Solutions to a Few Problems in A Second Course in Ordinary Differential Equations by Mayer Humi and William Miller

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To my family.

Introduction

As a final comment, I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

Review

Ordinary and Singular Points

Exercise 1.1

Part (a): All points are ordinary points.

Part (b): All points are ordinary points.

Part (c): Write this differential equation as

$$y'' - \frac{1}{x} y = 0 \,.$$

Now $x = 0$ is a singular point and all others are ordinary points.

Part (d): Write this differential equation as

$$y'' + \frac{x+1}{x(x-1)} y' + \frac{x+2}{x(x-1)} y = 0 \,.$$

Now $x = 0$ and $x = 1$ are a singular points and all others are ordinary points.

Part (e): Write this differential equation as

$$y'' + \left(\frac{x^2 + x + 1}{x^2 + x - 6} \right) y' + \left(\frac{x - 1}{x^2 + x - 6} \right) y = 0 \,,$$

or

$$y'' + \left(\frac{x^2 + x + 1}{(x+3)(x-2)} \right) y' + \left(\frac{x - 1}{(x+3)(x-2)} \right) y = 0 \,.$$

Now $x = -3$ and $x = 2$ are a singular points and all others are ordinary points.

Exercise 1.2

Part (a): As both $\sin(x)$ and e^x are analytic for all x all points are ordinary points.

Part (b): Write this differential equation as

$$y'' - \frac{x}{3\sqrt{x+1}} y' + \frac{\sqrt{x-1}}{\sqrt{x+1}} y = 0 \,.$$

We can expand the coefficients of y' and y in a Taylor series only when $x > -1$. Thus all these points are ordinary points and all others are singular points.

Part (c): Write this differential equation as

$$y'' + \frac{\cos(x)}{\sin(x)}y' + \frac{x}{\sin(x)}y = 0\,.$$

The coefficients of y' and y are singular when $x = n\pi$ for $n \in \mathbb{Z}$. For $x \in [-2\pi, 2\pi]$ the only points singular points are then $x \in \{-2\pi, -\pi, 0, \pi, 2\pi\}$ all other points are ordinary points.

Part (d): The coefficient of y' is the Taylor series for e^x and the coefficient of y is the Taylor series of $\cosh(x)$ both of which are analytic for all x. Thus there are no singular points of this differential equation and all points are ordinary points.

Boundary Value Problems

Introduction

Exercise 1.1

The reduced homogeneous system sets the right-hand-sides of both the differential equation and the boundary conditions to zero.

Part (a): The homogeneous system takes the form

$$\begin{aligned} x^2y'' + xy' + y &= 0 \\ y(0) + y'(0) &= 0 \\ y(0) - 3y(5) &= 0 \\ y'(0) &= 0\,. \end{aligned}$$

Part (a): The homogeneous system takes the form

$$\begin{aligned} y^{(iv)} + 6y'' + 9y &= 0 \\ y(a) - 2y'(a) + 3y(b) &= 0 \\ y'''(b) &= 0 \\ y''(a) - y'''(a) - 2y(b) - 3y'(b) &= 0\,. \end{aligned}$$

Exercise 1.2

This would be $2n$ with n the order of the differential equation. Thus in this case this is $2(5) = 10$.

Exercise 1.3

Part (a): Our boundary value problem is

$$\begin{aligned} y''(x) + 4y(x) &= 0 \\ y(0) + y'\left(\frac{\pi}{2}\right) &= 0\,. \end{aligned}$$

Here $m = 1$ (that is there is one boundary condition). Two linear independent solutions to the homogeneous equation are

$$\begin{aligned} y_1(x) &= \cos(2x) \\ y_2(x) &= \sin(2x)\,. \end{aligned}$$

The boundary condition matrix U (defined in the book) is

$$U = \begin{bmatrix} U_1(y_1) & U_1(y_2) \end{bmatrix} = \begin{bmatrix} U_1(\cos(2x)) & U_1(\sin(2x)) \end{bmatrix}.$$

Now

$$\begin{aligned} U_1(\cos(2x)) &= 1 - 2\sin(2x)|_{x=\frac{\pi}{2}} = 1 - 2\sin(\pi) = 1 \\ U_1(\sin(2x)) &= 0 + 2\cos(2x)|_{x=\frac{\pi}{2}} = 2\cos(\pi) = -2\,. \end{aligned}$$

Using these two we have

$$U = \begin{bmatrix} 1 & -2 \end{bmatrix}.$$

This matrix has rank of one thus the system is $n - r = 2 - 1 = 1$-ply i.e. there is one linearly independent solution to this boundary value problem. To find the constraints we must impose on $y_1(x)$ and $y_2(x)$ we want to solve

$$U\mathbf{c} = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \quad \text{so} \quad c_1 = 2c_2\,.$$

Taking $c_2 = 1$ a solution is

$$y_A(x) = 2\cos(2x) + \sin(2x)\,.$$

For this function we see that

$$y_A'(x) = -4\sin(2x) + 2\cos(2x)\,.$$

Thus we have that

$$y_A(0) + y_A'\left(\frac{\pi}{2}\right) = 2 - 2 = 0\,,$$

as it should.

Part (b): This is the same differential equation as in Part (a) thus we will use the same linear independent solutions to the homogeneous differential equation as above. Now we have we have $m = 2$ boundary conditions and the matrix U looks like

$$U = \begin{bmatrix} U_1(\cos(2x)) & U_1(\sin(2x)) \\ U_2(\cos(2x)) & U_2(\sin(2x)) \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ U_2(\cos(2x)) & U_2(\sin(2x)) \end{bmatrix}.$$

In my version of the book the second boundary condition could be

$$y\left(\frac{\pi}{2}\right) = 0\,,$$

in which case we would compute

$$\begin{aligned} U_2(\cos(2x)) &= \cos(2x)|_{x=\frac{\pi}{2}} = \cos(\pi) = -1 \\ U_2(\sin(2x)) &= \sin(\pi) = 0\,, \end{aligned}$$

so that

$$U = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}.$$

This matrix has a rank of two. Thus this system is $n-r = 2-2 = 0$-ply. The only solution to $U\mathbf{c} = 0$ is $\mathbf{c} = 0$. This means that the only solution is $y(x) = 0$.

If the the second boundary condition is be

$$y'\left(\frac{\pi}{2}\right) = 0\,,$$

in which case we would compute

$$\begin{aligned} U_2(\cos(2x)) &= -2\sin(2x)|_{x=\frac{\pi}{2}} = 0 \\ U_2(\sin(2x)) &= 2\cos(\pi) = -2\,, \end{aligned}$$

so that

$$U = \left[\begin{array}{cc} 1 & -2 \\ 0 & -2 \end{array}\right].$$

This matrix also has a rank of two, so the system is $n-r = 2-2 = 0$-ply. Again the only solution to $U\mathbf{c} = 0$ is $\mathbf{c} = 0$. This means that the only solution is $y(x) = 0$.

Part (c): For the homogeneous differential equation two linearly independent solutions are

$$\begin{aligned} y_1(x) &= e^{-3x} \quad \text{so} \quad y_1'(x) = -3e^{-3x} \\ y_2(x) &= e^{3x} \quad \text{so} \quad y_2'(x) = 3e^{3x}\,. \end{aligned}$$

Here we have $m = 2 < 2n = 2(2) = 4$. For these two boundary values the matrix U is given by

$$U = \left[\begin{array}{cc} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{array}\right] = \left[\begin{array}{cc} 1+e^{-3} & 1+e^{3} \\ -3+3e^{-3} & 3-3e^{3} \end{array}\right].$$

Using elementary row operations we can transform U into the matrix

$$\left[\begin{array}{cc} 1 & \frac{1+e^3}{1+e^{-3}} \\ 0 & 0 \end{array}\right],$$

thus this matrix has rank one. This system is then $n-r = 2-1 = 1$-ply. To find that one linearly independent solution we need to enforce the constraint

$$c_1 + \frac{1+e^3}{1+e^{-3}}c_2 = 0\,.$$

If we take $c_2 = -(1+e^{-3})$ then we should take

$$c_1 = 1+e^3\,,$$

so that the solution is

$$y_A(x) = (1+e^3)e^{-3x} - (1+e^{-3})e^{3x}\,.$$

As a check using the above we find

$$\begin{aligned} y_A(0) + y_A(1) &= (1+e^3) - (1+e^{-3}) + (1+e^3)e^{-3} - (1+e^{-3})e^3 \\ &= e^3 - e^{-3} + e^{-3} + 1 - e^3 - 1 = 0\,, \end{aligned}$$

and that

$$y_A'(0) - y_A'(1) = -3(1+e^3) - 3(1+e^{-3}) - (-3(1+e^3)e^{-3} - 3(1+e^{-3})e^3) = 0\,,$$

when we simplify.

Part (d): Here the degree is $n = 3$ and to find linearly independent solution to the homogeneous equation we let $y(x) = e^{rx}$ to get the characteristic equation of

$$r^3 - r^2 + r - 1 = 0\,.$$

This has roots $r = \pm i$ and $r = 1$ so the homogeneous solutions are

$$\begin{aligned} y_1(x) &= e^x \\ y_2(x) &= \cos(x) \\ y_3(x) &= \sin(x)\,. \end{aligned}$$

The matrix U for these boundary conditions is

$$U = \left[\begin{array}{ccc} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{array}\right] = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1+e^\pi & 1-\cos(\pi) & 0-\sin(\pi) \\ e^\pi & -\cos(\pi) & -\sin(\pi) \end{array}\right] = \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1+e^\pi & 2 & 0 \\ e^\pi & 1 & 0 \end{array}\right].$$

This matrix has a rank of two. This system is then $n - r = 3 - 2 = 1$-ply. To determine how to constrain the solutions to the homogeneous equation so that they satisfy the boundary conditions we next need to solve $U\mathbf{c} = 0$. Using elementary row operations we can reduce the system $U\mathbf{c} = 0$ to the system

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]\mathbf{c} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right].$$

This means that $c_1 = c_2 = 0$ and c_3 is arbitrary. Thus if we take $c_3 = 1$ we get the solution

$$y_A(x) = \sin(x)\,.$$

For this solution we have $y_A''(x) = -\sin(x)$ from which we see that this solution *does* satisfy all of the required boundary conditions.

Part (e): The degree of this equation is $n = 4$ and to find linearly independent solution to the homogeneous equation we will let $y(x) = e^{rx}$. Putting this into the homogeneous equation we get

$$r^4 - r^3 = r^3(r-1) = 0\,,$$

for the characteristic equation. This equation has roots $r = 0$ (with an algebraic multiplicity of three) and $r = 1$ so a set of linearly independent homogeneous solutions are

$$\begin{aligned} y_1(x) &= 1 \\ y_2(x) &= x \\ y_3(x) &= x^2 \\ y_4(x) &= e^x\,. \end{aligned}$$

With only one boundary condition the matrix U is a row vector and takes the form

$$U = \left[\begin{array}{cccc} 2 & 0+1 & 0+1^2 & 1+e^1 \end{array} \right] = \left[\begin{array}{cccc} 2 & 1 & 1 & 1+e^1 \end{array} \right].$$

This "matrix" has a rank of one. Thus our system is $n - r = 4 - 1 = 3$-ply. To find these three linearly independent solutions to our boundary value problem we need to solve $U\mathbf{c} = 0$ for $\mathbf{c}$. This is the equation

$$2c_1 + c_2 + c_3 + (1+e)c_4 = 0\,.$$

We can get three linearly independent solutions to this constraint by taking

$$c_2 = 2\,, c_3 = 0\,, c_4 = 0 \quad \text{so} \quad c_1 = -1\,,$$

$$c_2 = 0\,, c_3 = 2\,, c_4 = 0 \quad \text{so} \quad c_1 = -1\,,$$

$$c_2 = 0\,, c_3 = 0\,, c_4 = 2 \quad \text{so} \quad c_1 = -1 - e\,.$$

Using these constants, three linearly independent solutions are then

$$\begin{aligned} y_A(x) &= -1 + 2x \\ y_B(x) &= -1 + 2x^2 \\ y_C(x) &= -1 - e + 2e^x\,. \end{aligned}$$

We next check that each of these satisfies the boundary conditions

$$\begin{aligned} y_A(0) + y_A(1) &= -1 + 1 = 0 \\ y_B(0) + y_B(1) &= -1 + 1 = 0 \\ y_C(0) + y_C(1) &= (-1 - e + 2) + (-1 - e + 2e) = 0\,, \end{aligned}$$

as they should. Using these three functions the general solution to this boundary value problem is

$$y(x) = c_1 y_A(x) + c_2 y_B(x) + c_3 y_C(x)\,.$$

Part (f): For the homogeneous differential equation $y^{(iv)}(x) = 0$ if we let $y(x) = e^{rx}$ then we get the characteristic equation $r^4 = 0$. We can take $n = 4$ linearly independent homogeneous solutions to be (I also compute the x derivatives of each since I'll need them later)

$$\begin{aligned} & y_1(x) = 1 \quad y_1'(x) = 0 \\ & y_2(x) = x \quad y_2'(x) = 1 \quad y_2''(x) = 0 \\ & y_3(x) = x^2 \quad y_3'(x) = 2x \quad y_3''(x) = 2 \quad y_3'''(x) = 0 \\ & y_4(x) = x^3 \quad y_4'(x) = 3x^2 \quad y_4''(x) = 6x \quad y_4'''(x) = 6 \quad y_4^{(4)}(x) = 0\,. \end{aligned}$$

For the four boundary conditions given and the homogeneous solutions above the matrix U takes the form

$$U = \left[\begin{array}{cccc} 1-0 & 0-1 & 0-2 & 0-3 \\ 0 & 0 & 2 & 0 \\ 0+0 & 1+0 & 0+2 & 0+6 \\ 0-0 & 0-0 & 0-0 & 6-6 \end{array} \right] = \left[\begin{array}{cccc} 1 & -1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

We can show that this matrix has a rank of three. Thus this system is $n-r = 4-3 = 1$-ply. To find the constraint needed to apply to the functions $y_1(x), y_2(x), y_3(x), y_4(x)$ we need to solve $U\mathbf{c} = 0$ for $\mathbf{c}$. Using elementary row operations we can write the system $\mathbf{Uc} = 0$ as

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{c} = 0 \,.$$

This means that $c_1 = -3c_4$, $c_2 = -6c_4$, $c_3 = 0$, and c_4 is arbitrary. Taking $c_4 = 1$ this means that the only solution to the boundary value problem is

$$y_A(x) = -3 - 6x + x^3 \,.$$

The derivatives of this are given by

$$\begin{aligned} y_A'(x) &= -6 + 3x^2 \\ y_A''(x) &= 6x \\ y_A'''(x) &= 6 \,. \end{aligned}$$

Thus we see that $y_A(x)$ satisfies all of the boundary conditions as

$$\begin{aligned} y_A(0) - y_A'(1) &= -3 - (-3) = 0 \\ y_A''(0) &= 0 \\ y_A'(0) + y_A''(1) &= -6 + 6 = 0 \\ y_A'''(0) - y_A'''(1) &= 6 - 6 = 0 \,, \end{aligned}$$

as it should.

Part (g): Putting $y(x) = x^r$ into the differential equation we get

$$r(r-1) + r + 1 = 0 \,,$$

which has solutions $r = \pm i$. This means that two linearly independent solutions to the homogeneous problem are (here I also compute their derivatives)

$$\begin{aligned} y_1(x) &= \cos(\ln(x)) \quad y_1'(x) = -\frac{\sin(\ln(x))}{x} \\ y_2(x) &= \sin(\ln(x)) \quad y_2'(x) = \frac{\cos(\ln(x))}{x} \,. \end{aligned}$$

With only one boundary condition the matrix U is a single row and takes the form

$$U = \left[\begin{array}{cc} \cos(\ln(1)) - \frac{\sin(\ln(1))}{1} & \sin(0) + \frac{\cos(0)}{1} \end{array} \right] = \left[\begin{array}{cc} 1 & 1 \end{array} \right] .$$

This has a rank of one. Thus this system is $n - r = 2 - 1 = 1$-ply. To find our one linearly independent solution we need to solve $U\mathbf{c} = 0$ for $\mathbf{c}$. This is the requirement that $c_1 = -c_2$. We can get our one solution to this by taking $c_2 = 1$ so that $c_1 = -1$. Then one solution to our boundary value problem is then

$$y_A(x) = -\cos(\ln(x)) + \sin(\ln(x)) \,.$$

For this function we have

$$y_A'(x) = \frac{\sin(\ln(x))}{x} + \frac{\cos(\ln(x))}{x}.$$

Thus we have

$$y_A(1) + y_A'(1) = -\cos(0) + \sin(0) + \sin(0) + \cos(0) = 0,$$

as it should.

Exercise 1.4

Part (a): The solution to our homogeneous differential equation $y'(x) = 0$ is $y_1(x) = 1$. A particular solution is $y_p(x) = x$. As there are two boundary conditions the U matrix is $U = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and is rank one. The matrix U_γ is

$$U_\gamma = \begin{bmatrix} 1 & 0-0 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

which has rank two. As the rank of U and U_γ are not equal there no solutions to this system.

This made me wonder if there is a typo on this problem. Note that the general solution to this differential equation is

$$y(x) = c_1 + x.$$

Now to have $y(0) = 0$ we have $c_1 = 0$ so that $y(x) = x$. Note that in that case the second boundary condition won't hold true as $y(2) = 2 \neq 3$. Lets assume that there is a typo in the book and the boundary condition should have been $y(2) = 2$ (for then we know there will be a solution). In that case we have

$$U_\gamma = \begin{bmatrix} 1 & 0 \\ 1 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},$$

which has rank one. Following the book we now want to solve the linear system of $U\mathbf{c}$ set equal to the last column of U_γ. This is the system

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To make this true we must take $c_1 = 0$ and we get $y_A(x) = x$.

Part (b): By putting $y(x) = e^{rx}$ into the homogeneous equation we get $r^2 + 16 = 0$ which means that two solutions to the homogeneous solutions are given by

$$\begin{aligned} y_1(x) &= \cos(4x) \quad \text{so} \quad y_1'(x) = -4\sin(4x) \\ y_2(x) &= \sin(4x) \quad \text{so} \quad y_2'(x) = 4\cos(4x). \end{aligned}$$

A particular solution to this differential will take the form $y_p(x) = Ax + B$. For this form we have $y_p'(x) = A$ and $y_P''(x) = 0$. Putting this into our differential equation we require

$$16(Ax + B) = x\,.$$

This means that $A = \frac{1}{16}$ and $B = 0$ and we have shown that

$$y_p(x) = \frac{1}{16}x\,.$$

For these two boundary conditions the U matrix looks like

$$U = \begin{bmatrix} 1 - 4\sin(4\pi) & 0 + 4\cos(4\pi) \\ -4\sin(0) & 4\cos(0) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 4 \end{bmatrix}.$$

This matrix has rank two. Next we have

$$U_\gamma = \begin{bmatrix} 1 & 4 & 0 - U_1(y_p) \\ 0 & 4 & 2 - U_2(y_p) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 - (0 + \frac{1}{16}) \\ 0 & 4 & 2 - \frac{1}{16} \end{bmatrix} = \begin{bmatrix} 1 & 4 & -\frac{1}{16} \\ 0 & 4 & \frac{31}{16} \end{bmatrix},$$

which also has rank two. Next using elementary row operations we can transform the system where we have $U\mathbf{c}$ set equal to the last column of U_γ into

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ \frac{31}{64} \end{bmatrix}.$$

This means that $c_1 = -2$ and $c_2 = \frac{31}{64}$ so the solution looks like

$$y_A(x) = -2\cos(4x) + \frac{31}{64}\sin(4x) + \frac{1}{16}x\,.$$

For this expression we find

$$y_A'(x) = 8\sin(4x) + \frac{31}{16}\cos(4x) + \frac{1}{16}\,.$$

This means that

$$\begin{aligned} y_A(0) + y_A'(\pi) &= -2 + \frac{31}{16} + \frac{1}{16} = 0 \\ y_A'(0) &= \frac{31}{16} + \frac{1}{16} = 2\,, \end{aligned}$$

as it should.

Part (c): For the homogeneous solution we have

$$\begin{aligned} y_1(x) &= \cos(3x) \quad \text{so} \quad y_1'(x) = -3\sin(3x) \\ y_2(x) &= \sin(3x) \quad \text{so} \quad y_2'(x) = 3\cos(3x)\,. \end{aligned}$$

A particular solution takes the form

$$y_p(x) = A\cos(x) + B\sin(x)\,.$$

This means that

$$y_p''(x) = -A\cos(x) - B\sin(x)\,.$$

If we put this into the differential equation we get

$$(-1+9)A\cos(x) + (-1+9)B\sin(x) = \sin(x)\,.$$

Solving we have that

$$A = 0 \quad \text{and} \quad B = \frac{1}{8}\,,$$

so we get

$$y_p(x) = \frac{1}{8}\sin(x)\,.$$

Next consider the U matrix which for this boundary condition looks like

$$U = \left[\begin{array}{cc} \cos(0) - 3\sin(0) & \sin(0) + 3\cos(0) \end{array}\right] = \left[\begin{array}{cc} 1 & 3 \end{array}\right]\,,$$

which has rank one. Next consider

$$\begin{aligned} U_\gamma &= \left[\begin{array}{ccc} 1 & 3 & 1 - U_1\left(\frac{1}{8}\sin(x)\right) \end{array}\right] \\ &= \left[\begin{array}{ccc} 1 & 3 & 1 - \frac{1}{8} \end{array}\right] \\ &= \left[\begin{array}{ccc} 1 & 3 & \frac{7}{8} \end{array}\right]\,. \end{aligned}$$

Which also has a rank of one. Thus this system is $2 - 1 = 1$-ply. The system where we have $U\mathbf{c}$ set equal to the last column of U_γ gives

$$\left[\begin{array}{cc} 1 & 3 \end{array}\right]\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] = \frac{7}{8}\,.$$

This means that

$$c_1 = -3c_2 + \frac{7}{8}\,.$$

Thus taking $c_2 = \left(\frac{1}{3}\right)\left(\frac{7}{8}\right) = \frac{7}{24}$ gives $c_1 = 0$ and we get

$$y_A(x) = \frac{7}{24}\sin(3x) + \frac{1}{8}\sin(x)\,.$$

For this expression we have

$$y_A'(x) = \frac{7}{8}\cos(3x) + \frac{1}{8}\cos(x)\,.$$

Thus

$$y_A(0) + y_A'(0) = 0 + \frac{7}{8} + \frac{1}{8} = 1\,,$$

as it should.

Part (d): To find homogeneous solutions we take $y(x) = e^{rx}$ and put this into the differential equation where we get

$$r^2 - 2r - 3 = (r-3)(r+1) = 0\,.$$

This means that $r = 3$ and $r = -1$ so that two linearly independent solutions to the homogeneous equation are

$$\begin{aligned} y_1(x) &= e^{-x} \quad \text{so} \quad y_1'(x) = -e^{-x} \\ y_2(x) &= e^{3x} \quad \text{so} \quad y_2'(x) = 3e^{3x}\,. \end{aligned}$$

A particular solution is given by $y_p(x) = Ae^x$. If we put this into the differential equation we get

$$Ae^x - 2Ae^x - 3Ae^x = e^x \quad \text{so} \quad A = -\frac{1}{4}\,.$$

This means that

$$y_p(x) = -\frac{1}{4}e^x\,.$$

Next the U matrix (with three boundary conditions) looks like

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 3 \\ e^{-1} - e^{-1} & e^3 + 3e^3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \\ 0 & 4e^3 \end{bmatrix}.$$

This matrix has rank of two. Next consider

$$U_\gamma = \begin{bmatrix} 1 & 1 & 1 - \left(-\frac{1}{4}\right) \\ -1 & 3 & 0 - \left(-\frac{1}{4}\right) \\ 0 & 4e^3 & 2 - \left(-\frac{1}{4}e - \frac{1}{4}e\right) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{5}{4} \\ -1 & 3 & \frac{1}{4} \\ 0 & 4e^3 & 2 + \frac{1}{2}e \end{bmatrix}.$$

This matrix has a rank of three. As this is not equal to the rank of U there are no solutions to this boundary value problem. To create a problem that has a solution if we *change* the third boundary condition so that the last row in the U_γ matrix is exactly the sum of the two previous rows (when multiplied by e^3) that is

$$\gamma_3 - \left(-\frac{1}{4}e - \frac{1}{4}e\right) = \left(\frac{5}{4} + \frac{1}{4}\right)e^3 = \frac{3}{2}e^3\,.$$

This means that

$$\gamma_3 = \frac{3}{2}e^3 - \frac{1}{2}e\,.$$

Thus lets assume that the third boundary condition should be

$$y(1) + y'(1) = \frac{3}{2}e^3 - \frac{1}{2}e\,. \tag{1}$$

Then the matrix U_γ is

$$U_\gamma = \begin{bmatrix} 1 & 1 & \frac{5}{4} \\ -1 & 3 & \frac{1}{4} \\ 0 & 4e^3 & \frac{3}{2}e^3 - \frac{1}{2}e - \left(-\frac{1}{4}e - \frac{1}{4}e\right) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{5}{4} \\ -1 & 3 & \frac{1}{4} \\ 0 & 4e^3 & \frac{3}{2}e^3 \end{bmatrix}.$$

As the last row is a linear combination of the previous two rows this matrix has a rank of two. This system is then $2 - 2 = 0$-ply compatible which means there is only *one* solution but that linear multiples of it are *not* solutions. To find the solution we need to solve the system where we have $U\mathbf{c}$ set equal to the last column of U_γ gives which is

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \\ 0 & 4e^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/4 \\ (3/2)e^3 \end{bmatrix}.$$

Applying elementary row operations we get the following sequence of manipulations

$$\begin{bmatrix} 1 & 1 \\ 0 & 4 \\ 0 & 4e^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 3/2 \\ (3/2)e^3 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 4e^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 3/8 \\ (3/2)e^3 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 3/8 \\ 0 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7/8 \\ 3/8 \\ 0 \end{bmatrix}.$$

This means that the only solution is

$$y_A(x) = \frac{7}{8}e^{-x} + \frac{3}{8}e^{3x} - \frac{1}{4}e^{x}.$$

Lets check that this solution satisfies the boundary conditions (as it should)

$$y_A(0) = \frac{7}{8} + \frac{3}{8} - \frac{1}{4} = 1.$$

as it should be. The derivative of $y_A(x)$ is given by

$$y_A'(x) = -\frac{7}{8}e^{-x} + \frac{9}{8}e^{3x} - \frac{1}{4}e^{x},$$

so that

$$y_A'(0) = -\frac{7}{8} + \frac{9}{8} - \frac{1}{4} = 0.$$

Finally for the third boundary condition see that

$$y_A(1) + y_A'(1) = \frac{7}{8}e^{-1} + \frac{3}{8}e^{3} - \frac{1}{4}e^{1} + \left(-\frac{7}{8}e^{-1} + \frac{9}{8}e^{3} - \frac{1}{4}e^{1}\right) = \frac{3}{2}e^3 - \frac{1}{2}e.$$

which is the modified boundary condition we constructed in Equation 1.