0, \quad H_2^0(\xi_2) = 1, \quad H_2^{0\prime}(\xi_2) = 0$

$\Rightarrow \quad a = 0 \qquad \Rightarrow b = 0 \qquad c + d = 1 =$

$2c + 3d = 0 \Rightarrow d = -\frac{2}{3}c$

$\Rightarrow \underbrace{c - \frac{2}{3}c}_{} = 1$

$\frac{1}{3}c = 1 \Rightarrow c = 3$

$H_2^0(\xi) = 3\xi^2 - 2\xi^3 \qquad d = -2$

$= \xi^2(3 - 2\xi)$

- - -

$H_2^1(\xi) \rightarrow \quad H_2^1(\xi_1) = 0, \; H_2^{1\prime}(\xi_1) = 0, \; H_2^1(\xi_2) = 0, \; H_2^{1\prime}(\xi_2) = 1$

$a = 0 \qquad b = 0 \qquad c + d = 0 \qquad 2c + 3d = 1$

$c = -d$

$+ d = 1 \Rightarrow c = -1$

$H_2^1(\xi) = -\xi^2 + \xi^3 = \xi^2(\xi - 1)$

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$$-\int_{\Omega} k\nabla\tilde{u}\circ\nabla w \, d\Omega + \oint_{\Gamma} k \frac{\partial \tilde{u}}{\partial n} w \, d\Gamma = \int_{\Omega} q w \, d\Omega$$

$$\Rightarrow \text{w/} \quad \tilde{u}(\bar{x}) = \sum_{I} u_I(t) N_I(\bar{x}) \quad \& \quad w = N_J(\vec{x})$$

$$\Rightarrow -\sum_{I} u_I(t) \int_{\Omega} k \nabla N_I(\bar{x}) \circ \nabla N_J \, d\Omega + \oint_{\Gamma} k \frac{\partial \tilde{u}}{\partial n} N_J \, d\Gamma$$

$$= \int_{\Omega} q N_J \, d\Omega$$

$$\parallel$$

$$\int_{\Omega_J} q_{\bar{x}} N_J(\vec{x}) \, d\Omega$$

Note:

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In eq 5, 3, 6a

let $W = N_J(\vec{x})$ Then As $W(\vec{x})$ vanishes outside of the "triangles" containing global node J as an index

$N_J(\vec{x})$ is one at node J + goes to zero ~~in~~ some distance from node J. Note: the tent fn N_J may be made of smaller piecewise fns

$$\int_{\Omega} k \nabla \tilde{U} \circ \nabla N_J(\vec{x})\, d\vec{x} = \boxed{\int_{\Omega_J} k \nabla \tilde{U} \circ \nabla N_J(\vec{x})\, dx}$$

↑ integration region becomes restricted to only those elements containg J as an internal point.

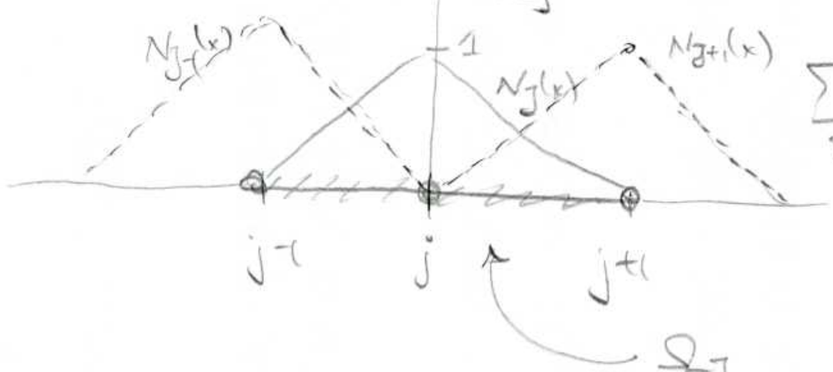
Then when $\tilde{U}(\vec{x}) = \sum_I U_I N_I(\vec{x})$ is put in the above one gets

$$\sum_I U_I \int_{\Omega_J} k \nabla N_I \circ \nabla N_J \, d\vec{x}$$

will further be restricted to including only the I's that are in ~~domain~~ ~~Ω_J~~ The triangles containing Node J.

$$-\sum_I U_I \int_{\Omega_J} k \nabla N_I \circ \nabla N_J \, d\vec{x} + \oint_{\Gamma} k \frac{\partial U}{\partial n} N_J \, d\Gamma = \int_{\Omega_J} q N_J \, d\Omega$$

eq 5.3.9 is
$$-\sum_I u_I \int_{\Omega_J} k\nabla N_I \cdot \nabla N_J \, d\Omega + \oint_\Gamma k \frac{\partial \tilde{u}}{\partial n} N_J \, d\Gamma = \int_{\Omega_J} q N_J \, d\Omega$$



$\sum_I$ = sum over all ~~elements~~ nodes contained in Ω_J

$$\Rightarrow -\sum_{i=j-1}^{j+1} u_i \int_{x_{j-1}}^{x_{j+1}} k(x) \frac{dN_i}{dx} \cdot \frac{dN_j}{dx} \, dx + \underbrace{k(x) \frac{\partial \tilde{u}}{\partial n} N_j(x) \Big|_{x_{j-1}}^{x_{j+1}}} = \int_{x_{j-1}}^{x_{j+1}} q N_j(x) \, dx$$

$N_j(x_{j-1}) = 0$
$N_j(x_{j+1}) = 0$
$\therefore$ this term drops out.

we need to evaluate

$$\int_{x_{j-1}}^{x_{j+1}} k(x) \frac{dN_{j-1}}{dx} \cdot \frac{dN_j}{dx} \, dx$$

$$\int_{x_{j-1}}^{x_{j+1}} k(x) \left(\frac{dN_j}{dx}\right)^2 dx \quad + \quad \int_{x_{j-1}}^{x_{j+1}} k(x) \frac{dN_{j+1}}{dx} \cdot \frac{dN_j}{dx} \, dx$$

w/ linear elements
$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{\Delta x_i} \\[2ex] \dfrac{x_{i+1} - x}{\Delta x_{i+1}} \end{cases} \Rightarrow \frac{dN_i}{dx} = \begin{cases} \dfrac{1}{\Delta x_i} & \text{in left elt.} \\[2ex] \dfrac{-1}{\Delta x_{i+1}} & \text{in Right elt.} \end{cases}$$

$\Rightarrow$ evaluating the integrals one at a time

$$\int_{x_{j-1}}^{x_{j+1}} k(x) \frac{dN_{j-1}}{dx} \cdot \frac{dN_j}{dx} = \int_{x_{j-1}}^{x_j} k(x) \frac{(-1)}{\Delta x_j} \cdot \frac{1}{\Delta x_j} \, dx = \frac{-1}{(\Delta x_j)^2} \int_{x_{j-1}}^{x_j} k(x) \, dx$$

$$\int_{x_{j-1}}^{x_{j+1}} k(x)\left(\frac{dN_j}{dx}\right)^2 dx = \int_{x_{j-1}}^{x_j} k(x)\left(\frac{dN_j}{dx}\right)^2 dx + \int_{x_j}^{x_{j+1}} k\left(\frac{dN_j}{dx}\right)^2 dx$$

$$= \left(\frac{1}{\Delta x_j}\right)^2 \int_{x_{j-1}}^{x_j} k\,dx + \frac{1}{(\Delta x_{j+1})^2}\int_{x_j}^{x_{j+1}} k\,dx$$

$$\int_{x_{j-1}}^{x_{j+1}} k(x)\,\frac{dN_{j+1}}{dx}\cdot\frac{dN_j}{dx} = \int_{x_j}^{x_{j+1}} k(x)\,\frac{dN_{j+1}}{dx}\cdot\frac{dN_j}{dx}\,dx = \int_{x_j}^{x_{j+1}} k(x)\frac{1}{(\Delta x_{j+1})}\left(\frac{-1}{\Delta x_{j+1}}\right)dx$$

$$= -\frac{1}{\Delta x_{j+1}^2}\int_{x_j}^{x_{j+1}} k\,dx$$

w/ $\Delta x_j = \Delta x \ \forall j$

$$\therefore \ -\left[-\frac{U_{j-1}}{\Delta x^2}\int_{x_{j-1}}^{x_j} k\,dx + \frac{U_j}{\Delta x^2}\int_{x_{j-1}}^{x_{j+1}} k\,dx - \frac{U_{j+1}}{\Delta x^2}\int_{x_j}^{x_{j+1}} k\,dx\right] = \int_{x_{j-1}}^{x_{j+1}} q\,N_j(x)\,dx$$

$$= \frac{U_{j-1}}{\Delta x^2}\int_{x_{j-1}}^{x_j} k\,dx - \frac{U_j}{\Delta x^2}\int_{x_{j-1}}^{x_j} k\,dx - \frac{U_j}{\Delta x^2}\int_{x_j}^{x_{j+1}} k\,dx + \frac{U_{j+1}}{\Delta x^2}\int_{x_j}^{x_{j+1}} k\,dx = \int_{x_{j-1}}^{x_{j+1}} q\,N_j\,dx$$

$$= \frac{(U_{j-1}-U_j)}{\Delta x}\,\frac{1}{\Delta x}\int_{x_{j-1}}^{x_j} k\,dx + \frac{U_{j+1}-U_j}{\Delta x}\,\frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k\,dx = \int_{x_{j-1}}^{x_{j+1}} q\,N_j\,dx$$

$\Rightarrow$ define $K_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} k\,dx$

$$= -\frac{(u_j - u_{j-1})}{\Delta x} k_{j-1/2} + \frac{(u_{j+1} - u_j)}{\Delta x} k_{j+1/2} = \int_{x_{j-1}}^{x_{j+1}} q N_j\,dx$$

$$\rightarrow \quad k_{j+1/2}\frac{(u_{j+1} - u_j)}{\Delta x} - k_{j-1/2}\frac{(u_j - u_{j-1})}{\Delta x} = \int_{x_{j-1}}^{x_{j+1}} q N_j\,dx$$

Using a 3 pt stencil to evaluate $\int_{x_{j-1}}^{x_{j+1}} qN$

$$\int_{x_{j-1}}^{x_{j+1}} f\,dx = c_1 f(x_{j-1}) + c_2 f(x_j) + c_3 f(x_{j+1})$$

$\underline{f = 1}$

$$x_{j+1} - x_{j-1} = c_1 + c_2 + c_3$$

$\underline{f = x}$

$$\frac{x_{j+1}^2}{2} - \frac{x_{j-1}^2}{2} = c_1 x_{j-1} + c_2 x_j + c_3 x_{j+1}$$

$\underline{f = x^2}$

$$\frac{x_{j+1}^3}{3} - \frac{x_{j-1}^3}{3} = c_1 x_{j-1}^2 + c_2 x_j^2 + c_3 x_{j+1}^2$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 \\ x_{j-1} & x_j & x_{j+1} \\ x_{j-1}^2 & x_j^2 & x_{j+1}^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_{j+1} - x_{j-1} \\ \frac{1}{2}(x_{j+1}^2 - x_{j-1}^2) \\ \frac{1}{3}(x_{j+1}^3 - x_{j-1}^3) \end{pmatrix}$$

w/ MMA:

$$c_1 = -\frac{(+x_{j+1} - x_{j-1})(x_{j+1} - 3x_j + 2x_{j-1})}{6(x_j - x_{j-1})}$$

$$c_2 = -\frac{(x_{j+1} - x_{j-1})^3}{6(x_j - x_{j-1})(x_{j+1} - x_j)}$$

$$c_3 = \frac{(x_{j+1} - x_{j-1})(2x_{j+1} - 3x_j + x_{j-1})}{6(x_{j+1} - x_j)}$$

w/ $\Delta x_j = x_j - x_{j-1}$ $\qquad \Delta x_{j+1} = x_{j+1} - x_j$

$$c_1 = -\frac{(x_{j+1} - x_j + x_j - x_{j-1})(x_{j+1} - x_j - 2(x_j - x_{j-1}))}{6(x_j - x_{j-1})}$$

$$c_2 = -\frac{(\Delta x_{j+1} + \Delta x_j)^3}{6\,\Delta x_j\,\Delta x_{j+1}}$$

$$c_3 = \frac{(\Delta x_{j+1} + \Delta x_j)(2(x_{j+1} - x_j) - (x_j - x_{j-1}))}{6(x_{j+1} - x_j)}$$

$\Rightarrow$

$$c_1 = -\frac{(\Delta x_{j+1} + \Delta x_j)(\Delta x_{j+1} - 2\Delta x_j)}{6\,\Delta x_j}$$

$$c_2 = -\frac{(\Delta x_{j+1} + \Delta x_j)^3}{6\,\Delta x_j\,\Delta x_{j+1}}$$

$$c_3 = (\Delta x_{j+1} + \Delta x_j)(2\Delta x_{j+1} - \Delta x_j) / 6\,\Delta x_{j+1}$$

w/ $\Delta x_{j+1} = \Delta x_j = \Delta x$

$$\Rightarrow C_1 = \frac{-2\Delta x(-\Delta x)}{6\Delta x} = \frac{\Delta x}{3}$$

$$C_2 = -\frac{8\Delta x}{6} = -4\frac{\Delta x}{3}$$

$$C_3 = \frac{2\Delta x^2}{6\Delta x} = \frac{\Delta x}{3}$$

$$\therefore \int_{x_{j-1}}^{x_{j+1}} f(x)\,dx \cong \frac{\Delta x}{3} f(x_{j-1}) - \frac{4\Delta x}{3} f(x_j) + \frac{\Delta x}{3} f(x_{j+1})$$

Thus R.H.S becomes

$$\int_{x_{j-1}}^{x_{j+1}} q N_j(x)\,dx = \frac{\Delta x}{3} q_{j-1} \cancelto{0}{N_j(x_{j-1})} + -\frac{4\Delta x}{3} q_j + \frac{\Delta x}{3} q_{j+1} \cancelto{0}{N_j(x_{j+1})}$$

Not correct?

look at exactly

$$= \int_{x_{j-1}}^{x_j} q(x)\left(\frac{x - x_{j-1}}{\Delta x_j}\right) dx + \int_{x_j}^{x_{j+1}} q(x)\left(\frac{x_{j+1} - x}{\Delta x_{j+1}}\right) dx$$

$$= \frac{1}{\Delta x_j}\int_{x_{j-1}}^{x_j} q(x)(x - x_{j-1})\,dx + \frac{1}{\Delta x_{j+1}}\int_{x_j}^{x_{j+1}} (x_{j+1} - x)\,q(x)\,dx$$

Assuming a linear variation in each region for q

$q = q_0 + q_1 x$ in each Δ

$$= \frac{1}{\Delta x_j} \int_{x_{j-1}}^{x_j} (q_0^L + q_1^L x)(x - x_{j-1})\, dx + \frac{1}{\Delta x_{j+1}} \int_{x_j}^{x_{j+1}} (q_0^R + q_1^R x)(x_{j+1} - x)\, dx$$

$$= \frac{1}{\Delta x_j} \int_{x_{j-1}}^{x_j} \left(q_0^L x - q_0^L x_{j-1} + q_1^L x^2 - x_{j-1} q_1^L x \right) dx + \quad ''$$

$$= \frac{1}{\Delta x_j} \Big[q_0^L \frac{(x_j^2 - x_{j-1}^2)}{2} - q_0^L x_{j-1} \Delta x_j + \frac{q_1^L}{3}(x_j^3 - x_{j-1}^3)$$

$$- \frac{x_{j-1}}{2} q_1^L (x_j^2 - x_{j-1}^2) \Big]$$

$+ \quad ''$ (increment j by 1 + switch L w/ R)

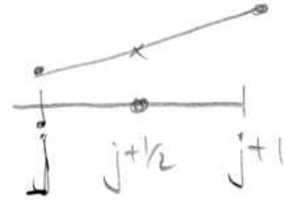
$$= q_0^L \frac{(x_j + x_{j-1})}{2} - q_0^L x_{j-1} + \frac{1}{3} q_1^L (x_j^2 + x_j x_{j-1} + x_{j-1}^2)$$

$$- \frac{x_{j-1} q_1^L}{2} (x_j + x_{j-1})$$

= How do I recombine to give E 5.3.3 ?

From
$$\int_{x_{i-1}}^{x_{i+1}} q(x)\, dx \cong \frac{\Delta x}{3} (q_{i-1} - 4q_i + q_{i+1})$$

w/ linear variation in $k \Rightarrow$ $\quad k_{j+1/2} = k_0 + k_1 x_{j+1/2}$



$$k_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} (k_0 + k_1 x)\,dx = k_0 \frac{(x_{j+1} - x_j)}{\Delta x} + \frac{k_1\left(\frac{x_{j+1}^2}{2} - \frac{x_j^2}{2}\right)}{\Delta x}$$

$$= k_0 + \frac{1}{2}(x_{j+1} + x_j)\,k_1$$

$$= \frac{1}{2}\left(2k_0 + (x_{j+1} + x_j)k_1\right) = \frac{1}{2}(k_0 + k_1 x_j) + \frac{1}{2}(k_0 + k_1 x_{j+1})$$

$$= \frac{1}{2}(k_j + k_{j+1}) \quad \text{eq E5.3.5}$$

$$\cancel{k_j = \frac{1}{\Delta x}\int_{x_{j+1/2}}^{x_{j+3/2}} (k_0 + k_1 x) = \frac{1}{\Delta x}(\Delta x)\,k_0 + \frac{k_1}{\Delta x}\left(x_{j+3/2}^2 - x_{j+1/2}^2\right)}$$

Can I show this going from k_j

$$\cancel{= k_0 + k_1 \left(x_{j+3/2} + x_{j+1/2}\right)}$$

$$\cancel{k_{j+1} = \frac{1}{\Delta x}}$$

Pg 219 Hirsh Vol I

$\Delta U = q \qquad U = U_0$ on Γ

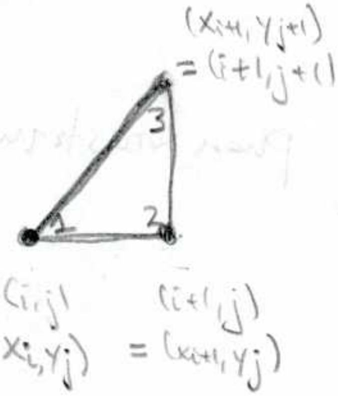
$\circ$ As $N_J = 0$ on $\partial\Omega_J$

$$-\sum_I U_I \int_{\Omega_J} \left(\frac{\partial N_I}{\partial x}\cdot\frac{\partial N_J}{\partial x} + \frac{\partial N_I}{\partial y}\frac{\partial N_J}{\partial y}\right) dx\,dy \;\cancel{\mp \oint_\Gamma k\frac{\partial \hat{U}}{\partial n} N_J \, d\Gamma} = \int_{\Omega_J} q N_J \, d\Omega$$

$$N_{ij}^{(1)} \equiv N_J^{(1)} = ax + by + c \quad \text{w/} \quad a = \frac{y_2 - y_3}{2A} = +\frac{(y_j - y_{j+1})}{2(\frac{1}{2}\Delta x \Delta y)} = +\frac{\cancel{\Delta y}}{\Delta x \cancel{\Delta y}}$$

$\parallel$

tent function for global node i,j restricted to triangle 1



$$= \frac{-1}{\Delta x}$$

$$\& \quad b = \frac{x_3 - x_2}{2A} = \frac{0}{\Delta x \Delta y} = 0$$

$$c = \frac{(x_2 y_3 - x_3 y_2)}{2A}$$

$$\text{w/ } A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\left[x_1(y_2 - y_3) - y_1(x_2 - x_3) + (x_2 y_3 - x_3 y_2)\right]$$

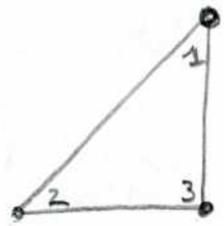
$$\Rightarrow \quad x_2 y_3 - x_3 y_2 = 2A - x_1(y_2 - y_3) + y_1(x_2 - x_3)$$

$$\Rightarrow \quad c = 1 - \frac{x_1(y_2 - y_3)}{2A} + \frac{y_1(x_2 - x_3)}{2A} = 1 + \frac{x_i(\Delta y)}{\cancel{2}(\frac{1}{\cancel{2}}\Delta x \Delta y)} + y_j \cdot 0$$

$$= 1 + \frac{x_i}{\Delta x}$$

$$\therefore \quad N_{ij}^{(1)} = -\frac{x}{\Delta x} + 1 + \frac{x_i}{\Delta x} = 1 - \frac{(x - x_i)}{\Delta x}$$

$$N^{(1)}_{i+1,j+1} = ax + by + c$$



place 1 at location of $(i+1, j+1)$
& order node labels counter clockwise.

$$a = \frac{y_2 - y_3}{2A} = 0$$

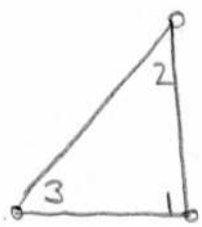
$$b = \frac{x_3 - x_2}{2A} = \frac{\Delta x}{\Delta x \Delta y} = \frac{1}{\Delta y}$$

$$c = \frac{x_2 y_3 - x_3 y_2}{2A} = \frac{2A - x_1(y_2 - y_3) + y_1(x_2 - x_3)}{2A} = \frac{2A - 0 - y_1 \Delta x}{2A}$$

$$= 1 - \frac{y_{j+1}}{\Delta y}$$

$$\therefore N^{(1)}_{i+1,j} = \frac{y}{\Delta y} + 1 - \frac{y_{j+1}}{\Delta y} = 1 + \frac{y - y_{j+1}}{\Delta y}$$

$$N^{(1)}_{i+1,j} = ax + by + c$$



place 1 at node location $(i+1, j)$

$$a = \frac{y_2 - y_3}{2A} = \frac{\Delta y}{\Delta x \Delta y} = \frac{1}{\Delta x}$$

$$b = \frac{x_3 - x_2}{2A} = \frac{-\Delta x}{\Delta x \Delta y} = \frac{-1}{\Delta y}$$

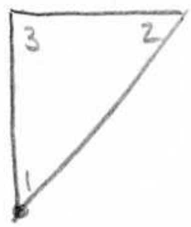
$$c = 1 - \frac{x_1(y_2 - y_3)}{2A} + \frac{y_1(x_2 - x_3)}{2A} = 1 - \frac{x_{i+1}\cancel{\Delta y}}{\Delta x \cancel{\Delta y}} + y_j \frac{\cancel{\Delta x}}{\cancel{\Delta x}\Delta y}$$

$$= 1 - \frac{x_{i+1}}{\Delta x} + \frac{y_j}{\Delta y}$$

$$N^{(1)}_{i+1,j} = \frac{x}{\Delta x} - \frac{y}{\Delta y} + 1 - \frac{x_{i+1}}{\Delta x} + \frac{y_j}{\Delta y}$$

$$= 1 + \frac{x - x_{i+1}}{\Delta x} - \frac{(y - y_j)}{\Delta y}$$

$\Delta 2$



$N^{(2)}_{ij}$

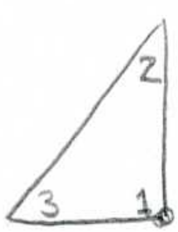
$$N^{(2)}_{ij} = ax + by + c$$

$$a = \frac{y_2 - y_3}{2A} = 0, \quad b = \frac{x_3 - x_2}{2A} = -\frac{\cancel{\Delta x}}{\cancel{\Delta x}\Delta y} = -\frac{1}{\Delta y}$$

$$c = 1 - \frac{x_1(y_2 - y_3)}{2A} + \frac{y_1(x_2 - x_3)}{2A} = 1 - 0 + \frac{\Delta x}{\Delta x \Delta y} y_j = 1 + \frac{y_j}{\Delta y}$$

$$\therefore N^{(2)}_{ij} = -\frac{y}{\Delta y} + 1 + \frac{y_j}{\Delta y} = 1 - \frac{(y - y_j)}{\Delta y}$$

$\Delta 3$



$$N^{(3)}_{ij} = ax + by + c$$

$$a = \frac{y_2 - y_3}{2A} = \frac{\Delta y}{\Delta x \Delta y} = \frac{1}{\Delta x}$$

$$b = \frac{x_3 - x_2}{\Delta x \Delta y} = \frac{-\Delta x}{\Delta x \Delta y} = -\frac{1}{\Delta y}$$

$$C = 1 - \frac{x_1(y_2 - y_3)}{2A} + \frac{y_1(x_2 - x_3)}{2A} = 1 - \frac{x_1}{\Delta x} + \frac{y_1}{\Delta y} = 1 - \frac{x_i}{\Delta x} + \frac{y_j}{\Delta y}$$

$$N_{ij}^{(3)} = \frac{x}{\Delta x} - \frac{y}{\Delta y} + 1 - \frac{x_i}{\Delta x} + \frac{y_j}{\Delta y} = 1 + \frac{x - x_i}{\Delta x} - \frac{(y - y_j)}{\Delta y}$$

To obtain coefficients of stiffness matrix k_{JJ} = diagonal elts $\equiv K_{ij}^{ij}$

k_{IJ}

restricted to Δ 1 restricted to Δ 1.

$K_{ij}^{i+1,j\,(1)}$ = interaction of Global basis fn $N_{ij}^{(1)}$ w/ global basis fn $N_{i+1,j}^{(1)}$

$$= \int_1 \left(\frac{\partial N_{ij}^{(1)}}{\partial x} \cdot \frac{\partial N_{i+1,j}^{(1)}}{\partial x} + \frac{\partial N_{ij}^{(1)}}{\partial y} \cdot \frac{\partial N_{i+1,j}^{(1)}}{\partial y} \right) dx\,dy$$

$$= \int_1 \left(\frac{-1}{\Delta x}\right)\left(\frac{1}{\Delta x}\right) dx\,dy = \frac{-1}{\Delta x^2}\,\frac{1}{2}\Delta x \Delta y = -\frac{\Delta y}{2\Delta x} = -\frac{1}{2}$$

$$K_{ij}^{i+1,j+1\,(1)} = \int_1 \left(\frac{\partial N_{ij}^{(1)}}{\partial x} \overset{0}{\cancel{\frac{\partial N_{i+1,j+1}^{(1)}}{\partial x}}} + \overset{0}{\cancel{\frac{\partial N_{ij}^{(1)}}{\partial y}}} \frac{\partial N_{i+1,j+1}^{(1)}}{\partial y} \right) dx\,dy = 0$$

$$K_{ij}^{ij\,(1)} = \int_1 \left(\frac{1}{\Delta x}\right)^2 dx\,dy = \frac{1}{2}$$

$$K_{ij}^{ij} = \sum_{t=1}^{6} K_{ij}^{ij(t)} = K_{ij}^{ij(1)} + K_{ij}^{ij(2)} + K_{ij}^{ij(3)} + K_{ij}^{ij(4)} + K_{ij}^{ij(5)} + K_{ij}^{ij(6)}$$

Sum over Δ's

$$= \frac{1}{2} + \frac{1}{2} \qquad \frac{1}{2} + \frac{1}{2} = \text{How get } 4?$$

Brute Force method compute each interaction exactly

$$K_{ij}^{ij(2)} = \int_2 \left(\frac{\partial N_{ij}^{(2)}}{\partial x} \cdot \frac{\partial N_{ij}^{(2)}}{\partial x} + \cdots \right) dx\,dy = \frac{1}{\Delta y^2} A = \frac{1}{2}$$

$$K_{ij}^{ij(3)} = \int_3 \left(\frac{\partial N_{ij}^{(3)}}{\partial x} \cdot \frac{\partial N_{ij}^{(3)}}{\partial x} + \left(\frac{\partial N_{ij}^{(3)}}{\partial y} \right)^2 \right) dx\,dy = \int_3 \left(\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2 \right) dx\,dy = \frac{2}{\Delta x^2} \cdot \frac{\Delta x^2}{2} = 1$$

Thus

$$K_{ij}^{ij} = \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{2} + 1 = 4 \quad ✓$$

$$K_{ij}^{i+1,j(1)} = -\frac{1}{2} \qquad K_{ij}^{i+1,j(2,3,4,5)} = 0$$

$$K_{ij}^{i+1,j(6)} = \int_6 \left(\frac{\partial N_{i+1,j}^{(6)}}{\partial x} \cdot \frac{\partial N_{ij}^{(6)}}{\partial x} + \frac{\partial N_{i+1,j}^{(6)}}{\partial y} \frac{\partial N_{ij}^{(6)}}{\partial y} \right) dx\,dy$$

$N_{ij}^{(6)}$ would look just like $N_{ij}^{(3)}$ w/ a change in sign

+ $N_{i+1,j}^{(6)}$ would look just like N ...

We can figure slopes at from figure only $\frac{\partial N_{i+1,j}^{(6)}}{\partial x} = +\frac{1}{\Delta x}$ $\frac{\partial N_{i+1,j}^{(6)}}{\partial y} = 0$

$$\therefore \int_6 \cdots = \int_6 \left(\frac{1}{\Delta x}\right)\left(\frac{-1}{\Delta x}\right) dx\,dy = -\frac{1}{2}$$

$$\therefore k_{ij}^{i+1,j} = -\frac{1}{2} + -\frac{1}{2} = -1$$

$k_{ij}^{i-1,j} = -1$ by symmetry w/ $k_{ij}^{i+1,j}$

$$k_{ij}^{i,j+1} = -\frac{1}{2} - \frac{1}{2} = -1$$

$k_{ij}^{i,j-1} = -1$ by symmetry w/ $k_{ij}^{i,j+1}$

$$k_{ij}^{i+1,j+1} = \int_1 \left(\left(\frac{-1}{\Delta x}\right)(0) + (0)\left(\frac{1}{\Delta y}\right)\right) dx\,dy + \int_2 \left(0\left(\frac{1}{\Delta x}\right) + \left(-\frac{1}{\Delta y}\right)0\right) dx\,dy = 0$$

$k_{ij}^{i-1,j-1} = 0$ by symmetry w/ $k_{ij}^{i+1,j+1}$ Also note:

$k_{i,j+1}^{ij} = k_{ij}^{i,j+1}$ i.e. the top + bottom indices can be interchanged

Then eq 5.3.9 Becomes

$$-\Big[u_{ij}(4) + u_{i+1,j}(-1) + u_{i+1,j+1}(0) + u_{i,j+1}(-1) + u_{i-1,j}(-1)$$
$$+ u_{i-1,j-1}(0) + u_{i,j-1}(-1)\Big] = \int_{\Omega_J} q N_J \, d\Omega$$

$$\Rightarrow \; -4u_{ij} + (u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}) = \int_{\Omega_J} q N_J \, d\Omega$$

For $\displaystyle\int_{\Omega_J} q N_J \, d\Omega$

$$= \sum_{t=1}^{6} \int_{t} q N_J \, d\Omega \cong \sum_{t=1}^{6} \Big[$$

Sum over triangles

$$N_J(\text{centroid}) = N_{ij}^{(3)}($$

Example 8.3.2 cont

evaluation of

Approximate q using linear basis fns, or

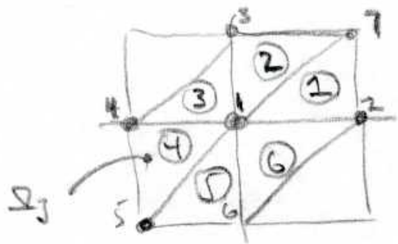
$$\int_{\Omega_J} q N_J \, dx \, dy \text{ term.} \quad \hat{q} \simeq q_{ij} N_{ij} + q_{i+1,j} N_{i+1,j} + q_{i,j+1} N_{i,j+1}$$

$$+ q_{i-1,j} N_{i-1,j} + q_{i-1,j-1} N_{i-1,j-1} + q_{i,j-1} N_{i,j-1}$$

$$+ q_{i+1,j+1} N_{i+1,j+1}$$

Then integrating against $N_J(x,y)$ gives w/ Ω_J compact support of (x_J, y_J)

$$\int_{\Omega_J} q N_J \, dx \, dy = \int_{\Omega_J} q_{ij} N_{ij} + \cdots$$

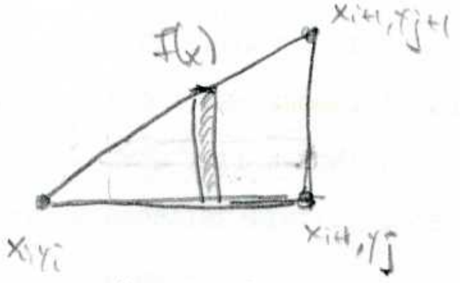


$$= q_{ij} \int_{\Omega_J} N_{ij}^2 \, dA + q_{i+1,j} \int_{\Omega_J} N_{i+1,j} N_{ij} \, dA + q_{i,j+1} \int_{\Omega_J} N_{i,j+1} N_{ij} \, dA$$

$$+ q_{i-1,j} \int_{\Omega_J} N_{i-1,j} N_{ij} \, dA + q_{i-1,j-1} \int_{\Omega_J} N_{i-1,j-1} N_{ij} \, dA + q_{i,j-1} \int_{\Omega_J} N_{i,j-1} N_{ij} \, dA$$

$$+ q_{i+1,j+1} \int_{\Omega_J} N_{i+1,j+1} N_{ij} \, dA$$

$$\int_{\Omega_J} N_{ij}^2 \, dA = 6 \cdot \int_{\Delta 1} N_{ij}^2 \, dA = 6 \iint \left(1 - \frac{x - x_{ij}}{\Delta x}\right)^2 dx \, dy$$



$$f(x) = y_i + \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)}(x - x_i)$$

$$= 6\int_{x_i}^{x_{i+1}}\int_{y_i}^{f(x)} \left(1 - \frac{x - x_i}{\Delta x}\right)^2 dy\,dx = 6\int_{x_i}^{x_{i+1}} \left(1 - \frac{x - x_i}{\Delta x}\right)^2 \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)}(x - x_i)\,dx$$

$$= 6\frac{\Delta y}{\Delta x}\int_{x_i}^{x_{i+1}} \left(1 - \frac{x - x_i}{\Delta x}\right)^2 (x - x_i)\,dx = \frac{6\Delta y}{\Delta x}\left[\frac{1}{2}\Delta x^2 + \frac{\Delta x^4}{4\Delta x^2} - \frac{2}{3}\frac{\Delta x^3}{\Delta x}\right]$$

$$= \frac{6\Delta y\,\Delta x^{\cancel{2}}}{\cancel{\Delta x}}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right] = 6\Delta x\,\Delta y\,\frac{1}{12} = \frac{6\Delta x\,\Delta y}{12}$$ term in RHS of E5.3.13

- - - -

Now integrals 2 + 4 must have the same value.
" " 3 + 6 " " " " "
" 7 + 5 " " " " "

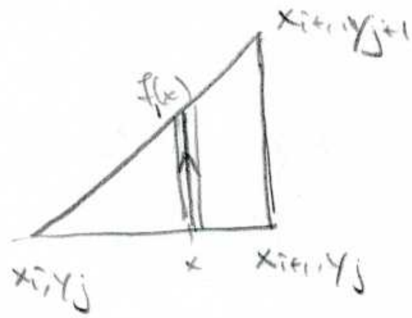
But also 2 + 3 must have the same value since in integral 2
The two Δ's we integrate over are 1 + 6
where in integral 3 we integrate over 2 + 3 +
the correspondence is shown w/ $\Leftrightarrow$ symbol. Thus all integrals must have the same value, to evaluate use integral 2

$$\int_{\Omega_J} N_{i+1,j}(x,y)\,N_{i,j}(x,y)\,dA = \int_{\Delta_1} N_{i+1,j}\,N_{i,j}\,dA + \int_{\Delta_6} N_{i+1,j}\,N_{i,j}\,dA$$

$$= \int_{\Delta_1} \left(1 + \frac{x - x_{i+1}}{\Delta x} - \frac{y - y_j}{\Delta y}\right)\left(1 - \frac{x - x_i}{\Delta x}\right) dA$$

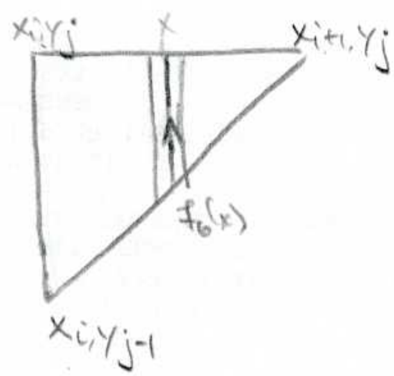
$$+ \int_{\Delta_6} \left(1 + \frac{x - x_{i+1}}{\Delta x}\right)\left(1 - \frac{x - x_i}{\Delta x} + \frac{y - y_j}{\Delta y}\right) dA$$

$\underline{\Delta_1}$:



$$f_1(x) = y_j + \left(\frac{y_{j+1} - y_j}{x_{i+1} - x_i}\right)(x - x_i)$$

$\underline{\Delta_6}$:



$$f_6(x) = y_{j-1} + \frac{(y_j - y_{j-1})}{(x_{i+1} - x_i)}(x - x_i)$$

Then the Above integrals become:

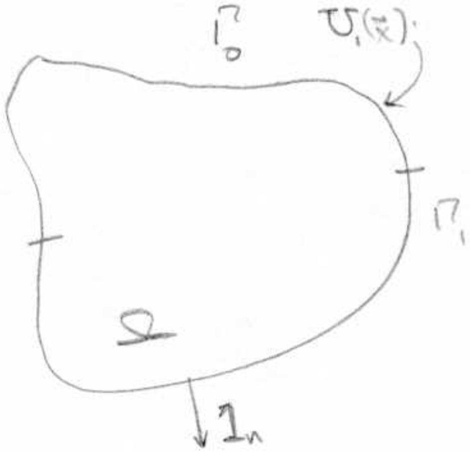
$$= \int_{x_i}^{x_{i+1}} \int_{y_j}^{f_1(x)} \left(1 + \frac{x - x_{i+1}}{\Delta x} - \frac{y - y_j}{\Delta y}\right)\left(1 - \frac{x - x_i}{\Delta x}\right) dy\, dx$$

$$+ \int_{x_i}^{x_{i+1}} \int_{f_6(x)}^{y_j} \left(1 + \frac{x - x_{i+1}}{\Delta x}\right)\left(1 - \frac{x - x_i}{\Delta x} + \frac{y - y_j}{\Delta y}\right) dy\, dx$$

$$= \frac{1}{24\Delta x^3} \Delta y \, \Delta x^2 (5\Delta x^2 - 16\Delta x^2 + 12\Delta x^2)$$

$$+ \frac{-1}{24\Delta x^3 \Delta y} (-\Delta x^4 \Delta y^2)$$

$$= \frac{\Delta y \, \Delta x^4}{24\Delta x^3} 1 + \frac{1}{24} \frac{\Delta x^4 \Delta y}{\Delta x^3} = \frac{2}{24} \Delta x \Delta y = \frac{1}{12} \Delta x \Delta y.$$ terms in E5.3.13



Let $W = 0$ on Dirichlet B.C. mult eq by W + integrate over entire domain

$$\frac{\partial \vec{U}}{\partial t} + \vec{\nabla}\cdot F = \vec{Q}$$

Shallow H_2O in 2D $\Rightarrow$

$$\begin{pmatrix} \rho \\ \rho U \\ \rho V \end{pmatrix}_t + \frac{\partial}{\partial x}\begin{pmatrix} \rho U \\ \rho U^2 \\ \rho UV \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho V \\ \rho UV \\ \rho V^2 \end{pmatrix} = 0$$

$$\int_\Omega \frac{\partial \vec{U}}{\partial t} w \, d\Omega + \int_\Omega (\vec{\nabla}\circ F) W \, d\Omega = \int_\Omega Q w \, d\Omega \qquad \text{I.B.P} \Rightarrow$$

$$\int_\Omega \frac{\partial \vec{U}}{\partial t} w + \oint_{\partial\Omega} W \vec{F}\circ\hat{n} \, d\Sigma - \int_\Omega \vec{F}\circ\nabla W \, d\Omega = \int_\Omega Q w \, d\Omega$$

$$= \int_{\Gamma_1} W \vec{F}\circ\hat{n} \, d\Sigma$$

let $W = N_J(\vec{x}) \Rightarrow$

$$\int_{\Omega_J} \frac{\partial \vec{U}}{\partial t} N_J(\vec{x}) \, d\Omega + \int_{\Gamma_1} N_J \vec{F}\circ\hat{n} \, d\Sigma - \int_{\Omega_J} \vec{F}\circ\nabla N_J \, d\Omega = \int_{\Omega_J} Q N_J \, d\Omega$$

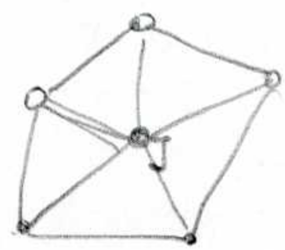
Let $\vec{U} = \sum_{I} \vec{U}_I(t) N_I(\vec{x})$ & $\vec{F} = \sum_{I} \vec{F}_I(t) N_I(\vec{x})$ then

get

$$\sum_I \frac{dU_I(t)}{dt} \int_{\Omega_J} N_I(\vec{x}) N_J(\vec{x})\, d\Omega + \underbrace{\sum_I \int_{\Gamma_1} N_I N_J\, \underbrace{F_I \cdot \hat{n}}\, d\Sigma}_{\int_{\Gamma_1} g N_J\, d\Gamma} - \sum_I \int_{\Omega_J} \vec{F}_I \cdot \nabla N_J\, N_I\, d\Omega = \int_{\Omega_J} Q N_J\, d\Omega$$

$$\Rightarrow \quad '' \quad + \quad '' \quad - \sum_I \vec{F}_I \cdot \int_{\Omega_J} \nabla N_J\, N_I\, d\Omega = \int_{\Omega_J}$$

define $M_{IJ} = \int_{\Omega_J} N_I N_J\, d\Omega \qquad \vec{k}_{IJ} = \int_{\Omega_J} N_I \nabla N_J\, d\Omega$



$$\Rightarrow \underbrace{\sum_{I \in \Omega_J} \frac{d\vec{U}_I}{dt} M_{IJ}}_{\text{5 terms + JJ term}} - \underbrace{\sum_{I \in \Omega_J} \vec{F}_I \cdot \vec{k}_{IJ}}_{\text{5 terms + JJ term}}$$

Now $\vec{k}_{IJ}$ is a ~~[illegible]~~ vector

J is the row of the matrix

$$[\vec{k}_{IJ}] = \Big(\underbrace{(\ )(\ )(\ ) \cdots}_{\text{columns representing matrix of vectors.}} \Big)$$

If the flux has a dissipative term $\Rightarrow$ $\vec{F} = \vec{F}_{convective} - K\vec{\nabla}U$ then

$$\frac{\partial U}{\partial t} + \nabla \cdot \vec{F} = 0$$

$$\Leftrightarrow \int_{\Omega} \frac{\partial U}{\partial t} W \, d\Omega - \underbrace{\int_{\Omega} (\vec{F} \cdot \vec{\nabla}) W \, d\Omega} + \int_{\Gamma} W \vec{F} \cdot d\vec{S} = \int_{\Omega} Q \, d\Omega$$

$$\left[\underbrace{(F_{conv} \cdot \nabla)} - (k \nabla U \cdot \nabla)\right] W$$

$$\sum_I N_I (F_{conv} \cdot \nabla) W - k \sum_I U_I (\nabla N_I \cdot \nabla) W$$

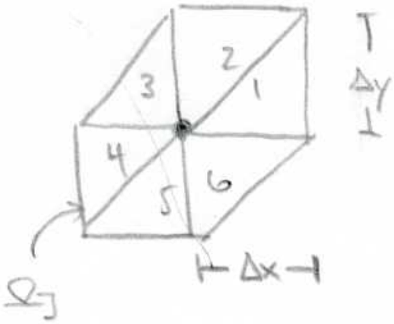
$$= \sum_I N_I (F_{I\,conv} \cdot \nabla) W -$$

we get an Additional term $- \sum_I U_I \int_{\Omega_J} k (\nabla N_I \cdot \nabla N_J) \, d\Omega$

& eq 5.3.17 would become

$$\sum_I \frac{dU_I}{dt} \int_{\Omega_J} N_I N_J \, d\Omega - \sum_I \vec{F}_I \cdot \int_{\Omega_J} N_I \nabla N_J \, d\Omega + \underbrace{\sum_I U_I \int_{\Omega_J} k (\nabla N_I \cdot \nabla N_J) \, d\Omega}_{\text{new term}}$$

$$= \int_{\Omega_J} Q N_J \, d\Omega - \int_{\Gamma_1} g N_J \, d\Gamma$$



$$\int_{\Omega_J} \vec{F} \circ \nabla N_J \, d\Omega = \int_{\Omega_J} \sum_I \vec{F}_I N_I \circ \bar{\nabla} N_J \, d\Omega$$

$$= \sum_I \vec{F}_I \circ \int_{\Omega_J} N_I \nabla N_J \, d\Omega$$

in 2D $\nabla = (\partial_x, \partial_y)$

& $(\vec{F}_I)_x = f$ component

$(F_I)_y = g$

$$= \sum_I f_I \int_{\Omega_J} N_I \frac{\partial N_J}{\partial x} \, d\Omega$$

$$+ \sum_I g_I \int_{\Omega_J} N_I \frac{\partial N_J}{\partial y} \, d\Omega$$

For Δ J12 $\quad \dfrac{\partial N_J}{\partial x} = a = \dfrac{(y_2 - y_3)}{2A_{12}}$ & $\dfrac{\partial N_J}{\partial y} = \dfrac{(x_3 - x_2)}{2A_{12}}$

For Δ w/ vertices labled 1,2,3 in the above Δ. our vertices go $J \Leftrightarrow 1$

$2 \Leftrightarrow 1$

$3 \Leftrightarrow 2$ &

The formula becomes : $\dfrac{\partial N_J}{\partial x} = \dfrac{(Y_1 - Y_2)}{2A_{12}}$ & $\dfrac{\partial N_J}{\partial y} = \dfrac{X_2 - X_1}{2A_{12}}$

Thus $\int_{\Omega_J} \bar{F} \circ \nabla N_J \, d\Omega = \sum_I f_I \frac{(y_1 - y_2)}{2A_{12}} \int_{\Omega_J} N_I \, d\Omega$

$+ \sum_I g_I \frac{(x_2 - x_1)}{2A_{12}} \int_{\Omega_J} N_I \, d\Omega$

Now $\int_{\Omega_J} N_I \, d\Omega = \int_{\Omega_{ref}} N_I(\vec{\xi}) \frac{d\Omega_{orig}}{d\Omega_{ref}} d\Omega_{ref} = \ldots$

From chart

$$\int_\Delta L_i' \, d\Omega = \frac{1}{3!} 2A = \frac{A}{3}.$$

$d\Omega_{ref} = \ldots$

$\frac{d\Omega_{orig}}{d\Omega_{ref}}$? ...

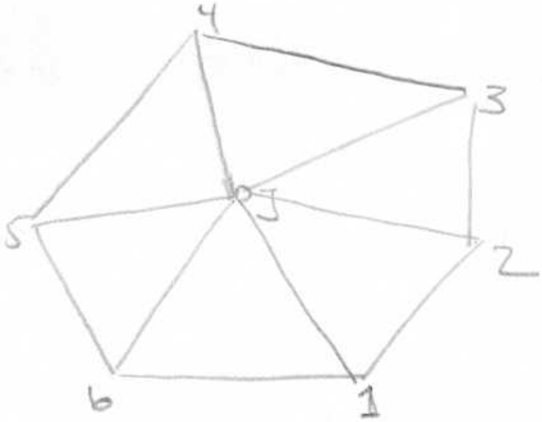
If integral is $\frac{A_{12}}{3}$ get

$$\int_{\Omega_J} \bar{F} \circ \nabla N_J \, d\Omega = \sum_I f_I \frac{(y_1 - y_2)}{2A_{12}} \frac{A_{12}}{3} + \sum_I g_I \frac{(x_2 - x_1)}{2A_{12}} \frac{A_{12}}{3}$$

$$\int_{J12} \bar{F} \circ \nabla N_J \, d\Omega = (y_1 - y_2) \frac{1}{6}(f_1 + f_2 + f_J) + \frac{1}{6}(x_2 - x_1)(g_1 + g_2 + g_J)$$

$f_1, f_2, f_J, g_1, g_2, + g_J$ defined as $x + y$ fluxes at the nodes of Δ J12.

$\sum$ All Δ's gives



$$\frac{1}{6}\cdot\Big[(y_1-y_2)(f_1+f_2+f_J)+(y_2-y_3)(f_2+f_3+f_J)+(y_3-y_4)(f_3+f_4+f_J)$$

$$+(y_4-y_5)(f_4+f_5+f_J)+(y_5-y_6)(f_5+f_6+f_J)+(y_6-y_1)(f_6+f_1+f_J)$$

$$+(x_2-x_1)(g_1+g_2+g_J)+(x_3-x_2)(g_2+g_3+g_J)+(x_4-x_3)(g_4+g_3+g_J)$$

$$+(x_5-x_4)(g_4+g_5+g_J)+(x_6-x_5)(g_5+g_6+g_J)+(x_1-x_6)(g_1+g_6+g_J)\Big]$$

$$=\frac{1}{6}\Big[f_J(\cancel{y_1}-y_2+y_2-y_3+y_3-y_4+y_4-y_5+y_5-y_6+y_6-\cancel{y_1})$$

$$+g_J(\cancel{x_2}-\cancel{x_1}+\cancel{x_3}-\cancel{x_2}+\cancel{x_4}-\cancel{x_3}+\cancel{x_5}-\cancel{x_4}+\cancel{x_6}-\cancel{x_5}+\cancel{x_1}-\cancel{x_6})$$

$$+f_1(y_1-y_2+y_6-y_1)+f_2(y_1-y_2+y_2-y_3)+f_3(y_2-y_3+y_3-y_4)$$

$$+f_4(y_3-y_4+y_4-y_5)+f_5(y_4-y_5+y_5-y_6)+f_6(y_5-y_6+y_6-y_1)$$

$$+g_1(x_2-x_1+x_1-x_6)+g_2(x_2-x_1+x_3-x_2)+g_3(x_3-x_2+x_4-x_3)$$

$$+g_4(x_4-x_3+x_5-x_4)+g_5(x_5-x_4+x_6-x_5)+g_6(x_6-x_5+x_1-x_6)\Big]$$

$$= \frac{1}{6}\Big[f_1(-y_2+y_6) + f_2(y_1-y_3) + f_3(y_2-y_4)$$

$$+ f_4(y_3-y_5) + f_5(y_4-y_6) + f_6(-y_1+y_5)$$

$$+ g_1(x_2-x_6) + g_2(-x_1+x_3) + g_3(-x_2+x_4)$$

$$+ g_4(-x_3+x_5) + g_5(-x_4+x_6) + g_6(x_1-x_5) \Big]$$

$$= \frac{1}{6}\Big[f_1(-y_2+y_6) + g_1(x_2-x_6)$$

$$+ f_2(y_1-y_3) - g_2(x_1-x_3)$$

$$+ f_3(y_2-y_4) - g_3(x_2-x_4)$$

$$+ f_4(y_3-y_5) - g_4(x_3-x_5)$$

$$+ f_5(y_4-y_6) - g_5(x_4-x_6)$$

$$+ f_6(-y_1+y_5) - g_6(-x_1+x_5) \Big] = \sum_{\text{All } \Delta\text{'s}} \int_{J12} \bar{F} \circ \nabla N_j \, d\Omega$$

in eq 5.3.17 need $-\sum_{\text{All } \Delta\text{'s}} \int_{J12} \bar{F} \circ \nabla N_j \, d\Omega$

E 5.3.17 is 

$$= (y_1 - y_2)\frac{1}{6}(f_1 + f_2 + f_J) - \frac{1}{6}(x_1 - x_2)(g_1 + g_2 + g_J)$$

$$= (y_1 - y_2)\frac{1}{3}\frac{(f_1 + f_2)}{2} + \frac{f_J}{6}(y_1 - y_2) - \frac{1}{3}(x_1 - x_2)\frac{(g_1 + g_2)}{2} - \frac{1}{6}g_J(x_1 - x_2)$$

$$\equiv -\frac{\Delta y_{12} f_{12}}{3} + \frac{\Delta x_{12}}{3} g_{12} - \frac{f_J \Delta y_{12}}{6} + \frac{g_J \Delta x_{12}}{6}$$

$$\therefore \; -\int_{\Omega_J} \vec{F} \cdot \vec{\nabla} N_J \, d\Omega = \frac{1}{3} \sum_{\substack{\text{All } \Delta\text{'s} \\ = \text{All sides}}} (\Delta y_{12} f_{12} - \Delta_{12} g_{12}) + \frac{1}{6} f_J \underbrace{\sum_{\substack{\text{All } \Delta\text{'s} \\ = \text{All sides}}} \Delta y_{12}}_{=0} - \frac{1}{6} g_J \underbrace{\sum_{\substack{\text{All } \Delta\text{'s} \\ = \text{All sides}}} \Delta x_{12}}_{=0}$$

change in y after one full circuit around the loop

change in x after one full circuit around the loop.

$$= \frac{1}{3} \sum_{\text{All sides}} (\Delta y_{12} f_{12} - \Delta_{12} g_{12})$$

5.3.17 is

negative sign comes from integration by parts.

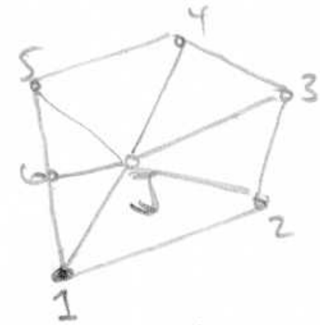
$$\sum_I \frac{dU_I}{dt} \int_{\Omega_J} N_I N_J \, d\Omega - \underbrace{\sum_I \vec{F}_I \cdot \int}_{\int_{\Omega_J} \vec{F} \cdot \vec{\nabla} N_J \, d\Omega} + \int_{\Gamma_1} g N_J \, d\Gamma = \int_{\Omega_J} Q N_J \, d\Omega$$

w/ Mass lumping:

$$\Rightarrow \sum_I \frac{dU_I}{dt} \frac{\Omega_J \delta_{IJ}}{3} + \frac{1}{3} \sum_{\text{sides}} (f_{12} \Delta y_{12} - \Delta x_{12} g_{12}) = 0$$

$$\frac{\Omega_J}{3}\frac{dU_J}{dt} + \frac{1}{3}\sum_{\text{sides}}\left(f_{12}\Delta y_{12} - \Delta x_{12}\, g_{12}\right) = 0$$

$$\Rightarrow \quad \Omega_J \frac{dU_J}{dt} + \sum_{\text{sides}}\left(f_{12}\Delta y_{12} - \Delta x_{12}\, g_{12}\right) = 0$$



From pg 4 these nodes one gets

$$\frac{\Omega_J}{3}\frac{dU_J}{dt} - \frac{1}{3}\cdot\frac{1}{2}\Big[f_1(-y_2+y_6) + f_2(-y_3+y_1) + f_3(-y_4+y_2)$$
$$+ f_4(-y_5+y_3) + f_5(-y_6+y_4) + f_6(-y_1+y_5)$$
$$+ g_1(x_2-x_6) + g_2(x_3-x_1) + g_3(x_4-x_2)$$
$$+ g_4(x_5-x_3) + g_5(x_6-x_4) + g_6(x_1-x_5)\Big] = 0$$

$$\Rightarrow \quad \Omega_J \frac{dU_J}{dt} - \frac{1}{2}\left[\sum_{I=1}^{6}\left[f_I(-y_{I+1}+y_{I-1}) + g_I(x_{I+1}-x_{I-1})\right]\right]$$

w/ $y_0 = y_6$, $y_7 = y_1$, $x_0 = x_6$, $x_7 = x_1$

or

$$\Omega_J \frac{dU_J}{dt} + \frac{1}{2}\sum_{I=1}^{6}\left[f_I(y_{I+1}-y_{I-1}) - g_I(x_{I+1}-x_{I-1})\right] \quad \text{ES.3.22}$$

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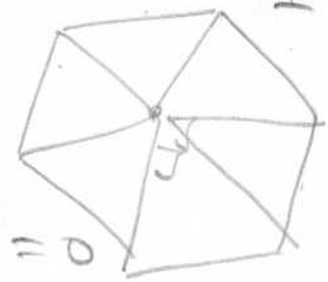


$$\Omega_J \frac{du_J}{dt} + \frac{1}{2}\sum_{I=1}^{6} f_I(y_{I+1}-y_{I-1}) - g_I(x_{I+1}-x_{I-1}) = 0$$

$$\Rightarrow \Omega_J \frac{du_J}{dt} + \frac{1}{2}\sum_{I=1}^{6}(f_I - f_J) \;\ldots$$

if $$\sum_{I=1}^{6} f_J(y_{I+1}-y_{I-1}) = 0$$

$$= f_J \sum_{I=1}^{6} y_{I+1} - y_I + y_I - y_{I-1}$$

$$= \;'' \sum_{I=1}^{6} \Delta y_I + \Delta y_{I-1} = y_I \Big|_1^{6+1} + y_{I-1} \Big|_1^{6+1}$$

$$= \overset{=0}{y_7 - y_1} + \overset{=0}{y_6 - y_0}$$

$$\cancel{y_2} - y_1 + \cancel{y_3} - \cancel{y_2} + \cancel{y_4} - \cancel{y_3} + \cancel{y_5} - \cancel{y_4} + \cancel{y_6} - \cancel{y_5} + y_7 - \cancel{y_6}$$

$$y_7 - y_1 = 0$$

Then

$$\Omega_J \frac{dw_J}{dt} + \frac{1}{2}\sum_{I=1}^{6} (f_I - f_J)(y_{I+1} - y_{I-1}) - (g_I - g_J)(x_{I+1} - x_{I-1}) = 0$$

All fluxes are relative to flux at center

Back to eq E 5.3.22

$$\Rightarrow \Omega_J \frac{dw_J}{dt} + \frac{1}{2}\sum_{I=1}^{6} f_I y_{I+1} - \frac{1}{2}\sum_{I=1}^{6} f_I y_{I-1}$$

$$- \frac{1}{2}\sum_{I=1}^{6} g_I x_{I+1} + \frac{1}{2}\sum_{I=1}^{6} g_I x_{I-1} = 0$$

$$\Rightarrow \Omega_J \frac{dw_J}{dt} + \frac{1}{2}\sum_{I=2}^{7} f_{I-1} y_I - \frac{1}{2}\sum_{I=0}^{5} f_{I+1} y_I$$

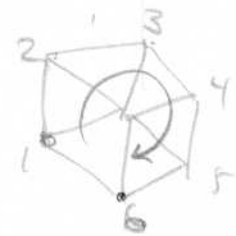
$$- \frac{1}{2}\sum_{I=} g_{I-1} x_I \quad \ldots = 0$$ same as for f w/ Negative sign

Now

$y_7 = y_1$

$$\Rightarrow \Omega_J \frac{dw_J}{dt} + \frac{1}{2} f_6 y_7 + \sum_{I=2}^{6} f_{I-1} y_I$$

$$- \frac{1}{2} f_1 y_0 - \frac{1}{2}\sum_{I=1}^{5} f_{I+1} y_I + \ldots$$ Same for g



$$\Rightarrow \quad \Omega_J \frac{dU_J}{dt} + \frac{1}{2} f_6 Y_1 + \frac{1}{2}\sum_{I=2}^{6} f_{I-1} Y_I$$

$$- \frac{1}{2} f_1 Y_2 - \frac{1}{2} \sum_{I=1}^{5} f_{I+1} Y_I + \cdots \quad = 0$$

$$+ \frac{1}{2} f_6 Y_1 + \frac{1}{2} \sum_{I=1}^{6} f_{I-1} Y_I - \frac{1}{2} f_0 Y_1$$

$$- \frac{1}{2} f_1 Y_2 - \frac{1}{2} \sum_{I=1}^{6} f_{I+1} Y_I + \frac{1}{2} f_7 Y_6 + \cdots = 0$$

define $f_0 \equiv f_6$ ✓ & $f_7 = f_1$

$$\Rightarrow \quad \Omega_J \frac{dU_J}{dt} + \frac{1}{2} \sum_{I=1}^{6} \left[(f_{I-1} - f_{I+1}) Y_I - (g_{I-1} - g_{I+1}) X_I \right] = 0$$

$$- \frac{1}{2} \sum_{I=1}^{6} \left[(f_{I+1} - f_{I-1}) Y_I - (g_{I+1} - \right.$$

$\Rightarrow$ Adding Y_J As $\displaystyle\sum_{I=1}^{6} Y_J (f_{I+1} - f_{I-1}) = Y_J \sum_{1}^{6} (f_{I+1} - f_{I-1})$

$$= 0$$

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$$\bar{\bar{J}} = \frac{\partial(\xi,\eta)}{\partial(x,y)} \qquad \bar{\bar{J}}^{-1} = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{pmatrix} = \frac{1}{\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right)} \begin{pmatrix} \frac{\partial \eta}{\partial y} & -\frac{\partial \xi}{\partial y} \\ -\frac{\partial \eta}{\partial x} & \frac{\partial \xi}{\partial x} \end{pmatrix}$$

From $\quad \bar{x} = \sum_I \bar{x}_I N_I(\bar{\xi})$

$$\bar{\bar{J}}^{-1} = ? \qquad \frac{\partial x}{\partial \xi} = \sum_I x_I \frac{\partial N_I(\xi,\eta)}{\partial \xi} \qquad \frac{\partial x}{\partial \eta} = \sum_I x_I \frac{\partial N_I(\xi,\eta)}{\partial \eta}$$

$$\frac{\partial y}{\partial \xi} = \sum_I y_I \frac{\partial N_I(\xi,\eta)}{\partial \xi} \qquad \frac{\partial y}{\partial \eta} = \sum y_I \frac{\partial N_I(\xi,\eta)}{\partial \eta}$$

$$\bar{\bar{J}}^{-1} = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$\bar{\bar{J}} = J \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{pmatrix} \quad w/ \quad J = \frac{1}{|\bar{\bar{J}}^{-1}|}$$

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If $\nabla_x N_I(\bar{x}) = \bar{\bar{J}}\ \nabla_\xi N_I(\bar{\xi})$

$$\Rightarrow \begin{pmatrix} \dfrac{\partial N_I}{\partial x} \\ \\ \dfrac{\partial N_I}{\partial y} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{pmatrix}}_{\bar{\bar{J}}} \begin{pmatrix} \dfrac{\partial N_I}{\partial \xi} \\ \\ \dfrac{\partial N_I}{\partial \eta} \end{pmatrix}$$

Then $\bar{\bar{J}}^{-1}$ = inverse transformation from (ξ, η) space to (x, y) space

$$\Rightarrow \bar{\bar{J}}^{-1} = \begin{pmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{pmatrix}$$

Then
$$\left.\begin{aligned} \frac{\partial x}{\partial \xi} &= \sum_I x_I \frac{\partial N_I}{\partial \xi} & \frac{\partial y}{\partial \xi} &= \sum_I y_I \frac{\partial N_I}{\partial \xi} \\ \frac{\partial x}{\partial \eta} &= \sum_I x_I \frac{\partial N_I}{\partial \eta} & \frac{\partial y}{\partial \eta} &= \sum_I y_I \frac{\partial N_I}{\partial \eta} \end{aligned}\right\} \text{known \& thus } \bar{\bar{J}}^{-1} \text{ is known}$$

Then $\bar{\bar{J}}$ = inverse of $\bar{\bar{J}}^{-1}$

$$= \frac{1}{|\bar{\bar{J}}^{-1}|} \begin{pmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{pmatrix} = J \begin{pmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{pmatrix}$$

w/ $J = \frac{1}{|J^{-1}|}$

$$\frac{\partial N_I}{\partial x} = \frac{\partial N_I}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_I}{\partial \eta}\frac{\partial \eta}{\partial x}$$

= From Matching sympolic meaning of $\bar{\bar{J}}$ =

$$\bar{\bar{J}} = \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} = J \begin{pmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{pmatrix} \stackrel{sym}{=} \frac{\partial(\xi,\eta)}{\partial(x,y)}$$

w/ computed value

$$\Rightarrow \frac{\partial N_I}{\partial x} = J\left(\frac{\partial N_I}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial N_I}{\partial \eta}\frac{\partial y}{\partial \xi}\right) \quad \text{eq } 5.4.7$$

$$\& \quad \frac{\partial N_I}{\partial y} = \frac{\partial N_I}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial N_I}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}$$

$$= J\left(-\frac{\partial N_I}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial N_I}{\partial \eta}\frac{\partial x}{\partial \xi}\right)$$

$$\Rightarrow dx\,dy = \frac{1}{J}\,d\xi\,d\eta$$

$$k_{IJ} = \int_\Omega \nabla_x N_I \circ \nabla_x N_J \, dx\,dy = \int \underbrace{\underline{\underline{J}}}_{\in \mathbb{R}^{2\times 2}} \, \underbrace{\nabla_\xi N_I}_{\in \mathbb{R}^2} \circ \underline{\underline{J}} \nabla_\xi N_J \frac{1}{J} d\xi d\eta$$

$\underline{\underline{J}} \nabla_\xi N_I \in \mathbb{R}^2$

$v^T w = (x\,x\,x)\begin{pmatrix} x \\ x \\ x \end{pmatrix}$

$$= \int (\underline{\underline{J}} \nabla_\xi N_I)^T \underline{\underline{J}} \nabla_\xi N_J \frac{1}{J} d\xi d\eta$$

$$= \int (\nabla_\xi N_I)^T \underbrace{\underline{\underline{J}}^T \underline{\underline{J}}}_{\text{matrix product}} \nabla_\xi N_J \frac{1}{J} d\xi d\eta$$

$$= \int (\partial_\xi N_I, \partial_\eta N_I) \underbrace{\underline{\underline{J}}^T \underline{\underline{J}}}_{} \begin{pmatrix} \partial_\xi N_J \\ \partial_\eta N_J \end{pmatrix} \frac{1}{J} d\xi d\eta$$

$$\cdot \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} \partial_\xi N_J \\ \partial_\eta N_J \end{pmatrix}$$

$$= \int (\partial_\xi N_I, \partial_\eta N_I) \begin{pmatrix} g_{11} \partial_\xi N_J + g_{12} \partial_\eta N_J \\ g_{21} \partial_\xi N_J + g_{22} \partial_\eta N_J \end{pmatrix} \frac{1}{J} d\xi d\eta$$

$$= \int \left(\partial_\xi N_I \, g_{11} \, \partial_\xi N_J + \partial_\xi N_I \, g_{12} \, \partial_\eta N_J + \partial_\xi N_I \, g_{21} \, \partial_\xi N_J + \partial_\eta N_I \, g_{22} \, \partial_\eta N_J \right)$$

$$\cdot \frac{1}{J} d\xi d\eta$$

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$$\int_{-1}^{1} f(\xi)\,d\xi = \sum_{i=1}^{n} H_i f(\xi_i)$$

$n=1$ does ~~0th~~ $\Rightarrow 1$ order polynomials (constants) exactly

$n=2$ ~~1st~~ $\Rightarrow 3$ " "

3 ~~2nd~~ 5

4 ~~3rd~~ 7

$n=n$ $2n-1$ order polys exactly.

$$\int_{-1}^{1} (ax+b)\,dx = \frac{ax^2}{2} + bx \Big|_{-1}^{1} = \cancel{a}^{\,0}$$

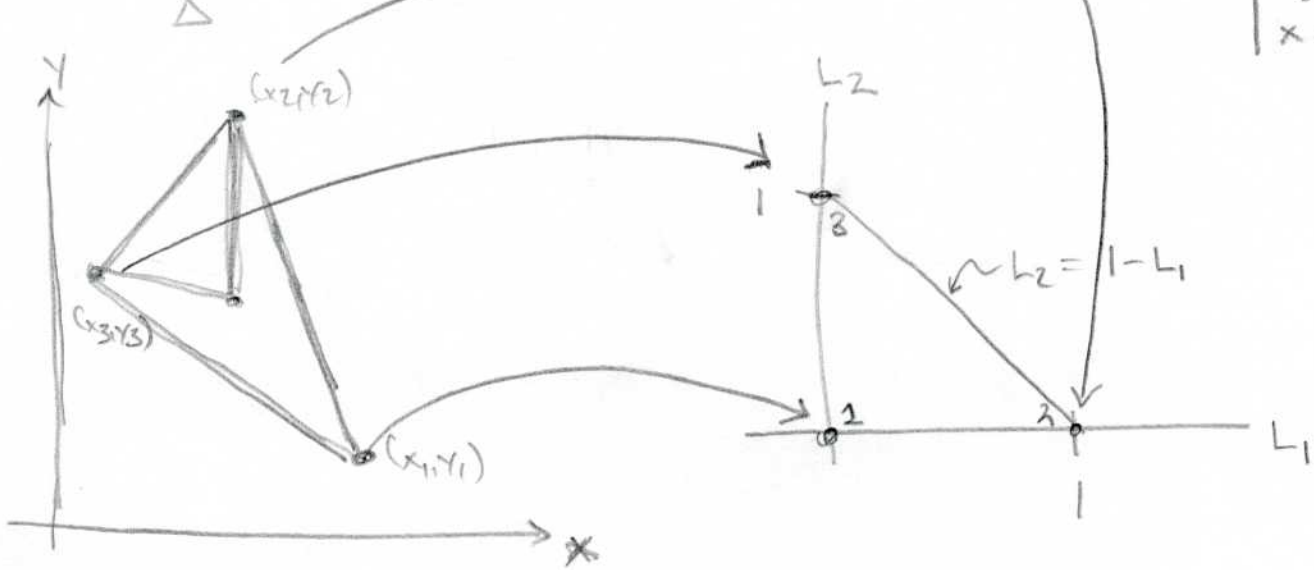
on interval $-1,+1$ every odd power vanishes thus get corresponding odd powers for free

$$= \int$$

$$\int_{\Delta} f(x,y)\,d\Omega$$

$$L_1 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix}$$

$$L_2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix}$$



$$d\Omega = \underbrace{\frac{\partial(x,y)}{\partial(L_1,L_2)}}\, dL_1\, dL_2$$ call work out ...

Estimating $\frac{1}{J}$ $\frac{1}{J}$ from

$dL_1 = 1$, $dL_2 = 1$

$d\Omega = A$

$A = \frac{1}{J} \Rightarrow J = \frac{1}{A}$.

$\therefore$ checking ... get $x = x(L_1, L_2)$
$y = y(L_1, L_2)$.

$$L_1 = \frac{1}{2}\left[x \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} - y \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right]$$

$$L_2 = \frac{1}{2}\left[x \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} - y \begin{vmatrix} x_1 & 1 \\ x_3 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} \right]$$

$$\Leftrightarrow \quad L_1 = \frac{1}{2}\left(c_1 x - c_2 y + c_3 \right)$$

$$L_2 = \frac{1}{2}\left(c_4 x - c_5 y + c_6 \right)$$

Mult 1st eq by $-c_4$ + 2nd by c_1 + Add

$$\Rightarrow \quad -c_4 L_1 + c_1 L_2 = \frac{1}{2}(c_2 c_4 - c_5 c_1) y + \frac{1}{2}(-c_3 c_4 + c_1 c_6)$$

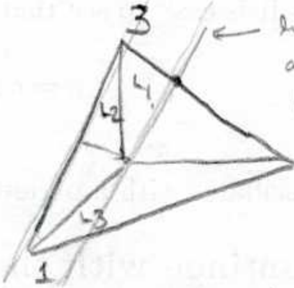
$$\Rightarrow \quad y = \frac{2(-c_4 L_1 + c_1 L_2)}{(c_2 c_4 - c_1 c_5)} + \frac{(c_3 c_4 - c_1 c_6)}{(c_2 c_4 - c_1 c_5)}$$

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{+1}\sum_{i=1}^{n} H_i f(\xi_i,\eta)\,d\eta$$

$$= \sum_{i=1}^{n} H_i \int_{-1}^{+1} f(\xi_i,\eta)\,d\eta$$

$$= \sum_{i=1}^{n} H_i \sum_{j=1}^{n} H_j f(\xi_i,\eta_j)$$

$$= \sum_{ij=1}^{n} H_i H_j f(\xi_i,\eta_j) \qquad \text{eq} \quad 5.4.12$$



← line ∥ to side 1-3 at a given value of L_2. Then limit of L_1 should be from side 2-3 to side 1-2

For Δ coord

$$\int f(L_1,L_2,L_3)\,d\Omega = \iint$$

to integrate over entire Δ here

$0 < L_2 < 1$ while $0 < L_1 < 1-L_2$

$$d\Omega = |G|\,dL_1\,dL_2$$

$$G = \frac{\partial(x,y)}{\partial(L_1,L_2)} \qquad x = x(L_1,L_2) = x_1L_1 + x_2L_2 + x_3(1-L_1-L_2)$$
$$y = y(L_1,L_2) = y_1L_1 + y_2L_2 + y_3(1-L_1-L_2)$$

at side 2-3 $L_1 = 0$

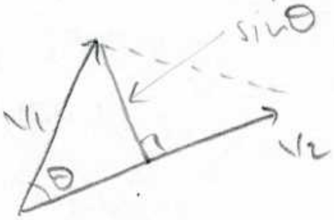
at side 1-2 $L_1 = 1-L_2$

see next pg

$$\frac{\partial(x,y)}{\partial(L_1,L_2)} = \begin{pmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial y}{\partial L_1} \\ \frac{\partial x}{\partial L_2} & \frac{\partial y}{\partial L_2} \end{pmatrix} = \begin{pmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{pmatrix} \Rightarrow \left|\frac{\partial(x,y)}{\partial(L_1,L_2)}\right| =$$

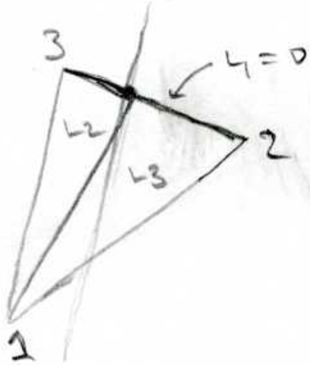
$$(x_1-x_3)(y_2-y_3) - (x_2-x_3)(y_1-y_3)$$
$$\| $$
$$2A$$

Now Area of triangle Ω =

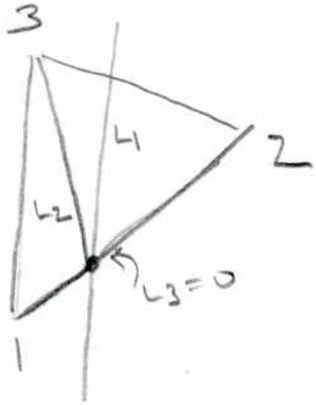


$$|\bar{v}_1 \times \bar{v}_2| = |v_1||v_2|\sin\theta = 2\,\text{Area of}\ \Delta$$

when L_1 is at side 2-3 Δ looks like



When L_1 is on side 1-2 Δ looks like



Thus $L_1 + L_2 = 1$ in this case & $L_1 = 1 - L_2$

Thus $\vec{V}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

$\vec{V}_2 = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j}$

$$\therefore \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \hat{k}\left((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right)$$

$x_2 \to x_1 \quad y_3 \to y_2$

$x_1 \to x_3 \quad y_1 \to y_3$

$x_3 \to x_2 \quad y_2 \to y_1$

$x_1 \to x_3 \quad y_1 \to y_3$

$x_2 \to x_1 \quad y_2 \to y_1$

$x_3 \to x_2 \quad y_3 \to y_2$

$$\therefore \iint_\Omega f(x,y)\,d\Omega = \int_0^1 \int_0^{1-L_2} f(L_1, L_2, L_3)\,dL_1\,dL_2 \cdot 2A$$

$$= 2A \int_0^1 \int_0^{1-L_2} f(L_1, L_2, L_3)\,dL_1\,dL_2$$

or eq 5.4.13

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$$\int L_1^m L_2^n L_3^p \, d\Omega \underset{\uparrow}{=} 2A \int_0^1 \int_0^{1-L_2} L_1^m L_2^n (1-L_1-L_2)^p \, dL_1 dL_2$$

By 5.4.13

$$= 2A \int_0^1 L_2^n \int_0^{1-L_2} L_1^m (1-L_1-L_2)^p \, dL_1 \quad dL_2$$

$$\left\{ \text{Beta Fn } B(p,q) = \int_0^1 t^p (1-t)^q dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q+1)} \right.$$

Check.

Inner integral is $\int_0^{1-L_2} L_1^m (1-L_2-L_1)^p \, dL_1 \qquad p$ integer ≥ 0

let $v = \dfrac{L_1}{1-L_2}$ $\qquad v(L_1=0) = 0$

$\qquad\qquad\qquad\qquad v(L_1 = 1-L_2) = 1$

$L_1 = (1-L_2)v$

Then $\;$ & $dL_1 = (1-L_2)dv$ & $1-L_2-L_1 = 1-L_2-(1-L_2)v = (1-L_2)(1-v)$

$$= \int_0^1 (1-L_2)^m v^m (1-L_2)^p (1-v)^p (1-L_2) \, dv$$

$$= (1-L_2)^{m+p+1} \int_0^1 v^m (1-v)^p dv$$

let $x = v - \frac{1}{2}$ $\quad v = x + 1/2$

$dx = dv \qquad 1 - v = 1 - x - 1/2$

$\qquad\qquad\qquad = 1/2 - x$

$$= (1-L_2)^{m+p+1} \int_{-1/2}^{1/2} (x+1/2)^m (1/2-x)^p dv$$ now its symmetric

Do just $\int_{-1/2}^{1/2} (x+1/2)^m (1/2-x)^p \, dx$ part.

Assume $m \geq p \Rightarrow m = p + r \quad r \geq 0$

$$= \int_{-1/2}^{1/2} (1/2 + x)^{p+r} (1/2 - x)^p \, dx = \int_{-1/2}^{1/2} (1/4 - x^2)^p (1/2 + x)^r \, dx$$

$v = \frac{1}{4} - x^2$

$dv = -2x \, dx \quad \ldots$ How Finish?

For tetrahedral elements: $L_4 = 1 - L_1 - L_2 - L_3$

$\int L_1^m L_2^n L_3^p L_4^q \, d\Omega = \quad + \; d\Omega = dx\,dy\,dz$. How compute?

Prob 5.1 For $j = i$ $\quad N_i(x) = 1 - \xi^2$

Then $$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{1}{\sum_k x_k \frac{\partial N_k}{\partial \xi}} = -2\xi \frac{1}{\xi(\Delta x_{i+1} - \Delta x_i) + \frac{1}{2}(\Delta x_{i+1} + \Delta x_i)}$$

$$\frac{\partial N_i}{\partial x} = \frac{-4\xi}{2\xi(\Delta x_{i+1} - \Delta x_i) + \Delta x_{i+1} + \Delta x_i}$$

$$\left.\frac{\partial N_i}{\partial x}\right|_{i-1} = \left.\frac{-4\xi}{2\xi(\Delta x_{i+1} - \Delta x_i) + \Delta x_{i+1} + \Delta x_i}\right|_{\xi=-1} = \frac{4}{-2\Delta x_{i+1} + 2\Delta x_i + \Delta x_{i+1} + \Delta x_i}$$

$$= \frac{4}{-\Delta x_{i+1} + 3\Delta x_i} = \frac{-4}{\Delta x_{i+1} - 3\Delta x_i}$$

$$\left.\frac{\partial N_i}{\partial x}\right|_{i} = \text{"} \Big|_{\xi=0} = 0$$

$$\left.\frac{\partial N_i}{\partial x}\right|_{i+1} = \text{"} \Big|_{\xi=+1} = \frac{-4}{3\Delta x_{i+1} - \Delta x_i}$$

For $j = i+1$ $\quad N_j(\xi) = \frac{1}{2}\xi(1+\xi)$

$$\frac{\partial N_{i+1}}{\partial x} = \frac{\partial N_{i+1}}{\partial \xi} \frac{1}{\sum_k x_k \frac{\partial N_k}{\partial \xi}} = \frac{\left(\frac{1}{2}(1+\xi) + \frac{1}{2}\xi\right)}{\xi(\Delta x_{i+1} - \Delta x_i) + \frac{1}{2}(\Delta x_{i+1} + \Delta x_i)}$$

$$\frac{\partial N_{i+1}}{\partial x} = \frac{1+2\xi}{2\xi(\Delta x_{i+1} - \Delta x_i) + (\Delta x_{i+1} + \Delta x_i)}$$

$$\left.\frac{\partial N_{i+1}}{\partial x}\right|_{i-1} = \left.''\ \right|_{\xi=-1} = \frac{-1}{-\Delta x_{i+1} + 3\Delta x_i} = \frac{1}{\Delta x_{i+1} - 3\Delta x_i}$$

$$\left.\frac{\partial N_{i+1}}{\partial x}\right|_{i} = \left.''\ \right|_{\xi=0} = \frac{1}{\Delta x_{i+1} + \Delta x_i}$$

$$\left.\frac{\partial N_{i+1}}{\partial x}\right|_{i+1} = \left.''\ \right|_{\xi=+1} = \frac{3}{3\Delta x_{i+1} - \Delta x_i}$$

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Prob 5.2

$$u_x = u_{i-1}\frac{\partial N_{i-1}}{\partial x} + u_i \frac{\partial N_i}{\partial x} + u_{i+1}\frac{\partial N_{i+1}}{\partial x}$$

$$(u_x)_{i+1} = u_{i-1}\left.\frac{\partial N_{i-1}}{\partial x}\right|_{i+1} + u_i \left.\frac{\partial N_i}{\partial x}\right|_{i+1} + u_{i+1}\left.\frac{\partial N_{i+1}}{\partial x}\right|_{i+1}$$ From Prob 5.1

$$= \frac{u_{i-1}}{3\Delta x_{i+1} - \Delta x_i} + \frac{-4u_i}{3\Delta x_{i+1} - \Delta x_i} + \frac{3u_{i+1}}{3\Delta x_{i+1} - \Delta x_i}$$

$$= \frac{u_{i-1} - 4u_i + 3u_{i+1}}{3\Delta x_{i+1} - \Delta x_i}$$ eq 4.2.21 on uniform grid or Backwards differencing.

$$(u_x)_i = \frac{-u_{i-1}}{\Delta x_{i+1} + \Delta x_i} + 0 + \frac{u_{i+1}}{\Delta x_{i+1} + \Delta x_i} = \frac{u_{i+1} - u_{i-1}}{\Delta x_{i+1} + \Delta x_i}$$ eq 4.2.31 Central differencing

$$(u_x)_{i-1} = \frac{-3u_{i-1}}{3\Delta x_i - \Delta x_{i+1}} + \frac{4u_i}{3\Delta x_i - \Delta x_{i+1}} - \frac{u_{i+1}}{3\Delta x_i + \Delta x_{i+1}}$$

$$= \frac{-3u_{i-1} + 4u_i - u_{i+1}}{3\Delta x_i - \Delta x_{i+1}}$$ eq 4.2.19 on uniform grid or forward differencing

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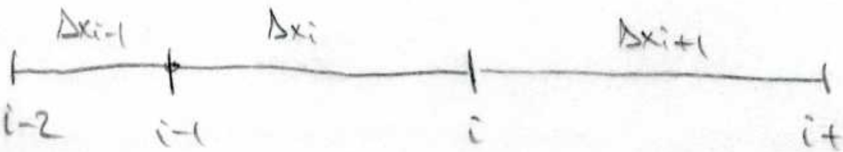

expand every term about the point where its derivative is evaluated

$$(u_x)_{i+1} = \frac{u_{i-1} - 4u_i + 3u_{i+1}}{3\Delta x_{i+1} - \Delta x_i}$$ expand about x_{i+1}

$$\Leftrightarrow (u_x)_i = \frac{u_{i-2} - 4u_{i-1} + 3u_i}{3\Delta x_i - \Delta x_{i-1}}$$ expand about x_i

$$u_{i-2} \equiv u(x_i - \Delta x_i - \Delta x_{i-1}) = u(x_i - (\Delta x_i + \Delta x_{i-1}))$$

$$= u(x_i) - u'(x_i)(\Delta x_i + \Delta x_{i-1}) + \frac{u''(x_i)}{2}(\Delta x_i + \Delta x_{i-1})^2$$

$$- \frac{u'''(x_i)}{6}(\Delta x_i + \Delta x_{i-1})^3 + O((\Delta x_i + \Delta x_{i-1})^4)$$

$$\text{\&}\ u_{i-1} \equiv u(x_i - \Delta x_i) = u(x_i) - u'(x_i)\Delta x_i + \frac{u_i''}{2}\Delta x_i^2$$

$$- \frac{u_i'''}{6}\Delta x_i^3 + O(\Delta x_i^4)$$

Then $(u_x)_i$ Above becomes

$$(u_x)_i = \frac{\begin{aligned} &\cancel{u_i} - u_i'(\Delta x_i + \Delta x_{i-1}) + u_i''\frac{(\Delta x_i + \Delta x_{i-1})^2}{2} - \frac{u_i'''}{6}(\Delta x_i + \Delta x_{i-1})^3 + O(\ ) \\ &- 4\cancel{u_i} + 4u_i'\Delta x_i - \frac{4u_i''}{2}\Delta x_i^2 + \frac{4u_i'''}{6}\Delta x_i^3 + O(\Delta x^4) \\ &+ \cancel{3}u_i \end{aligned}}{(3\Delta x_i - \Delta x_{i-1})}$$

$$(u_x)_i = \frac{-\Delta x_i u_i' - \Delta x_{i-1} u_i' + \frac{u_i''}{2}(\Delta x_i^2 + 2\Delta x_i \Delta x_{i-1} + \Delta x_{i-1}^2) - \frac{u_i'''}{6}(\Delta x_i + \Delta x_{i-1})^3 + O((\Delta x_i + \Delta x_{i-1})^4) + 4u_i'\Delta x_i - 4\frac{u_i''}{2}\Delta x_i^2 + \frac{4}{6}u_i'''\Delta x_i^3 + O(\Delta x_i^4)}{3\Delta x_i - \Delta x_{i-1}}$$

$$= \left(\frac{3\Delta x_i - \Delta x_{i-1}}{3\Delta x_i - \Delta x_{i-1}}\right) u_i' +$$

$$\frac{u_i''}{2}\left\{\frac{\Delta x_i^2 + 2\Delta x_i \Delta x_{i-1} + \Delta x_{i-1}^2 - 4\Delta x_i^2}{3\Delta x_i - \Delta x_{i-1}}\right\} +$$

$$= u_i' + \frac{u_i''}{2}\left\{\frac{-3\Delta x_i^2 + 2\Delta x_i \Delta x_{i-1} + \Delta x_{i-1}^2}{3\Delta x_i - \Delta x_{i-1}}\right\} +$$

$$+ \frac{u_i'''}{6}\left\{\frac{-(\Delta x_i + \Delta x_{i-1})^3 + 4\Delta x_i^3}{3\Delta x_i - \Delta x_{i-1}}\right\} + O\left(\frac{(\Delta x_i + \Delta x_{i-1})^4}{3\Delta x_i - \Delta x_{i-1}}\right) + O\left(\frac{\Delta x_i^4}{3\Delta x_i - \Delta x_{i-1}}\right)$$

if 2nd order terms are to vanish one must have

$$-3\Delta x_i^2 + 2\Delta x_i \Delta x_{i-1} + \Delta x_{i-1}^2 = 0$$

$$\Rightarrow 3\left(\frac{\Delta x_i}{\Delta x_{i-1}}\right)^2 - 2\left(\frac{\Delta x_i}{\Delta x_{i-1}}\right) - 1 = 0 \qquad \lambda = \frac{\Delta x_i}{\Delta x_{i-1}}$$

$$(3\lambda + 1)(\lambda - 1) = 0$$

$$\Rightarrow \lambda = -1/3 \quad \text{or} \quad \boxed{\lambda = 1} \quad \text{equivalent spacing.}$$

Does the $O(\Delta x^3)$ term vanish as well? Put $\Delta x = \Delta x_i = \Delta x_{i+1}$

$$-8\Delta x^3 + 4\Delta x_i^3 \neq 0 \quad \text{No.}$$

For $(u_x)_i = \dfrac{u_{i+1} - u_{i-1}}{\Delta x_{i+1} + \Delta x_i}$ expand u about x_i

$$= \frac{1}{\Delta x_{i+1} + \Delta x_i}\Big[\cancel{u_i} + \Delta x_{i+1} u_i' + \frac{\Delta x_{i+1}^2}{2} u_i'' + \frac{\Delta x_{i+1}^3}{6} u_i''' + O(\Delta x_{i+1}^4)$$

$$- \cancel{u_i} + \Delta x_i u_i' - \frac{\Delta x_i^2}{2} u_i'' + \frac{\Delta x_i^3}{6} u_i''' + O(\Delta x_i^4)\Big]$$

$$= u_i' + \frac{1}{2}\left(\frac{\Delta x_{i+1}^2 - \Delta x_i^2}{\Delta x_{i+1} + \Delta x_i}\right) u_i'' + \frac{1}{6}\left(\frac{\Delta x_{i+1}^3 + \Delta x_i^3}{\Delta x_{i+1} + \Delta x_i}\right) u_i''' + O(\Delta x_i^3, \Delta x_{i+1}^3)$$

$$= u_i' + \underbrace{(\Delta x_{i+1} - \Delta x_i)}_{=0} u_i'' + \frac{1}{6}\big(\Delta x_{i+1}^2 - \Delta x_{i+1}\Delta x_i + \Delta x_i^2\big) u_i''' + O(\;)$$

check.

$= 0$ & order is 2nd order if grid is uniform.

For 3rd formula (shifted)

$$(u_x)_i = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{3\Delta x_{i+1} - \Delta x_{i+2}}$$ taylor expanding gives

$$u_{i+1} = u(x_i + \Delta x_{i+1}) = u(x_i) + \Delta x_{i+1}(u_x)_i + \frac{\Delta x_{i+1}^2}{2}(u_{xx})_i + \frac{\Delta x_{i+1}^3}{6}(u_{xxx})_i$$

$$+ \frac{\Delta x_{i+1}^4}{24}(u_{xxxx})_i + O(\Delta x_{i+1}^5)$$

$$+\ u_{i+2} = u(x_i + \Delta x_{i+1} + \Delta x_{i+2})$$

$$= u(x_i) + (\Delta x_{i+1} + \Delta x_{i+2})(u_x)_i + \frac{(\Delta x_{i+1} + \Delta x_{i+2})^2}{2}(u_{xx})_i +$$

$$+ \frac{(\Delta x_{i+1} + \Delta x_{i+2})^3}{6}(u_{xxx})_i + \frac{(\Delta x_{i+1} + \Delta x_{i+2})^4}{24}(u_{xxxx})_i$$

$$+ O((\Delta x_{i+1} + \Delta x_{i+2})^5)$$

$$\therefore\ -3u_i + 4u_{i+1} - u_{i+2} = -\cancel{3u_i} + \cancel{4u_i} + 4\Delta x_{i+1}(u_x)_i + 4\frac{\Delta x_{i+1}^2}{2}(u_{xx})_i$$

$$+ 4\frac{\Delta x_{i+1}^3}{6}(u_{xxx})_i + 4\frac{\Delta x_{i+1}^4}{24}(u_{xxxx})_i + O(\Delta x_{i+1}^5)$$

$$-\cancel{u_i} - (\Delta x_{i+1} + \Delta x_{i+2})(u_x)_i - \frac{(\Delta x_{i+1} + \Delta x_{i+2})^2}{2}(u_{xx})_i$$

$$- \frac{(\Delta x_{i+1} + \Delta x_{i+2})^3}{6}(u_{xxx})_i - \frac{(\Delta x_{i+1} + \Delta x_{i+2})^4}{24}(u_{xxxx})_i$$

$$+ O((\Delta x_{i+1} + \Delta x_{i+2})^5)$$

$$= (3\Delta x_{i+1} - \Delta x_{i+2})(u_x)_i + \frac{1}{2}(4\Delta x_{i+1}^2 - (\Delta x_{i+1} + \Delta x_{i+2})^2)(u_{xx})_i$$

$$+ \frac{1}{6}(4\Delta x_{i+1}^3 - (\Delta x_{i+1} + \Delta x_{i+2})^3)(u_{xxx})_i + O(\Delta x_{i+1}^4, \Delta x_{i+2}^4)$$

$$\therefore\ \frac{-3u_i + 4u_{i+1} - u_{i+2}}{3\Delta x_{i+1} - \Delta x_{i+2}} = (u_x)_i + \frac{1}{2}\frac{(3\Delta x_{i+1}^2 - 2\Delta x_{i+1}\Delta x_{i+2} - \Delta x_{i+2}^2)}{3\Delta x_{i+1} - \Delta x_{i+2}}(u_{xx})_i$$

$$+ \frac{1}{6}\frac{(4\Delta x_{i+1}^3 - (\Delta x_{i+1} + \Delta x_{i+2})^3)}{3\Delta x_{i+1} - \Delta x_{i+2}}(u_{xxx})_i$$

If scheme is to be 2nd order

$$3\Delta x_{i+1}^2 - 2\Delta x_{i+1}\Delta x_{i+2} - \Delta x_{i+2}^2 = 0$$

$$\Rightarrow (3\Delta x_{i+1} + \Delta x_{i+2})(\Delta x_{i+1} - \Delta x_{i+2}) = 0$$

$\Rightarrow \quad \Delta x_{i+1} = \Delta x_{i+2}$ or a uniform grid.

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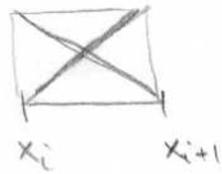
Prob 5.3



$$a\frac{\partial U}{\partial x} = q \qquad (*)$$

linear elements

Mult (*) by $W(x)$ + integrate over the support of one tent fn.

$$a\int_{x_{i-1}}^{x_{i+1}} W(x)\frac{\partial U}{\partial x} = \int_{x_{i-1}}^{x_{i+1}} W(x)\, q\, dx$$

this is the weak form of the equations.
Why don't we integrate by parts?

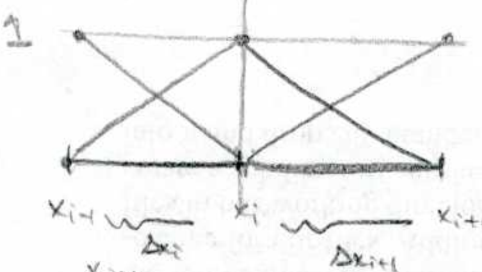
let $U(x) \cong \sum_J U_J N_J(x)$

let $W(x) = N_I(x)$ + pick limits of integration to correspond to

$$\Rightarrow \quad a\int_{x_{i-1}}^{x_{i+1}} N_i(x)\big(\underbrace{U_{i-1}N_{i-1}'(x) + U_i N_i'(x) + U_{i+1}N_{i+1}'(x)}_{\text{only non vanishing terms}}\big)\, dx$$

again approximate $q(x) \simeq \sum_J q_J N_J$ only 3 terms contribute $i-1, i, i+1$

$$= \int_{x_{i-1}}^{x_{i+1}} N_i(x)\big(\underbrace{q_{i-1}N_{i-1}(x) + q_i N_i(x) + q_{i+1}N_{i+1}(x)}_{\text{Again the only non vanishing terms}}\big)\, dx$$



$$\int_{x_{i-1}}^{x_{i+1}} q(x) N_i(x)\, dx \qquad \text{Now } q(x) \simeq q_{i-1} N_{i-1}(x) + q_i N_i(x) + q_{i+1} N_i(x)$$

to the accuracy of the scheme

So that one evaluates the above integral in the following manner.

$$\int_{x_{i-1}}^{x_{i+1}} \left(q_{i-1} N_{i-1}(x) N_i(x) + q_i N_i(x)^2 + q_{i+1} N_{i+1}(x) N_i(x) \right) dx$$

Now the normals become $N_{i-1}(x) = \dfrac{x_i - x}{x_i - x_{i-1}} = \dfrac{x_i - x}{\Delta x_i}$

$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}} = \dfrac{x - x_{i-1}}{\Delta x_i} \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i} = \dfrac{x_{i+1} - x}{\Delta x_{i+1}} \end{cases}$$

$$N_{i+1}(x) = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{\Delta x_{i+1}}$$

Then

$$\int_{x_{i-1}}^{x_{i+1}} N_{i-1}(x) N_i(x)\, dx = \int_{x_{i-1}}^{x_i} \left(\frac{x_i - x}{\Delta x_i}\right)\left(\frac{x - x_{i-1}}{\Delta x_i}\right) dx = +\frac{1}{\Delta x_i^2} \int_{x_{i-1}}^{x_i} (x_i - x)(x - x_{i-1})\, dx$$

$$= +\frac{1}{6} \Delta x_i$$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x)^2\,dx = \int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{\Delta x_i}\right)^2 dx + \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{\Delta x_{i+1}}\right)^2 dx = \frac{1}{3}\Delta x_i + \frac{1}{3}\Delta x_{i+1}$$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x)N_{i+1}(x)\,dx = \int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{\Delta x_{i+1}}\right)\left(\frac{x-x_i}{\Delta x_{i+1}}\right) dx = \frac{1}{6}\Delta x_{i+1}$$

$$\therefore \int_{x_{i-1}}^{x_{i+1}} \sum_{j=i-1}^{i+1} q_j N_j\, N_i(x)\,dx = +\frac{q_{i-1}}{6}\Delta x_i + \frac{q_i}{3}\Delta x_i + \frac{q_i}{3}\Delta x_{i+1} + \frac{q_{i+1}}{6}\Delta x_{i+1}$$

w/ $\Delta x_i = \Delta x_{i+1} = \Delta x$ $\quad = \frac{\Delta x}{6}\left(q_{i-1} + 4q_i + q_{i+1}\right)$ RHS of E 5.3.3

Δx must be $\doteq$ ct

Then For derivative integrals see following pages

$$N_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{\Delta x_i} & x_{i-1} < x < x_i \\ \dfrac{x_{i+1} - x}{\Delta x_{i+1}} & x_i < x < x_{i+1} \end{cases} \qquad \{\Delta x_i = x_i - x_{i-1}\} \qquad \frac{dN_i}{dx} = \begin{cases} \dfrac{1}{\Delta x_i} \\ \dfrac{-1}{\Delta x_{i+1}} \end{cases}$$

$$N_{i-1}(x) = \begin{cases} \dfrac{x_i - x}{\Delta x_i} & x_{i-1} < x < x_i \\ 0 & x_i < x < x_{i+1} \end{cases} \qquad \frac{dN_{i-1}}{dx} = \begin{cases} -\dfrac{1}{\Delta x_i} \\ 0 \end{cases}$$

$$N_{i+1}(x) = \begin{cases} 0 & x_{i-1} < x < x_i \\ \dfrac{x - x_i}{\Delta x_{i+1}} & x_i < x < x_{i+1} \end{cases} \qquad \frac{dN_{i+1}}{dx} = \begin{cases} 0 \\ \dfrac{1}{\Delta x_{i+1}} \end{cases}$$

Then integrals to compute A_{ne}:

$$\int_{x_{i-1}}^{x_{i+1}} N_i \frac{dN_{i-1}}{dx}\,dx = \int_{x_{i-1}}^{x_i} \frac{-1}{(\Delta x_i)} \frac{(x - x_{i-1})}{\Delta x_i}\,dx = -\frac{1}{\Delta x_i^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})\,dx$$

$$= \frac{-1}{\Delta x_i^2} \frac{(x - x_{i-1})^2}{2} \Bigg|_{x_{i-1}}^{x_i} = \frac{-1}{2\Delta x_i^2} \Delta x_i^2 = \frac{-1}{2}$$

$$\int_{x_{i-1}}^{x_{i+1}} N_i \frac{dN_i}{dx}\cdot dx = \int_{x_{i-1}}^{x_i} N_i \frac{dN_i}{dx}\,dx + \int_{x_i}^{x_{i+1}} N_i \frac{dN_i}{dx}\cdot dx = \frac{1}{\Delta x_i^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})\,dx + \frac{(-1)(-1)}{\Delta x_{i+1}\,\Delta x_i} \int_{x_i}^{x_{i+1}} (x_{i+1} - x)\,dx$$

$$= \frac{1}{2\Delta x_i^2} (x - x_{i-1})^2 \Bigg|_{x_{i-1}}^{x_i} + \frac{(-1)}{\Delta x_i\,\Delta x_{i+1}} \frac{(x_{i+1} - x)^2}{2}(-1) \Bigg|_{x_i}^{x_{i+1}}$$

$$= \frac{1}{2\Delta x_i^2}\Delta x_i^2 - 0 + \frac{1}{2\Delta x_i \Delta x_{i+1}}\left(0 - \Delta x_{i+1}^2\right)$$

$$= \frac{1}{2} - \frac{\Delta x_{i+1}}{2\Delta x_i} = \frac{1}{2}\left(1 - \frac{\Delta x_{i+1}}{\Delta x_i}\right) = \frac{1}{2\Delta x_i}\left(\Delta x_i - \Delta x_{i+1}\right)$$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x)\frac{dN_{i+1}}{dx}\,dx = \int_{x_i}^{x_{i+1}} \frac{x_{i+1}-x}{\Delta x_{i+1}}\cdot\frac{1}{\Delta x_{i+1}}\,dx = \frac{1}{\Delta x_{i+1}\Delta x_{i+1}}\int_{x_i}^{x_{i+1}} (x_{i+1}-x)\,dx$$

$$= \frac{1(-1)}{\Delta x_{i+1}\Delta x_{i+1}}\left.\frac{(x_{i+1}-x)^2}{2}\right|_{x_i}^{x_{i+1}} = \frac{-1}{2\Delta x_{i+1}\Delta x_{i+1}}\left(0 - \Delta x_{i+1}^2\right)$$

$$= \frac{1}{2}$$

Then FEM scheme is

$$a\left[U_{i-1}\left(\frac{-1}{2}\right) + \frac{U_i}{2\Delta x_i}\left(\Delta x_i + \Delta x_{i+1}\right) + \frac{U_{i+1}}{2}\right]$$

$$= \frac{q_{i-1}}{6}\Delta x_i + \frac{q_i}{3}\Delta x_i + \frac{q_i}{3}\Delta x_{i+1} + \frac{q_{i+1}}{6}\Delta x_{i+1}$$

$$\Rightarrow \frac{a}{2}\left[U_{i+1} + \frac{(\Delta x_i - \Delta x_{i+1})}{\Delta x_i}U_i - U_{i-1}\right] = \frac{1}{6}\left[\Delta x_i q_{i-1} + 2\Delta x_i q_i + 2\Delta x_{i+1} q_i + \Delta x_{i+1} q_{i+1}\right]$$

w/ $\Delta x_i = \Delta x_{i+1} = \Delta x$ we obtain

$$\frac{a}{2}\left[u_{i+1} - u_{i-1}\right] = \frac{\Delta x}{6}\left[q_{i-1} + 2q_i + 2q_i + q_{i+1}\right]$$

$$= \frac{\Delta x}{6}\left[q_{i-1} + 4q_i + q_{i+1}\right]$$

$$\Rightarrow \quad \underbrace{a\left(\frac{u_{i+1} - u_{i-1}}{2\Delta x}\right)}_{\text{Central diff}} = \frac{q_{i-1} + 4q_i + q_{i+1}}{6}$$

Approx to $\frac{du}{dx}$

pg 232 Hirsh Vol I

Prob 5.4 eq E5.3.1 is (w/ constant k)

$$k\frac{\partial^2 U}{\partial x^2} = q$$ w/ Legrangian (2nd order) Elements.

$$\Rightarrow \quad -k\frac{\partial^2 U}{\partial x^2} + q = 0$$

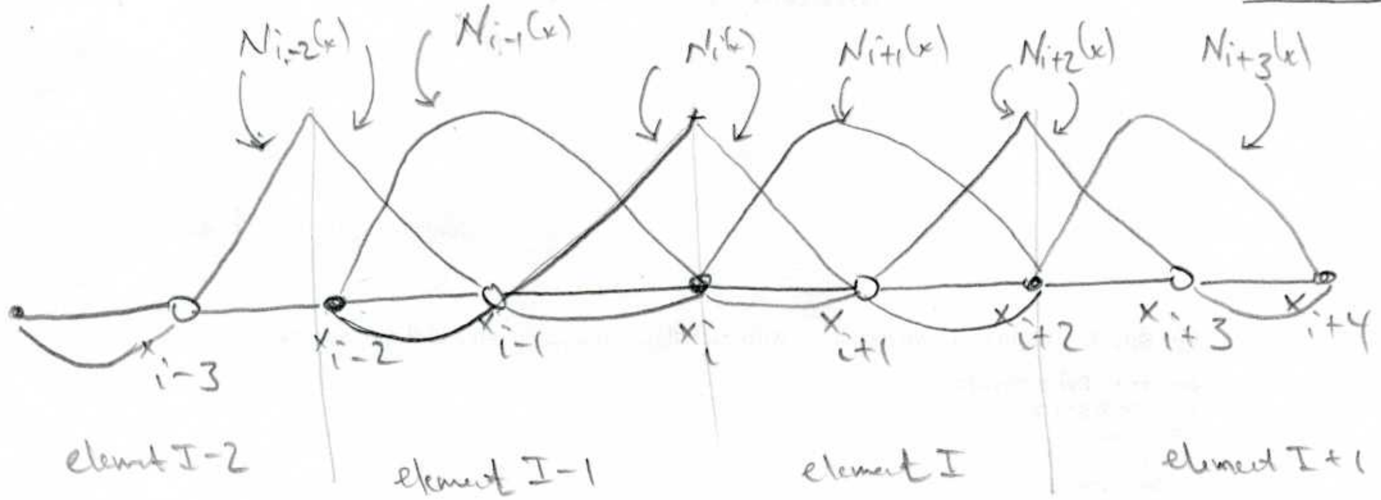
1st get Residual Formulation

$$\int_0^L W(x)\left(-k\frac{d^2 U}{dx^2}\right)dx + \int_0^L W(x)\,q\,dx = 0$$

$$= W(x)\left(-k\frac{dU}{dx}\right)\Big|_0^L + k\int_0^L \frac{dW}{dx}\cdot\frac{dU}{dx}\cdot dx + \int_0^L W(x)\,q\,dx = 0$$

For general internal element flux terms $W(x)\left(-k\frac{dU}{dx}\right)\Big|_0^L = 0$

$$\Rightarrow k\int_0^L \frac{dW}{dx}\ \frac{dU}{dx}\,dx + \int_0^L W(x)\,q\,dx = 0$$



For quadratic Lagrangian elements: Note global tent fn $N_i(x)$ is supported in two elements + global tent fn's $N_{i-1}(x)$ + $N_{i+1}(x)$ are supported entirly in one element.

Then to obtain the Finite Element representation of the difference eqs let $W(x) = N_{i-1}(x)$ + expand q + U in this element i.e.

$$U(x) \cong U_{i-2}N_{i-2}(x) + U_{i-1}N_{i-1}(x) + U_iN_i(x)$$ quadratic fn in this element

$$+\ q \cong q_{i-2}N_{i-2}(x) + q_{i-1}N_{i-1}(x) + q_iN_i(x)$$ "

Then let $W(x) = N_{i-1}(x)$ + integrate over element I-1

$$\Rightarrow K\int_{x_{i-2}}^{x_i} N_{i-1}'(x)\tilde{U}'(x)\,dx + \int_{x_{i-2}}^{x_i} N_{i-1}(x)\hat{q}(x)\,dx = 0$$

$$\Rightarrow k\int_{x_{i-2}}^{x_i} N_{i-1}'(x)\left(u_{i-2}N_{i-2}'(x) + u_{i-1}N_{i-1}'(x) + u_iN_i'(x)\right)dx$$

$$+ \int_{x_{i-2}}^{x_i} N_{i-1}(x)\left(q_{i-2}N_{i-2}(x) + q_{i-1}N_{i-1}(x) + q_iN_i(x)\right)dx = 0$$

$$\Rightarrow k\left[u_{i-2}\int_{x_{i-2}}^{x_i} N_{i-1}'(x)N_{i-2}'(x)\,dx + u_{i-1}\int_{x_{i-2}}^{x_i} N_{i-1}'(x)N_{i-1}'(x)\,dx + \right.$$

$$\left. u_i\int_{x_{i-2}}^{x_i} N_{i-1}'(x)N_i'(x)\,dx\right]$$

$$+ q_{i-2}\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_{i-2}(x)\,dx + q_{i-1}\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_{i-1}(x)\,dx$$

$$+ q_i\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_i(x)\,dx = 0$$

$$N_{i-2}(x) = \frac{(x-x_{i-1})(x-x_i)}{\left(-\frac{\Delta x}{2}\right)(-\Delta x)} = \frac{2}{\Delta x^2}(x-x_{i-1})(x-x_i)$$

$$N_{i-1}(x) = \frac{(x-x_{i-2})(x-x_i)}{\left(\frac{\Delta x}{2}\right)\left(-\frac{\Delta x}{2}\right)} = \frac{-4}{\Delta x^2}(x-x_{i-2})(x-x_i)$$

$$N_i(x) = \frac{(x-x_{i-2})(x-x_{i-1})}{\Delta x\left(\frac{\Delta x}{2}\right)} = \frac{2}{\Delta x^2}(x-x_{i-2})(x-x_{i-1})$$

Then evaluating all of the integrals on pg 3 gives:

$$k\left[u_{i-2}\left(\frac{-8}{3\Delta x}\right) + u_{i-1}\left(\frac{16}{3\Delta x}\right) + u_i\left(\frac{-8}{3\Delta x}\right)\right]$$

$$+ q_{i-2}\left(\frac{\Delta x}{15}\right) + q_{i-1}\left(\frac{8\Delta x}{15}\right) + q_i\left(\frac{\Delta x}{15}\right) = 0$$

$$\Rightarrow -\frac{8}{3\Delta x}k\left(u_{i-2} - 2u_{i-1} + u_i\right) + \frac{\Delta x}{15}\left(q_{i-2} + 8q_{i-1} + q_i\right) = 0$$

$$\Rightarrow \frac{k}{\Delta x^2}\left(u_{i-2} - 2u_{i-1} + u_i\right) - \frac{1}{5 \cdot 8}\left(q_{i-2} + 8q_{i-1} + q_i\right) = 0$$

$$\frac{k}{\Delta x^2}\left(u_{i-2} - 2u_{i-1} + u_i\right) = \frac{1}{10}\,\frac{1}{4}\left(q_{i-2} + 8q_{i-1} + q_i\right)$$

but my Δx is $2 \cdot \Delta\tilde{x}$ in Book

$\Delta x = 2\Delta\tilde{x}$

$$\rightarrow \quad \frac{k}{\Delta\tilde{x}^2}(u_{i-2} - 2u_{i-1} + u_i) = \frac{1}{10}(q_{i-2} + 8q_{i-1} + q_i)$$ 1st eq in 5.4

For a boundry point as i we get integrating over element $I-1$ + element I.

$$k\int_{x_{i-2}}^{x_{i+2}} N_i'(x)\tilde{u}'(x)\,dx + \int_{x_{i-2}}^{x_{i+2}} N_i(x)\tilde{q}(x)\,dx = 0$$

$$\Rightarrow k\int_{x_{i-2}}^{x_i} N_i'(x)\tilde{u}'(x)\,dx + k\int_{x_i}^{x_{i+2}} N_i'(x)\tilde{u}'(x)\,dx + \int_{x_{i-2}}^{x_i} N_i(x)\tilde{q}(x)\,dx + \int_{x_i}^{x_{i+2}} N_i(x)\tilde{q}(x)\,dx = 0$$

$$\Rightarrow k\left[u_{i-2}\int_{x_{i-2}}^{x_i} N_i'(x)N_{i-2}'(x)\,dx + u_{i-1}\int_{x_{i-2}}^{x_i} N_i'(x)N_{i-1}'(x)\,dx + u_i\int_{x_{i-2}}^{x_i} N_i'(x)N_i'(x)\,dx\right]$$

$$+ k\left[u_i\int_{x_i}^{x_{i+2}} N_i'(x)N_i'(x)\,dx + u_{i+1}\int_{x_i}^{x_{i+2}} N_i'(x)N_{i+1}'(x)\,dx + u_{i+2}\int_{x_i}^{x_{i+2}} N_i'(x)N_{i+2}'(x)\,dx\right]$$

$$+ q_{i-2}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-2}(x)\,dx + q_{i-1}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-1}(x) + q_i\int_{x_{i-2}}^{x_i} N_i(x)N_i(x)\,dx$$

$$+ q_i\int_{x_i}^{x_{i+2}} N_i(x)N_i(x)\,dx + q_{i+1}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+1}(x)\,dx + q_{i+2}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+2}(x)\,dx$$

$$= 0$$

Here

$$N_i(x) = \frac{2}{\Delta x^2}(x - x_{i-2})(x - x_{i-1}) \qquad x_{i-2} < x < x_i$$

$$N_i(x) = \frac{(x - x_{i+1})(x - x_{i+2})}{(-\frac{\Delta x}{2})(-\Delta x)} = \frac{2}{\Delta x^2}(x - x_{i+1})(x - x_{i+2}) \qquad x_i < x < x_{i+2}$$

$$N_{i+1}(x) = \frac{(x - x_i)(x - x_{i+2})}{(\frac{\Delta x}{2})(-\frac{\Delta x}{2})} = \frac{-4}{\Delta x^2}(x - x_i)(x - x_{i+2})$$

$$N_{i+2}(x) = \frac{(x - x_i)(x - x_{i+1})}{\Delta x\,(\frac{\Delta x}{2})} = \frac{2}{\Delta x^2}(x - x_i)(x - x_{i+1})$$

Then integrating we get:

$$k\left[u_{i-2}\left(\frac{1}{3\Delta x}\right) + u_{i-1}\left(-\frac{8}{3\Delta x}\right) + u_i\left(\frac{7}{3\Delta x}\right)\right]$$

$$+k\left[u_i\left(\frac{7}{3\Delta x}\right) + u_{i+1}\left(-\frac{8}{3\Delta x}\right) + u_{i+2}\left(\frac{1}{3\Delta x}\right)\right]$$

$$+ q_{i-2}\left(-\frac{\Delta x}{30}\right) + q_{i-1}\left(\frac{\Delta x}{15}\right) + q_i\left(\frac{2\Delta x}{15}\right)$$

$$+ q_i\left(\frac{2\Delta x}{15}\right) + q_{i+1}\left(\frac{\Delta x}{15}\right) + q_{i+2}\left(-\frac{\Delta x}{30}\right) = 0$$

$$\Rightarrow \frac{k}{3\Delta x}\left[v_{i-2} - 8v_{i-1} + 14v_i - 8v_{i+1} + v_{i+2} \right]$$

$$+ \frac{\Delta x}{15}\left(-\frac{q_{i-2}}{2} + q_{i-1} + 2q_i + 2q_i + q_{i+1} - \frac{q_{i+2}}{2} \right) = 0$$

$$\Rightarrow -\frac{k}{\Delta x^2}\left(-v_{i-2} + 8v_{i-1} - 14v_i + 8v_{i+1} - v_{i+2} \right)$$

$$+ \frac{1}{10}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2} \right) = 0$$

$$\Rightarrow \frac{k}{\Delta x^2}\left(-v_{i-2} + 8v_{i-1} - 14v_i + 8v_{i+1} - v_{i+2} \right)$$

$$= -\frac{1}{10}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2} \right)$$

But again my $\Delta x = 2\hat{\Delta x}$ $\qquad \hat{\Delta x} = \Delta x$ used in the book

$$\rightarrow \frac{k}{4\Delta x^2}\left(-v_{i-2} + 8v_{i-1} - 14v_i + 8v_{i+1} - v_{i+2} \right)$$

$$= -\frac{1}{10}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2} \right)$$

Same As 2nd eq in Probl 5.4

left hand side $\frac{k}{\Delta x^2}(U_{i-2}-2U_{i-1}+U_i)$ in one case

$+ \quad \frac{k}{4\Delta x^2}(-U_{i-2}+8U_{i-1}-14U_i+8U_{i+1}-U_{i+2})$ in the other

eq 4.2.47 is

$$(U_{xx})_i \cong \frac{1}{12\Delta x^2}(-U_{i+2}+16U_{i+1}-30U_i+16U_{i-1}-U_{i-2})$$

$$+ \frac{\Delta x^4}{90}\frac{\partial^6 U}{\partial x^6}.$$

Taylor expanding

$\frac{1}{\Delta x^2}(U_{i-2}-2U_{i-1}+U_i)$ about x_{i-1} gives

$$(U_{xx})_{i-1} = \frac{1}{\Delta x^2}(U_i-2U_{i-1}+U_{i-2}) - \frac{\Delta x^2}{12}\left.\frac{\partial^4 U}{\partial x^4}\right|_{x_{i-1}}$$ From eq 4.2.23

Taylor expanding

$\frac{1}{4\Delta x^2}(-U_{i-2}+8U_{i-1}-14U_i+8U_{i+1}-U_{i+2})$ about x_i gives

w/ MMA gives

$$= U''(x_i) - \frac{1}{6}U''''(x_i)\Delta x^2 + O(\Delta x^4)$$ Don't see comparison

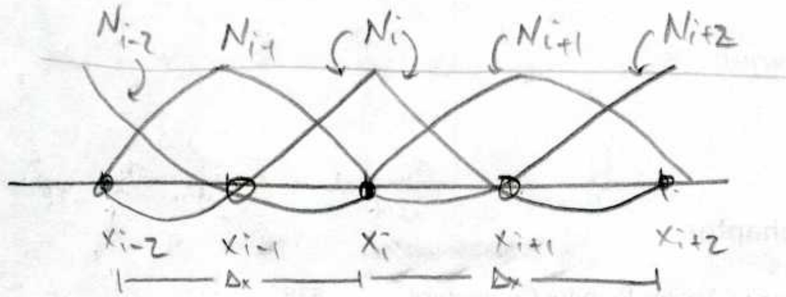
Prob 5.5 03-31-01

$$a \frac{\partial U}{\partial x} = q$$

Assuming book means quadratic Lagrangian elements

Multiply by $\overline{W}(x)$ + integrate

$$\Rightarrow \quad a \int \overline{W}(x) \frac{\partial U}{\partial x}\, dx = \int \overline{W}(x)\, q(x)\, dx$$



To evaluate the finite difference scheme at node $i-1$ (interior node)

Let $\overline{W}(x) = N_{i-1}(x)$ + integrate over element $I-1$ (x_{i-2}, x_i)

$$U(x) \cong U_{i-2} N_{i-2}(x) + U_{i-1} N_{i-1}(x) + U_i N_i(x)$$

$$q(x) \cong q_{i-2} N_{i-2}(x) + q_{i-1} N_{i-1}(x) + q_i N_i(x)$$

Then integrating + passing integral through constants

$$\Rightarrow \quad a \Big[U_{i-2} \int_{x_{i-2}}^{x_i} N_{i-1}(x) N_{i-2}'(x)\, dx + U_{i-1} \int_{x_{i-2}}^{x_i} N_{i-1}(x) N_{i-1}'(x)\, dx$$

$$+ U_i \int_{x_{i-2}}^{x_i} N_{i-1}(x) N_i'(x)\, dx \Big] =$$

$$q_{i-2}\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_{i-2}(x)\,dx + q_{i-1}\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_{i-1}(x)\,dx + q_i\int_{x_{i-2}}^{x_i} N_{i-1}(x)N_i(x)\,dx$$

integrating

$$\Rightarrow a\left[v_{i-2}\left(-\frac{2}{3}\right) + v_{i-1}(0) + v_i\left(\frac{2}{3}\right)\right]$$

$$= q_{i-2}\left(\frac{\Delta x}{15}\right) + q_{i-1}\left(\frac{8\Delta x}{15}\right) + q_i\left(\frac{\Delta x}{15}\right)$$

$$\Rightarrow -\frac{2}{3}a(v_{i-2}-v_i) = \frac{\Delta x}{15}(q_{i-2}+8q_{i-1}+q_i)$$

$$\Rightarrow \frac{a}{\Delta x}(v_{i-2}-v_i) = -\frac{1}{10}(q_{i-2}+8q_{i-1}+q_i)$$

Again my $\Delta x = 2\tilde{\Delta x}$ $\quad \tilde{\Delta x}$ spacing used in the book

$$\Rightarrow -\frac{a}{2\tilde{\Delta x}}(v_{i-2}-v_i) = +\frac{1}{10}(q_{i-2}+8q_{i-1}+q_i)$$

got a different sign but it must have this minus or

At an boundary point i, $w(x) = N_i(x)$

& integrate over two elements $I-1$ & I

$$\Rightarrow a\left[U_{i-2}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-2}'(x)\,dx + U_{i-1}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-1}'(x)\,dx + U_i\int_{x_{i-2}}^{x_i} N_i(x)N_i'(x)\,dx\right.$$

$$\left. + U_i\int_{x_i}^{x_{i+2}} N_i(x)N_i'(x)\,dx + U_{i+1}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+1}'(x)\,dx + U_{i+2}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+2}'(x)\,dx\right]$$

$$= q_{i-2}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-2}(x)\,dx + q_{i-1}\int_{x_{i-2}}^{x_i} N_i(x)N_{i-1}(x)\,dx + q_i\int_{x_{i-2}}^{x_i} N_i(x)N_i(x)\,dx$$

$$+ q_i\int_{x_i}^{x_{i+2}} N_i(x)N_i(x)\,dx + q_{i+1}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+1}(x)\,dx + q_{i+2}\int_{x_i}^{x_{i+2}} N_i(x)N_{i+2}(x)\,dx$$

Integrating w/ MMA gives.

$$\Rightarrow a\left[U_{i-2}\left(\frac{1}{6}\right) - U_{i-1}\left(\frac{2}{3}\right) + \cancel{U_i\left(\frac{1}{2}\right)} + \cancel{U_i\left(\frac{-1}{2}\right)} + U_{i+1}\left(\frac{2}{3}\right) + U_{i+2}\left(\frac{-1}{6}\right)\right]$$

$$= q_{i-2}\left(-\frac{\Delta x}{30}\right) + q_{i-1}\left(\frac{\Delta x}{15}\right) + q_i\left(\frac{2\Delta x}{15}\right) + q_i\left(\frac{2\Delta x}{15}\right) + q_{i+1}\left(\frac{\Delta x}{15}\right)$$

$$+ q_{i+2}\left(-\frac{\Delta x}{30}\right)$$

$$\Rightarrow \frac{a}{6}\left[u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right]$$

$$= \frac{\Delta x}{30}\left[-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2}\right]$$

$$\Rightarrow \frac{a}{6\Delta x}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right)$$

$-2+4+4-2$
$= 4$ not $\underline{6}$!!

$$= \frac{1}{30}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2}\right)$$

But as before $\Delta x = 2\tilde{\Delta x}$

Coefficient of Δx in Denominator shall be As this is a 1st derivative approximation

$$\Rightarrow \frac{a}{6\Delta x}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right)$$

$-2-4(-1)+4-2$
$= 4$ not $\underline{6}$!!

$$= \frac{1}{15}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2}\right)$$

$$\Rightarrow \frac{a}{4\Delta x}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right)$$

$$= \frac{1}{10}\left(-q_{i-2} + 2q_{i-1} + 8q_i + 2q_{i+1} - q_{i+2}\right)$$

equation asked for

Prob 5.6   Cubic Hermitian interpolation fns pg 211

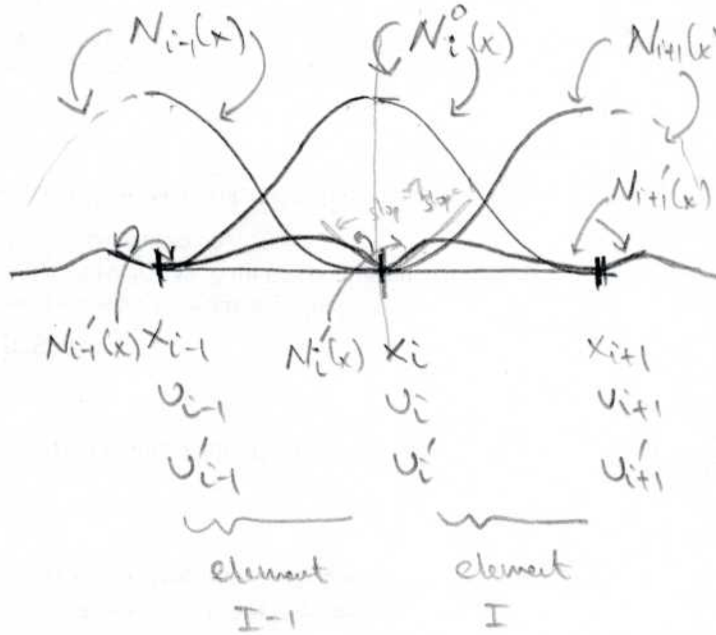
element on pg 210

$$U(x) \cong U_1 H_1^0 + U_2 H_2^0 + (U_x)_1 H_1^1 + (U_x)_2 H_2^1$$ x restricted to an element I.

Then

$$\int_{\substack{\text{support of}\\ N_J(x)}} U_x N_J \, dx = \int_{\text{element}} \left(\sum_{I=1}^{8} U_I N_I'(x) \right) N_J \, dx$$

$$= \sum_{I=1}^{8} U_I \left(N_I'(x), N_J \right)$$



$$(U_x, H_i^0) = ?$$

$$U \cong U_{i-1} N_{i-1}^0(x) + U_{i-1}' N_{i-1}^1(x) + U_i N_i^0(x) + U_i' N_i^1(x)$$

2 elements

$$+ U_{i+1} N_{i+1}^0(x) + U_{i+1}' N_{i+1}^1(x)$$ ✓

$$U_x \cong U_{i-1} \frac{dN_{i-1}^0(x)}{dx} + U_{i-1}' \frac{dN_{i-1}^1(x)}{dx} + U_i \frac{dN_i^0(x)}{dx} + U_i' \frac{dN_i^1(x)}{dx}$$

$$+ U_{i+1} \frac{dN_{i+1}^0(x)}{dx} + U_{i+1}' \frac{dN_{i+1}^1(x)}{dx}$$ ✓

Then $(U_x, N_J) = \int_{x_{i-1}}^{x_i} dx + \int_{x_i}^{x_{i+1}} dx$

$\xi : (a,b) \rightarrow (0,1)$

$\xi = \frac{1-0}{b-a}(x-a)$

$\xi(x) = \frac{x-a}{b-a}$

$x_{i-1} \quad x_i \quad x_{i+1}$

$\Downarrow \quad \Downarrow \quad \Downarrow$

$0 \quad 1,0 \quad 1$

$\xi(x) = \frac{x - x_{i-1}}{\Delta x} \qquad \xi(x) = \frac{x - x_i}{\Delta x}$

Then from pg 211

$H_1^0(\xi) = 1 - 3\xi^2 + 2\xi^3 \qquad H_2^0(\xi) = \xi^2(3-2\xi)$

$H_1^1(\xi) = (1-\xi)^2\xi \qquad H_2^1(\xi) = \xi^2(\xi-1)$

Then in terms of x

element to left (x_{i-1}, x_i) $\qquad H_j^k(x)$

$H_1^0(x) = 1 - \frac{3}{\Delta x^2}(x-x_{i-1})^2 + 2\frac{(x-x_{i-1})^3}{\Delta x^3} \qquad H_2^0(x) = \left(\frac{x-x_{i-1}}{\Delta x}\right)^2\left(3 - 2\left(\frac{x-x_{i-1}}{\Delta x}\right)\right)$

$H_1^1(x) = \left(1 - \frac{x-x_{i-1}}{\Delta x}\right)^2\left(\frac{x-x_{i-1}}{\Delta x}\right) \qquad H_2^1(x) = \frac{(x-x_{i-1})^2}{\Delta x^2}\left(\frac{x-x_{i-1}}{\Delta x} - 1\right)$

& for element I (x_i, x_{i+1})

$H_1^0(x) = 1 - \frac{3}{\Delta x^2}(x-x_i)^2 + \frac{2}{\Delta x^3}(x-x_i)^3 \qquad H_2^0(x) = \left(\frac{x-x_i}{\Delta x}\right)^2\left(3 - 2\left(\frac{x-x_i}{\Delta x}\right)\right)$

$H_1^1(x) = \left(1 - \frac{x-x_i}{\Delta x}\right)^2\left(\frac{x-x_i}{\Delta x}\right) \qquad H_2^1(x) = \frac{(x-x_i)^2}{\Delta x^2}\left(\frac{x-x_i}{\Delta x} - 1\right)$

Check ~~global~~ expressions for $H_1^0(x)$ etc

$$H_1^0(x) = 1 - \frac{3}{\Delta x^2}(x - x_{i-1})^2 + \frac{2(x - x_{i-1})^3}{\Delta x^3}$$

$$H_1^0(x_{i-1}) = 1 \qquad H_1^0$$

$\vdots$ etc

Now derivatives transform as..

$$H_1^1(x) = \left(1 - \left(\frac{x - x_{i-1}}{\Delta x}\right)^2\right)\left(\frac{x - x_{i-1}}{\Delta x}\right)$$

$$H_1^1(x_{i-1}) = 0 \checkmark \qquad H_1^1(x_i) = 0 \checkmark$$

$$\frac{dH_1^1(x)}{dx} = \frac{1}{\Delta x}\left(1 - \left(\frac{x - x_{i-1}}{\Delta x}\right)^2\right) + -2\left(\frac{x - x_{i-1}}{\Delta x}\right)\frac{1}{\Delta x}\left(\frac{x - x_{i-1}}{\Delta x}\right)$$

$$\frac{dH_1^1}{dx}(x_{i-1}) = \frac{1}{\Delta x}$$

$$\frac{dH_1^1}{dx}(x_i) = 0$$

$$H_2'(x) = \frac{(x-x_{i-1})^2}{\Delta x^2}\left(\frac{x-x_{i-1}}{\Delta x} - 1\right)$$

$$H_2'(x_{i-1}) = 0 \qquad H_2'(x_i) = 0$$

$$\frac{dH_2'}{dx} = 2\frac{(x-x_{i-1})}{\Delta x^2}\left(\frac{x-x_{i-1}}{\Delta x} - 1\right) + \frac{(x-x_{i-1})^2}{\Delta x^2}\frac{1}{\Delta x}$$

$$\frac{dH_2'}{dx}(x_{i-1}) = 0 + 0 = 0$$

$$\frac{dH_2'}{dx}(x_i) = \frac{2}{\Delta x}\cdot 0 + \frac{1}{\Delta x} = \frac{1}{\Delta x}$$

Thus we see the need to modify

$$H_1'(x) = \left(1 - \frac{x-x_i}{\Delta x}\right)^2 \left(\frac{x-x_i}{\Delta x}\right) \qquad H_2'(x) = \frac{(x-x_i)^2}{\Delta x^2}\left(\frac{x-x_i}{\Delta x} - 1\right)$$

Then

$$(U_x, H_i^0) = \int_{x_{i-1}}^{x_i} U_{i-1} \frac{dN_{i-1}^0(x)}{dx} N_i^0(x)\, dx + U_{i-1}' \int_{x_{i-1}}^{x_i} \frac{dN_{i-1}^1(x)}{dx} N_i^0(x)\, dx$$

$$+ U_i \int_{x_{i-1}}^{x_i} \frac{dN_i^0(x)}{dx} N_i^0(x)\, dx + U_i' \int_{x_{i-1}}^{x_i} \frac{dN_i^1(x)}{dx} N_i^0(x)\, dx$$

$$+ U_i \int_{x_i}^{x_{i+1}} \frac{dN_i^0(x)}{dx} N_i^0(x)\, dx + U_i' \int_{x_i}^{x_{i+1}} \frac{dN_i^1(x)}{dx} N_i^0(x)\, dx$$

$$+ U_{i+1} \int_{x_i}^{x_{i+1}} \frac{dN_{i+1}^0}{dx} N_i^0(x)\, dx + U_{i+1}' \int_{x_i}^{x_{i+1}} \frac{dN_{i+1}^1}{dx} N_i^0(x)\, dx$$

$$\Rightarrow (U_x, H_i^0) = U_{i-1} \underbrace{\int_{x_{i-1}}^{x_{i-1}+\Delta x} \frac{dN_{i-1}^0(x)}{dx} N_i^0(x)\, dx}_{\text{How to integrate.}} + U_{i-1}' \int_{x_{i-1}}^{x_{i-1}+\Delta x}$$

$$(u_x, H_i^0) = -\frac{u_{i-1}}{2} - \frac{u_{i-1}'}{10} + \cancel{\frac{u_i}{2}} + \frac{u_i'}{10}$$

$$- \cancel{\frac{u_i}{2}} + \frac{u_i'}{10} + \frac{u_{i+1}}{2} - \frac{u_{i+1}'}{10}$$

$$= -\frac{\left[u_{i-1}' - 2u_i' + u_{i+1}'\right]}{10} + \frac{(u_{i+1} - u_{i-1})}{2}$$

Do taylor series of the above & check it. Seems O.k

Why dont we have the Δx?

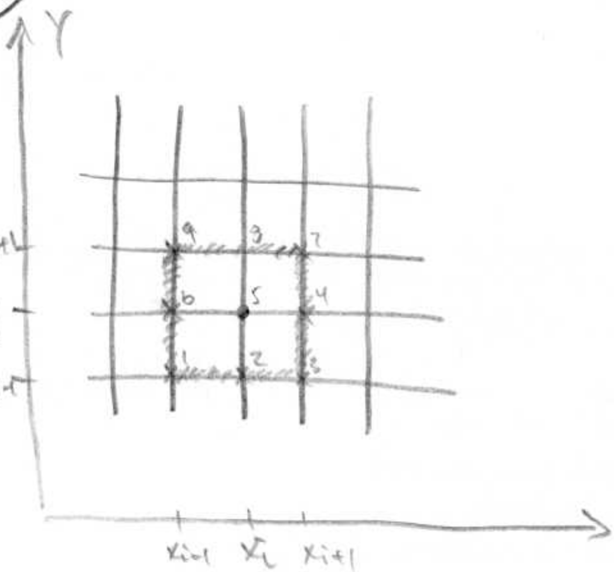
$$(u_x, H_i^1) = u_{i-1} \int_{x_{i-1}}^{x_i} \frac{dN_{i-1}^0}{dx} \cdot N_i^1(x)\,dx + u_{i-1}' \int_{x_{i-1}}^{x_i} \frac{dN_{i-1}^1}{dx} \cdot N_i^1(x)\,dx$$

$$+ u_i \int_{x_{i-1}}^{x_i} \frac{dN_i^0}{dx} N_i^1\,dx + u_i' \int_{x_{i-1}}^{x_i} \frac{dN_i^1}{dx} \cdot N_i^1\,dx$$

$$+ u_i \int_{x_i}^{x_{i+1}} \frac{dN_i^0}{dx} N_i^1\,dx + u_i' \int_{x_i}^{x_{i+1}} \frac{dN_i^1}{dx} \cdot N_i^1\,dx$$

$$+ u_{i+1} \int_{x_i}^{x_{i+1}} \frac{dN_{i+1}^0}{dx} N_i^1\,dx + u_{i+1}' \int_{x_i}^{x_{i+1}} \frac{dN_{i+1}^1}{dx} N_i^1\,dx.$$

Prob 5.8



$$\nabla^2 u = 0$$

Multiply by $W(\bar{x})$ $\bar{x} \in \mathbb{R}^2$

& integrate by parts

$$\int_{elt} W(\bar{x}) \nabla^2 u \, d\bar{x} = 0$$

$$= \oint_{\partial E} W(\bar{x}) \nabla u \circ \hat{n} \, dS$$

$$- \int_E \nabla W \circ \nabla u \, d\Omega = 0$$

Then $\displaystyle\oint_{\partial E} \bar{W}(\bar{x}) \nabla u \circ \hat{n} \, dS = 0$

$\Rightarrow$ pick $\bar{W}(x) = N_{i,j}(\bar{x})$

$$\int_E \nabla N_{ij}(\bar{x}) \circ \nabla u \, d\Omega = 0$$

Approximate $u(x,y)$ on this element

$$u(x,y) \cong u_{i-1,j-1} N_{i-1,j-1}(x,y) + u_{i,j-1} N_{i,j-1}(x,y) + u_{i+1,j-1} N_{i+1,j-1}(x,y)$$

$$+ u_{i+1,j} N_{i+1,j}(x,y) + u_{i,j} N_{i,j}(x,y) + u_{i-1,j} N_{i-1,j}(x,y)$$

$$+ u_{i+1,j+1} N_{i+1,j+1}(x,y) + u_{i,j+1} N_{i,j+1}(x,y) + u_{i-1,j+1} N_{i-1,j+1}(x,y)$$

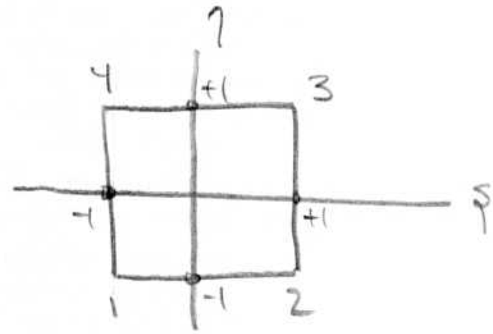
in support of element of $N_{i,j}(x,y)$

Then $\int_E \nabla N_{ij}(\bar{x}) \cdot \nabla u \, d\Omega = 0$

$\Rightarrow$ integrate $\nabla N_{ij}(x,y)$ against each $\nabla N_{\square\square}$ in turn.

Bilinear quadrilateral elements:

$$\vec{x} = \sum_{I=1}^{4} \vec{x}_I N_I(\xi,\eta) \qquad N_I(\xi,\eta) = \tfrac{1}{4}(1+\xi\xi_I)(1+\eta\eta_I)$$



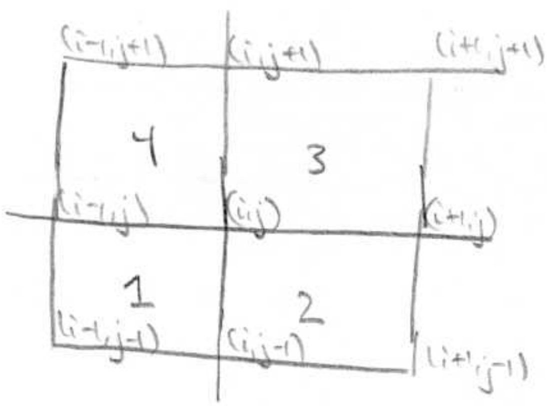
$$\int_E \nabla N_I \cdot \nabla N_J \, dx\, dy =$$

From problem 5.9

$$\int_\Omega f(x,y)\, dx\, dy = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)|J|\, d\xi\, d\eta$$

$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$

$$\vec{x} = \sum_{I=1}^{4} \vec{x}_I N_I$$



Then the difference scheme is computed as

$$U_{i-1,j-1}\int_E \nabla N_{i,j}\circ\nabla N_{i-1,j-1}\,d\Omega + U_{i,j-1}\int_E \nabla N_{i,j}\circ\nabla N_{i,j-1}\,d\Omega$$

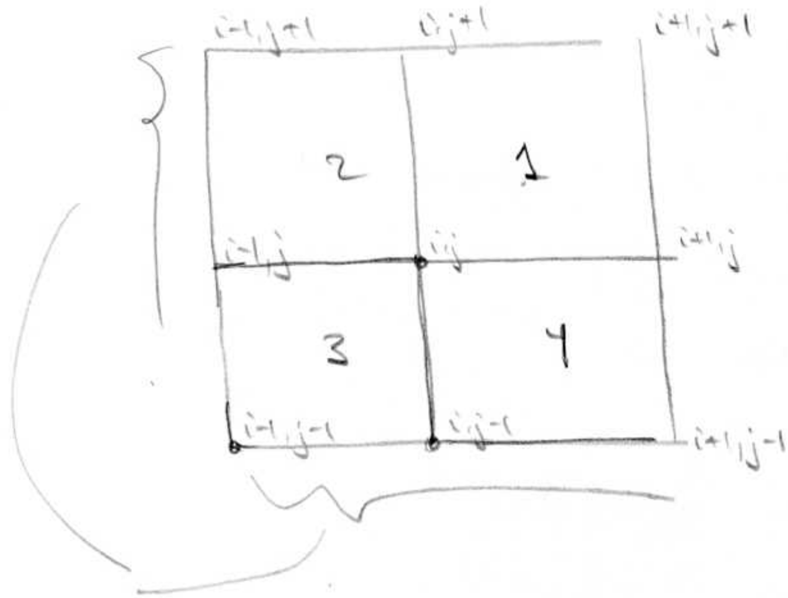
$$+U_{i+1,j-1}\int_E \nabla N_{i,j}\circ\nabla N_{i+1,j-1}\,d\Omega +$$

$$U_{i-1,j}\int_E \nabla N_{i,j}\circ\nabla N_{i-1,j}\,d\Omega + U_{i,j}\int_E \nabla N_{i,j}\circ\nabla N_{i,j}\,d\Omega$$

$$+U_{i+1,j}\int_E \nabla N_{i,j}\circ\nabla N_{i+1,j}\,d\Omega +$$

$$U_{i-1,j+1}\int_E \nabla N_{i,j}\circ\nabla N_{i-1,j+1}\,d\Omega + U_{i,j+1}\int_E \nabla N_{i,j}\circ\nabla N_{i,j+1}\,d\Omega$$

$$+U_{i+1,j+1}\int_E \nabla N_{i,j}\circ\nabla N_{i+1,j+1}\,d\Omega = 0$$



Square of 4 pieces is element E $E_1, E_2, E_3,$ & E_4

$$= U_{i-1,j-1} \int_{E_3} \nabla N_{i,j} \circ \nabla N_{i-1,j-1} \, d\Omega + U_{i,j-1} \int_{E_3 \cup E_4} \nabla N_{i,j} \circ \nabla N_{i,j-1} \, d\Omega$$

$$+ U_{i+1,j-1} \int_{E_4} \nabla N_{i,j} \circ \nabla N_{i+1,j} \, d\Omega +$$

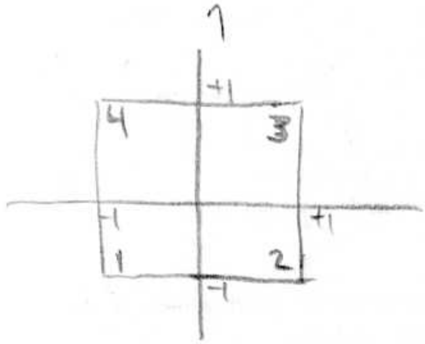
$$U_{i-1,j} \int_{E_3 \cup E_2} \nabla N_{i,j} \circ \nabla N_{i-1,j} \, d\Omega + U_{i,j} \int_{E_1 \cup E_2 \cup E_3 \cup E_4} \nabla N_{i,j} \circ \nabla N_{i,j} \, d\Omega$$

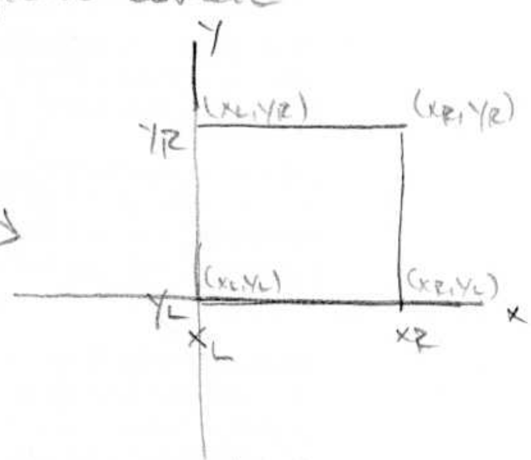
$$+ U_{i+1,j} \int_{E_1 \cup E_4} \nabla N_{i,j} \circ \nabla N_{i+1,j} \, d\Omega +$$

$$U_{i-1,j+1}\int_{E_2}\nabla N_{i,j}\cdot\nabla N_{i-1,j+1}\,d\Omega + U_{i,j+1}\int_{E_2\cup E_1}\nabla N_{i,j}\cdot\nabla N_{i,j+1}\,d\Omega$$

$$+\ U_{i+1,j+1}\int_{E_1}\nabla N_{i,j}\cdot\nabla N_{i+1,j+1}\,d\Omega \quad = 0$$

Now discription of tent fns in each respectin domain





$x : (-1,+1) \longrightarrow (x_L, x_R)$ $\qquad x = \dfrac{x_R - x_L}{+1-(-1)}(\xi+1) + x_L$

$y : (-1,+1) \longrightarrow (y_L, y_R)$ $\qquad y = \dfrac{y_R - y_L}{2}(\eta+1) + y_L$

Then inverse of these maps are:

$$\xi = \frac{2(x - x_L)}{x_R - x_L} - 1$$

$$\eta = \frac{2(y - y_L)}{y_R - y_L} - 1$$

Then $N_1(x,y) = \frac{1}{4}\left(1-\left(\frac{2(x-x_L)}{x_R-x_L}-1\right)\right)\left(1-\left(\frac{2(y-y_L)}{y_R-y_L}-1\right)\right)$

$$N_2(x,y) = \frac{1}{4}\left(1+\left(\frac{2(x-x_L)}{x_R-x_L}-1\right)\right)\left(1-\left(\frac{2(y-y_L)}{y_R-y_L}-1\right)\right)$$

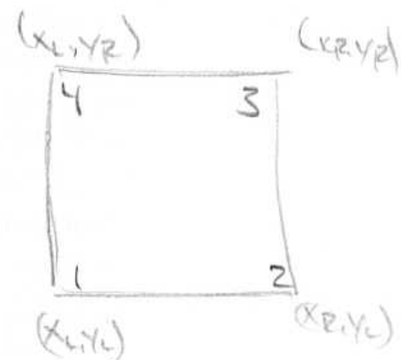
$$N_3(x,y) = \frac{1}{4}\left(1+\left(\frac{2(x-x_L)}{x_R-x_L}-1\right)\right)\left(1+\left(\frac{2(y-y_L)}{y_R-y_L}-1\right)\right)$$

$$N_4(x,y) = \frac{1}{4}\left(1-\left(\frac{2(x-x_L)}{x_R-x_L}-1\right)\right)\left(1+\left(\frac{2(y-y_L)}{y_R-y_L}-1\right)\right)$$

$x_R - x_L = \Delta x$

$y_R - y_L = \Delta y$

$\Rightarrow$ $N_1(x,y) = \left(1-\frac{x-x_L}{\Delta x}\right)\left(1-\frac{y-y_L}{\Delta y}\right)$ ✓



$N_2(x,y) = \left(\frac{x-x_L}{\Delta x}\right)\left(1-\frac{y-y_L}{\Delta y}\right)$ ✓

$N_3(x,y) = \left(\frac{x-x_L}{\Delta x}\right)\left(\frac{y-y_L}{\Delta y}\right)$ ✓

$N_4(x,y) = \left(1-\frac{x-x_L}{\Delta x}\right)\left(\frac{y-y_L}{\Delta y}\right)$ ✓

By symmetry

$$\int_{E_3} \nabla N_{ij} \circ \nabla N_{i-1,j-1}\, d\Omega = \int_{E_4} \nabla N_{ij} \cdot \nabla N_{i+1,j}\, d\Omega = \int_{E_2} \nabla N_{ij} \circ \nabla N_{i-1,j+1}\, d\Omega$$

$$= \int_{E_1} \nabla N_{ij} \circ \nabla N_{i+1,j+1}\, d\Omega \equiv I_1$$

$$+ \int_{E_3 \cup E_4} \nabla N_{ij} \circ \nabla N_{i,j-1}\, d\Omega = \int_{E_3 \cup E_2} \nabla N_{ij} \circ \nabla N_{i-1,j}\, d\Omega = \int_{E_1 \cup E_4} \nabla N_{ij} \cdot \nabla N_{i+1,j}\, d\Omega$$

$$= \int_{E_2 \cup E_1} \nabla N_{ij} \circ \nabla N_{i,j+1}\, d\Omega \equiv I_2$$

Thus $\exists$ 3 integrals we need to do

$$I_1 = \int_{E_3} \nabla N_{ij} \cdot \nabla N_{i-1,j-1}\, d\Omega = \int_{E_3} \nabla \left(\frac{x - x_{i-1}}{\Delta x}\right)\left(\frac{y - y_{j-1}}{\Delta y}\right) \circ \nabla \left(1 - \frac{x - x_{i-1}}{\Delta x}\right)\left(1 - \frac{y - y_{j-1}}{\Delta y}\right) d\Omega$$

$$= \int_{E_3} \left(\frac{1}{\Delta x}\left(\frac{y - y_{j-1}}{\Delta y}\right)\hat{i} + \frac{1}{\Delta y}\left(\frac{x - x_{i-1}}{\Delta x}\right)\hat{j} \right) \circ \left(\frac{-1}{\Delta x}\left(1 - \frac{y - y_{j-1}}{\Delta y}\right)\hat{i} - \frac{1}{\Delta y}\left(1 - \frac{x - x_{i-1}}{\Delta x}\right)\hat{j} \right) d\Omega$$

$$\Rightarrow I_1 = \int_{E_3} \left[\frac{-1}{\Delta x^2} \left(\frac{y - y_{j-1}}{\Delta y} \right) \left(1 - \frac{y - y_{j-1}}{\Delta y} \right) - \frac{1}{\Delta y^2} \left(\frac{x - x_{i-1}}{\Delta x} \right) \left(1 - \frac{x - x_{i-1}}{\Delta x} \right) \right] d\Omega$$

w/ $\Delta x = \Delta y$

$$I_1 = \frac{-1}{\Delta x^2} \int_{y_{j-1}}^{y_j} \int_{x_{i-1}}^{x_i} \left[\frac{y - y_{j-1}}{\Delta x} \left(1 - \frac{(y - y_{j-1})}{\Delta x} \right) + \left(\frac{x - x_{i-1}}{\Delta x} \right) \left(1 - \frac{(x - x_{i-1})}{\Delta x} \right) \right] dx\, dy$$

$$= \frac{-1}{\Delta x^2} \int_{y_{j-1}}^{y_{j-1}+\Delta x} \int_{x_{i-1}}^{x_{i-1}+\Delta x} \left\{ \frac{y - y_{j-1}}{\Delta x} \left(1 - \frac{y - y_{j-1}}{\Delta x} \right) + \left(\frac{x - x_{i-1}}{\Delta x} \right) \left(1 - \frac{x - x_{i-1}}{\Delta x} \right) \right\} dx\, dy$$

$$= -\frac{1}{3}$$

$$I_2 = \int_{E_3 \cup E_4} \nabla N_{ij} \circ \nabla N_{i,j-1}\, d\Omega = \int_{E_3} \nabla N_{ij} \circ \nabla N_{i,j-1}\, d\Omega + \int_{E_4} \nabla N_{ij} \circ \nabla N_{i,j-1}\, d\Omega$$

$$= \int_{E_3} \nabla \left(\frac{x - x_{i-1}}{\Delta x} \right) \left(\frac{y - y_{j-1}}{\Delta y} \right) \circ \nabla \left(\frac{x - x_{i-1}}{\Delta x} \right) \left(1 - \frac{y - y_{j-1}}{\Delta y} \right) d\Omega \quad \checkmark$$

$$+ \int_{E_4} \nabla \left(1 - \frac{x - x_i}{\Delta x} \right) \left(\frac{y - y_{j-1}}{\Delta y} \right) \circ \nabla \left(1 - \frac{x - x_i}{\Delta x} \right) \left(1 - \frac{y - y_{j-1}}{\Delta y} \right) d\Omega \quad \checkmark$$

$$\Rightarrow \dot{I}_2 = \int_{E_3} \left(\frac{1}{\Delta x}\left(\frac{y-y_{j-1}}{\Delta y}\right)\hat{\imath} + \frac{1}{\Delta y}\left(\frac{x-x_{i-1}}{\Delta x}\right)\hat{\jmath}\right) \circ \left(\frac{1}{\Delta x}\left(1-\frac{y-y_{j-1}}{\Delta y}\right)\hat{\imath} - \frac{1}{\Delta y}\left(\frac{x-x_{i-1}}{\Delta x}\right)\hat{\jmath}\right) d\Omega$$

$$+ \int_{E_4} \left(\frac{-1}{\Delta x}\left(\frac{y-y_{j-1}}{\Delta y}\right)\hat{\imath} + \frac{1}{\Delta y}\left(1-\frac{x-x_i}{\Delta x}\right)\hat{\jmath}\right) \circ \left(\frac{-1}{\Delta x}\left(1-\frac{y-y_{j-1}}{\Delta y}\right)\hat{\imath} + \frac{-1}{\Delta y}\left(1-\frac{x-x_i}{\Delta x}\right)\hat{\jmath}\right) d\Omega \quad \checkmark$$

$$\Rightarrow I_2 = \int_{E_3} \left[\frac{1}{\Delta x^2}\left(\frac{y-y_{j-1}}{\Delta y}\right)\left(1-\frac{y-y_{j-1}}{\Delta y}\right) - \frac{1}{\Delta y^2}\left(\frac{x-x_{i-1}}{\Delta x}\right)^2\right] d\Omega$$

$$+ \int_{E_4} \left[\frac{1}{\Delta x^2}\left(\frac{y-y_{j-1}}{\Delta y}\right)\left(1-\frac{y-y_{j-1}}{\Delta y}\right) - \frac{1}{\Delta y^2}\left(1-\frac{x-x_i}{\Delta x}\right)^2\right] d\Omega$$

$\Delta x = \Delta y$

$$= \frac{1}{\Delta x^2}\int_{y_{j-1}}^{y_{j-1}+\Delta x}\int_{x_{i-1}}^{x_{i-1}+\Delta x} \left[\left(\frac{y-y_{j-1}}{\Delta x}\right)\left(1-\frac{y-y_{j-1}}{\Delta x}\right) - \left(\frac{x-x_{i-1}}{\Delta x}\right)^2\right] dx\, dy$$

$$+ \frac{1}{\Delta x^2}\int_{y_{j-1}}^{y_{j-1}+\Delta y}\int_{x_i}^{x_i+\Delta x} \left[\left(\frac{y-y_{j-1}}{\Delta x}\right)\left(1-\frac{y-y_{j-1}}{\Delta y}\right) - \left(1-\frac{x-x_i}{\Delta x}\right)^2\right] dx\, dy$$

$$= 1$$

$$I_3 = \underset{\substack{\uparrow \\ \text{By symmetry}}}{4} \iint_{E_3} \nabla N_{i,j} \circ \nabla N_{i,j} \, d\Omega$$

$$= 4 \iint_{E_3} \left[\nabla \left(\frac{x - x_{i-1}}{\Delta x}\right)\left(\frac{y - y_{j-1}}{\Delta y}\right)\right]^2 d\Omega$$

$$= \frac{4}{\Delta x^2} \int_{y_{j-1}}^{y_{j-1}+\Delta y} \int_{x_{i-1}}^{x_{i-1}+\Delta x} \left[\left(\frac{y - y_{j-1}}{\Delta y}\right)^2 + \left(\frac{x - x_{i-1}}{\Delta x}\right)^2\right] dx\, dy = \frac{8}{3}$$

$\therefore$ I get

$$-\frac{U_{i-1,j-1}}{3} + U_{i,j-1} - \frac{U_{i+1,j-1}}{3} + U_{i-1,j} + \frac{8}{3} U_{i,j}$$ Where is the error?

$$+ U_{i+1,j} - \frac{U_{i-1,j+1}}{3} + U_{i,j+1} - \frac{U_{i+1,j+1}}{3} = 0$$

$$\Rightarrow -\left(U_{i-1,j-1} - U_{i+1,j-1} - U_{i-1,j+1} - U_{i+1,j+1}\right)$$

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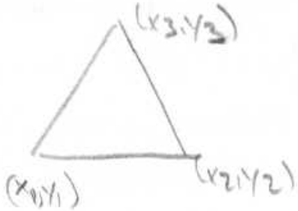
(Prob 5.9) Show $\frac{1}{\Omega}\int_{\Omega} v \, d\Omega = \frac{1}{3}\sum_{I=1}^{3} v_I$

$$= \frac{1}{4}\sum_{I=1}^{4} v_I$$

linear Δ's.



$$v = U_1 N_1(x,y) + U_2 N_2(x,y) + U_3 N_3(x,y)$$

$$\frac{1}{\Omega}\int_{\Omega} \sum_{i=1}^{3} U_i N_i(x,y)\, dx\, dy = \frac{1}{A}\sum_{i=1}^{3} U_i \int_{\Omega} N_i(x,y)\, dx\, dy$$

From Table 5.1 $N_i = L_i$ (local coordinates)

$$= \frac{1}{A}\sum_{i=1}^{3} U_i \int_0^1 \int_0^{1-L_2} L_i (2A)\, dL_1\, dL_2 \quad \text{From eq 5.4.13}$$

$$= 2\left[U_1 \int_0^1 \int_0^{1-L_2} L_1\, dL_1\, dL_2 + U_2 \int_0^1 \int_0^{1-L_2} L_2\, dL_1\, dL_2 + U_3 \int_0^1 \int_0^{1-L_2} (1-L_1-L_2)\, dL_1\, dL_2 \right]$$

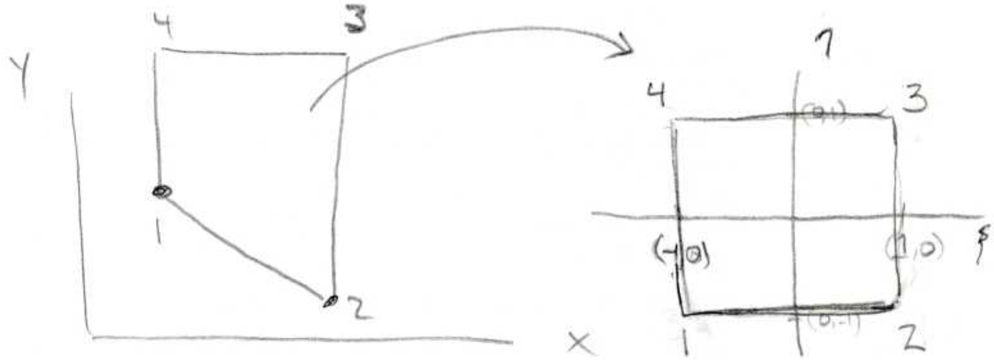
$$= 2\left[U_1 \int_0^1 \frac{L_1^2}{2}\Big|_0^{1-L_2} dL_2 + U_2 \int_0^1 L_2(1-L_2)\, dL_2 + U_3 \int_0^1 \left((1-L_2)L_1 - \frac{L_1^2}{2}\right)\Big|_0^{1-L_2} dL_2 \right]$$

$$= 2\left[\frac{U_1}{2} \int_0^1 (1-L_2)^2\, dL_2 + U_2 \int_0^1 (L_2 - L_2^2)\, dL_2 + U_3 \int_0^1 \left((1-L_2)^2 - \frac{1}{2}(1-L_2)^2\right) dL_2 \right.$$

$$= 2\left[\frac{U_1}{2}\underbrace{\int_0^1 (1-L_2)^2\, dL_2}_{\left(-\frac{(1-L_2)^3}{3}\right)\Big|_0^1 \;=\; +\frac{1}{3}} + U_2 \underbrace{\int_0^1 (L_2 - L_2^2)\, dL_2}_{\left(\frac{L_2^2}{2} - \frac{L_2^3}{3}\right)\Big|_0^1 \;=\; \frac{1}{2}-\frac{1}{3} = \frac{3}{6}-\frac{2}{6} = \frac{1}{6}} + \frac{U_3}{2}\underbrace{\int_0^1 (1-L_2)^2\, dL_2}_{\left(-\frac{(1-L_2)^3}{3}\right)\Big|_0^1 \;=\; \frac{1}{3}} \right]$$

$$= \frac{U_1}{3} + \frac{U_2}{3} + \frac{U_3}{3} = \frac{1}{3}\sum_{I=1}^{3} U_I$$

For Bilinear quadrilaterals:



$$\vec{x} = \sum_{I=1}^{4} \vec{x}_I N_I$$

$$N_I = \frac{1}{4}(1+\xi\xi_I)(1+\eta\eta_I)$$ pg 213 Hish Vol I.

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

derivatives computed on pg 3-4 where more space permits.

$$\frac{1}{\Omega}\int_{\Omega} u \, d\Omega = \frac{1}{\Omega}\int_{\Omega} \sum_{I=1}^{4} u_I N_I(x,y) \, d\Omega$$

$$= \frac{1}{\Omega} \sum_{I=1}^{4} u_I \int_{\Omega} N_I(x,y) \, dx\,dy$$

$$dx\,dy = |J| \, d\xi \, d\eta \quad w/ \ J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}$$

Now $\quad x = \sum_{I=1}^{4} x_I N_I(\xi,\eta) \qquad \frac{\partial x}{\partial \xi} = \sum_{I=1}^{4} x_I \frac{\partial N_I}{\partial \xi} \qquad \frac{\partial x}{\partial \eta} = \sum_{I=1}^{4} x_I \frac{\partial N_I}{\partial \eta}$

$\qquad y = \sum_{I=1}^{4} y_I N_I(\xi,\eta) \qquad \frac{\partial y}{\partial \xi} = \sum_{I=1}^{4} y_I \frac{\partial N_I}{\partial \xi} \qquad \frac{\partial y}{\partial \eta} = \sum_{I=1}^{4} y_I \frac{\partial N_I}{\partial \eta}$

Now $\quad \frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \qquad \frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \qquad \cdots \qquad ✓$

$$\therefore \frac{\partial x}{\partial \xi} = -\frac{x_1}{4}(1-\eta) + \frac{x_2}{4}(1-\eta) + \frac{x_3}{4}(1+\eta) - \frac{x_4}{4}(1+\eta) \qquad ✓$$

$$= \frac{(x_2-x_1)}{4}(1-\eta) + \frac{(x_3-x_4)}{4}(1+\eta) \qquad ✓$$

$$\frac{\partial x}{\partial \eta} = -\frac{x_1}{4}(1-\xi) - \frac{x_2}{4}(1+\xi) + \frac{x_3}{4}(1+\xi) + \frac{x_4}{4}(1-\xi) \quad ✓$$

$$= \frac{(x_4 - x_1)}{4}(1-\xi) + \frac{(x_3 - x_2)}{4}(1+\xi) \quad ✓$$

Similarly:

$$\frac{\partial y}{\partial \xi} = \frac{(y_2 - y_1)}{4}(1-\eta) + \frac{(y_3 - y_4)}{4}(1+\eta) \quad ✓$$

$$\frac{\partial y}{\partial \eta} = \frac{(y_4 - y_1)}{4}(1-\xi) + \frac{(y_3 - y_2)}{4}(1+\xi) \quad ✓$$

Then

$$|J| = \frac{\partial x}{\partial \xi} \cdot \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \cdot \frac{\partial x}{\partial \eta} \quad ✓$$

$$= \left(\frac{(x_2 - x_1)}{4}(1-\eta) + \frac{(x_3 - x_4)}{4}(1+\eta) \right)\left(\frac{(y_4 - y_1)}{4}(1-\xi) + \frac{(y_3 - y_2)}{4}(1+\xi) \right)$$

$$- \left(\frac{(y_2 - y_1)}{4}(1-\eta) + \frac{(y_3 - y_4)}{4}(1+\eta) \right)\left(\frac{(x_4 - x_1)}{4}(1-\xi) + \frac{(x_3 - x_2)}{4}(1+\xi) \right) \quad ✓$$

$$= \frac{(x_2 - x_1)(y_4 - y_1)(1-\eta)(1-\xi)}{4^2} + \frac{(x_2 - x_1)(y_3 - y_2)}{4^2}(1-\eta)(1+\xi)$$

$$+ \frac{(x_3 - x_4)(y_4 - y_1)}{4^2}(1+\eta)(1-\xi) + \frac{(x_3 - x_4)(y_3 - y_2)}{4^2}(1+\eta)(1+\xi) \quad ✓$$

$$- \frac{(y_2-y_1)(x_4-x_1)}{4^2}(1-\eta)(1-\xi) - \frac{(y_2-y_1)(x_3-x_2)}{4^2}(1-\eta)(1+\xi)$$

$$- \frac{(y_3-y_4)(x_4-x_1)}{4^2}(1+\eta)(1-\xi) - \frac{(y_3-y_4)(x_3-x_2)}{4^2}(1+\eta)(1+\xi) \checkmark$$

$$\Rightarrow \frac{1}{4}\left[(x_2-x_1)(y_4-y_1)-(x_4-x_1)(y_2-y_1)\right]N_1(\xi,\eta) \checkmark$$

$$+ \frac{1}{4}\left[(x_2-x_1)(y_3-y_2)-(x_3-x_2)(y_2-y_1)\right]N_2(\xi,\eta) \checkmark$$

$$+ \frac{1}{4}\left[(x_3-x_4)(y_3-y_2)-(x_3-x_2)(y_3-y_4)\right]N_3(\xi,\eta) \checkmark$$

$$+ \frac{1}{4}\left[(x_3-x_4)(y_4-y_1)-(x_4-x_1)(y_3-y_4)\right]N_4(\xi,\eta) \checkmark$$

$$\Rightarrow \frac{1}{4}C_1N_1 + \frac{1}{4}C_2N_2 + \frac{1}{4}C_3N_3 + \frac{1}{4}C_4N_4$$

Then

$$\frac{1}{\Omega}\int_{\Omega} f(x,y)\,d\Omega = \frac{1}{\Omega}\int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)J(\xi,\eta)\,d\xi\,d\eta$$

Thus we need to exclude to evaluate $\sum_{I} u_I \int_{\Omega} N_I \, d\Omega$

$$\frac{1}{\Omega}\int_{\Omega} N_I(x,y)\, d\Omega \qquad I = 1,2,3,4 \qquad \underset{\uparrow}{=} \quad \frac{1}{4}$$

want to show Then done.

$$= \frac{1}{\Omega}\int_{-1}^{+1}\int_{-1}^{+1} N_I(\xi,\eta)\, J(\xi,\eta)\, d\xi\, d\eta$$

$$= \frac{1}{\Omega}\int_{-1}^{+1}\int_{-1}^{+1} N_I(\xi,\eta) \sum_{J=1}^{4} \frac{c_J}{4} N_J(\xi,\eta)\, d\xi\, d\eta$$

$$= \frac{1}{4\Omega}\sum_{J=1}^{4} c_J \underbrace{\int_{-1}^{+1}\int_{-1}^{+1} N_I N_J \, d\xi\, d\eta}_{} \qquad I = 1,2,3,4$$

$= A\, \delta_{IJ}$ (Not true see below for the evaluation of the integrals)

Lets check that for Piecewise Bilinear polynomials that this is indeed true.

$$\int_{-1}^{+1}\int_{-1}^{1} N_1$$

$I \neq J$ count of # of integrals $= \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot 3}{2} = 6$

$$\begin{pmatrix} \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \\ & & & \times \end{pmatrix}_{4\times 4}$$

+ 4 integrals along diagonal.

Based on this geometric fact $\qquad \frac{4^2}{2} = \frac{16}{2} = 8$

$$\binom{n}{2} + n = \frac{n^2}{2}$$

\# off diagonal elts of a symmetric matrix (unique) $\uparrow$

\# diagonal element $\uparrow$

$\uparrow$ 1/2 total # of elements

Doesn't seem to work. What's wrong?

$$\int_{-1}^{+1}\int_{-1}^{+1} N_1^2(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{+1}\int_{-1}^{+1}\frac{1}{16}(1-\xi)^2(1-\eta)^2\,d\xi\,d\eta$$

$$= \frac{1}{16}\,\frac{(1-\xi)^3}{3}(-1)\Big|_{-1}^{+1}\ \frac{(1-\eta)^3}{3}(-1)\Big|_{-1}^{+1} = \frac{1}{9\cdot 16}(0-2^3)(0-2^3)$$

$$= \frac{4}{9} \checkmark$$

$$\int_{-1}^{+1}\int_{-1}^{+1} N_2^2(\xi,\eta)\,d\xi\,d\eta = \frac{1}{16}\int_{-1}^{+1}\int_{-1}^{+1}(1+\xi)^2(1-\eta)^2\,d\xi\,d\eta = \frac{1}{16}\,\frac{(1+\xi)^3}{3}\Big|_{-1}^{+1}\frac{(1-\eta)^3}{3}(-1)\Big|_{-1}^{+1}$$

$$= \frac{1}{9\cdot 16}(8)(8) = \frac{4}{9} \checkmark$$

By symmetry
$$\int_{-1}^{+1}\int_{-1}^{+1} N_3^2 \, d\xi \, d\eta = \frac{4}{9} \quad +$$
$$\int_{-1}^{+1}\int_{-1}^{+1} N_4^2 \, d\xi \, d\eta = \frac{4}{9} .$$

Then
$$\int_{-1}^{+1}\int_{-1}^{+1} N_1 N_2 \, d\xi \, d\eta = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{16}(1-\xi)(1-\eta)^2(1+\xi) \, d\xi \, d\eta$$

$$= \frac{1}{16} \frac{(1-\eta)^3}{3}(-1) \Big|_{-1}^{+1} \int_{-1}^{+1} (1-\xi)(1+\xi) \, d\xi = \frac{1}{2\cdot 3} \int_{-1}^{+1} 1-\xi^2 \, d\xi$$

$$= \frac{1}{2.3}\left(\xi - \frac{\xi^3}{3}\right) \Big|_{-1}^{+1} = \cancel{\frac{1}{6}\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right)}$$

$$= \frac{2}{2\cdot 3}\left(1-\tfrac{1}{3}\right) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9} . \checkmark$$

$$\int_{-1}^{+1}\int_{-1}^{+1} N_1 N_3 \, d\xi \, d\eta = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{16}(1-\xi)(1-\eta)(1+\xi)(1+\eta) \, d\xi \, d\eta$$

$$= \frac{1}{16} \int_{-1}^{+1} (1-\xi^2) \, d\xi \int_{-1}^{+1} (1-\eta^2) \, d\eta = \frac{4}{16}\left[\xi - \frac{\xi^3}{3} \Big|_0^1 \right]^2$$

$$= \frac{1}{4}\left(\tfrac{2}{3}\right)^2 = \frac{1}{9} \quad \checkmark$$

$$\int_{-1}^{+1}\int_{-1}^{+1} N_1 N_4 \, d\xi \, d\eta = \int_{-1}^{+1}\int_{-1}^{+1} \frac{1}{16}(1-\xi)(1-\eta)(1-\xi)(1+\eta) \, d\xi \, d\eta$$

$$= \frac{1}{16} \int_{-1}^{+1} (1-\xi)^2 \, d\xi \int_{-1}^{+1} (1-\eta)(1+\eta) \, d\eta$$

$$= \frac{1}{16} \frac{(1-\xi)^3}{3}(-1) \Big|_{-1}^{+1} \int_{-1}^{+1} (1-\eta^2) \, d\eta$$

$$= \frac{1}{16} \cdot (0 - 8/3)(-1) \, 2 \left(\eta - \frac{\eta^3}{3} \right|_{0}^{1}$$

$$= \frac{1}{\cancel{2} \cdot 3} \cancel{2} \left(1 - 1/3\right) = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9} \quad ✓$$

$$\int_{-1}^{+1}\int_{-1}^{+1} N_2 N_3 \, d\xi \, d\eta = \frac{1}{16} \int_{-1}^{+1}\int_{-1}^{+1} (1+\xi)(1-\eta)(1+\xi)(1+\eta) \, d\xi \, d\eta$$

$$= \frac{1}{16} \int_{-1}^{+1}\int_{-1}^{+1} (1+\xi)^2 (1-\eta^2) \, d\xi \, d\eta$$

$$= \frac{1}{16} \frac{(1+\xi)^3}{3} \Big|_{-1}^{+1} \left(\eta - \frac{\eta^3}{3} \right|_{-1}^{+1} = \frac{1}{16} \cdot \frac{(8-0)}{3} \, 2 \left(1 - \frac{1}{3}\right) = \frac{2}{9}. \quad ✓$$

$$\int_{-1}^{+1}\int_{-1}^{+1} N_2 N_4 \, d\xi \, d\eta = \frac{1}{9} \qquad \int_{-1}^{+1}\int_{-1}^{+1} N_3 N_4 \, d\xi \, d\eta = \frac{2}{9}$$

$$=$$

From pg 6 let $I = 1$

& we evaluate: $\frac{1}{4\Omega} \sum_{J=1}^{4} c_J \int_{-1}^{+1} \int_{-1}^{+1} N_1 N_J \, d\xi \, d\eta$

$$= \frac{1}{4\Omega} \left(c_1 \frac{4}{9} + c_2 \frac{2}{9} + c_3 \frac{1}{9} + c_4 \frac{2}{9} \right)$$

$$= \frac{1}{36\Omega} \left(4c_1 + 2c_2 + c_3 + 2c_4 \right)$$

let $I = 2$

$$\frac{1}{4\Omega} \sum_{J=1}^{4} c_J \int_{-1}^{+1} \int_{-1}^{+1} N_2 N_J \, d\xi \, d\eta$$

$$= \frac{1}{4\Omega} \left(c_1 \frac{2}{9} + c_2 \frac{4}{9} + c_3 \frac{2}{9} + c_4 \frac{1}{9} \right)$$

$$= \frac{1}{36\Omega} \left(2c_1 + 4c_2 + 2c_3 + c_4 \right)$$

let $I = 3$

$$\frac{1}{4\Omega} \sum_{J=1}^{4} c_J \int_{-1}^{+1} \int_{-1}^{+1} N_3 N_J \, d\xi \, d\eta = \frac{1}{4\Omega} \left(c_1 \frac{1}{9} + c_2 \frac{2}{9} + c_3 \frac{4}{9} + c_4 \frac{2}{9} \right)$$

$$= \frac{1}{36\Omega} \left(c_1 + 2c_2 + 4c_3 + 2c_4 \right)$$

let $I = 4$

$$\frac{1}{4\Omega}\sum_{J=1}^{4} C_J \int_{-1}^{+1}\int_{-1}^{+1} N_4 N_J \, d\xi \, d\eta$$

$$= \frac{1}{4\Omega}\left(C_1 \frac{2}{9} + C_2 \frac{1}{9} + C_3 \frac{2}{9} + C_4 \frac{4}{9} \right)$$

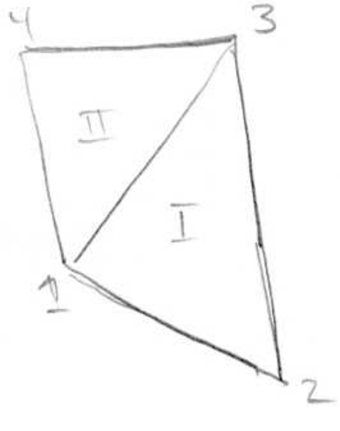
$$= \frac{1}{36\Omega}\left(2C_1 + C_2 + 2C_3 + 4C_4 \right)$$

Now each expression that is the sum of C's can be evaluated using MMA to give

Thus

$$4C_1 + 2C_2 + C_3 + 2C_4 =$$

I would imagine that Ω or the area of the 2D Figure follows from Addition of two Δ areas



$$2A_{I} = (x_3-x_1)(y_2-y_1) - (y_3-y_1)(x_2-x_1)$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_3-x_1 & x_2-x_1 & 0 \\ y_3-y_1 & y_2-y_1 & 0 \end{vmatrix} \right\| \quad \checkmark$$

$+ \quad 2A_{II} =$ let $3 \to 1$, $2 \to 4$, $1 \to 3$

$$= (x_1-x_3)(y_4-y_3) - (y_1-y_3)(x_4-x_3)$$

$$= \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1-x_3 & x_4-x_3 & 0 \\ y_1-y_3 & y_4-y_3 & 0 \end{vmatrix} \right\|$$

so $\Omega = \frac{1}{2}(2A_I + 2A_{II})$

$$\Rightarrow 2\Omega = (x_3-x_1)(y_2-y_1) - (y_3-y_1)(x_2-x_1)$$
$$+ (x_1-x_3)(y_4-y_3) - (y_1-y_3)(x_4-x_3)$$

$$= (x_1-x_3)(y_1-y_2-y_3+y_4)$$
$$- (y_1-y_3)(x_1-x_2-x_3+x_4)$$

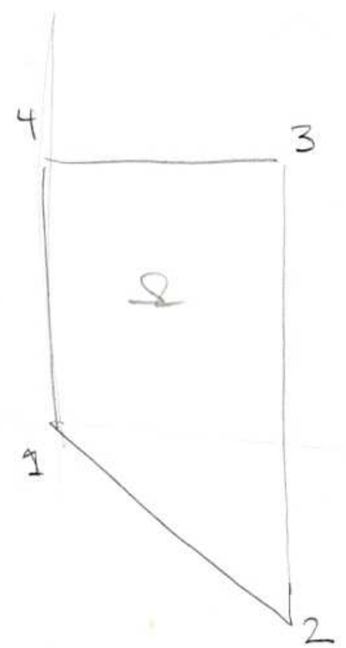
$$\therefore \quad 2\Omega = (x_1-x_3)(y_1-y_2-y_3+y_4) - (y_1-y_3)(x_1-x_2-x_3+x_4)$$

$$= (x_1-x_3)(y_1-(y_2+y_3)+y_4) - (y_1-y_3)(x_1+x_4-(x_2+x_3))$$

$$(y_1+y_4-(y_2+y_3))$$

$$2\Omega = \begin{vmatrix} x_1 - x_3 & x_1 + x_4 - x_2 - x_3 \\ y_1 - y_3 & y_1 + y_4 - y_2 - y_3 \end{vmatrix}$$

+ obviously any permutation of this antisymmetric form.



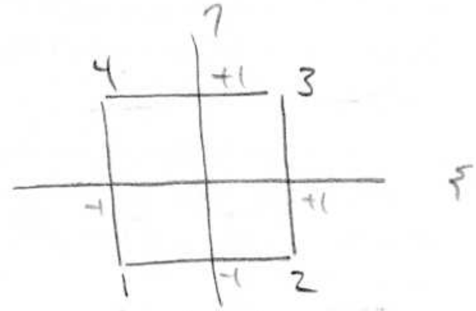
What does this represent geometrically?

How is Ω cut up into pieces we can find the area of?

Prob 5.9 continued

Try a different approach to 2nd part of problem. i.e. show:

$$\frac{1}{\Omega}\int_{\Omega} v \, d\Omega = \frac{1}{4}\sum_{I=1}^{4} v_I$$



$$\vec{x} = \sum_{I=1}^{4} \vec{x}_I N_I$$

$$N_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$

$$\Rightarrow \quad x = \sum x_I N_I \qquad dx = \sum x_I \frac{\partial N_I}{\partial \xi} d\xi + \sum x_I \frac{\partial N_I}{\partial \eta} d\eta$$
$$y = \sum y_I N_I \qquad dy = \sum y_I \frac{\partial N_I}{\partial \xi} d\xi + \sum y_I \frac{\partial N_I}{\partial \eta} d\eta$$

Now on an element $\displaystyle \frac{1}{\Omega}\int_{\Omega} v \, d\Omega = \frac{1}{\Omega}\sum_{I=1}^{4} v_I \int_{\Omega} N_I \, d\Omega$

$$= \frac{1}{\Omega}\sum_{I=1}^{4} v_I \iint N_I \, |J| \, d\xi \, d\eta \qquad dx\,dy = |J| \, d\xi \, d\eta$$

$$\Rightarrow J = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}$$

$$|J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$= \left(\sum x_I \frac{\partial N_I}{\partial \xi}\right)\left(\sum y_I \frac{\partial N_I}{\partial \eta}\right) - \left(\sum x_I \frac{\partial N_I}{\partial \eta}\right)\left(\sum y_I \frac{\partial N_I}{\partial \xi}\right)$$

$$= \left(x_1\left(-\frac{(1-\eta)}{4}\right) + x_2\left(\frac{1-\eta}{4}\right) + x_3\left(\frac{1+\eta}{4}\right) + x_4\left(-\frac{(1+\eta)}{4}\right)\right) \cdot$$

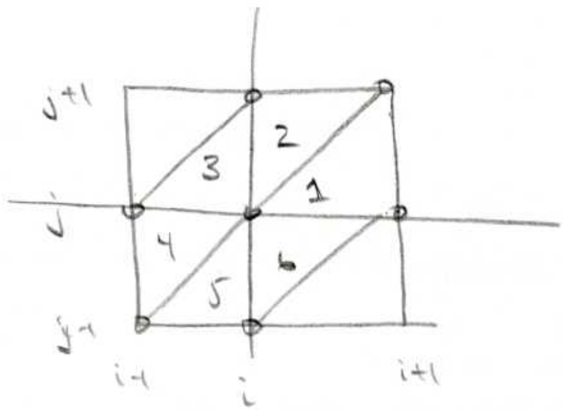
$$\left(y_1\left(-\frac{(1-\xi)}{4}\right) + y_2\left(-\frac{(1+\xi)}{4}\right) + y_3\left(\frac{1+\xi}{4}\right) + y_4\left(\frac{1-\xi}{4}\right)\right)$$

$$- \left(x_1\left(-\frac{(1-\xi)}{4}\right) + x_2\left(-\frac{(1+\xi)}{4}\right) + x_3\left(\frac{1+\xi}{4}\right) + x_4\left(\frac{1-\xi}{4}\right)\right) \cdot$$

04-09-01 2

$$\left(y_1 \left(-\frac{(1+\eta)}{4} \right) + y_2 \left(\frac{1-\eta}{4} \right) + y_3 \left(\frac{1+\eta}{4} \right) + y_4 \left(-\frac{1}{4}(1+\eta) \right) \right)$$

Prob 5.10

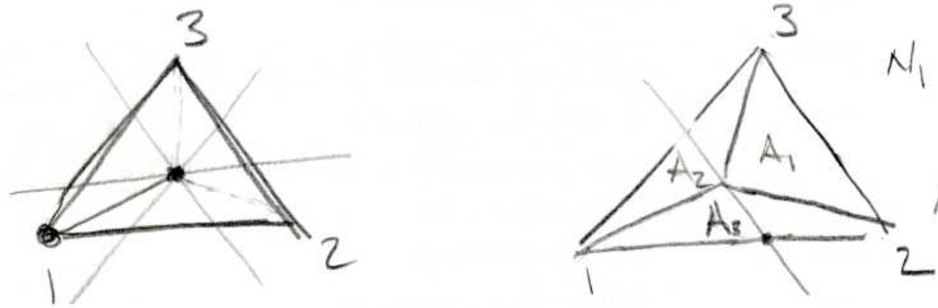


6 overlapping nodes (at least their foot prints)

$\Rightarrow [M_{IJ}] \in \mathbb{R}^7$ th?

$$M_{IJ} = \int_{\Omega_J} N_I N_J \, d\Omega$$

Not entirely sure but think just looking at 1 Δ.



$N_1 = \frac{A_1}{A}$

$N_2 = \frac{A_2}{A}$

$$\iint_\Delta N_1 N_1 \, d\Omega = 2A \int_0^1 \int_0^{1-L_1} L_1^2 \, dL_2 \, dL_1 = 2A \int_0^1 L_1^2 (1-L_1) \, dL_1$$

$$= 2A \left(\frac{L_1^3}{3} - \frac{L_1^4}{4} \right|_0^1 = 2A\left(\frac{1}{3} - \frac{1}{4}\right) = 2A\left(\frac{4}{12} - \frac{3}{12}\right) = \frac{2A}{12}$$

$$= \frac{A}{6}$$

$$\iint\limits_{\Delta} N_1 N_2 \, d\Omega = 2A \int_0^1 \int_0^{1-L_1} L_1 L_2 \, dL_2 \, dL_1 = 2A \int_0^1 \frac{L_1}{2} \left(L_2^2 \right|_0^{1-L_1} dL_1$$

$$= \frac{2A}{2} \int_0^1 L_1 (1-L_1)^2 \, dL_1 = \frac{2A}{2} \int_0^1 L_1 (1 - 2L_1 + L_1^2) \, dL_1$$

$$= A \int_0^1 (L_1 - 2L_1^2 + L_1^3) \, dL_1 = A \left(\frac{L_1^2}{2} - \frac{2L_1^3}{3} + \frac{L_1^4}{4} \right|_0^1$$

$$= A \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = A \left(\frac{6}{12} - \frac{8}{12} + \frac{3}{12} \right) = \frac{A}{12}$$

$$\iint\limits_{\Delta} N_1 N_3 \, d\Omega = 2A \int_0^1 \int_0^{1-L_1} L_1 L_3 \, dL_2 \, dL_1 = 2A \int_0^1 \int_0^{1-L_1} L_1 (1 - L_1 - L_2) \, dL_2 \, dL_1$$

$$= 2A \int_0^1 \int_0^{1-L_1} (L_1 - L_1^2 - L_1 L_2) \, dL_2 \, dL_1 = 2A \int_0^1 \left(L_1 L_2 - L_1^2 L_2 - L_1 \frac{L_2^2}{2} \right|_0^{1-L_1} dL_1$$

$$= 2A \int_0^1 \left(L_1 (1-L_1) - L_1^2 (1-L_1) - \frac{L_1}{2} (1-L_1)^2 \right) dL_1$$

$$= 2A \int_0^1 \left(L_1 - L_1^2 - L_1^2 + L_1^3 - \frac{L_1}{2} (1 - 2L_1 + L_1^2) \right) dL_1$$

$$= 2A \int_0^1 \left(L_1 - 2L_1^2 + L_1^3 - \frac{L_1}{2} + \frac{2L_1^2}{2} - \frac{L_1^3}{2} \right) dL_1$$

$$= 2A \int_0^1 \left(\frac{L_1}{2} - L_1^2 + \frac{L_1^3}{2} \right) dL_1 = A \left(\frac{L_1^2}{2} - \frac{2L_1^3}{3} + \frac{L_1^4}{4} \right) \Big|_0^1$$

$$= \frac{A}{12}$$

Seem to be off by a factor of 2, check   Found error & updated All of the above lines

$$\iint_\Delta f(x,y)\, d\Omega \overset{?}{=} A \int_0^1 \int_0^{1-L_1} f(L_1, L_2)\, dL_2\, dL_1$$

$$d\Omega = |J|\, dL_2\, dL_1 \qquad J = \begin{pmatrix} \frac{\partial x}{\partial L_1} & \frac{\partial x}{\partial L_2} \\ \frac{\partial y}{\partial L_1} & \frac{\partial y}{\partial L_2} \end{pmatrix}$$

$$x = x(L_1, L_2)$$
$$y = y(L_1, L_2)$$

$$dx = x_{L_1} dL_1 + x_{L_2} dL_2 \qquad x = x_1 L_1 + x_2 L_2 + x_3(1 - L_1 - L_2)$$
$$dy = y_{L_1} dL_1 + y_{L_2} dL_2 \qquad y = y_1 L_1 + y_2 L_2 + y_3(1 - L_1 - L_2)$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \underbrace{\begin{pmatrix} x_{L_1} & x_{L_2} \\ y_{L_1} & y_{L_2} \end{pmatrix}}_{J} \begin{pmatrix} dL_1 \\ dL_2 \end{pmatrix} \qquad \frac{\partial x}{\partial L_1} = x_1 - x_3 \quad \frac{\partial x}{\partial L_2} = x_2 - x_3$$
$$\frac{\partial y}{\partial L_1} = y_1 - y_3 \quad \frac{\partial y}{\partial L_2} = y_2 - y_3$$

✓ factor of 2

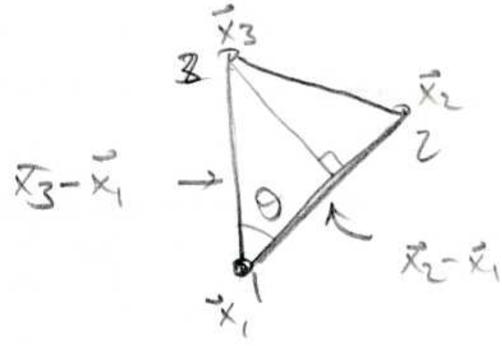
$$|J| = (x_1 - x_3)(y_2 - y_3) - (x_2 - x_3)(y_1 - y_3) = \underline{\underline{2A}}$$

$$A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1(y_2 - y_3) - y_1(x_2 y_3 - x_3 y_2) + (x_2 y_3 - y_2 x_3)$$

$$\rightarrow A = (y_2 - y_3)(x_1) + \underline{x_3}\, y_1 \underline{y_2} - x_3 y_2$$

$$+ x_2 y_3 - y_1 x_2 y_3$$

$$= (y_2 - y_3)$$



$$M_{21} = M_{12}$$

$$M_{22} = \iint_\Delta N_2 N_2 \, d\Omega = 2A \iint_\Delta L_2^2 \, dL_2 \, dL_1$$

$$= 2A \int_0^1 \int_0^{1-L_1} L_2^2 \, dL_2 \, dL_1$$

$$= 2A \int_0^1 \frac{L_2^3}{3} \Big|_0^{1-L_1} dL_1$$

$$= \frac{2A}{3} \int_0^1 (1-L_1)^3 \, dL_1$$

$$= \frac{2A}{3} \int_0^1 (1 - 3L_1 + 3L_1^2 - L_1^3) \, dL_1$$

$$= \frac{2A}{3} \left(L_1 - \frac{3L_1^2}{2} + \frac{3L_1^3}{3} - \frac{L_1^4}{4} \right) \Big|_0^1 = \frac{2A}{3}\left(1 - \frac{3}{2} + 1 - \frac{1}{4}\right)$$

$$= \frac{2A}{3}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{2A}{3}\left(+\frac{1}{4}\right) = \frac{A}{6}$$

$\therefore$

$$2A = |\vec{x}_2 - \vec{x}_1| \sin\theta \, |\vec{x}_3 - \vec{x}_1|$$

$$\parallel$$

$$(\vec{x}_2 - \vec{x}_1) \times (\vec{x}_3 - \vec{x}_1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & x_3 - x_1 & 0 \\ y_2 - y_1 & y_3 - y_1 & 0 \end{vmatrix}$$

$$= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

$$M_{23} = \iint_{\Delta} N_2 N_3 \, d\Omega = 2A \int_0^1 \int_0^{1-L_1} L_2 L_3 \, dL_2 \, dL_1$$

$$= 2A \int_0^1 \int_0^{1-L_1} L_2(1-L_2-L_1)\, dL_2 \, dL_1 = 2A \int_0^1 \int_0^{1-L_1} (L_2 - L_2^2 - L_1 L_2)\, dL_2 \, dL_1$$

$$= 2A \int_0^1 \left(\frac{L_2^2}{2} - \frac{L_2^3}{3} - L_1 \frac{L_2^2}{2} \right)\Bigg|_0^{1-L_1} dL_1$$

$$= \frac{2A}{2} \int_0^1 \left((1-L_1)^2 - \frac{2}{3}(1-L_1)^3 - L_1(1-L_1)^2 \right) dL_1$$

$$= A \int_0^1 \left(1 - 2L_1 + L_1^2 - \frac{2}{3}(1 - 3L_1 + 3L_1^2 - L_1^3) - L_1(1 - 2L_1 + L_1^2) \right) dL_1$$

$$= A \int_0^1 \left(1 - \frac{2}{3} - \cancel{2L_1} + \cancel{2L_1} - L_1 + L_1^2 - \cancel{2L_1^2} + \cancel{2L_1^2} + \frac{2}{3}L_1^3 - L_1^3 \right) dL_1$$

$$= A \int_0^1 \left(\frac{1}{3} - L_1 + L_1^2 - \frac{L_1^3}{3} \right) dL_1$$

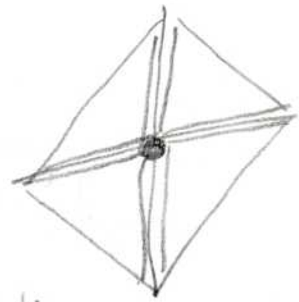
$$= \frac{A}{3} \int_0^1 (1 - 3L_1 + 3L_1^2 - L_1^3)\, dL_1 = \frac{A}{3} \int_0^1 (1-L_1)^3 \, dL_1$$

$$= -\frac{A}{3} \int_1^0 v^3 \, dv = -\frac{A}{12} v \Big|_1^0 = \frac{A}{12}$$

$M_{31} = M_{13}$ & $M_{32} = M_{23}$

04-06-01

$$M_{33} = \iint_{\Delta} N_3^{\,2}\, d\Omega = 2A \iint_{\Delta} L_3^{\,2}\, dL_2\, dL_1$$



$$= 2A \int_0^1 \int_0^{1-L_1} (1-L_1-L_2)^2\, dL_2\, dL_1$$

$$= 2A \int_0^1 \int_0^{1-L_1} \left((1-L_1)^2 - 2(1-L_1)L_2 + L_2^{\,2} \right) dL_2\, dL_1$$

$$= 2A \int_0^1 \left((1-L_1)^2 L_2 - (1-L_1)L_2^{\,2} + \frac{1}{3} L_2^{\,3} \right|_0^{1-L_1} dL_1$$

$$= 2A \int_0^1 \cancel{(1-L_1)^3} - \cancel{(1-L_1)^3} + \frac{1}{3}(1-L_1)^3\, dL_1$$

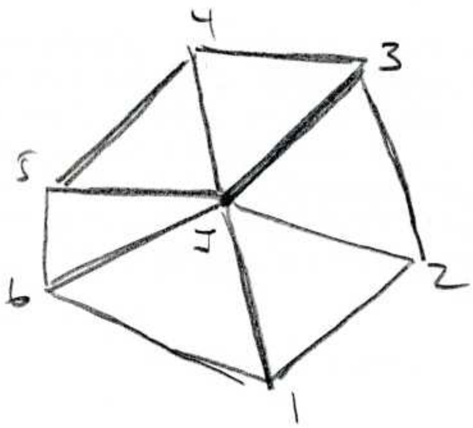
$$= \frac{2}{3} \cdot \frac{A}{4} = \frac{A}{6}$$

$$M = \frac{A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \text{Mass lumped} = \frac{A}{12} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 4$$

$$= \frac{A}{3}\, \delta_{IJ}$$

Prob 5.11

Fig 5.3.1



Show

$$\left(\overline{\frac{\partial f}{\partial x}}\right) = \frac{1}{\Omega_J}\int_{\Omega_J} \frac{\partial f}{\partial x}\, d\Omega \quad + \text{ sim } \quad \left(\overline{\frac{\partial g}{\partial y}}\right) = \frac{1}{\Omega_J}\int_{\Omega_J} \frac{\partial g}{\partial y}\, d\Omega$$

$$= \frac{1}{2\Omega_J}\sum_I f_I (y_{I+1} - y_{I-1}) \qquad = \frac{-1}{2\Omega_J}\sum_I g_I (x_{I+1} - x_{I-1})$$

Following hint $f = \sum_I f_I N_I(x,y)$ w/ $N_I(x,y)$ being linear tent fns

for Δ_{J12}

$$\int_{J12} \frac{\partial f}{\partial x}\, d\Omega = \int_{J12} \left(f_J \frac{\partial N_J}{\partial x} + f_1 \frac{\partial N_1}{\partial x} + f_2 \frac{\partial N_2}{\partial x} \right) d\Omega$$

$$N_J(x,y) = \frac{(y-y_1)(x_2-x_1) - (y_2-y_1)(x-x_1)}{(y_J-y_1)(x_2-x_1) - (y_2-y_1)(x_J-x_1)} \quad ✓$$

line between 2 pts is

$$y - y_1 = \frac{(y_2-y_1)}{x_2-x_1}(x-x_1)$$

$$\Rightarrow \quad y - y_1 - \frac{(y_2-y_1)}{x_2-x_1}(x-x_1) = 0$$

$$\rightarrow \quad (x_2-x_1)(y-y_1) - (y_2-y_1)(x-x_1) = 0$$

Then

$$N_1(x,y) = \text{switch } P_3 \leftrightarrow P_1$$

$$= \frac{(y-y_3)(x_2-x_3)-(y_2-y_3)(x-x_3)}{(y_1-y_3)(x_2-x_3)-(y_2-y_3)(x_1-x_3)}$$ ✓

&

$$N_2(x,y) = \text{switch } P_3 \Leftrightarrow P_2$$

$$= \frac{(y-y_1)(x_3-x_1)-(y_3-y_1)(x-x_1)}{(y_2-y_1)(x_3-x_1)-(y_3-y_1)(x_2-x_1)}$$ ✓

Now Area of Δ_{123} is

$$= \frac{1}{2}\left|\bar{x}_{12}\times\bar{x}_{13}\right| = \frac{1}{2}\left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ x_2-x_1 & y_2-y_1 & 0 \\ x_3-x_1 & y_3-y_1 & 0 \end{matrix} \right\|$$

$$= \left| \frac{\hat{k}}{2}\left[(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)\right] \right|$$

$$= \frac{1}{2}\left[(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)\right] = A_{123}$$

$\therefore$ denominator of all expressions is $2A_{123}$.

$$\frac{\partial N_J}{\partial x} = \frac{-(y_2 - y_1)}{2A} \qquad \frac{\partial N_J}{\partial y} = \frac{(x_2 - x_1)}{2A}$$

$$\frac{\partial N_1}{\partial x} = \frac{-(y_2 - y_J)}{2A} \qquad \frac{\partial N_1}{\partial y} = \frac{(x_2 - x_J)}{2A}$$

$$\frac{\partial N_2}{\partial x} = \frac{-(y_J - y_1)}{2A} \qquad \frac{\partial N_2}{\partial y} = \frac{x_J - x_1}{2A}$$

Then

$$\int_{J12} \frac{\partial f}{\partial x}\, d\Omega = \int_{J12} \left(f_J \frac{\partial N_J}{\partial x} + f_1 \frac{\partial N_1}{\partial x} + f_2 \frac{\partial N_2}{\partial x} \right) d\Omega$$

$$= \int_{J12} f_J \left(\frac{-(y_2 - y_1)}{2A} \right) + f_1 \left(-\frac{(y_2 - y_J)}{2A} \right) + f_2 \left(-\frac{(y_J - y_1)}{2A} \right) d\Omega$$

$$= -\frac{1}{2A} \left[f_J (y_2 - y_1) + f_1 (y_2 - y_J) + f_2 (y_J - y_1) \right] A$$

$$= \frac{1}{2} \left[f_1 (y_J - y_2) + f_2 (y_1 - y_J \right.$$

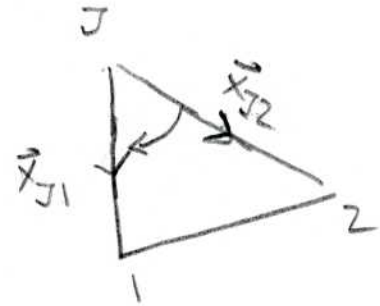
· line between 1 + J $\quad (x_J - x_1)(y - y_1) - (y_J - y_1)(x - x_1) = 0$

Note: there is an error in sign in the definitions of $N_1(x,y)$ + $N_2(x,y)$

Denominator of $N_1(x,y)$ is

$$(y_1 - y_J)(x_2 - x_J) - (y_2 - y_J)(x_1 - x_J) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_J & y_2 - y_J & 0 \\ x_1 - x_J & y_1 - y_J & 0 \end{vmatrix} \cdot \hat{k}$$

But crossing $\vec{x}_{J2}$ into $\vec{x}_{J1}$ gives the wrong sign. This vector points in the $-\hat{k}$ direction $\Rightarrow$



$\vec{x}_{J2} = \vec{r}_2 - \vec{r}_J$

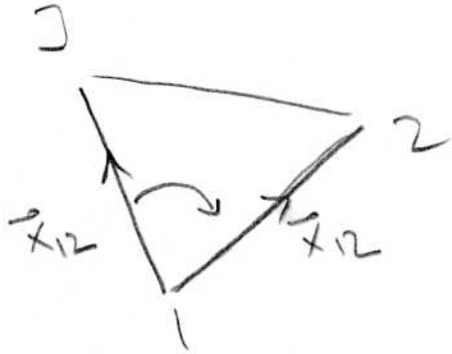
$\vec{x}_{1J} = \vec{x}_J - \vec{x}_1$

What About Denominator of $N_2(x,y)$

$$(x_J - x_1)(y_2 - y_1) - (y_J - y_1)(x_2 - x_1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_J - x_1 & y_J - y_1 & 0 \\ x_2 - x_1 & y_2 - y_1 & 0 \end{vmatrix} \cdot \hat{k}$$

Again cross gives wrong sign

Correcting the results on pg 3 gives



$$\frac{\partial N_3}{\partial x} = \frac{-(y_2-y_1)}{2A} \qquad \frac{\partial N_3}{\partial y} = \frac{x_2-x_1}{2A}$$

$$\frac{\partial N_1}{\partial x} = \frac{-(y_3-y_2)}{2A} \qquad \frac{\partial N_1}{\partial y} = \frac{(x_3-x_2)}{2A}$$

$$\frac{\partial N_2}{\partial y} = \frac{-(y_1-y_3)}{2A} \qquad \frac{\partial N_2}{\partial y} = \frac{(x_1-x_3)}{2A}$$

Then

$$\int_{312} \frac{\partial f}{\partial x}\, d\Omega = -\frac{1}{2A}\left[f_3(y_2-y_1) + f_1(y_3-y_2) + f_2(y_1-y_3)\right] A$$

$$= \frac{1}{2}\left[f_1(y_2-y_3) + f_2(y_3-y_1) + f_3(y_1-y_2)\right]$$

By Anlgy switch x's w/ y's & take negative

$$\int_{312} \frac{\partial g}{\partial y}\, d\Omega = -\frac{1}{2}\left[g_1(x_2-x_3) + g_2(x_3-x_1) + g_3(x_1-x_2)\right]$$

Then

$$\frac{1}{\Omega_3}\int_{\Omega_3} \frac{\partial f}{\partial x}\, d\Omega = \frac{1}{\Omega_3}\sum_{\Delta's}\int_{\Delta} \frac{\partial f}{\partial x}\, d\Omega$$

$$= \frac{1}{2\Omega_3}\left[f_1(y_2-y_3) + f_2(y_3-y_1) + f_3(y_1-y_2)\right]$$

$+$ see next pg

$$\frac{1}{\Omega_J}\int_{\Omega_J} \frac{\partial f}{\partial x}\, d\Omega = \frac{1}{2\Omega_J}\Big[f_1(y_2 - y_J) + f_2(y_J - y_1) + f_J(y_1 - y_2)$$

$$+ f_2(y_3 - y_J) + f_3(y_J - y_2) + f_J(y_2 - y_3)$$

$$+ f_3(y_4 - y_J) + f_4(y_J - y_3) + f_J(y_3 - y_4)$$

$$+ f_4(y_5 - y_J) + f_5(y_J - y_4) + f_J(y_4 - y_5)$$

$$+ f_5(y_6 - y_J) + f_6(y_J - y_5) + f_J(y_5 - y_6)$$

$$+ f_6(y_1 - y_J) + f_1(y_J - y_6) + f_J(y_6 - y_1)\Big]$$

$$= \frac{1}{2\Omega_J}\Big[f_J\cancelto{0}{(y_1 - y_2 + y_2 - y_3 + y_3 - y_4 + y_4 - y_5 + y_5 - y_6 + y_6 - y_1)}$$

$$+ f_1(y_2 - y_6) + f_2(y_3 - y_1) + f_3(y_4 - y_2)$$

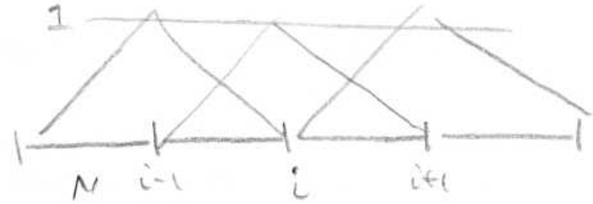
$$+ f_4(y_5 - y_3) + f_5(y_6 - y_4) + f_6(y_1 - y_5)\Big]$$

$$= \frac{1}{2\Omega_J}\sum_i f_i(y_{i+1} - y_{i-1})$$

For $\frac{1}{\Omega_J}\int_{\Omega_J} \frac{\partial g}{\partial y}\, d\Omega$ switch x w/ y & introduce a minus sign

$$\frac{1}{\Omega_J}\int_{\Omega_J} \frac{\partial g}{\partial y}\, d\Omega = \frac{-1}{2\Omega_J}\sum_i g_i(x_{i+1} - x_{i-1})$$

Prob 5.44



(*) $\dfrac{\partial U}{\partial t} + \dfrac{\partial f}{\partial x} = 0$

$$U(x,t) = \sum_{i=1}^{N} U_i(t) N_i(x) \quad \text{Also} \quad f(x,t) = \sum_{i=1}^{N} f_i(t) N_i(x)$$

Mult eq * by $w(x)$ & integrate w.r.t. x

$N_{i-1}(x) = \dfrac{x_i - x}{\Delta x}$

$N_i(x) = \dfrac{x - x_{i-1}}{\Delta x}$ $x_{i-1} < x < x_i$

$$\int w \frac{\partial U}{\partial t}\, dx + \int w \frac{\partial f}{\partial x}\, dx = 0$$

$N_i(x) = \dfrac{x_{i+1} - x}{\Delta x}$ $x_i < x < x_{i+1}$

$N_{i+1}(x) = \dfrac{x - x_i}{\Delta x}$

put in assumed expression for $U(x,t)$ & $f(x,t)$ & $w(x) = N_i(x)$ & integrate over elements $(i-1, i)$ & $(i, i+1)$

$$\Rightarrow \int_{x_{i-1}}^{x_{i+1}} N_i(x) \sum_{j=i-1}^{i+1} U_j'(t) N_j(x)\, dx + \int_{x_{i-1}}^{x_{i+1}} N_i(x) \sum_{j=i-1}^{i+1} f_j N_j'(x)\, dx = 0$$

$$\Rightarrow \sum_{j=i-1}^{i+1} U_j'(t) \int_{x_{i-1}}^{x_{i+1}} N_i(x) N_j(x)\, dx + \sum_{j=i-1}^{i+1} f_j \int_{x_{i-1}}^{x_{i+1}} N_i(x) N_j'(x)\, dx = 0$$

$j = i-1$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N_{i-1}(x)\, dx = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{\Delta x}\right)\left(\frac{x_i - x}{\Delta x}\right) dx = \frac{\Delta x}{6}$$

04-03-01

$\underline{j=i}$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N_i(x)\, dx = \int_{x_{i-1}}^{x_i} \frac{(x-x_{i-1})^2}{\Delta x^2}\, dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-x)^2}{\Delta x^2}\, dx$$

$$= \frac{2\Delta x}{3}$$

$\underline{j=i+1}$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N_{i+1}(x)\, dx = \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1}-x}{\Delta x}\right)\left(\frac{x-x_i}{\Delta x}\right) dx = \frac{\Delta x}{6}$$

$\underline{j=i-1}$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N_{i-1}'(x)\, dx = \int_{x_{i-1}}^{x_i} N_i(x)\left(\frac{-1}{\Delta x}\right) dx = \frac{-1}{\Delta x}\int_{x_{i-1}}^{x_i} \frac{x-x_{i-1}}{\Delta x}\, dx = -\frac{1}{2}$$

$\underline{j=i}$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N_i'(x)\, dx = \int_{x_{i-1}}^{x_i} \frac{(x-x_{i-1})}{\Delta x}\left(\frac{1}{\Delta x}\right) dx + \int_{x_i}^{x_{i+1}} \frac{(x_{i+1}-x)}{\Delta x}\left(\frac{-1}{\Delta x}\right) dx$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

$$\int_{x_{i-1}}^{x_{i+1}} N_i(x) N'_{i+1}(x)\,dx = \int_{x_i}^{x_{i+1}} \frac{(x_{i+1} - x)}{\Delta x}\left(\frac{1}{\Delta x}\right) dx = +\frac{1}{2}$$

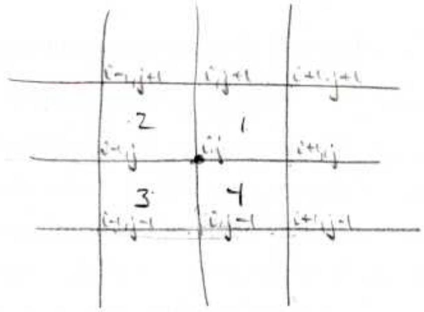
Then numerical scheme becomes:

$$\frac{dw_{i-1}}{dt}\frac{\Delta x}{6} + \frac{dw_i}{dt}\frac{2\Delta x}{3} + \frac{dw_{i+1}}{dt}\frac{\Delta x}{6} + f_{i-1}\left(\frac{-1}{2}\right) + \frac{1}{2} f_{i+1} = 0$$

$$\Rightarrow \frac{1}{6}\left[\frac{dw_{i-1}}{dt} + 4\frac{dw_i}{dt} + \frac{dw_{i+1}}{dt}\right] = \frac{1}{2\Delta x}\left[f_{i+1} - f_{i-1}\right]$$

Prob 5.16

$$\left(\frac{\partial u}{\partial x}\right)_{ij} \equiv \frac{\displaystyle\int_{\Omega_I} \frac{\partial u}{\partial x} N_I \, d\Omega}{\displaystyle\int_{\Omega_I} N_I \, d\Omega}$$



Bilinear elements: $N_I = \frac{1}{4}(1+\xi\xi_I)(1+\eta\eta_I)$

$$\int_{\Omega_I} N_I \, d\Omega = 4 \int_{E_3} N_{ij}(\vec{x}) \, d\Omega = 4 \int_{x_{i-1}}^{x_{i-1}+\Delta x} \int_{y_{j-1}}^{y_{j-1}+\Delta y} \left(\frac{x-x_{i-1}}{\Delta x}\right)\left(\frac{y-y_{j-1}}{\Delta y}\right) d\Omega$$ ✓

$$= 4 \int_{x_{i-1}}^{x_{i-1}+\Delta x} \left(\frac{x-x_{i-1}}{\Delta x}\right) dx \int_{y_{j-1}}^{y_{j-1}+\Delta y} \left(\frac{y-y_{j-1}}{\Delta y}\right) dy$$

$$= \frac{\cancel{4}}{\Delta x \Delta y} \frac{(x-x_{i-1})^2}{\cancel{2}} \Bigg|_{x_{i-1}}^{x_{i-1}+\Delta x} \frac{(y-y_{j-1})^2}{\cancel{2}} \Bigg|_{y_{j-1}}^{y_{j-1}+\Delta y}$$

$$= \frac{\Delta x^2 \, \Delta y^2}{\Delta x \Delta y} = \Delta x \Delta y = \Delta x^2 \quad \text{if } \Delta x = \Delta y.$$

$$\int_{\Omega_I} \frac{\partial u}{\partial x} N_I \, d\Omega \qquad u(x) \cong U_{i..}$$

w/ $u(\bar{x}) \cong U_{i-1,j-1} N_{i-1,j-1}(\bar{x}) + U_{i,j-1} N_{i,j-1}(\bar{x}) + U_{i+1,j-1} N_{i+1,j-1}(\bar{x}) +$

$U_{i-1,j} N_{i-1,j}(\bar{x}) + U_{i,j} N_{i,j}(\bar{x}) + U_{i+1,j} N_{i+1,j}(x) +$

$U_{i-1,j+1} N_{i-1,j+1}(\bar{x}) + U_{i,j+1} N_{i,j+1}(\bar{x}) + U_{i+1,j+1} N_{i+1,j+1}(\bar{x})$

$$\frac{\partial u}{\partial x}(\bar{x}) \cong U_{i-1,j-1} \frac{\partial N_{i-1,j-1}}{\partial x} + U_{i,j-1} \frac{\partial N_{i,j-1}}{\partial x} + \cdots$$

$$\Rightarrow \quad U_{i-1,j-1} \int_{E_3} \frac{\partial N_{i-1,j-1}}{\partial x} N_{i,j} \, d\Omega + U_{i,j-1} \int_{E_3 \cup E_4} \frac{\partial N_{i,j-1}}{\partial x} N_{i,j} \, d\Omega$$

$$+ U_{i+1,j-1} \int_{E_4} \frac{\partial N_{i+1,j-1}}{\partial x} N_{i,j} \, d\Omega + U_{i-1,j} \int_{E_3 \cup E_2} \frac{\partial N_{i-1,j}}{\partial x} N_{i,j} \, d\Omega$$

$$+ U_{i,j} \int_{E_1 \cup E_2 \cup E_3 \cup E_4} \frac{\partial N_{i,j}}{\partial x} N_{i,j} \, d\Omega + U_{i+1,j} \int_{E_4 \cup E_1} \frac{\partial N_{i+1,j}}{\partial x} N_{i,j} \, d\Omega$$

$$+ U_{i-1,j+1} \int_{E_2} \frac{\partial N_{i-1,j+1}}{\partial x} N_{i,j} \, d\Omega + U_{i,j+1} \int_{E_2 \cup E_1} \frac{\partial N_{i,j+1}}{\partial x} N_{i,j} \, d\Omega$$

$$+ U_{i+1,j+1} \int_{E_1} \frac{\partial N_{i+1,j+1}}{\partial x} N_{ij} \, d\Omega .$$

Integrating each expression one by one

$$\int_{E_3} \frac{\partial N_{i,j-1}}{\partial x} N_{ij} \, d\Omega = \int_{E_3} \frac{\partial}{\partial x}\left[\left(1 - \frac{x - x_{i-1}}{\Delta x}\right)\left(1 - \frac{y - y_{j-1}}{\Delta y}\right)\right]\left(\frac{x - x_{i-1}}{\Delta x}\right)\left(\frac{y - y_{j-1}}{\Delta y}\right) d\Omega$$

$$= -\frac{1}{\Delta x} \int_{E_3} \left(\frac{x - x_{i-1}}{\Delta x}\right)\left(\frac{y - y_{j-1}}{\Delta y}\right)\left(1 - \frac{y - y_{j-1}}{\Delta y}\right) d\Omega$$

$\Delta x = \Delta y$

$$= -\frac{1}{\Delta x^3} \int_{y_{j-1}}^{y_{j-1}+\Delta y} \int_{x_{i-1}}^{x_{i-1}+\Delta x} (x - x_{i-1})(y - y_{j-1})\left(1 - \frac{y - y_{j-1}}{\Delta y}\right) dx \, dy$$

$$= -\frac{\Delta x}{12}$$

$$\int_{E_3 \cup E_4} \frac{\partial N_{i,j-1}}{\partial x} N_{ij} \, d\Omega = \int_{E_3} \quad + \int_{E_1}$$

$$= \int_{E_3} \frac{\partial}{\partial x}\Big($$

Prob 5.17

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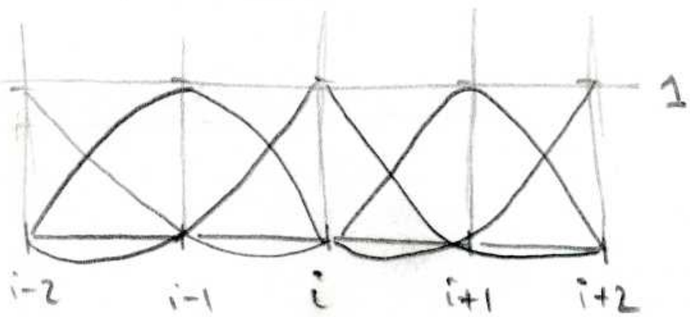

$U_t + aU_x = 0$

$N_1 = -\frac{1}{2}\xi(1-\xi)$ $N_1'(\xi) = -\frac{1}{2}(1-2\xi)$

$N_2 = 1-\xi^2$ $N_2'(\xi) = -2\xi$

$N_3 = \frac{1}{2}\xi(1+\xi)$ $N_3'(\xi) = \frac{1}{2}(1+2\xi)$

Multiply by $\varpi(x)$ & integrate

$$\int \varpi(x) U_t \, dx + a\int \varpi(x) U_x \, dx = 0$$

Now $U(x,t) \cong \sum_{i=0}^{N} U_i(t) N_i(x)$ $U_t(x,t) \cong \sum_{i=0}^{N} U_i'(t) N_i(x)$

$$\Rightarrow \sum_{i=0}^{N} U_i'(t) \int \varpi(x) N_i(x) dx + a\sum_{i=0}^{N} U_i(t) \int \varpi(x) N_i'(x) \, dx = 0$$

Pick $\varpi(x) = N_i(x)$ Then integrate over element to the left & element to the Right

$$\Rightarrow U_{i-2}'(t)\int_{x_{i-2}}^{x_i} N_i(x) N_{i-2}(x)\,dx + U_{i-1}'(t)\int_{x_{i-2}}^{x_i} N_i(x)N_{i-1}(x)\,dx + U_i'(t)\int_{x_{i-2}}^{x_i} N_i(x)N_i(x)\,dx$$

$$+ U_i'(t)\int_{x_i}^{x_{i+2}} N_i(x)N_i(x)\,dx + U_{i+1}'(t)\int_{x_i}^{x_{i+2}} N_i(x)N_{i+1}(x)\,dx + U_{i+2}'(t)\int_{x_i}^{x_{i+2}} N_i(t)N_{i+2}(x)\,dx$$

$$a U_{i-2}(t) \int_{x_{i-2}}^{x_i} N_i(x) N_{i-2}'(x)\,dx + a U_{i-1}(t) \int_{x_{i-2}}^{x_i} N_i(x) N_{i-1}'(x)\,dx + a U_i(t) \int_{x_{i-2}}^{x_i} N_i(x) N_i'(x)\,dx$$

$$+ a U_i(t) \int_{x_i}^{x_{i+2}} N_i(x) N_i'(x)\,dx + a U_{i+1}(t) \int_{x_i}^{x_{i+2}} N_i(x) N_{i+1}'(x) + a U_{i+2}(t) \int_{x_i}^{x_{i+2}} N_i(x) N_{i+2}'(x)\,dx$$

$$= 0$$

$\xi : (-1,+1) \to (a,b)$

$$x - a = \frac{(b-a)}{(2)}(\xi+1)$$

$$x = \frac{(b-a)}{2}(\xi+1) + a$$

$$dx = \frac{\Delta x}{2}\,d\xi$$

Note:
$b - a = x_i - x_{i-2}$
$= 2\Delta x$
w/
$\Delta x \equiv x_i - x_{i-1}$

$$\int_a^b f(x)\,dx = \int_{-1}^{+1} f(\xi)\frac{\Delta x}{2}\,d\xi = \frac{\Delta x}{2}\int_{-1}^{+1} f(\xi)\,d\xi$$

$\Rightarrow$

$$\frac{\Delta x}{2}\Big[U_{i-2}'(t)\int_{-1}^{1} N_i(\xi) N_{i-2}(\xi)\,d\xi + U_{i-1}'(t)\int_{-1}^{+1} N_i(\xi) N_{i-1}(\xi)\,d\xi + U_i'(t)\int_{-1}^{+1} N_i(\xi) N_i(\xi)\,d\xi$$

$$+ U_i'(t)\int_{-1}^{+1} N_i(\xi) N_i(\xi)\,d\xi + U_{i+1}'(t)\int_{-1}^{+1} N_i(\xi) N_{i+1}(\xi)\,d\xi + U_{i+2}'(t)\int_{-1}^{+1} N_i(\xi) N_{i+2}(\xi)\,d\xi\Big]$$

$$\int_a^b f'(x)\,dx = \int_a^b \frac{df}{dx}\cdot dx = \int_{-1}^{+1} \frac{df}{d\xi}\cdot\frac{d\xi}{dx}\cdot dx$$

How ~~deriv.~~ integrals w/ derivatives will transform.

$$= \int_{-1}^{+1} \frac{df}{d\xi}\underbrace{\left(\frac{d\xi}{dx}\cdot\frac{dx}{d\xi}\right)}_{=1} d\xi = \int_{-1}^{+1}\frac{df}{d\xi}\,d\xi$$

$$+ aU_{i-2}(t)\int_{-1}^{+1} N_i(\xi)N_{i-2}'(\xi)\,d\xi + aU_{i-1}(t)\int_{-1}^{+1} N_i(\xi)N_{i-1}'(\xi)\,d\xi + aU_i(t)\int_{-1}^{+1} N_i(\xi)N_i'(\xi)\,d\xi$$

$$+ aU_i(t)\int_{-1}^{+1} N_i(\xi)N_i'(\xi)\,d\xi + aU_{i+1}(t)\int_{-1}^{+1} N_i(\xi)N_{i+1}'(\xi)\,d\xi + aU_{i+2}\int_{-1}^{+1} N_i(\xi)N_{i+2}'(\xi)\,d\xi$$

$$= 0$$

Evaluating each integral in turn gives:

$$\int_{-1}^{+1} \frac{1}{2}(\xi+\xi^2)\left(-\frac{1}{2}\right)(\xi-\xi^2)\,d\xi = -\frac{1}{4}\int_{-1}^{+1}(\xi^2-\xi^4)\,d\xi$$

w/ MMA gives.

$$= \frac{\Delta x}{2}\Big[U_{i-2}'(t)\left(-\frac{1}{15}\right) + U_{i-1}'(t)\left(\frac{2}{15}\right) + U_i'(t)\left(\frac{4}{15}\right)$$

$$+ U_i'(t)\left(\frac{4}{15}\right) + U_{i+1}'(t)\left(\frac{2}{15}\right) + U_{i+2}'(t)\left(-\frac{1}{15}\right)\Big]$$

$$+ aU_{i-2}(t)\left(\frac{1}{6}\right) + aU_{i-1}(t)\left(\frac{-2}{3}\right) + aU_i\cancel{(t)}\left(\frac{1}{2}\right)$$

$$+ a u_i\cancel{(t)}\left(\frac{-1}{2}\right) + a u_{i+1}(t)\left(\frac{2}{3}\right) + a u_{i+2}\left(\frac{-1}{6}\right) = 0$$

$$\Rightarrow \quad \frac{\Delta x}{30}\left(-\frac{du_{i-2}}{dt} + 2\frac{du_{i-1}}{dt} + 8\frac{du_i}{dt} + 2\frac{du_{i+1}}{dt} - \frac{du_{i+2}}{dt}\right)$$

$$+ \frac{a}{6}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right) = 0$$

$$\Rightarrow \quad \frac{1}{10}\left(-\frac{du_{i-2}}{dt} + 2\frac{du_{i-1}}{dt} + 8\frac{du_i}{dt} + 2\frac{du_{i+1}}{dt} - \frac{du_{i+2}}{dt}\right) \qquad -\cancel{1}+\cancel{2}+8+2-\cancel{1} = 10 \checkmark$$

$$+ \frac{a}{2\Delta x}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right) = 0$$

Thus again the books $\Delta x_{Book} = \frac{\Delta x_{min}}{2} \Rightarrow \Delta x_{min} = 2\Delta x_{Book}$ &

Above formula becomes

$$\frac{1}{10}\left(-\frac{du_{i-2}}{dt} + 2\frac{du_{i-1}}{dt} + 8\frac{du_i}{dt} + 2\frac{du_{i+1}}{dt} - \frac{du_{i+2}}{dt}\right)$$

$$+ \frac{a}{4\Delta x}\left(u_{i-2} - 4u_{i-1} + 4u_{i+1} - u_{i+2}\right) = 0 \quad \checkmark$$

let $W(x) = N_{i+1}(x)$ then integrate only over one element.

$$\Rightarrow U_i'(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_i(x)\,dx + U_{i+1}'(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_{i+1}(x)\,dx + U_{i+2}'(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_{i+2}(x)\,dx$$

$$+ aU_i(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_i'(x)\,dx + aU_{i+1}(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_{i+1}'(x)\,dx + aU_{i+2}(t)\int_{x_i}^{x_{i+2}} N_{i+1}(x)N_{i+2}'(x)\,dx$$

$$= 0$$

$$\Rightarrow \frac{\Delta x}{2}\left[U_i'(t)\int_{-1}^{+1} N_{i+1}(\xi)N_i(\xi)\,d\xi + U_{i+1}'(t)\int_{-1}^{+1} N_{i+1}(\xi)N_{i+1}(\xi)\,d\xi + U_{i+2}'(t)\int_{-1}^{+1} N_{i+1}(\xi)N_{i+2}(\xi)\,d\xi\right]$$

$$+ aU_i(t)\int_{-1}^{+1} N_{i+1}(\xi)N_i'(\xi)\,d\xi + aU_{i+1}(t)\int_{-1}^{+1} N_{i+1}(\xi)N_{i+1}'(\xi)\,d\xi$$

$$+ aU_{i+2}(t)\int_{-1}^{+1} N_{i+1}(\xi)N_{i+2}'(\xi)\,d\xi = 0.$$

integrating each term by MMA gives

$$\Rightarrow \frac{\Delta x}{2}\left[U_i'(t)\left(\frac{2}{15}\right) + U_{i+1}'(t)\left(\frac{16}{15}\right) + U_{i+2}'(t)\left(\frac{2}{15}\right)\right]$$

$$+ a\left[U_i(t)\left(-\frac{2}{3}\right) + U_{i+1}(t)\cdot 0 + U_{i+2}(t)\left(\frac{2}{3}\right)\right] = 0$$

$$\Rightarrow \frac{2\Delta x}{30}\left[\frac{dv_{i+2}}{dt} + 8\frac{dv_{i+1}}{dt} + \frac{dv_i}{dt}\right] + \frac{2a}{3}\left[v_{i+2}(t) - v_i(t)\right] = 0$$

$$\Rightarrow \frac{1}{10}\left[\frac{dv_{i+2}}{dt} + 8\frac{dv_{i+1}}{dt} + \frac{dv_i}{dt}\right] + \frac{a}{\Delta x}\left[v_{i+2}(t) - v_i(t)\right] = 0$$

Again My $\Delta x = 2\Delta x_{Book}$

$$\Rightarrow \frac{1}{10}\left[\frac{dv_{i+2}}{dt} + 8\frac{dv_{i+1}}{dt} + \frac{dv_i}{dt}\right] + \frac{a}{2\Delta x}\left[v_{i+2}(t) - v_i(t)\right] = 0$$