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$$\nabla\circ(\rho\,\vec{v}\otimes\vec{v} + p\bar{\bar{I}}) = \nabla\circ(\rho\,\vec{v}\otimes\vec{v}) + \nabla\circ(p\,\bar{\bar{I}})$$

$$= \nabla\circ(\rho v v^T) + \nabla\circ\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

$$= \nabla\circ\left(\rho\begin{pmatrix} u \\ v \\ w \end{pmatrix}\begin{pmatrix} u & v & w \end{pmatrix}\right) + \nabla\circ\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

$$= \nabla\circ\left(\rho\begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}\right) + \nabla\circ\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

$$= \nabla\circ\left(\begin{pmatrix} \rho u^2 & \rho uv & \rho uw \\ \rho vu & \rho v^2 & \rho vw \\ \rho wu & \rho wv & \rho w^2 \end{pmatrix}\right) + \nabla\circ\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$$

Idea behind $\nabla\circ\bar{\bar{M}}$ is that $\nabla\cdot\bar{\bar{M}}v = \nabla\circ(\bar{\bar{M}}v)$

= Regular divergence.

Thus $\nabla\circ\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \nabla\circ\begin{pmatrix} p\bar{x} \\ p\bar{y} \\ p\bar{z} \end{pmatrix} = \frac{\partial}{\partial x}(p\bar{x}$

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$$[\vec{F}] \cdot \vec{1}_n - \vec{C}[U] \cdot \vec{1}_n = 0 \quad *$$

$$\frac{\partial \Sigma}{\partial t} + \vec{C} \cdot \nabla \Sigma = 0$$

$$\vec{1}_n = \frac{\vec{\nabla}\Sigma}{|\vec{\nabla}\Sigma|}$$

$\therefore * \Rightarrow$ ~~[illegible]~~

$$[\vec{F}] \cdot \vec{\nabla}\Sigma - \underbrace{|\vec{\nabla}\Sigma| \vec{C}[U] \cdot \vec{1}_n}_{} = 0$$

$$- \vec{C}[U] \cdot \nabla \Sigma$$

How?

$$[F] \cdot \vec{\nabla}\Sigma - \underbrace{\vec{C} \cdot \nabla \Sigma}_{} [U] = 0$$

$$\leftarrow \frac{\partial \Sigma}{\partial t}$$

$$[\vec{F}] \cdot \vec{\nabla}\Sigma + \frac{\partial \Sigma}{\partial t}[U] = 0$$

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$[\rho \vec{v} \cdot \vec{1}_n] = 0$ eq 16.1.25 w/ $\vec{C} = 0$

$[(\rho \vec{v} \cdot \vec{1}_n)\vec{v}] + [p \vec{1}_n] = 0$ eq 16.1.26 $\vec{C} = 0$

$(\rho \vec{v} \cdot \vec{1}_n)[\vec{v}] + [p]\vec{1}_n = 0$

eq 16.1.27 w/ $\vec{C} = 0$

$[H \rho \vec{v} \cdot \vec{1}_n] = 0$

$\rho \vec{v} \cdot \vec{1}_n [H] = 0 \Rightarrow [H] = 0$

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It $\frac{\partial(\rho s)}{\partial t} + \nabla \cdot (\rho \vec{v} s) \geq 0$

$\rho_t s + \rho s_t + \nabla s \cdot (\rho \vec{v}) + \nabla \cdot (\rho \vec{v}) s \geq 0$

$\cancel{\rho} s_t + \rho \nabla s \cdot \vec{v} \geq 0$

$s_t + \nabla s \cdot \vec{v} \geq 0$

$s_t + (\vec{v} \cdot \nabla) s \geq 0$

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Check $F(\lambda U) = \lambda F(U)$ $\quad$ $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$ $\quad$ if $p = \rho f(e)$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

Now:

$$E = e + \frac{\vec{V}^2}{2}$$

$$\Rightarrow e = E - \frac{V^2}{2} = E - \frac{1}{2}\frac{(u_2^2 + u_3^2 + u_4^2)}{\rho^2}$$

$$\Rightarrow e = E - \frac{1}{2}\left(\frac{u_2^2 + u_3^2 + u_4^2}{u_1^2}\right)$$

Check $f(\lambda U) = \lambda f(U)$

$$p = u_1 f\left(\frac{u_5}{u_1} - \frac{1}{2}\frac{(u_2^2 + u_3^2 + u_4^2)}{u_1^2}\right)$$

$$f = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ \rho u w \\ \rho u\left(E + \frac{p}{\rho}\right) \end{pmatrix}$$

$$= \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + u_1 f\left(\frac{u_5}{u_1} - \frac{1}{2}\frac{(u_2^2 + u_3^2 + u_4^2)}{u_1^2}\right) \\ \frac{u_2 u_3}{u_1} \\ \frac{u_2 u_4}{u_1} \\ u_2\left(\frac{u_5}{u_1} + f\left(\frac{u_5}{u_1} - \frac{1}{2}\frac{(u_2^2 + u_3^2 + u_4^2)}{u_1^2}\right)\right) \end{pmatrix}$$

Now $f(\lambda U) = \begin{pmatrix} \lambda U_2 \\ \lambda \frac{U_2^2}{U_1} + \lambda U_1 f\left( \frac{U_5}{U_1} - \frac{1}{2}\frac{(U_2^2+U_3^2+U_4^2)}{U_1^2} \right) \\ \lambda \frac{U_2 U_3}{U_1} \\ \lambda \frac{U_2 U_4}{U_1} \\ \lambda U_2 \left( \ldots \right) \end{pmatrix}$

$= \lambda f(U)$ ✓

$$g = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ \rho v H \end{pmatrix} = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ \rho v \left( E + \frac{p}{\rho} \right) \end{pmatrix} = \begin{pmatrix} U_3 \\ \frac{U_2 U_3}{U_1} \\ U_3^2/U_1 + U_1 f(\underbrace{\qquad}_{\text{Homo degree 0 in } U}) \\ U_4 U_3 / U_1 \\ U_3 \left( \frac{U_5}{U_1} + f(\underbrace{\qquad}_{\text{Homo degree 0 in } U}) \right) \end{pmatrix}$$

This is Homo. degree 1 in $U$.

$$h = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ \rho w H \end{pmatrix} =$$

Note $H$ is homo degree 0 in $U$.

$$p = \rho f(e)$$

For perfect gas $\quad p = \rho r T$

w/ $T = C_v e \quad \therefore \quad p = \rho r C_v e = \rho (r C_v) e$

$$F(\lambda v) = \lambda F(v)$$

$\frac{d}{d\lambda} \Rightarrow$

$$\Rightarrow \quad \frac{dF(v)}{dv} \frac{dv}{d\lambda} = F(v)$$

$$\Rightarrow \quad \frac{dF}{dv} \frac{d(\lambda v)}{d\lambda} = F(v)$$

$$\Rightarrow \quad \frac{dF}{dv} \cdot v = F(v)$$

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$$(AU)_x = AU_x$$

$$A_x U + A U_x = A U_x$$

$$A_x U = 0$$

$$\vec{m} = \rho \vec{V}$$

$$\mathcal{E} = \rho E$$

$$H = E + \frac{P}{\rho}$$

$$U = \begin{pmatrix} \rho \\ \vec{m} \\ \mathcal{E} \end{pmatrix} \quad \vec{m} = \begin{pmatrix} m \\ n \\ l \end{pmatrix}$$

$$f = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho u v \\ \rho u w \\ \rho u H \end{pmatrix}$$

$$\therefore f = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho u v \\ \rho u w \\ \rho u E + uP \end{pmatrix}$$

putting in the def of stagnation entholphy per unit mass

$$f = \begin{pmatrix} m \\ \frac{m^2}{\rho} + P \\ \frac{mn}{\rho} \\ \frac{ml}{\rho} \\ \frac{m}{\rho}(\mathcal{E} + P) \end{pmatrix}$$

$$P = (\gamma - 1)\left(\mathcal{E} - \frac{\vec{m}^2}{2\rho}\right)$$

$$g = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ \rho v w \\ \rho v H \end{pmatrix} = \begin{pmatrix} n \\ \frac{mn}{\rho} \\ \frac{n^2}{\rho} + p \\ \frac{nl}{\rho} \\ \frac{n\varepsilon}{\rho} + \frac{n}{\rho} p \end{pmatrix}$$

$$\rho v H = \rho v (E + p/\rho) = \rho v E + v p$$

$$h = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ \rho w H \end{pmatrix} = \begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ \rho w^2 + p \\ \rho w E + w p \end{pmatrix} = \begin{pmatrix} l \\ \frac{ml}{\rho} \\ \frac{nl}{\rho} \\ \frac{l^2}{\rho} + p \\ \frac{l}{\rho}\varepsilon + \frac{l}{\rho} p \end{pmatrix}$$

$$\rho w H = \rho w (E + p/\rho) = \rho w E + w p$$

---

$$\varepsilon = \rho E = \rho\left(e + \frac{v^2}{2}\right) = \rho e + \frac{\vec{m}^2}{2\rho}$$

internal energy per unit mass

K.E per unit mass

Ideal gas: Pg 141 Hirsh Vol II

$p = \rho r T$ EOS + internal energy per unit mass $e = c_v T$

$\left\{ T = \frac{e}{c_v} \right\}$

$p = \rho r \frac{e}{c_v} = \rho e \frac{r}{c_v}$ w/ $\frac{r}{c_v} = \gamma - 1$

$\Rightarrow p = (\gamma - 1)\rho e$ + 16.2.15

$\Rightarrow p = (\gamma-1)\left(\varepsilon - \frac{\vec{m}^2}{2\rho}\right)$

$$\frac{\partial f}{\partial \rho} = \begin{pmatrix} 0 \\ -\frac{m^2}{\rho^2} + \frac{\partial p}{\partial \rho} \\ -\frac{mn}{\rho^2} \\ -\frac{ml}{\rho^2} \\ -\frac{m}{\rho^2}(\varepsilon + p) + \frac{m}{\rho}\frac{\partial p}{\partial \rho} \end{pmatrix}$$

w/

$\frac{\partial p}{\partial \rho} = (\gamma-1)\left(\frac{\vec{m}^2}{2\rho^2}\right)$

$= \frac{(\gamma-1)}{2} \vec{v}^2$

$\frac{\partial p}{\partial \vec{m}} = (\gamma-1)\left(\frac{-1}{2\rho}\right) 2\vec{m}$

$= -(\gamma-1)\frac{\vec{m}}{\rho} = -(\gamma-1)\vec{v}$

$\frac{\partial p}{\partial \varepsilon} = (\gamma-1)$

$$\therefore \frac{\partial f}{\partial \rho} = \begin{pmatrix} 0 \\ -\frac{m^2}{\rho^2} + (\gamma-1)\frac{\vec{m}^2}{2\rho^2} \\ -\frac{mn}{\rho^2} \\ -\frac{ml}{\rho^2} \\ -\frac{m}{\rho}\left(\varepsilon + (\gamma-1)\varepsilon - (\gamma-1)\frac{\vec{m}^2}{2\rho}\right) + \frac{m}{\rho}\frac{(\gamma-1)}{2}\frac{\vec{m}^2}{2\rho^2} \end{pmatrix}$$

last component of $\frac{\partial f}{\partial \rho}$ simplifies to

$$= \frac{-m}{\rho^2}\left(\gamma E - (\gamma-1)\frac{m^2}{2\rho} - \frac{(\gamma-1)}{2}\frac{m^2}{2\rho}\right)$$

$$= \frac{-m}{\rho^2}\left(\gamma E - (\gamma-1)\frac{m^2}{\rho}\right)$$

---

$\frac{\partial f}{\partial m} = ?$ $\qquad$ $\frac{\partial f}{\partial m} = ?$

$$\frac{\partial f}{\partial m} = \begin{pmatrix} 1 \\ \frac{2m}{\rho} + (\gamma-1)\left(\frac{-1}{\cancel{2}\rho}\right)\cancel{2}m \\ \frac{n}{\rho} \\ \frac{\ell}{\rho} \\ \frac{1}{\rho}(E+p) + \frac{m}{\rho}(\gamma-1)\left(-\frac{m}{\rho}\right) \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{2m}{\rho} - (\gamma-1)\frac{m}{\rho} \\ \frac{n}{\rho} \\ \ell/\rho \\ \frac{1}{\rho}\left(E + (\gamma-1)\left(E - \frac{m^2}{2\rho}\right)\right) - \frac{m^2}{\rho^2}(\gamma-1) \end{pmatrix}$$

last component simplifies to:

$$\frac{1}{\rho}\left(\cancel{E} + \gamma E - \frac{\gamma}{2\rho}m^2 - \cancel{E} + \frac{m^2}{2\rho}\right) - \frac{m^2}{\rho^2}(\gamma-1)$$

$$= \frac{\gamma}{\rho}E + \frac{m^2}{2\rho^2}(-\gamma+1) - \frac{m^2}{\rho^2}(\gamma-1)$$

$$= \frac{r\varepsilon}{\rho} + -\frac{\vec{m}^2}{2\rho^2}(r-1) - \frac{m^2}{\rho^2}(r-1)$$

$$= \frac{r\varepsilon}{\rho} - \frac{(r-1)}{2\rho^2}\left(\vec{m}^2 + 2m^2\right)$$ = lost component of 1st column in 16.2.23

$$\frac{\partial f}{\partial n} = \begin{pmatrix} 0 \\ 0 - (r-1)\frac{n}{\rho} \\ \frac{m}{\rho} \\ 0 \\ \frac{m}{\rho}\left(-(r-1)\frac{n}{\rho}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -(r-1)\frac{n}{\rho} \\ \frac{m}{\rho} \\ 0 \\ -(r-1)\frac{mn}{\rho^2} \end{pmatrix}$$

$$\frac{\partial f}{\partial l} = \begin{pmatrix} 0 \\ 0 - (r-1)\frac{l}{\rho} \\ 0 \\ \frac{m}{\rho} \\ \frac{m}{\rho}\left(-(r-1)\frac{l}{\rho}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -(r-1)l/\rho \\ 0 \\ \frac{m}{\rho} \\ -(r-1)\frac{lm}{\rho^2} \end{pmatrix}$$ lost column in 16.2.23

Simplifying eq 16.2.23 gives.

$$\frac{\partial f}{\partial \vec{m}} =$$

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Simplifying eq 16.2.23 gives.

$$\frac{\partial f}{\partial \vec{m}} = \begin{pmatrix} 1 & 0 & 0 \\ (3-\gamma)u & -(\gamma-1)v & -(\gamma-1)w \\ v & u & 0 \\ w & 0 & u \\ \gamma E - \frac{(\gamma-1)}{2}(\vec{v}^2 + 2u^2) & -(\gamma-1)uv & -(\gamma-1)wu \end{pmatrix}$$

Now:

$$\frac{\partial f}{\partial E} = \begin{pmatrix} 0 \\ \gamma-1 \\ 0 \\ 0 \\ \frac{m}{\rho}(1+\gamma-1) \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma-1 \\ 0 \\ 0 \\ \gamma\frac{m}{\rho} \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma-1 \\ 0 \\ 0 \\ \gamma u \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -u^2 + \frac{(\gamma-1)}{2}\vec{v}^2 & (3-\gamma)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ -u(\gamma E - (\gamma-1)\vec{v}^2) & \gamma E - \frac{(\gamma-1)}{2}(\vec{v}^2 + 2u^2) & -(\gamma-1)uv & -(\gamma-1)u & \gamma u \end{pmatrix}$$

Cyclic permutations of $u, v, w \Rightarrow$

$\cancel{(u,v,w) \to (v,w,u)}$ & ~~switch columns 2,3,4~~ rotate
& ~~rows 2,3,4~~.

For B switch columns 2+3 & rows 2+3 & $u \leftrightarrow v$ in matrix A.

Switching columns 2+3

$$A' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -u^2 + \frac{\gamma-1}{2}\tilde{V}^2 & -(\gamma-1)v & (3-\gamma)u & -(\gamma-1)w & \gamma-1 \\ -uv & u & v & 0 & 0 \\ -uw & 0 & w & u & 0 \\ -u\left[\gamma E - (\gamma-1)\tilde{V}^2\right] & -(\gamma-1)uv & \gamma E - \frac{(\gamma-1)}{2}(\tilde{V}^2 + 2u^2) & -(\gamma-1)uw & \gamma u \end{pmatrix}$$

Switching rows 2+3 & changing $u \leftrightarrow v$ & $v \leftrightarrow u$

$$A'' = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -uv & v & u & 0 & 0 \\ -v^2 + \frac{(\gamma-1)}{2}\tilde{V}^2 & -(\gamma-1)u & (3-\gamma)v & -(\gamma-1)w & \gamma-1 \\ -vw & 0 & w & v & 0 \\ -v\left[\gamma E - (\gamma-1)\tilde{V}^2\right] & -(\gamma-1)uv & \gamma E - \frac{(\gamma-1)}{2}(\tilde{V}^2 + 2v^2) & -(\gamma-1)vw & \gamma v \end{pmatrix}$$

To get matrix C switch columns 2 & 4
& rows 2 & 4 & $v \leftrightarrow w$.

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -wu & w & 0 & u & 0 \\ -wv & 0 & w & v & 0 \\ -w^2 + \frac{r-1}{2}\vec{v}^2 & -(r-1)u & -(r-1)v & (3-r)w & r-1 \\ -w[rE-(r-1)\vec{v}^2] & -(r-1)wu & -(r-1)wv & j & rw \end{pmatrix}$$

$$rE - \frac{(r-1)}{2}(\vec{v}^2 + 2w^2)$$

$$E = \frac{1}{2}v^2 + e$$

$$\left.\frac{\partial p}{\partial \rho}\right|_s = c^2$$

$e = C_v T \qquad p = \rho r T$

$$e = \frac{C_v}{r}\frac{p}{\rho} = \frac{1}{r-1}\frac{p}{\rho} \quad \text{eq } \underline{2.1.14}.$$

$$= \frac{(r-1)}{2}\vec{v}^2$$

$p = P(\rho)$

$P(\rho) = ?$

taylor expanding $p$ at constant entropy $\rho_0 = 0$.

$$P = \cancel{\left.\frac{\partial p}{\partial \rho}\right|_s (\rho_0)} + \partial^2 p$$

$$p = \left.\frac{\partial p}{\partial \rho}\right|_s (\rho_0 = 0)\,\rho + \left.\frac{\partial^2 p}{\partial \rho^2}\right|_s \left(\frac{\rho^2}{2}\right) + \cdots$$

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$$E = e + \frac{\vec{v}^2}{2} = \frac{c_v}{r}\frac{p}{\rho} + \frac{\vec{v}^2}{2}$$ $\left\{ \begin{array}{l} \text{Using } e = c_v T \\ + \text{ EOS } p = \rho r T \end{array} \right.$

Now From eq 2.1.21 for isentropic flows $\frac{p}{\rho} = \frac{c^2}{\gamma}$

Then $E = \frac{c_v}{r}\frac{c^2}{\gamma} + \frac{\vec{v}^2}{2}$

Now $\frac{c_v}{r} = ?$

From 2.1.12 $\frac{c_p}{r} = \frac{\gamma}{\gamma - 1}$ + 2.1.13 $\frac{c_p}{c_v} = \gamma$

$$\Rightarrow \frac{c_v}{r} = \frac{1}{\gamma - 1}$$

Thus $E = \frac{c^2}{\gamma(\gamma - 1)} + \frac{\vec{v}^2}{2}$

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$$\rho u H = \rho u\left(E + \frac{p}{\rho}\right) = u(\rho E + p) = \frac{m}{\rho}(\mathcal{E} + p)$$

Dropping 3rd 4th columns & 3&4 rows

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -u^2 + \frac{\gamma-1}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -u\left[\gamma E - (\gamma-1)u^2\right] & \gamma E - \frac{\gamma-1}{2}(u^2+2u^2) & \gamma u \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{u^2}{2}(-2+\gamma-1) & (3-\gamma)u & \gamma-1 \\ -u\gamma E + (\gamma-1)u^3 & \gamma E - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix} = \begin{matrix} 0 & 1 & 0 \\ -(3+\gamma)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ '' & '' & \gamma u \end{matrix}$$

---

removing 4th row & column of A & B

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \left(-1+\frac{(\gamma-1)}{2}\right)u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ -u\left[\gamma E - (\gamma-1)(u^2+v^2)\right] & \gamma E - \frac{\gamma-1}{2}(3u^2+v^2) & -(\gamma-1)uv & \gamma u \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{(\gamma-3)}{2}u^2 + \frac{(\gamma-1)}{2}v^2 & (3-\gamma)u & -(\gamma-1)v \\ -uv & v & u \\ -\gamma u E + (\gamma-1)u(u^2+v^2) & \gamma E - \frac{(\gamma-1)}{2}(v^2+3u^2) & \end{pmatrix}$$

removing 4-th row & column of B

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \left(-1+\frac{\gamma-1}{2}\right)v^2 + \frac{\gamma-1}{2}u^2 & -(\gamma-1)u & (3-\gamma)v & \gamma-1 \\ -v\left[\gamma E - (\gamma-1)(u^2+v^2)\right] & -(\gamma-1)uv & \gamma E - \frac{(\gamma-1)}{2}(u^2+3v^2) & \gamma v \end{pmatrix}$$

$$= \begin{matrix} 0 \\ -uv \\ \frac{(\gamma-3)}{2}v^2 + \frac{\gamma-1}{2}u^2 \\ -v\gamma E + (\gamma-1)v(u^2+v^2) \end{matrix}$$

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(16.2.16) For ideal gas

$$p = (\gamma-1)\left(\varepsilon - \frac{\vec{m}^2}{2\rho}\right) \qquad M = \frac{\partial U}{\partial V} \qquad U = \begin{pmatrix} \rho \\ \vec{m} \\ \varepsilon \end{pmatrix};\ V = \begin{pmatrix} \rho \\ \vec{v} \\ p \end{pmatrix}$$

$$U = \begin{pmatrix} \rho \\ \rho\vec{v} \\ \frac{\rho \vec{v}^2}{2} + \frac{p}{\gamma-1} \end{pmatrix}$$

From eq 16.2.16

$$\frac{p}{\gamma-1} + \frac{\vec{m}^2}{2\rho} = \varepsilon$$

$$\Rightarrow \quad \varepsilon = \rho\frac{\vec{v}^2}{2} + \frac{p}{\gamma-1}$$

$$M = \begin{pmatrix} \frac{\partial U}{\partial \rho} & \frac{\partial U}{\partial \vec{v}} & \frac{\partial U}{\partial p} \end{pmatrix}$$

$$\frac{\partial U}{\partial \rho} = \begin{pmatrix} 1 \\ \vec{v} \\ \frac{\vec{v}^2}{2} \end{pmatrix}, \quad \frac{\partial U}{\partial \vec{v}} = ? \quad \frac{\partial U}{\partial u} = \begin{pmatrix} 0 \\ \rho \\ 0 \\ 0 \\ \rho u \end{pmatrix} \quad \frac{\partial U}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ \rho \\ 0 \\ \rho v \end{pmatrix} \quad \frac{\partial U}{\partial w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \rho \\ \rho w \end{pmatrix}$$

$$\frac{\partial U}{\partial p} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\gamma-1} \end{pmatrix}$$

$$M = \begin{vmatrix} 1 & 0 & 0 \\ \vec{v} & \rho\bar{\bar{I}} & 0 \\ \frac{\vec{v}^2}{2} & \rho\vec{v}^T & \frac{1}{\gamma-1} \end{vmatrix}$$

$$M^{-1} = \frac{\partial V}{\partial U} \qquad V = \begin{pmatrix} \rho \\ \frac{\vec{m}}{\rho} \\ (\gamma-1)\left(\mathcal{E} - \frac{\vec{m}^2}{2\rho}\right) \end{pmatrix}$$

(write $U$ in terms of components of $U$)

$$= \left( \frac{\partial V}{\partial \rho}, \frac{\partial V}{\partial m}, \frac{\partial V}{\partial n}, \frac{\partial V}{\partial l}, \frac{\partial V}{\partial \mathcal{E}} \right)$$

$$= \begin{pmatrix} 1 & 0_{1\times 3} & 0 \\ -\frac{\vec{m}}{\rho^2} & \frac{1}{\rho} \bar{\bar{I}}_{3\times 3} & 0_{3\times 1} \\ (\gamma-1)\left(\frac{\vec{m}^2}{2\rho^2}\right) & (\gamma-1)\left(-\frac{1}{\rho}\vec{m}^T\right)_{1\times 3} & (\gamma-1) \end{pmatrix} = \begin{pmatrix} 1 & 0_{1\times 3} & 0 \\ -\frac{\vec{V}}{\rho}{}_{3\times 1} & \frac{1}{\rho}\bar{\bar{I}}_{3\times 3} & 0_{3\times 1} \\ (\gamma-1)\frac{\vec{V}^2}{2} & -(\gamma-1)\vec{V}^T_{1\times 3} & \gamma-1 \end{pmatrix}$$

$$\det M^{-1} = \begin{vmatrix} \frac{1}{\rho}\bar{\bar{I}}_{3\times 3} & 0 \\ -(\gamma-1)\vec{V}^T_{1\times 3} & \gamma-1 \end{vmatrix} = \begin{vmatrix} \frac{1}{\rho} & 0 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 1/\rho & 0 \\ -(\gamma-1)u & -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \end{vmatrix}$$

$$= \frac{1}{\rho} \begin{vmatrix} \frac{1}{\rho} & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 \\ -(\gamma-1)v & -(\gamma-1)w & \gamma-1 \end{vmatrix} + (\gamma-1)u \underbrace{\begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{\rho} & 0 & 0 \\ 0 & 1/\rho & 0 \end{vmatrix}}_{=0}$$

$$= \frac{1}{\rho}\left[\left(\frac{1}{\rho}\right) \begin{vmatrix} 1/\rho & 0 \\ -(\gamma-1)w & \gamma-1 \end{vmatrix} - (\gamma-1)v \underbrace{\begin{vmatrix} 0 & 0 \\ 1/\rho & 0 \end{vmatrix}}_{=0} \right]$$

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$$\frac{1}{\rho^2}\left(\frac{1}{\rho}\right)(\gamma-1) = \frac{\gamma-1}{\rho^3}$$

16.2.1 $$\frac{\partial U}{\partial t} + \frac{\partial \vec{F}}{\partial U} \circ \nabla U = Q$$

Now $\partial_\square U = M \partial_\square V$ for $\square = t, x, y, z$

$$\Rightarrow M\frac{\partial V}{\partial t} + \frac{\partial \vec{F}}{\partial U} M \circ \nabla V = 0$$

$$\Rightarrow M\frac{\partial V}{\partial t} + \vec{A} M \cdot \nabla V = Q$$

$$\frac{\partial V}{\partial t} + (M^{-1}\vec{A}M)\nabla V = Q$$

$$M^{-1}\vec{A}\cdot(M\nabla V)$$

$(A \circ \nabla)U$

$$\underset{5\times5}{A}\partial_x U + \underset{5\times5}{B}\partial_y U + \underset{5\times5}{C}\partial_z U$$

$$= AM\partial_x V + BM\partial_y V + CM\partial_z V$$

$$= \underset{3\cdot5\times5}{\vec{A}}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} M\nabla V$$

$$k = \vec{A} \circ \vec{k}$$

$$= Ak_x + Bk_y + Ck_z$$

Think

$M^{-1}AM$

$M^{-1}BM$

$M^{-1}CM$

Then $M^{-1}kM$

$$= M^{-1}AMk_x + M^{-1}BMk_y + M^{-1}CMk_z$$

$$= \tilde{A}k_x + \tilde{B}k_y + \tilde{C}k_z$$

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$$T\,ds = de + p\,d(1/\rho)$$ 1st law of thermo $e$ = ~~[illegible]~~ specific energy per unit mass.

$$d(p/\rho) = \frac{dp}{\rho} + p\,d(1/\rho)$$

$$\therefore\ p\,d(1/\rho) = d(p/\rho) - dp/\rho$$

$$\therefore\ T\,ds = de + d(p/\rho) - dp/\rho = d\underbrace{\left(e + \frac{p}{\rho}\right)}_{h} - \frac{dp}{\rho}$$

$$= dh - \frac{dp}{\rho}$$

if $ds = 0$

$$dh - \frac{dp}{\rho} = de + p\,d(1/\rho)$$

$$\left.\frac{dh}{dp}\right|_s - \frac{1}{\rho} = \left.\frac{de}{dp}\right|_s + p\frac{d(1/\rho)}{dp} = 0$$

$$\left.\frac{dh}{dp}\right|_s = \frac{1}{\rho} \qquad \left.\frac{\partial e}{\partial p}\right|_s = \left.\frac{\partial h}{\partial p}\right|_s - \frac{1}{\rho} - p\left.\frac{d(1/\rho)}{dp}\right|_s$$

$$= \underbrace{\left.\frac{\partial h}{\partial p}\right|_s - \frac{1}{\rho}}_{=0} + \frac{p}{\rho^2}\left.\frac{\partial \rho}{\partial p}\right|_s$$

$$= \frac{p}{\rho^2}\frac{1}{\partial p/\partial\rho|_s} = \frac{p}{\rho^2 c^2}$$

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mmt

$$\frac{\partial \vec V}{\partial t} + (\vec V\circ\nabla)\vec V + \frac{\nabla P}{\rho} = \vec f_e$$

$\vec V\circ \Rightarrow$

$$\frac{1}{2}\frac{\partial(\vec V\circ\vec V)}{\partial t} + \vec V\circ(\vec V\circ\nabla)\vec V + \vec V\circ\frac{\nabla P}{\rho} = \vec V\circ\vec f_e$$

Note: $\vec V\circ(\vec V\circ\nabla)\vec V$ simplifies & is done below

Now energy eq w/ $E = e + \frac{V^2}{2}$ becomes:

$$\frac{\partial e}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}(V^2) + \vec V\circ\nabla\left(\frac{V^2}{2}+e\right) + \frac{1}{\rho}\nabla\circ(\vec V p) = \vec f_e\circ V$$

Simplify $\nabla\circ(\vec V p) = (\nabla\circ\vec V)p + \vec V\circ\nabla P$

$\Rightarrow$ Energy becomes

$$\frac{\partial e}{\partial t} + \underline{\frac{1}{2}\frac{\partial}{\partial t}V^2} + \vec V\circ\nabla e + \underline{\vec V\circ\nabla\left(\frac{V^2}{2}\right)} + \frac{(\nabla\circ\vec V)P}{\rho} + \underline{\frac{1}{\rho}\vec V\circ\nabla P} = \underline{\vec f_e\circ V}$$

— = common to $\vec V\circ$ mmt eq

Expand $\vec V\circ\nabla\left(\frac{V^2}{2}\right) = \vec V\circ\left(\frac{\partial}{\partial x}\frac{V^2}{2}, \partial_y\frac{V^2}{2}, \partial_z\frac{V^2}{2}\right)$

$= \vec V\circ\left(\vec V\circ\partial_x\vec V, \vec V\circ\partial_y\vec V, \vec V\circ\partial_z\vec V\right) = u\,\vec V\circ\partial_x\vec V + v\,\vec V\circ\partial_y\vec V + w\,\vec V\circ\partial_z\vec V.$

& consider

$\vec V\circ(\vec V\circ\nabla)\vec V = \vec V\circ\left((u\partial_x + v\partial_y + w\partial_z)\vec V\right) = \vec V\circ(u\partial_x u + v\partial_y u + w\partial_z u,$

$u\partial_x v + v\partial_y v + w\partial_z v,$

$u\partial_x w$

Now: $(\vec{v}\circ\nabla)\vec{v} = \nabla\left(\frac{v^2}{2}\right) - \vec{v}\times(\nabla\times\vec{v})$ is used in $\vec{v}\circ$ mmt eq

$\therefore\ \vec{v}\circ(\vec{v}\circ\nabla)\vec{v} = \vec{v}\circ\nabla\left(\frac{v^2}{2}\right)$ As $\vec{v}\circ(\vec{v}\times(\nabla\times\vec{v})) = 0.$

Then dot of mmt eq becomes

$$\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + \vec{v}\circ\nabla\left(\frac{v^2}{2}\right) + \vec{v}\circ\frac{\nabla P}{\rho} = \vec{v}\circ\vec{f}_e$$

Removing from the energy eq one obtains.

$$\frac{\partial e}{\partial t} + \vec{v}\circ\nabla e + \frac{1}{\rho}(\nabla\circ\vec{v})P = 0.$$

Assuming isentropic flow. In general there wont be an expansion for the pressure unless I nail down another one of the thermodynamic quantities.

$$\left(\frac{\partial e}{\partial \rho}\right)_s = \frac{P}{\rho^2 c^2}$$

$$\Rightarrow\ de|_s = \frac{P}{\rho^2 c^2}\, d\rho|_s$$

$$\frac{P}{\rho^2 c^2}\frac{\partial \rho}{\partial t} + \frac{P}{\rho^2 c^2}\vec{v}\circ\nabla\rho + \frac{1}{\rho}(\nabla\circ\vec{v})P = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{v}\circ\nabla\rho + \rho c^2\,\nabla\circ\vec{v} = 0 \qquad \text{eq } 16.2.38$$

---

Check statement on pg 146:

eq 16.2.38 is alternate form of

$$d\rho = c^2\, d\rho \qquad \text{w/} \quad d\rho = \partial_t + \vec{v}\circ\nabla$$

$$\partial_t p + \vec{v}\cdot\nabla p = c^2\left(\rho_t + (\vec{v}\cdot\nabla)\rho\right)$$

But conservation of mass $\rho_t + \vec{v}\cdot\nabla\rho + \rho\nabla\cdot\vec{v} = 0$

$$\Rightarrow \quad \rho_t + v\cdot\nabla\rho = -\rho\nabla\cdot\vec{v}$$

$$\Rightarrow \partial_t p + \vec{v}\cdot\nabla p = -c^2\rho\nabla\cdot\vec{v} \qquad \text{eq } 16.2.38$$

---

$p = ?$

($\mathcal{E} = \rho E$ 3rd component in conservation vector formulation

$$\mathcal{E} = \rho E = \rho\left(e + \frac{v^2}{2}\right)$$

For Ideal gas $e = C_v T$ w/ EOS $\frac{p}{\rho} = rT \Rightarrow T = \frac{p}{\rho r}$

$$\therefore \mathcal{E} = \rho\left(\frac{C_v p}{\rho r} + \frac{v^2}{2}\right) \Rightarrow p = \left(\frac{\mathcal{E}}{\rho} - \frac{v^2}{2}\right)\frac{\rho r}{C_v} \Rightarrow$$

$$p = \frac{r}{C_v}\left(\mathcal{E} - \frac{\rho v^2}{2}\right)$$

Now $r = \frac{(\gamma-1)c_p}{\gamma}$ $\therefore \frac{r}{C_v} = \frac{(\gamma-1)}{\gamma}\frac{c_p}{C_v} = \gamma - 1$

$$\therefore p = (\gamma-1)\left(\mathcal{E} - \frac{\rho v^2}{2}\right) = (\gamma-1)\left(\mathcal{E} - \frac{m^2}{2\rho}\right)$$

$$\frac{\partial \rho}{\partial t} + u\rho_x + v\rho_y + w\rho_z + \rho(u_x + v_y + w_z) = 0$$

$$u_t + uu_x + vu_y + wu_z + \frac{p_x}{\rho} = f_{ex}$$

$$v_t + uv_x + vv_y + wv_z + \frac{p_y}{\rho} = f_{ey}$$

$$w_t + uw_x + vw_y + ww_z + \frac{p_z}{\rho} = f_{ez}$$

$$p_t + up_x + vp_y + wp_z + \rho c^2(u_x + v_y + w_z) = 0$$

$$\therefore \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}_t + \begin{pmatrix} u & \rho & \cdot & \cdot & \cdot \\ \cdot & u & \cdot & \cdot & 1/\rho \\ \cdot & \cdot & u & \cdot & \cdot \\ \cdot & \cdot & \cdot & u & \cdot \\ \cdot & \rho c^2 & \cdot & \cdot & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}_x + \begin{pmatrix} v & \cdot & \rho & \cdot & \cdot \\ \cdot & v & \cdot & \cdot & \cdot \\ \cdot & \cdot & v & \cdot & 1/\rho \\ \cdot & \cdot & \cdot & v & \cdot \\ \cdot & \cdot & \rho c^2 & \cdot & v \end{pmatrix} \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}_y$$

$$+ \begin{pmatrix} w & \cdot & \cdot & \rho & \cdot \\ \cdot & w & \cdot & \cdot & \cdot \\ \cdot & \cdot & w & \cdot & \cdot \\ \cdot & \cdot & \cdot & w & 1/\rho \\ \cdot & \cdot & \cdot & \rho c^2 & w \end{pmatrix} \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}_z = \begin{pmatrix} 0 \\ f_{ex} \\ f_{ey} \\ f_{ez} \\ 0 \end{pmatrix}$$

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$K = \vec{A} \cdot \vec{k}$

$\tilde{K} = \vec{\tilde{A}} \cdot \vec{k}$

$$M^{-1} K M = M^{-1} (\vec{A} \cdot \vec{k}) M$$

$$= M^{-1} (A k_x + B k_y + C k_z) M$$

$$= \underbrace{M^{-1} A M}_{\hat{A}} k_x + \underbrace{M^{-1} B M}_{\hat{B}} k_y + \underbrace{M^{-1} C M}_{\hat{C}} k_z = \vec{\tilde{A}} \cdot \vec{k}.$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{pmatrix} ; \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u & \gamma-1 \end{pmatrix}$$ eq E16.2,8

eq E16.2,7

$$\tilde{A} = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix}$$ from dropping col. 3 & 4 & rows 3 & 4 from $\tilde{A}$

or multiplying $\tilde{A} = M^{-1} A M$ ?

$$= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & 1/\rho & 0 \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u & \gamma-1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -(3-\gamma)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma u E & \gamma E - 3\frac{(\gamma-1)}{2}u^2 & \gamma u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \frac{u^2}{2} & \rho u & \frac{1}{\gamma-1} \end{pmatrix}$$ red

$$= \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & 1/\rho & 0 \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u & \gamma-1 \end{pmatrix} \begin{pmatrix} u & \rho \\ -(3-\gamma)\frac{u^2}{2} + (3-\gamma)u^2 + \frac{(\gamma-1)u^2}{2} & (3-\gamma)u\rho + (\gamma-1)\rho u \\ (\gamma-1)u^3 - \cancel{\gamma u E} + \cancel{\gamma u E} - \frac{3(\gamma-1)u^3}{2} + \frac{\gamma u^3}{2} & \rho \gamma E - \frac{3}{2}(\gamma-1)\rho u^2 + \gamma \rho u^2 \end{pmatrix}$$

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$$\begin{pmatrix} 0 \\ 1 \\ \frac{\gamma u}{\gamma-1} \end{pmatrix}$$

Simplifying the terms of the 2nd matrix gives:   position

$$\frac{(3-\gamma)}{2}u^2 + \frac{(\gamma-1)}{2}u^2 = \frac{u^2}{2}(3-\cancel{\gamma}+\cancel{\gamma}-1) = u^2 \qquad (2,1)$$

$$\frac{u^3}{2}(\cancel{2}\gamma-2-\cancel{3}\gamma+3+\cancel{\gamma}) = \frac{u^3}{2} \qquad (3,1)$$

$$\rho u(3-\gamma+\gamma-1) = 2\rho u \qquad (2,2)$$

$$\rho\gamma E + \rho u^2\left(\gamma - \frac{3}{2}\gamma + \frac{3}{2}\right) = \rho\gamma E + \rho u^2\left(-\frac{\gamma}{2}+\frac{3}{2}\right) \qquad (3,2)$$

$$= \rho\gamma E + \frac{\rho u^2}{2}(3-\gamma)$$

$$M^{-1}AM = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{u}{\rho} & 1/\rho & 0 \\ (\gamma-1)\frac{u^2}{2} & -(\gamma-1)u & \gamma-1 \end{pmatrix}\begin{pmatrix} u & \rho & 0 \\ u^2 & 2\rho u & 1 \\ \frac{u^3}{2} & \rho\gamma E + \frac{\rho u^2}{2}(3-\gamma) & \frac{\gamma u}{\gamma-1} \end{pmatrix}$$

$$= \begin{pmatrix} u & \rho \\ -\frac{u^{\cancel{2}}}{\cancel{\rho}} + \frac{u^{\cancel{2}}}{\cancel{\rho}} & -u+2u \\ (\gamma-1)\frac{u^3}{\cancel{2}} - (\gamma-1)u^3 + (\gamma-1)\frac{u^3}{\cancel{2}} & (\gamma-1)\rho\frac{u^2}{2} - (\gamma-1)2\rho u^2 + (\gamma-1)\rho\gamma E + \frac{\rho u^2}{2}(3-\gamma)(\gamma-1) \end{pmatrix}$$

$$\begin{matrix} 0 \\ 1/\rho \\ -(\gamma-1)u + \gamma u \end{matrix} \Bigg)$$

Simplifying position (3,2) gives w/ $E = \frac{c^2}{\gamma(\gamma-1)} + \frac{u^2}{2}$ which is true for isentropic flows (This assumption is needed for the primitive variables representation)

Simplifying (3,2)

$$\frac{\rho u^2}{2}\Big[\underbrace{(\gamma-1) - 4(\gamma-1) + (3-\gamma)(\gamma-1)}_{-\cancel{3(\gamma-1)} + \cancel{3(\gamma-1)} - \gamma(\gamma-1)}\Big] + \underbrace{\rho\gamma(\gamma-1)\left(\frac{c^2}{\gamma(\gamma-1)} + \frac{u^2}{2}\right)}_{= \rho c^2 + \rho\gamma(\gamma-1)\frac{u^2}{2}}$$

$$= -\gamma(\gamma-1)\frac{\rho u^2}{2} + \rho c^2 + \gamma(\gamma-1)\frac{\rho u^2}{2} = \rho c^2$$

Then

$$M^{-1}AM = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix}$$

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$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2+v^2}{2} & \rho u & \rho v & \frac{1}{\gamma-1} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{u}{\rho} & \frac{1}{\rho} & 0 & 0 \\ -\frac{v}{\rho} & 0 & \frac{1}{\rho} & 0 \\ \frac{(\gamma-1)}{2}(u^2+v^2) & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \end{pmatrix}$$

$\tilde{A} =$ By elliminating rows + columns from big on $\hat{A}$ or

$\tilde{A} = M^{-1} A M$

1st do $AM = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{(\gamma-1)}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ -\gamma uE + (\gamma-1)uv^2 & \gamma E - \frac{(\gamma-1)}{2}(v^2+3u^2) & -(\gamma-1)uv & \gamma u \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2+v^2}{2} & \rho u & \rho v & \frac{1}{\gamma-1} \end{pmatrix}$

$$= \begin{pmatrix} u \\ \frac{\gamma-3}{2}u^2 + \frac{(\gamma-1)}{2}v^2 + (3-\gamma)u^2 - (\gamma-1)v^2 + (\gamma-1)\left(\frac{u^2+v^2}{2}\right) \\ -uv + uv + uv \\ -\gamma uE + (\gamma-1)uv^2 + \gamma uE - \frac{(\gamma-1)}{2}(uv^2+3u^3) - (\gamma-1)uv^2 + \frac{\gamma}{2}(u^3+uv^2) \end{pmatrix}$$

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$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial}{\partial x}(\rho u S) = 0 \qquad S = S(x)$$

$$\rho_t S + (\rho u)_x S + (\rho u) S_x = 0$$

$$\rho_t + (\rho u)_x = -\frac{\rho u}{S} S_x$$

1st eq on Hint Prob 16.8 pg 215

---

$$\frac{\partial}{\partial t}(\rho u S) + \frac{\partial}{\partial x}((\rho u^2 + p) S) = p\frac{dS}{dx}$$

$$S\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) S + (\rho u^2 + \cancel{p})\frac{dS}{dx} = \cancel{p\frac{dS}{dx}}$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial(\rho u^2 + p)}{\partial x} = -\frac{\rho u^2}{S}\frac{dS}{dx}$$

2nd eq on Hint Prob 16.8 pg 215

---

$$\frac{\partial(\rho E S)}{\partial t} + \frac{\partial}{\partial x}(\rho u H S) = 0$$

$$S\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}(\rho u H) S + (\rho u H) S_x = 0$$

$$\Rightarrow \frac{\partial(\rho E)}{\partial t} + \frac{\partial}{\partial x}(\rho u H) = -\frac{1}{S}\frac{dS}{dx}\rho u H$$

3rd eq on Hint Prob 16.8 pg 215.

✓ed

---

1st eq in 16.4.2 is expanded $\rho_t + (\rho u)_x = -\frac{\rho u}{S}\frac{dS}{dx}$

taking $(\rho u)_t + (\rho u^2 + p)_x = -\frac{\rho u^2}{S}\frac{dS}{dx}$

$$\rho_t u + \rho u_t + \rho_x u^2 + 2uu_x\rho + p_x = -\frac{\rho u^2}{S}\frac{dS}{dx}$$

put in $\rho_t = -u\rho_x - \rho u_x - \frac{\rho u}{S}\frac{dS}{dx}$

$$\Rightarrow -\cancel{u^2\rho_x} - \cancel{\rho u u_x} - \cancel{\frac{\rho u^2}{S}\frac{dS}{dx}} + \rho u_t + \cancel{\rho_x u^2} + \cancel{2}uu_x\rho + p_x = -\cancel{\frac{\rho u^2}{S}\frac{dS}{dx}}$$

$$\Rightarrow \rho u_t + u\rho u_x + p_x = 0$$

$$\Rightarrow u_t + uu_x + \frac{p_x}{\rho} = 0$$ 2nd eq in 16.4.2

---

Taking $\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x}(\rho u H) = -\frac{\rho u H}{S}\frac{dS}{dx}$

$$\rho_t E + \rho E_t + (\rho u)_x H + \rho u H_x = -\frac{\rho u H}{S}\frac{dS}{dx}$$

$H = E + \frac{p}{\rho}$

∴ Above becomes

$$\cancel{\rho_t E} + \rho E_t + \cancel{(\rho u)_x E} + (\rho u)_x \frac{p}{\rho} + \rho u H_x = -\frac{\rho u H}{S}\frac{dS}{dx}$$

$\times \quad \rho_t + (\rho u)_x = \left(-\frac{\rho u}{S}\frac{dS}{dx}\right)$

$$E_t + (\rho u)_x \frac{P}{\rho^2} + uH_x = -\frac{uH}{S}\frac{dS}{dx} + \frac{\rho u}{S\rho} E \frac{dS}{dx}$$

$$H = E + \frac{P}{\rho}$$

$$\Rightarrow E_t + \underbrace{(\rho u)_x \frac{P}{\rho^2}}_{\parallel} + uE_x + \left(\frac{P}{\rho}\right)_x u = -\frac{uH}{S}\frac{dS}{dx} + \frac{uE}{S}\frac{dS}{dx}$$

$$(\cancel{\rho_x u} + u_x \rho)\frac{P}{\rho^2} \qquad \left(\frac{P_x}{\rho} - \cancel{\frac{P}{\rho^2}\rho_x}\right)u$$

$$\Rightarrow E_t + uE_x + \frac{P}{\rho}u_x + \frac{uP_x}{\rho} = -\frac{uH}{S}\frac{dS}{dx} + \frac{uE}{S}\frac{dS}{dx}$$

But $E = e + \frac{u^2}{2}$

$$e_t + \underline{\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right)} + u\frac{\partial e}{\partial x} + \underline{u\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)} + \frac{Pu_x}{\rho} + \underline{\frac{uP_x}{\rho}} = -\frac{uH}{S}\frac{dS}{dx} + \frac{uE}{S}\frac{dS}{dx} \quad *$$

$\underline{\quad}$ = mmt eq multiplied by the velocity $u$.

Now for isentropic flow $\quad Tds = de + pd(1/\rho) = 0$

$$de - \frac{P}{\rho^2}d\rho = 0$$

$\{ de = Tds - pdV \}$

$$\left.\frac{de}{d\rho}\right|_s = \frac{P}{\rho^2}$$ doesnt involve pressure!

$$\frac{de}{d\rho}\Big|_s \cdot \underbrace{\frac{dp}{d\rho}\Big|_s}_{c^2} = \frac{p}{\rho^2}$$

$$\frac{de}{dp}\Big|_s = \frac{p}{\rho^2 c^2}$$

Thus eq * becomes

$$\frac{\cancel{p}}{\rho^2 c^2}\frac{\partial p}{\partial t} + \frac{u\cancel{p}}{\rho^2 c^2}\frac{\partial p}{\partial x} + \frac{\cancel{p} u_x}{\rho} = -\frac{uH}{\rho S}\frac{dS}{dx} + \frac{uE}{S\rho}\frac{dS}{dx}$$

$$\Rightarrow \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2 u_x = -\frac{\rho^2 c^2 uH}{pS}\frac{dS}{dx} + \frac{\rho^2 c^2 uE}{Sp}\frac{dS}{dx}$$

$$= \frac{-\rho^2 c^2 u}{pS}\frac{dS}{dx}\underbrace{(H - E)}_{\frac{p}{\rho}}$$

$$= -\frac{\rho c^2 u}{S}\frac{dS}{dx}$$

---

$$V = \begin{pmatrix}\rho \\ u \\ p\end{pmatrix}$$

$$\begin{pmatrix}\rho \\ u \\ p\end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u\end{pmatrix}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -c^2\end{pmatrix}\frac{\rho u}{S}\frac{dS}{dx}$$

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$$|\tilde{A} - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & 1/\rho \\ 0 & \rho c^2 & u-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (u-\lambda)\left[(u-\lambda)^2 - c^2\right] = 0$$

$$\Rightarrow \lambda = u \quad \text{or} \quad (u-\lambda)^2 = c^2$$

$$\lambda = u \pm c$$

Left eigenvectors for $\lambda_{(i)}$ $i = 1, 2, 3$ are

$$\tilde{\ell}^{(i)T} \tilde{A} = \lambda_{(i)} \tilde{\ell}^{(i)T}$$ taking the transpose of both sides gives

$$\tilde{A}^T \tilde{\ell}^{(i)} = \lambda_{(i)} \hat{\ell}^{(i)}$$

$$\Rightarrow (\tilde{A}^T - \lambda_{(i)} I)\, \hat{\ell}^{(i)} = 0$$

For $i = 1$ $\lambda_{(1)} = u$

$$\begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & 1/\rho \\ 0 & \rho c^2 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\ell}^{(1)}_1 \\ \hat{\ell}^{(1)}_2 \\ \tilde{\ell}^{(1)}_3 \end{pmatrix} = \vec{0}$$

X No transpose taken.

$i = 1$ w/ transpose

$$\begin{pmatrix} 0 & 0 & 0 \\ \rho & 0 & \rho c^2 \\ 0 & 1/\rho & 0 \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1^{(1)} \\ \hat{\ell}_2^{(1)} \\ \hat{\ell}_3^{(1)} \end{pmatrix} = \bar{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\ell}_1^{(1)} \\ \hat{\ell}_2^{(1)} \\ \hat{\ell}_3^{(1)} \end{pmatrix} = \vec{0}$$

$\Rightarrow \quad \hat{\ell}_2^{(2)} = 0 \quad , \quad \hat{\ell}_3^{(1)}$ arb $\quad \tilde{\ell}_1^{(1)} = -c^2 \hat{\ell}_3^{(1)}$

take $\tilde{\ell}_3^{(1)} = \alpha$

$$\Rightarrow \hat{\ell}^{(1)} = \begin{pmatrix} -c^2\alpha \\ 0 \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ -\frac{\alpha}{c^2} \end{pmatrix} \qquad \alpha' = \alpha(-c^2)$$

For $i = 2 \quad \lambda_2 = u + c$

$$\begin{pmatrix} -c & 0 & 0 \\ \rho & -c & \rho c^2 \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \tilde{\ell}_1^{(2)} \\ \tilde{\ell}_2^{(2)} \\ \tilde{\ell}_3^{(2)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -c & \rho c^2 \\ 0 & 1/\rho & -c \end{pmatrix} \begin{pmatrix} \hat{l}_1^{(2)} \\ \hat{l}_2^{(2)} \\ \hat{l}_3^{(2)} \end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\rho c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{l}_1^{(2)} \\ \tilde{l}_2^{(2)} \\ \hat{l}_3^{(2)} \end{pmatrix} = \vec{0}$$

$$-\frac{1}{\rho}(-\rho c) - c = 0$$

$$\Rightarrow \quad \hat{l}_1^{(2)} = 0, \quad \hat{l}_2^{(2)} = \rho c\, \tilde{l}_3^{(2)}, \quad \hat{l}_3^{(2)} \text{ arb} \overset{\text{set}}{=} B$$

$$\Rightarrow \quad \hat{l}^{(2)} = \begin{pmatrix} 0 \\ \rho c B \\ B \end{pmatrix} \overset{\text{eq}}{=} \begin{pmatrix} 0 \\ B \\ \frac{B}{\rho c} \end{pmatrix}$$

<u>$i = 3$</u>

$$\begin{pmatrix} c & 0 & 0 \\ \rho & c & \rho c^2 \\ 0 & 1/\rho & c \end{pmatrix} \begin{pmatrix} \hat{l}_1^{(3)} \\ \hat{l}_2^{(3)} \\ \hat{l}_3^{(3)} \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & \rho c^2 \\ 0 & 1/\rho & c \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho c \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{\ell}_1^{(3)} \\ \hat{\ell}_2^{(3)} \\ \hat{\ell}_3^{(3)} \end{pmatrix} = \vec{0} \qquad -\frac{1}{\rho}(\rho c) + c$$

$\hat{\ell}_1^{(3)} = 0$, $\hat{\ell}_2^{(3)} = -\rho c\, \hat{\ell}_3^{(3)}$, $\hat{\ell}_3^{(3)}$ arb $= \delta$

$$\hat{\ell}^{(3)} = \begin{pmatrix} 0 \\ -\rho c \delta \\ \delta \end{pmatrix} \stackrel{\text{e.g.}}{=} \begin{pmatrix} 0 \\ \delta \\ -\frac{1}{\rho c}\delta \end{pmatrix}$$

---

Taking $\alpha = \beta = \delta = 1$

$L^{-1}$ = rows are made up of the left eigenvectors of $\tilde{A}$.

$$\Rightarrow L^{-1} = \begin{pmatrix} \hat{\ell}^{(1)T} \\ \hat{\ell}^{(2)T} \\ \hat{\ell}^{(3)T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}$$

Find $L = R$: ?

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1/c^2 & 1 & 0 & 0 \\ 0 & 1 & 1/\rho c & 0 & 1 & 0 \\ 0 & 1 & -1/\rho c & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1/c^2 & 1 & 0 & 0 \\ 0 & 1 & 1/\rho c & 0 & 1 & 0 \\ 0 & 0 & -2/\rho c & 0 & -1 & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -1/c^2 & 1 & 0 & 0 \\ 0 & 1 & 1/\rho c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \frac{\rho c}{2} & -\frac{\rho c}{2} \end{array}\right)$$

$$\frac{\rho c}{2}\left(\frac{1}{c^2}\right) + 0 = \frac{\rho}{2c}$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{\rho}{2c} & -\frac{\rho}{2c} \\ 0 & 1 & 0 & 0 & 1/2 & +1/2 \\ 0 & 0 & 1 & 0 & \frac{\rho c}{2} & -\frac{\rho c}{2} \end{array}\right)$$

$$= L$$

q 16.4.15

$$L^{-1}\frac{\partial V}{\partial t} + \Lambda L^{-1}\frac{\partial V}{\partial x} = L^{-1}\tilde{Q}$$

$$V = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} \quad \Lambda = \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix} \quad L^{-1} = \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}$$

$$+ \quad \tilde{Q} = \begin{pmatrix} -\rho u \\ 0 \\ -\rho c^2 u \end{pmatrix} \frac{1}{S}\frac{dS}{dx}$$

$$\begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}\begin{pmatrix} \rho_t \\ u_t \\ p_t \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix}\begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}\begin{pmatrix} \rho_x \\ u_x \\ p_x \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}\begin{pmatrix} -\rho u \\ 0 \\ -\rho c^2 u \end{pmatrix}\frac{1}{S}\frac{dS}{dx}$$

$$= \begin{pmatrix} \rho_t - \frac{1}{c^2}p_t \\ u_t + \frac{p_t}{\rho c} \\ u_t - \frac{p_t}{\rho c} \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix}\begin{pmatrix} \rho_x - \frac{p_x}{c^2} \\ u_x + p_x/\rho c \\ u_x - p_x/\rho c \end{pmatrix}$$

$$= \frac{1}{\delta}\frac{dS}{dx}\begin{pmatrix} -\rho u + \rho u \\ -cu \\ +cu \end{pmatrix} = \frac{1}{\delta}\frac{dS}{dx}\begin{pmatrix} 0 \\ -cu \\ +cu \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \rho_t - \frac{p_t}{c^2} \\ u_t + \frac{p_t}{\rho c} \\ u_t - \frac{p_t}{\rho c} \end{pmatrix} + \begin{pmatrix} u\left(\rho_x - \frac{p_x}{c^2}\right) \\ (u+c)\left(u_x + \frac{p_x}{\rho c}\right) \\ (u-c)\left(u_x - \frac{p_x}{\rho c}\right) \end{pmatrix} = \frac{1}{\delta}\frac{dS}{dx}\begin{pmatrix} 0 \\ -cu \\ +cu \end{pmatrix}$$

---

16.3.34a $\quad \delta W = L^{-1}\delta V$

$$\begin{pmatrix} \delta w_1 \\ \delta w_2 \\ \delta w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1 & 1/\rho c \\ 0 & 1 & -1/\rho c \end{pmatrix}\begin{pmatrix} \delta \rho \\ \delta u \\ \delta p \end{pmatrix}$$

$$\delta w_1 = \delta\rho - \frac{\delta p}{c^2}$$

$$\delta w_2 = \delta u + \frac{\delta p}{\rho c}$$

$$\delta w_3 = \delta u - \frac{\delta p}{\rho c}$$

$$\delta w_2 = \delta u + \frac{1}{\rho c}\,\delta p$$

How?

$$= \delta u + \frac{1}{\rho c}\frac{\delta p}{\delta \rho}\cdot\delta\rho = \delta u + \frac{c^2}{\rho c}\,\delta\rho = \delta u + \frac{c}{\rho}\,\delta\rho$$

eq $16.4.7_1$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \frac{-1}{c^2}\left(\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x}\right) = 0$$

$$d^{(0)}\rho - \frac{1}{c^2}\,d^{(0)}p = 0 \qquad\qquad d^{(0)} = \partial_t + u\,\partial_x$$

eq $16.4.7_2$

$$\frac{\partial u}{\partial t} + (u+c)\frac{\partial u}{\partial x} + \frac{1}{\rho c}\left(\frac{\partial p}{\partial t} + (u+c)\frac{\partial p}{\partial x}\right) = -\frac{uc}{S}\frac{dS}{dx}$$

$$d^{(+)}u + \frac{1}{\rho c}\,d^{(+)}p = -\frac{uc}{S}\frac{dS}{dx} \qquad\qquad d^{(+)} = \partial_t + (u+c)\,\partial_x$$

eq $16.4.7_3$

$$\frac{\partial u}{\partial t} + (u-c)\frac{\partial u}{\partial x} + \frac{-1}{\rho c}\left(\frac{\partial p}{\partial t} + (u-c)\frac{\partial p}{\partial x}\right) = \frac{uc}{S}\frac{dS}{dx}$$

$$d^{(-)}u - \frac{1}{\rho c}\,d^{(-)}p = \frac{uc}{S}\frac{dS}{dx} \qquad\qquad d^{-} = \partial_t + (u-c)\,\partial_x$$

$$dw_2 = du + \frac{1}{\rho c}\, dp$$

$$w_2 = u + \int \frac{dp}{\rho c} = u + \int \frac{1}{\rho c}\frac{dp}{d\rho} \cdot d\rho = u + \int \frac{c^2}{\rho c}\, d\rho$$

$$w_2 = u + \int \frac{c(\rho)}{\rho}\, d\rho$$

$$c^2 = k r \rho^{r-1}$$

$$c = \sqrt{kr}\, \rho^{r/2 - 1/2}$$

$$\frac{c}{\rho} = \sqrt{kr}\, \rho^{r/2 - 3/2} \quad \text{then}$$

$$w_2 = u + \int \sqrt{kr}\, \rho^{r/2 - 3/2}\, d\rho = u + \sqrt{kr}\, \frac{\rho^{r/2 - 1/2}}{r/2 - 1/2}$$

$$= u + \frac{2}{(r-1)} \underbrace{\sqrt{kr}\, \rho^{r/2 - 1/2}}_{c}$$

$$= u + \frac{2}{r-1} c$$

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eqs 16.2.40

$$u\rho_x + v\rho_y + \rho(u_x + v_y) = 0$$

$$uu_x + vu_y + \frac{1}{\rho}p_x = 0$$

$$uv_x + vv_y + \frac{1}{\rho}p_y = 0$$

$$up_x + vp_y + \rho c^2(u_x + v_y) = 0$$ Using ~~conservation of mass to replace~~

the ~~$p(u_x+v_y)$ term one gets~~

~~$up_x + vp_y + c^2($~~ $up_x$ dropping out =

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \rho c^2\frac{\partial u}{\partial x} + \rho c^2\frac{\partial v}{\partial y} = 0$$

$$V = \begin{pmatrix} u \\ v \\ p \end{pmatrix}$$ w/o $\rho$)

$$u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$u\frac{\partial v}{\partial x} + \quad v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

$$U\frac{\partial p}{\partial x} + \rho c^2 \frac{\partial U}{\partial x} + V\frac{\partial p}{\partial y} + \rho c^2 \frac{\partial V}{\partial y} = 0$$

$$= \underbrace{\begin{pmatrix} U & 0 & 1/\rho \\ 0 & U & 0 \\ \rho c^2 & 0 & U \end{pmatrix}}_{A_1} \begin{pmatrix} U_x \\ V_x \\ p_x \end{pmatrix} + \underbrace{\begin{pmatrix} V & 0 & 0 \\ 0 & V & 1/\rho \\ 0 & \rho c^2 & V \end{pmatrix}}_{A_2} \begin{pmatrix} U_y \\ V_y \\ P_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then $\dfrac{\partial V}{\partial x} + A_1^{-1} A_2 \dfrac{\partial V}{\partial y} = 0$

$$\left(\begin{array}{ccc|ccc} U & 0 & 1/\rho & 1 & 0 & 0 \\ 0 & U & 0 & 0 & 1 & 0 \\ \rho c^2 & 0 & U & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ U & 0 & 0 & 0 & 1 & 0 \\ \rho c^2 & 0 & U & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ 0 & 0 & -1/\rho & -1 & 1 & 0 \\ 0 & 0 & -c^2/U & -\rho c^2/U & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ 0 & U & 0 & 0 & 1 & 0 \\ \rho c^2 & 0 & U & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ 0 & U & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{c^2}{U}+U & -\frac{\rho c^2}{U} & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/U & 0 \\ 0 & 0 & -\frac{c^2}{U}+U & -\frac{\rho c^2}{U} & 0 & 1 \end{array}\right)$$

$$-\frac{c^2}{U}+U = \frac{-c^2+U^2}{U} = \frac{U^2-c^2}{U}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/\rho U & 1/U & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/U & 0 \\ 0 & 0 & 1 & -\frac{\rho c^2}{U}\cdot\frac{U}{U^2-c^2} & 0 & \frac{U}{U^2-c^2} \end{array}\right) \quad -\frac{1}{\rho U}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{U}+\frac{c^2}{(U^2-c^2)U} & 0 & -\frac{\cancel{U}}{\cancel{\rho U}(U^2-c^2)} \\ 0 & 1 & 0 & 0 & \frac{1}{U} & 0 \\ 0 & 0 & 1 & -\frac{\rho c^2}{U^2-c^2} & 0 & \frac{U}{U^2-c^2} \end{array}\right)$$

$$\therefore \quad \cancel{A^{-1} = \frac{1}{u} + \frac{c}{u^2-c^2} \quad 0 \quad \frac{-u}{\rho c}}$$

Check :

$$\begin{pmatrix} u & 0 & 1/\rho \\ 0 & u & 0 \\ \rho c^2 & 0 & u \end{pmatrix} \begin{pmatrix} \frac{1}{u} + \frac{c^2}{(u^2-c^2)u} & 0 & \frac{-u}{\rho(u^2-c^2)} \\ 0 & \frac{1}{u} & 0 \\ \frac{-\rho c^2}{u^2-c^2} & 0 & \frac{u}{u^2-c^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{c^2}{u^2-c^2} - \frac{c^2}{u^2-c^2} & 0 & \frac{-u}{\rho(u^2-c^2)} + \frac{u}{\rho(u^2-c^2)} \\ 0 & 1 & 0 \\ \frac{\rho c^2}{u} + \frac{\rho c^4}{u(u^2-c^2)} - \frac{\rho u c^2}{u^2-c^2} & 0 & \frac{-\rho c^2}{\rho(u^2-c^2)} + \frac{u^2}{u^2-c^2} \end{pmatrix}$$

$$= \cancel{\frac{(u-c)}{(u-c)(u+c)}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \rho c^2 \left( \frac{1}{u} + \frac{c^2}{u(u^2-c^2)} - \frac{u}{u^2-c^2} \right) & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \rho c^2 \left( \frac{u^2-c^2+c^2-u^2}{u(u^2-c^2)} \right) & 0 & 1 \end{pmatrix} = I \checkmark$$

Then $A_1^{-1} A_2 = ?$

$$A_1^{-1} = \begin{pmatrix} \frac{1}{u} + \frac{c^2}{u(u^2-c^2)} & 0 & \frac{-1}{\rho(u^2-c^2)} \\ 0 & \frac{1}{u} & 0 \\ \frac{-\rho c^2}{u^2-c^2} & 0 & \frac{u}{u^2-c^2} \end{pmatrix}$$

$$\frac{1}{u}\left(1 + \frac{c^2}{u^2-c^2}\right)$$

$$\frac{1}{u}\left(\frac{(u^2-\cancel{c^2}) + \cancel{c^2}}{u^2-c^2}\right) = \frac{u}{u^2-c^2}$$

$$= \begin{pmatrix} \frac{u}{u^2-c^2} & 0 & \frac{-1}{\rho(u^2-c^2)} \\ 0 & \frac{1}{u} & 0 \\ \frac{-\rho c^2}{u^2-c^2} & 0 & \frac{u}{u^2-c^2} \end{pmatrix}$$

$$= \frac{1}{u^2-c^2} \begin{pmatrix} u & 0 & -1/\rho \\ 0 & \frac{u^2-c^2}{u} & 0 \\ -\rho c^2 & 0 & u \end{pmatrix}$$

so

$$A_1^{-1} A_2 = \frac{1}{u^2-c^2} \begin{pmatrix} u & 0 & -1/\rho \\ 0 & \frac{u^2-c^2}{u} & 0 \\ -\rho c^2 & 0 & u \end{pmatrix} \begin{pmatrix} v & 0 & 0 \\ 0 & v & 1/\rho \\ 0 & \rho c^2 & v \end{pmatrix}$$

$$= \frac{1}{U^2-c^2}\begin{pmatrix} Uv & -c^2 & -v/\rho \\ 0 & (U^2-c^2)\frac{v}{U} & \frac{U^2-c^2}{\rho U} \\ -\rho v c^2 & U\rho c^2 & Uv \end{pmatrix}$$ E16.4.9

Find the Eigenvalues of this system

$|A - \lambda I| =$

$$\begin{vmatrix} \frac{Uv}{U^2-c^2} - \lambda & \frac{-c^2}{U^2-c^2} & \frac{-v}{\rho(U^2-c^2)} \\ 0 & \frac{v}{U} - \lambda & \frac{1}{\rho U} \\ -\frac{\rho v c^2}{U^2-c^2} & \frac{U\rho c^2}{U^2-c^2} & \frac{Uv}{U^2-c^2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\frac{Uv}{U^2-c^2} - \lambda\right)\left[\left(\frac{v}{U}-\lambda\right)\left(\frac{Uv}{U^2-c^2}-\lambda\right) - \frac{c^2}{U^2-c^2}\right]$$

$$+ -\frac{\rho v c^2}{U^2-c^2}\left[\frac{-c^2}{U^2-c^2}\frac{1}{\rho U} + \frac{v}{\rho(U^2-c^2)}\left(\frac{v}{U}-\lambda\right)\right] = 0$$

$$\Rightarrow \left(\frac{v}{U}-\lambda\right)\left(\frac{Uv}{U^2-c^2}-\lambda\right)^2 - \frac{c^2}{U^2-c^2}\left(\frac{Uv}{U^2-c^2}-\lambda\right) + \frac{vc^4}{U(U^2-c^2)^2}$$

$$- \frac{v^2c^2}{(U^2-c^2)^2}\left(\frac{v}{U}-\lambda\right) = 0$$

$$\Rightarrow \left(\frac{v}{u}-\lambda\right)\left(\frac{uv}{u^2-c^2}-\lambda\right)^2 - \frac{v^2c^2}{(u^2-c^2)^2}\left(\frac{v}{u}-\lambda\right)$$

$$+ \frac{c^2}{(u^2-c^2)}\left[-\frac{uv}{u^2-c^2}+\lambda+\frac{vc^2}{u(u^2-c^2)}\right] = 0$$

$$\Rightarrow \left(\frac{v}{u}-\lambda\right)\left(\frac{uv}{u^2-c^2}-\lambda\right)^2 - \frac{v^2c^2}{(u^2-c^2)^2}\left(\frac{v}{u}-\lambda\right)$$

$$+ \frac{c^2}{(u^2-c^2)}\left[\lambda+\frac{-u^2v+vc^2}{u(u^2-c^2)}\right] = 0$$

$$\Rightarrow \left(\frac{v}{u}-\lambda\right)\left(\frac{uv}{u^2-c^2}-\lambda\right)^2 - \frac{v^2c^2}{(u^2-c^2)^2}\left(\frac{v}{u}-\lambda\right) + \frac{c^2}{(u^2-c^2)}\left[\lambda-\frac{v}{u}\right] = 0$$

$\lambda = \frac{v}{u}$ or

$$-\left(\frac{uv}{u^2-c^2}-\lambda\right)^2 + \frac{v^2c^2}{(u^2-c^2)^2} + \frac{c^2}{(u^2-c^2)} = 0$$

$$\Rightarrow \left(\lambda-\frac{uv}{u^2-c^2}\right)^2 = \frac{v^2c^2}{(u^2-c^2)^2} + \frac{c^2(u^2-c^2)}{(u^2-c^2)^2} = \frac{c^2(u^2+v^2-c^2)}{(u^2-c^2)^2}$$

$$\Rightarrow \lambda = \frac{uv}{u^2-c^2} \pm \frac{c\sqrt{u^2+v^2-c^2}}{|u^2-c^2|} = \frac{uv}{u^2-c^2} \pm \frac{c^2\sqrt{\frac{u^2+v^2}{c^2}-1}}{u^2-c^2}$$

$$= \frac{uv \pm c^2\alpha}{u^2-c^2} \qquad \text{w/ } \alpha = \sqrt{M^2-1}, \quad M^2 = \frac{u^2+v^2}{c^2}$$

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Find left-eigenvectors of primitive var pseudolinear form

$$\tilde{l}^{(j)T}\tilde{A} = \lambda_{(j)}\tilde{l}^{(j)T}$$

taking the transpose of both sides

$$\tilde{A}^T\tilde{l}^{(j)} = \lambda_{(j)}\tilde{l}^{(j)}$$

$$\Rightarrow \left(\tilde{A}^T - \lambda_{(j)} I\right)\tilde{l}^{(j)} = 0$$

$\therefore$ for $\lambda_{(1)} = \frac{V}{U}$

$$\begin{pmatrix} \frac{UV}{U^2-c^2} - \frac{V}{U} & 0 & \frac{-\rho V c^2}{U^2-c^2} \\ -\frac{c^2}{U^2-c^2} & 0 & \frac{\rho U c^2}{U^2-c^2} \\ \frac{-V}{\rho(U^2-c^2)} & \frac{1}{\rho U} & \frac{UV}{U^2-c^2} - \frac{V}{U} \end{pmatrix}\begin{pmatrix} l_1^{(1)} \\ l_2^{(1)} \\ l_3^{(1)} \end{pmatrix} = \vec{0} \qquad \checkmark ed$$

$$= \begin{pmatrix} \frac{U^2V - V(U^2-c^2)}{U(U^2-c^2)} & 0 & \frac{-\rho V c^2}{U^2-c^2} \\ \frac{-c^2}{U^2-c^2} & 0 & \frac{\rho U c^2}{U^2-c^2} \\ \frac{-V}{\rho(U^2-c^2)} & \frac{1}{\rho U} & \frac{U^2V - VU^2 + Vc^2}{U(U^2-c^2)} \end{pmatrix}\begin{pmatrix} l_1^{(1)} \\ \\ \\ \end{pmatrix} = \vec{0} \qquad \checkmark ed$$

$$= \begin{pmatrix} \frac{vc^2}{u(u^2-c^2)} & 0 & -\frac{\rho v c^2}{u^2-c^2} \\ -\frac{c^2}{u^2-c^2} & 0 & \frac{\rho u c^2}{u^2-c^2} \\ \frac{-v}{\rho(u^2-c^2)} & \frac{1}{\rho u} & \frac{vc^2}{u(u^2-c^2)} \end{pmatrix} \begin{pmatrix} \ell_1^{(1)} \\ \ell_2^{(1)} \\ \ell_3^{(1)} \end{pmatrix} = \vec{0}$$

$$= \begin{pmatrix} 1 & 0 & -\rho u \\ -\frac{c^2}{u^2-c^2} & 0 & \frac{\rho u c^2}{u^2-c^2} \\ \frac{-v}{\rho(u^2-c^2)} & \frac{1}{\rho u} & \frac{vc^2}{u(u^2-c^2)} \end{pmatrix} \begin{pmatrix} \ell_1^{(1)} \\ \ell_2^{(1)} \\ \ell_3^{(1)} \end{pmatrix} = \vec{0}$$

$$= \begin{pmatrix} 1 & 0 & -\rho u \\ 0 & 0 & 0 \\ 0 & \frac{1}{\rho u} & \underbrace{\frac{-\rho u v}{\rho(u^2-c^2)} + \frac{vc^2}{u(u^2-c^2)}} \end{pmatrix} \begin{pmatrix} \ell_1^{(1)} \\ \ell_2^{(1)} \\ \ell_3^{(1)} \end{pmatrix} = \vec{0}$$

$$= \frac{-u^2 v + vc^2}{u(u^2-c^2)}$$

$$= \frac{-v}{u}$$

$$\begin{pmatrix} 1 & 0 & -\rho U \\ 0 & 1 & -\frac{V}{U}\rho U \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1^{(1)} \\ \ell_2^{(1)} \\ \ell_3^{(1)} \end{pmatrix} = \vec{0}$$

$$\ell_1^{(1)} = \rho U \ell_3^{(1)}$$

$$\ell_2^{(1)} = +V\rho\, \ell_3^{(1)}$$

Thus $\vec{\ell}^{(1)} = \begin{pmatrix} \rho U \\ +V\rho \\ 1 \end{pmatrix} \ell_3^{(1)}$ Normalizing $\ell_3^{(1)} = 1$

$$\vec{\ell}^{(1)} = \begin{pmatrix} \rho U \\ +V\rho \\ 1 \end{pmatrix}$$

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For $\lambda_2 = \dfrac{UV + c^2\alpha}{U^2 - c^2}$

$$\left(\tilde{A}^T - \lambda_2 I\right)\tilde{\ell}^{(2)} = 0$$

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$$\Rightarrow \begin{pmatrix} \frac{UV}{U^2-c^2} - \frac{UV+c^2\alpha}{U^2-c^2} & 0 & \frac{-\rho V c^2}{U^2-c^2} \\ \frac{-c^2}{U^2-c^2} & \frac{V}{U} - \frac{(UV+c^2)}{U^2-c^2} & \frac{\rho U c^2}{U^2-c^2} \\ -\frac{V}{\rho(U^2-c^2)} & \frac{1}{\rho U} & \frac{UV}{U^2-c^2} - \frac{(UV+c^2\alpha)}{U^2-c^2} \end{pmatrix} \begin{pmatrix} \ell_1^{(2)} \\ \ell_2^{(2)} \\ \ell_3^{(3)} \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} \frac{-c^2\alpha}{u^2-c^2} & 0 & \frac{-\rho v c^2}{v^2-c^2} \\ \frac{-c^2}{u^2-c^2} & \frac{v(\cancel{v^2}-c^2)-u(\cancel{v}+c^2)}{u(v^2-c^2)} & \frac{\rho u c^2}{u^2-c^2} \\ \frac{-v}{\rho(u^2-c^2)} & \frac{1}{\rho u} & \frac{-c^2\alpha}{u^2-c^2} \end{pmatrix} \begin{pmatrix} l_1^{(2)} \\ l_2^{(2)} \\ l_3^{(2)} \end{pmatrix} = \vec{0}$$

✓ed

$$= \begin{pmatrix} 1 & 0 & \frac{+\rho v}{\alpha} \\ \frac{-c^2}{u^2-c^2} & \frac{-vc^2-uc^2}{u(u^2-c^2)} & \frac{\rho u c^2}{u^2-c^2} \\ \frac{-v}{\rho(u^2-c^2)} & \frac{1}{\rho u} & \frac{-c^2\alpha}{u^2-\cancel{\alpha}c^2} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

✓ed

$$= \begin{pmatrix} 1 & 0 & \frac{\rho v}{\alpha} \\ 0 & -\frac{c^2(v+u)}{u(u^2-c^2)} & \frac{\rho u c^2}{u^2-c^2} + \frac{c^2}{u^2-c^2}\left(\frac{\rho v}{\alpha}\right) \\ 0 & \frac{1}{\rho u} & \frac{v^2}{\alpha(u^2-c^2)} - \frac{c^2\alpha}{(u^2-c^2)} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

✓ed

Ved.

$$\begin{pmatrix} 1 & 0 & \frac{\rho V}{\alpha} \\ 0 & \frac{-c^2(V+U)}{U(U^2-c^2)} & \frac{\alpha \rho U c^2 + \rho V c^2}{\alpha(U^2-c^2)} \\ 0 & \frac{1}{\rho U} & \frac{V^2-\alpha^2 c^2}{\alpha(U^2-c^2)} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

Ved

$$= \begin{pmatrix} 1 & 0 & \frac{\rho V}{\alpha} \\ 0 & \frac{-c^2(V+U)}{U(U^2-c^2)} & \rho \frac{c^2(\alpha U + V)}{\alpha(U^2-c^2)} \\ 0 & \frac{1}{\rho U} & \frac{V^2-\alpha^2 c^2}{\alpha(U^2-c^2)} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

Ved

$$= \begin{pmatrix} 1 & 0 & \frac{\rho V}{\alpha} \\ 0 & 1 & -\frac{(\alpha U + V)\rho U}{\alpha(V+U)} \\ 0 & \frac{1}{\rho U} & \frac{V^2-\alpha^2 c^2}{\alpha(U^2-c^2)} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

Ved

$$\begin{pmatrix} 1 & 0 & \rho V/\alpha \\ 0 & 1 & -\frac{\rho U(\alpha U+V)}{\alpha(V+U)} \\ 0 & 0 & \underbrace{\frac{(\alpha U+V)}{\alpha(V+U)} + \frac{V^2-\alpha^2 c^2}{\alpha(U^2-c^2)}} \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \vec{0}$$

$\alpha^2 = M^2 - 1$

$$\frac{(\alpha U+V)(U^2-c^2) + (V^2-\alpha^2c^2)(V+U)}{\alpha(V+U)(U^2-c^2)}$$

$$= \frac{\alpha U^3 - \alpha c^2 U + VU^2 - Vc^2 + (V^2 - (M^2c^2 - c^2))(V+U)}{\alpha(V+U)(U^2-c^2)}$$

$$= \frac{\alpha U^3 - \alpha c^2 U + VU^2 - Vc^2 + (V^2 - (U^2+V^2-c^2))(U+V)}{\alpha(V+U)(U^2-c^2)}$$

$$= \alpha U^3 - \alpha c^2 U + VU^2 - Vc^2 +$$