

# Mathematical Models of the Real World by Lancaster

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To my brother Doug.

# Introduction

Here you'll find solutions to the problems that I wrote up as I worked through this excellent book. I would say that the problems you will find here are more challenging than the ones found in a typical a first year calculus course. They are a great introduction to some more advanced techniques. For some of the problems I used **R** to perform any needed calculations or plots. Any code snippets for various exercises can be found at the following location:

[http://www.waxworksmath.com/Authors/A\\_F/Ferrar/ferrar.html](http://www.waxworksmath.com/Authors/A_F/Ferrar/ferrar.html)

I've worked hard to make these notes as good as I can, but I have no illusions that they are perfect. If you feel that that there is a better way to accomplish or explain an exercise or derivation presented in these notes; or that one or more of the explanations is unclear, incomplete, or misleading, please tell me. If you find an error of any kind – technical, grammatical, typographical, whatever – please tell me that, too. I'll gladly add to the acknowledgments in later printings the name of the first person to bring each problem to my attention.

# Chapter 1: Mathematical Models

## An Illustrative Example of Inventory Control

### Exercise 1

The answers to these would really depend on

- Whether a solvable mathematical model could be developed under the assumptions.
- The requirement that the mathematical model gives solutions that are “close” to observed reality.

**Part (a):** As a point particle circling another point particle each with different masses.

**Part (b):** As a flat surface on which the ship sails.

**Part (c):** As a large sphere with the helicopter hovering above it.

**Part (d):** As a large sphere with a point particle circling above it.

### Exercise 2

**Part (a):** In this case the new total cost function is

$$c(k) = 100 \left[ 100 + 9k + \frac{140}{k} \right].$$

**Part (b):** In this case the new total cost function is

$$c(k) = 100 \left[ 100 + 9k + \frac{81}{k} \right].$$

The following R code evaluate this expression at different values of  $k$  is given by

```
ks = seq(1, 11)
Cs = 100 * ( 100 + 9*ks + 81/ks )
print(data.frame(k=ks, C=Cs))
```

which gives

	k	C
1	1	19000.00
2	2	15850.00
3	3	15400.00
4	4	15625.00
5	5	16120.00
6	6	16750.00
7	7	17457.14
8	8	18212.50
9	9	19000.00
10	10	19810.00
11	11	20636.36

This has a minimum at  $k = 3$  as stated in the problem. As  $\frac{100}{k} = 33.33333$  is not an integer we might ship 33 out of the warehouse every 33 days and then ship 34 out of the warehouse the last time to still make a total of 100. This should have a cost approximately equal to that of the optimal strategy but allows integer numbers.

### Exercise 3

We would need to know the current distribution of the population i.e. what mix people do we currently have and what is the distribution of people coming into the country. Our mathematical model would be one of modeling the change from the initial population distribution to one that would more closely match the incoming population distributing. Models of these types are commonly called "dynamic population models". A simple model of this type is the logistic equation.

### Exercise 4

Recall that an egg is made up of the "egg yoke" and the "egg white" with the yoke a mostly spherical region in the center of the egg. Assuming a spherical egg, one might model an egg cooking as the change that takes place to the spherical yoke and white outside the yoke but inside the spherical shell of the egg. In this formulation, the modeling is specifically constructed to model things like the density change in the various regions as the cooking takes place. Obviously the chemical process that takes place during the cooking process is complicated and could involve as many modeling assumptions as desired.

# Chapter 2: Linear Programming

## Introductory Example

### Exercise 1

For this expression for  $f$  we find

$$f(kx, ky, kz) = akx + bky + ckz = k(ax + by + cz) = kf(x, y, z),$$

and that

$$\begin{aligned} f(x_1, y_1, z_1) + f(x_2, y_2, z_2) &= ax_1 + by_1 + cz_1 + ax_2 + by_2 + cz_2 \\ &= a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) \\ &= f(x_1 + x_2, y_1 + y_2, z_1 + z_2). \end{aligned}$$

### Exercise 2

**Part (a):** Similar to the above we would have functions of the form

$$\sum_{i=1}^n a_i x_i.$$

**Part (b):** Note that for this  $f$  we have

$$f(kx, ky) = akx + bky + c = k(ax + by) + c \neq kf(x, y),$$

and thus is not a linear function.

**Part (c):** We have

$$f(kx) = (kx)^2 = k^2x^2 \neq kf(x).$$

## Linear Inequalities

### Exercise 1

In each of these plots we have drawn the  $x$ - $y$  Cartesian coordinate axis along with lines representing the decision boundaries. The solution set for each part is drawn in gray.

**Part (a-b):** See Figure 1.

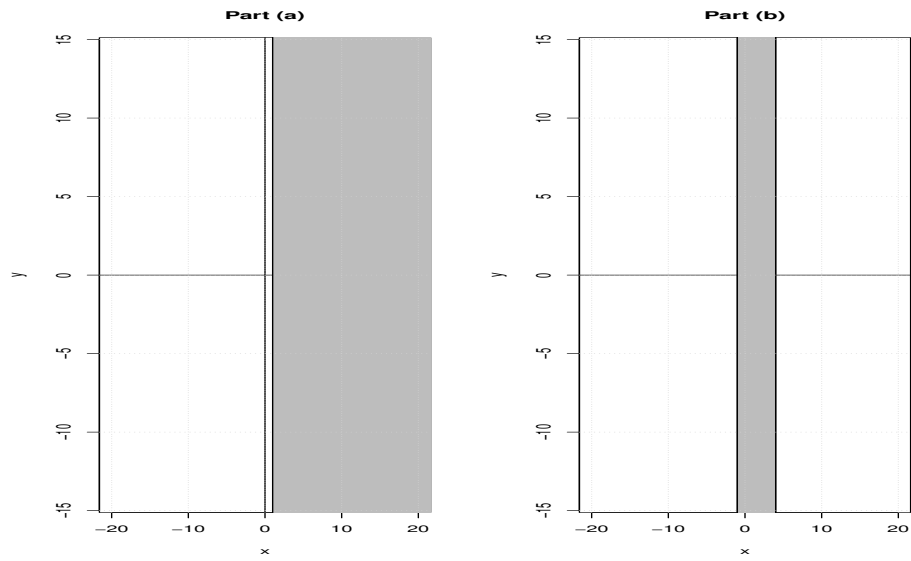


Figure 1: **Left:** The solution set for Part (a). **Right:** The solution set for Part (b).

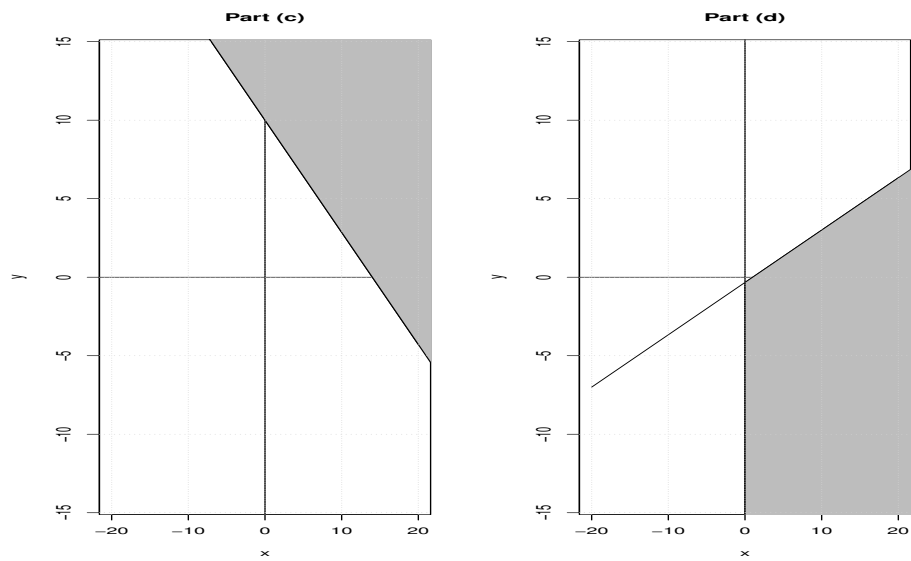


Figure 2: **Left:** The solution set for Part (c). **Right:** The solution set for Part (d).



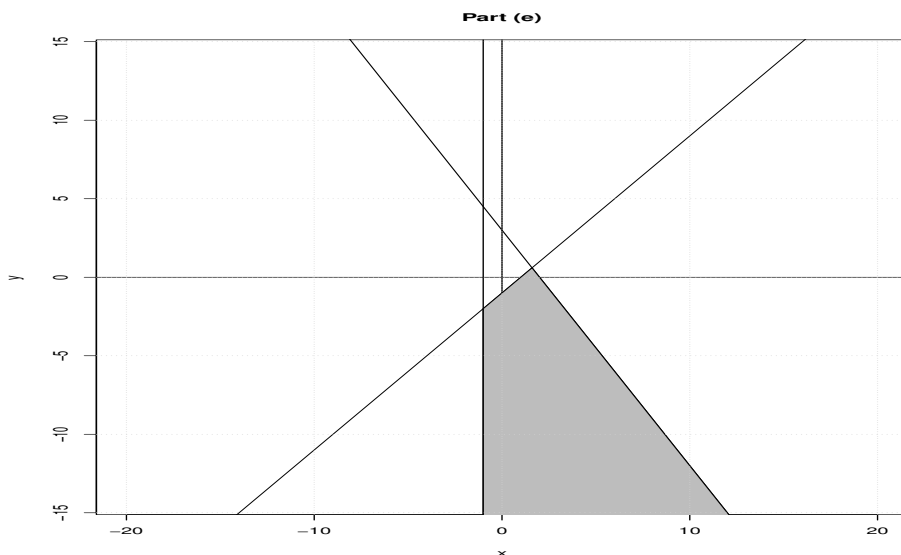


Figure 3: The solution set for Part (e).

**Part (c-d):** See Figure 2.

**Part (e):** See Figure 3.

## Exercise 2

Lets denote these points as  $A = (0, 0)$ ,  $B = (1, 2)$ , and  $C = (-2, 4)$  then

- The line connecting the points  $A$  and  $B$  is given by  $y = \frac{2}{1}x = 2x$ .
- The line connecting the points  $B$  and  $C$  is given by

$$y - 2 = \frac{4 - 2}{(-2 - 1)}(x - 1),$$

which becomes

$$y = -\frac{2}{3}x + \frac{8}{3},$$

when we simplify.

- The line connecting the points  $C$  and  $A$  is given by  $y = \left(\frac{4-0}{-2-0}\right)x = -2x$ .

To denote the *interior* of the triangle we would need to specify inequalities on the "lines" above to be

$$\begin{aligned} y - 2x &\geq 0 \\ 3y + 2x &\leq 0 \\ y + 2x &\geq 0. \end{aligned}$$

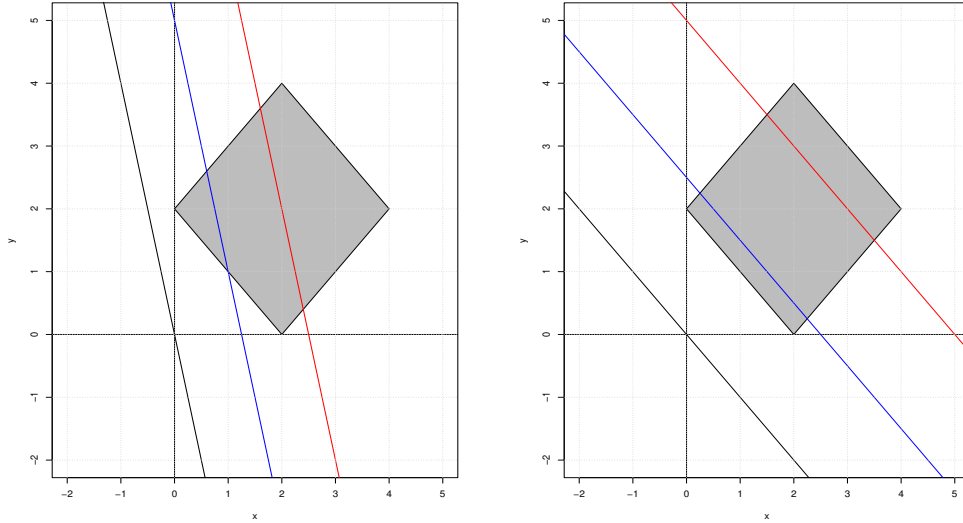


Figure 4: **Left:** The solution set for Part (c). **Right:** The solution set for Part (d).

### Exercise 3

These would be

$$\begin{aligned} x^2 + y^2 &\leq 2^2 \\ x &\leq 0. \end{aligned}$$

## Two Examples

### Exercise 1

**Part (a):** See Figure 4 (left). There we increase the value of the contours of  $P(x, y) = 4x + y = a$  for different values of  $a$  we get the black, blue, and red lines. These lines step to the right as  $a$  increases. From this behavior it looks like the largest value for  $P(x, y)$  will be when it passes through the "right most" corner of the square which is the point  $(x, y) = (4, 2)$ . There the value is  $P(4, 2) = 18$ .

**Part (b):** See Figure 4 (right). There we increase the value of the contours of  $P(x, y) = x + y = a$  for different values of  $a$  we get the black, blue, and red lines. These lines step to the right as  $a$  increases. From this behavior it looks like the smallest value for  $P(x, y)$  will be when it passes through the "bottom-left" side of the square which is a line segment going through the points  $(0, 2)$  and  $(2, 0)$ . That line segment is the line  $y = 2 - x$  for  $0 \leq x \leq 2$  and there is no unique optimum location but the smallest value on this segment is  $P(x, y) = 2$ .

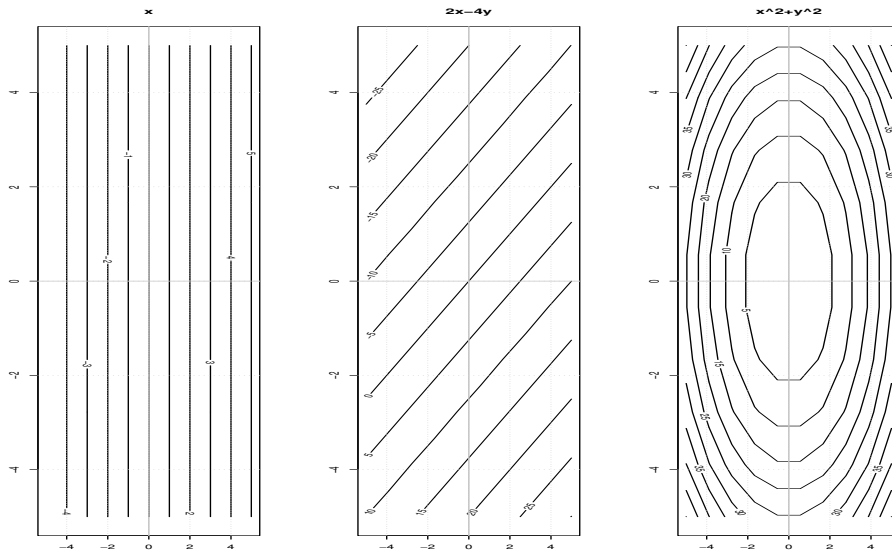


Figure 5: The level curves for  $x$ ,  $2x - 4y$ , and  $x^2 + y^2$ .

## Exercise 2

This can often be done using the `contour` function in R. In the R code `chap_2_sect_4_prob_2.R` we do this for each of the five given expressions. The first three are plotted in Figure 5 and the last two are plotted in Figure 6. Notice that level curves for the first three are vertical lines, diagonal lines, and circles respectively.

Level curves for the second two would be  $\frac{x}{y} = k_0$  (or  $x = k_0y$ ) and  $\frac{x}{y^2} = k_0$  (or  $x = k_0y^2$ ). Notice that in their "ratio" form there will be singularities when  $y = 0$ . Thus the two forms  $x = k_0y$  and  $x = k_0y^2$  would be preferable for analytic purposes.

## Exercise 3

**Part (a):** See Figure 7 where the set  $S$  is plotted in gray.

**Part (b):** In that figure the lines  $f(x, y) = 6x + 2y = k_0$  are plotted for  $k_0 = 7$  (green),  $k_0 = 10$  (blue), and  $k_0 = 15$  (red). Based on this plot we see that the smallest value for  $f$  will be when this line passes through  $(0, 0)$  giving a value of  $f = 0$ . The largest value for this function is when  $(x, y) = (4, 1)$  to give a value of  $f = 26$ .

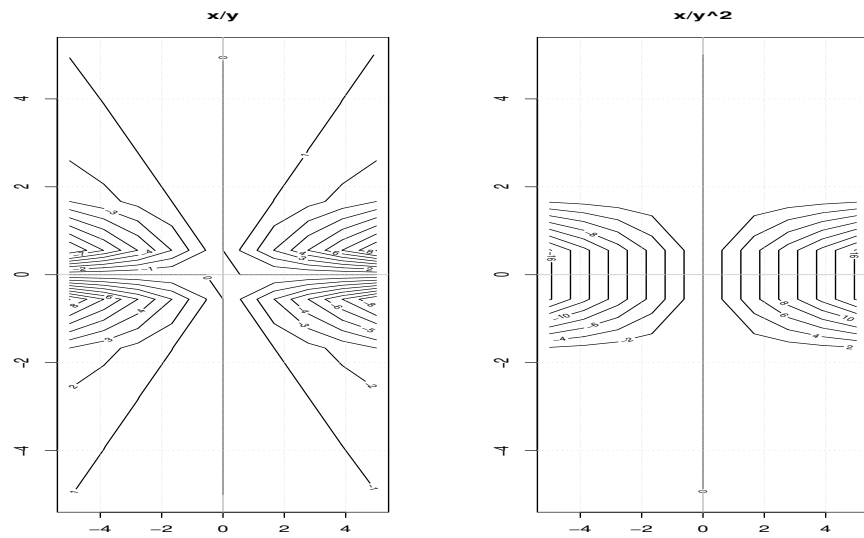


Figure 6: The level curves for  $x/y$  and  $x/y^2$ .

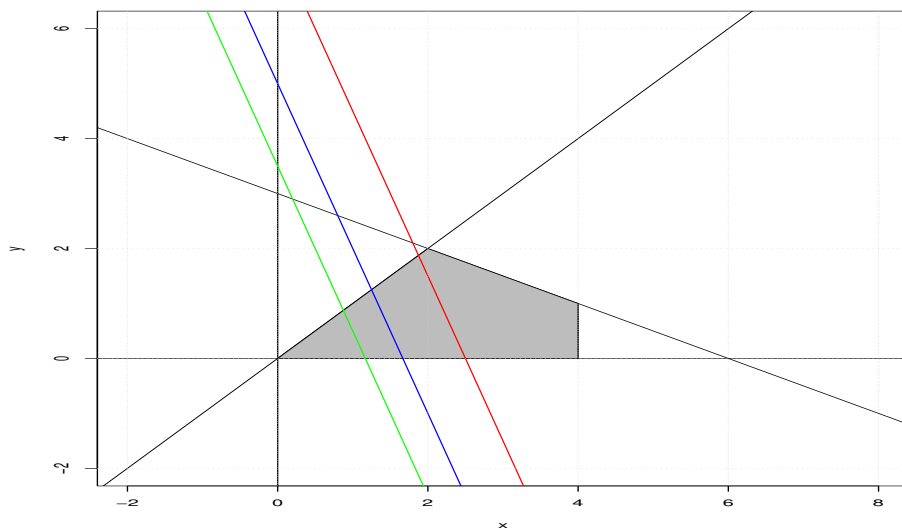


Figure 7: The feasible region and several level set of the function  $f(x, y)$  for Exercise 3.

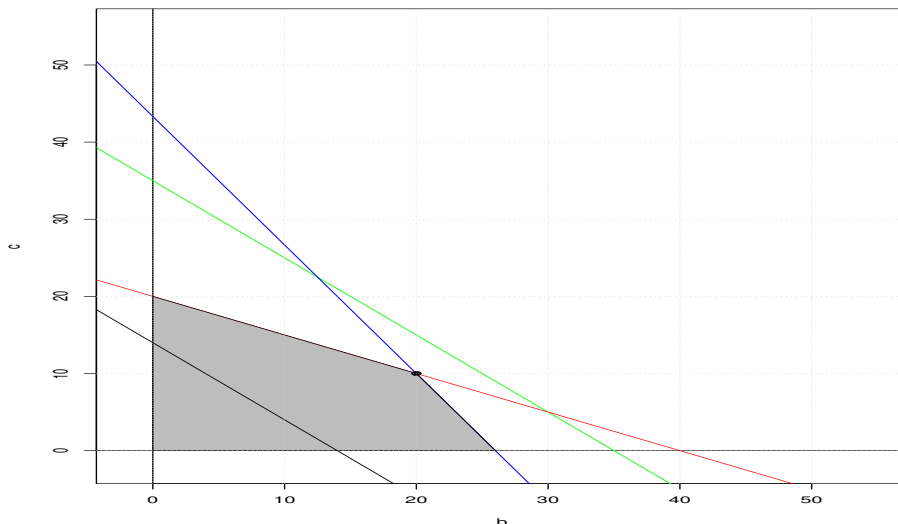


Figure 8: The feasible region and several level set for profit  $P$  for Exercise 4.

#### Exercise 4

Let  $b$  be the acres planted in beets and  $c$  the acres planted in corn. Then from the description of the problem we have the constraints

$$b + c \leq 35 \quad (1)$$

$$50b + 30c \leq 1300 \quad (2)$$

$$2b + 4c \leq 80. \quad (3)$$

Under these we seek to maximize the profit  $P = 100b + 100c$ .

See Figure 8 where this region is plotted. Notice that the green line corresponding to the constraint  $b + c \leq 35$  is never an actual constraint as the feasible region (in gray) is entirely below this line. A single level curve  $P(b, c) = 1400$  is plotted as a black line. Increasing the value of  $P(b, c)$  will correspond to moving that line upwards (as we are increasing the  $c$  intercept).

From this discussion the largest profit corresponds to the  $(b, c)$  at the “corner” of the gray region. This is given by the intersection of the red and blue lines or

$$2b + 4c = 80$$

$$50b + 30c = 1300.$$

The intersection of these two lines is  $(b, c) = (20, 10)$  (shown as a black dot) where we find  $P(20, 10) = 3000$ .

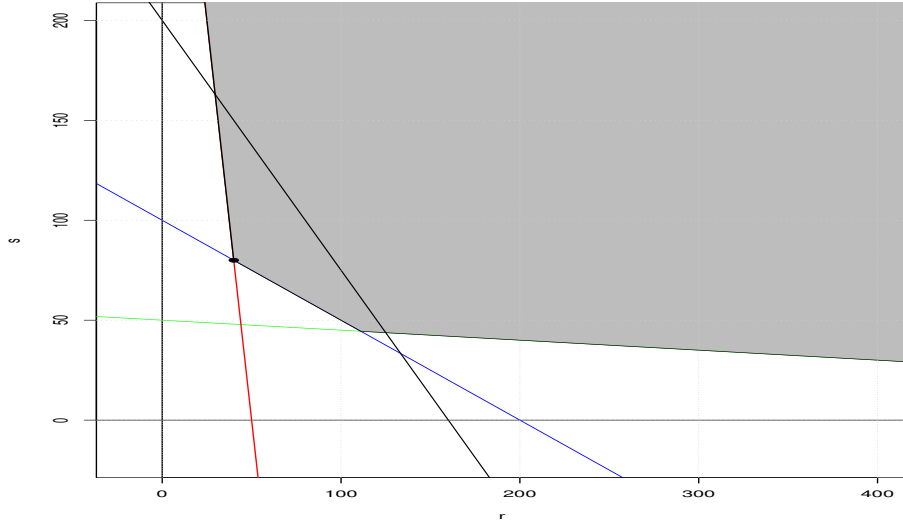


Figure 9: The feasible region and several level set for cost  $C$  for Exercise 5.

### Exercise 5

Let  $r$  be the amount of material  $R_1$  (in tons) and  $s$  be the amount of material  $R_2$  (also in tons). Then from the problem description into how a ton of  $R_1$  (and  $R_2$ ) is broken down into  $M_1$ ,  $M_2$ , and  $M_3$  we have

$$0.03r + 0.6s \geq 30 \quad (4)$$

$$0.125r + 0.25s \geq 25 \quad (5)$$

$$0.4r + 0.05s \geq 20. \quad (6)$$

Under these we seek to minimize the cost  $C = 250r + 200s$ .

See Figure 9 where this region is plotted. In this case the feasible region is the region in the "upper-right" corner of the graph. A single level curve of  $c(r, s) = 40000$  is plotted as a black line. Increasing the value of  $c(r, s)$  will correspond to moving that line upwards (we are increasing the  $s$  intercept). From this discussion the smallest cost corresponds to the  $(r, s)$  at one of the "corners" of the gray region. This is given by the intersection of the red and blue lines or

$$0.125r + 0.25s = 25$$

$$0.4r + 0.05s = 20.$$

The intersection of these two lines is  $(r, s) = (40, 80)$  (shown in black). At that point we find  $c(40, 80) = 26000$ .

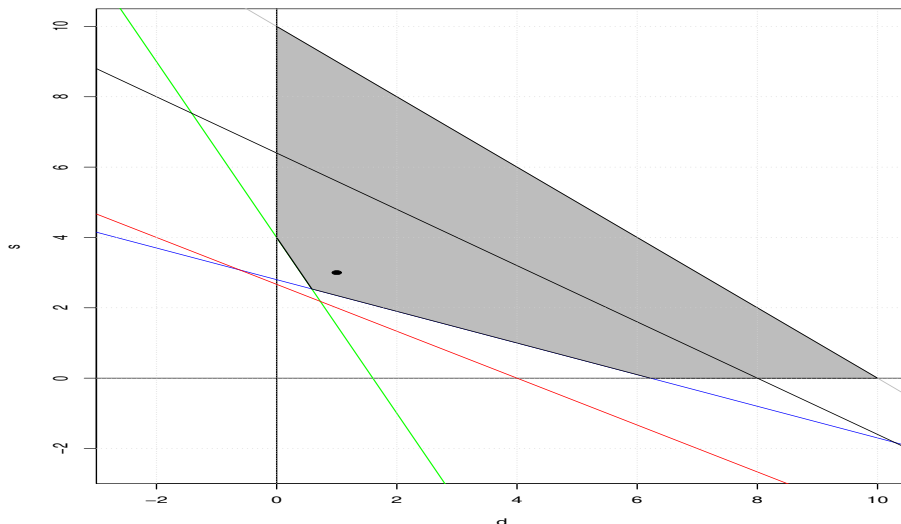


Figure 10: The feasible region for Exercise 6.

### Exercise 6

Let  $s$  be the amount (in boxes) of Scumptious cookies and  $d$  the amount (in boxes) of Delicious cookies. Then as Jimmy can only carry a total of ten boxes we have  $s + d \leq 10$ . The constraints on the number of types of cookies are given by

$$20s + 50d \geq 4(20) = 80 \quad (7)$$

$$40s + 18d \geq 4(28) = 112 \quad (8)$$

$$15s + 10d \geq 4(10) = 40. \quad (9)$$

Under these we seek to minimize the cost  $C(s, d) = 2.5s + 2d$ . See Figure 10 where this region is plotted.

In this case the feasible region is the region in the "upper-right" corner of the graph. A single level curve of  $c(d, s) = 16$  is plotted as a black line through the feasible region. Increasing the value of  $c(d, s)$  will correspond to moving that line upwards (we are increasing the  $y$  intercept). From this discussion the smallest cost corresponds to the  $(d, s)$  at one of the "corners" of the gray region. This is given by the intersection of the green and blue lines. This happens at the point  $(d^*, s^*) = (0.5853659, 2.5365854)$  with an optimal cost of  $C = 7.512195$ . This is not an integer solution as required by the understanding of what  $d$  and  $s$  represent (you can't bring 0.5853659 of a box of Delicious cookies). Because of this we need to evaluate  $C$  for integer values of  $(d, s)$  around the point  $(d^*, s^*)$  that are still feasible. Based on the feasible region the point closest to  $(d^*, s^*)$  and still in the feasible region would be  $(d, s) = (1, 3)$  (drawn in the graph as a black point). The cost of this solution is  $c(1, 3) = 9.5$ .

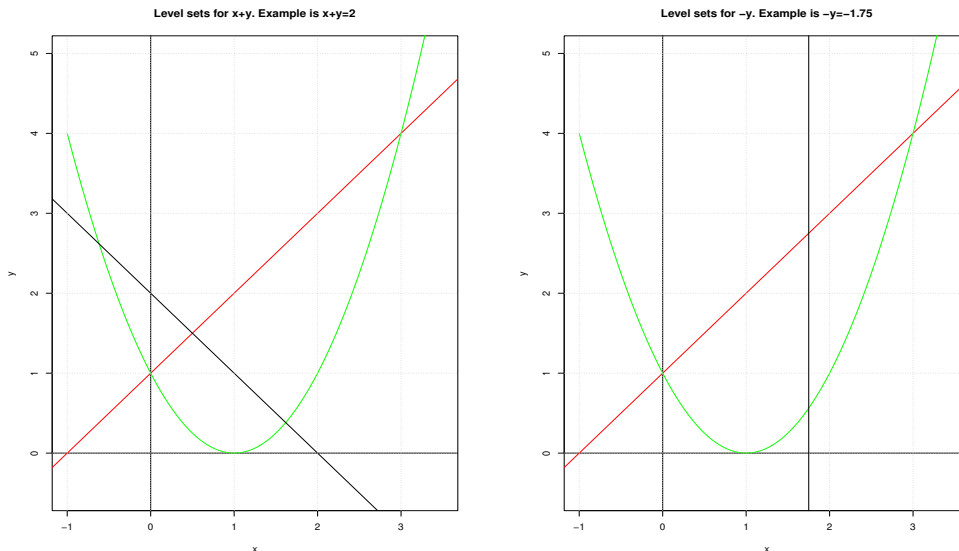


Figure 11: **Left:** Level sets for  $x + y$ . **Right:** Level sets for  $-y$ .

### Exercise 7

**Part (b-i):** See Figure 11 (left) where this region is plotted. As the level sets of  $x + y$  increase the black line in that graph moves upwards (we are increasing the  $y$  intercept). Eventually this line will go through the point that is the intersection of the two curves

$$\begin{aligned} y &= x + 1 \\ y &= x^2 - 2x + 1. \end{aligned}$$

This point of intersection is  $(x, y) = (3, 4)$  to give a maximum of  $x + y = 75$ .

**Part (b-ii):** See Figure 11 (right) where this region is plotted. As the level sets of  $-y$  increase the black vertical line in that graph moves to the left. Eventually this line will go through the point that is the *other* intersection of the two curves above. This is the point  $(x, y) = (0, 1)$  to give a maximum of  $-y$  of  $-1$ .

### Exercise 8

See Figure 12 where the curves defining this region are plotted. The feasible region is then the region below the red line, inside the green circle, and with  $x \geq 0$ . A level set of the objective  $3x + 2y = k_0$  (for  $k_0 = 12$ ) is also plotted as a black line. As  $k_0$  increase the black line in that graph moves upwards (we are increasing the  $y$  intercept). Eventually this line will pass through the red line that it inside the green circle. The largest value it can take is when it passes through the "upper" point that is the intersection of the two curves

$$\begin{aligned} y + 2x &= 10 \\ y &= \sqrt{25 - x^2}. \end{aligned}$$



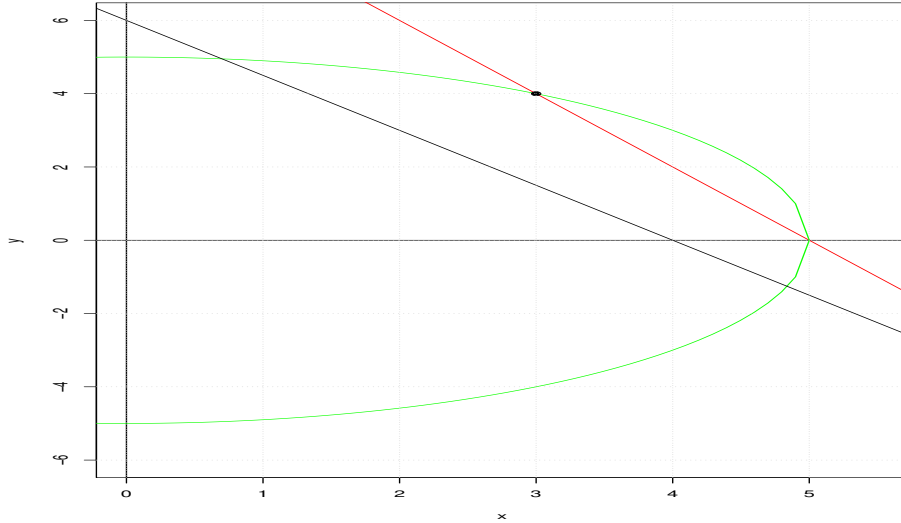


Figure 12: The feasible region for Exercise 8.

This point of intersection is  $(x, y) = (3, 4)$  (shown as a black dot) to give a maximum of  $3x + 2y = 17$ .

### Exercise 9

Imagine our rectangle with a horizontal length of  $x$  and a vertical length of  $y$  then the perimeter is  $P = 2x + 2y$ . The constraint conditions can be written as

$$\sqrt{x^2 + y^2} \leq 5 \quad (10)$$

$$\frac{y}{x} \geq 4. \quad (11)$$

See Figure 13 where the curves defining this region are plotted.

The feasible region is then the region above the red line, inside the green circle, with  $x > 0$  and  $y > 0$ . A level set of the objective  $2x + 2y = k_0$  (for  $k_0 = 8$ ) is also plotted as a black line. As  $k_0$  increase the black line in that graph moves upwards (we are increasing the  $y$  intercept). We can move this line upwards until eventually it will pass through the intersection of the red line and the green circle (shown as a black dot). This is the point where

$$\begin{aligned} y &= 4x \\ \sqrt{x^2 + y^2} &= 5. \end{aligned}$$

This point of intersection is  $(x, y) = \left(\frac{5}{\sqrt{17}}, \frac{20}{\sqrt{17}}\right)$  to give a maximum of  $2x + 2y = \frac{50}{\sqrt{17}}$ .

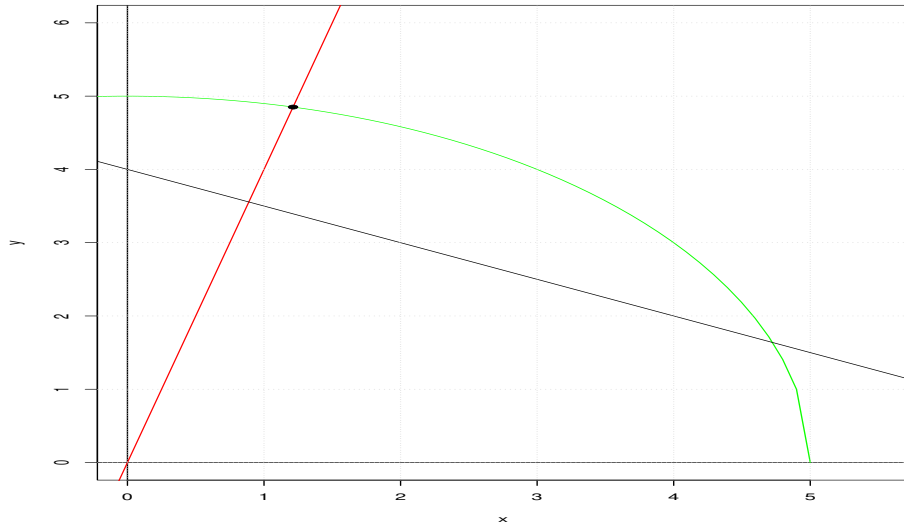


Figure 13: The feasible region for Exercise 9.

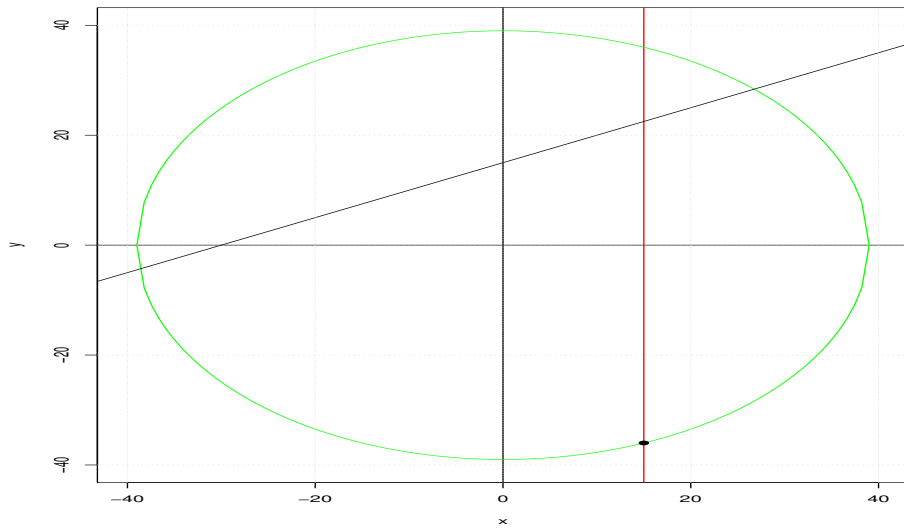


Figure 14: The feasible region for Exercise 10.

## Exercise 10

See Figure 14 where the curves defining this region are plotted. The feasible region is then the region inside the green circle and to the left of the red line. As the grass is "better" when  $3y - x$  is "smallest" we also plot a level set of the objective  $3y - x = k_0$  (for  $k_0 = 45$ ) as a black line. As  $k_0$  decreases the black line in that graph moves downwards (we are decreasing the  $y$  intercept). We can move this line downwards until eventually it will pass through the bottom intersection of the red line and the green circle (shown as a black dot). This is the point where

$$\begin{aligned}x &= 15 \\ \sqrt{x^2 + y^2} &= 39.\end{aligned}$$

This point of intersection is  $(x, y) = (15, -36)$  to give a minimum of  $3y - x = -123$ .

## A Metamorphosis

### Exercise 1

Exercise 1(e) are the linear inequalities

$$\begin{aligned}3x + 2y &\leq 6 \\ x - y &\geq 1 \\ x &\geq -1.\end{aligned}\tag{12}$$

If we introduce three slack variables  $u \geq 0$ ,  $v \geq 0$  and  $w \geq 0$  we get the three equations

$$\begin{aligned}3x + 2y + u &= 6 \\ x - y &= 1 + v \\ x &= -1 + w.\end{aligned}$$

### Exercise 2

The fertilizer problem has two constraints given by

$$\begin{aligned}0.55x + 0.2y &\geq 300 \\ 0.4x + 0.7y &\geq 400.\end{aligned}\tag{13}$$

If we introduce two slack variables  $u \geq 0$  and  $v \geq 0$  we get the two equations

$$\begin{aligned}0.55x + 0.2y &= 300 + u \\ 0.4x + 0.7y &= 400 + v.\end{aligned}$$

## The Elimination Method

### Exercise 1

Two lines will not intersect (have no solution) if they are parallel i.e. have the same slope but different intercepts. Two lines will have an infinite number of solutions if they “are the same” i.e. have the same slope and intercept. Two lines will have one point of intersection if they have different slopes (all other cases).

### Exercise 2

For this we have the transformations

$$\left[ \begin{array}{ccc|c} 4 & 2 & 2 & 4 \\ -8 & 4 & 16 & 7 \\ 12 & -6 & 3 & 4 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ -8 & 4 & 16 & 7 \\ 12 & -6 & 3 & 4 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ -16 & 0 & 12 & -1 \\ 24 & 0 & 9 & 16 \end{array} \right].$$

### Exercise 3

One sequence of transformations might be

$$\begin{aligned} \left[ \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 2 & 8 & 16 & 8 \\ 1 & 6 & 17 & 9 \end{array} \right] &\Rightarrow \left[ \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 0 & -6 & -6 \\ 0 & 2 & 6 & 2 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & 4 & 11 & 7 \\ 0 & 0 & -6 & -6 \\ 0 & 1 & 3 & 1 \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 0 & -6 & -6 \\ 0 & 1 & 3 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \end{array} \right] \Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array} \right]. \end{aligned}$$

This means that the only solution is  $(x, y, z) = (4, -2, 1)$ . We can “check” this solution with some simple R code

```
x = 4
y = -2
z = 1
c(x + 4*y + 11*z, 2*x + 8*y + 16*z, x + 6*y + 17*z)
```

## Exercise 4

**Part (a):** Here the number of equations is  $m = 3$  and the number of variables is  $n = 4$ . Following the suggested pivots we have

$$\begin{aligned} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & 3 & 7 \\ 3 & 3 & 5 & 4 & 15 \end{array} \right] &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 3 \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 3 \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{array} \right] \\ &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{array} \right]. \end{aligned}$$

If we let  $x_4 = \alpha$  then a general solution is given by

$$\begin{aligned} x_1 &= 1 + \alpha \\ x_2 &= \frac{3}{2} - \frac{3}{2}\alpha \\ x_3 &= \frac{3}{2} - \frac{1}{2}\alpha. \end{aligned}$$

A basic solution is when  $\alpha = 0$  or

$$(x_1, x_2, x_3, x_4) = \left( 1, \frac{3}{2}, \frac{3}{2}, 0 \right).$$

We can “check” this solution with some simple R code

```
alpha = 0 ## other alphas give the same right-hand-side
x1 = 1 + alpha
x2 = 3/2 - (3/2)*alpha
x3 = 3/2 - (1/2)*alpha
x4 = alpha
c(x1 + x2 + x3 + x4, x1 + 2*x2 + 2*x3 + 3*x4, 3*x1 + 3*x2 + 5*x3 + 4*x4)
```

**Part (b):** Here the number of equations is  $m = 3$  and the number of variables is  $n = 4$ .

Following the suggested pivots we have

$$\begin{aligned}
 \left[ \begin{array}{cccc|c} 2 & 1 & 3 & 5 & 6 \\ 3 & 2 & 4 & 6 & 8 \\ -1 & 3 & 2 & 7 & -3 \end{array} \right] &\Rightarrow \left[ \begin{array}{cccc|c} 2 & 1 & 3 & 5 & 6 \\ -1 & 0 & -2 & -4 & -4 \\ -7 & 0 & -7 & -8 & -21 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 2 & 1 & 3 & 5 & 6 \\ 1 & 0 & 2 & 4 & 4 \\ -7 & 0 & -7 & -8 & -21 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 0 & 1 & -1 & -3 & -2 \\ 1 & 0 & 2 & 4 & 4 \\ 0 & 0 & 7 & 20 & 7 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 0 & 1 & -1 & -3 & -2 \\ 1 & 0 & 2 & 4 & 4 \\ 0 & 0 & 1 & \frac{20}{7} & 1 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 0 & 1 & 0 & -\frac{1}{7} & -1 \\ 1 & 0 & 0 & -\frac{12}{7} & 2 \\ 0 & 0 & 1 & \frac{20}{7} & 1 \end{array} \right].
 \end{aligned}$$

If we let  $x_4 = \alpha$  then a general solution is given by

$$\begin{aligned}
 x_1 &= 2 + \frac{12}{7}\alpha \\
 x_2 &= -1 + \frac{1}{7}\alpha \\
 x_3 &= 1 - \frac{20}{7}\alpha.
 \end{aligned}$$

A basic solution is when  $\alpha = 0$  or

$$(x_1, x_2, x_3, x_4) = (2, -1, 1, 0).$$

We can “check” this solution with some simple R code

```

alpha = 0.0 ## other alphas give the same right-hand-side
x1 = 2 + (12/7)*alpha
x2 = -1 + (1/7)*alpha
x3 = 1 - (20/7)*alpha
x4 = alpha
c(2*x1 + x2 + 3*x3 + 5*x4, 3*x1 + 2*x2 + 4*x3 + 6*x4, -x1 + 3*x2 + 2*x3 + 7*x4)

```

**Part (c):** Here the number of equations is  $m = 3$  and the number of variables is  $n = 4$ .

Applying the elimination method we get

$$\begin{aligned}
 \left[ \begin{array}{cccc|c} 4 & 0 & 6 & -2 & 4 \\ 3 & -7 & 1 & 2 & 3 \\ 0 & 1 & -7 & 1 & 2 \end{array} \right] &\Rightarrow \left[ \begin{array}{cccc|c} 4 & 0 & 6 & -2 & 4 \\ 3 & 0 & -48 & 9 & 17 \\ 0 & 1 & -7 & 1 & 2 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 3 & 0 & -48 & 9 & 17 \\ 0 & 1 & -7 & 1 & 2 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{105}{2} & \frac{21}{2} & 14 \\ 0 & 1 & -7 & 1 & 2 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{4}{15} \\ 0 & 1 & -7 & 1 & 2 \end{array} \right] \\
 &\Rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & 1 & -\frac{1}{5} & -\frac{4}{15} \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{2}{15} \end{array} \right].
 \end{aligned}$$

If we let  $w = \alpha$  we get a general solution given by

$$\begin{aligned}
 x &= \frac{7}{5} + \frac{1}{5}\alpha \\
 y &= \frac{2}{15} + \frac{2}{5}\alpha \\
 z &= -\frac{4}{15} + \frac{1}{5}\alpha.
 \end{aligned}$$

A basic solution is where  $\alpha = 0$  and gives

$$(x, y, z, w) = \left( \frac{7}{5}, \frac{2}{15}, -\frac{4}{15}, 0 \right).$$

We can “check” this solution with some simple R code

```

alpha = 0.0 ## other alphas give the same right-hand-side
x = 7/5 + (1/5)*alpha
y = 2/15 + (2/5)*alpha
z = -4/15 + (1/5)*alpha
w = alpha
c(4*x + 6*z - 2*w, 3*x - 7*y + z + 2*w, y - 7*z + w)

```

# The Simplex Method for a Special Problem

## Exercise 1

**Part (a):** This exercise was discussed above on Page 13. We start by introducing slack variables  $u \geq 0$ ,  $v \geq 0$ , and  $w \geq 0$  such that

$$b + c + u = 35 \tag{14}$$

$$50b + 30c + v = 1300 \tag{15}$$

$$2b + 4c + w = 80. \tag{16}$$

Let the maximum profit be  $M$  such that  $M = 100b + 100c$ . In normal form these can be written

$$\left[ \begin{array}{cccccc|c} b & c & u & v & w & M & \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 35 \\ 50 & 30 & 0 & 1 & 0 & 0 & 1300 \\ 2 & 4 & 0 & 0 & 1 & 0 & 80 \\ 100 & 100 & 0 & 0 & 0 & -1 & 0 \end{array} \right].$$

The basic solution in this form is

$$(b_1, c_1, u_1, v_1, w_1, M_1) = (0, 0, 35, 1300, 80, 0).$$

As  $M = 100b + 100c$  lets try to adjust  $b$  such that  $c = c_1 = 0$  and the needed constraints on  $u$ ,  $v$ , and  $w$  hold. From the above we have

$$u = 35 - b - c = 35 - b \geq 0$$

$$v = 1300 - 50b - 30c = 1300 - 50b \geq 0$$

$$w = 80 - 2b - 4c = 80 - 2b \geq 0.$$

These give the requirements

$$b \leq 35$$

$$b \leq 26$$

$$b \leq 40.$$

To make all of these true we need to take the most restrictive condition or that  $b \leq 26$ . Thus  $b_2 = 26$  and we must pivot around the (2, 1) location. This gives the following normal form

$$\left[ \begin{array}{cccccc|c} b & c & u & v & w & M & \\ \hline 0 & 0.4 & 1 & -0.02 & 0 & 0 & 9 \\ 1 & 0.6 & 0 & 0.02 & 0 & 0 & 26 \\ 0 & 2.8 & 0 & -0.04 & 1 & 0 & 28 \\ 0 & 40 & 0 & -2 & 0 & -1 & -2601 \end{array} \right].$$

The basic solution in this form is

$$(b_2, c_2, u_2, v_2, w_2, M_2) = (26, 0, 9, 0, 28, 2601).$$



The above equation for  $M$  is

$$M = 2601 + 40c - 2v,$$

thus we can increase  $M$  by increasing  $c$ . We thus ask if we can increase  $c$  while keeping needed constraints on the “unit variables” of  $b \geq 0$ ,  $u \geq 0$ , and  $w \geq 0$ . Solving for these three variables in the above we have (taking  $v = v_2 = 0$ )

$$\begin{aligned} b &= -0.6c - 0.02v + 26 = -0.6c + 26 \geq 0 & \text{so } c &\leq 43.3333 \\ u &= -0.4c + 0.02v + 9 = -0.4c + 9 \geq 0 & \text{so } c &\leq 22.5 \\ w &= -2.8c + 0.04v + 28 = -2.8c + 28 \geq 0 & \text{so } c &\leq 10. \end{aligned}$$

The most restrictive of these is  $c \leq 10$  and so we take  $c_3 = 10$  and we must pivot around the (3, 2) location. This gives the following normal form

$$\left[ \begin{array}{cccccc|c} b & c & u & v & w & M & \\ \hline 0 & 0 & 1 & -0.01428571 & -0.1428571 & 0 & 5 \\ 1 & 0 & 0 & 0.02857143 & -0.2142857 & 0 & 20 \\ 0 & 1 & 0 & -0.01428571 & 0.3571429 & 0 & 10 \\ 0 & 0 & 0 & -1.42857143 & -14.2857143 & -1 & -3000 \end{array} \right].$$

The basic solution in this form is

$$(b_3, c_3, u_3, v_3, w_3, M_3) = (20, 10, 5, 0, 0, 3000).$$

The above equation for  $M$  is

$$M = 3000 - 1.42857143v - 14.2857143w,$$

and we cannot change  $b$  or  $c$  to make this larger. Thus we can stop this procedure and we have found the optimal solution  $(b, c) = (20, 10)$  with  $M = 3000$ . Note that this agrees with the solution found on Page 13. Note that the given pivots above were computed in the following R code

```
M = matrix(c(1, 1, 1, 0, 0, 0, 35, 50, 30, 0, 1, 0, 0, 1300,
            2, 4, 0, 0, 1, 0, 80, 100, 100, 0, 0, 0, -1, 0), nrow=4, ncol=7, byrow=T)

## Pivot on (2, 1)
M[2, ] = M[2, ]/M[2, 1]
M[1, ] = M[1, ] + (-M[1, 1])*M[2, ]
M[3, ] = M[3, ] + (-M[3, 1])*M[2, ]
M[4, ] = M[4, ] + (-M[4, 1])*M[2, ]

## Pivot on (3, 2)
M[3, ] = M[3, ]/M[3, 2]
M[1, ] = M[1, ] + (-M[1, 2])*M[3, ]
M[2, ] = M[2, ] + (-M[2, 2])*M[3, ]
M[4, ] = M[4, ] + (-M[4, 2])*M[3, ]
```

**Part (b):** We are given the constraints in Equation 13 and seek to minimize the cost

$$C(x, y) = 10x + 7y.$$

Introducing slack variables  $u \geq 0$  and  $v \geq 0$  we have the system of equations

$$\begin{aligned} 0.55x + 0.2y &= 300 + u \\ 0.4x + 0.7y &= 400 + v \\ -10x - 7y + C &= 0. \end{aligned}$$

In normal form these can be written

$$\left[ \begin{array}{ccccc|c} x & y & u & v & C & \\ \hline -0.55 & -0.2 & 1 & 0 & 0 & -300 \\ -0.4 & -0.7 & 0 & 1 & 0 & -400 \\ -10 & -7 & 0 & 0 & 1 & 0 \end{array} \right].$$

The basic solution in this form is

$$(x_1, y_1, u_1, v_1, C_1) = (0, 0, -300, -400, 0).$$

At this point we note that something seems a bit strange. In this basic solution neither of the values for  $u_1$  and  $v_1$  are positive and thus  $(x_1, y_1) = (0, 0)$  will not satisfy the required constraints of the problem. Rather than worry about this too much I'll attempt to follow something similar to the books arguments and see where they lead me. If anyone sees how to make this problem more equivalent to a maximization problem (as was presented in the book's examples) please contact me.

We ask if we can change  $C$  to a larger value (I thought we wanted to minimize  $C$ ?) by moving  $x$  away from  $x_1 = 0$ , keeping  $y$  fixed at  $y_1 = 0$  and while still keeping the constraints  $u \geq 0$  and  $v \geq 0$  valid. From the above,  $u$  and  $v$  are given by

$$u = -300 + 0.55x + 0.2y = -300 + 0.55x \geq 0 \quad (17)$$

$$v = -400 + 0.4x + 0.7y = -400 + 0.4x \geq 0. \quad (18)$$

These constraints require

$$\begin{aligned} x &\geq 545.4545 \\ x &\geq 1000. \end{aligned}$$

As the second of these will satisfy the first we will take  $x_2 = 1000$ . Now to "impute" this value into the basic solution, in the above normal form we will pivot on the (2, 1) location. This gives

$$\left[ \begin{array}{ccccc|c} x & y & u & v & C & \\ \hline 0 & 0.7625 & 1 & -1.375 & 0 & 250 \\ 1 & 1.75 & 0 & -2.5 & 0 & 1000 \\ 0 & 10.5 & 0 & -25 & 1 & 10000 \end{array} \right].$$

The basic solution in the above is now when  $y_2 = 0$  and  $v_2 = 0$  to get

$$(x_2, y_2, u_2, v_2, C_2) = (1000, 0, 250, 0, 10000).$$

In the above normal form the equation for  $C$  is

$$C = 10000 - 10.5y + 25v,$$

and we can make  $C$  smaller by increasing  $y$ . We desire to move  $y$  away from  $y_2 = 0$  while still keeping the constraints on the “unit variables” of  $x \geq 0$  and  $u \geq 0$  valid. From the normal form above we have

$$\begin{aligned}x &= -1.75y + 2.5v + 1000 = -1.75y + 1000 \geq 0 \quad \text{so} \quad y \leq 571.4286 \\u &= -0.7625y + 1.375v + 250 = -0.7625y + 250 \geq 0 \quad \text{so} \quad y \leq 327.8689.\end{aligned}$$

Here I have taken  $v = v_2 = 0$ . As the second of these will also satisfy the first we will take  $y_3 = 327.8689$ . Now to “impute” this value into the basic solution, in the above normal form we will pivot on the  $(1, 2)$  location. This gives

$$\left[ \begin{array}{cccc|c} x & y & u & v & C \\ \hline 0 & 1 & 1.311475 & -1.8032787 & 0 & 327.8689 \\ 1 & 0 & -2.295082 & 0.6557377 & 0 & 426.2295 \\ 0 & 0 & -13.770492 & -6.0655738 & 1 & 6557.3770 \end{array} \right].$$

The basic solution in the above is now when  $u_3 = 0$  and  $v_3 = 0$  to get

$$(x_3, y_3, u_3, v_3, C_3) = (426.2295, 327.8689, 0, 0, 6557.3770).$$

In the above normal form the equation for  $C$  is

$$C = 6557.3770 + 13.770492u + 6.0655738v,$$

and we cannot change  $C$  by changing  $x$  or  $y$ . Thus we can stop this procedure and we have found the optimal solution  $(x, y) = (426.2295, 327.8689)$  with  $C = 6557.3770$ . Note that this agrees with the solution given in the book for this problem.

## Preliminary Exercise

### Exercise 1

**Part (a):** Lets assume that  $x_1 < x_2$  and  $y_1 < y_2$ . Draw the segment connecting  $A \equiv (x_1, y_1)$  and  $B \equiv (x_2, y_2)$ . Next drop a perpendicular from  $(x_2, y_2)$  and draw a horizontal from  $(x_1, y_1)$ . These two lines meet at  $C \equiv (x_2, y_1)$ . Then the triangle  $\triangle ACB$  is a right triangle with  $\theta = \angle BAC$ . We have the lengths  $AC = x_2 - x_1$  and  $BC = y_2 - y_1$ . From the definition of  $\tan(\theta)$  we have then

$$\tan(\theta) = \frac{BC}{AC} = \frac{y_2 - y_1}{x_2 - x_1}.$$

**Part (b):** These would be

$$\frac{4-2}{2-1} = 2, \quad \frac{-1-5}{-2+3} = -6, \quad \frac{2-2}{-4-13} = 0, \quad \frac{3+1}{1-1} = \infty, \quad \frac{0-9}{0-3} = 3,$$

using the definition that  $m = \infty$  if  $x_2 = x_1$ .

**Part (c):** This equation is the matching of slopes between the segments  $(x, y) \rightarrow (x_1, y_1)$  and  $(x_1, y_1) \rightarrow (x_2, y_2)$ . Taking  $m = \frac{y_2 - y_1}{x_2 - x_1}$  gives the given equation.

**Part (d):** Here we are given  $(x_1, y_1)$  and  $m$  for each line. We get the equations

$$\begin{aligned}y - (-1) &= 6(x - 2) \\y - 0 &= -\frac{1}{2}(x - 4) \\y - 1 &= \infty.\end{aligned}$$

The last line can be written as  $x = 2$ .

**Part (e):** Note that we can write this as

$$\frac{y - d}{x - 0} = m,$$

which is another way of stating the desired condition.

**Part (f):** We have three slopes  $m \in \{6, -5, 0\}$  and two  $y$  intercepts  $d \in \{1, -3\}$ . These give the lines

$$\begin{aligned}y &= 6x + 1 \\y &= 6x - 3 \\y &= -5x + 1 \\y &= -5x - 3 \\y &= 1 \\y &= -3.\end{aligned}$$

## Exercise 2

**Part (a):** This is a line with a  $y$ -intercept of  $-1$  and a slope of  $m$ . If  $m > 0$  the line slants upwards to the right. If  $m < 0$  the line slants downwards to the right. If  $m = +\infty$  the line is the vertical line  $x = 0$ .

**Part (b):** This is  $y = -x + k$  which is a line with a  $y$ -intercept of  $k$  and a slope of  $-1$ .

## Exercise 4

**Part (a):** Using  $x = 2$  in the second equation gives  $-4y = 0$  so  $y = 0$  thus the solution is  $(2, 0)$ .

## Chapter 3: Growth of Populations

### Rate of Growth Dependent on Population

#### Exercise 1

Plots of  $X_n$  as a function of  $n$  are plotted when one runs the R code `chap_3_sect_2_prob_1.R`. I also plot the “equilibrium points”  $X = 2$  and  $x = 8$  as black horizontal lines. There we see that the first “case” has a population that converges to zero while the others seem to converge to a value of 8.

From the plot of  $R = R(X)$  and the discussions in the book if we start with  $X_0 \leq 2$  we see that  $R(X_0) < 1$  and  $X_n$  starts decreasing and continues to do so. If we start with  $2 \leq X_0 \leq 8$  we have  $R(X_0) > 1$  and  $X_n$  increases until it gets to a value  $X_n > 8$  at which point  $R(X_n) < 1$  where  $X_n$  will decrease again. Thus  $X = 2$  is an unstable equilibrium point and  $X = 8$  is a stable equilibrium point. These conclusions are demonstrated in the plots.

#### Exercise 2

**Part (a):** This looks like steady population growth to an equilibrium point followed by a stable population at  $X = 2.5$ .

**Part (b):** The slope of this curve seems to start at zero, increase to a positive number, and then decrease back to zero. For the given expression for  $X(t)$  we have

$$X'(t) = \frac{5\pi}{4T} \sin\left(\frac{\pi t}{T}\right).$$

This is zero at  $t = 0$  and  $t = T$  and increases from zero to a maximum before decreasing back to zero.

#### Exercise 3

This graph is plotted with the R code `chap_3_sect_2_prob_3.R`. The equilibrium points will be when  $R(X) = 1$  or when

$$X^2 - 21X + 20 = -8X.$$

Solving this gives the solutions

[1] 1.783009 11.216991

for the equilibrium values for  $X$ .

## The Linearized Problem

### Exercise 1

This exercise is worked in the R code `chap_3_sect_4_prob_1.R`. Running that code gives

```
[1] "Part (a):"  
  n    x_n  chi_n  
1 0  0.0500  0.0500  
2 1 -0.0350 -0.0250  
3 2  0.0126  0.0125  
4 3 -0.0069 -0.0063  
5 4  0.0033  0.0031  
[1] "Part (b):"  
  n    x_n  chi_n  
1 0  0.0500  0.0500  
2 1  0.0275  0.0375  
3 2  0.0176  0.0281  
4 3  0.0120  0.0211  
5 4  0.0084  0.0158
```

## Errors in the Approximate Solution

### Exercise 1

**Part (a):** Since  $x_i^2 \leq \epsilon$  we have

$$|\xi_n - x_n| \leq m\epsilon \sum_{k=0}^{n-1} |\alpha|^k \leq m\epsilon \sum_{k=0}^{\infty} |\alpha|^k = \frac{m\epsilon}{1 - |\alpha|}.$$

**Part (b):** Using

$$\xi_{n+1} - x_{n+1} = \alpha(\xi_n - x_n) + mx_n^2, \quad (19)$$

if  $m > 0$  and  $\xi_n - x_n > 0$  then using the above we will have  $\xi_{n+1} - x_{n+1} > 0$  and by induction thus for all  $n$ .

Using Equation 19 and the assumption that  $\xi_n - x_n < \frac{\epsilon}{X_B}$  we have

$$\xi_{n+1} - x_{n+1} < \alpha \left( \frac{\epsilon}{X_B} \right) + m\epsilon = (1 - mX_B) \frac{\epsilon}{X_B} + m\epsilon = \frac{\epsilon}{X_B},$$

thus the induction step is proved.

## Exercise 2

This exercise is worked in the R code `chap_3_sect_5_prob_2.R`. Running that code gives

```
[1] "alpha= 0.900000"
  n    x_n  xi_n error_n
1 0 0.0200 0.0200 0.0000
2 1 0.0160 0.0180 0.0020
3 2 0.0131 0.0162 0.0031
4 3 0.0109 0.0146 0.0036
5 4 0.0093 0.0131 0.0039
6 5 0.0079 0.0118 0.0039
[1] "eps= 0.000400; err_bound= 0.020000"
```

## Exercise 3

**Part (a):** Here we have  $X_B = 1$ ,  $m = 1.5$ , and  $X_0 = 1.05$  so that

$$\begin{aligned}x_0 &= X_0 - X_B = 0.05 \\ \epsilon &= x_0^2 = 0.0025 \\ \alpha &= 1 - mX_B = 1 - 1.5 = -0.5.\end{aligned}$$

Thus an upper bound on  $|\xi_n - x_n|$  is given by

$$\frac{m\epsilon}{1 - |\alpha|} = 0.075.$$

**Part (b):** Here  $m = 0.25$  and an upper bound on  $|\xi_n - x_n|$  is given by

$$\frac{m\epsilon}{1 - |\alpha|} = 0.0125.$$

## Interaction of Predator-Prey Species

### Exercise 1

This is implemented in the R code `chap_3_sect_6_prob_1.R`. Running that gives

```
[1] "alpha= -0.100000"
  n          x_n      xi_n      error_n
1  0 0.000000e+00 0.0000 0.000000e+00
```

```

2  1 -5.000000e-02 -0.0500 0.000000e+00
3  2 -7.000000e-02 -0.0450 2.500000e-02
4  3 -9.200000e-02 -0.0455 4.650000e-02
5  4 -1.254000e-01 -0.0454 8.000000e-02
6  5 -1.948000e-01 -0.0455 1.494000e-01
7  6 -4.100000e-01 -0.0455 3.646000e-01
8  7 -1.690200e+00 -0.0455 1.644700e+00
9  8 -2.844760e+01 -0.0455 2.840210e+01
10 9 -8.089867e+03 -0.0455 8.089821e+03
11 10 -6.544586e+08 -0.0455 6.544586e+08
[1] "eps= 0.015735; err_bound= 0.174835"
[1] "X_s= 0.995000; (X_B + last_xis)= 0.954545"

```

Notice that in this case the solution for  $x_n$  is unbounded and the steady state population  $X_s$  is close to  $X_B + \xi_{10}$ .

## Exercise 2

**Part (a):** The book's equation 16 is

$$X_n = (1 - mX_{n-1} + mX_B)X_{n-1} - N. \quad (20)$$

In steady state and with the numbers given here this is

$$X_s = X_s - X_s^2 + 1.05X_s - 0.05,$$

or

$$X_s^2 - 1.05X_s + 0.05 = 0.$$

This has a solution  $X_s = 1.0$ .

**Part (b):** The estimate is given by

$$X_B - \frac{N}{mX_B} = 1.05 - \frac{0.05}{1.05} = 1.00238.$$

## Exercise 3

Taking the limit as  $n \rightarrow \infty$  we have

$$4p = 3p^2 - 2p + 3.$$

This is a quadratic equation that can be written as  $(p - 1)^2 = 0$  so has a solution  $p = 1$ .

Taking  $p_n = 1 + x_n$  we have  $p_n^2 = 1 + 2x_n + x_n^2$  and so our original equation becomes

$$4 + 4x_n = 3 + 6x_{n-1} + 3x_{n-1}^2 - 2 + 2x_{n-1} + 3.$$



We can solve this for  $x_n$  in terms of  $x_{n-1}$  to get

$$x_n = 2x_{n-1} + \frac{3}{2}x_{n-1}^2.$$

The linear relationship is obtained by dropping the term  $\frac{3}{2}x_{n-1}^2$  in the above to get

$$x_n = 2x_{n-1}.$$

This has the solution  $x_n = 2^n x_0$ .

#### Exercise 4

I would expect that the values for  $\xi_n$  would trend to zero which would mean that  $X_n \rightarrow X_s$ .

#### Exercise 5

Recall the book's equation 16 is given by Equation 20 or

$$X_n = (1 - mX_{n-1} + mX_B)X_{n-1} - N,$$

and have  $X_s$  satisfy

$$mX_s^2 - mX_B X_s + N = 0. \quad (21)$$

Next replace  $X_n = y_n + X_s$  into Equation 20 to get

$$y_n + X_s = (1 - my_{n-1} - mX_s + mX_B)(y_{n-1} + X_s) - N.$$

Expanding this we have

$$y_n + X_s = y_{n-1} + X_s - my_{n-1}^2 - mX_s y_{n-1} - mX_s y_{n-1} - mX_s^2 + mX_B y_{n-1} + mX_B X_s - N.$$

Using Equation 21 this becomes

$$\begin{aligned} y_n &= y_{n-1} - my_{n-1}^2 - mX_s y_{n-1} - mX_s y_{n-1} + mX_B y_{n-1} \\ &= (1 - mX_B - 2mX_s)y_{n-1} - my_{n-1}^2, \end{aligned}$$

which is the expression we were trying to show.

## Mutually Dependent Predator-Prey Species

### Notes on the Text: The derivation of $X_E$ and $Y_E$

Starting with

$$Y_n = \frac{k}{X_B} X_{n-1} Y_{n-1} \quad (22)$$

$$X_n - X_{n-1} = mX_B X_{n-1} - mX_{n-1}^2 - cX_{n-1} Y_{n-1}, \quad (23)$$

If we take  $X_n = X_E$  and  $Y_n = Y_E$  in Equation 22 we get  $X_E = \frac{X_B}{k}$ . If we take  $X_n = X_E$  and  $Y_n = Y_E$  in Equation 23 (and divide by  $X_E$ ) we get

$$0 = mX_B - mX_E - cY_E \quad \text{so} \quad Y_E = \frac{m}{c}(X_B - X_E) = \frac{m}{c}X_B \left(1 - \frac{1}{k}\right),$$

which are the expressions given in the book.

### Notes on the Text: The derivation of equations for $x_n$ and $y_n$

To start we will put  $x_n = X_n - X_E$  and  $y_n = Y_n - Y_E$  into Equation 22 to get

$$y_n + Y_E = \frac{k}{X_B}(x_{n-1} + X_E)(y_{n-1} + Y_E),$$

or expanding the right-hand-side we get

$$\begin{aligned} y_n + Y_E &= \frac{k}{X_B}(x_{n-1}y_{n-1} + Y_Ex_{n-1} + X_Ey_{n-1} + X_EY_E) \\ &= \frac{k}{X_B} \left( x_{n-1}y_{n-1} + Y_Ex_{n-1} + \frac{X_B}{k}y_{n-1} + \frac{X_B}{k}Y_E \right) \\ &= \frac{k}{X_B}x_{n-1}y_{n-1} + \frac{kY_E}{X_B}x_{n-1} + y_{n-1} + Y_E. \end{aligned}$$

Replacing  $X_B$  with its expression in terms of  $X_E$  this becomes

$$y_n = \left(\frac{1}{X_E}\right) x_{n-1}y_{n-1} + y_{n-1} + \left(\frac{Y_E}{X_E}\right) x_{n-1}, \quad (24)$$

which is the book's Eq. 23.

Next we will put  $x_n = X_n - X_E$  and  $y_n = Y_n - Y_E$  into Equation 23 to get

$$\begin{aligned} x_n - x_{n-1} &= mX_Bx_{n-1} + mX_BX_E - m(x_{n-1}^2 + 2X_Ex_{n-1} + X_E^2) \\ &\quad - c(x_{n-1}y_{n-1} + Y_Ex_{n-1} + X_Ey_{n-1} + X_EY_E) \\ &= mX_Bx_{n-1} + mX_BX_E - mx_{n-1}^2 - \frac{2mX_B}{k}x_{n-1} - mX_E^2 \\ &\quad - cx_{n-1}y_{n-1} - cY_Ex_{n-1} - \frac{cX_B}{k}y_{n-1} - cX_EY_E \\ &= \left(mX_B - \frac{2mX_B}{k} - cY_E\right) x_{n-1} + (mX_BX_E - mX_E^2 - cX_EY_E) \\ &\quad - \frac{cX_B}{k}y_{n-1} - mx_{n-1}^2 - cx_{n-1}y_{n-1}. \end{aligned}$$

Now writing  $Y_E$  in terms of  $X_B$  the coefficient of the  $x_{n-1}$  is given by

$$mX_B - \frac{2m}{k}X_B - m \left(1 - \frac{1}{k}\right) X_B = -\frac{m}{k}X_B = -mX_E.$$

In the same way the constant term on the right-hand-side of the above is given by

$$\begin{aligned} mX_B X_E - mX_E^2 - cX_E Y_E &= X_E (mX_B - mX_E - cY_E) \\ &= X_E \left( mX_B - \frac{mX_B}{k} - \frac{m(k-1)}{k} X_B \right) = 0. \end{aligned}$$

This means that the equation for  $x_n$  can be written as

$$x_n - x_{n-1} = mX_E x_{n-1} - \frac{cX_B}{k} y_{n-1} - mx_{n-1}^2 - cx_{n-1} y_{n-1},$$

or

$$x_n = (1 - mX_E)x_{n-1} - cX_E y_{n-1} - mx_{n-1}^2 - cx_{n-1} y_{n-1}, \quad (25)$$

which is the book's Eq. 22.

### Notes on the Text: The derivation of the equation for $\eta_n$

Starting with  $\xi_{n-1}$  in terms of  $\eta_n$  as

$$\xi_{n-1} = \frac{X_E}{Y_E} (\eta_n - \eta_{n-1}),$$

Then putting this into the equation for  $\xi_n$  (the linearized version of Equation 25) to get

$$\frac{X_E}{Y_E} (\eta_{n+1} - \eta_n) = (1 - mX_E) \frac{X_E}{Y_E} (\eta_n - \eta_{n-1}) - cX_E \eta_{n-1},$$

or

$$\begin{aligned} \eta_{n+1} - \eta_n &= (1 - mX_E)(\eta_n - \eta_{n-1}) - cY_E \eta_{n-1} \\ &= \eta_n - \eta_{n-1} - mX_E \eta_n + mX_E \eta_{n-1} - cY_E \eta_{n-1}. \end{aligned}$$

Solving for  $\eta_{n+1}$  this is

$$\eta_{n+1} = (2 - mX_E)\eta_n - (1 - mX_E + cY_E)\eta_{n-1}, \quad (26)$$

which is the book's Eq. 26.

### Exercise 1

This is done in these notes on Page 34.

## Exercise 2

In these equations we will take  $X_n \rightarrow X_E$  and  $Y_n \rightarrow Y_E$ . The first one gives

$$0 = \frac{1}{2}X_E - \frac{1}{2}X_E^2 - \frac{1}{8}X_E Y_E,$$

or assuming  $X_E \neq 0$  and dividing by it

$$0 = \frac{1}{2} - \frac{1}{2}X_E - \frac{1}{8}Y_E. \quad (27)$$

This substitution in the second equation gives

$$Y_E = 2X_E Y_E - Y_E^2,$$

or assuming  $Y_E \neq 0$  and dividing by it

$$1 = 2X_E - Y_E. \quad (28)$$

Solving the second equation for  $Y_E$  and putting that expression into Equation 27 gives a single equation for  $X_E$ . Solving this gives  $X_E = \frac{5}{6}$ . We then find  $Y_E = 2X_E - 1 = \frac{5}{3} - 1 = \frac{2}{3}$ .

## Exercise 4

If I understand the problem correctly we start each year (from 1970) with  $X_n$  staff members where  $n = 0$  is January 1st 1970 and we are told that  $X_0 = \frac{4}{5}N$  with  $N$  the desired number of staff. Then in July each year our recruiting officer will add  $N - X_n$  staff in an attempt to bring the staff up to  $N$ . Then in December each year 10% of the staff will resign. From this logic at the end of 1970 (the beginning of 1971) we should have

$$X_1 = \frac{4}{5}N + \frac{1}{5}N - 0.1N = \frac{9}{10}N.$$

Then at the end of 1971 (the beginning of 1972) we should have

$$X_2 = \frac{9}{10}N + \frac{1}{10}N - 0.1N = \frac{9}{10}N,$$

the same as  $X_1$ . Thus the staff is the same at the end of each year and the goal of having a complete staff will never be obtained.

If the recruiting officer instead orders  $\alpha_0 N$  (during 1970) then at the end of 1970 he will have

$$X_1 = \frac{4}{5}N + \alpha_0 N - 0.1 \left( \frac{4}{5}N + \alpha_0 N \right) = 0.9 \left( \frac{4}{5}N + \alpha_0 N \right).$$

If we want this to be  $N$  then we can solve for  $\alpha_0$  to find  $\alpha_0 = \frac{14}{45} = 0.311111$ .

At this point we have  $X_1 = N$  and if the recruiting officer orders  $\alpha_1 N$  (during 1971) the next year he will have

$$X_2 = N + \alpha_1 N - 0.1(N + \alpha_1 N) = 0.9(N + \alpha_1 N) = 0.9(1 + \alpha_1)N.$$

If we want this to be  $N$  then we must have

$$\alpha_1 = \frac{1}{0.9} - 1 = \frac{1}{9} = 0.111111.$$

The recruiting officer should hire  $\alpha_1 N$  in all subsequent years.

## Volterra's Principle

### Exercise 1

The linearized equations assume that the values of  $x_n = X_n - X_E$  and  $y_n = Y_n - Y_E$  are small. This means that  $X_n$  and  $Y_n$  are close to  $X_E$  and  $Y_E$ . In the example from this section this is not true.

### Exercise 2

**Part (a):** This is implemented in the R code `chap_3_sect_8_prob_2.R`. Running that gives

```
[1] "n= 0; xi_n= -0.5000; eta_n= -1.8750; X_n= 0.5000; Y_n= 1.8750"
[1] "n= 1; xi_n= 0.0625; eta_n= -3.7500; X_n= 1.0625; Y_n= 0.0000"
[1] "n= 2; xi_n= 0.3906; eta_n= -3.5156; X_n= 1.3906; Y_n= 0.2344"
[1] "n= 3; xi_n= 0.4492; eta_n= -2.0508; X_n= 1.4492; Y_n= 1.6992"
[1] "n= 4; xi_n= 0.3174; eta_n= -0.3662; X_n= 1.3174; Y_n= 3.3838"
```

Notice from this output that  $Y_1 = 0$ .

**Part (b):** If  $y_{n-1} = 0$  for some  $n$  (here  $n = 2$ ) then this means that  $y_{n-1} = -Y_E$  and putting that into Equation 24 we see that

$$y_n = \left( \frac{Y_E}{X_E} \right) x_{n-1} - Y_E - \frac{Y_E}{X_E} x_{n-1} = -Y_E,$$

and  $y_n$  is a constant for all time going forward.

Taking  $y_{n-1} = -Y_E$  in Equation 25 we get that

$$\begin{aligned} x_n &= (1 - mX_E)x_{n-1} + cX_EY_E - mx_{n-1}^2 + cY_Ex_{n-1} \\ &= (1 - mX_E + cY_E)x_{n-1} - mx_{n-1}^2 + cX_EY_E. \end{aligned}$$

Notice that this is same as the book's Eq. 12 which is

$$x_n = \alpha x_{n-1} - m x_{n-1}^2 - N, \quad (29)$$

with  $\alpha = 1 - mX_E + cY_E$  and  $N = -cX_EY_E$ . From the discussion in the book we expect the solution for  $x_n$  to “converge” at a rate of  $\alpha$  to a limiting value that is below the equilibrium point  $X_B$  by something proportional to  $N$ . Here since  $N < 0$  our the equilibrium for the prey when there are no predators will be above what it was before.

## Other Forms of Biological Association

**Notes on the Text: The derivation of the equilibrium populations  $X_E$  and  $Y_E$**

We start by taking  $X_n \rightarrow X_E$  and  $Y_n \rightarrow Y_E$  in

$$X_n = R(X_{n-1}, Y_{n-1})X_{n-1} \quad \text{and} \quad Y_n = S(X_{n-1}, Y_{n-1})Y_{n-1}, \quad (30)$$

to get

$$X_E = (1 - m(X_E - X_B) - kY_E)X_E \quad (31)$$

$$Y_E = (1 - \mu(Y_E - Y_B) - lX_E)Y_E. \quad (32)$$

Now if we divide the first by  $X_E$  and the second by  $Y_E$  (both assumed to be nonzero) we get

$$1 = 1 - m(X_E - X_B) - kY_E$$

$$1 = 1 - \mu(Y_E - Y_B) - lX_E,$$

or

$$mX_E + kY_E = mX_B$$

$$lX_E + \mu Y_E = \mu X_B.$$

We can solve this linear system for  $X_E$  and  $Y_E$ . We find

$$D = \begin{vmatrix} m & k \\ l & \mu \end{vmatrix} = m\mu - kl,$$

and then

$$X_E = \frac{\begin{vmatrix} mX_B & k \\ \mu Y_B & \mu \end{vmatrix}}{D} = \frac{m\mu X_B - k\mu Y_B}{m\mu - kl} = \left( \frac{mX_B - kY_B}{m\mu - kl} \right) \mu$$

$$Y_E = \frac{\begin{vmatrix} m & mX_B \\ l & \mu Y_B \end{vmatrix} \mu}{D} = \frac{m\mu Y_B - lmX_B}{m\mu - kl} = \left( \frac{\mu Y_B - lX_B}{m\mu - kl} \right) m,$$

which are the expressions given in the book.

## Exercise 1

We start with Equation 30 with  $R_{n-1}$  and  $S_{n-1}$  expanded or

$$\begin{aligned}X_n &= (1 - m(X_{n-1} - X_B) - kY_{n-1})X_{n-1} \\Y_n &= (1 - \mu(Y_{n-1} - Y_B) - lX_{n-1})Y_{n-1},\end{aligned}$$

Let  $X_n = x_n + X_E$  and  $Y_n = y_n + Y_E$  in the above where we get

$$\begin{aligned}x_n + X_E &= (1 - m(x_{n-1} + X_E - X_B) - k(y_{n-1} + Y_E))(x_{n-1} + X_E) \\y_n + Y_E &= (1 - \mu(y_{n-1} + Y_E - Y_B) - l(x_{n-1} + X_E))(y_{n-1} + Y_E).\end{aligned}$$

We will write the first of these by "expanding"  $(x_{n-1} + X_E)$  as

$$\begin{aligned}x_n + X_E &= (1 - m(x_{n-1} + X_E - X_B) - k(y_{n-1} + Y_E))x_{n-1} \\&\quad + (1 - m(x_{n-1} + X_E - X_B) - k(y_{n-1} + Y_E))X_E,\end{aligned}$$

and then by "extracting"  $x_{n-1}$  and  $y_{n-1}$  from the coefficient of  $X_E$  to get

$$\begin{aligned}x_n + X_E &= (1 - m(x_{n-1} + X_E - X_B) - k(y_{n-1} + Y_E))x_{n-1} \\&\quad + (-mx_{n-1} - ky_{n-1})X_E + (1 - m(X_E - X_B) - kY_E)X_E.\end{aligned}$$

Using Equation 31 we can cancel  $X_E$  on the left and end with

$$x_n = (1 - m(x_{n-1} + X_E - X_B) - k(y_{n-1} + Y_E))x_{n-1} + (-mx_{n-1} - ky_{n-1})X_E.$$

This gives coefficients for  $x_{n-1}$  and  $y_{n-1}$  of

$$x_n = (1 - mx_{n-1} - mX_E + mX_B - ky_{n-1} - kY_E - mX_E)x_{n-1} - kX_Ey_{n-1}.$$

Extracting the two nonlinear terms this becomes

$$x_n = -mx_{n-1}^2 - kx_{n-1}y_{n-1} + (1 - mX_E + mX_B - kY_E - mX_E)x_{n-1} - kX_Ey_{n-1}.$$

In the above we now consider the coefficient of  $x_{n-1}$ .

WWX: Working here.

## Exercise 2

WWX: DP.

## Exercise 3

WWX: DP.

#### **Exercise 4**

WWX: DP.

#### **Exercise 5**

WWX: DP.

### **Three-Term Recurrence Relations**

#### **Exercise 1**

WWX: DP.

### **Preliminary Exercise**

#### **Exercise 1**



# Chapter 4: Searching for a Maximum

## Introduction

### Exercise 1