pg 3 ince...

$\lambda = 1$

$$\frac{dy}{dx}\frac{dx}{dy} = 1$$

$$\frac{d^3}{dx^3} \to X \qquad \text{No.}$$

$$\frac{d^2}{dx^2} \to \qquad \text{yes}$$

gives w/ MMA...

$$\Rightarrow 3\frac{dy}{dx}\frac{d^2x}{dy^2}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\frac{d^3x}{dy^3} + \frac{dx}{dy}\frac{d^3y}{dx^3} = 0$$

mult by $\frac{dx}{dy}$

$$\Rightarrow 3\frac{d^2x}{dy^2}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\frac{d^3x}{dy^3} + \left(\frac{dx}{dy}\right)^2\frac{d^3y}{dx^3} = 0.$$

pg 5 Ince

$$y^2 - (a+b)yx + abx^2 = 0$$

$$\Leftrightarrow (y-ax)(y-bx) = 0.$$

$$y - ax = 0 \qquad y - bx = 0 \qquad \Rightarrow xy' - y = 0$$

$$y' - a = 0 \qquad y' - b = 0$$

$$y' = a = \frac{y}{x}$$

$$2yy' - (a+b)y'x - (a+b)y + 2abx = 0.$$

$$\cancel{2y\left(\frac{y}{x}\right) - 2(a+b)(}$$

$$2\cancel{x}y'^2 - (a+b)y'\cancel{x} - (a+b)\cancel{x}y' + 2ab\cancel{x} = 0$$

$$\cancel{2}y'^2 - \cancel{2}(a+b)y' + \cancel{2}ab = 0$$

$$y'^2 - (a+b)y' + ab = 0$$

$$(y' - a)(y' - b) = 0$$

pg 175 Ince

$$y'' + (a - 2\theta\cos 2x)y = 0 \qquad p(x) = a - 2\theta\cos 2x$$
of period π.

$G(x)$ sol.

$y(x)$ even then $y'(x=0) = 0$

$$C_0 = \sum_{r=0}^{\infty} C_r \cos(2r+1)x \quad \text{periodic of period } 2\pi.$$

$$C_0'' = \sum_{r=0}^{\infty} -(2r+1)^2 C_r \cos(2r+1)x$$

put in

$$\sum_{r=0}^{\infty} -(2r+1)^2 C_r \cos(2r+1)x + a\sum_{r=0}^{\infty} C_r \cos(2r+1)x$$

$$-2\theta \sum_{r=0}^{\infty} \cos 2x \cos(2r+1)x = 0$$

But $\cos 2x \cos(2r+1)x = \frac{1}{2}\left[\cos((2r+3)x) + \cos((2r-1)x)\right]$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

Then

$$\sum_{r=0}^{\infty} C_r\left(-(2r+1)^2 + a\right)\cos(2r+1)$$

Pg 223 Ince

Regular Sturmian theory.

10.2 $a < x_n < b \quad \forall n$: Bolzano-Weierstrass.

limit pt $y(x_n) = 0 \quad \forall n$ why $y(c) = 0$ not $y(c) = \epsilon \ll 1$

$$y(c+h) = \underset{\overset{\|}{0}}{y(c)} + h y'(c+\theta h) \qquad 0 \le \theta < 1$$

continuity of y gives ~~use~~ us $y(c) = 0$ existence gives 1 cont. sol w/ 1 cont derivative.

$y'(c+\theta h) = 0$ h small $h \ll 1$

If y_2 vanished at $y_1(x_1)$ $L(y) = 0$

$y(x_1) = 0$

& for y_2 $y'(x_1) = r_1$

$L(y) = 0$
$y(x_2) = 0$
$y'(x_1) = r_2$
$\Rightarrow y_1 + y_2$ multiples of each other

$\frac{y_1}{y_2}$ vanishes $x_1 + x_2$ $\exists$ point in (x_1, x_2)

when $\frac{d}{dx}\left(\frac{y_1}{y_2}\right) = \frac{y_1'}{y_2} - \frac{y_1 y_2'}{y_2^2} = \frac{y_1' y_2 - y_1 y_2'}{y_2^2}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

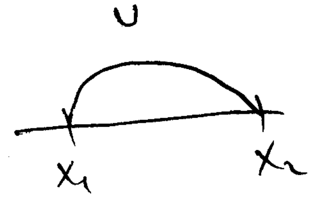
$$\begin{vmatrix} A\sin x + B\cos x & C\sin x + D\cos x \\ A\cos x - B\sin x & C\cos x - D\sin x \end{vmatrix}$$

$$= \cancel{AC \sin x \cos x} - DA\sin^2 x + BC\cos^2 x - \cancel{DB\cos x \sin x}$$

$$-\left(\cancel{AC\sin x \cos x} + AD\cos^2 x - BC\sin^2 x - \cancel{BD\sin x \cos x}\right)$$

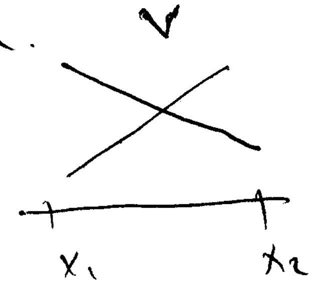
$$= -DA + BC \neq 0$$

"$k(x) \underset{\neq}{>} 0 \quad a \leq x \leq b$" pg 225 Ince



x_1 & x_2 consec zeros of u.

& Assume v has no zero between.



(we hope to reach a $\rightarrow\leftarrow$)

W.O.L.O.G. u & $v \geq 0$ on (x_1, x_2)

$$G_1 - G_2 \underset{\neq}{>} 0$$

$$\text{L.H.S.} = k(x_2)\left[u'(x_2)v(x_2) - 0\right] - k(x_1)\left[u'(x_1)v(x_1)\right]$$

$$= k(x_2)u'(x_2)\underbrace{v(x_2)}_{\geq 0} - k(x_1)u'(x_1)\underbrace{v(x_1)}_{\geq 0}$$

$u'(x_2) < 0 \qquad u'(x_1) > 0$

$\therefore$ LHS neg $-$ pos $=$ ~~neg~~ ~~neg~~ neg

Wher as the right hand side is positive.

$\ast$

if $u + v = 0$ & x_2 next zero of u
$x = x_1$.

Then LHS

$$\Rightarrow k(x_2)[u'(x_2)v(x_2) - \cancel{u(x_2)}]$$
$$- k(x_1)[\cancel{u'(x_1)}] = 0$$

$$\Rightarrow \underbrace{k(x_2)u'(x_2)v(x_2)}_{< 0} \quad \ast.$$

$\therefore \exists$ zero of v before the next zero of $u(x)$.

Prob 5: 18.075 Summer 95?

Sol to example problem is

$$u = A\cos mx + B\sin mx$$

$$v = A'\cos nx + B'\sin nx$$

if $m>n$ u ossilates more rapidly than v.

I wanted something like if $G_1(x) > G_2(x)$

Then $u(x)$ will osscilate less ~~more~~ rapidly than $v(x)$.

or $v(x)$ will osscilate more rapidly than $u(x)$.

---

mult by v in 1st & $u(x)$ in 2nd & integrate from x_1 to x_2.

to get given expression

x_1 & x_2 are cons. zeros of $u(x)$

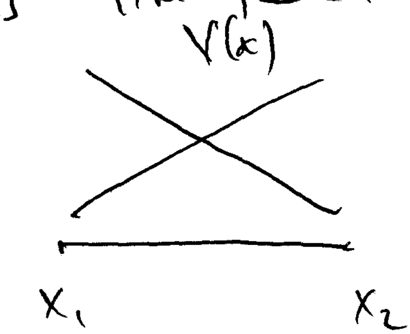
$u(x)$ & $v(x)$ can both be pos. because if they are negative $-u(x)$ is positive & is also a solution of the ODE.

Integral is positive.

If $u(x)$ looks like the picture to the right then

$u'(x_1) > 0$ & $u'(x_2) < 0$

Also if $v(x)$ does not have a zero between x_1 & x_2 it looks like the following, & both $v(x_1)$ & $v(x_2)$ are positive



LHS becomes

$$\underbrace{k(x_2)(u'(x_2)v(x_2))}_{\text{neg.}} - \underbrace{k(x_1)u'(x_1)v(x_1)}_{\text{pos.}}$$

$\therefore$ neg − pos is ~~[illegible]~~ negative &

we have a contradiction if $v(x)$ does not

cross the x-axis between x_1 & x_2.

Thus $v(x)$ must vanish between x_1 & x_2

if both $U(x)$ & $v(x)$ vanish at x_1 then

$v(x)$ must vanish again before x_2. $\therefore$ $v(x)$

osscilates more then $U(x)$.

As P_n satisfies

$$\frac{d}{dx}\left((1-x^2)y'\right) + n(n+1)y = 0$$

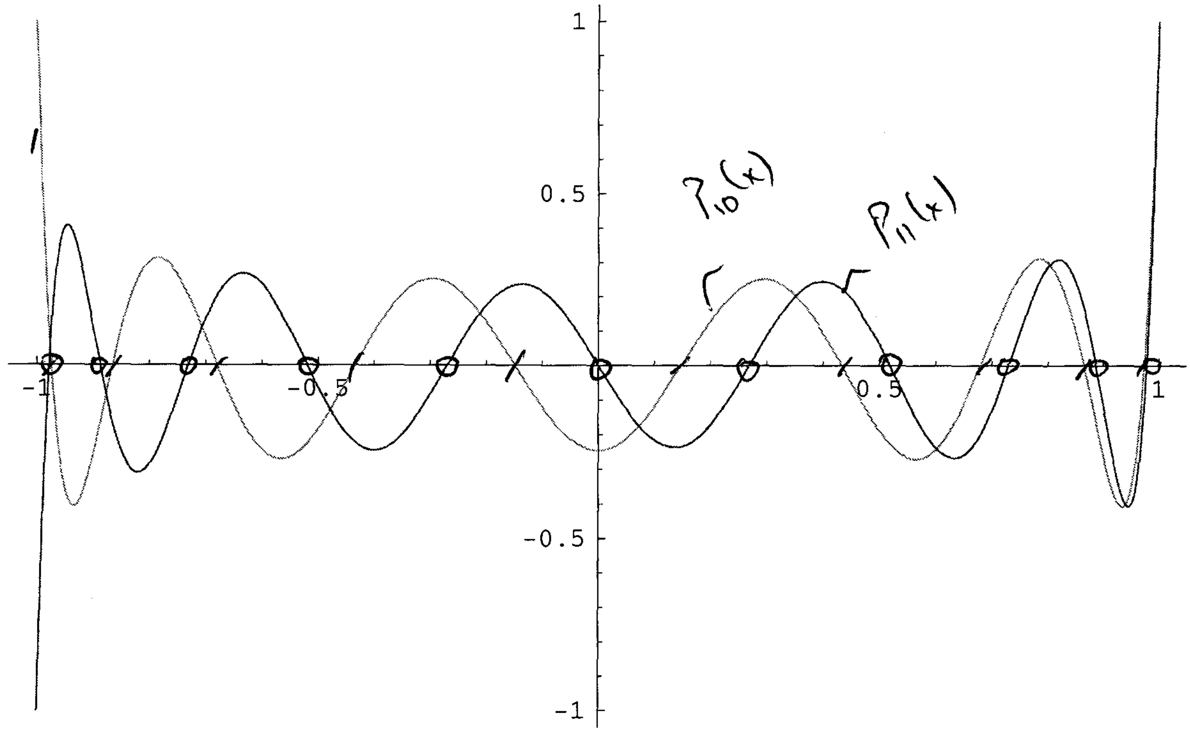
& P_{n+1} satisfies

$$\frac{d}{dx}\left((1-x^2)y'\right) + (n+1)(n+2)y = 0$$

P_{n+1} must ossicate more ~~[illegible]~~ rapidly than P_n

this can be seen by looking at plots of P_n & P_{n+1} for $n = 10$ say.

```
In[36]:= Plot[{LegendreP[10, x], LegendreP[11, x]}, {x, -1, 1},
         PlotStyle -> {RGBColor[0, 1, 0], RGBColor[1, 0, 0]}]
```




```
Out[36]= - Graphics -
```

$P_{10}(x)$ crosses 10 x's

$P_{11}(x)$ crosses 11 x's > 10 As

predicted.

pg 381 - 384

why want sol $w(z+2\omega) = s\,u(z)$?

No loss in gen assuming $\omega \in \mathbb{R}$?

If s_1 repeated

$u_1(z), \ldots, u_\mu(z) \ni$

$$u_1(z+2\omega) = s_1 u_1(z)$$
$$u_2(z+2\omega) = s_1\{u_2(z) + u_1(z)\}$$

$u_3 \cdots$

$$u_\mu(z+2\omega) = s_1\{u_\mu(z) + u_{\mu-1}(z)\} \quad ?$$

$$e^{-a(z+2\omega)}\underline{u_1(z+2\omega)} = s_1 u_1(z)\, e^{-az} e^{-2a\omega}$$
$$= \left[s_1 e^{-2a\omega}\right] e^{-az} u_1(z)$$

$\therefore$

$\therefore \quad e^{-az} u_1(z)$

if $\quad s_1 = e^{2a\omega}$

if n roots of charact eq are

pg 382 Ince

$\cancel{V_1(z+2w) = V_1(z)}$ pg 381 gives

$\cancel{V_2(z+2w) = V_2(z)+}$

$$U_1(z+2w) = S_1 U_1(z)$$
$$U_2(z+2w) = S_1\{U_2(z)+U_1(z)\}$$
$$\vdots$$
$$U_\mu(z+2w) = S_1\{U_\mu(z)+U_{\mu-1}(z)\}$$

w/ $U_\nu(z) = e^{a_1 z} V_\nu(z)$ w/ $\cancel{V_\nu(z) \text{ } 2w \text{ periodic} \ldots}$

$$\Rightarrow \quad e^{a_1(z+2w)} V_1(z+2w) = S_1 \cancel{e^{a_1 z}} V_1(z)$$
$$e^{a_1(z+2w)} V_2(z+2w) = S_1 e^{a_1 z}\{V_1(z)+V_2(z)\}$$
$$\vdots$$
$$e^{a_1 z + 2wa_1} V_\mu(z+2w) = S_1 e^{a_1 z}\{V_\mu(z)+V_{\mu-1}(z)\}$$

$\Rightarrow$ w/ $e^{2wa_1} = S_1$

$$V_1(z+2w) = V_1(z)$$
$$V_2(z+2w) = V_1(z) + V_2(z)$$

$$V_\mu(z+2\omega) = V_\mu(z) + V_{\mu-1}(z)$$

$$\therefore \quad \frac{V_2(z+2\omega)}{V_1(z+2\omega)} = \frac{V_2(z)}{V_1(z)} + 1$$

3. consider $\dfrac{V_2(z)}{V_1(z)} - \dfrac{z}{2\omega}$.

$$\frac{V_2(z+2\omega)}{V_1(z+2\omega)} - \frac{(z+2\omega)}{2\omega}$$

$$= \frac{V_2(z)}{V_1(z)} + 1 - \frac{z}{2\omega} - 1 =$$

$$\frac{V_2(z)}{V_1(z)} - \frac{z}{2\omega} \quad \checkmark.$$

$\therefore$ distinct $\Rightarrow \exists\, n$ sol

$u_i(x) = e^{\alpha}$

Pg 383

$$\frac{d^2 w}{dz^2} = p(z) w$$

$$\begin{vmatrix} a_{11} - s & a_{12} \\ a_{21} & a_{22} - s \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - s)(a_{22} - s) - a_{12}a_{21} = 0$$

$$a_{11}a_{22} - (a_{11} + a_{22})s + s^2 - a_{12}a_{21} = 0$$

$$\Rightarrow s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$\Rightarrow$ $\tilde{a}_{11} \times \tilde{a}_{22}$ we can scale, $|A| = 1$

As $|A| \neq 0$

$$s^2 - As + 1 = 0$$

$\Rightarrow$

- idea... given

$$\frac{d^n w}{dz^n} + p_1(z)\frac{d^{n-1} w}{dz^{n-1}} + p_2(z)\frac{d^{n-2} w}{dz^{n-2}} + \cdots + p_n(z) w = 0$$

w/ $p_1, p_2, \ldots p_n$ periodic w/ period 2ω..
(p for periodic)
what type of solutions do we get? i.e. is the solution periodic?

Ans. depends on n characteristic #'s of the

eg $u_i(z) = e^{a_i z}\phi_i(z)$

w/ ϕ_i periodic of period 2ω if

All characteristic #'s distinct. if not ...

Challenge: write a code to calculate these Floquet multipliers ...
or find one that does it...
Hint: look in the dynamical systems litterature...

pg 383 Ince

compute Floquet Exponents for $\frac{d^2w}{dz^2} = p(z)w$

($\exists$ 2 #'s) sol to $s^2 - As + 1 = 0$

w/ $p(z) = \ldots$

If $p(z) > 0 \quad \forall z \in \mathbb{R}$ char exponents real

$\Rightarrow$ unstable...

$\therefore p(z)$ must be negative

Try $p(z) = \sin z$. not neg $\forall$ values > 0.

$\cancel{2\pi \int_0^{2\pi} \sin z \, dz = -2\pi \cos z \Big|_0^{2\pi}}$

$\cancel{= \ldots}$

i.e. solve $\frac{d^2w}{dz^2} = \sin z \, w$.

Solve for $A = ?$ $\quad A = f(2\omega) + g'(2\omega)$ $\quad \omega = \pi$ in this case...

$$= 2 + \sum_{n=1}^{\infty} \left[f_n(2\pi) + g_n'(2\pi) \right]$$

Approx made in MMA

$$A \cong 2 + \sum_{n=1}^{10} \left[f_n(2\pi) + g_n'(2\pi) \right].$$

pg 383 Ince ..

If $f(z,\lambda) = 1 + \lambda f_1(z) + \cdots + \lambda^n f_n(z) + \cdots = \sum_{n=0}^{\infty} \lambda^n f_n(z)$

& $g(z,\lambda) = z + \lambda g_1(z) + \cdots + \lambda^n g_n(z) + \cdots = \sum_{n=0}^{\infty} \lambda^n g_n(z)$

w/ $f(z,\lambda)$ sol to $\frac{d^2w}{dz^2} = \lambda p(z) w$

$f(0) = 1 \quad f'(0) = 0$

& $g(z,\lambda)$ sol to $\frac{d^2w}{dz^2} = \lambda p(z) w$

$g(0) = 0 \quad g'(0) = 1$

Then $\frac{d^2f}{dz^2} = \lambda p(z)$

$$\sum_{n=1}^{\infty} \lambda^n f_n''(z) = \lambda p(z) \sum_{n=0}^{\infty} \lambda^n f_n$$

As $n=0$ is constants

$$= \sum_{n=0}^{\infty} \lambda^{n+1} p(z) f_n(z)$$

$$= \sum_{n=1}^{\infty} \lambda^n p(z) f_{n-1}(z)$$

$\Rightarrow \quad f_n''(z) = p(z) f_{n-1}(z)$

$g(z)$ is similar...

pg 394 Ince

(4) $w'' + pw' + qw = 0$

let $w = \bar{W} e^{-\frac{1}{2}\int p\,dz}$

then $w' = \bar{W}' e^{-\frac{1}{2}\int p\,dz} + \bar{W} e^{-\frac{1}{2}\int p\,dz}\left(-\frac{1}{2}p\right)$

$$w'' = \bar{W}'' e^{-\frac{1}{2}\int p\,dz} + \bar{W}'\left(-\frac{1}{2}p\right)e^{-\frac{1}{2}\int p} + \bar{W}'\left(-\frac{1}{2}p\right)e^{-\frac{1}{2}\int p\,dz}$$

$$+ \bar{W}\left(-\frac{1}{2}p'\right)e^{-\frac{1}{2}\int p\,dz} + \bar{W} e^{-\frac{1}{2}\int p\,dz}\left(\frac{1}{4}p^2\right)$$

$$= \left(\bar{W}'' - p\bar{W}' + \left(-\frac{1}{2}p' + \frac{1}{4}p^2\right)\bar{W}\right)e^{-\frac{1}{2}\int p\,dz}$$

Thus Adding together + ÷ by $e^{-\frac{1}{2}\int p\,dz}$ gives

$$\cancel{\bar{W}'' - p\bar{W}' + \left(-\frac{p'}{2} + \frac{p^2}{2}\right)\bar{W} + p\bar{W}' + q\bar{W} = 0}$$

$$\bar{W}'' - \cancel{p\bar{W}'} + \left(-\frac{p'}{2} + \frac{p^2}{4}\right)\bar{W} + \cancel{p\bar{W}'} - \frac{p^2}{2}\bar{W} + q\bar{W} = 0$$

$$\bar{W}'' + \left(q - \frac{p'}{2} + \frac{p^2}{4} - \frac{p^2}{2}\right)\bar{W} = 0$$

$$\Rightarrow \bar{W}'' + \left(q - \frac{p^2}{4} - \frac{p'}{2}\right)\bar{W} = 0$$