Solutions To Selected Exercises In: Difference Equations: An Introduction with Applications by Walter G. Kelly and Allan C. Peterson

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Chapter 3: Linear Difference Equations

Nonlinear equations that can be linearized: Ricatti Equations

The Ricatti *difference* equation is given by

$$y(t+1)y(t) + p(t)y(t+1) + q(t)y(t) + r(t) = 0\,. \tag{1}$$

To solve this let $y(t) = \frac{z(t+1)}{z(t)} - p(t)$, determine an equation for $z(t)$. When we put this into Equation 1, we find

$$\begin{aligned}
&\left(\frac{z(t+2)}{z(t+1)} - p(t+1)\right)\left(\frac{z(t+1)}{z(t)} - p(t)\right) + p(t)\left(\frac{z(t+2)}{z(t+1)} - p(t+1)\right) \\
+ \quad & q(t)\left(\frac{z(t+1)}{z(t)} - p(t)\right) + r(t) = 0\,.
\end{aligned}$$

On expanding this expression we have

$$\begin{aligned}
&\frac{z(t+2)}{z(t)} - p(t)\frac{z(t+2)}{z(t+1)} - p(t+1)\frac{z(t+1)}{z(t)} + p(t+1)p(t) \\
+ \quad & p(t)\frac{z(t+2)}{z(t+1)} - p(t)p(t+1) + q(t)\frac{z(t+1)}{z(t)} - p(t)q(t) + r(t) = 0\,.
\end{aligned}$$

On multiplying by $z(t)$ and grouping terms gives

$$z(t+2) + (q(t) - p(t+1))z(t+1) + (r(t) - p(t)q(t))z(t) = 0\,, \tag{2}$$

which is linear in $z(t)$ and can be solved by the methods previously discussed for linear equations.

Example 3.27

Comparing to Equation 1, the given Ricatti equation has $p(t) = 2$, $q(t) = 4$, and $r(t) = 9$ so that the corresponding linear equation for this example is then given by Equation 2 or

$$z(t+2) + (4-2)z(t+1) + (9-8)z(t) = 0\,,$$

or

$$z(t+2) + 2z(t+1) + z(t) = 0\,.$$

This equation has a solution given by

$$z(t) = A(-1)^t + Bt(-1)^t\,.$$

So that the solution $y(t)$ we want is given by

$$y(t) = \frac{z(t+1)}{z(t)} - 2 = \frac{A(-1)^{t+1} + B(t+1)(-1)^{t+1}}{A(-1)^t + Bt(-1)^t} - 2 = \frac{-A - B(t+1)}{A+Bt} - 2\,.$$

Assuming $A \neq 0$ we can divide by it to obtain

$$y(t) = \frac{-1 - C(t+1)}{1 + Ct} - 2 = \frac{-3 - C(3t+1)}{1 + Ct},$$

where $C \equiv \frac{B}{A}$. If $A = 0$ we cannot divide by A and the above expression for $y(t)$ in this case becomes

$$y(t) = -\frac{t+1}{t} - 2,$$

both of which agree with the solutions given in the book.

Problem Solutions

Problem 86 (Solving Ricatti equations)

Part (a): For the Ricatti equation

$$y(t+1)y(t) + 2y(t+1) + 7y(t) + 20 = 0,$$

comparing to Equation 1 we have $p(t) = 2$, $q(t) = 7$, and $r(t) = 20$ so the corresponding linear equation given in Equation 2 then becomes

$$z(t+2) + (7-2)z(t+1) + (20-14)z(t) = 0,$$

or

$$z(t+2) + 5z(t+1) + 6z(t) = 0.$$

This latter equation has characteristic roots given by -2 and -3. Thus the general solution for $z(t)$ in this linear problem is given by

$$z(t) = A(-2)^t + B(-3)^t.$$

Thus, $y(t)$, the solution to the Ricatti equation of interest is given by

$$y(t) = \frac{A(-2)^{t+1} + B(-3)^{t+1}}{A(-2)^t + B(-3)^t} - 2.$$

If $A \neq 0$ we can obtain (by dividing by A) the following

$$y(t) = \frac{-2^{t+2} - 5C3^t}{2^t + C3^t}.$$

If $A = 0$ we can not divide by it and we find a particular solution of $y(t) = -3 - 2 = -5$, which can be easily verified.

Chapter 10 Clarifications

The eigensystem for the discrete diffusion equation (Page 403)

The eigenvalues and eigenvectors of our matrix must satisfy

$$\alpha w(t+1) + (1-2\alpha)w(t) + \alpha w(t-1) = \lambda w(t) \tag{3}$$

with boundary conditions that $w(0) = 0$ and $w(N) = 0$. Then the above equation can be written as

$$w(t+1) + (\frac{1-\lambda}{\alpha} - 2)w(t) + w(t-1) = 0 \tag{4}$$

Defining $\mu = \frac{1-\lambda}{\alpha}$ and substituting $w(t) = m^t$ into the above we get

$$m^2 + (\mu - 2)m + 1 = 0 \tag{5}$$

Solving this quadratic equation for m gives

$$m = \frac{-(\mu-2) \pm \sqrt{(\mu-2)^2 - 4}}{2} \tag{6}$$

From this expression if $|\mu - 2| \geq 2$ the expression under the square root is positive and the two roots are both *real*. With two real roots, the only solution that satisfies the boundary conditions is the trivial one ($w(t) = 0$). If $|\mu - 2| < 2$ then m is a complex number and the boundary conditions can be satisfied non-trivially. To further express this, define θ such that

$$\mu - 2 = -2\cos(\theta)$$

then the expression for m (in terms of θ) becomes

$$m = \frac{2\cos(\theta) \pm \sqrt{4\cos(\theta)^2 - 4}}{2} = \cos(\theta) \pm \sqrt{\cos(\theta)^2 - 1} \tag{7}$$

or

$$m = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta} \tag{8}$$

so the solution $w(t)$ is a linear combination of the two fundamental solutions or

$$w(t) = Ae^{i\theta t} + Be^{-i\theta t} \tag{9}$$

Imposing the two homogeneous boundary condition we have the following system that must be solved for A and B

$$A + B = 0 \tag{10}$$

$$Ae^{i\theta N} + Be^{-i\theta N} = 0 \tag{11}$$

Putting the first equation into the second gives

$$B(e^{i\theta N} - e^{-i\theta N}) = 0 \tag{12}$$

Since B cannot be zero (else the eigenfunction is identically zero) we must have θ satisfy

$$\sin(\theta N) = 0 \tag{13}$$

Thus $\theta N = \pi n$ or

$$\theta = \frac{\pi n}{N} \qquad \text{for} \qquad n = 1, 2, \ldots, N-1$$

Tracing θ back to the definition of μ we have that

$$\mu = 2 - 2\cos(\theta) = 2 - 2\cos(\frac{\pi n}{N}) \tag{14}$$

Using the trigonometric identity

$$1 - \cos(\psi) = 2\sin(\frac{\psi}{2})^2$$

we get

$$\mu = 2 \cdot 2\sin(\frac{\pi n}{2N})^2 \qquad \text{for} n = 1, 2, 3, \ldots, N-1 \tag{15}$$

Further tracing μ back to the definition of λ we have

$$\lambda_n = 1 - \alpha\mu_n = 1 - 4\alpha\sin(\frac{\pi n}{2N})^2 \qquad \text{for} n = 1, 2, 3, \ldots, N-1 \tag{16}$$

With this expression for the eigenvalues we can explicitly solve for the unknowns $y(i, \cdot)$ at every time-level j. Expressing the unknowns at every time level in a vector $v(j)$ as

$$v(j) = y(\cdot, j) = \begin{bmatrix} y(1,j) \\ y(2,j) \\ \vdots \\ y(N-1,j) \end{bmatrix} \tag{17}$$

Then by decomposing the coefficient matrix A into a basis spanned by its eigenvalues as $A = M^{-1}\Lambda M$ and defining $b(j) = Mv(j)$ we see that $b(j)$ satisfies

$$b(j+1) = \Lambda b(j)\,. \tag{18}$$

Since Λ is diagonal the solution to the above difference equation is given by

$$\begin{bmatrix} b(1,j) \\ b(2,j) \\ \vdots \\ b(N-1,j) \end{bmatrix} = \begin{bmatrix} \lambda_1^j b(1,0) \\ \lambda_2^j b(2,0) \\ \vdots \\ \lambda_{N-1}^j b(N-1,0) \end{bmatrix} \tag{19}$$

we can obtain the solution to the components of $v(j)$ by premultiplying by M or

$$\begin{bmatrix} y(1,j) \\ y(2,j) \\ \vdots \\ y(N-1,j) \end{bmatrix} = M \begin{bmatrix} b(1,0)\lambda_1^j \\ b(2,0)\lambda_2^j \\ \vdots \\ b(N-1,0)\lambda_{N-1}^j \end{bmatrix} \tag{20}$$

To guarantee stability of these matrix iterations we require $|\lambda_n| < 1$ which will be true if

$$|1 - 4\alpha \sin(\frac{n\pi}{2N})^2| < 1 \quad \text{for} \quad n = 1, 2, \ldots, N-1 \tag{21}$$

which is equivalent to

$$-1 \leq 1 - 4\alpha \sin(\frac{n\pi}{2N})^2 \leq 1 \tag{22}$$

or

$$-2 \leq -4\alpha \sin(\frac{n\pi}{2N})^2 \leq 0 \tag{23}$$

or

$$\frac{k}{h^2} \sin(\frac{n\pi}{2N})^2 < \frac{1}{2} \quad \text{for} \quad n = 1, 2, \ldots, N-1 \tag{24}$$

Since the maximum of $\sin(\frac{n\pi}{2N})$ over n is when $n = N - 1$ we see that for stability we must have

$$\frac{k}{h^2} \sin(\frac{\pi}{2} \frac{(N-1)}{N})^2 < \frac{1}{2} \tag{25}$$

Which is equation 10.8 in the book.

Problem Solutions Chapter 10

Problem 1

We begin by noting that the expression

$$y(i, j+1) = \frac{1}{2} y(i, j) + \frac{1}{4}(y(i+1, j) + y(i-1, j)) \tag{26}$$

with $y(0, j) = y(4, j) = 0$ and $y(i, 0) = \sin(\frac{i\pi}{4})$ is a special case of the problem considered on Page 403 (equation 10.6) of the book with $\alpha = \frac{1}{4}$ and $N = 4$. Now defining all the unknowns $y(i, j)$ for $i = 1, 2, 3$ at a given time-level j as the vector unknown $v(j)$ we have

$$v(j) = \begin{bmatrix} y(1, j) \\ y(2, j) \\ y(3, j) \end{bmatrix} \tag{27}$$

and

$$A = \begin{bmatrix} 1/2 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/4 & 1/2 \end{bmatrix} \tag{28}$$

Then the unknowns at a new time level j in terms of the previous time level $j - 1$ is given by $v(j) = Av(j-1)$. This vector difference equation has solution

$$v(j) = A^j v^0 \tag{29}$$

which can be simplified with an eigendecomposition of A (i.e. $A = M^{-1} \Lambda M$) as follows. Here M is a matrix with columns representing the eigenvalues of A and Λ is a diagonal matrix who's diagonal elements are the eigenvalues of A. Assuming this decomposition of A Eq. 29 becomes

$$v(j) = (M^{-1}\Lambda M)(M^{-1}\Lambda M)(M^{-1}\Lambda M)\dots(M^{-1}\Lambda M)v^0 = M^{-1}\Lambda^j M v^0 \tag{30}$$

Where we have j products in the above expression. For this problem the initial vector v^0 is given by evaluating $y(i,0) = \sin(\frac{i\pi}{4})$ for $i = 1, 2, 3$ giving

$$v^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{31}$$

From the analogy with the equation on page 404 we see that the eigenvalues of A are given by (using $\alpha = 1/4$ and $N = 4$)

$$\begin{aligned} \lambda_n &= 1 - 4\alpha\sin(\frac{n\pi}{2N})^2 && (32)\\ &= 1 - \sin(\frac{n\pi}{8})^2 && (33)\\ &= \frac{1}{2} + \frac{1}{2}\cos(\frac{n\pi}{4}) \qquad \text{for} \qquad n = 1, 2, 3 && (34) \end{aligned}$$

Where the last expression follows from the trigonometric identity

$$\sin(\theta)^2 = \frac{1 - \cos(2\theta)}{2}$$

which upon evaluation gives

$$\begin{aligned} \lambda_1 &= \frac{1}{2}(1 + \cos(\frac{\pi}{4})) = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) = \frac{(\sqrt{2}+1)}{2\sqrt{2}} \\ \lambda_2 &= \frac{1}{2}(1 + \cos(\frac{2\pi}{4})) = \frac{1}{2}(1 + 0) = \frac{1}{2} \\ \lambda_3 &= \frac{1}{2}(1 + \cos(\frac{3\pi}{4})) = \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) = \frac{(\sqrt{2}-1)}{2\sqrt{2}} && (35) \end{aligned}$$

while a matrix M with eigenvectors as columns is given by

$$\begin{aligned} M &= \begin{bmatrix} \sin(\frac{\pi}{4}) & \sin(\frac{2\pi}{4}) & \sin(\frac{3\pi}{4}) \\ \sin(\frac{2\pi}{4}) & \sin(\frac{4\pi}{4}) & \sin(\frac{6\pi}{4}) \\ \sin(\frac{3\pi}{4}) & \sin(\frac{6\pi}{4}) & \sin(\frac{9\pi}{4}) \end{bmatrix} && (36)\\ &= \begin{bmatrix} \sin(\frac{\pi}{4}) & \sin(\frac{\pi}{2}) & \sin(\frac{3\pi}{4}) \\ \sin(\frac{\pi}{2}) & \sin(\pi) & \sin(\frac{3\pi}{2}) \\ \sin(\frac{3\pi}{4}) & \sin(\frac{3\pi}{2}) & \sin(\frac{9\pi}{4}) \end{bmatrix} && (37)\\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} && (38) \end{aligned}$$

Now defining $b(j) = Mv(j)$ we see from Eq. 30 that the vector $b(j)$ satisfies $b(j) = \Lambda^j b^0$. So calculating b^0 we obtain

$$b^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

so from Eq. 30 and Eqs. 35 $b(j)$ is given by

$$b(j) = \begin{bmatrix} \frac{(\sqrt{2}+1)^j}{2^j\sqrt{2}^j} & 0 & 0 \\ 0 & \frac{1}{2^j} & 0 \\ 0 & 0 & \frac{(\sqrt{2}-1)^j}{2^j\sqrt{2}^j} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{(\sqrt{2}+1)^j}{2^{j-1}\sqrt{2}^j} \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

then $v(j)$ is obtained from $b(j)$ by premultiplying by M^{-1}. Since M^{-1} is given by

$$M^{-1} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \tag{41}$$

so we get

$$v(j) = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} & \frac{1}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{(\sqrt{2}+1)^j}{2^{j-1}\sqrt{2}^j} \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

$$= \frac{(\sqrt{2}+1)^j}{2^{j-1}\sqrt{2}^j} \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2} \\ \frac{1}{2\sqrt{2}} \end{bmatrix} \tag{43}$$

$$= \frac{(\sqrt{2}+1)^j}{2^j\sqrt{2}^j} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \tag{44}$$

Which is equivalent to the expression given at the back of the book.

Problem 2

The continuous equation to discretize is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{45}$$

defining a discrete representation of u as $y(i,j) = u(x_i, t_j) = u(ih, jk)$ and using the approximations for the derivatives provided in section 10.1 of the book we get

$$\frac{y(i,j) - y(i,j-1)}{k} + O(k) = \frac{y(i+1,j) - 2y(i,j) + y(i-1,j)}{h^2} + O(h^2) \tag{46}$$

dropping the order symbols and solving for $y(i, j-1)$ we obtain

$$y(i,j-1) = (1 + \frac{2k}{h^2})y(i,j) - \frac{k}{h^2}(y(i+1,j) + y(i-1,j)) \tag{47}$$

which is equation 10.9 in our book.

Problem 3

In problem 2 above we derived the partial difference equation $y(i, j)$ must satisfy. In that expression defining $\alpha = \frac{k}{h^2}$, $v(j)$, and $v(0)$ as

$$v(j) = \begin{bmatrix} y(1,j) \\ y(2,j) \\ \vdots \\ y(N-1,j) \end{bmatrix} \tag{48}$$

and

$$v(0) = \begin{bmatrix} y(1,0) \\ y(2,0) \\ \vdots \\ y(N-1,0) \end{bmatrix} = \begin{bmatrix} f(1) \\ f(2) \\ \cdots \\ f(N-1) \end{bmatrix} \tag{49}$$

we obtain the matrix difference equation $v(j-1) = Bv(j)$ or $v(j) = B^{-1}v(j-1)$ for $j = 1, 2, 3, \cdots$.

In general, the matrix difference equation $v(i-1) = Bv(i)$, or $v(i) = B^{-1}v(i-1)$ has explicit solutions depending on the eigenvalues of B^{-1}. Since for nonsingular matrices B the eigenvalues of B^{-1} are the reciprocals of the eigenvalues of B. In this problem rather than start with the expression for B^{-1} we will first consider the eigenvalues of B. From the solution above the eigenvalues of B must satisfy

$$-\alpha w(t+1) + (1+2\alpha)w(t) - \alpha w(t-1) = \lambda w(t) \tag{50}$$

with boundary conditions that $w(0) = 0$ and $w(N) = 0$. Here we have written the components of the vector $v(j)$ as $w(t)$. Then the above equation can be written as

$$w(t+1) - (2 - \frac{\lambda - 1}{\alpha})w(t) + w(t-1) = 0 \tag{51}$$

Defining $2\mu = \frac{\lambda-1}{\alpha}$ and substituting $w(t) = m^t$ into the above we get a characteristic equation of

$$m^2 - 2(1-\mu)m + 1 = 0\,. \tag{52}$$

Solving this quadratic equation for m gives

$$m = \frac{2(1-\mu) \pm \sqrt{4(1-\mu)^2 - 4}}{2} = 1 - \mu \pm \sqrt{(1-\mu)^2 - 1}\,. \tag{53}$$

From this expression if $|1-\mu| \geq 1$ the expression under the square root is positive and the two roots are both *real*. With two real roots, the only solution that satisfies the boundary conditions is the trivial one ($w(t) = 0$). If $|1-\mu| < 1$ then m is a complex number and the boundary conditions can be satisfied non-trivially. To further express this, define θ such that

$$1 - \mu = \cos(\theta)$$

then the expression for m (in terms of θ) becomes

$$m = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta} \tag{54}$$

so the solution $w(t)$ is a linear combination of the two fundamental solutions or

$$w(t) = Ae^{i\theta t} + Be^{-i\theta t} \tag{55}$$

Imposing the two homogeneous boundary conditions ($w(0) = 0$ and $w(N) = 0$) we have the following system that must be solved for A and B

$$A + B = 0 \tag{56}$$

$$Ae^{i\theta N} + Be^{-i\theta N} = 0 \tag{57}$$

Putting the first equation into the second gives

$$B(e^{i\theta N} - e^{-i\theta N}) = 0 \tag{58}$$

Since the coefficient B cannot be zero (else the eigenfunction is identically zero) we must have θ satisfy

$$\sin(\theta N) = 0 \tag{59}$$

Thus $\theta N = \pi n$ or

$$\theta = \frac{\pi n}{N} \quad \text{for} \quad n = 1, 2, \ldots, N-1$$

A point that is often confusing is the range of n in the above expression. Note that if the range of n was any larger than $1, 2, 3, \ldots, N-1$ due to the periodicity of the $\sin(\cdot)$ function eigenvalues would start to repeat. Thus the range specified above is maximal. Tracing θ back to the definition of μ we have that

$$\mu = 1 - \cos(\theta) = 1 - \cos(\frac{\pi n}{N}) \tag{60}$$

Using the trigonometric identity

$$1 - \cos(\psi) = 2\sin(\frac{\psi}{2})^2$$

we get

$$\mu = 2\sin(\frac{\pi n}{2N})^2 \quad \text{for} \quad n = 1, 2, 3, \ldots, N-1 \tag{61}$$

Further tracing μ back to the definition of λ we have

$$\lambda_n(B) = 1 + 2\alpha\mu_n = 1 + 4\alpha\sin(\frac{\pi n}{2N})^2 \quad \text{for} \quad n = 1, 2, 3, \ldots, N-1 \tag{62}$$

Which we can see never is zero. Implying that our matrix B is not singular. In addition, the eigenvalues of B^{-1} are the reciprocals of those of B.[1]

$$\lambda_n(B^{-1}) = \frac{1}{1 + 4\alpha\sin(\frac{\pi n}{2N})^2} \quad \text{for} \quad n = 1, 2, 3, \ldots, N-1 \tag{63}$$

[1]This is a good trick if you are ever asked to compute the eigenvalues of the inverse of a matrix.

In the above two expressions I have explicitly include an argument of B or B^{-1} to indicate which matrix the eigenvalues correspond to.

With this expression for the eigenvalues of B^{-1} we can explicitly solve for the unknowns $y(i,\cdot)$ at every time-level j. By eigendecomposing the coefficient matrix B^{-1} as $B^{-1} = M^{-1}\Lambda M$ and defining the vector $b(j)$ as $b(j) = Mv(j)$ we see that $b(j)$ satisfies

$$b(j+1) = \Lambda(B^{-1})b(j)\,. \tag{64}$$

Since Λ is diagonal the solution to the above difference equation is given by

$$\begin{bmatrix} b(1,j) \\ b(2,j) \\ \vdots \\ b(N-1,j) \end{bmatrix} = \begin{bmatrix} \lambda_1^{-j} b(1,0) \\ \lambda_2^{-j} b(2,0) \\ \vdots \\ \lambda_{N-1}^{-j} b(N-1,0) \end{bmatrix} \tag{65}$$

we can obtain the solution to the components of $v(j)$ by premultiplying by M or

$$\begin{bmatrix} y(1,j) \\ y(2,j) \\ \vdots \\ y(N-1,j) \end{bmatrix} = M \begin{bmatrix} b(1,0)\lambda_1^{-j} \\ b(2,0)\lambda_2^{-j} \\ \vdots \\ b(N-1,0)\lambda_{N-1}^{-j} \end{bmatrix} \tag{66}$$

We have yet expressed the matrix of eigenvectors M. Again B^{-1} is the inverse of B and as such has the *same* eigenvectors as B. As such since the eigenvector solutions $w(t)$ were found to be

$$w(t) = e^{i\frac{\pi n}{N}t} - e^{-i\frac{\pi n}{N}t} \propto \sin(\frac{\pi n}{N}) \tag{67}$$

We have that the eigenvectors of B (and B^{-1}) are

$$w_n(t) = \sin(\frac{\pi n}{N}t) \quad \text{for} \quad i = 1, 2, 3, \ldots, N-1 \tag{68}$$

To guarantee stability of these matrix iterations we require $|\lambda_n^{-1}| < 1$ which is the same as

$$\frac{1}{1 + 4\alpha \sin(\frac{\pi n}{2N})^2} < 1 \quad \text{for} \quad n = 1, 2, \ldots, N-1 \tag{69}$$

which is always true. As such this method is called unconditionally stable.

Problem 4

Using the result from equation 10.3 in the text we obtain

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{z(i+1,j) - 2z(i,j) + z(i-1,j)}{h^2} + O(h^2) \tag{70}$$

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{z(i,j+1) - 2z(i,j) + z(i,j-1)}{k^2} + O(k^2) \tag{71}$$

with $z(i,j) \equiv u(ih, jk)$ which upon substitution into Laplace's equation gives

$$\frac{1}{h^2}\left[z(i+1,j) - 2z(i,j) + z(i-1,j)\right] + \frac{1}{k^2}\left[z(i,j+1) - 2z(i,j) + z(i,j-1)\right] = 0 \quad (72)$$

Solving for $z(i,j)$ we obtain

$$2\left[\left(\frac{h}{k}\right)^2 + 1\right] z(i,j) = z(i+1,j) + z(i-1,j) + \left(\frac{h}{k}\right)^2 (z(i,j+1) + z(i,j-1)) \quad (73)$$

Which is the equation 10.11 in the text.

Problem 5

Part (a): Since

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_j) = \frac{y(i,j+1) - 2y(i,j) + y(i,j-1)}{k^2} + O(k^2) \quad (74)$$

and

$$\frac{\partial^2 u}{\partial x^2}(x_i, t_j) = \frac{y(i+1,j) - 2y(i,j) + y(i-1,j)}{h^2} + O(h^2) \quad (75)$$

with $y(i,j) \equiv u(x_i, t_j) = u(ih, jk)$. Putting these two discrete approximations into the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (76)$$

we obtain

$$y(i,j+1) = 2y(i,j) - y(i,j-1) + \left(\frac{k}{h}\right)^2 (y(i+1,j) - 2y(i,j) + y(i-1,j)) \quad (77)$$

Solving for $y(i,j+1)$, the next time level in terms of the previous timelevels and defining $\alpha = \frac{k}{h}$ we obtain

$$y(i,j+1) = 2(1-\alpha^2)y(i,j) + \alpha^2(y(i+1,j) + y(i-1,j)) - y(i,j-1) \quad (78)$$

Which is the equation requested.

Part (b): This computational molecule looks identical to the one given for Laplace's equation in Figure 10.5 of the book.

Problem 6

As suggested in the text for equations of the form

$$y(i,j) = p(i)y(i+a, j+b) \quad (79)$$

we can try a substitution of the following form

$$y(i,j) = z(i)f(aj - bi) \tag{80}$$

In the problem given here we have $a = 2$ and $b = 1$ giving the substitution to make of $y(i,j) = z(i)f(2j - i)$. When this is inserted into the given difference equation and the common function f canceled from both sides the following ordinary difference equation results

$$z(i) = 4z(i+2) \qquad \text{or} \qquad z(i+2) = \frac{1}{4}z(i) \tag{81}$$

Which can be solved by iteration. This difference equation has two linearly independent solutions given by

$$z_1(i) = \begin{cases} \frac{1}{4}^{i/2} & i = 0, 2, 4, \ldots \\ 0 & i = 1, 3, 5, \ldots \end{cases} \tag{82}$$

$$z_2(i) = \begin{cases} 0 & i = 0, 2, 4, \ldots \\ \frac{1}{4}^{(i-1)/2} & i = 1, 3, 5, \ldots \end{cases} \tag{83}$$

Thus our total solution is given by

$$y(i,j) = Az_1(i)f(2j - i) + Bz_2(i)f(2j - i) \tag{84}$$

with A and B arbitrary constants and f an arbitrary function.

Problem 7

Consider the given difference equation

$$y(i,j) = 2y(i-1, j-1) + 3^i \tag{85}$$

We first find a solution to the homogeneous equation

$$y(i,j) = 2y(i-1, j-1) \tag{86}$$

As such we can use the substitution

$$y(i,j) = z(i)f(aj - bi) \tag{87}$$

Which in our case is $y(i,j) = z(i)f(-j + i)$ since $a = -1$ and $b = 1$. Putting this in the above and canceling the common f on both sides we obtain the following ordinary difference equation

$$z(i) = 2z(i-1) \tag{88}$$

Which has fundamental solution $z(i) = 2^i$. Thus a solution to the homogeneous equation above is given by

$$y(i,j) = 2^i f(-j + i) \tag{89}$$

for an arbitrary function f. To find a non-homogeneous solution we use the observation that the right hand side is a function of only i and thus look for solutions of the form $y(i,j) = z(i)$. Putting this into our difference equation 85 we obtain

$$z(i) = 2z(i-1) + 3^i \quad \text{or} \quad z(i+1) - 2z(i) = 3^{i+1} \tag{90}$$

This can be solved by inspection by noting that if $z(i+1) = 3^{i+2}$ the above equation is satisfied. Another method is to define the operator E as $Ez(i) = z(i+1)$ and then the above equation becomes

$$(E-2)z(i) = 3^{i+1} \tag{91}$$

or

$$\begin{aligned} z(i) &= \frac{1}{E-2}3^{i+1} = -\frac{1}{2}\frac{1}{1-\frac{1}{2}E}3^{i+1} && (92)\\ &= -\frac{1}{2}\sum_{k=0}^{\infty}\frac{E^k}{2^k}3^{i+1} = -\frac{1}{2}\sum_{k=0}^{\infty}\frac{3^{i+k+1}}{2^k} && (93)\\ &= -\frac{3^{i+1}}{2}\sum_{k=0}^{\infty}\left(\frac{3}{2}\right)^k = -\frac{3^{i+1}}{2}\frac{1}{1-\frac{3}{2}} = 3^{i+1} && (94) \end{aligned}$$

Where we have *formally* summed the infinite series above. With this particular solution we obtain a total solution of

$$y(i,j) = 2^i f(-j+i) + 3^{i+1} \tag{95}$$

Problem 8

Given the difference equation

$$y(i,j) = p(i)y(i+a, j+b) + q(i)y(i+c, j+d) \tag{96}$$

the substitution

$$y(i,j) = z(i)f(aj - bi) \tag{97}$$

will reduce the given partial difference equation (for $y(i,j)$) into a ordinary difference equation (for $z(i)$) if a,b,c, and d satisfy $ad - bc = 0$. For the two problems given we have the following:

Part (a): For this specific equation $a = -1$, $b = 3$, $c = -2$, $d = +6$, so $ad - bc = 0$. Applying the above substitution and ignoring the f term which will cancel from both sides i.e. substituting $y(i,j) = z(i)f(-j-3i)$ into the homogeneous equation we obtain the following ordinary difference equation for $z(i)$

$$z(i) = 2z(i-1) - z(i-2) \tag{98}$$

or

$$z(i) - 2z(i-1) - z(i-2) = 0 \tag{99}$$

which has a characteristic equation of

$$m^2 - 2m - 1 = 0 \tag{100}$$

from which we see that $m = 1$ is a double root. Thus two linearly independent solutions to this difference equation are given by

$$\begin{aligned} z_1(i) &= 1 & (101) \\ z_2(i) &= i & (102) \end{aligned}$$

Thus we have a total solution to the above difference equation of

$$y(i,j) = f_1(-j - 3i) + if_2(-j - 3i) \tag{103}$$

For arbitrary functions f_1 and f_2. Since these functions are arbitrary we can absorb the negative sign in the above expressions obtaining

$$y(i,j) = f_1(j + 3i) + if_2(j + 3i) \tag{104}$$

Part (b): For this specific equation $a = -1$, $b = +1$, $c = -2$, $d = +2$, so $ad - bc = 0$. Applying the above substitution to the homogeneous equation and ignoring the f term which will cancel from both sides i.e. substituting $y(i,j) = z(i)f(-j - i)$ we obtain the following ordinary difference equation for $z(i)$

$$z(i) - 5z(i-1) + 6z(i-2) = 0 \tag{105}$$

which has a characteristic equation of

$$m^2 - 5m + 6 = 0 \tag{106}$$

or

$$(m-2)(m-3) = 0 \tag{107}$$

so two linearly independent solutions are given by

$$\begin{aligned} z_1(i) &= 2^i & (108) \\ z_2(i) &= 3^i & (109) \end{aligned}$$

Thus a homogeneous solution to this difference equation is given by

$$y(i,j) = 2^i f_1(i+j) + 3^i f_2(i+j) \tag{110}$$

To find a particular solution we note that since the right hand side is a function of only i we will try a particular solution that is a function of only i. Motivated by the method of undetermined coefficients we attempt a particular solution of the following form

$$y(i,j) = Ai + B \tag{111}$$

when substituted into the given partial difference equation we obtain

$$Ai + B - 5(A(i-1) + B) + 6(A(i-2) + B) = 3i \tag{112}$$

Collecting coefficients of i^1 and i^0 we have the following system of equations to be solved for A and B.

$$\begin{aligned} 2A &= 3 \quad (113) \\ 2B - 7A &= 0\,. \quad (114) \end{aligned}$$

Which gives $A = \frac{3}{2}$ and $B = \frac{21}{4}$. Thus the entire solution to this problem is given by

$$y(i,j) = 2^i f_1(i+j) + 3^i f_2(i+j) + \frac{3}{2}i + \frac{21}{4} \quad (115)$$

Problem 9

Given the difference equation

$$W(n,k) = rW(n-1,k-1) + gW(n-1,k) \quad (116)$$

with initial condition of $W(n,0) = g^n$ we will solve this problem using operator methods in two ways. Defining the operators E_1 and E_2 as

$$\begin{aligned} E_1 W(n,k) &= W(n+1,k) \quad (117) \\ E_2 W(n,k) &= W(n,k+1) \quad (118) \end{aligned}$$

we can write our partial difference equation as

$$W = rE_1^{-1}E_2^{-1}W + gE_1^{-1}W \quad (119)$$

The first method we will use to solve this equation is the simpler of the two and results from recognizing that since our boundary conditions are given when the variable $k = 0$, we desire an to solve for $E_2 W(n,k)$ in terms of the operator E_1 if possible. From the above expression we obtain (after multiplying by E_1 on both sides) the equation

$$E_1 W = rE_2^{-1}W + gW = r(\frac{g}{r} + E_2^{-1})W \quad (120)$$

the solution of which is given by

$$W(n,k) = r^n(\frac{g}{r} + E_2^{-1})^n \tilde{W}(k) \quad (121)$$

Where $\tilde{W}(k)$ is (at this point) an arbitrary function of the variable k. Using the binomial theorem to expand the term $(\cdot)^n$ we obtain

$$W(n,k) = r^n \sum_{l=0}^{n} \binom{n}{l} \left(\frac{g}{r}\right)^l (E_2^{-1})^{n-l}\, \tilde{W}(k) \quad (122)$$

or performing the E_2^{-1} operation we obtain

$$W(n,k) = r^n \sum_{l=0}^{n} \binom{n}{l} \left(\frac{g}{r}\right)^l \tilde{W}(k-n+l) \quad (123)$$

Evaluating this expression at $k = 0$ and assigning to the known initial conditions gives

$$W(n,0) = g^n = r^n \sum_{l=0}^{n} \binom{n}{l} \left(\frac{g}{r}\right)^l \tilde{W}(-n+l) \tag{124}$$

or

$$\left(\frac{g}{r}\right)^n = \sum_{l=0}^{n} \binom{n}{l} \left(\frac{g}{r}\right)^l \tilde{W}(-n+l) \tag{125}$$

Since in this sum the *last* term (when $l = n$) is the same as the left hand side we can obtain an equality if we take $\tilde{W}$ to be a delta function picking out this last element. Specifically let

$$\tilde{W}(-n+l) = \delta_{0,-n+l} \tag{126}$$

This gives for $W(n,k)$ the following

$$W(n,k) = r^n \sum_{l=0}^{n} \binom{n}{l} \left(\frac{g}{r}\right)^l \delta_{0,k-n+l} \tag{127}$$

Letting $l = n - k$ (the only nonzero term in the above) we obtain

$$W(n,k) = r^n \binom{n}{n-k} \left(\frac{g}{r}\right)^{n-k} = r^k g^{n-k} \binom{n}{n-k} . \tag{128}$$

Which is the same expression given in the book.

We might be motivated to solve this equation in a slightly different way with the following observation. Since we are given our initial condition with respect to k i.e. $W(n,0) = g^n$ it might be better to derive an "increment" equation in the k variable rather than the n variable. Doing so would give an expression of the following form

$$E_2 W(n,k) = \mathcal{A} W(n,k) \qquad \text{or} \qquad W(n,k) = \mathcal{A}^k f(n)$$

for some "object" $\mathcal{A}$ and some function $\tilde{W}(\cdot)$. The initial condition we are given then imply that would have $\tilde{W}(n) = g^n$ and for $W(n,k)$ the expression

$$W(n,k) = \mathcal{A}^k g^n$$

This alternative method can be formulated as follows. Solving Eq. 119 for $E_2 W$ we obtain

$$E_2 W(n,k) = r(1 - gE_1^{-1})^{-1} E_1^{-1} W(n,k) \tag{129}$$

or

$$E_2 W(n,k) = r(E_1 - g)^{-1} W(n,k) \tag{130}$$

Which has as its solution the following

$$W(n,k) = r^k (E_1 - g)^{-k} \tilde{W}(n) \tag{131}$$

which since $W(n,0) = g^n$ we obtain $\tilde{W}(n) = g^n$ and thus

$$W(n,k) = r^k (E_1 - g)^{-k} g^n = r^k \frac{1}{(E_1 - g)^k} g^n \tag{132}$$

We must now determine how to evaluate expressions like

$$\left(\frac{1}{E_1 - g}\right)^k g^n \tag{133}$$

We will derive a general expression for such expressions. To determine the solution ($X(n)$) to this (for $k = 1$) we recognized that it must satisfy (by definition)

$$\left(\frac{1}{E_1 - g}\right) g^n = X(n)$$

which is the same as

$$E_1 X(n) - g X(n) = g^n \tag{134}$$

and we are seeking a particular solution to the above equation. Since the homogeneous equation has solution g^n which is the same as the in homogeneous term the particular solution will be proportional to ng^n. With this ansatz we see that a particular solution is given by $X(n) = ng^{n-1}$. Thus we have the following

$$\left(\frac{1}{E_1 - g}\right) g^n = ng^{n-1} \tag{135}$$

Now for the second application of the operator $\frac{1}{E_1 - g}$ we see that

$$\left(\frac{1}{E_1 - g}\right)^2 g^n = \left(\frac{1}{E_1 - g}\right) ng^{n-1} \tag{136}$$

which has the same type of solutions as before (proportional to ng^n). We can see that in this case that

$$\frac{n(n-1)}{2} g^{n-2} = \binom{n}{2} g^{n-2}$$

is the solution $X(n)$ to

$$(E_1 - g)^2 X(n) = g^n \tag{137}$$

Generalizing these results by induction we conclude that

$$\left(\frac{1}{E_1 - g}\right)^k g^n = \binom{n}{k} g^{n-k} . \tag{138}$$

With this we get for the solution $W(n, k)$ of

$$W(n, k) = \binom{n}{k} r^k g^{n-k} . \tag{139}$$

The same as before. In general, the technique of writing the inverse of a difference operator (e.g. Eq. 134) as the solution to an inhomogeneous difference equation can be a quite powerful technique that comes up rather often.

Problem 10

Given the partial differential equation

$$y(i+1,j) = ay(i,j+1) + by(i,j)\,. \tag{140}$$

We can solve this with operator methods as follows. Defining E_1 and E_2 as

$$\begin{aligned} E_1 y(n,k) &= y(n+1,k) & (141)\\ E_2 y(n,k) &= y(n,k+1) & (142) \end{aligned}$$

our partial difference equation becomes

$$E_1 y(i,j) = (aE_2 + b) y(i,j) \tag{143}$$

which has solution of

$$y(i,j) = (aE_2+b)^i f(j) = b^i (1 + \frac{a}{b} E_2)^i f(j) \tag{144}$$

for an arbitrary function $f(j)$. Expanding the sum using the binomial expansion we obtain

$$y(i,j) = b^i \sum_{n=0}^{i} \left(\begin{array}{c} i \\ n \end{array} \right) \left(\frac{a}{b}\right)^n E_2^n f(j) \tag{145}$$

or

$$y(i,j) = b^i \sum_{n=0}^{i} \left(\begin{array}{c} i \\ n \end{array} \right) \left(\frac{a}{b}\right)^n f(j+n) \tag{146}$$

Which is the desired result.

Problem 11

Part (a): Under the given problem assumption P wins a point with probability p and Q wins a point with probability $q = 1-p$. We define the function $y(i,j)$ as the probability P wins the game when he/she needs i more points to win, while playing against Q which needs j more points to win. We can derive a difference equation for $y(i,j)$ by noting that after the next play if P has won a point (which happens with probability p) he/she will now need only $i-1$ points to win, while if Q wins the point (with a probability q) then Q will need $j-1$ points to win. This is represented mathematically by

$$y(i,j) = py(i-1,j) + qy(i,j-1) \quad \text{for} \quad i \geq 1 \quad \text{and} \quad j \geq 1 \tag{147}$$

With initial conditions of $y(i,0) = 0$ for $i \geq 1$, and $y(0,j) = 1$ for $j \geq 1$ which says that the probability P wins when Q has needs no more points to win is zero (since Q has already won) and that the probability P wins he/she requires no more points, while Q requires j points is one.

Part (b): Skipped

Problem 12

Given the partial difference equation

$$y(i+1,j+1) + y(i,j) = 2ij\,. \tag{148}$$

The homogeneous equation is given by

$$y(i+1,j+1) + y(i,j) = 0\,, \tag{149}$$

which can be solved by a great number of methods. To solve by the operator method we define

$$E_1 y(i,j) = y(i+1,j) \tag{150}$$
$$E_2 y(i,j) = y(i,j+1) \tag{151}$$

and our our original equation becomes

$$E_1 E_2 y(i,j) = -y(i,j) \tag{152}$$

which has solution

$$y(i,j) = (-E_2)^{-i} f(j) = (-1)^i E_2^{-i} f(j) = (-1)^i f(j-i) \tag{153}$$

One can also use the method on Page 409 since our equation is of the form

$$y(i,j) = p(i) y(i+a, j+b)$$

and we would substitute with $y(i,j) = z(i) f(aj - bi)$ to obtain the same solution. To find a particular solution as suggested in the text we substitute the trial solution

$$y(i,j) = aij + bi + cj + d$$

to obtain the following (here we have grouped the coefficients of ij, i, j, and constant terms together)

$$2aij + (a+2b)i + (a+2c)j + (a+b+c+2d) = 2ij \tag{154}$$

Which enforcing equality among the coefficients gives

$$2a = 2 \tag{155}$$
$$a + 2b = 0 \tag{156}$$
$$a + 2c = 0 \tag{157}$$
$$a + b + c + 2d = 0 \tag{158}$$

The solution of which is

$$a = 1 \tag{159}$$
$$b = -\frac{1}{2} \tag{160}$$
$$c = -\frac{1}{2} \tag{161}$$
$$d = 0 \tag{162}$$

giving for the final solution the expression

$$y(i,j) = (-1)^i f(j-i) + ij - \frac{1}{2}i - \frac{1}{2}j \tag{163}$$

Problem 13

We desire the first few values to the following partial difference equation

$$y(i+1, j+1) = iy(i, j+1) + y(i, j) \tag{164}$$

with initial conditions $y(i, 0) = \delta_{i0}$ and $y(0, j) = \delta_{0j}$. As such, we can iterate the above equation to obtain any number of terms. For instance to obtain $y(1, 1)$ we compute

$$y(1, 1) = 0 \cdot y(0, 1) + y(0, 0) = 1 \tag{165}$$

This procedure for all of the requested i's and j's gives the following grid of values (the i index corresponds to the row and the j index corresponds to the columns each starting from 0)

	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	2	0	0
4	0	1	6	11	6	0
5	0	1	10	35	50	24

Problem 14

Part (a): We desire to show that $Z(2^j \left[{i \atop j}\right]) = \prod_{k=0}^{i-1} \left(k + \frac{2}{z}\right)$, where the Z-transform is taken with respect to j. By definition the Z-transform of this expression is

$$Z\left(2^j \begin{bmatrix} i \\ j \end{bmatrix}\right) = \sum_{j\geq 0} 2^j \begin{bmatrix} i \\ j \end{bmatrix} z^{-j} = \sum_{j\geq 0} \begin{bmatrix} i \\ n \end{bmatrix} \left(\frac{z}{2}\right)^{-n} = Z_{\mathcal{S}}(\frac{z}{2}) \tag{166}$$

where $Z_{\mathcal{S}}$ is the Z transform of the Stirling numbers of the second kind with respect to j. Since we know that

$$Z\left(\begin{bmatrix} i \\ j \end{bmatrix}\right) = Z_{\mathcal{S}} = \prod_{k=0}^{i-1}(k + \frac{1}{z}) \tag{167}$$

we have from the above that

$$Z_{\mathcal{S}}(\frac{z}{2}) = \prod_{k=0}^{i-1}(k + \frac{2}{k}) \tag{168}$$

as requested.

Part (b): Our partial difference equation to solve is given by

$$y(i+1, j+1) = (i-1)y(i, j+1) + 2y(i, j) \quad i \geq 1, j \geq 0 \tag{169}$$

with initial conditions given by $y(i,0) = \delta_{i1}$ and $y(1,j) = \delta_{j0}$. The Z-transform of this equation with respect to j gives

$$zY(i+1,z) - zy(i+1,0) = (i-1)\left(zY(i,z) - zy(i,0)\right) + 2Y(i,z) \tag{170}$$

since $y(i+1,0) = 0$ when $i \geq 1$ since $y(i,0) = \delta_{i1}$ the above becomes

$$Y(i+1,z) = (i-1)Y(i,z) - (i-1)\delta_{i1} + \frac{2}{z}Y(i,z) \quad \text{for} \quad i \geq 1\,. \tag{171}$$

Note that the term $(i-1)\delta_{i1} = 0$ for all $i \geq 1$, giving

$$Y(i+1,z) = \left(i - 1 + \frac{2}{z}\right)Y(i,z) \quad \text{for} \quad i \geq 1 \tag{172}$$

Iterating the above equation a few times gives

$$Y(2,z) = \frac{2}{z}Y(1,z) \tag{173}$$

$$Y(3,z) = \left(1+\frac{2}{z}\right)Y(2,z) = \left(1+\frac{2}{z}\right)\frac{2}{z}Y(1,z) \tag{174}$$

$$Y(4,z) = \left(2+\frac{2}{z}\right)Y(3,z) = \left(2+\frac{2}{z}\right)\left(1+\frac{2}{z}\right)\frac{2}{z}Y(1,z) \tag{175}$$

$$Y(5,z) = \left(3+\frac{2}{z}\right)Y(4,z) = \left(3+\frac{2}{z}\right)\left(2+\frac{2}{z}\right)\left(1+\frac{2}{z}\right)\frac{2}{z}Y(1,z) \tag{176}$$

So by induction we see that

$$Y(i,z) = \prod_{n=0}^{i-2}(n + \frac{2}{z})Y(1,z) \quad \text{for} \quad i \geq 2 \tag{177}$$

To evaluate $Y(1,z)$ we have from its definition

$$\sum_{j\geq 0} y(1,j)z^{-j} = \sum_{j\geq 0}\delta_{j0}z^{-j} = 1$$

So the above becomes

$$Y(i,z) = \prod_{n=0}^{i-2}(n + \frac{2}{z}) \tag{178}$$

Since this is the expression is so similar to the one part (a) of this problem we know that the solution to this difference equation is given by the inverse Z-transform of the above expression. Since this is computed in part (a) of this problem we have

$$y(i,j) = 2^j \begin{bmatrix} i-1 \\ j \end{bmatrix} \quad \text{for} \quad i \geq 2 \quad \text{and} \quad j \geq 0\,. \tag{179}$$