Solutions to Selected Problems in Linear Integral Equations by Rainer Kress

John Weatherwax

Chapter 9

Problem 1

Part (a): From the text Eq. 9.8 is

$$\int_{\mathbb{R}^m} e^{-|z|^2} dz \quad (1)$$

In that expression writing $dz = \prod_{i=1}^m dx_i$ and $|z|^2 = \sum_{i=1}^m x_i^2$ we see that the m nested integral expression defined above can be broken in to m product integrals. Thus the above simplifies to

$$\int_{\mathbb{R}^m} e^{-|z|^2} dz = \prod_{i=1}^m \int_{\mathbb{R}} e^{-x^2} dx \quad (2)$$

The individual integrals above can be recognized as the integral of a Gaussian with the standard result that

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

This combined with Eq. **??** gives

$$\int_{\mathbb{R}^m} e^{-|z|^2} dz = \sqrt{\pi}^m \quad (3)$$

Part (b): From the text Eq. 9.25 is

$$\int_{\sigma}^{t} \frac{d\tau}{(t-\tau)^{\alpha}(\tau-\sigma)^{\beta}} \tag{4}$$

Defining the substitution

$$s = \frac{\tau - \sigma}{t - \sigma}$$

with the differential of

$$ds = \frac{d\tau}{t - \sigma}$$

one obtains $\tau - \sigma = (t - \sigma)s$ and so the integral above becomes

$$\begin{aligned} \int_{\sigma}^{t} \frac{d\tau}{(t-\tau)^{\alpha}(\tau-\sigma)^{\beta}} &= \int_{0}^{1} \frac{(t-\sigma)ds}{(t-\sigma)^{\alpha}(1-s)^{\alpha}(t-\sigma)^{\beta}s^{\beta}} && (5) \\ &= \frac{1}{(t-\sigma)^{\alpha+\beta+1}} \int_{0}^{1} \frac{ds}{s^{\beta}(1-s)^{\alpha}} && (6) \end{aligned}$$

Which is the desired result. Since we have $\beta \leq \alpha < 1$ we have

$$\frac{1}{s^{\beta}} < \frac{1}{s^{\alpha}}$$

so a bound to this integral is given as

$$\frac{1}{(t-\sigma)^{\alpha+\beta+1}} \int_{0}^{1} \frac{ds}{s^{\beta}(1-s)^{\alpha}} < \frac{1}{(t-\sigma)^{\alpha+\beta+1}} \int_{0}^{1} \frac{ds}{(s(1-s))^{\alpha}} \tag{7}$$

Problem 2

The inequality in question is

$$s^{\beta} e^{-2} \leq \beta^{\beta} e^{-\beta} \tag{8}$$

For $0 < s, \beta < \infty$. Under a sequence of transformation all equivalent to each other we have that the above inequality is equivalent to

$$\begin{aligned} (\frac{s}{\beta})^{\beta} &\leq e^{-\beta+s} && (9) \\ \frac{s}{\beta} &\leq e^{-1+\frac{s}{\beta}} && (10) \\ \log(\frac{s}{\beta}) &\leq -1 + \frac{s}{\beta} && (11) \\ 1 &\leq \frac{s}{\beta} - \log(\frac{s}{\beta}) && (12) \end{aligned}$$

Defining the variable $\xi = \frac{s}{\beta}$ and the function $f(\xi) = \xi - \log(\xi)$, from the limits of s and β we see that $0 < \xi < \infty$. Now the function $f(\cdot)$ has limiting values of

$$\begin{aligned} f(0) &= +\infty \quad (13) \\ f(\infty) &= +\infty \quad (14) \end{aligned}$$

and any extremal values at the locations $f'(\xi) = 0$. Computing this first derivative and setting equal to zero we have

$$f'(\xi) = 1 - \frac{1}{\xi} = 0 \qquad \Longrightarrow \qquad \xi = 1 \tag{15}$$

Evaluating the second derivative of $f(\xi)$ we have

$$f''(\xi) = \frac{1}{\xi^2}$$

which evaluated at $\xi = 1$ gives $f''(1) = 1 > 0$. A positive second derivative means that $\xi = 1$ is a local minimum. Thus our function must satisfy

$$f(\xi) \geq 1 \tag{16}$$

or equivalently

$$\frac{s}{\beta} - \log(\frac{s}{\beta}) \geq 1 \tag{17}$$

which was what we desired to show.

Problem 3

Problem skipped

Problem 4

Equation 9.12 in the text is

$$\begin{aligned} u(x,t) &= \int_0^t \frac{a-x}{4\sqrt{\pi}\sqrt{(t-\tau)^3}} e^{-\frac{(x-a)^2}{4(t-\tau)}} \phi(a,\tau) d\tau \quad (18) \\ &+ \int_0^t \frac{x-b}{4\sqrt{\pi}\sqrt{(t-\tau)^3}} e^{-\frac{(x-b)^2}{4(t-\tau)}} \phi(b,\tau) d\tau \quad (19) \end{aligned}$$

Evaluating at $t = 0$ we see that $u(x, 0) = 0$, so this solution satisfies the homogeneous initial conditions. Evaluating at $x = a$ and using the result of Theorem 9.5 we have

$$\int_0^t \frac{a-b}{4\sqrt{\pi}\sqrt{(t-\tau)^3}} e^{-\frac{(a-b)^2}{4(t-\tau)}} \phi(b,\tau) d\tau - \frac{1}{2}\phi(a,\tau) = f(a,t) \tag{20}$$

Now defining $h(t, \tau)$ as suggested (and multiplying by -2) we obtain

$$\phi(a,t) - \int_0^t h(t,\tau)\phi(b,\tau)d\tau = -2f(a,t) \tag{21}$$

Similarly evaluating equation 9.12 at $x = b$ and using the result from Theorem 9.5 we have

$$\int_0^t \frac{a-b}{4\sqrt{\pi}\sqrt{(t-\tau)^3}} e^{-\frac{(a-b)^2}{4(t-\tau)}} \phi(a,\tau) - \frac{1}{2}\phi(b,\tau) = f(b,t) \tag{22}$$

which gives

$$\phi(b,\tau) - \int_0^t h(t,\tau)\phi(a,\tau)d\tau = -2f(b,t) \tag{23}$$

Problem 5

Equation 9.11 in the text is

$$\begin{aligned} u(x,t) &= \int_0^t \frac{a-x}{4\sqrt{\pi}\sqrt{(t-\tau)}} e^{-\frac{(x-a)^2}{4(t-\tau)}} \phi(a,\tau) d\tau & (24) \\ &+ \int_0^t \frac{x-b}{4\sqrt{\pi}\sqrt{(t-\tau)}} e^{-\frac{(x-b)^2}{4(t-\tau)}} \phi(b,\tau) d\tau & (25) \end{aligned}$$

Evaluating at $t = 0$ we see that $u(x, 0) = 0$, so this solution satisfies the homogeneous initial conditions. Performing the required x derivative of this expression gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-a)^2}{4(t-\tau)}} \left(\frac{-2(x-a)}{4(t-\tau)}\right) \phi(a,\tau) d\tau + & (26) \\ & \int_0^t \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-b)^2}{4(t-\tau)}} \left(\frac{-2(x-b)}{4(t-\tau)}\right) \phi(b,\tau) d\tau & (27) \end{aligned}$$

$$= \int_0^t \frac{a-x}{\sqrt{4\pi}(t-\tau)^{3/2}} e^{-\frac{(x-a)^2}{4(t-\tau)}} \phi(a,\tau) d\tau + \tag{28}$$

$$\int_0^t \frac{b-x}{\sqrt{4\pi}(t-\tau)^{3/2}} e^{-\frac{(x-b)^2}{4(t-\tau)}} \phi(b,\tau) d\tau \tag{29}$$

Evaluating this expression at $x = a$ and using the results like that from Theorem 9.5 we obtain

$$\frac{\partial u(a,t)}{\partial x} = \frac{\phi(a,t)}{2} - \int_0^t \frac{1}{2} h(t,\tau)\phi(b,\tau) d\tau = -g(a,t) \tag{30}$$

which upon multiplying by -2 is the first integral equation given. Similarly evaluating this expression at $x = b$ and using results like in Theorem 9.5 we obtain

$$\lim_{x \to b^-} \frac{\partial u(x,t)}{\partial x} = +\frac{1}{2} \int_0^t h(t,\tau)\phi(a,\tau) + \frac{\phi(b,t)}{2} \tag{31}$$

giving

$$\frac{1}{2} \int_0^t h(t,\tau)\phi(a,\tau) + \frac{\phi(b,t)}{2} = g(b,t) \tag{32}$$

which upon multiplying by 2 is the second integral equation given.