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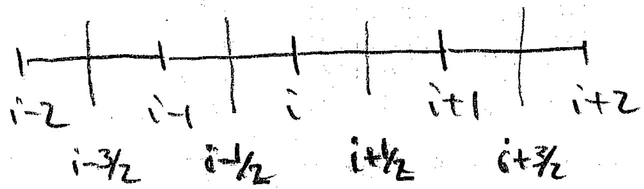
04-14-01

$u_t + a(u)u_x = q$

$\frac{du_i}{dt}$



$$(a(u)u_x)_i \cong a_i \frac{(u_{i+1/2} - u_{i-1/2})}{\Delta x} \quad \text{w/ } a_i = \frac{1}{2}(a_{i+1/2} + a_{i-1/2})$$

To what order is this an approximation to $au_x|_{x_i}$?

All even powers will vanish + all odd powers will double

$$\bar{a}_i \frac{(u_{i+1/2} - u_{i-1/2})}{\Delta x} = \frac{1}{2}\underbrace{(a_{i+1/2} + a_{i-1/2})}\frac{(u_{i+1/2} - u_{i-1/2})}{\Delta x}$$

All odd powers will vanish + All even powers will double.

$$= \frac{1}{2}\left[a_i + \frac{\Delta x}{2}a' + \frac{\Delta x^2}{4 \cdot 2}a'' + \frac{\Delta x^3}{6\, 2^3}a''' + O(\Delta x^4)\right)$$

$$+ \left(a_i - \frac{\Delta x}{2}a' + \frac{\Delta x^2}{4 \cdot 2}a'' - \Delta x\right.$$

Based on observations above

$= \frac{1}{2}\big(2$ taylor expansion of $a_{i+1/2}$ w/ only even powers $\big)$ $\frac{1}{\Delta x}$ 2 taylor expansion of $u_{i+1/2}$ w/ only odd powers

$$= \left(a_i + \frac{\Delta x^2}{2}a_i'' + \frac{\Delta x^4}{24 \cdot 2^4}a_i^{IV} + O(\Delta x^6)\right)\frac{2}{\Delta x}\left(u_i'\frac{\Delta x}{2} + \frac{u_i'''}{6}\frac{\Delta x^3}{2^3} + \frac{u_i^{V}}{5!}\frac{\Delta x^5}{2^5} + O(\Delta x^7)\right)$$

$$= \left(a_i + \frac{\Delta x^2}{2} a_i'' + \frac{\Delta x^4}{24 \cdot 2^4} a_i^{IV} + O(\Delta x^6)\right)\left(U_i' + \frac{U_i'''}{6 \cdot 2^2}\Delta x^2 + \frac{U_i^{V}}{5!\,2^4}\Delta x^4 + O(\Delta x^6)\right)$$

$$= a_i U_i' + \left(\frac{a_i U_i'''}{24} + \frac{U_i' a_i''}{2}\right)\Delta x^2 + O(\Delta x^4)$$

$$\therefore \quad a U_x \cong \frac{1}{2}(a_{i+1/2} + a_{i-1/2})\frac{(U_{i+1/2} - U_{i-1/2})}{\Delta x} + O(\Delta x^2)$$

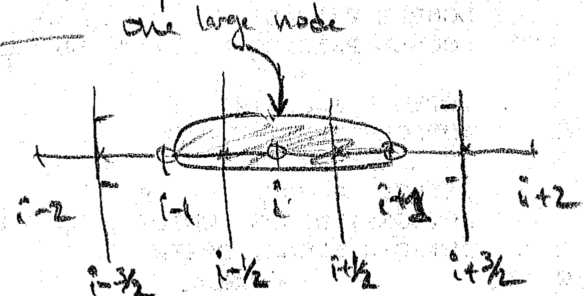
Then writing the discrete approximate eq for $U_t + aU_x = q$ at nodes $i-1, i, \& \, i+1$

we get

$$\frac{dU_i}{dt} + \bar{a}_i\frac{(U_{i+1/2} - U_{i-1/2})}{\Delta x} = q_i$$

$$\frac{dU_{i-1}}{dt} + \bar{a}_{i-1}\frac{(U_{i-1/2} - U_{i-3/2})}{\Delta x} = q_{i-1}$$

$$\frac{dU_{i+1}}{dt} + \bar{a}_{i+1}\frac{(U_{i+3/2} - U_{i+1/2})}{\Delta x} = q_{i+1}$$



When we average 3 locations we get one large node whose contributions in terms of the flux terms come from nodes $i+3/2$ & $i-3/2$.

& Averaging these 3 eqs gives

$$\frac{d}{dt}\left(\frac{1}{3}(U_{i-1} + U_i + U_{i+1})\right) + \frac{1}{2}(a_{i+3/2} + a_{i-3/2})\frac{(U_{i+3/2} - U_{i-3/2})}{3\Delta x}$$

$$- \frac{1}{3}(q_{i-1} + q_i + q_{i+1}) = + \frac{1}{2}(a_{i+3/2} + a_{i-3/2})\frac{(U_{i+3/2} - U_{i-3/2})}{3\Delta x}$$

$$- \frac{1}{2}(a_{i+1/2} + a_{i-1/2})\frac{(U_{i+1/2} - U_{i-1/2})}{3\Delta x}$$

$$- \frac{1}{2}(a_{i-3/2} + a_{i-1/2})\frac{(v_{i-1/2} - v_{i-3/2})}{3\Delta x}$$

$$- \frac{1}{2}(a_{i+3/2} + a_{i+1/2})\frac{(v_{i+3/2} - v_{i+1/2})}{3\Delta x}$$

RHS of this expression simplifies to:

$$= \frac{1}{6\Delta x}\Big[a_{i+3/2}\cancel{v_{i+3/2}} - \underset{1}{a_{i+3/2}v_{i-3/2}} + \underset{2}{a_{i-3/2}v_{i+3/2}} - a_{\cancel{i-3/2}}v_{i-3/2}$$

$$- a_{i+1/2}\cancel{v_{i+1/2}} + \underset{3}{a_{i+1/2}v_{i-1/2}} - \underset{4}{a_{i-1/2}v_{i+1/2}} + a_{\cancel{i-1/2}}v_{i-1/2}$$

$$- \underset{3}{a_{i-3/2}v_{i-1/2}} + \cancel{a_{i-3/2}v_{i-3/2}} - \cancel{a_{i-1/2}v_{i-1/2}} + \underset{1}{a_{i-1/2}v_{i-3/2}}$$

$$- a_{i+3/2}\cancel{v_{i+3/2}} + \underset{4}{a_{i+3/2}v_{i+1/2}} - \underset{2}{a_{i+1/2}v_{i+3/2}} + \cancel{a_{i+1/2}v_{i+1/2}} \Big]$$

$$= \frac{1}{6\Delta x}\Big[\underset{1}{(-a_{i+3/2} + a_{i-1/2})}v_{i-3/2} + \underset{2}{(-a_{i+1/2} + a_{i-3/2})}v_{i+3/2}$$

$$+ \underset{2}{(a_{i+1/2} - a_{i-3/2})}v_{i-1/2} + \underset{1}{(a_{i+3/2} - a_{i-1/2})}v_{i+1/2} \Big]$$

$$= \frac{1}{6\Delta x}\Big[+ (a_{i+3/2} - a_{i-1/2})(v_{i+1/2} - v_{i-3/2})$$

$$- (a_{i+1/2} - a_{i-3/2})(v_{i+3/2} - v_{i-1/2}) \Big]$$

$$= -(a_{i+1/2} - a_{i-3/2})\frac{(v_{i+3/2} - v_{i-1/2})}{6\Delta x} + (a_{i+3/2} - a_{i-1/2})\frac{(v_{i+1/2} - v_{i-3/2})}{6\Delta x}$$

R.H.S of eq 6.1.10

Taylor series expanding R.H.S gives

$$-\Big[\cancel{a_i} + \frac{\Delta x}{2} a_i' + \frac{\Delta x^2}{2 \cdot 2^2} a_i'' + \frac{\Delta x^3}{6 \cdot 2^3} a_i''' + O(\Delta x^4)$$

$$-\Big(\cancel{a_i} - \frac{3\Delta x}{2} a_i' + \frac{9}{4(2)}\Delta x^2 a_i'' - \frac{27}{8(6)}\Delta x^3 a_i''' + O(\Delta x^4)\Big)\Big] \cdot$$

$$\frac{1}{6\Delta x}\Big[\cancel{\upsilon_i} + \frac{3}{2}\Delta x\, \upsilon_i' + \frac{9}{4}\frac{\Delta x^2}{2}\upsilon_i'' + \frac{27}{8(6)}\Delta x^3 \upsilon_i''' + O(\Delta x^4)$$

$$-\Big(\cancel{\upsilon_i} - \frac{\Delta x}{2}\upsilon_i' + \frac{\Delta x^2}{4(2)}\upsilon_i'' - \frac{\Delta x^3}{8(6)}\upsilon_i''' + O(\Delta x^4)\Big)\Big]$$

$$+\Big[\cancel{a_i} + \frac{3}{2}\Delta x\, a_i' + \frac{9}{4}\frac{\Delta x^2}{2} a_i'' + \frac{27}{8(6)}\Delta x^3 a_i''' + O(\Delta x^4)$$

$$-\Big(\cancel{a_i} - \frac{\Delta x}{2} a_i' + \frac{\Delta x^2}{4(2)} a_i'' - \frac{\Delta x^3}{8(6)} a_i''' + O(\Delta x^4)\Big)\Big] \cdot$$

$$\frac{1}{6\Delta x}\Big[\cancel{\upsilon_i} + \frac{\Delta x}{2}\upsilon_i' + \frac{\Delta x^2}{4 \cdot 2}\upsilon_i'' + \frac{\Delta x^3}{8 \cdot 6}\upsilon_i''' + O(\Delta x^4)$$

$$-\Big(\cancel{\upsilon_i} - \frac{3}{2}\Delta x\, \upsilon_i' + \frac{9}{4 \cdot 2}\Delta x^2 \upsilon_i'' - \frac{27}{86}\Delta x^3 \upsilon_i''' + O(\Delta x^4)\Big)\Big]$$

$$= -\left[\Delta x\left(\frac{1}{2}+\frac{3}{2}\right)a_i' + \Delta x^2\left(\frac{1}{8}-\frac{9}{8}\right)a_i'' + \Delta x^3\left(\frac{1}{48}+\frac{27}{48}\right)a_i''' + O(\Delta x^4)\right]\cdot$$

$$\frac{1}{6\Delta x}\left[\Delta x\left(\frac{3}{2}+\frac{1}{2}\right)u_i' + \Delta x^2\left(\frac{9}{8}-\frac{1}{8}\right)u_i'' + \Delta x^3\left(\frac{27}{48}+\frac{1}{48}\right)u_i''' + O(\Delta x^4)\right]$$

$$+ \left[\Delta x\left(\frac{3}{2}+\frac{1}{2}\right)a_i' + \Delta x^2\left(\frac{9}{8}-\frac{1}{8}\right)a_i'' + \Delta x^3\left(\frac{27}{48}+\frac{1}{48}\right)a_i''' + O(\Delta x^4)\right]\cdot$$

$$\frac{1}{6\Delta x}\left[\Delta x\left(\frac{1}{2}+\frac{3}{2}\right)u_i' + \Delta x^2\left(\frac{1}{8}-\frac{9}{8}\right)u_i'' + \Delta x^3\left(\frac{1}{48}+\frac{27}{48}\right)u_i''' + O(\Delta x^4)\right]$$

$$\frac{28}{48} \quad \frac{14}{24} = \frac{7}{12}$$

$$= -\left[2\Delta x\, a_i' - \Delta x^2 a_i'' + \Delta x^3\left(\frac{7}{12}\right)a_i''' + O(\Delta x^4)\right]\cdot$$

$$\frac{1}{6\Delta x}\left[2\Delta x\, u_i' + \Delta x^2 u_i'' + \Delta x^3\left(\frac{7}{12}\right)u_i''' + O(\Delta x^4)\right]$$

$$+ \left[2\Delta x\, a_i' + \Delta x^2 a_i'' + \Delta x^3\left(\frac{7}{12}\right)a_i''' + O(\Delta x^4)\right]\cdot$$

$$\frac{1}{6\Delta x}\left[2\Delta x\, u_i' + \Delta x^2 u_i'' + \Delta x^3\left(\frac{7}{12}\right)u_i''' + O(\Delta x^4)\right]$$

$$= O(\Delta x) \quad \rightarrow\leftarrow$$ what Book gives.

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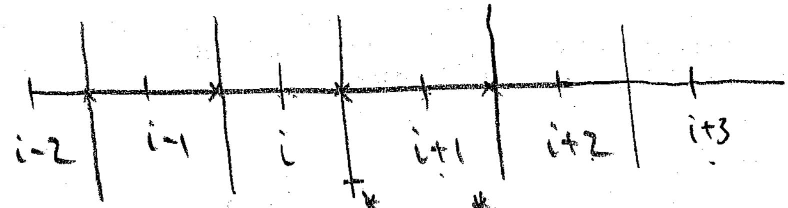


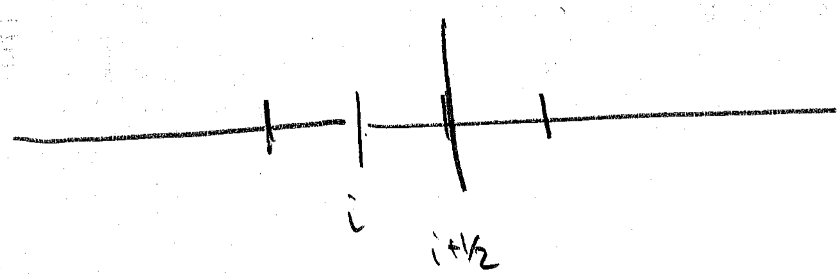
$f^*_{i+1/2} = f^*(U_{i+1}, U_i)$ $k=1$ 1 pt besides U_i U_{i+1}

potential flux dependencies

$= f^*(U_{i+2}, U_{i+1}, U_i, U_{i-1})$ $k=2$ 3 pts besides U_i $U_{i+2}, U_{i+1}, U_{i-1}$

$= f^*(U_{i+3}, U_{i+2}, U_{i+1}, U_i, U_{i-1}, U_{i-2})$

$k=3$ $5 = 2(3)-1$ points besides U_i $U_{i+3}, U_{i+2}, U_{i+1}, \dots, U_{i-1}, U_{i-2}$



Split k points on each side of $i+1/2$ $k=+1$ pts $i+1, i$

$k=+2$ pts $i+1, i+2,$
$i, i-1$

$k=+3$ pts $i+1, i+2, i+3$
$i, i-1, i-2$

general k pts $i+1, i+2, \dots, i+k$
$i, i-1, i-2, \dots, i-k+1$

eq 6.1.12

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04-16-01

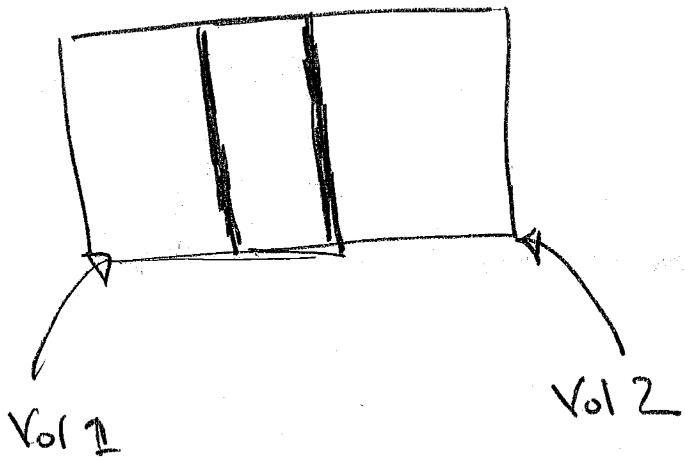
eq 5.3.14

Think W goes here look up derivation.

$$\int_{\Omega} \frac{\partial U}{\partial t} W \, d\Omega - \int_{\Omega} (\bar{F} \cdot \bar{\nabla}) W \, d\Omega + \int_{\Gamma} W \bar{F} \cdot d\bar{S} = \int_{\Omega} W Q \, d\Omega$$

$W = 1$

$$\rightarrow \int_{\Omega} \frac{\partial U}{\partial t} d\Omega + \int_{\Gamma} \bar{F} \cdot d\bar{S} = \int_{\Omega} Q \, d\Omega$$



Vol 1 + Vol 2 overlap

can we use these two as control volumns ?

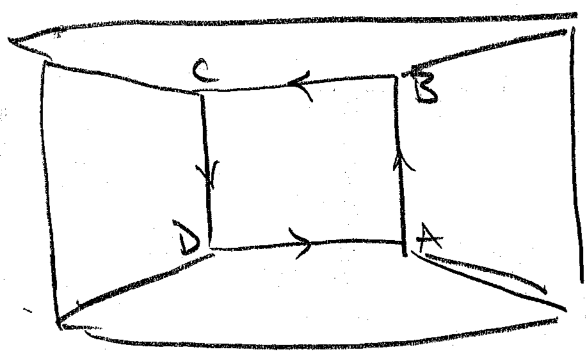
eq 6.21 is

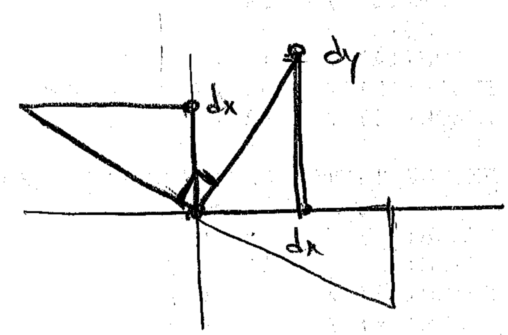
$$\frac{\partial}{\partial t}\int_{\Omega} U\,d\Omega + \oint_{S} \vec{F}\circ d\vec{S} = \int_{\Omega} Q\,d\Omega$$

reduced to 2D

$\Rightarrow \quad \vec{F} = (\vec{f}_1, \vec{f}_2) = (\vec{f}, \vec{g})$

$d\vec{S} = dx\,\hat{i} + dy\,\hat{j}$ dx, dy independent





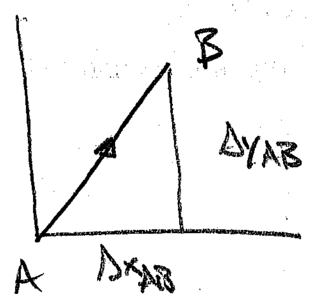
$\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$\vec{r}_{\perp}{}' = -dy\,\hat{i} + dx\,\hat{j}$ inward normal

$\vec{r}_{\perp}{}'' = dy\,\hat{i} - dx\,\hat{j}$ outward normal

$$\oint_{S} \vec{F}\circ d\vec{S} = \oint_{S} \vec{F}\circ(dy\,\hat{i} - dx\,\hat{j})$$

$$= \oint_{ABCD} (f\,dy - g\,dx)$$

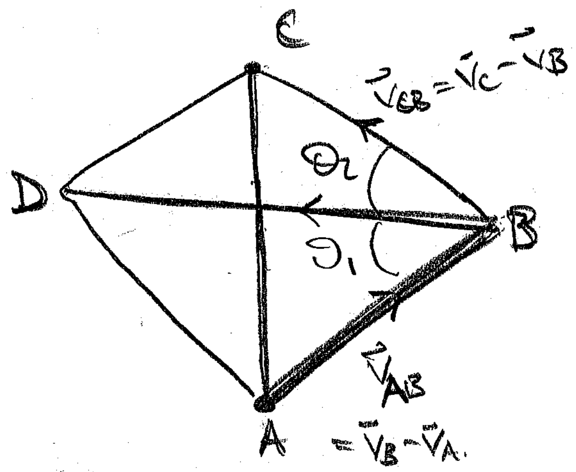


$\vec{R}_{AB} = \Delta x_{AB}\,\hat{i} + \Delta y_{AB}\,\hat{j}$

$\vec{R}_{AB\ \text{outward normal}} = \Delta y_{AB}\,\hat{i} - \Delta x_{AB}\,\hat{j}$

$$\Rightarrow \frac{\partial}{\partial t}(\Omega_{ij} U_{ij}) + \sum_{ABCD}(f_{AB}\,\Delta y_{AB} - g_{AB}\,\Delta x_{AB}) = \Omega_{ij} Q_{ij}$$

Assuming f + g constant along leg AB



$$\left|\vec{V}_{AB} \times \vec{V}_{DB}\right| = |\vec{V}_{AB}||\vec{V}_{DB}||\sin\theta_{AB,DB}|$$
$$= 2A_{ABD}$$

$$\left|\vec{V}_{BC} \times \vec{V}_{DB}\right| = |\vec{V}_{BC}||\vec{V}_{DB}||\sin\theta_{BC,DB}|$$

$\Omega_{ABCD} =$

$$= 2A_{DCB}$$

$$\therefore\ 2(\Omega_{ABCD}) = |\vec{V}_{AB} \times \vec{V}_{DB}| + |\vec{V}_{BC} \times \vec{V}_{DB}|$$

Choose orientation of cross products to have sign come out positive.

Now $\hat{k}\cdot(\vec{V}_{BA} \times \vec{V}_{DB}) > 0$ as $\theta_1 = \theta_{BA,DB} < \frac{\pi}{2}$

$\hat{k}\cdot(\vec{V}_{CB} \times \vec{V}_{DB}) > 0$ as $\theta_2 = \theta_{CB,DB} < \frac{\pi}{2}$

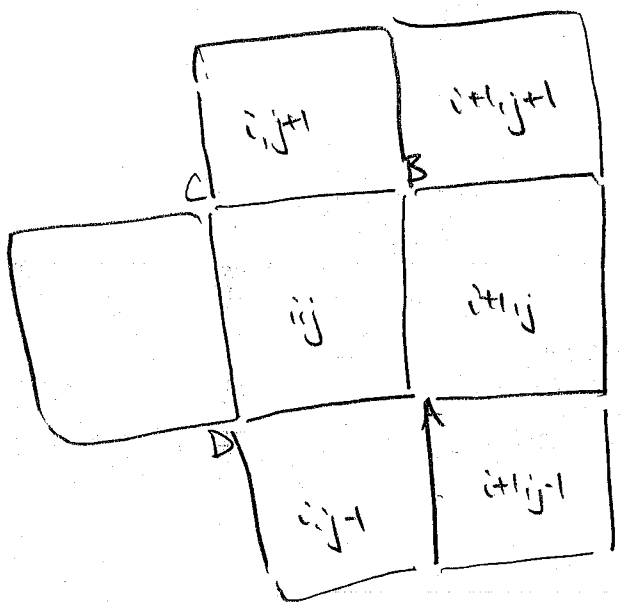
Then

$$2\Omega_{ABCD}\hat{k} = (\vec{V}_{BA} + \vec{V}_{CB}) \times \vec{V}_{DB}$$
$$= (\vec{V}_A - \cancel{\vec{V}_B} + \cancel{\vec{V}_B} - \vec{V}_C) \times \vec{V}_{DB}$$
$$= (\vec{V}_A - \vec{V}_C) \times \vec{V}_{DB} = \vec{V}_{CA} \times \vec{V}_{DB} \quad \text{w/ } \hat{k} \text{ + norms.}$$

$$\Omega_{ABCD} = \left| \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_A - x_C & x_B - x_D & 0 \\ y_A - y_C & y_B - y_D & 0 \end{vmatrix} \right| = \left| \frac{1}{2}\hat{k}\left[(y_B - y_D)(x_A - x_C) - (x_B - x_D)(y_A - y_C)\right] \right|$$

$$\Omega_{ABCD} = \frac{1}{2}\left[(x_C - x_A)(y_D - y_B) - (y_C - y_A)(x_D - x_B)\right]$$

$$= \frac{1}{2}\left[\Delta x_{AC}\,\Delta y_{BD} - \Delta x_{BD}\,\Delta y_{AC}\right] \qquad \text{eq } 6.2.7$$



$$f_{AB} = \frac{1}{2}(f_{i,j} + f_{i+1,j})$$

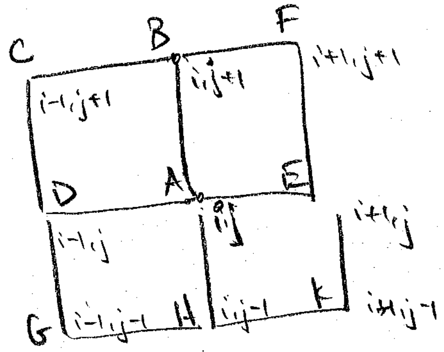
$$U_A = \frac{1}{4}(U_{ij} + U_{i+1,j} + U_{i,j-1} + U_{i+1,j-1})$$

$$U_B = \frac{1}{4}(U_{ij} + U_{i+1,j} + U_{i,j+1} + U_{i+1,j+1})$$

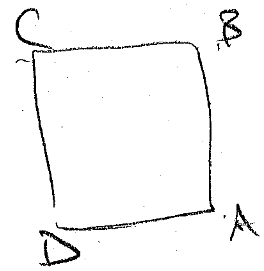
Then $f_{AB} = \frac{1}{2}(f_A + f_B) = \frac{1}{2}(f(U_A) + f(U_B))$

or

$$f_A = \frac{1}{4}(f_{ij} + f_{i+1,j-1} + f_{i+1,j} + f_{i,j-1})$$



$$\int_{AB} f\,dy \cong \frac{1}{2}(f_A+f_B)(y_B-y_A)$$



$$\Omega_{ij}\frac{dU}{dt} + \underbrace{\sum_{ABCD}\left[f_{AB}(y_B-y_A) - g_{AB}(x_B-x_A)\right]} = \Omega_{ij}Q_{ij}$$

$$+ \; f_{AB}\Delta y_{AB} - g_{AB}\Delta x_{AB}$$

w/ Approximate $f_{AB} = \frac{1}{2}(f_A+f_B)$

$$+ \; f_{BC}\Delta y_{BC} - g_{BC}\Delta x_{BC}$$

$$+ \; f_{CD}\Delta y_{CD} - g_{CD}\Delta x_{CD}$$

$$+ \; f_{DA}\Delta y_{DA} - g_{DA}\Delta x_{DA}$$

$$= \frac{1}{2}\Big[(f_A+f_B)(y_B-y_A) - (g_A+g_B)(x_B-x_A)$$

$$+ (f_B+f_C)(y_C-y_B) - (g_B+g_C)(x_C-x_B)$$

$$+ (f_C+f_D)(y_D-y_C) - (g_D-g_C)(x_D-x_C)$$

$$+ (f_D+f_A)(y_A-y_D) - \underbrace{(g_D+g_A)(x_A-x_D)}_{\text{exact same as } f \text{ + } y\text{'s}}\Big]$$

$$= \frac{1}{2}\Big[f_A y_B - \cancel{f_A y_A} + \cancel{f_B y_B} - f_B y_A$$

$$+ f_B y_C - \cancel{f_B y_B} + \cancel{f_C y_C} - f_C y_B$$

$$+ f_C y_D - \cancel{f_C y_C} + \cancel{f_D y_D} - f_D y_C$$

$$+ f_D y_A - \cancel{f_D y_D} + \cancel{f_A y_A} - f_A y_D \Big]$$

$$= \frac{1}{2}\Big[f_A(y_B - y_D) - f_B(y_A - y_C)$$

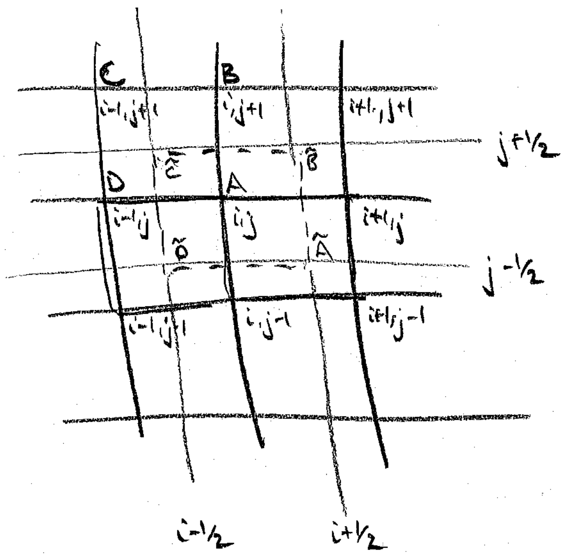
$$- f_C(y_B - y_D) - f_D(y_C - y_A) \Big]$$

$$= \frac{1}{2}\Big[(f_A - f_C)(y_B - y_D) + \underbrace{(f_D - f_B)(y_A - y_C)}_{(f_B - f_D)(y_C - y_A)} \Big]$$

$\therefore$

$$\oint_{ABCD} (f\,dy - g\,dx) \cong \frac{1}{2}\Big[(f_A - f_C)\Delta y_{DB} + (f_B - f_D)\Delta y_{AC}$$

$$- (g_A - g_C)\Delta x_{DB} - (g_B - g_D)\Delta x_{AC} \Big] \quad \text{eq } 6.2.15$$



$\Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$

To update node (i,j) dropping tilde's for convenience

$$\Delta y_{AB} = y_B - y_A = y_{i+1/2,j+1/2} - y_{i+1/2,j-1/2} = \Delta y \qquad \Delta x_{AB} = x_B - x_A = 0$$

$$\Delta x_{BC} = x_C - x_B = -\Delta x \qquad \Delta y_{BC} = 0$$

$$\Delta x_{CD} = 0 \qquad \Delta y_{CD} = y_D - y_C = -\Delta y$$

$$\Delta x_{DA} = x_A - x_D = \Delta x \qquad \Delta y_{DA} = 0$$

} 4 sides of rectangle.

Then 6.2.6 becomes w/ $f_{AB} = f_{i+1/2,j}$

$$\frac{d}{dt}(U_{ij}\Delta x \Delta y) + \left[f_{i+1/2,j}\Delta y - g_{i+1/2,j}\cdot 0 \right]$$
$$+ \left[f_{i,j+1/2}\, 0 - g_{i,j+1/2}(-\Delta x) \right]$$
$$+ \left[f_{i-1/2,j}(-\Delta y) - g_{i-1/2,j}(0) \right]$$
$$+ \left[f_{i,j-1/2}\, 0 - g_{i,j-1/2}\cdot \Delta x \right] = \Delta x \Delta y\, Q_{ij}$$

$$\Rightarrow \frac{d}{dt}(v_{ij}\Delta x \Delta y) + (f_{i+1/2,j} - f_{i-1/2,j})\Delta y + (g_{i,j+1/2} - g_{i,j-1/2})\Delta x$$

$$= \Delta x \Delta y\, Q_{ij} \qquad \text{eq E6.2.2}$$

$$\Rightarrow \frac{dv_{ij}}{dt} + \frac{f_{i+1/2,j} - f_{i-1/2,j}}{\Delta x} + \frac{g_{i,j+1/2} - g_{i,j-1/2}}{\Delta y} = Q_{ij}$$

Choosing 6.2.8 $f_{AB} = \frac{1}{2}(f_{ij} + f_{i+1,j})$

$$= \frac{dv_{ij}}{dt} + \frac{(\cancel{f_{ij}} + f_{i+1,j} - f_{i-1,j} - \cancel{f_{ij}})}{2\Delta x} + \frac{(g_{i,j+1} + \cancel{g_{ij}} - \cancel{g_{ij}} - g_{i,j-1})}{2\Delta y} \quad Q_{ij}$$

$$\Rightarrow \frac{dv_{ij}}{dt} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = Q_{ij} \qquad \text{eq E6.2.4}$$

eq 6.2.11 $f_{AB} = \frac{1}{2}(f_A + f_B)$ w/

$$f_A = \frac{1}{4}(f_{ij} + f_{i+1,j} + f_{i+1,j-1} + f_{i,j-1}) \qquad \text{gives}$$

$$\frac{dv_{ij}}{dt} + \frac{1}{2\Delta x}\Big[(f_{ij} + f_{i+1,j} + f_{i+1,j-1} + f_{i,j-1} + f_{ij} + f_{i+1,j} + f_{i+1,j-1} + f_{i,j-1})\frac{1}{4}$$

$$- \frac{1}{4}(f_{i-1,j+1} + f_{i,j+1} + f_{ij} + f_{i-1,j} + f_{ij} + f_{i-1,j} + f_{i,j-1} + f_{i-1,j-1})\Big]$$

$$+ \frac{1}{2\Delta y}\Big[(g_{i-1,j+1} + g_{i,j+1} + g_{ij} + g_{i-1,j} + g_{i,j+1} + g_{i+1,j+1} + g_{i+1,j} + g_{ij})\frac{1}{4}$$

$$- \frac{1}{4}\left(g_{i-1,j} + g_{i,j} + g_{i-1,j-1} + g_{i,j-1} + g_{i,j} + g_{i+1,j} + g_{i+1,j-1} + g_{i,j-1}\right)\Big] = Q_{i,j}$$

$\Rightarrow$

$$\frac{dw_{i,j}}{dt} + \frac{1}{8\Delta x}\Big[\cancel{f_{i,j+1}} + f_{i+1,j+1} + 2\cancel{f_{i,j}} + 2f_{i+1,j} + \cancel{f_{i,j-1}} + f_{i+1,j-1}$$

$$-\left(f_{i-1,j-1} + \cancel{f_{i,j-1}} + 2f_{i-1,j} + 2\cancel{f_{i,j}} + f_{i-1,j+1} + \cancel{f_{i,j+1}}\right)\Big]$$

$$+ \frac{1}{8\Delta y}\Big[g_{i-1,j+1} + 2g_{i,j+1} + g_{i+1,j+1} + \cancel{g_{i+1,j}} + 2\cancel{g_{i,j}} + \cancel{g_{i-1,j}}$$

$$-\left(g_{i-1,j-1} + 2g_{i,j-1} + g_{i+1,j-1} + \cancel{g_{i+1,j}} + 2\cancel{g_{i,j}} + \cancel{g_{i-1,j}}\right)\Big] = Q_{i,j}$$

$$\Rightarrow \frac{dw_{i,j}}{dt} + \frac{1}{8\Delta x}\Big[f_{i+1,j+1} - f_{i-1,j+1} + 2(f_{i+1,j} - f_{i-1,j}) + f_{i+1,j-1} - f_{i-1,j-1}\Big]$$

$$+ \frac{1}{8\Delta y}\Big[g_{i-1,j+1} - g_{i-1,j-1} + 2(g_{i,j+1} - g_{i,j-1}) + g_{i+1,j+1} - g_{i+1,j-1}\Big]$$

$$= Q_{i,j}$$

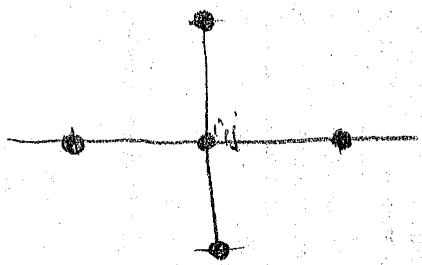
$$\Rightarrow \frac{dw_{i,j}}{dt} + \frac{1}{4}\left[2\left(\frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}\right) + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x} + \frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta x}\right]$$

Eq Eb.2.5

$$+ \frac{1}{4}\left[2\left(\frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y}\right) + \frac{g_{i+1,j+1} - g_{i+1,j-1}}{2\Delta y} + \frac{g_{i-1,j+1} - g_{i-1,j-1}}{2\Delta y}\right] = Q_{i,j}$$

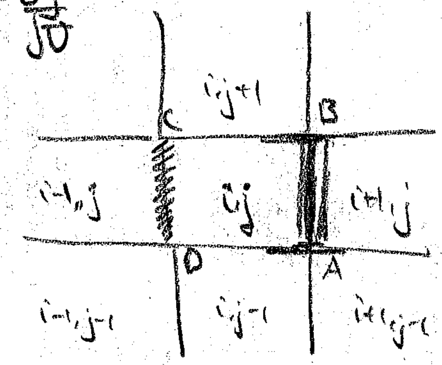
$$\frac{du_{ij}}{dt} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = Q_{ij}$$

Has stencil



Similar to $\Delta^{(2)}$ scheme for Laplacian operator pg 189 Hirsh Vol I

$$\overline{A}(U) = \frac{\partial F}{\partial U} = a 1_x + b 1_y \qquad a = \frac{\partial f}{\partial U} \qquad b = \frac{\partial g}{\partial U}$$



$$(\overline{F}\cdot\vec{S})_{AB} = (\overline{F}\cdot\vec{S})_{ij} \quad \text{if } (\overline{A}\cdot\vec{S})_{AB} > 0$$

$$(F\cdot\vec{S})_{AB} = (F\cdot\vec{S})_{i+1,j} \quad \text{if } (\overline{A}\cdot\vec{S})_{AB} < 0$$

Ex 6.2.2 Same picture as Above

$a > 0$ & $b > 0 \Rightarrow$ flow is up & to the Right

$$(\overline{F}\cdot\vec{S})_{AB} = (F\cdot\vec{S})_{ij} \quad \text{As } (\overline{A}\cdot\vec{S})_{AB} > 0 \quad \text{then}$$

$\vec{S}$ is defined as the outward normal

$$(\overline{F}\cdot\vec{S})_{AB} = (\overline{F}\cdot\vec{S})_{ij} = (aU, bU)\cdot(\Delta y, 0)_{ij} = a\Delta y U_{ij}$$

$$(\overline{F}\cdot\vec{S})_{CD} = (\overline{F}\cdot\vec{S})_{i-1,j} \quad \text{As } (\vec{A}\cdot\vec{S})_{CD} > 0$$

$$= (aU, bU)_{i-1,j}\cdot(-\Delta y, 0) = -a\Delta y U_{i-1,j}$$

$$(\bar{F}\circ\bar{S})_{BC} = (\bar{F}\circ\bar{S})_{ij} \quad \text{As} \quad (\bar{A}\circ\bar{S}) > 0$$

$$= (aU, bU)_{ij}\circ(0, +\Delta x) = b\Delta x U_{ij}$$

$$(\bar{F}\circ\bar{S})_{DA} = (\bar{F}\circ\bar{S})_{i,j-1} \quad \text{As} \quad (A\circ\bar{S})_{DA} > 0$$

$$\| \quad (a, b)\circ(0, -\Delta x) = -b\Delta x < 0 \ ?$$

$$= (aU, bU)_{ij-1}\circ(0, -\Delta x)$$

why is this sign not correct?

$$= -bU_{ij-1}\Delta x$$

$\Rightarrow$ Flux outward is negative $\Rightarrow$ there is really a flux inward, evaluate the flux at Node $i,j-1$

Then $\Omega_{ij}\dfrac{\partial U_{ij}}{\partial t} + \sum_{ABCD}(\bar{F}\circ\bar{S}) = 0$

$$= \Delta x\Delta y\frac{\partial U_{ij}}{\partial t} + a\Delta y U_{ij} - a\Delta y U_{i-1,j} + b\Delta x U_{ij} - b\Delta x U_{ij-1} = 0$$

$$\Rightarrow \quad \frac{dU_{ij}}{dt} + \frac{a}{\Delta x}(U_{ij} - U_{i-1,j}) + \frac{b}{\Delta y}(U_{ij} - U_{i,j-1}) = 0 \quad \text{eq}$$

E6.2.10

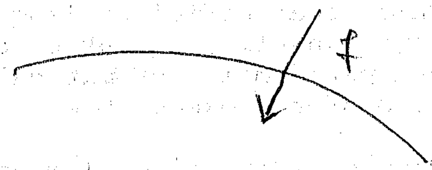
If I perform the same calculation w/ $\bar{S}$ the inward normal I get

$$\frac{dU_{ij}}{dt} + \frac{a}{\Delta x}(-U_{ij} + U_{i-1,j}) + \frac{b}{\Delta y}(-U_{ij} + U_{ij-1}) = 0$$

which seems to have the wrong sign & is not correct, why?

The formulation of the conservation eqs pg 10 Hirsh Vol I

Are derived for outward pointing unit normals in the integration ~~@~~ over the surface volume,



$$f = (f \circ \hat{n}_{In}) = -f \circ \hat{n}_{out}$$

$$\frac{d\sigma}{dt} = -\int_{S} f \circ d\hat{S}$$

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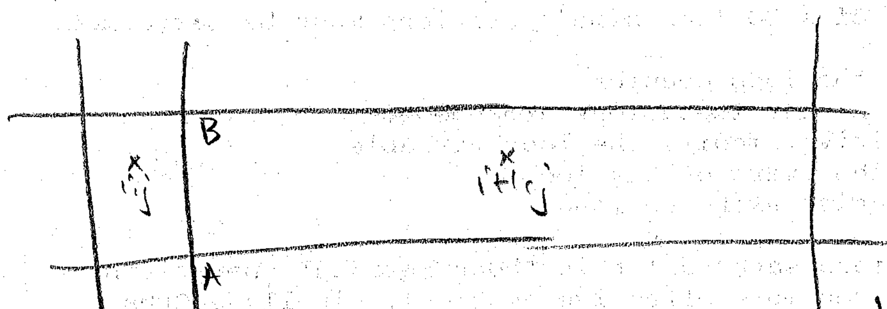
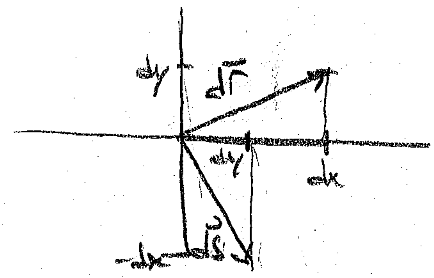
On a Non uniform grid



one should prob do linear interpolation of the cell centered values (i,j) & (i+1,j) if one hopes to get a 2nd order approximation to the flux through AB

For 2-D volumes $d\vec{S} = dy\,\hat{i} - dx\,\hat{j}$ $\|d\vec{S}\| = \sqrt{dx^2 + dy^2}$

Then



$$\overline{\left(\frac{\partial U}{\partial x}\right)}_{\Omega} = \frac{1}{\Omega}\int_{\Omega} \frac{\partial U}{\partial x} = \frac{1}{\Omega}\oint_S U\,\vec{1}_x \cdot d\vec{S}$$

$$= \frac{1}{\Omega}\oint_S U\,dy = \frac{1}{\Omega}\Big[\cancel{Uy\Big|}^{0} - \oint_S y\,dU\Big]$$

$d\vec{S} = +dy\,\hat{i} - dx\,\hat{j}$

diff endpoints of contour
is this mathematically correct?

$$= \frac{-1}{\Omega}\oint_S y\,dU \qquad \text{eq } 6.2.25a$$

$$\overline{\left(\frac{\partial U}{\partial y}\right)}_{\Omega} = \frac{1}{\Omega}\int_{\Omega} \frac{\partial U}{\partial y}\,d\Omega = \frac{1}{\Omega}\oint_S U\,\vec{1}_y \cdot d\vec{S} = -\frac{1}{\Omega}\oint_S U\,dx$$

$$= \frac{-1}{\Omega}\Big[Ux\Big| - \oint_S x\,dU\Big]$$

diff. endpoints of contour.
Again is this mathematically correct?

$$= +\frac{1}{\Omega}\oint_S x\,dU \qquad \text{eq } 6.2.25b$$

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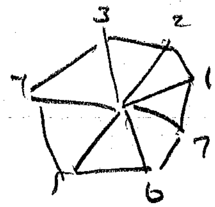
$$\overline{\left(\frac{\partial v}{\partial x}\right)}_\Omega \equiv \frac{1}{\Omega}\int_\Omega \frac{\partial v}{\partial x}\, d\Omega = \frac{1}{\Omega}\oint_S v\, dy = -\frac{1}{\Omega}\oint_S y\, dv$$

$$= \frac{1}{\Omega}\sum_{\substack{\text{edges}\\ \text{of } \partial\Omega}}\int v\, dy = -\frac{1}{\Omega}\sum_{\substack{\text{edges}\\ \text{of } \partial\Omega}}\int y\, dv$$



Now trapezoidal fn is

$$\int_a^b f(x)\,dx \cong \frac{(f(a)+f(b))}{2}(b-a)$$

$$\overset{(1)}{=} \frac{1}{\Omega}\sum_i \frac{(v_i+v_{i+1})}{2}(y_{i+1}-y_i) \overset{(2)}{=} -\frac{1}{\Omega}\sum_i \frac{(y_i+y_{i+1})}{2}(v_{i+1}-v_i)$$

working out 1st equality gives:

$$= -\frac{1}{2\Omega}\left(\sum_i v_i(y_i-y_{i+1}) + \sum_i v_{i+1}(y_i-y_{i+1})\right)$$

$$= -\frac{1}{2\Omega}\left(\sum_i v_i(y_i-y_{i+1}) + \sum_i v_i(y_{i-1}-y_i)\right)$$

$$= -\frac{1}{2\Omega}\left(\sum_i v_i(y_{i-1}-y_{i+1})\right) = \frac{1}{2\Omega}\sum_i v_i(y_{i+1}-y_{i-1})$$

Working out 2nd equality gives:

$$= +\frac{1}{2\Omega}\left(\sum y_i(v_i-v_{i+1}) + \sum y_{i+1}(v_i-v_{i+1})\right)$$

$$= +\frac{1}{2\Omega}\left(\sum y_i(v_i-v_{i+1}) + \sum y_i(v_{i-1}-v_i)\right)$$

$$= \frac{+1}{2\Omega}\left(\sum y_i(v_{i-1} - v_{i+1})\right) = \frac{-1}{2\Omega}\sum y_i(v_{i+1} - v_{i-1}) \quad \text{eqs } 6.2.26a$$

Then

$$\overline{\left(\frac{\partial v}{\partial y}\right)}_\Omega = -\frac{1}{\Omega}\oint_S v\, dx = \frac{1}{\Omega}\oint x\, dv$$

Thus to obtain the corresponding formulas for the y-derivative the 1st eq has a negative sign & $y \leftrightarrow x$

& the 2nd eq has a negative sign & $x \leftrightarrow y$ ∴ Using eq 6.2.26a as a marker

$$= -\frac{1}{2\Omega}\sum_l (v_l + v_{l+1})(x_{l+1} - x_l)$$

$$= +\frac{1}{2\Omega}\sum_l (x_l + x_{l+1})(v_{l+1} - v_l)$$

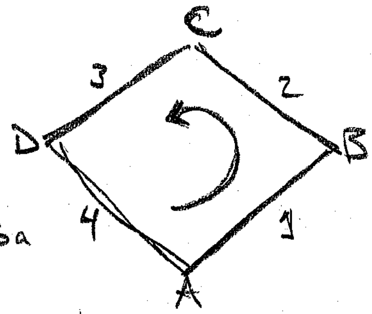
$$= \frac{-1}{2\Omega}\sum_l v_l(x_{l+1} - x_{l-1})$$

$$= +\frac{1}{2\Omega}\sum_l x_l(v_{l+1} - v_{l-1})$$

To evaluate the area

$$\Omega = \frac{1}{2}\sum_l (x_l + x_{l+1})(y_{l+1} - y_l) = \frac{-1}{2}\sum_l (y_l + y_{l+1})(x_{l+1} - x_l)$$

$$= \frac{1}{2}\sum x_l(y_{l+1} - y_{l-1}) = \frac{-1}{2}\sum y_l(x_{l+1} - x_{l-1})$$



$$\oint_{ABCD} v\,dx = \frac{1}{2}\sum_{\ell} v_\ell (y_{\ell+1} - y_{\ell-1})$$

from 3rd row of 6.2.26a eq.

$$= \frac{1}{2}\left[v_A(\underline{y_B} - y_D) + v_B(\underline{y_C} - y_A) + v_C(\underline{y_D} - y_B) + v_D(\underline{y_A} - y_C) \right]$$

$$= \frac{1}{2}\left[(v_A - v_C)(y_B - y_D) + (v_B - v_D)(y_C - y_A) \right]$$

$$= \frac{1}{2}\left[(v_A - v_C)(y_B - y_D) - (v_B - v_D)(y_A - y_C) \right] \quad \text{eq } 6.2.28$$

$$\left(\overline{\frac{\partial v}{\partial x}}\right)_{ABCD} = \frac{1}{2\Omega}\sum_{\ell} v_\ell (y_{\ell+1} - y_{\ell-1})$$

By 3rd eq in eq 6.2.26a
same as above

$$= \frac{1}{2\Omega}\left[(v_A - v_C)(y_B - y_D) - (v_B - v_D)(y_A - y_C) \right]$$

$$\& \quad \Omega = \frac{1}{2}\sum_{\ell} x_\ell (y_{\ell+1} - y_{\ell-1})$$

By 3rd eq in eq 6.27
same as 3rd eq in 6.2.26a w/
$v \Rightarrow x$.

$$= \frac{1}{2}\left[(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C) \right]$$

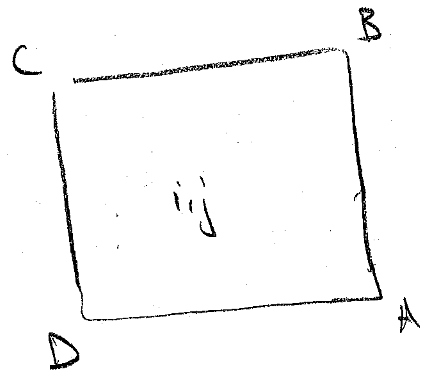
$$\therefore \quad \left(\overline{\frac{\partial v}{\partial x}}\right)_{ABCD} = \frac{(v_A - v_C)(y_B - y_D) - (v_B - v_D)(y_A - y_C)}{(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C)} \quad \text{eq } 6.2.29a$$

$$\left(\frac{\partial v}{\partial y}\right)_{ABCD} = -\frac{1}{2\Omega}\sum_{\ell} v_\ell (x_{\ell+1} - x_{\ell-1})$$

$$= \frac{-1}{2\Omega}\left[\begin{matrix}\text{same as before}\\ + y \leftrightarrow x\end{matrix}\right]$$

$$= \frac{-\left[(v_A - v_C)(x_B - x_D) - (v_B - v_D)(x_A - x_C)\right]}{(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C)}$$

$$= \frac{(x_A - x_C)(v_B - v_D) - (x_B - x_D)(v_A - v_C)}{\cdots} \quad \text{eq } 6.2.29b$$



$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial v}{\partial y}\right) = 0$$

Finite Volume

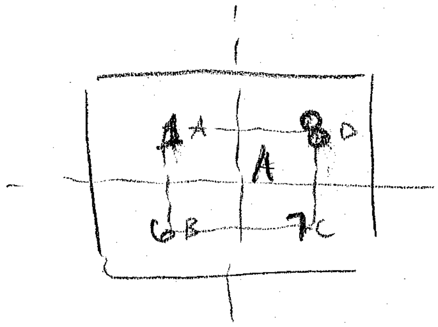
$$\int_\Omega \frac{\partial v}{\partial t}\, d\Omega + \int_\Omega \frac{\partial f}{\partial x}\, d\Omega + \int_\Omega \frac{\partial g}{\partial y}\, d\Omega = 0$$

$$\Rightarrow \Omega \frac{\partial \bar{v}}{\partial t} + \int f(x^+)\, dy - \int f(x^-)\, dy + \int g(y^+)\, dx - \int g(y^-)\, dx = 0$$

$$\Rightarrow \Omega \frac{\partial \bar{v}}{\partial t} + \Delta y(f_{AB} - f_{CD}) + \Delta x(g_{BC} - g_{DA}) = 0$$

$$\Rightarrow \Delta x \Delta y \frac{\partial \bar{v}}{\partial t} + \Delta y(f_{AB} - f_{CD}) + \Delta x(g_{BC} - g_{DA}) = 0$$

$$\tau_A = k\left(\frac{\partial v}{\partial x}\right)_A =$$



$$\left(\frac{\partial v}{\partial x}\right)_{1678} = \frac{(v_1 - v_7)(y_6 - y_8) - (v_6 - v_8)(y_1 - y_7)}{(x_1 - x_7)(y_6 - y_8) - (x_6 - x_8)(y_1 - y_7)}$$

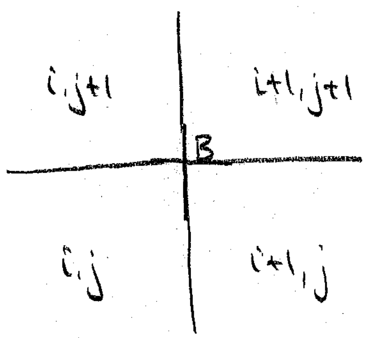
$$= \frac{(v_1 - v_7)(-\Delta y) - (v_6 - v_8)\Delta y}{(-\Delta x)(-\Delta y) - (-\Delta x)(\Delta y)}$$

$$= \frac{1\Delta y}{2\Delta x \Delta y}\left(v_7 - v_1 - v_6 + v_8\right)$$

$$= \frac{1}{2\Delta x}\left(v_{i+1,j-1} - v_{i,j} - v_{i,j-1} + v_{i+1,j}\right)$$

$$\therefore \tau_A = \frac{k}{2\Delta x}\left(\qquad \right) \quad \text{eq E6.2.14}$$

For pt B:

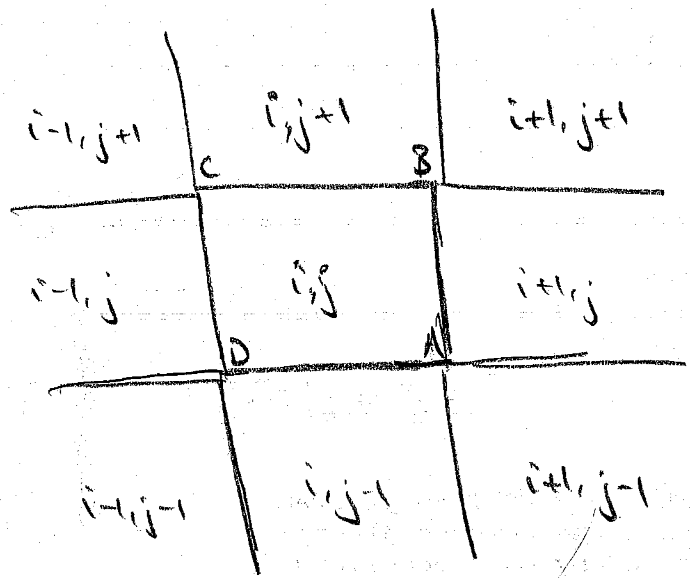


Then

$$k\left(\frac{\partial v}{\partial x}\right)_B = \text{Average at center of cell surrounding pt B}$$

$$= k\left[\frac{(v_{i+1,j} - v_{i,j+1})\Delta y - (v_{i+1,j+1} - v_{i,j})(-\Delta y)}{\Delta x \Delta y - \Delta x(-\Delta y)}\right]$$

$$k\left(\frac{\partial v}{\partial x}\right)_B = \frac{k}{2\Delta x \Delta y}\Delta y\left[v_{i+1,j} - v_{i,j+1} + v_{i+1,j+1} - v_{i,j}\right]$$



Then $f_{AB} = \frac{1}{2}(f_A + f_B) = \frac{k}{4\Delta x}\left[2v_{i+1,j} + v_{i+1,j-1} + v_{i+1,j+1} - 2v_{i,j} - v_{i,j-1} - v_{i,j+1}\right]$

Based on this f_{CD} can be written down

$$f_{CD} = \frac{1}{2}(f_C + f_D) = \frac{k}{4\Delta x}\left[2(v_{i,j} - v_{i-1,j}) + v_{i,j+1} - v_{i-1,j+1} + v_{i,j-1} - v_{i-1,j-1}\right]$$

$$g_B = k\left(\frac{\partial v}{\partial y}\right)_B = k\left[\frac{\Delta x(v_{i+1,j+1} - v_{i,j}) - \Delta x(v_{i+1,j} - v_{i,j+1})}{\Delta x\,\Delta y - \Delta x(-\Delta y)}\right]$$

$$= \frac{k}{2\Delta y}\left[v_{i+1,j+1} - v_{i,j} - v_{i+1,j} + v_{i,j+1}\right]$$

$g_C \left\{ \begin{array}{l} i \to i-1 \\ j \to j \end{array} \right\}$ $\quad g_C = \frac{k}{2\Delta y}\left[v_{i,j+1} - v_{i-1,j} - v_{i,j} + v_{i-1,j+1}\right]$

Then E6.2.13 $\Rightarrow$

$$\left(\frac{\partial U}{\partial t}\right)_{ij} \Delta x \Delta y + (f_{AB} - f_{CD})\Delta y + (g_{BC} - g_{DA})\Delta x = 0 \quad *$$

$$f_{AB} = \frac{k}{4\Delta x}\left(2U_{i+1,j} + U_{i+1,j-1} + U_{i+1,j+1} - 2U_{i,j} - U_{i,j-1} - U_{i,j+1}\right)$$

$$f_{CD} = \frac{k}{4\Delta x}\left(2(U_{i,j} - U_{i-1,j}) + U_{i,j+1} - U_{i-1,j+1} + U_{i,j-1} - U_{i-1,j-1}\right)$$

Then

$$g_{BC} = \frac{k}{4\Delta y}\left(2U_{i,j+1} - 2U_{i,j} + U_{i+1,j+1} - U_{i+1,j} - U_{i-1,j} + U_{i-1,j+1}\right)$$

$$= \frac{k}{4\Delta y}\left(2(U_{i,j+1} - U_{i,j}) + U_{i-1,j+1} - U_{i-1,j} + U_{i+1,j+1} - U_{i+1,j}\right)$$

Then By inspection

$$g_{DA} = \frac{k}{4\Delta y}\left(2(U_{i,j} - U_{i,j-1}) + U_{i+1,j} - U_{i+1,j-1} + U_{i-1,j} - U_{i-1,j-1}\right)$$

Then eq * Above gives

$$\left(\frac{\partial U}{\partial t}\right)_{ij} \Delta x \Delta y + \frac{k\Delta y}{4\Delta x}\Big(2\cancel{U}_{i+1,j} + \overset{1}{U}_{i+1,j-1} + \overset{2}{U}_{i+1,j+1} - 4\overset{3}{U}_{i,j} - 2\cancel{U}_{i,j-1} - 2\cancel{U}_{i,j+1}$$

$$+ 2\cancel{U}_{i-1,j} + \overset{4}{U}_{i-1,j+1} + \overset{5}{U}_{i-1,j-1}\Big)$$

$$+ \frac{k}{4\Delta y}\Delta x\Big(2\cancel{U}_{i,j+1} - 4\overset{3}{U}_{i,j} + \overset{4}{U}_{i-1,j+1} - 2\cancel{U}_{i-1,j} + \overset{2}{U}_{i+1,j+1} - 2\cancel{U}_{i+1,j}$$

$$+ 2\cancel{U}_{i,j-1} + \overset{1}{U}_{i+1,j-1} + \overset{5}{U}_{i-1,j-1}\Big)$$

$\Delta x = \Delta y$

$$\Rightarrow \left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{k}{4\Delta x^2}\left(2\overset{2}{U}_{i+1,j-1} + 2\overset{1}{U}_{i+1,j+1} - 8U_{i,j} + 2U_{i-1,j+1} + 2U_{i-1,j-1}\right) = 0$$

$$\Rightarrow \left(\frac{\partial U}{\partial t}\right)_{ij} + \frac{k}{2\Delta x^2}\left(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} - 4U_{i,j}\right) = 0$$

$$f_{AB} = k\left(\frac{\partial v}{\partial x}\right)_{AB} = \frac{k}{\Delta x}\left(v_{i+1,j} - v_{i,j}\right)$$

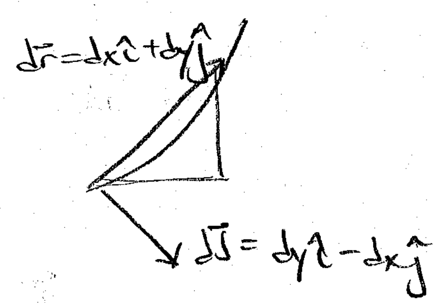
gives Then

$$\Delta x \Delta y \left(\frac{\partial v}{\partial t}\right)_{ij} + \frac{k\Delta y}{\Delta x}\left(v_{i+1,j} - v_{ij} - (v_{ij} - v_{i-1,j})\right)$$

$$+ \frac{k\Delta x}{\Delta y}\left(v_{i,j+1} - v_{i,j} - (v_{i,j} - v_{i,j-1})\right) = 0$$

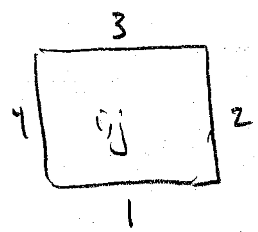
$$\Rightarrow \quad \left(\frac{\partial v}{\partial t}\right)_{ij} + \frac{k}{\Delta x^2}\left(v_{i+1,j} + v_{i,j+1} + v_{i-1,j} + v_{i,j-1} - 4v_{i,j}\right) = 0$$

$$\int_\Omega \nabla \circ \bar{a} \, d\Omega = \oint_S \bar{a} \circ d\bar{S} \qquad a = x \in \mathbb{R}^2 \qquad \nabla \circ x = 2$$



$$\Rightarrow 2\Omega = \oint_S \bar{x} \circ d\bar{S} = \oint_S x\,dy - y\,dx$$

For A rectangular region



+ Applying the trapezoid rule one gets

$$\oint_S x\,dy - y\,dx = \sum_{\text{sides}} \int x\,dy - y\,dx = \int_1 -y\,dx + \int_2 x\,dy + \int_3 -y\,dx$$

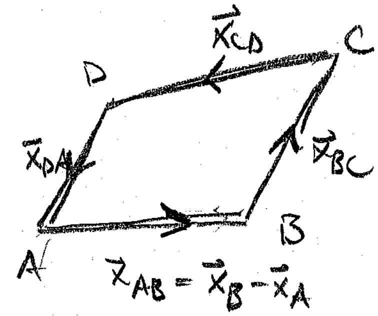
$$+ \int_4 x\,dy$$

Then $2\Omega = -\left(y_{i+1/2,j-1/2} + y_{i+1/2,j+1/2}\right)\left(x_{i+1/2} - x_{i-1/2}\right)$

$+ \quad \cdots$

This should replicate eq 6.2.27.

$$\vec{S}_{ABCD} = \frac{1}{2}\left[(\vec{x}_{AB} \times \vec{x}_{BC}) + (\vec{x}_{CD} \times \vec{x}_{DA})\right]$$



$$+ \quad \vec{S}_{ABCD} = \frac{1}{2}\left[(\vec{x}_{BC} \times \vec{x}_{CD}) + (\vec{x}_{DA} + \vec{x}_{AB})\right]$$

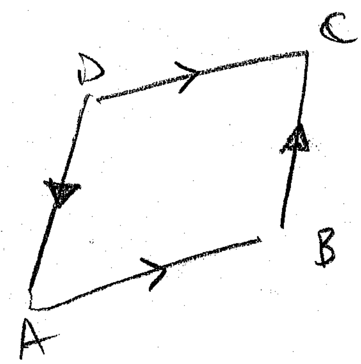
$$\vec{S}_{ABCD} = \frac{1}{4}\left[\vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{BC} \times \vec{x}_{CD}\right.$$

$$\left. + \vec{x}_{CD} \times \vec{x}_{DA} + \vec{x}_{DA} \times \vec{x}_{AB}\right]$$

$$= \frac{1}{4}\left[(\vec{x}_{AB} - \vec{x}_{CD}) \times \vec{x}_{BC} + \vec{x}_{DA} \times (-\vec{x}_{CD} + \vec{x}_{AB})\right]$$

$$= \frac{1}{4}\left[(\vec{x}_{AB} - \vec{x}_{CD}) \times (\vec{x}_{BC} + \vec{x}_{DA})\right]$$

$$= \frac{1}{4}(\vec{x}_{AB} + \vec{x}_{DC}) \times (\vec{x}_{BC} + \vec{x}_{DA})$$



or Eq 6.2.37

$$\mathcal{V}_{PABC} = \frac{1}{6}\bar{x}_{PA}\circ(\bar{x}_{AB}\times\bar{x}_{BC}) = \frac{1}{6}\bar{x}_{PA}\circ((\bar{x}_{AB}+\bar{x}_{BC})\times\bar{x}_{BC})$$

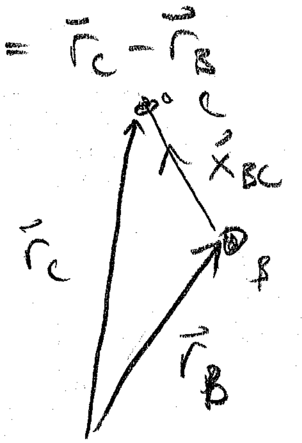
Is?

$$= \frac{1}{6}\bar{x}_{PA}\circ(\bar{x}_{AC}\times\bar{x}_{BC})$$

$$= \frac{1}{6}\bar{x}_{PA}\circ(\bar{x}_{BC}\times\bar{x}_{CA}) \quad \text{eq } 6.2.40$$

$$= \frac{1}{6}\bar{x}_{PA}\circ\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_C-x_B & y_C-y_B & z_C-z_B \\ x_A-x_C & y_A-y_C & z_A-z_C \end{vmatrix}$$

$\bar{x}_{BC} = \vec{r}_C - \vec{r}_B$



$$= \frac{1}{6}\bar{x}_{PA}\circ\Big[\hat{i}\big[(y_C-y_B)(z_A-z_C)-(y_A-y_C)(z_C-z_B)\big]$$

$$-\hat{j}\big[(x_C-x_B)(z_A-z_C)-(x_A-x_C)(z_C-z_B)\big]$$

$$+\hat{k}\big[(x_C-x_B)(y_A-y_C)-(x_A-x_C)(y_C-y_B)\big]$$

$\vec{r}_B + \vec{x}_{BC} = \vec{r}_C$

$\bar{x}_{BC} = \vec{r}_C - \vec{r}_B$

$$= \frac{1}{6}\Big[(x_A-x_P)\big((y_C-y_B)(z_A-z_C)-(y_A-y_C)(z_C-z_B)\big)$$

$$-(y_A-y_P)\big((x_C-x_B)(z_A-z_C)-(x_A-x_C)(z_C-z_B)\big)$$

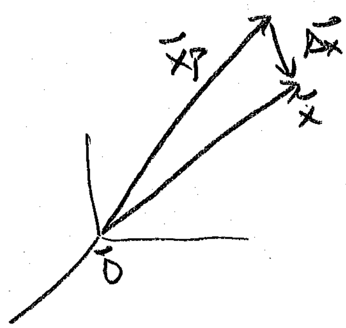
$$+(z_A-z_P)\big((x_C-x_B)(y_A-y_C)-(x_A-x_C)(y_C-y_B)\big)\Big]$$

How show eq 6.2.41?

$$\Omega_{PABCD} = \frac{1}{3}\oint \vec{x} \circ d\vec{S}$$ surface integral over pyramid in Figure 6.2.8

1st can I translate this integral for $\vec{x}$ being relative to origin $\vec{O}$ to the point $\vec{P}$ instead?

$$\vec{x} = \vec{x}_p + \vec{\Delta x}$$



Then $\Omega = \frac{1}{3}\oint(\vec{x}_p \circ \vec{\Delta x}) \circ d\vec{S} = \frac{1}{3}\int \vec{x}_p \circ d\vec{S} + \frac{1}{3}\int \vec{\Delta x} \circ d\vec{S}$

$$= \frac{1}{3}\vec{x}_p \circ \cancel{\oint d\vec{S}} + \frac{1}{3}\oint \vec{\Delta x} \circ d\vec{S}$$

0 But is $\oint d\vec{S}$ not the surface area?

Anyway, $\vec{\Delta x}$ is $\perp$ to 3 of the 4 faces of the pyramid & as such

$$\Omega = \frac{1}{3}\int_{ABCD} \vec{\Delta x} \circ d\vec{S} \cong \frac{1}{3}\vec{\Delta x} \circ \vec{S}_{ABCD}$$ eq 6.2.42

Vector from P to plane ABCD

look up how to do this correctly.

$$= \frac{1}{6}\left[\begin{array}{c} 16 \\ 3 \\ \hline 48 \end{array}\right. \text{ terms}$$

How show easily?
From 6.2.41

$$\vec{x}_{(P)} = \frac{1}{4}(\vec{x}_{PA} + \vec{x}_{PB} + \vec{x}_{PC} + \vec{x}_{PD})$$

$$\Omega_{PABCD} = \frac{1}{3}\left(\frac{1}{4}(\vec{x}_{PA} + \vec{x}_{PB} + \vec{x}_{PC} + \vec{x}_{PD})\right)\circ\left(\frac{1}{2}(\vec{x}_{AC} \times \vec{x}_{BD})\right)$$

$$= \frac{1}{24}(\vec{x}_{PA} + \vec{x}_{PB} + \vec{x}_{PC} + \vec{x}_{PD})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$ eq 6.2.44 1st half

$$= \frac{1}{24}(\vec{x}_{PA} + \vec{x}_{PB})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$

$$+ \frac{1}{24}(\vec{x}_{PC} + \vec{x}_{PD})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$

But $\vec{x}_{PC} = \vec{x}_{PB} + \vec{x}_{BC}$ + $\vec{x}_{PD} = \vec{x}_{PA} + \vec{x}_{AD}$

Then

$$\Omega_{PABCD} = \frac{1}{24}(\vec{x}_{PA} + \vec{x}_{PB})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$

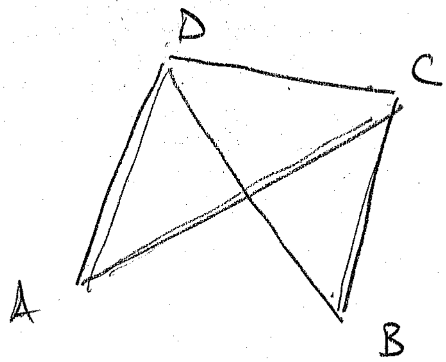
$$+ \frac{1}{24}(\vec{x}_{PA} + \vec{x}_{PB})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$

$$+ \frac{1}{24}(\vec{x}_{BC} + \vec{x}_{AD})\circ(\vec{x}_{AC} \times \vec{x}_{BD})$$

$$\Omega_{PABCD} = \frac{1}{12}(\bar{x}_{PA} + \bar{x}_{PB}) \circ (\bar{x}_{AC} \times \bar{x}_{BD})$$

$$+ \frac{1}{24}(\bar{x}_{BC} + \bar{x}_{AD}) \circ (\bar{x}_{AC} \times \bar{x}_{BD})$$

Show $(\bar{x}_{BC} + \bar{x}_{AD}) \circ (\bar{x}_{AC} \times \bar{x}_{BD}) = 0.$



But $\bar{x}_{AC} = \bar{x}_{AD} + \bar{x}_{DC}$

& $\bar{x}_{BD} = \bar{x}_{BC} + \bar{x}_{CD}$

Then

$$(\bar{x}_{BC} + \bar{x}_{AD}) \circ (\bar{x}_{AC} \times \bar{x}_{BD})$$

$$= (\bar{x}_{BC} + \bar{x}_{AD}) \circ \underbrace{((\bar{x}_{AD} + \bar{x}_{DC}) \times (\bar{x}_{BC} + \bar{x}_{CD}))}$$

$$\bar{x}_{AD} \times \bar{x}_{BC} + \bar{x}_{AD} \times \bar{x}_{CD} + \bar{x}_{DC} \times \bar{x}_{BC} + \cancel{\bar{x}_{DC} \times \bar{x}_{CD}}^{\,0}$$

$$= \bar{x}_{BC} \circ (\bar{x}_{AD} \times \bar{x}_{CD}) + \underbrace{\bar{x}_{AD} \circ (\bar{x}_{DC} \times \bar{x}_{BC})}$$

$$+ \bar{x}_{BC} \circ (\bar{x}_{AD} \times \bar{x}_{DC})$$ w/ vector identity.

↓

changing sign gives $= 0$

If ABCD is coplaner Then

$$\bar{x}_{PB} = \vec{x}_{PA} + \bar{x}_{BA}$$

Thus

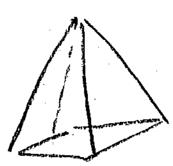
$$\Omega_{PABCD} = \frac{1}{12}\left(2\vec{x}_{PA} + \bar{x}_{BA}\right) \circ \left(\bar{x}_{AC} \times \bar{x}_{BD}\right)$$

& $\bar{x}_{BA} \circ (\bar{x}_{AC} \times \bar{x}_{BD}) = 0$ if All pts ABCD lie in the same plane

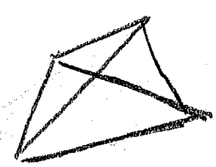
$$\Rightarrow \Omega_{PABCD} = \frac{1}{6}\bar{x}_{PA} \circ (\bar{x}_{AC} \times \bar{x}_{BD}) \qquad \text{eq } 6.2.45$$

Note: tetrahedra are 4 sided 3 dimentional objects, while pyramids are 5 sided

pyramid

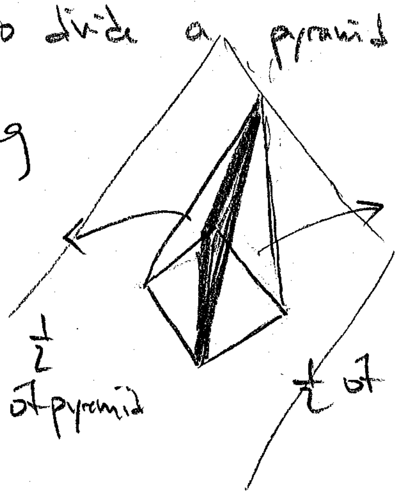


tetrahedra



To divide a pyramid into 2 tetrahedra slice it down one "vertex" or along a diagonal of the face w/ 4 edges.

eg



05-08-01 6

Note: The 5 faces of the pyramid w/ D as summit in Figure 6.2.8 are

Δ_{DEB}, Δ_{DGE}, Δ_{DGB}, Δ_{GFE}, Δ_{GFB}

Then the oth

pyramids in Figure 6.2.8(b) can be obtained from "cutting" along the diagonal DF in the following

$$\Omega_{HEX} = \Omega_{DBFEA} + \Omega_{DBFGC} + \Omega_{DFGEH} \quad \text{eq } 6.2.46$$

Then Breaking each pyramid up into tetrahedron results in

$$\Omega_{HEX} = \underbrace{\Omega_{DFCB} + \Omega_{DFCG}}_{\text{From "Right" pyramid}} + \underbrace{\Omega_{DBFA} + \Omega_{DFAE}}_{\text{From "bottom" pyramid}}$$

{Book chooses $\Omega_{DBCG} + \Omega_{DBGF}$} {Book chooses $\Omega_{DBEF} + \Omega_{DBAE}$}

$$+ \underbrace{\Omega_{DFEH} + \Omega_{DHFG}}_{\text{From "Left" pyramid}}$$

Book chooses $\Omega_{DGHE} + \Omega_{DFGE}$

This is non unique depending on the diagonal choosen on the base of the pyramid eq 6.2.48 is determined in the following way

one picks a corner like corner D in Figure 6.2.8

then tesselates Ω_{DGBC}, Ω_{DBAE}, Ω_{DGHE}

+ is left w/ Ω_{DBFGE} a 6 sided figure.

Then chop this in 1/2 along in this case the

plane GBE,

giving

$$\Omega_{HEX} = \Omega_{DGBC} + \Omega_{DBAE} + \Omega_{GHED}$$

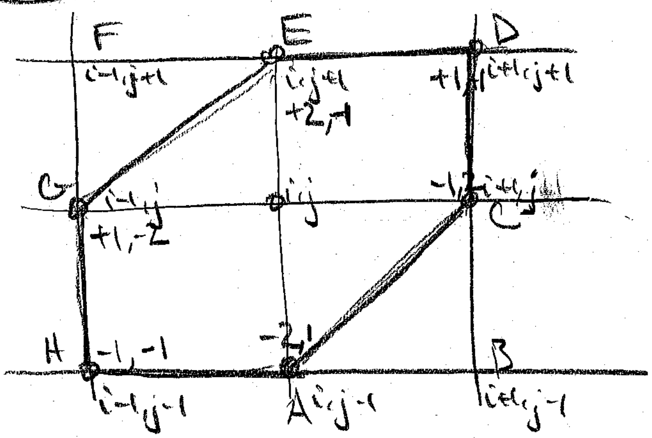
$$+ \Omega_{DGBE} + \Omega_{FGBE} \qquad \text{eq } 6.2.48$$

The 2nd decomposition gives

$$\Omega_{HEX} = \Omega_{FAHE} + \Omega_{FABC} + \Omega_{FHCG} + \Omega_{FHCA} + \Omega_{DHCA}$$

eq 6.2.49

Prob 6.1 Use $f_{AC} = \frac{1}{2}(f_A + f_C)$ on grid



Assume rectangular grid w/ $\Delta x = \Delta y$

$$\frac{\partial}{\partial t}(U_{ij}\,\Omega_{ij}) + \sum_{\text{sides}} (\vec{F} \circ \vec{S}) = Q_{ij}\,\Omega_{ij}$$

$\vec{F} = (\vec{f}, \vec{g}) \circ \vec{S}$

$\|\vec{S}\| =$ length of side

Note: Here I work the problem on a general cartesian mesh.

$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \left(\tfrac{1}{2}(f_H + f_A), \tfrac{1}{2}(g_H + g_A)\right) \circ (0, -\Delta x)$$

$$+ \left(\tfrac{1}{2}(f_A + f_C), \tfrac{1}{2}(g_A + g_C)\right) \circ (\Delta y, -\Delta x)$$

$$+ \left(\tfrac{1}{2}(f_C + f_D), \tfrac{1}{2}(g_C + g_D)\right) \circ (\Delta y, 0)$$

$$+ \left(\tfrac{1}{2}(f_E + f_D), \tfrac{1}{2}(g_E + g_D)\right) \circ (0, \Delta x)$$

$$+ \left(\tfrac{1}{2}(f_E + f_G), \tfrac{1}{2}(g_E + g_G)\right) \circ (-\Delta y, \Delta x)$$

$$+ \left(\tfrac{1}{2}(f_G + f_H), \tfrac{1}{2}(g_G + g_H)\right) \circ (-\Delta y, 0) = \Omega_{ij}\, Q_{ij}$$

$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} - \frac{\Delta x}{2}(g_H + g_A) + \frac{\Delta y}{2}(f_A + f_C) - \frac{\Delta x}{2}(g_A + g_C)$$

$$+ \frac{\Delta y}{2}(f_C + f_D) + \frac{\Delta x}{2}(g_E + g_D)$$

$$- \frac{\Delta y}{2}(f_E + f_G) + \frac{\Delta x}{2}(g_E + g_G)$$

$$- \frac{\Delta y}{2}(f_G + f_H) = \Omega_{ij} Q_{ij}$$

$$\Omega_{ij} = 2\Delta x \Delta y + \frac{2}{2}\Delta x \Delta y = 3\Delta x \Delta y.$$

$$\frac{\partial U_{ij}}{\partial t} + \frac{\Delta x}{2\Omega_{ij}}(-g_H - g_A - g_A - g_C + g_E + g_D + g_E + g_G)$$

$$+ \frac{\Delta y}{2\Omega_{ij}}(f_A + f_C + f_C + f_D - f_E - f_G - f_G - f_H) = Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{\Delta x}{2\Omega_{ij}}(-g_H - 2g_A - g_C + 2g_E + g_D + g_G)$$

$$+ \frac{\Delta y}{2\Omega_{ij}}(f_A + 2f_C + f_D - f_E - 2f_G - f_H) = Q_{ij}$$

$$\rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{6\Delta y}(-2g_{i,j-1} - g_{i+1,j} + g_{i+1,j+1} + 2g_{i,j+1} + g_{i-1,j} - g_{i-1,j-1})$$

$$+ \frac{1}{6\Delta x}(f_{i,j-1} + 2f_{i+1,j} + f_{i+1,j+1} - f_{i,j+1} - 2f_{i-1,j} - f_{i-1,j-1}) = Q_{ij}$$

$$= \frac{\partial U_{ij}}{\partial t} + \frac{1}{3}\left[\frac{2(g_{i,j+1} - g_{i,j-1})}{2\Delta y} + \frac{1}{2}\frac{(g_{i+1,j+1} - g_{i+1,j})}{\Delta y} + \frac{1}{2}\frac{(g_{i-1,j} - g_{i-1,j-1})}{\Delta y} \right]$$

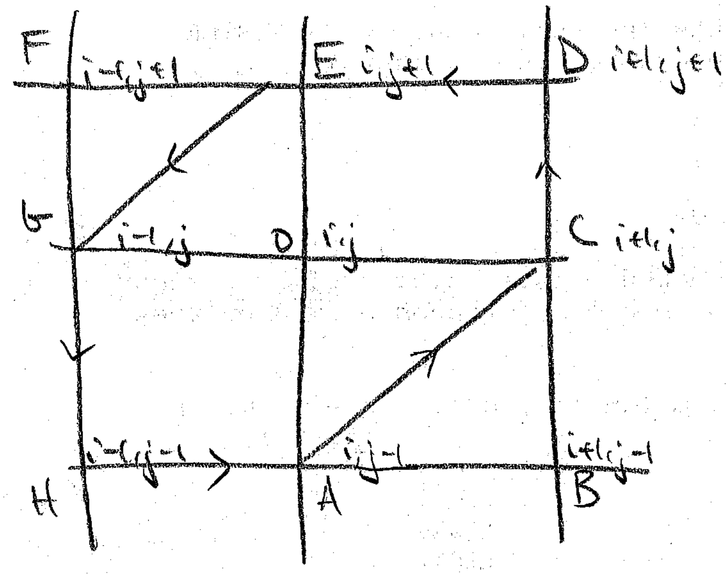
$$+ \frac{1}{3}\left[\frac{2(f_{i+1,j} - f_{i-1,j})}{2\Delta x} + \frac{1}{2}\frac{(f_{i+1,j+1} - f_{i,j+1})}{\Delta x} + \frac{1}{2}\frac{(f_{i,j-1} - f_{i-1,j-1})}{\Delta x} \right]$$

$$= Q_{ij}$$

Check to 1st order:

$$\frac{1}{3}\left(2g_y + \frac{1}{2}g_y + \frac{1}{2}g_y\right) \cong g_y \quad \checkmark$$

Now if the mesh is not cartesian:



$$\frac{\partial(\Omega_{ij}\sigma_{ij})}{\partial t} + \sum_{ACDEGH}\left(f_{AC}\,\Delta y_{AC} - g_{AC}\,\Delta x_{AC}\right) = \Omega_{ij} Q_{ij}$$

Using

$f_{AC} = \frac{1}{2}(f_A + f_C)$ to evaluate the fluxes we get

$$\Omega_{ij}\frac{\partial \sigma_{ij}}{\partial t} + \frac{1}{2}(f_A + f_C)(y_C - y_A) - \frac{1}{2}(g_A + g_C)(x_C - x_A)$$

$$+ \frac{1}{2}(f_C + f_D)(y_D - y_C) - \frac{1}{2}(g_C + g_D)(x_D - x_C)$$

$$+ \frac{1}{2}(f_E + f_D)(y_E - y_D) - \frac{1}{2}(g_E + g_D)(x_E - x_D)$$

$$+ \frac{1}{2}(f_G + f_E)(y_G - y_E) - \frac{1}{2}(g_G + g_E)(x_G - x_E)$$

$$+ \frac{1}{2}(f_G + f_H)(y_H - y_G) - \frac{1}{2}(g_G + g_H)(x_H - x_G)$$

$$+\frac{1}{2}(f_H+f_A)(y_A-y_H)-\frac{1}{2}(g_H+g_A)(x_A-x_H)=\Omega_{ij}Q_{ij}$$

$$\rightarrow \frac{\partial U_{ij}}{\partial t}+\frac{1}{2\Omega_{ij}}\Big[f_A[y_C-\cancel{y_A}+\cancel{y_A}-y_H]+f_C[\cancel{y_C}-y_A+y_D-\cancel{y_C}]$$

$$+f_D[\cancel{y_D}-y_C+y_E-\cancel{y_D}]+f_E[\cancel{y_E}-y_D+y_G-\cancel{y_E}]$$

$$+f_G[\cancel{y_G}-y_E+y_H-\cancel{y_G}]+f_H[\cancel{y_H}-y_G+y_A-\cancel{y_H}]\Big]$$

$$-\frac{1}{2\Omega_{ij}}\Big[g_A[\quad\cdots\quad]\Big]$$ same as above but w/ $y \leftrightarrow x$ & $f \leftrightarrow g$.

$$=Q_{ij}$$

$$\frac{\partial U_{ij}}{\partial t}+\frac{1}{2\Omega_{ij}}\Big[f_A(y_C-y_H)+f_C(y_D-y_A)+f_D(y_E-y_C)+f_E(y_G-y_D)$$

$$+f_G(y_H-y_E)+f_H(y_A-y_G)\Big]$$

$$-\frac{1}{2\Omega_{ij}}\Big[g_A(x_C-x_H)+\cdots\Big]=Q_{ij}$$

$$\Omega_{ij} = \Omega_{AOGH} + \Omega_{OEG} + \Omega_{OCDE} + \Omega_{OCA}$$

$$\tfrac{1}{2}|\vec{x}_{OH} \times \vec{x}_{AG}| + \tfrac{1}{2}|\vec{x}_{OE} \times \vec{x}_{OG}| + \tfrac{1}{2}|\vec{x}_{OC} \times \vec{x}_{OA}| + \tfrac{1}{2}|\vec{x}_{OD} \times \vec{x}_{EC}|$$

See eq 6.2.7
pg 247

Then w/ i's + j's $\Rightarrow$

$$\frac{\partial U_{ij}}{\partial t} + \frac{1}{2\Omega_{ij}}\Big[f_{i,j-1}(y_j - y_{j-1}) + f_{i+1,j}(y_{j+1} - y_{j-1}) + f_{i+1,j+1}(y_{j+1} - y_j)$$

$$+ f_{i,j+1}(y_j - y_{j+1}) + f_{i-1,j}(y_{j-1} - y_{j+1}) + f_{i-1,j-1}(y_{j-1} - y_j)\Big]$$

$$- \frac{1}{2\Omega_{ij}}\Big[g_{i,j-1}(x_{i+1} - x_{i-1}) + g_{i+1,j}(x_{i+1} - x_i) + g_{i+1,j+1}(x_i - x_{i+1})$$

$$+ g_{i,j+1}(x_{i-1} - x_{i+1}) + g_{i-1,j}(x_{i-1} - x_i) + g_{i-1,j-1}(x_i - x_{i-1})\Big]$$

$$= Q_{ij}$$

Restricting to a cartesian grid $\Rightarrow$ $\Omega_{ij} = 3\Delta x \Delta y$ as before

$$\frac{\partial U_{ij}}{\partial t} + \frac{1}{6\Delta x \Delta y}\Big[f_{i,j-1}\Delta y + f_{i+1,j} 2\Delta y + f_{i+1,j+1}\Delta y$$

$$+ f_{i,j+1}(-\Delta y) - f_{i-1,j} 2\Delta y + - f_{i-1,j-1}\Delta y\Big]$$

$$-\frac{1}{6\Delta x\Delta y}\Big[g_{i,j-1}\, 2\Delta x + g_{i+1,j}\,\Delta x - g_{i+1,j+1}\,\Delta x - 2\Delta x\, g_{i,j+1}$$

$$+ -\Delta x\, g_{i-1,j} + \Delta x\, g_{i-1,j+1} \Big] = Q_{ij}$$

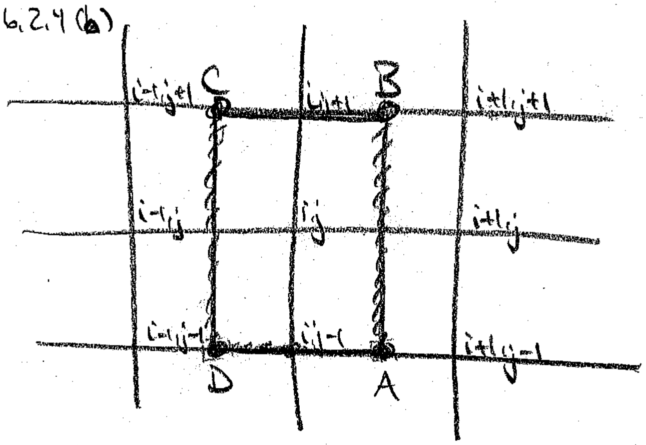
$$\Rightarrow \frac{\partial v_{ij}}{\partial t} + \frac{1}{3}\Big[\frac{2(f_{i+1,j} - f_{i-1,j})}{2\Delta x} + \frac{1}{2}($$

gives same result as on pg 04-24-01 3.

Problem 6.2

$$\Omega_{ij}\frac{\partial w_{ij}}{\partial t} + \sum_{\text{sides}}(\vec{F}\circ\vec{S}) = Q_{ij}\Omega_{ij}$$

Figure 6.2.4 (b)



Sides BC + AD lie along grid lines
Sides CD + BA do not.
∴ we need to construct different ways of evaluating the fluxes along either direction.

Note: since this is assumed a non-uniform grid both indices on x + y

I am assuming that $x_A = \frac{1}{2}(x_{i+1,j-1} + x_{i,j-1})$ etc. are required.

Then method 1: $f_{AB} = \frac{1}{2}(f_{i+1,j} + f_{ij})$ + $f_{DA} = f_{i,j-1}$ give

$\{\vec{S} = \Delta y\,\hat{i} - \Delta x\,\hat{j}\}$

$$\Omega_{ij}\frac{\partial w_{ij}}{\partial t} + \left(\tfrac{1}{2}(f_{i+1,j} + f_{ij}), \tfrac{1}{2}(g_{i+1,j} + g_{ij})\right)\circ\left(y_B - y_A, -(x_B - x_A)\right)$$

$$+ \left(f_{i,j+1}, g_{i,j+1}\right)\circ\left(y_C - y_B, -(x_C - x_B)\right)$$

$$+ \left(\tfrac{1}{2}(f_{i-1,j} + f_{ij}), \tfrac{1}{2}(g_{i-1,j} + g_{ij})\right)\circ\left(y_D - y_C, -(x_D - x_C)\right)$$

$$+ \left(f_{i,j-1}, g_{i,j-1}\right)\circ\left(y_A - y_D, -(x_A - x_D)\right) = \Omega_{ij}Q_{ij}$$

replacing $y_A = \frac{1}{2}(y_{i+1,j-1} + y_{i,j-1})$, $x_B = \frac{1}{2}(x_{i+1,j+1} + x_{i,j+1})$ etc

This would be one method.

Ω_{ij} = Area of 4 quadrilateral ABCD

$$\Omega_{ij} = \frac{1}{2}\left|\vec{x}_{BD} \times \vec{x}_{CA}\right| = \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_D - x_B & y_D - y_B & 0 \\ x_A - x_C & y_A - y_C & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2}(x_{i-1,j-1} + x_{i,j-1} - x_{i,j+1} - x_{i+1,j+1}) & \frac{1}{2}(y_{i-1,j-1} + y_{i,j-1} - y_{i,j+1} - y_{i+1,j+1}) & 0 \\ \frac{1}{2}(x_{i,j-1} + x_{i+1,j-1} - x_{i-1,j+1} - x_{i,j+1}) & \frac{1}{2}(y_{i,j-1} + y_{i+1,j-1} - y_{i-1,j+1} - y_{i,j+1}) & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2}\left| \frac{\hat{k}}{4}\Big[(x_{i-1,j-1} + x_{i,j-1} - x_{i,j+1} - x_{i+1,j+1})(\qquad) - (\qquad)(\qquad) \right|$$

= ... known expression in terms of the grid points at node i,j.

Method 2:
$$f_{AB} = \tfrac{1}{2}(f_{i+1,j} + f_{i,j})$$
$$f_{DA} = \tfrac{1}{4}(2f_{i,j-1} + f_{i-1,j-1} + f_{i+1,j-1})$$

Then
$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \sum_{\text{Sides}}(\vec{F}\circ\vec{S}) = Q_{ij}\Omega_{ij} \qquad \vec{F} = (f,g)$$

sum over DA, AB, BC, CD

$$\Rightarrow \Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \left(\tfrac{1}{4}(f_{i-1,j-1} + 2f_{i,j-1} + f_{i+1,j-1}), \tfrac{1}{4}(g_{i-1,j-1} + 2g_{i,j-1} + g_{i+1,j-1})\right)$$
$$\circ\left(y_A - y_D, -(x_A - x_D)\right)$$
$$+ \left(\tfrac{1}{2}(f_{i,j} + f_{i+1,j}), \tfrac{1}{2}(g_{i,j} + g_{i+1,j})\right)\circ\left(y_B - y_A, -(x_B - x_A)\right)$$
$$+ \left(\tfrac{1}{4}(f_{i-1,j+1} + 2f_{i,j+1} + f_{i+1,j+1}), \tfrac{1}{4}(g_{i-1,j+1} + 2g_{i,j+1} + g_{i+1,j+1})\right)$$
$$\circ\left(y_C - y_B, -(x_C - x_B)\right)$$
$$+ \left(\tfrac{1}{2}(f_{i-1,j} + f_{i,j}), \tfrac{1}{2}(g_{i-1,j} + g_{i,j})\right)\circ\left(y_D - y_C, -(x_D - x_C)\right)$$
$$= \Omega_{ij} Q_{ij}$$

. Method 3:

$$f_{AB} = \frac{1}{6}\left(f_{i,j-1} + f_{i+1,j-1} + f_{i+1,j} + f_{i,j} + f_{i+1,j+1} + f_{i,j+1}\right)$$

& $f_{DA} = f_{i,j-1}$

summing faces in the order of AB, BC, CD, DA

$$S_{ij}\frac{\partial U_{ij}}{\partial t} + \Big(\frac{1}{6}(f_{i,j-1} + f_{i+1,j-1} + f_{i,j} + f_{i+1,j} + f_{i+1,j+1} + f_{i,j+1}),$$

$$\frac{1}{6}(g_{i,j-1} + g_{i+1,j-1} + g_{i,j} + g_{i+1,j} + g_{i+1,j+1} + g_{i,j+1})\Big)$$

$$\circ\left(y_B - y_A, -(x_B - x_A)\right)$$

$$+ \left(f_{i,j+1}, g_{i,j+1}\right)\circ\left(y_C - y_B, -(x_C - x_B)\right)$$

$$+ \Big(\frac{1}{6}(f_{i-1,j+1} + f_{i,j+1} + f_{i-1,j} + f_{i,j} + f_{i-1,j-1} + f_{i,j-1}),$$

$$\frac{1}{6}(g_{i-1,j+1} + g_{i,j+1} + g_{i-1,j} + g_{i,j} + g_{i-1,j-1} + g_{i,j-1})\Big)$$

$$\circ\left(y_D - y_C, -(x_D - x_C)\right)$$

$$+ \left(f_{i,j-1}, g_{i,j-1}\right)\circ\left(+y_A - y_D, -(x_A - x_D)\right)$$

$$= S_{ij}Q_{ij}$$

Method 4:

$$f_{AB} = \frac{1}{6}\left(f_{i+1,j} + f_{i+1,j+1} + f_{i+1,j-1} + f_{ij} + f_{i+1,j-1} + f_{i,j-1}\right)$$

$$+ \quad f_{DA} = \frac{1}{4}\left(f_{i-1,j-1} + 2f_{i,j-1} + f_{i+1,j-1}\right)$$

Then summing faces in the order of AB, BC, CD, DA

$$\Rightarrow \Omega_{ij}\frac{dQ_{ij}}{dt} + \Big(\tfrac{1}{6}\left(f_{i+1,j} + f_{i+1,j+1} + f_{i+1,j-1} + f_{ij} + f_{i,j+1} + f_{i,j-1}\right),$$
$$\tfrac{1}{6}\left(g_{i+1,j} + g_{i+1,j+1} + g_{i+1,j-1} + g_{ij} + g_{i+1,j-1} + g_{i,j-1}\right)\Big)$$
$$\circ \left(y_B - y_A, -(x_B - x_A)\right)$$

$$+ \Big(\tfrac{1}{4}\left(f_{i-1,j+1} + 2f_{i,j+1} + f_{i+1,j+1}\right), \tfrac{1}{4}\left(g_{i-1,j+1} + 2g_{i,j+1} + g_{i+1,j+1}\right)\Big)$$
$$\circ \left(y_C - y_B, -(x_C - x_B)\right)$$

$$+ \Big(\tfrac{1}{6}\left(f_{i-1,j+1} + f_{i,j+1} + f_{i-1,j} + f_{ij} + f_{i-1,j-1} + f_{i,j-1}\right),$$
$$\tfrac{1}{6}\left(g_{i-1,j+1} + g_{i,j+1} + g_{i-1,j} + g_{ij} + g_{i-1,j-1} + g_{i,j-1}\right)\Big)$$

$$\circ \left(y_D - y_C, -(x_D - x_C)\right)$$

$$+ \Big(\tfrac{1}{4}\left(f_{i-1,j-1} + 2f_{i,j-1} + f_{i+1,j-1}\right), \tfrac{1}{4}\left(g_{i-1,j-1} + 2g_{i,j-1} + g_{i+1,j-1}\right)\Big)$$

$$\circ \left(y_A - y_D, -(x_A - x_D)\right) = \Omega_{ij} Q_{ij}$$

Prob 6.3　If the mesh is cartesian, then $x_{i+1} - x_i = \Delta x$

$y_{j+1} - y_j = \Delta y.$

$$\Omega_{ij} = \frac{1}{2} \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\Delta x & -2\Delta y & 0 \\ \Delta x & -2\Delta y & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \left| \hat{k} (2\Delta x \Delta y + 2\Delta x \Delta y) \right| = 2\Delta x \Delta y$$

Then method 1 becomes:

$$\Delta y \Delta x\, 2 \frac{\partial U_{ij}}{\partial t} + \frac{2\Delta y}{2} (f_{i+1,j} + f_{ij}) - (-\Delta x) g_{i,j+1}$$

$$+ -\frac{\Delta y 2}{2} (f_{i-1,j} + f_{ij}) + -\Delta x\, g_{i,j-1} = \Delta x^2 Q_{ij}$$

$$\Delta y \Delta x (2) \frac{\partial U_{ij}}{\partial t} + \Delta y (f_{i+1,j} + \cancel{f_{ij}} - \cancel{f_{ij}} - f_{i-1,j}) + \Delta x (g_{i,j+1} - g_{i,j-1}) = \Delta x^2 Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = Q_{ij}$$　1st method.

2nd method:

$$2\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + -\frac{\Delta x}{4}\left(g_{i-1,j-1} + 2g_{i,j-1} + g_{i+1,j-1}\right)$$

$$+ \frac{2\Delta y}{2}\left(f_{i,j} + f_{i+1,j}\right)$$

$$+ \frac{\Delta x}{4}\left(g_{i-1,j+1}\ 2g_{i,j+1} + g_{i+1,j+1}\right)$$

$$+ -\frac{\Delta y(2)}{2}\left(f_{i-1,j} + f_{i,j}\right) = 2\Delta x \Delta y Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} - \frac{1}{8\Delta y}\left(\underset{3}{g_{i-1,j-1}} + \underset{2}{2g_{i,j-1}} + \underset{1}{g_{i+1,j-1}} - \underset{3}{g_{i-1,j+1}} - \underset{2}{2g_{i,j+1}} - \underset{1}{g_{i+1,j+1}}\right)$$

$$+ \frac{1}{2\Delta x}\left(\cancel{f_{i,j}} + f_{i+1,j} - f_{i-1,j} - \cancel{f_{i,j}}\right) = Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{1}{4}\left[\frac{g_{i+1,j+1} - g_{i+1,j-1}}{2\Delta y} + 2\frac{(g_{i,j+1} - g_{i,j-1})}{2\Delta y}\right.$$

$$\left. + \frac{g_{i-1,j+1} - g_{i-1,j-1}}{2\Delta y}\right] = Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{1}{4}\left[\frac{g_{i-1,j+1} - g_{i-1,j-1}}{2\Delta y} + 2\left(\frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y}\right) + \frac{g_{i+1,j+1} - g_{i+1,j-1}}{2\Delta y}\right] = Q_{ij}.$$

Method 3:

$$2\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{2\Delta y}{6}\left(f_{i,j-1} + f_{i+1,j-1} + f_{ij} + f_{i+1,j} + f_{i+1,j+1} + f_{i,j+1}\right)$$

$$+ \Delta x\, g_{i,j+1} + -\frac{2\Delta y}{6}\left(f_{i-1,j+1} + f_{i,j+1} + f_{i-1,j} + f_{i,j} + f_{i-1,j-1} + f_{i,j-1}\right)$$

$$- \Delta x\, g_{i,j-1} = 2\Delta x \Delta y\, Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{6\Delta x}\Big(\cancel{f_{i,j-1}} + \overset{1}{f_{i+1,j-1}} + \cancel{f_{ij}} + \overset{2}{f_{i+1,j}} + f_{i+1,j+1} + \cancel{f_{i,j+1}}$$

$$- f_{i-1,j+1} - \cancel{f_{i,j+1}} - \overset{2}{f_{i-1,j}} - \cancel{f_{ij}} - \overset{1}{f_{i-1,j-1}} - \cancel{f_{i,j-1}}\Big)$$

$$+ \frac{1}{2\Delta y}\left(g_{i,j+1} - g_{i,j-1}\right) = Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{3}\left[\frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta x} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x}\right]$$

$$+ \frac{1}{2\Delta y}\left(g_{i,j+1} - g_{i,j-1}\right) = Q_{ij}$$

Method 4:

$$2\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{2\Delta y}{6}\left(\overset{2}{f}_{i+1,j} + \overset{3}{f}_{i+1,j+1} + \overset{1}{f}_{i+1,j-1} + \cancel{f_{i,j}} + \cancel{f_{i,j+1}} + \cancel{f_{i,j-1}}\right)$$

$$+ \frac{\Delta x}{4}\left(\overset{1}{g}_{i-1,j+1} + 2\overset{2}{g}_{i,j+1} + g_{i+1,j+1}\right)$$

$$- \frac{2\Delta y}{6}\left(f_{i-1,j+1} + \cancel{f_{i,j+1}} + \overset{2}{f}_{i-1,j} + \cancel{f_{i,j}} + \overset{1}{f}_{i-1,j-1} + \cancel{f_{i,j-1}}\right)$$

$$- \frac{\Delta x}{4}\left(\overset{1}{g}_{i-1,j-1} + 2\overset{2}{g}_{i,j-1} + g_{i+1,j-1}\right) = 2\Delta x \Delta y\, Q_{ij}$$

$$\Rightarrow 2\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{2\Delta y}{6}\left(f_{i+1,j-1} - f_{i-1,j-1} + f_{i+1,j} - f_{i-1,j} + f_{i+1,j+1} - f_{i-1,j+1}\right)$$

$$+ \frac{\Delta x}{4}\left(g_{i-1,j+1} - g_{i-1,j-1} + 2(g_{i,j+1} - g_{i,j-1}) + g_{i+1,j+1} - g_{i+1,j-1}\right)$$

$$= 2\Delta x \Delta y\, Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{3}\left[\frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta x} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x}\right]$$

$$+ \frac{1}{4}\left[\frac{g_{i-1,j+1} - g_{i-1,j-1}}{2\Delta y} + \frac{2(g_{i,j+1} - g_{i,j-1})}{2\Delta y} + \frac{(g_{i+1,j+1} - g_{i+1,j-1})}{2\Delta y}\right]$$

$$= Q_{ij}$$

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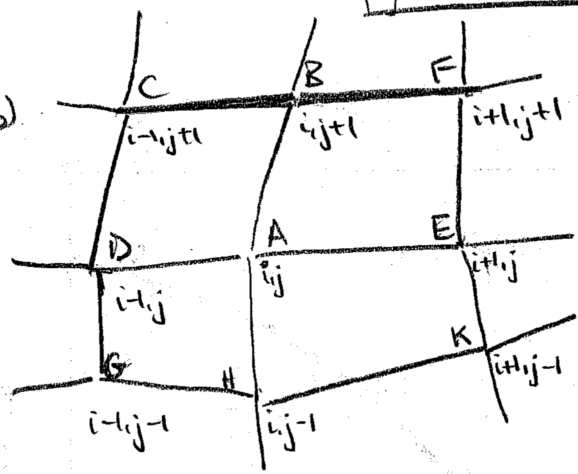
Prob 6.4

6.2.2(b)



$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \sum_{sides}(\vec{F}\cdot\vec{S}) = \Omega_{ij}Q_{ij}$$

sum over sides KF, FC, CG, GK

$$\Rightarrow \Omega_{ij}\frac{\partial U_{ij}}{\partial t} + (f_{KF}, g_{KF})\cdot(\Delta y_{KF}, -\Delta x_{KF})$$

$$+ (f_{CF}, g_{CF})\cdot(\Delta y_{CF}, -\Delta x_{CF})$$

$$+ (f_{GC}, g_{GC})\cdot(\Delta y_{GC}, -\Delta x_{GC})$$

$$+ (f_{KG}, g_{KG})\cdot(\Delta y_{KG}, -\Delta x_{KG}) = \Omega_{ij}Q_{ij}$$

Method 1:

$$f_{KF}\Delta y_{KF} = \frac{1}{2}(f_{i+1,j-1} + f_{i+1,j+1})(y_{i+1,j+1} - y_{i+1,j-1})$$

Then the method becomes

$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \frac{1}{2}(f_{i+1,j-1} + f_{i+1,j+1})(y_{i+1,j+1} - y_{i+1,j-1})$$

$$- \frac{1}{2}(g_{i+1,j-1} + g_{i+1,j+1})(x_{i+1,j+1} - x_{i+1,j-1})$$

$$+ \frac{1}{2}(f_{i+1,j+1} + f_{i-1,j+1})(y_{i-1,j+1} - y_{i+1,j+1})$$

$$- \frac{1}{2}(g_{i+1,j+1} + g_{i-1,j+1})(x_{i-1,j+1} - x_{i+1,j+1})$$

$$+ \frac{1}{2}(f_{i-1,j+1} + f_{i-1,j-1})(y_{i-1,j-1} - y_{i-1,j+1})$$

$$- \frac{1}{2}(g_{i-1,j+1} + g_{i-1,j-1})(x_{i-1,j+1} - x_{i-1,j+1})$$

$$+ \frac{1}{2}(f_{i+1,j-1} + f_{i-1,j-1})(y_{i+1,j-1} - y_{i-1,j-1})$$

$$- \frac{1}{2}(g_{i+1,j-1} + g_{i-1,j-1})(x_{i+1,j-1} - x_{i-1,j-1}) = \Omega_{ij} Q_{ij}$$

Method 2: $\Delta f_{kF} \Delta y_{kF} =$

$$\frac{1}{2}(f_{i+1,j} + f_{i+1,j-1})(y_{i+1,j} - y_{i+1,j-1}) + \frac{1}{2}(f_{i+1,j} + f_{i+1,j+1})(y_{i+1,j+1} - y_{i+1,j})$$

Then the method becomes..

$$\Omega_{ij} \frac{\partial U_{ij}}{\partial t} + \frac{1}{2}(f_{i+1,j} + f_{i+1,j-1})(y_{i+1,j} - y_{i+1,j-1}) + \frac{1}{2}(f_{i+1,j} + f_{i+1,j+1})(y_{i+1,j+1} - y_{i+1,j})$$

$$- \frac{1}{2}(g_{i+1,j} + g_{i+1,j-1})(x_{i+1,j} - x_{i+1,j-1}) - \frac{1}{2}(g_{i+1,j} + f_{i+1,j+1})(y_{i+1,j+1} - x_{i+1,j})$$

$$+\frac{1}{2}(f_{i+1,j+1}+f_{i,j+1})(y_{i,j+1}-y_{i+1,j+1})+\frac{1}{2}(f_{i-1,j+1}+f_{i,j+1})(y_{i-1,j+1}-y_{i,j+1})$$

$$-\frac{1}{2}(g_{i+1,j+1}+g_{i,j+1})(x_{i,j+1}-x_{i+1,j+1})-\frac{1}{2}(g_{i-1,j+1}+g_{i,j+1})(x_{i-1,j+1}-x_{i,j+1})$$

$$+\frac{1}{2}(f_{i-1,j+1}+f_{i-1,j})(y_{i-1,j}-y_{i-1,j+1})+\frac{1}{2}(f_{i-1,j}+f_{i-1,j-1})(y_{i-1,j-1}-y_{i-1,j})$$

$$-\frac{1}{2}(g_{i-1,j+1}+g_{i-1,j})(x_{i-1,j}-x_{i-1,j+1})-\frac{1}{2}(g_{i-1,j}+g_{i-1,j-1})(x_{i-1,j-1}-x_{i-1,j})$$

$$+\frac{1}{2}(f_{i-1,j-1}+f_{i,j-1})(y_{i,j-1}-y_{i-1,j-1})+\frac{1}{2}(f_{i+1,j-1}+f_{i,j-1})(y_{i+1,j-1}-y_{i,j-1})$$

$$-\frac{1}{2}(g_{i-1,j-1}+g_{i,j-1})(x_{i,j-1}-x_{i-1,j-1})-\frac{1}{2}(g_{i+1,j-1}+g_{i,j-1})(x_{i+1,j-1}-x_{i,j-1})$$

$$=\Omega_{ij}Q_{ij}$$

Method 3:

$$f_{kF}\Delta y_{kF}=f_E\Delta y_{kF}=f_{i+1,j}(y_{i+1,j+1}-y_{i+1,j-1})$$

Then the method Becomes:

$$\Omega_{ij}\frac{\partial U_{ij}}{dt} + f_{i+1,j}(y_{i+1,j+1} - y_{i+1,j-1}) - g_{i+1,j}(x_{i+1,j+1} - x_{i+1,j-1})$$

$$+ f_{i,j+1}(y_{i-1,j+1} - y_{i+1,j+1}) - g_{i,j+1}(x_{i-1,j+1} - x_{i+1,j+1})$$

$$+ f_{i-1,j}(y_{i-1,j-1} - y_{i-1,j+1}) - g_{i-1,j}(x_{i-1,j-1} - x_{i-1,j+1})$$

$$+ f_{i,j-1}(y_{i+1,j-1} - y_{i-1,j-1}) - g_{i,j-1}(x_{i+1,j-1} - x_{i-1,j-1})$$

$$= \Omega_{ij} Q_{ij}$$

& for a general non orthogonal grid where

$$\Omega_{ij} = \sum_{p=1}^{4} (\Omega_{ij})^{(p)} \quad w/ \quad (\Omega_{ij})^{p} = \frac{1}{2}\underbrace{|\vec{d}_1 \times \vec{d}_2|}_{\text{cross product of diagonals}}$$

Then for a general cartesian mesh one gets for each

Method 1: $\Omega_{ij} = 4\Delta x \Delta y$

Method 1:

$$4\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{1}{2}(f_{i+1,j-1} + f_{i+1,j+1})\, 2\Delta y$$

$$-\frac{1}{2}\left(g_{i+1,j+1}+g_{i-1,j+1}\right)(-2\Delta x)$$

$$+\frac{1}{2}\left(f_{i-1,j+1}+f_{i-1,j-1}\right)(-2\Delta y)$$

$$-\frac{1}{2}\left(g_{i+1,j-1}+g_{i-1,j-1}\right)(2\Delta x)=4\Delta x\Delta y\,Q_{ij}$$

$$\Rightarrow\quad 4\Delta x\Delta y\frac{\partial U_{ij}}{\partial t}+\Delta y\left[f_{i+1,j-1}-f_{i-1,j-1}+f_{i+1,j+1}-f_{i-1,j+1}\right]$$

$$+\Delta x\left[g_{i-1,j+1}-g_{i-1,j-1}+g_{i+1,j+1}-g_{i+1,j-1}\right]$$

$$=4\Delta x\Delta y\,Q_{ij}$$

$$\Rightarrow\quad \frac{\partial U_{ij}}{\partial t}+\frac{1}{2}\left[\frac{f_{i+1,j-1}-f_{i-1,j-1}}{2\Delta x}+\frac{f_{i+1,j+1}-f_{i-1,j+1}}{2\Delta x}\right]$$

$$+\frac{1}{2}\left[\frac{g_{i-1,j+1}-g_{i-1,j-1}}{2\Delta y}+\frac{g_{i+1,j+1}-g_{i+1,j-1}}{2\Delta y}\right]=Q_{ij}$$

+ Method 2 then becomes:

$$4\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{\Delta y}{2}\left(f_{i+1,j} + f_{i+1,j-1}\right) + \frac{\Delta y}{2}\left(f_{i+1,j} + f_{i+1,j+1}\right)$$

$$- \frac{(-\Delta x)}{2}\left(g_{i+1,j+1} + g_{i,j+1}\right) - \frac{(-\Delta x)}{2}\left(g_{i-1,j+1} + g_{i,j+1}\right)$$

$$+ \frac{1}{2}\left(f_{i-1,j+1} + f_{i-1,j}\right)(-\Delta y) + \frac{1}{2}\left(f_{i-1,j} + f_{i-1,j-1}\right)(-\Delta y)$$

$$- \frac{1}{2}\left(g_{i-1,j-1} + g_{i,j-1}\right)\Delta x - \frac{1}{2}\left(g_{i+1,j-1} + g_{i,j-1}\right)\Delta x$$

$$= 4\Delta x \Delta y Q_{ij}$$

$$\Rightarrow 4\Delta x \Delta y \frac{\partial U_{ij}}{\partial t} + \frac{\Delta y}{2}\left[2f_{i+1,j} + f_{i+1,j-1} + f_{i+1,j+1} - 2f_{i-1,j} - f_{i-1,j+1} - f_{i-1,j-1}\right]$$

$$+ \frac{\Delta x}{2}\left[2g_{i,j+1} + g_{i+1,j+1} + g_{i-1,j+1} - 2g_{i,j-1} - g_{i-1,j-1} - g_{i+1,j-1}\right]$$

$$= 4\Delta x \Delta y Q_{ij}$$

$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{8\Delta x}\left[f_{i+1,j-1} - f_{i-1,j-1} + 2\left(f_{i+1,j} - f_{i-1,j}\right) + f_{i+1,j+1} - f_{i-1,j+1}\right]$$

$$+ \frac{1}{8\Delta y}\left[g_{i-1,j+1} - g_{i-1,j-1} + 2\left(g_{i,j+1} - g_{i,j-1}\right) + g_{i+1,j+1} - g_{i+1,j-1}\right] = Q_{ij}$$

$$\frac{\partial U_{ij}}{\partial t} +$$

$$\frac{1}{4}\left[\frac{(f_{i+1,j-1} - f_{i-1,j-1})}{2\Delta x} + 2\left(\frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x}\right) + \frac{(f_{i+1,j+1} - f_{i-1,j+1})}{2\Delta x}\right] +$$

$$\frac{1}{4}\left[\frac{(g_{i-1,j+1} - g_{i-1,j-1})}{2\Delta y} + 2\left(\frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y}\right) + \frac{(g_{i+1,j+1} - g_{i+1,j-1})}{2\Delta y}\right] = Q_{ij}$$

Method 3:

$$\Rightarrow 4\Delta x\Delta y\frac{\partial U_{ij}}{\partial t} + f_{i+1,j}\,2\Delta y - g_{i,j+1}(-2\Delta x) - f_{i-1,j}\,2\Delta y - g_{i,j-1}\,2\Delta x$$

$$= 4\Delta x\,\Delta y\,Q_{ij}$$

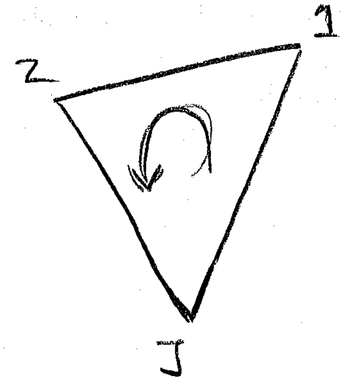
$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = Q_{ij}$$

Prob 6.5　eq 6.2.26

Become

$$\overline{\left(\frac{\partial U}{\partial x}\right)}_\Omega = \frac{-1}{2\Omega}\sum_\ell y_\ell (U_{\ell+1} - U_{\ell-1})$$

$$\overline{\left(\frac{\partial U}{\partial y}\right)}_\Omega = \frac{1}{2\Omega}\sum_\ell x_\ell (U_{\ell+1} - U_{\ell-1})$$



Then direct application of 6.2.26

$$\overline{\left(\frac{\partial f}{\partial x}\right)}_\Omega = \frac{-1}{2\Omega}\sum_\ell y_\ell (f_{\ell+1} - f_{\ell-1}) = \frac{-1}{2\Omega}\Big[y_1(f_2 - f_J) + y_2(f_J - f_1) + y_J(f_1 - f_2)\Big]$$

$$\overline{\left(\frac{\partial g}{\partial y}\right)}_\Omega = \frac{+1}{2\Omega}\sum_\ell x_\ell (g_{\ell+1} - g_{\ell-1}) = \frac{1}{2\Omega}\Big[x_1(g_2 - g_J) + x_2(g_J - g_1) + x_J(g_1 - g_2)\Big]$$

Consider PDE

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

$$\iint \cdot \, d\Omega$$

$$\frac{d}{dt}\iint U\,d\Omega + \iint_\Omega \frac{\partial f}{\partial x}\,d\Omega + \iint_\Omega \frac{\partial g}{\partial y}\,d\Omega = 0$$

$$\frac{d}{dt}U_J + \frac{-1}{2\Omega_J}\sum_\ell y_\ell (f_{\ell+1} - f_{\ell-1}) + \frac{1}{2\Omega_J}\sum_\ell x_\ell (g_{\ell+1} - g_{\ell-1}) = 0$$

I note that using the alternate formulation

$$\overline{\left(\frac{\partial v}{\partial x}\right)}_{\Omega} = \frac{+1}{2\Omega}\sum_{l} v_{e}(y_{e+1} - y_{e-1})$$

$$+ \overline{\left(\frac{\partial v}{\partial y}\right)}_{\Omega} = \frac{-1}{2\Omega}\sum v_{e}(x_{e+1} - x_{e-1})$$

One gets identically eqs E5,3,21 + E5,3,22

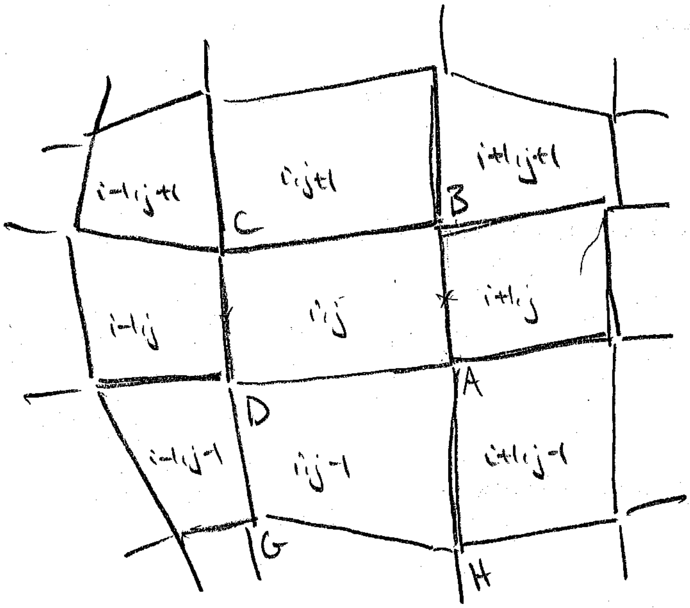
Why are these two results equivalent ~~with~~ with the Finite Element Formulation?

Prob 6.6 $\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial U}{\partial y}\right) = 0$

Mesh like 6.2.2(a)

$k = k(x,y)$ given.

Cartesian.



$f = k\frac{\partial U}{\partial x}$ $\quad g = k\frac{\partial U}{\partial y}$

$f_{AB} = \frac{1}{2}(f_A + f_B)$

Discretized eq $\Omega_J \frac{dU_J}{dt} + \sum_{sides} (f, g)\circ(S_x, S_y) = 0$ $\quad \bar{S}$ is outward normal.

$$\Rightarrow \Delta x \Delta y \frac{\partial U_J}{\partial t} + \left(\tfrac{1}{2}(f_A + f_B), \tfrac{1}{2}(g_A + g_B)\right)\circ(+\Delta y, 0)$$

$$+ \left(\tfrac{1}{2}(f_B + f_C), \tfrac{1}{2}(g_B + g_C)\right)\circ(0, \Delta x)$$

$$+ \left(\tfrac{1}{2}(f_C + f_D), \tfrac{1}{2}(g_C + g_D)\right)\circ(-\Delta y, 0)$$

$$+ \left(\tfrac{1}{2}(f_D + f_A), \tfrac{1}{2}(g_D + g_A)\right)\circ(0, -\Delta x) = 0$$

$$\Rightarrow \Delta x \Delta y \frac{\partial U_J}{\partial t} + \frac{\Delta y}{2}(f_A + f_B) - \frac{\Delta y}{2}(f_C + f_D)$$

$$+ \frac{\Delta x}{2}(g_B + g_C) - \frac{\Delta x}{2}(g_D + g_A) = 0$$

$$\Delta x \Delta y \frac{\partial U_J}{\partial t} + \frac{\Delta y}{2}\left(k_A\left(\frac{\partial U}{\partial x}\right)_A + k_B\left(\frac{\partial U}{\partial x}\right)_B\right) - \frac{\Delta y}{2}\left(k_C\left(\frac{\partial U}{\partial x}\right)_C + k_D\left(\frac{\partial U}{\partial x}\right)_D\right)$$

$$+ \frac{\Delta x}{2}\left(k_B\left(\frac{\partial U}{\partial y}\right)_B + k_C\left(\frac{\partial U}{\partial y}\right)_C\right) - \frac{\Delta x}{2}\left(k_D\left(\frac{\partial U}{\partial y}\right)_D + k_A\left(\frac{\partial U}{\partial y}\right)_A\right) = 0$$

Following Example 6.2.3

$$\left(\frac{\partial U}{\partial x}\right)_A = \frac{1}{2\Delta x}\left(U_{i+1,j} + U_{i+1,j-1} - U_{i,j} - U_{i,j-1}\right)$$

So Above discretization becomes: ✗ But one must also discretize k_A etc., wrong it is assumed that we know the values of k at the corners of a cell.

$$\Delta x \Delta y \frac{\partial U_J}{\partial t} + \frac{\Delta y}{2}\left(\frac{k_A}{2\Delta x}\left(U_{i+1,j} + U_{i+1,j-1} - U_{i,j} - U_{i,j-1}\right) + \frac{k_B}{2\Delta x}\left(U_{i+1,j+1} + U_{i+1,j} - U_{i,j+1} - U_{i,j}\right)\right)$$

$$- \frac{\Delta y}{2}\left(\frac{k_C}{2\Delta x}\left(U_{i,j+1} + U_{i,j} - U_{i-1,j+1} - U_{i-1,j}\right) + \frac{k_D}{2\Delta x}\left(U_{i,j} + U_{i,j-1} - U_{i-1,j} - U_{i-1,j-1}\right)\right)$$

$$+ \frac{\Delta x}{2}\left(\frac{k_B}{2\Delta y}\left(U_{i,j+1} + U_{i+1,j+1} - U_{i,j} - U_{i+1,j}\right) + \frac{k_C}{2\Delta y}\left(U_{i-1,j+1} + U_{i,j+1} - U_{i-1,j} - U_{i,j}\right)\right)$$

$$- \frac{\Delta x}{2}\left(\frac{k_D}{2\Delta y}\left(U_{i-1,j} + U_{i,j} - U_{i-1,j-1} - U_{i,j-1}\right) + \frac{k_A}{2\Delta y}\left(U_{i,j} + U_{i+1,j} - U_{i,j-1} - U_{i+1,j-1}\right)\right)$$

$$= 0$$

$$\Rightarrow \frac{\partial U_J}{\partial t} + \frac{1}{2\Delta x^2}\left[\left(\frac{k_A + k_B}{2}\right)\left(U_{i+1,j} - U_{i,j}\right) + \frac{k_A}{2} U_{i+1,j-1} - \frac{k_B}{2} U_{i,j+1} + \frac{k_B}{2} U_{i+1,j+1} - \frac{k_A}{2} U_{i,j-1}\right]$$

$$-\frac{1}{2\Delta x^2}\left[\frac{k_C+k_D}{2}(v_{ij}-v_{i-1,j})+\frac{k_C}{2}v_{i,j+1}-\frac{k_D}{2}v_{i-1,j-1}+\frac{k_D}{2}v_{i,j-1}-\frac{k_C}{2}v_{i-1,j+1}\right]$$

$$+\frac{1}{2\Delta y^2}\left[\frac{(k_B+k_C)}{2}(v_{i,j+1}-v_{i,j})+\frac{k_B}{2}v_{i+1,j+1}-\frac{k_C}{2}v_{i-1,j}+\frac{k_C}{2}v_{i-1,j+1}-\frac{k_B}{2}v_{i+1,j}\right]$$

$$-\frac{1}{2\Delta y^2}\left[\frac{(k_A+k_D)}{2}(v_{i,j}-v_{i,j-1})+\frac{k_D}{2}v_{i-1,j}-\frac{k_A}{2}v_{i+1,j-1}+\frac{k_A}{2}v_{i+1,j}-\frac{k_D}{2}v_{i-1,j-1}\right]$$

$$=0$$

$$\Rightarrow \frac{\partial v_j}{\partial t}+\frac{1}{2\Delta x}\left[k_{AB}\left(\frac{v_{i+1,j}-v_{i,j}}{\Delta x}\right)+\frac{1}{2}\,\frac{k_A v_{i+1,j-1}-k_B v_{i,j+1}}{\Delta x}+\frac{1}{2}\,\frac{k_B v_{i+1,j+1}-k_A v_{i,j-1}}{\Delta x}\right]$$

$$-\frac{1}{2\Delta x}\left[k_{CD}\left(\frac{v_{i,j}-v_{i-1,j}}{\Delta x}\right)+\frac{1}{2}\left(\frac{k_C v_{i,j+1}-k_D v_{i-1,j-1}}{\Delta x}\right)+\frac{1}{2}\left(\frac{k_D v_{i,j-1}-k_C v_{i-1,j+1}}{\Delta x}\right)\right]$$

$$+\frac{1}{2\Delta y}\left[k_{BC}\left(\frac{v_{i,j+1}-v_{i,j}}{\Delta y}\right)+\frac{1}{2}\left(\frac{k_B v_{i+1,j+1}-k_C v_{i-1,j}}{\Delta y}\right)+\frac{1}{2}\left(\frac{v_{i-1,j+1}-v_{i+1,j}}{\Delta y}\right)\right]$$

$$-\frac{1}{2\Delta y}\left[k_{AD}\left(\frac{v_{i,j}-v_{i,j-1}}{\Delta y}\right)+\frac{1}{2}\left(\frac{k_D v_{i-1,j}-k_A v_{i+1,j-1}}{\Delta y}\right)+\frac{1}{2}\left(\frac{k_A v_{i+1,j}-k_D v_{i-1,j-1}}{\Delta y}\right)\right]$$

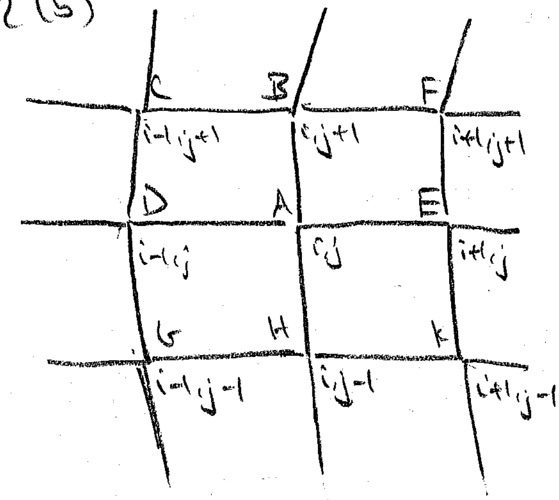
might have a factor of 2 off since think proper approximation is $= 0$

$$\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)\cong\frac{\left(k_{AB}\left(\frac{v_{i+1,j}-v_{i,j}}{\Delta x}\right)-k_{CD}\left(\frac{v_{i,j}-v_{i-1,j}}{\Delta x}\right)\right)}{\Delta x}\quad \text{Not}\quad \frac{(\quad)}{\underline{\underline{2\Delta x}}}$$

(Prob 6.7)

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(1\frac{\partial U}{\partial y}\right) = 0$$

Fig 6.2.2 (b)



Then Finite Vol method in 2D

$$\frac{\partial}{\partial t}\int_{\Omega_{ij}} U\,d\Omega + \int_{\text{contour}} f\,dy - g\,dx = 0$$

$$\Omega_{ij}\frac{\partial U_{ij}}{\partial t} + \sum_{\text{sides}}\left(f_{AB}\Delta y_{AB} - g_{AB}\Delta x_{AB}\right) = 0$$

For cartesian grid Method 1: $f_{KF}\Delta y_{KF} = \frac{1}{2}(f_{i+1,j-1} + f_{i+1,j+1})(2\Delta y)$

$$4\Delta x\Delta y\frac{\partial U_{ij}}{\partial t} + \frac{1}{2}(f_{i+1,j-1} + f_{i+1,j+1})2\Delta y - \frac{1}{2}(g_{i+1,j+1} + g_{i-1,j+1})(-2\Delta x)$$

$$+ \frac{1}{2}(f_{i-1,j+1} + f_{i-1,j-1})(-2\Delta y) - \frac{1}{2}(g_{i+1,j-1} + g_{i-1,j-1})2\Delta x = 0$$

$$\frac{\partial U_{ij}}{\partial t} + \frac{1}{4\Delta x}\left(f_{i+1,j+1} - f_{i-1,j+1} + f_{i+1,j-1} - f_{i-1,j-1}\right)$$

$$+\frac{1}{4\Delta y}\left(g_{i-1,j+1}-g_{i-1,j-1}+g_{i+1,j+1}-g_{i+1,j-1}\right)=0$$

$$\Rightarrow \frac{\partial v_{ij}}{\partial t}+\frac{1}{2}\left[\frac{f_{i+1,j+1}-f_{i-1,j+1}}{2\Delta x}+\frac{f_{i+1,j-1}-f_{i-1,j-1}}{2\Delta x}\right]$$

$$+\frac{1}{2}\left[\frac{g_{i-1,j+1}-g_{i-1,j-1}}{2\Delta y}+\frac{g_{i+1,j+1}-g_{i+1,j-1}}{2\Delta y}\right]=0$$

Now: $f=k\frac{\partial v}{\partial x}$ $\quad g=k\frac{\partial v}{\partial y}$ w/ k constant.

$$\Rightarrow \frac{\partial v_{ij}}{\partial t}+\frac{1}{2}\left[\frac{k}{2\Delta x}\left[\left(\frac{v_{i+1,j+1}-v_{i,j+1}}{\Delta x}\right)-\left(\frac{v_{i,j+1}-v_{i-1,j+1}}{\Delta x}\right)\right]+\right.$$

$$\left.\frac{k}{2\Delta x}\left[\left(\frac{v_{i+1,j-1}-v_{i,j-1}}{\Delta x}\right)-\left(\frac{v_{i,j-1}-v_{i-1,j-1}}{\Delta x}\right)\right]\right]$$

$$+\frac{1}{2}\left[\frac{k}{2\Delta y}\left[\left(\frac{v_{i-1,j+1}-v_{i-1,j}}{\Delta y}\right)-\left(\frac{v_{i-1,j}-v_{i-1,j-1}}{\Delta y}\right)\right]+\right.$$

$$\left.\frac{k}{2\Delta y}\left[\left(\frac{v_{i+1,j+1}-v_{i+1,j}}{\Delta y}\right)-\left(\frac{v_{i+1,j}-v_{i+1,j-1}}{\Delta y}\right)\right]\right]=0$$

$$\Rightarrow \frac{\partial v_{ij}}{\partial t}+\frac{k}{4\Delta x^2}\left[v_{i+1,j+1}-v_{i,j+1}-v_{i,j+1}+v_{i-1,j+1}+v_{i+1,j-1}-v_{i,j-1}-v_{i,j-1}+v_{i-1,j-1}\right]$$

$$+\frac{k}{4\Delta y^2}\left[v_{i-1,j+1}-v_{i-1,j}-v_{i-1,j}+v_{i-1,j-1}+v_{i+1,j+1}-v_{i+1,j}-v_{i+1,j}+v_{i+1,j-1}\right]=0$$

$$\frac{\partial \upsilon_{i,j}}{\partial t} + \frac{k}{4(\Delta x)^2}\left[\upsilon_{i+1,j+1} - 2\upsilon_{i,j+1} + \upsilon_{i-1,j+1} + \upsilon_{i+1,j-1} - 2\upsilon_{i,j-1} + \upsilon_{i-1,j-1} \right]$$

$$+ \frac{k}{4(\Delta y)^2}\left[\upsilon_{i-1,j+1} - 2\upsilon_{i-1,j} + \upsilon_{i-1,j-1} + \upsilon_{i+1,j+1} - 2\upsilon_{i+1,j} + \upsilon_{i+1,j-1} \right] = 0$$

For method 2: w/ $f_{KF}\Delta y_{KF} = f_{KE}\Delta y_{KE} + f_{EF}\Delta y_{EF}$

same result as Problem 6.4 will be obtained $\therefore$ using that

result we get.

$$\frac{\partial \upsilon_{i,j}}{\partial t} +$$

$$\frac{1}{4}\left[\frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta x} + \frac{2(f_{i+1,j} - f_{i-1,j})}{2\Delta x} + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x} \right] +$$

$$\frac{1}{4}\left[\frac{g_{i-1,j+1} - g_{i-1,j-1}}{2\Delta y} + \frac{2(g_{i,j+1} - g_{i,j-1})}{2\Delta y} + \frac{(g_{i+1,j+1} - g_{i+1,j-1})}{2\Delta y} \right] = 0$$

Now $f = k\frac{\partial \upsilon}{\partial x}$ & $g = k\frac{\partial \upsilon}{\partial y}$ w/ k = constant

$$\Rightarrow \frac{\partial \upsilon_{i,j}}{\partial t} +$$

$$\frac{1}{4}\left(\frac{k}{2\Delta x}\left[\left(\frac{\upsilon_{i+1,j-1} - \upsilon_{i,j-1}}{\Delta x}\right) - \left(\frac{\upsilon_{i,j-1} - \upsilon_{i-1,j-1}}{\Delta x}\right) + 2\left(\frac{\upsilon_{i+1,j} - \upsilon_{i,j}}{\Delta}\right) - 2\left(\frac{\upsilon_{i,j} - \upsilon_{i-1,j}}{\Delta x}\right) \right.\right.$$

$$+\left(\frac{v_{i+1,j+1}-v_{i,j+1}}{\Delta x}\right)-\left(\frac{v_{i,j+1}-v_{i-1,j+1}}{\Delta x}\right)\Bigg]\Bigg]+$$

$$\frac{1}{4}\Bigg[\frac{k}{2\Delta y}\Bigg[\left(\frac{v_{i-1,j+1}-v_{i-1,j}}{\Delta x}\right)-\left(\frac{v_{i-1,j}-v_{i-1,j-1}}{\Delta y}\right)$$

$$+2\left(\frac{v_{i,j+1}-v_{i,j}}{\Delta y}\right)-2\left(\frac{v_{i,j}-v_{i,j-1}}{\Delta y}\right)$$

$$+\left(\frac{v_{i+1,j+1}-v_{i+1,j}}{\Delta y}\right)-\left(\frac{v_{i+1,j}-v_{i+1,j-1}}{\Delta y}\right)\Bigg]\Bigg]$$

$$\rightarrow \frac{\partial v_{i,j}}{\partial t}+$$

$$\frac{1k}{8\Delta x^2}\Big(v_{i+1,j-1}-2v_{i,j-1}+v_{i-1,j-1}+2\left(v_{i+1,j}-2v_{i,j}+v_{i-1,j}\right)+$$

$$v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}\Big]$$

$$+\frac{k}{8\Delta y^2}\Big[v_{i-1,j+1}-2v_{i-1,j}+v_{i-1,j-1}+2\left(v_{i,j+1}-2v_{i,j}+v_{i,j-1}\right)$$

$$+v_{i+1,j+1}-2v_{i+1,j}+v_{i+1,j-1}\Big]=0$$

$$\frac{\partial U_{ij}}{\partial t} + \frac{k}{4 \cdot 2}\left[\frac{U_{i+1,j-1} - 2U_{i,j-1} + U_{i-1,j-1}}{\Delta x^2} + 2\frac{(U_{i+1,j} - 2U_{i,j} + U_{i-1,j})}{\Delta x^2} + \frac{U_{i+1,j+1} - 2U_{i,j+1} + U_{i-1,j+1}}{\Delta x^2} \right]$$

$$+ \underset{?}{\frac{k}{4.2}}\left[\frac{U_{i-1,j+1} - 2U_{i-1,j} + U_{i-1,j-1}}{\Delta y^2} + 2\frac{(U_{i,j+1} - 2U_{i,j} + U_{i,j-1})}{\Delta y^2} + \frac{U_{i+1,j+1} - 2U_{i+1,j} + U_{i+1,j-1}}{\Delta y^2} \right] = 0$$

Might be off by a factor of 2 in each expression, or this factor may

No it is Ok Because size of cell that we are updating is of volumn $4\Delta x \Delta y$

Not sure? Seems to be wrong but ?

Again Method 3 will follow the same lines of Problem 6.4 which gives in a cartesian system.

$$\frac{\partial U_{ij}}{\partial t} + \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta x} + \frac{g_{i,j+1} - g_{i,j-1}}{2\Delta y} = 0$$

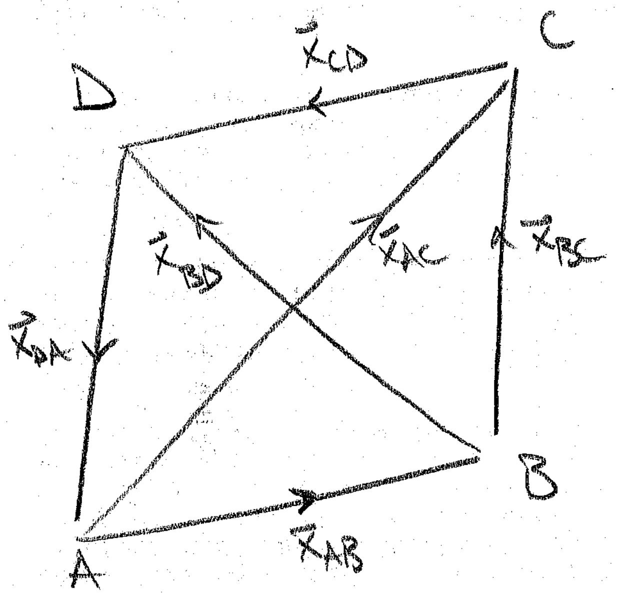
$$\Rightarrow \frac{\partial U_{ij}}{\partial t} + \frac{1}{2\Delta x}\left[\left(\frac{U_{i+1,j} - U_{i,j}}{\Delta x}\right) - \left(\frac{U_{i,j} - U_{i-1,j}}{\Delta x}\right)\right] +$$

$$\frac{1}{2\Delta y}\left[\left(\frac{U_{i,j+1} - U_{i,j}}{\Delta y}\right) - \left(\frac{U_{i,j} - U_{i,j-1}}{\Delta y}\right)\right] = 0$$

$$\frac{\partial v_{ij}}{\partial t} + \frac{1}{2}\left[\frac{v_{i+1,j} - 2v_{ij} + v_{i-1,j}}{\Delta x^2}\right] + \frac{1}{2}\left[\frac{v_{i,j+1} - 2v_{ij} + v_{i,j-1}}{\Delta y^2}\right] = 0$$

Prob 6.3

Eq 6.2.34 is $\vec{S}_{ABCD} = \frac{1}{2}(\vec{x}_{AC} \times \vec{x}_{BD})$



Now as we walk around the structure we get back to the same point.

$\Rightarrow \vec{x}_{AB} + \vec{x}_{BC} + \vec{x}_{CD} + \vec{x}_{DA} = 0$ *

we can write $\vec{x}_{AC} = \vec{x}_{AB} + \vec{x}_{BC}$ +

$\vec{x}_{BD} = \vec{x}_{BC} + \vec{x}_{CD}$

Then

$$\vec{S}_{ABCD} = \frac{1}{2}\left[(\vec{x}_{AB} + \vec{x}_{BC}) \times (\vec{x}_{BC} + \vec{x}_{CD})\right]$$

$$= \frac{1}{2}\left[\vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{AB} \times \vec{x}_{CD} + 0 + \vec{x}_{BC} \times \vec{x}_{CD}\right]$$

$$= \frac{1}{2}\left[\vec{x}_{AB} \times \vec{x}_{BC} + (\vec{x}_{AB} + \vec{x}_{BC}) \times \vec{x}_{CD}\right]$$

From eq * (take cross product w/ $\vec{x}_{CD}$

$$(\vec{x}_{AB} + \vec{x}_{BC}) \times \vec{x}_{CD} + 0 + \vec{x}_{DA} \times \vec{x}_{CD} = 0$$

$$\Rightarrow \quad (\vec{x}_{AB} + \vec{x}_{BC}) \times \vec{x}_{CD} = -(\vec{x}_{DA} \times \vec{x}_{CD})$$

Thus

$$\vec{S}_{ABCD} = \frac{1}{2}\left[\vec{x}_{AB} \times \vec{x}_{BC} - (\vec{x}_{DA} \times \vec{x}_{CD})\right]$$

$$= \frac{1}{2}\left[\vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{CD} \times \vec{x}_{DA}\right] \quad \text{eq } 6.2.35$$

Now take eq * & cross it w/ $\vec{x}_{BC}$ on the left giving

$$\vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{CD} \times \vec{x}_{BC} + \underline{\vec{x}_{DA} \times \vec{x}_{BC}} = 0$$

Take eq * & cross it w/ $\vec{x}_{DA}$ on the left giving

$$\vec{x}_{AB} \times \vec{x}_{DA} + \underline{\vec{x}_{BC} \times \vec{x}_{DA}} + \vec{x}_{CD} \times \vec{x}_{DA} = 0$$

Then Add notice terms w/ both $(\vec{x}_{BC} + \vec{x}_{DA})$ cancel

$$\Rightarrow \quad \vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{CD} \times \vec{x}_{DA} = -(\vec{x}_{CD} \times \vec{x}_{BC} + \vec{x}_{AB} \times \vec{x}_{DA})$$

$$= \vec{x}_{BC} \times \vec{x}_{CD} + \vec{x}_{DA} \times \vec{x}_{AB}$$

Thus $\vec{S}_{ABCD} = \frac{1}{2}$ (Left hand side of Above eq) $= \frac{1}{2}$ (Right hand side of Above eq) Eq 6.2.36

or

Eq 6.2.37 is obtained by Averaging Eq 6.2.35
& Eq 6.2.36

$$\vec{S}_{ABCD} = \frac{1}{2}\left[\frac{1}{2}\left(\vec{x}_{AB} \times \vec{x}_{BC} + \vec{x}_{CD} \times \vec{x}_{DA} \right) + \frac{1}{2}\left(\vec{x}_{BC} \times \vec{x}_{CD} + \vec{x}_{DA} \times \vec{x}_{AB} \right) \right]$$

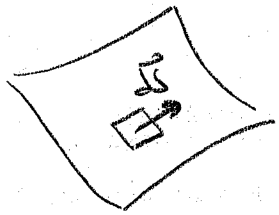
$$= \frac{1}{4}\left[\vec{x}_{AB} \times \left(\vec{x}_{BC} - \vec{x}_{DA} \right) + \vec{x}_{CD} \times \left(\vec{x}_{DA} - \vec{x}_{BC} \right) \right]$$

$$= \frac{1}{4}\left[\left(\vec{x}_{BC} - \vec{x}_{DA} \right) \times \left(\vec{x}_{AB} - \vec{x}_{CD} \right) \right]$$

$$= \frac{1}{4}\left[\left(\vec{x}_{AB} + \vec{x}_{DC} \right) \times \left(\vec{x}_{BC} + \vec{x}_{AD} \right) \right] \quad \text{Eq 6.2.37}$$

Prob 6.9

6.2.33 is $\vec{S}_{face} = \int_{face} d\vec{S} =$



6.2.37 is

$$\vec{S}_{ABCD} = \frac{1}{4}\left[(\bar{x}_{AB} + \bar{x}_{DC}) \times (\bar{x}_{BC} + \bar{x}_{AD})\right]$$

bilinear interpolation eq 5.4.8 $\quad d\Omega = \frac{1}{J} d\xi\, d\eta$

$$\vec{S}_{face} = \int_{face} d\vec{S} = \int_{face} (n_x \hat{i} + n_y \hat{j})\, d\Omega \qquad \|\hat{n}\| = \sqrt{n_x^2 + n_y^2} = 1$$

$n_x = n_x(x,y)$
$n_y = n_y(x,y)$

$$= \hat{i}\int_{face} n_x\, d\Omega + \hat{j}\int_{face} n_y\, d\Omega$$

Change to.

5.4.8 $\quad d\Omega = \frac{1}{J} d\xi\, d\eta$

Section 5.4

$$J \equiv \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{vmatrix}$$

$$\bar{S}_{face} = \int_{face} d\vec{S} = \int_{face} |d\vec{S}|\, \hat{dS} \qquad \frac{1}{J} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \end{vmatrix}$$

$$= \int_{face} \hat{dS}(\xi,\eta)\, \frac{1}{J}\, d\xi\, d\eta$$

2 point gauss quad

$$\int_{-1}^{+1} f(\xi)\, d\xi \cong \sum_{i=1}^{n} H_i f(\xi_i)$$

$$= \sum_{i=1}^{2} H_i f(\xi_i)$$

$H_i = +1.0 \quad \xi_i = \pm .577$

Bilinear element N_I

$$\bar{x} = \sum \bar{x}_I N_I(\xi,\eta) \qquad x = \sum x_I N_I(\xi,\eta)$$
$$y = \sum y_I N_I(\xi,\eta)$$

Do 2x2 planar case 1st:

$$\bar{S}_{face} = \hat{k} \int_{face} \frac{1}{J}\, d\xi\, d\eta = \hat{k} \int_{-1}^{+1}\int_{-1}^{+1} \frac{d\xi\, d\eta}{J} = \hat{k} \sum_{j=1}^{2}\sum_{i=1}^{2} H_i H_j \frac{1}{J(\xi_i,\eta_j)}$$

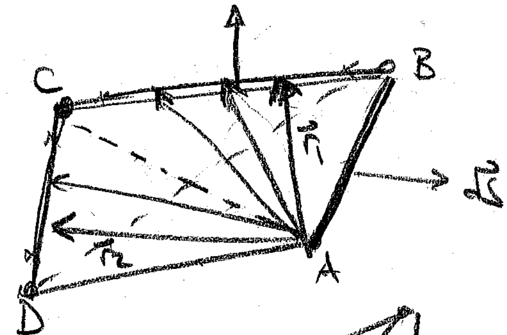
Then write $\frac{1}{J(\xi_i,\eta_j)}$ in terms of x's + y's.

$$\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{pmatrix} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial x}{}$$

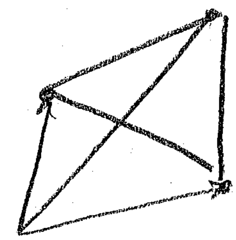
Prob 6.10 eq 6.2.31 $2\Omega = \oint_S \vec{x} \circ d\vec{S} = \oint_S x\,dy - y\,dx$

w/ A as the origin $\vec{x}_A \circ d\vec{S} = 0$

Along segments BA + DA



$$\Rightarrow 2\Omega = \cancel{\int_{AB}}^{0} + \int_{BC} + \int_{CD} + \cancel{\int_{DA}}^{0}$$

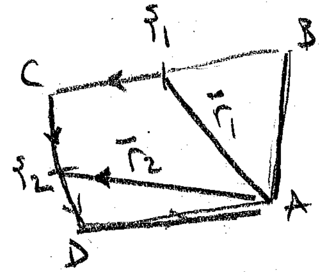


$$2\Omega = \int_{BC} \vec{x} \circ d\vec{S} + \int_{CD} \vec{x} \circ d\vec{S}$$

Claim $\forall\ \vec{r}_1 \ni$ $\vec{r}_1$ Between B & C on upper leg then $\exists$ $\vec{r}_2$ on segment CD $\ni$ $\vec{r}_1 + \vec{r}_2 = \vec{r}_{AC}$

$\vec{r}_2 = \vec{r}_{AC} - \vec{r}_1$ Must be shown that $\vec{r}_2$ is on segment CD

$$\underbrace{\vec{r}_{AB} + \vec{r}_{B\xi_1}}_{\vec{r}_1} + \vec{r}_{\xi_1 C} + \vec{r}_{C\xi_2} + \underbrace{\vec{r}_{\xi_2 D} + \vec{r}_{DA}}_{-\vec{r}_2} = 0$$

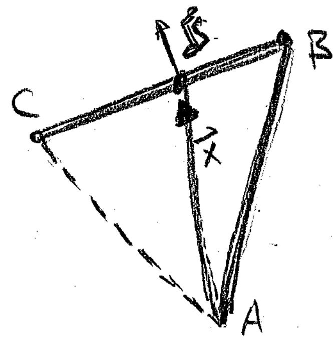


$$\Rightarrow \vec{r}_1 - \vec{r}_2 = -\vec{r}_{\xi_1 C} - \vec{r}_{C\xi_2}$$

By definition $\vec{x}_{AC} = \vec{x}_{AB} + \vec{x}_{AD}$

- idea that ∃ $\forall r_1$ Between B+C ∃ a r_2 ∋

Prob 6.11 Fig 6.2.5



$$2\Omega = \oint_S \vec{x} \circ \vec{ds}$$

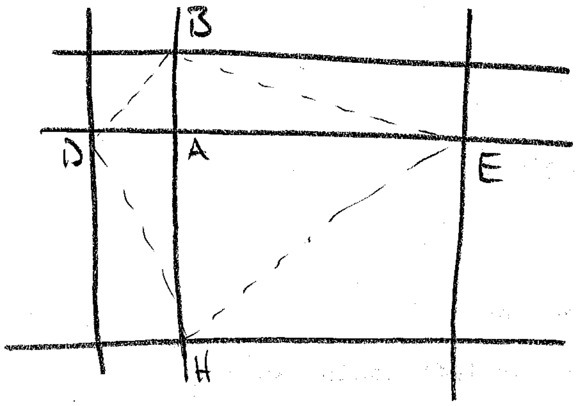
taking pt A as an origin

$$2\Omega = \cancel{\int_{AB} \vec{x} \circ \vec{ds}}^{\,0} + \int_{BC} + \cancel{\int_{CA}}^{\,0} = \int_{BC} \vec{x} \circ \vec{ds}$$

x + ds are normal

Prob 6.12

Fig 6.2.6 (b)



Eq 6.2.29 a

$$\left(\frac{\partial U}{\partial x}\right)_{ABD} = \frac{(U_A - U_C)(y_B - y_D) - (U_B - U_D)(y_A - y_C)}{(x_A - x_C)(y_B - y_D) - (x_B - x_D)(y_A - y_C)}$$

By Eq Above

$$\overline{\left(\frac{\partial U}{\partial x}\right)}_{BDHE} = \frac{(U_B - U_H)\cancelto{0}{(y_D - y_E)} - (U_D - U_E)(y_B - y_H)}{(x_B - x_H)\cancelto{0}{(y_D - y_E)} - (x_D - x_E)(y_B - y_H)}$$

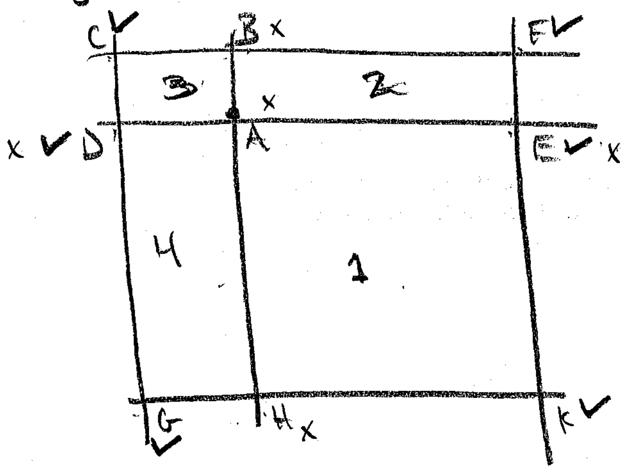
told $y_E = y_D$

$$\overline{\left(\frac{\partial U}{\partial x}\right)}_{BDHE} = \frac{U_D - U_E}{x_D - x_E} = \frac{U_E - U_D}{x_E - x_D}$$

would be more accurate than 1st order but not as accurate as 2nd order because A is not ness. at the midpoint of $\overline{DE}$.

Prob 6.13

Fig 6.2.6(b)



$Y_E = Y_D$

$+ x_B = x_H$.

Eqs 6.2.26a

$$\overline{\left(\frac{\partial U}{\partial x}\right)}_{\Omega} = \frac{-1}{2\Omega}\sum_{\ell}(U_\ell + U_{\ell+1})(y_{\ell+1} - y_\ell) =$$

$$= \frac{-1}{2\Delta y_{FK}\Delta x_{GK}}\Big[(U_K + \underline{U_E})(y_E - y_K) + (\underline{U_E} + U_F)(y_F - y_E)$$

$$+ (U_F + U_B)(y_B - y_F) + (U_B + U_C)(y_C - y_B)$$

$$+ (U_C + \underline{U_D})(y_D - y_C) + (\underline{U_D} + U_G)(y_G - y_D)$$

$$+ (U_G + U_H)(y_H - y_G) + (U_K + U_H)(y_K - y_H)\Big]$$

Try Eq

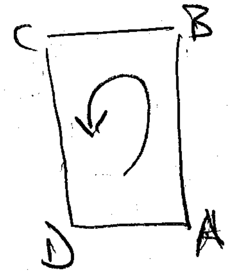
$$\overline{\left(\frac{\partial U}{\partial x}\right)}_{\Omega} = \frac{1}{2\Omega}\sum_{\ell} U_\ell (y_{\ell+1} - y_{\ell-1})$$

$$= \frac{1}{2\Delta y_{KG}\Delta x_{FK}}\Big[U_G(y_H - y_D) + U_H(y_K - y_G) + U_K(y_E - y_H)$$

$$+ U_E(y_F - y_K) + U_F(y_B - y_E) + U_B(y_C - y_F)$$

$$+ U_C(y_D - y_B) + U_D(y_G - y_C)\Big]$$

~~For Rectangle ... $A\left(\frac{\partial U}{\partial x}\right)$~~ =



ABCD KFBH

Rect BHKF

$$A_{BHKF}\left(\frac{\partial U}{\partial x}\right) = (U_K - U_B)(y_F - y_H) - (U_F - U_H)(y_K - y_B)$$

ABCD HBCG =

$$A_{HBCD}\left(\frac{\partial U}{\partial x}\right) = (U_H - U_C)(y_B - y_G) - (U_B - U_G)(y_H - y_C)$$

Adding

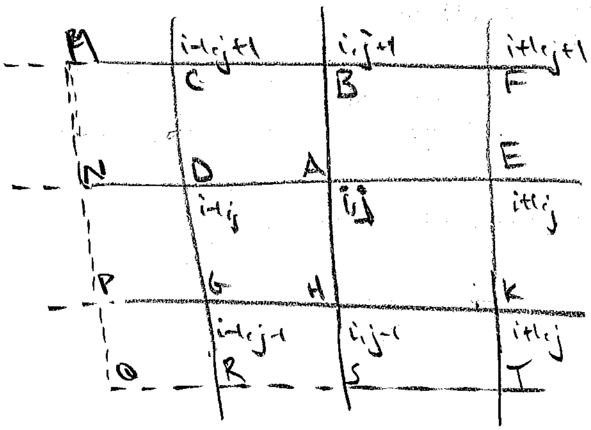
$$= U_H(y_B - y_G + y_K - y_B) + U_B(-\cancel{y_H} + y_C - y_F + \cancel{y_H})$$

Add top Rectangle & Bottom rectangle

EF

Prob 6.14

Fig 6.2.2 in cartesian.



Eq $\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} + b\frac{\partial U}{\partial y} = 0$ $\vec{F} = [f, g] = [aU, bU]$

$a, b > 0$ $\vec{A} = \frac{\partial f}{\partial U}\hat{x} + \frac{\partial g}{\partial U}\hat{y} = a\hat{i}_x + b\hat{i}_y$

$$\text{Then Flux } (\vec F \cdot \vec S)_{AB} = \begin{cases} (\vec F \cdot \vec S)_{ij} & \text{if } (A \cdot \vec S)_{AB} > 0 \\ (\vec F \cdot \vec S)_{EF} & \text{if } (A \cdot \vec S)_{AB} < 0 \end{cases}$$

Then Finite Vol method is

$$\Omega_{ij}\frac{dU_{ij}}{dt} + \sum_{\text{Sides}} (\vec F \cdot \vec S) = 0$$

$$\Rightarrow 4\Delta x \Delta y \frac{dU_{ij}}{dt} + (\vec F \cdot \vec S)_{KE} + (\vec F \cdot \vec S)_{EF} + (\vec F \cdot \vec S)_{FB} + (\vec F \cdot \vec S)_{BC} + (\vec F \cdot \vec S)_{CD} + (\vec F \cdot \vec S)_{DG}$$

$$+ (\vec F \cdot \vec S)_{GH} + (\vec F \cdot \vec S)_{HK} = 0.$$

$$\Rightarrow 4\Delta x \Delta y \frac{dU_{ij}}{dt} + (\vec F \cdot \vec S)_{HA} + (\vec F \cdot \vec S)_{AB} + (\vec F \cdot \vec S)_{EA} + (\vec F \cdot \vec S)_{AD} + (\vec F \cdot \vec S)_{MN}$$

$$+ (\vec F \cdot \vec S)_{NP} + (\vec F \cdot \vec S)_{RS} + (\vec F \cdot \vec S)_{ST} = 0$$

Note: Don't see this result from the Flux definition but this is the intuitive meaning of the upwind scheme.

$$\Rightarrow 4\Delta x \Delta y \frac{dv_{ij}}{dt} + \frac{1}{2}(f_{i,j-1} + f_{ij})\Delta y + \frac{1}{2}(f_{ij} + f_{i,j+1})\Delta y - \frac{1}{2}(g_{ij} + g_{i+1,j})(-\Delta x)$$

$$-\frac{1}{2}(g_{i-1,j} + g_{ij})(-\Delta x) + \frac{1}{2}(f_{i-2,j+1} + f_{i-2,j})(-\Delta y)$$

$$+ \frac{1}{2}(f_{i-2,j} + f_{i-2,j-1})(-\Delta y) - \frac{1}{2}(g_{i-1,j-2} + g_{i,j-2})\Delta x - \frac{1}{2}(g_{i,j-2} + g_{i+1,j-2})\Delta x$$

$$= 0$$

$$\Rightarrow 4\Delta x \Delta y \frac{dv_{ij}}{dt} + \frac{a}{2}\Delta y\left[f_{i,j-1} + 2f_{ij} + f_{i,j+1} - f_{i-2,j+1} - 2f_{i-2,j} - f_{i-2,j-1}\right]$$

$$+ \frac{b\Delta x}{2}\left[g_{i+1,j} + 2g_{ij} + g_{i-1,j} - g_{i-1,j-2} - 2g_{i,j-2} - g_{i+1,j-2}\right]$$

$$= 0$$

$$\Rightarrow \frac{dv_{ij}}{dt} + \frac{a}{8\Delta x}\left[f_{i,j-1} - f_{i-2,j-1} + 2(f_{ij} - f_{i-2,j}) + f_{i,j+1} - f_{i-2,j+1}\right]$$

$$+ \frac{b}{8\Delta y}\left[g_{i-1,j} - g_{i-1,j-2} + 2(g_{ij} - g_{i,j-2}) + g_{i+1,j} - g_{i+1,j-2}\right] = 0$$

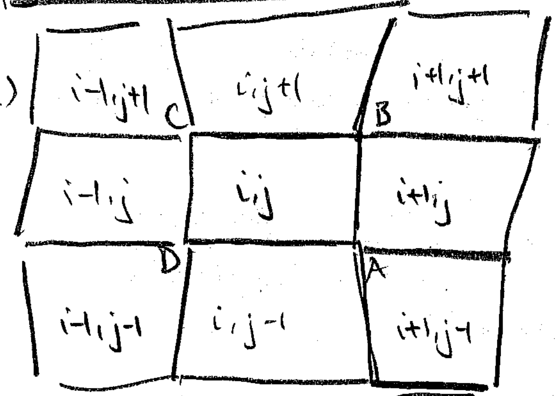
$$\Rightarrow \frac{dv_{ij}}{dt} + \frac{a}{4}\left[\frac{f_{i,j-1} - f_{i-2,j-1}}{2\Delta x} + 2\left(\frac{f_{ij} - f_{i-2,j}}{2\Delta x}\right) + \frac{f_{i,j+1} - f_{i-2,j+1}}{2\Delta x}\right]$$

$$+ \frac{b}{4}\left[\frac{g_{i-1,j} - g_{i-1,j-2}}{2\Delta y} + \frac{2(g_{ij} - g_{i,j-2})}{2\Delta y} + \frac{g_{i+1,j} - g_{i+1,j-2}}{2\Delta y}\right] = 0$$

05-19-01

Prob 6.15

6.2.2 (a)



Arbitrary mesh

$$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial U}{\partial y}\right) = 0$$

2D finite volume formulation gives.

$$\Omega_{ij}\frac{dU_{ij}}{dt} + \sum_{ABCD} f_{AB}(y_B - y_A) - g_{AB}(x_B - x_A) = 0$$

From 6.2.7

$$\Omega_{ij} = \frac{1}{2}\left(\Delta x_{AC}\,\Delta y_{BD} - \Delta x_{BD}\,\Delta y_{AC}\right)$$

$$= \frac{1}{2}\left((x_C - x_A)(y_D - y_B) - (x_D - x_B)(y_C - y_A)\right)$$

Then letting $f_{AB} = \frac{1}{2}(f_A + f_B)$ +

$$f_A = k\left.\frac{\partial U}{\partial x}\right|_A \cong \frac{K}{2\Delta x}\left(U_{i+1,j} + U_{i+1,j-1} - U_{ij} - U_{i,j-1}\right)$$

Prob 6.16

$$\frac{\partial U}{\partial t} + \frac{\partial f}{\partial x} = 0$$

$$\int_a^b \cdot \, dx \qquad a, b \text{ fixed}$$

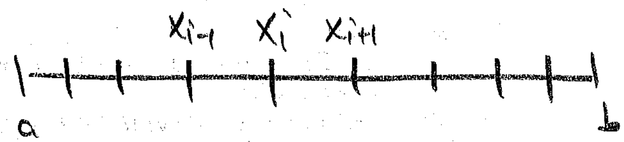
$$\frac{d}{dt}\int_a^b U \, dx + f(b) - f(a) = 0$$

$$\Rightarrow \frac{d}{dt}\int_a^b U \, dx = 0$$

$$\Rightarrow \frac{d}{dt} Q(t) = 0$$

$$\Rightarrow Q(t) = \text{constant}$$

$$\Rightarrow \int_a^b U \, dx = \text{constant}$$



$x_0 = a \qquad x_N = b$

x_i given $0 \le i \le N$

$$\int_{x_i}^{x_{i+1}} f(\xi)\, d\xi \cong \frac{1}{2}\left(f(x_i) + f(x_{i+1})\right)\Delta x_i$$

$$\Delta x_i = x_{i+1} - x_i$$

$$\Rightarrow \frac{1}{2}\left(U(x_0) + U(x_1)\right)\Delta x_1 + \cdots + \frac{1}{2}\left(U(x_{N-1}) + U(x_N)\right)\Delta x_{N-1} = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{N-1}\left(U(x_i) + U(x_{i+1})\right)\Delta x_i = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{N-1}\left(U(x_i) + U(x_{i+1})\right)\left(x_{i+1} - x_i\right) = 0$$

$$\frac{1}{2}\sum_{i=0}^{N-1} U(x_i)\, x_{i+1} - \underbrace{\frac{1}{2}\sum_{i=0}^{N-1} U(x_{i+1})\, x_i}_{} = 0$$

$$\frac{1}{2}\sum_{i=0}^{N-1} U(x_i)\, x_{i+1} - \frac{1}{2}\sum_{i=1}^{N} U(x_i)\, x_{i-1} = 0$$

$\Rightarrow$ Then $\frac{1}{2}u(x_0)x_1 - \frac{1}{2}u(x_N)x_{N-1} + \frac{1}{2}\sum_{i=1}^{N-1} u(x_i)(x_{i+1} - x_{i-1}) = 0$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{N-1} u(x_i)(x_{i+1} - x_{i-1}) = \frac{1}{2}u(x_N)x_{N-1} - \frac{1}{2}u(x_0)x_1$$

Consider

$$\frac{d}{dt}\int_a^b u(\xi,t)\,d\xi = 0$$

$$\frac{d}{dt}\frac{1}{2}\sum_{i=0}^{N-1}\left(u(x_i,t) + u(x_{i+1},t)\right)\Delta x_i = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{N-1}\left(\frac{du(x_i,t)}{dt} + \frac{du(x_{i+1},t)}{dt}\right)(x_{i+1} - x_i) = 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=0}^{N-1}\frac{du(x_i,t)}{dt}(x_{i+1} - x_i) + \underbrace{\frac{1}{2}\sum_{i=0}^{N-1}\frac{du(x_{i+1},t)}{dt}(x_{i+1} - x_i)} = 0$$

$$\frac{1}{2}\sum_{i=1}^{N}\frac{du(x_i,t)}{dt}(x_i - x_{i-1}) = 0$$

$$\frac{1}{2}\frac{du(x_0,t)}{dt}(x_1 - x_0) + \frac{1}{2}\frac{du(x_N,t)}{dt}(x_N - x_{N-1}) + \frac{1}{2}\sum_{i=1}^{N-1}\frac{du(x_i,t)}{dt}(x_{i+1} - x_i + x_i - x_{i-1})$$

$$= 0$$

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{N-1} \frac{du(x_i,t)}{dt}(x_{i+1}-x_{i-1}) + \frac{1}{2}\frac{du(x_0,t)}{dt}(x_1-x_0) + \frac{1}{2}\frac{du(x_N,t)}{dt}(x_N-x_{N-1}) = 0$$

$$\frac{du(x_i,t)}{dt} \approx \frac{\Delta u_i}{\Delta t}$$ & multiply by Δt.

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{N-1} \Delta u_i (x_{i+1}-x_{i-1}) + \underbrace{\frac{1}{2}\Delta u_0 (x_1-x_0) + \frac{1}{2}\Delta u_N (x_N-x_{N-1})}_{\text{How know are zero?}} = 0$$

If zero

$$\frac{1}{2}\sum_{i=1}^{N-1} \Delta u_i (x_{i+1}-x_{i-1}) = 0 .$$