Ex 3.1 eq 2.16 is

$$\int_{x_1}^{x_2} u(x,t_2)\,dx = \int_{x_1}^{x_2} u(x,t_1)\,dx + \int_{t_1}^{t_2} f(u(x_1,t))\,dt - \int_{t_1}^{t_2} f(u(x_2,t))\,dt \quad \forall\, x_1, x_2, t_1, t_2$$

w/ $f(u) = au$ we get

$$\int_{x_1}^{x_2} u_0(x-at_2)\,dx \overset{?}{=} \int_{x_1}^{x_2} u_0(x-at_1)\,dx + \int_{t_1}^{t_2} a u_0(x_1-at)\,dt - \int_{t_1}^{t_2} a u_0(x_2-at)\,dt$$

working on R.H.S we get

let $v = x - at_1$, $dv = dx$ let $v = x_1 - at$, $dv = -a\,dt$ let $v = x_2 - at$, $dv = -a\,dt$

$$= \int_{x_1-at_1}^{x_2-at_1} u_0(v)\,dv - \int_{x_1-at_1}^{x_1-at_2} u_0(v)\,dv + \int_{x_2-at_1}^{x_2-at_2} u_0(v)\,dv$$

Then

$$= \int_{x_1-at_1}^{x_2-at_1} u_0(v)\,dv + \int_{x_1-at_2}^{x_1-at_1} u_0(v)\,dv + \int_{x_2-at_1}^{x_2-at_2} u_0(v)\,dv$$

$$= \int_{x_1-at_2}^{x_1-at_1} + \int_{x_1-at_1}^{x_2-at_2} + \int_{x_2-at_1}^{x_2-at_2}$$

$$= \int_{x_1-at_2}^{x_2-at_2} u_0(v)\,dv \qquad \text{let } v = x - at_2, \; dv = dx$$

$$= \int_{x_1}^{x_2} u_0(x-at_2)\,dx \qquad \text{correct} \checkmark$$

Ex 3.2 $u_t + a u_x = \epsilon u_{xx}$

let $v(x,t) = u(x+at,t) \Rightarrow u(x,t) = v(x-at,t)$ see pg 2

$v_x = u_x \qquad v_t = u_x a + u_t \Rightarrow u_t = v_t - a u_x$

$v_{xx} = u_{xx} \qquad\qquad = v_t - a v_x$

$\Rightarrow \quad v_t - a\cancel{v_x} + a\cancel{v_x} = \epsilon v_{xx}$

$v_t = \epsilon v_{xx}$ eq 3.12

Solve w/ fourier transforms: let $v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{v}(\omega,t) e^{i\omega x}\, d\omega$

$\quad \& \quad \hat{v}(\omega,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} v(x,t) e^{-i\omega x}\, dx$

Thus eq above becomes

$$\frac{d\hat{v}}{dt} = \epsilon (i\omega)^2 \hat{v} = -\epsilon\omega^2 \hat{v}$$

$\hat{v}(\omega,t) = c_0 e^{-\epsilon\omega^2 t} \qquad \hat{v}(\omega,0) = c_0 = \hat{u}_0(\omega)$ F.T. of $u_0(x)$

Then $v(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}_0(\omega) e^{-\epsilon\omega^2 t} e^{i\omega x}\, d\omega$

so that see pg 2.

If $u(x,t) = v(x-at, t)$

$$u_x = v_x \qquad u_t = v_x(-a) + v_t$$

$\therefore$ $u_t + a u_x = \epsilon u_{xx} \Rightarrow -\cancel{a v_x} + v_t + \cancel{a v_x} = \epsilon v_{xx}$ Yes

$\therefore$

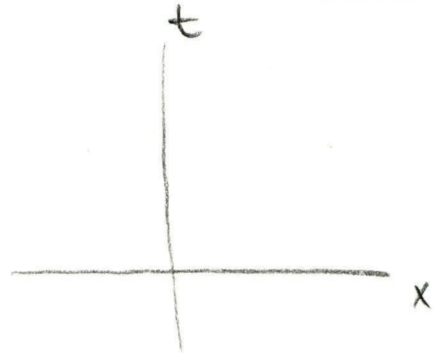
$$u^{\epsilon}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\omega) e^{-\epsilon\omega^2 t} e^{i\omega(x-at)} \, d\omega$$

$$\lim_{\epsilon \to 0} u^{\epsilon}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\omega) e^{i\omega(x-at)} \, d\omega. \equiv u_0(x-at)$$

Since $$\hat{u}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x) e^{-ix\omega} \, dx$$

$$u_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{u}_0(\omega) e^{i\omega x} \, d\omega$$

Ex 3.3 eq 3.14 is $u_t + uu_x = 0$



By method of characteristics $\frac{du(X(t),t)}{dt} = 0$

Along $\frac{dX(t)}{dt} = u(X(t),t)$ $u(x,0) = u_0(x)$ given

$\therefore$ evaluating at 0 $\frac{dX(0)}{dt} = u(X(0),0) = u(x_0,0) = u_0(x_0)$

Thus $\frac{du(X(t),t)}{dt} = 0$ along $\frac{dX(t)}{dt} = u_0(X(0))$

Breaking occurs iff characteristics cross. This happen if the characteristic from x_0 & $x_0 + dx$ intersect $\Rightarrow$ each characteristic is a st. line

$\Rightarrow$ $x - x_0 = u_0(x_0)t \Rightarrow x - x_0 = u_0(x_0)t$

$x - x_0 - dx = u_0(x_0 + dx)t$

& $\approx u_0(x_0)t + u_0'(x_0)dx\,t + O(dx^2)$

Subtracting to find x & t when the intersection occurs we obtain

$\cancel{d}x = -u_0'(x_0)\cancel{\Delta}x\,t \Rightarrow t = \frac{-1}{u_0'(x_0)} \quad \forall x_0.$

Thus the 1st characteristic crossing (t being minimum occurs when)

$$t = T_b = \frac{+1}{(-u_0'(x_0))} = \frac{+1}{\max(-u_0'(x_0))} = \frac{-1}{\min_{x_0 \in (-\infty,+\infty)}(u_0'(x_0))}$$

For arbitrary convex scaler eqs

$$u_t + (f(u))_x = 0 \qquad f(u) \text{ convex} \Rightarrow f''(u) > 0$$

smooth flows

$$\Leftrightarrow \quad u_t + f'(u)\, u_x = 0$$

everything follows as before w/

$$x - x_0 = f'(u_0(x_0))\, t$$

$$\& \quad x - x_0 - dx = f'(u_0(x_0 + dx))\, t$$

$$= f'(u_0(x_0) + u_0'(x_0)\, dx + O(dx^2))\, t$$

$$= \left(f'(u_0(x_0)) + f''(u_0(x_0))\, u_0'(x_0)\, dx + O(dx^2)\right) t$$

Subtracting $\quad \cancel{d}x = -f''(u_0(x_0))\, u_0'(x_0)\, \cancel{d}x\, t$

$$\Rightarrow \quad t = \frac{-1}{f''(u_0(x_0))\, u_0'(x_0)}$$

$$T_b = \frac{+1}{\max\limits_{-\infty < x_0 < +\infty} \left(-f''(u_0(x_0))\, u_0'(x_0)\right)} = \frac{-1}{\min\limits_{-\infty < x_0 < +\infty} \left(f''(u_0(x_0))\, u_0'(x_0)\right)}$$

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$$\int_0^\infty \int_{-\infty}^\infty \left(\phi u_t + \phi f(u)_x\right) dx\,dt = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\int_0^\infty \phi u_t\,dt\right)dx + \int_0^\infty\left(\int_{-\infty}^\infty \phi f(u)_x\,dx\right)dt = 0$$

$$\cancel{\int_{-\infty}^\infty \left(\phi u\Big|_0^\infty\right) dx} - \int_{-\infty}^\infty\int_0^\infty \phi_t u\,dt\,dx + \cancel{\int_0^\infty \phi f(u)\Big|_{-\infty}^{\infty} dt} - \int_{-\infty}^\infty\int_0^\infty \phi_x f(u)\,dx\,dt = 0$$

$$- \int_{-\infty}^\infty \phi(x,0)u(x,0)\,dx \quad \therefore \int_{-\infty}^\infty\int_0^\infty \left(\phi_t u + \phi_x f(u)\right) dt\,dx = -\int_{-\infty}^\infty \phi(x,0)u(x,0)\,dx$$

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Ex 3.4   3.25 $\Rightarrow$ $u(x,t) = \begin{cases} u_l & x < st \\ u_r & x > st \end{cases}$ $s = \frac{1}{2}(u_l + u_r)$

3.22

$$-\int_0^\infty \int_{-\infty}^{\infty} \left(\phi_t u + \phi_x f(u)\right) dx\, dt \stackrel{?}{=} -\int_{-\infty}^{\infty} \phi(x,0)\, u(x,0)\, dx$$

R.H.S eq w/ $u(x,0) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases}$ & $f(u) = \frac{u^2}{2}$

$$= -u_l \int_{-\infty}^{0} \phi(x,0)\, dx - u_r \int_{0}^{+\infty} \phi(x,0)\, dx$$ ✓

L.H.S.

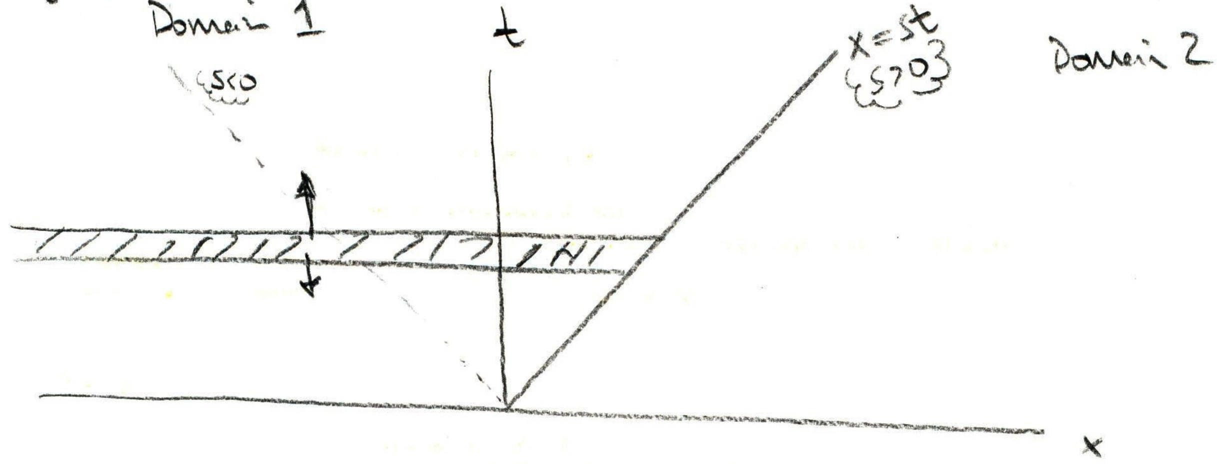
$$= \int_0^{+\infty} \int_{-\infty}^{st} \phi_t u\, dx\, dt + \int_0^{+\infty} \int_{st}^{+\infty} \phi_t u\, dx\, dt + \int_0^{\infty} \int_{-\infty}^{st} \phi_x f(u)\, dx\, dt$$

$$+ \int_0^{+\infty} \int_{st}^{+\infty} \phi_x f(u)\, dx\, dt$$ ✓

$$= u_l \int_0^{+\infty} \int_{-\infty}^{st} \phi_t\, dx\, dt + u_r \int_0^{+\infty} \int_{st}^{+\infty} \phi_t\, dx\, dt + \frac{u_l^2}{2} \int_0^{\infty} \int_{-\infty}^{st} \phi_x\, dx\, dt$$

$$\frac{u_r^2}{2} \int_0^{+\infty} \int_{st}^{+\infty} \phi_x\, dx\, dt$$ ✓

Changing the order of integration of the 1st term



Assume $s>0$ If $s<0$ procedure is the same

Then limits of integrand of 1st integral become:

$$-\infty \le x \le 0 \quad + \quad 0 \le x \le +\infty$$
$$0 \le t \le +\infty \qquad x/s \le t \le +\infty$$

The limits of the 2nd integral then becomes:

$$0 \le x \le +\infty$$
$$0 \le t \le x/s$$

L.H.S

$$\Rightarrow U_l \int_{-\infty}^{0} \int_{0}^{+\infty} \phi_t \, dt \, dx + U_l \int_{0}^{+\infty} \int_{x/s}^{+\infty} \phi_t \, dt \, dx$$

$$+ U_r \int_{0}^{+\infty} \int_{0}^{x/s} \phi_t \, dt \, dx$$

$$+ \frac{U_l^2}{2} \int_{0}^{\infty} \int_{-\infty}^{st} \phi_x \, dx \, dt + \frac{U_r^2}{2} \int_{0}^{\infty} \int_{st}^{+\infty} \phi_x \, dx \, dt$$

$$\Rightarrow U_l \int_{-\infty}^{0} (-\phi(x,0))\,dx + U_l \int_{0}^{+\infty} (-\phi(x,x/s))\,dx$$

$$+ U_r \int_{0}^{+\infty} (\phi(x,x/s) - \phi(x,0))\,dx$$

$$+ \frac{U_l^2}{2} \int_{0}^{\infty} \phi(st,t)\,dt + \frac{U_r^2}{2} \int_{0}^{\infty} (-\phi(st,t))\,dt$$

$$= -U_l \int_{-\infty}^{0} \phi(x,0)\,dx - U_r \int_{0}^{+\infty} \phi(x,0)\,dx$$

$$- U_l \int_{0}^{+\infty} \phi(x,x/s)\,dx + U_r \int_{0}^{+\infty} \phi(x,x/s)\,dx + \left(\frac{U_l^2}{2} - \frac{U_r^2}{2}\right) \int_{0}^{\infty} \phi(st,t)\,dt$$

$v = st$
$dv = dt\,s$

$$= -U_l \int_{-\infty}^{0} \phi(x,0)\,dx - U_r \int_{0}^{+\infty} \phi(x,0)\,dx$$

$$\underbrace{-(U_l - U_r) \int_{0}^{+\infty} \phi(x,x/s)\,dx + \frac{(U_l + U_r)(U_l - U_r)}{\cancel{2s}} \int_{0}^{+\infty} \phi(v,v/s)\,dv}_{=0}$$

$$\therefore = -U_l \int_{-\infty}^{0} \phi(x,0)\,dx - U_r \int_{0}^{+\infty} \phi(x,0)\,dx = \text{RHS} \quad \checkmark$$

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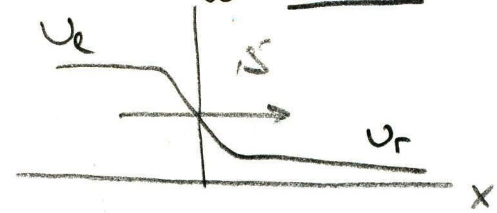


Ex 3.5

eq 3.15 is $u_t + uu_x = \epsilon u_{xx}$

and $u^\epsilon(x,t) = w(x-st)$ s constant, let $\theta = x - st$

$$-sw'(\theta) + ww'(\theta) = \epsilon w''(\theta) \quad ✓$$

$$\epsilon w''(\theta) + (s-w)w'(\theta) = 0 \quad ✓$$

$$\Rightarrow \epsilon w''(\theta) + sw'(\theta) - \frac{1}{2}\frac{d}{d\theta} w^2(\theta) = 0 \quad ✓$$

properties of solution we seek is that for fixed t

$$\lim_{x\to-\infty} u^\epsilon(x,t) = u_l$$

$$\lim_{x\to+\infty} u^\epsilon(x,t) = u_r$$

+ derivatives go to zero

$$\lim_{x\to-\infty} \frac{du^\epsilon(x,t)}{dx} = 0$$

$$\lim_{x\to+\infty} \frac{du^\epsilon(x,t)}{dx} = 0$$

integrate once

$$\epsilon w'(\theta) + sw(\theta) - \frac{1}{2} w^2(\theta) = C_1 \quad ✓$$

take $x \to \pm\infty \Rightarrow \theta \to \pm\infty$

$x \to -\infty$

$$0 + sw_l - \frac{1}{2} w_l^2 = C_1 \quad ✓$$

$x \to +\infty$

$$0 + sw_r - \frac{1}{2} w_r^2 = C_1 \quad ✓$$

Then subtracting these two gives

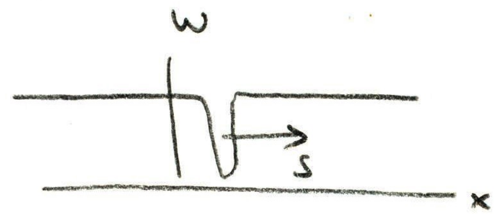
$$s(w_l - w_r) - \frac{1}{2}(w_l^2 - w_r^2) = 0 \quad ✓$$

$$(w_l - w_r)\left[s - \frac{1}{2}(w_l + w_r)\right] = 0 \quad ✓$$

$$\Rightarrow \quad w_l = w_r \quad \text{or} \quad s = \frac{1}{2}(w_l + w_r)$$

If $w_\ell = w_r \equiv W$

Then fn $v^\epsilon(x,t)$ would look like



But this would not correspond to
a "wave" shape & $\therefore$ we excluded this possibility,

How show that this is not a possibility otherwise?

$\therefore \quad s = \frac{1}{2}(w_\ell + w_r)$

& $C_1 = \frac{1}{2}(w_\ell + w_r)w_\ell - \frac{1}{2}w_\ell^2 = \frac{1}{2}w_\ell w_r$ ✓

Then our eq becomes

$$\epsilon w'(\theta) + \frac{1}{2}(w_\ell + w_r)w(\theta) - \frac{1}{2}w^2(\theta) - \frac{1}{2}w_\ell w_r = 0 \quad ✓$$

$$\epsilon w'(\theta) \quad - \frac{1}{2}\left(w^2(\theta) - (w_\ell + w_r)w(\theta) + w_\ell w_r\right) = 0 \quad ✓$$

$$\epsilon w'(\theta) \quad - \frac{1}{2}\left(w(\theta) - w_\ell\right)\left(w(\theta) - w_r\right) = 0 \quad ✓$$

$$\frac{dw}{d\theta} = \frac{1}{2\epsilon}(w - w_\ell)(w - w_r) \quad ✓$$

$$\frac{dw}{(w - w_\ell)(w - w_r)} = \frac{1}{2\epsilon}\, d\theta \quad ✓$$

integrate both sides

$$\frac{1}{2\epsilon}\theta + C_2 = \int \frac{dw}{(w-w_\ell)(w-w_r)} \quad \checkmark$$

$$\frac{1}{(w-w_\ell)(w-w_r)} = \frac{A}{(w-w_\ell)} + \frac{B}{(w-w_r)}$$

$$A = \frac{1}{w_\ell - w_r} \qquad B = \frac{1}{w_r - w_\ell} = \frac{-1}{w_\ell - w_r} \quad \checkmark$$

$$\Rightarrow \frac{1}{(w-w_\ell)(w-w_r)} = \frac{1}{(w_\ell-w_r)(w-w_\ell)} - \frac{1}{(w_\ell-w_r)(w-w_r)} \quad \checkmark$$

$$= \frac{1}{(w_\ell-w_r)}\left[\frac{1}{(w-w_\ell)} - \frac{1}{(w-w_r)}\right] \quad \checkmark$$

Then $\displaystyle\int \frac{dw}{(\ )(\ )} = \frac{1}{(w_\ell-w_r)}\left(\ln|w-w_\ell| - \ln|w-w_r|\right)$ ✓

$$= \frac{1}{(w_\ell-w_r)} \ln\left(\left|\frac{w-w_\ell}{w-w_r}\right|\right)$$

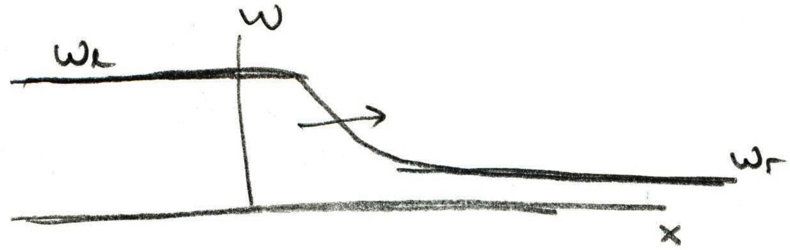
Generally, from the assumed profile solution $w < w_\ell$ & $w > w_r$

$$\therefore \ln\left(\left|\frac{w-w_\ell}{w-w_r}\right|\right) = (w_\ell - w_r)\left[\frac{\theta}{2\epsilon} + C_2\right] \quad \checkmark$$

let $x \to -\infty$ $\quad w \to w_\ell$ $\quad$ eq $\quad -\infty = -\infty$ ✓ if $w_\ell - w_r > 0$

$x \to +\infty$ $\quad w \to w_r$ $\quad$ eq $\quad +\infty = +\infty$ ✓ if $w_\ell - w_r > 0$

Assum



so $w_l > w_r$

+ $w_r \le w \le w_l$

$$\ln\left(\frac{w_l - w}{w - w_r}\right) = (w_l - w_r)\left[\frac{\theta}{2\epsilon} + c_2\right] \quad ✓$$

$$\frac{w_l - w}{w - w_r} = c_3 \exp\left[\frac{(w_l - w_r)}{2\epsilon}\theta\right] \quad ✓$$

$$w_l - w = (w - w_r)\, c_3 \exp\left[\frac{(w_l - w_r)}{2\epsilon}\theta\right] \quad ✓$$

$$\left[c_3 \exp[t] + 1\right] w = w_l + w_r c_3 \exp[t]$$

$$t \equiv \frac{w_l - w_r}{2\epsilon}\theta$$

$$w = \frac{w_l + w_r c_3 \exp(t)}{1 + c_3 \exp(t)} = w_r + \frac{-w_r(1 + c_3 \cancel{\exp(t)}) + w_l + w_r \cancel{c_3} \exp(t)}{1 + c_3 \exp(t)}$$

$$w = w_r + \frac{w_l - w_r}{1 + c_3 \exp(t)} \quad ✓$$

$$= w_r + \frac{(w_l - w_r)}{2} \frac{2}{1 + c_3 \exp(t)} \quad ✓$$

$$\omega(\theta \to -\infty) = \omega_\ell = \omega_r + \frac{(\omega_\ell - \omega_r)}{2} \frac{2}{(1+0)} = \omega_\ell \checkmark.$$

$$\omega(\theta \to +\infty) = \omega_r = \omega_r + \frac{(\omega_\ell - \omega_r)}{2} \left(\frac{1}{\infty}\right) = \omega_r \checkmark$$

Thus C_3 is undetermined pick C_3 ∋ How choose otherwise?

$$\omega(\theta = 0) = \frac{1}{2}(\omega_\ell + \omega_r) = \omega_r + \frac{(\omega_\ell - \omega_r)}{1 + C_3}$$

$$= \frac{\frac{1}{2}\omega_\ell + \frac{1}{2}\omega_r - \omega_r}{\omega_\ell - \omega_r} = \frac{1}{1+C_3}$$

$$= \frac{1}{2} = \frac{1}{1+C_3} \Rightarrow C_3 = 1$$

$\therefore$

$$\omega(\theta) = \omega_r + \frac{\omega_\ell - \omega_r}{2} \frac{2}{1 + \exp(\ast)}$$

$$= \omega_r + \frac{\omega_\ell - \omega_r}{2} \frac{2}{1+\exp(\ast)} \frac{\exp(-\ast/2)}{\exp(-\ast/2)}$$

$$= \omega_r + \frac{(\omega_\ell - \omega_r)}{2} \left[\frac{2\exp(-\ast/2)}{\exp(-\ast/2) + \exp(\ast/2)} \right]$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\frac{2e^{-\theta}}{e^{-\theta}+e^{\theta}} = \frac{e^{-\theta}-e^{\theta}+e^{-\theta}+e^{\theta}}{e^{-\theta}+e^{\theta}} = -\frac{(e^{+\theta}+e^{-\theta})}{e^{-\theta}+e^{\theta}}+1$$

$$= 1-\tanh(\theta)$$

Thus $w(\theta) = w_r + \frac{(w_\ell - w_r)}{2}\left[1-\tanh\left(+\frac{y}{2}\right)\right]$

$$= w_r + \frac{(w_\ell - w_r)}{2}\left[1-\tanh\left(\frac{w_\ell - w_r}{4\epsilon}\theta\right)\right]$$

w/ $\theta = x - st = x - \frac{1}{2}(w_\ell + w_r)t$

To plot we take $w_\ell = 2$ $w_r = 0$ $\Rightarrow s = 1$

Horizontal scale is given by?

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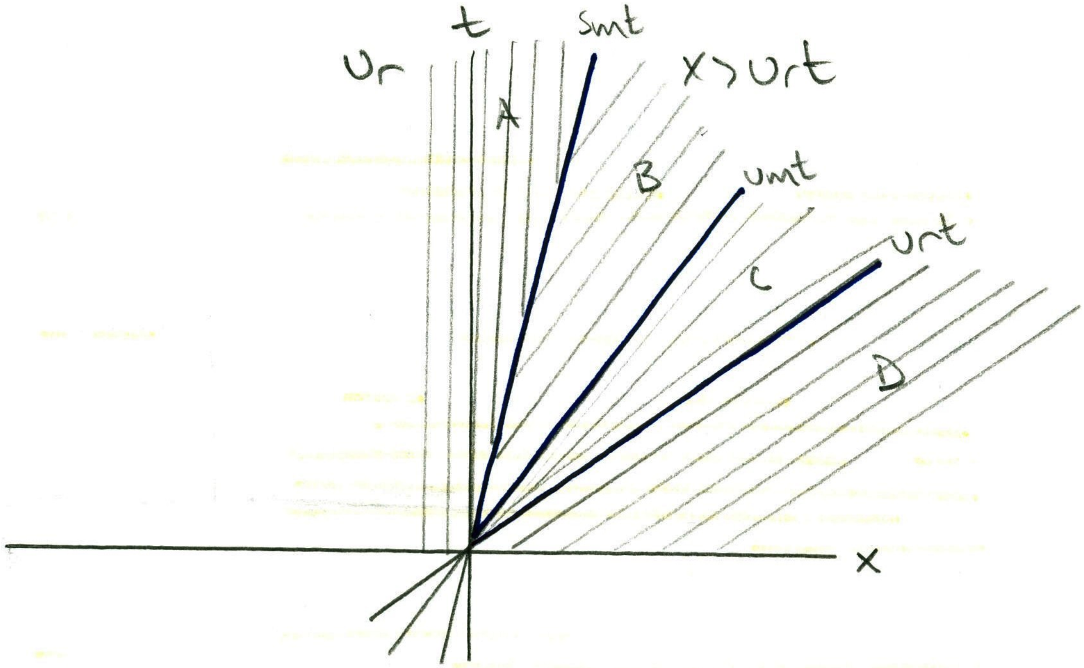
Ex 3.6 To be a weak solution $u(x,t)$ must satisfy

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\left(\phi_t u + \phi_x f(u)\right)dx\,dt = -\int_{-\infty}^{\infty}\phi(x,0)\,u(x,0)\,dx$$ ✓

$\forall\ \phi(x,t) \in C_0^1(\mathbb{R}\times\mathbb{R})$ $\qquad f(u) = u^2/2$

let $u(x,t) = \begin{cases} u_l & x < s_m t \\ u_m & s_m t \le x \le u_m t \\ x/t & u_m t \le x \le u_r t \\ u_r & x > u_r t \end{cases}$

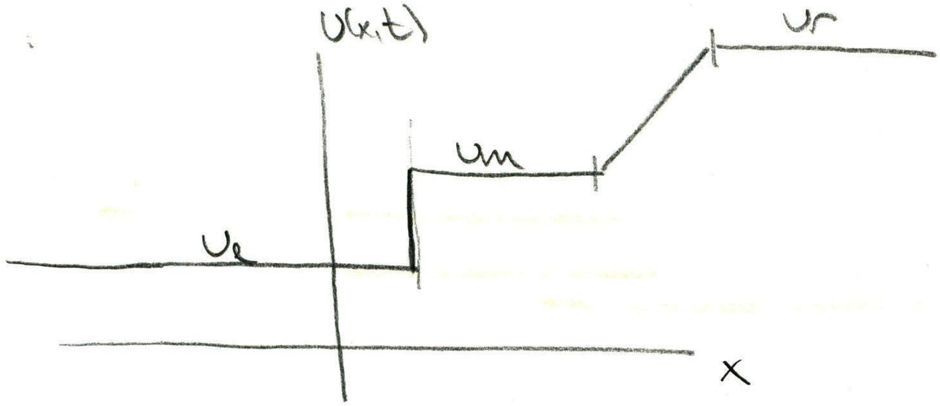


In Region A characteristics are slower than s_m but still positive

B " " faster than s_m & ∥ to u_m

C characteristics can be any speed between u_m & u_r respectively

D " are ∥ to u_r



Then RHS of 3.22 is

$$-U_\ell \int_{-\infty}^{0} \phi(x,0)\,dx - U_r \int_{0}^{+\infty} \phi(x,0)\,dx$$

True $\forall$ Riemann problem solutions. ✓

Then L.H.S of 3.22 becomes

$$\int_0^{+\infty} \left\{ \int_{-\infty}^{S_m t} \left(\phi_t U + \phi_x f(U)\right) dx + \int_{S_m t}^{U_m t} \left(\phi_t U + \phi_x f(U)\right) dx + \int_{U_m t}^{U_r t} \left(\phi_t U + \phi_x f(U)\right) dx \right.$$

$$\left. + \int_{U_r t}^{+\infty} \left(\phi_t U + \phi_x f(U)\right) dx \right\} dt$$ ✓

$$= \int_0^{+\infty} \left\{ \int_{-\infty}^{S_m t} \left(\phi_t U_\ell + \phi_x f(U_\ell)\right) dx + \int_{S_m t}^{U_m t} \left(\phi_t U_m + \phi_x f(U_m)\right) dx + \int_{U_m t}^{U_r t} \left(\phi_t \left(\frac{x}{t}\right) + \phi_x f(x/t)\right) dx \right.$$

$$\left. + \int_{U_r t}^{+\infty} \left(\phi_t U_r + \phi_x f(U_r)\right) dx \right\} dt$$ ✓

$$= \int_0^{+\infty} \left\{ \int_{-\infty}^{S_m t} \phi_t U_\ell \, dx + \int_{S_m t}^{U_m t} \phi_t U_m \, dx + \int_{U_m t}^{U_r t} \phi_t \left(\frac{x}{t}\right) dx + \int_{U_r t}^{+\infty} \phi_t U_r \, dx \right\} dt$$

$$+ \int_0^{+\infty} \left\{ \int_{-\infty}^{S_m t} \phi_x f(U_\ell) \, dx + \int_{S_m t}^{U_m t} \phi_x f(U_m) \, dx + \int_{U_m t}^{U_r t} \phi_x f(x/t) \, dx + \int_{U_r t}^{+\infty} \phi_x f(U_r) \, dx \right\} dt$$

Change limits of integration:

Assuming $S_m, U_m, U_r > 0$

region A_1 , region A_2 , region B , region C

$-\infty \le x \le 0$, $0 \le x \le +\infty$, $0 \le x \le +\infty$, $0 \le x \le +\infty$

$0 \le t \le +\infty$, $x/S_m \le t \le +\infty$, $\frac{x}{U_m} \le t \le \frac{x}{S_m}$, $\frac{x}{U_r} \le t \le \frac{x}{U_m}$

region D

$0 < x < +\infty$

$0 \le t \le x/U_r$

$$= \int_{-\infty}^{0} \int_{0}^{+\infty} \phi_t U_\ell \, dt \, dx + \int_{0}^{+\infty} \int_{x/S_m}^{+\infty} \phi_t U_\ell \, dt \, dx$$

$$+ \int_{0}^{+\infty} \int_{x/U_m}^{x/S_m} \phi_t U_m \, dt \, dx$$

$$+\int_0^{+\infty}\int_{x/u_r}^{x/u_m} \phi_t(x/t)\,dt\,dx$$

$$+\int_0^{+\infty}\int_0^{x/u_r} \phi_t u_r\,dt\,dx$$

$$+\int_0^{+\infty}\Big\{ f(u_\ell)\,\phi(s_m t, t) + f(u_m)\big(\phi(u_m t, t) - \phi(s_m t, t)\big) + \int_{u_m t}^{u_r t}\phi_x f(x/t)\,dx$$

$$+ f(u_r)\big(-\phi(u_r t, t)\big)\Big\}\,dt$$

✓

$$= u_\ell\int_{-\infty}^{0}\big(-\phi(x,0)\big)\,dx + \int_0^{\infty} u_\ell\big(-\phi(x, x/s_m)\big)\,dx$$

$$+\int_0^{+\infty} u_m\big(\phi(x, x/s_m) - \phi(x, x/u_m)\big)\,dx + \int_0^{+\infty}\int_{x/u_r}^{x/u_m}\phi_t(x/t)\,dt\,dx$$

$$+\int_0^{+\infty} u_r\big(\phi(x, x/u_r) - \phi(x,0)\big)\,dx$$

$$+ f(u_\ell)\int_0^{+\infty}\phi(s_m t, t)\,dt + f(u_m)\int_0^{+\infty}\phi(u_m t, t)\,dt - f(u_m)\int_0^{+\infty}\phi(s_m t, t)\,dt$$

$$+\int_0^{+\infty}\int_{u_m t}^{u_r t}\phi_x f(x/t)\,dx\,dt - f(u_r)\int_0^{+\infty}\phi(u_r t, t)\,dt$$ ✓

$$= -u_l\int_{-\infty}^{0}\phi(x,0)\,dx - u_r\int_0^{+\infty}\phi(x,0)\,dx \qquad = RHS \Big\}$$

$$- u_l\int_0^{+\infty}\phi(x, x/s_m)\,dx + f(u_l)\int_0^{+\infty}\phi(s_m t, t)\,dt$$

$$+ u_m\int_0^{+\infty}\phi(x, x/s_m)\,dx - f(u_m)\int_0^{+\infty}\phi(s_m t, t)\,dt$$

$$- u_m\int_0^{+\infty}\phi(x, x/u_m)\,dx + f(u_m)\int_0^{+\infty}\phi(u_m t, t)\,dt$$

$$+\int_0^{+\infty}\int_{x/u_r}^{x/u_m}\phi_t(x/t)\,dt\,dx + \int_0^{+\infty}\int_{u_m t}^{u_r t}\phi_x f(x/t)\,dx\,dt$$

$$+ u_r\int_0^{+\infty}\phi(x, x/u_r)\,dx - f(u_r)\int_0^{+\infty}\phi(u_r t, t)\,dt$$ ✓

$$= -u_l \int_{-\infty}^{0} \phi(x,0)\,dx - u_r \int_{0}^{+\infty} \phi(x,0)\,dx$$

$$+ \left(-u_l + \frac{f(u_l)}{s_m}\right) \int_{0}^{+\infty} \phi(x, x/s_m)\,dx$$

$$+ \left(u_m - \frac{f(u_m)}{s_m}\right) \int_{0}^{+\infty} \phi(x, x/s_m)\,dx$$

$$+ \left(-u_m + \frac{f(u_m)}{u_m}\right) \int_{0}^{+\infty} \phi(x, x/u_m)\,dx$$

$$+ \int_{0}^{+\infty} \int_{x/u_r}^{x/u_m} \phi_t(x/t)\,dt\,dx + \int_{0}^{+\infty} \int_{u_m t}^{u_r t} \phi_x f(x/t)\,dx\,dt$$

$$+ \left(u_r - \frac{f(u_r)}{u_r}\right) \int_{0}^{+\infty} \phi(x, x/u_r)\,dx$$

w/ $f(u) = u^2/2$ ✓

$$= -u_l \int_{-\infty}^{0} \phi(x,0)\,dx - u_r \int_{0}^{+\infty} \phi(x,0)\,dx$$

$$+ \left(-u_l + \frac{f(u_l)}{s_m} + u_m - \frac{f(u_m)}{s_m}\right) \int_{0}^{+\infty} \phi(x, x/s_m)\,dx$$

$$+ \left(-u_m + \frac{u_m}{2}\right) \int_{0}^{+\infty} \phi(x, x/u_m)\,dx$$

$$= \int_0^{+\infty}\int_{x/u_r}^{x/u_m} \phi_t(x/t)\,dt\,dx + \int_0^{+\infty}\int_{u_m t}^{u_r t} \phi_x f(x/t)\,dx\,dt$$

$$+ \frac{u_r}{2}\int_0^{+\infty} \phi(x, x/u_r)\,dx$$

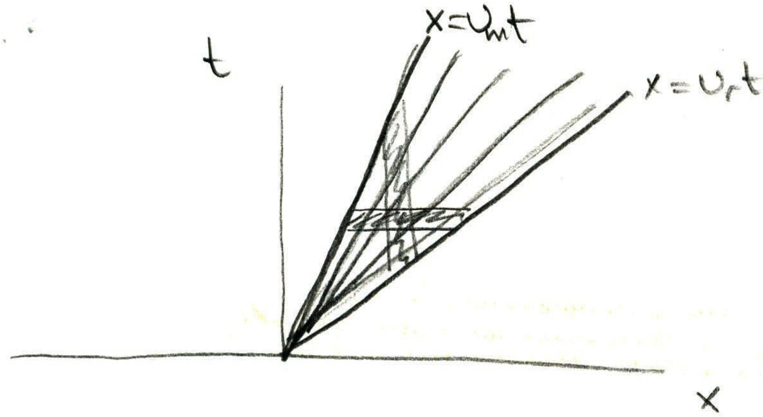
$$= -u_l\int_{-\infty}^{0} \phi(x,0)\,dx - u_r\int_0^{+\infty}\phi(x,0)\,dx$$

$$\underbrace{\left(u_m - u_l + \frac{1}{s_m}\left(f(u_l) - f(u_m)\right)\right)}_{\underbrace{\frac{2}{(s_l+u_m)}\,\frac{(u_l^2 - u_m^2)}{2}}_{(u_l+u_m)\;=\;0}}\int_0^{+\infty}\phi(x, x/s_m)\,dx$$

$$+ -\frac{u_m}{2}\int_0^{+\infty}\phi(x, x/u_m)\,dx + \int_0^{+\infty}\int_{x/u_r}^{x/u_m}\phi_t f'(x/t)\,dt\,dx + \int_0^{+\infty}\int_{u_m t}^{u_r t}\phi_x f(x/t)\,dx\,dt$$

$$+ \frac{u_r}{2}\int_0^{+\infty}\phi(x, x/u_r)\,dx$$

But now lets consider the two integrals



Now $$\int_0^{+\infty}\int_{x/v_r}^{x/v_m} \phi_t\left(\frac{x}{t}\right) dt\,dx + \underbrace{\int_0^{+\infty}\int_{v_m t}^{v_r t} \phi_x f(x/t)\,dx\,dt}$$

* Lets change the order of integration & see what we get.

* Currently x integration is done 1st w/ t integration done second

$0 \le x \le +\infty$

$x/v_r \le t \le x/v_m$

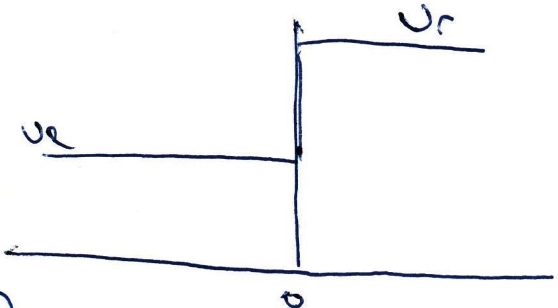
$$\Rightarrow \int_0^{+\infty}\int_{x/v_r}^{x/v_m} \left(\left(\frac{x}{t}\right)\phi_t + \frac{1}{2}\left(\frac{x}{t}\right)^2\phi_x\right) dt\,dx$$ How integrate?

$$= \int_{r=0}^{+\infty}\int_{\theta=}$$

Ex 3.7

LeVeque: $u_t + (f(u))_x = 0 \Rightarrow u_t + f'(u)u_x = 0$ 1

$$u(x,0) = \begin{cases} u_\ell & x<0 \\ u_r & x>0 \end{cases}$$



LeVeque: pg 31

Chars

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = f'(u) \qquad \frac{du}{ds} = 0$$

$$t = s + t_0(\xi) \qquad x = f'(u_0(\xi))s + x_0(\xi) \qquad u = u_0(\xi)$$

IC $\quad t(\xi) = 0$

$x(\xi) = \xi$

$u(\xi) = u_0(\xi, 0)$

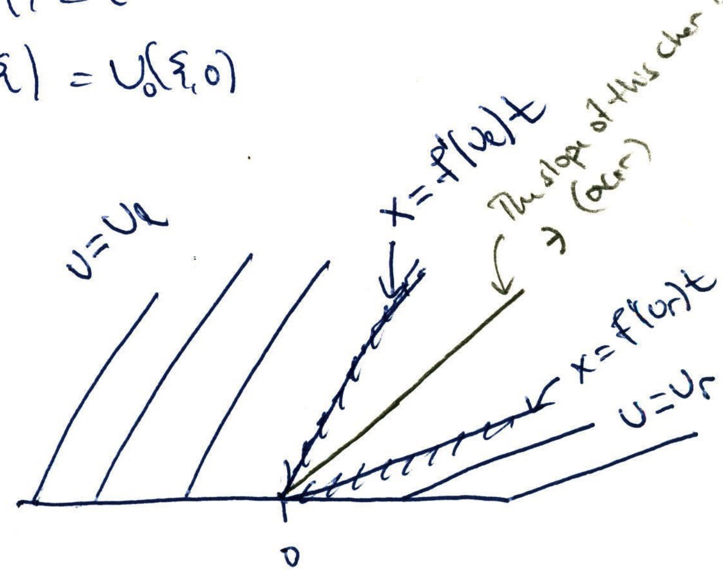
$\Rightarrow \; t = s \; ; \; x = f'(u_0(\xi))s + \xi \; ; \; u = u_0(\xi)$

$$x = f'(u_0(\xi))t + \xi \qquad u = u_0(\xi)$$

As

$$f'' > 0 \Rightarrow f'(u_\ell) < f'(u_r)$$

$$\frac{1}{f'(u_\ell)} > \frac{1}{f'(u_r)}$$



In fan region must have characteristic of all slope 2

Pg 291 Kevorkian goes by smoothing initial discontinuity to get all slopes from $f'(u_l)$ to $f'(u_r)$.

The slope of characteristic in the fan must be between $f'(u_l) < x/t < f'(u_r)$

Thus given $v \ni f'(u_l) < f'(v) < f'(u_r)$ The value on the characteristic w/ speed $x/t = f'(v)$ will be v.

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$$\int_{-M}^{M} u \, dx = u_l (x \Big|_{-M}^{st} + u_r (x \Big|_{st}^{M}$$

$$= u_l (st + M) + u_r (M - st)$$

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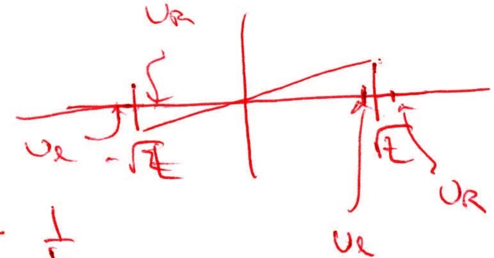
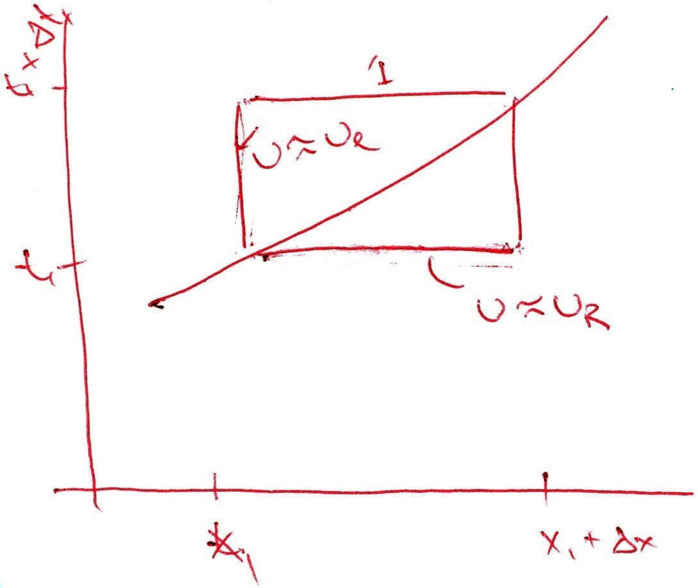
~~Sch~~ shock speed $S = \frac{[f]}{[u]} = \frac{\frac{1}{2}(f_u^2 - f_e^2)}{u_u - u_e}$

$$u = \begin{cases} x/t & -\sqrt{t} < x < \sqrt{t} \\ 0 & \text{else} \end{cases}$$



$$\therefore \quad f_u^2 - f_e^2 = \begin{cases} 0 - \left(\frac{x_e}{t}\right)^2 = -\frac{1}{t} \\ \left(\frac{x_u}{t}\right)^2 - 0 = \frac{1}{t} \end{cases}$$

$$\therefore \quad S = \frac{1}{2} \frac{\pm(1/t)}{\frac{1}{\sqrt{t}}} = \frac{\pm 1}{2\sqrt{t}}.$$



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3.38 1

$$\Rightarrow \cancel{\int_{x_1}^{x_1+\Delta x} u_R\, dx} \le \cancel{\int_{x_1}^{x_1+\Delta x}}$$

$$\Rightarrow \int_{x_1}^{x_1+\Delta x} \left(u_l(t_1) + O(\Delta t)\right) = \int_{x_1}^{x_1+\Delta x} u_r(t_1) + O(\Delta t)$$

$$+ \int_{t_1}^{t_1+\Delta t} f(u_l)\, dt - \int_{t_1}^{t_1+\Delta t} f(u(x_1+\Delta x), t)\, dt$$

$$\approx$$

$$\cancel{f(u_r) + f(u_r)}$$

$$\approx f(u_R(t_1) + O(\Delta t_1), t)$$

$$\approx f(u_R(t_1)) + f(u_R, t_1)\Delta t + f(u_R, t_1)\Delta t^2 + \cdots$$

$$\Rightarrow \int$$

$$U_\ell \Delta x + O(\cancel{\Delta t}\,\Delta x) = U_R \Delta x + O(\cancel{\Delta t}\,\Delta x) + f(U_\ell)\Delta t$$

$$- f(U_R)\Delta t + f(U_R, t_1)\Delta t^2 + \dots$$

$$\simeq O(\Delta t^2)$$

$$\therefore \quad \cancel{\frac{\Delta x}{\Delta t} \simeq s =}$$

$$\Delta x \simeq s(t_1)\Delta t + O(\Delta t^2)$$

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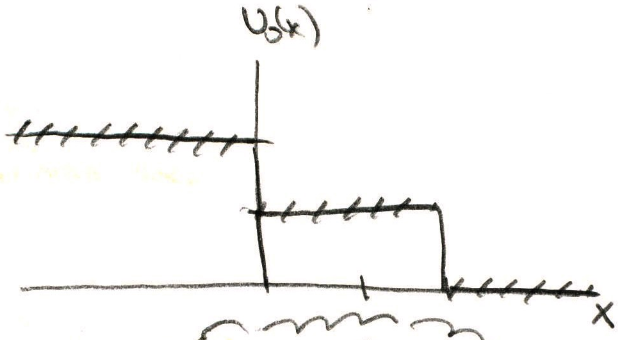
Exerc 3,8

Solve $u_t + uu_x = 0$ w/

$$u_0(x) = \begin{cases} 2 & x<0 \\ 1 & 0<x<2 \\ 0 & x>2 \end{cases}$$



2 Shocks form: Both Right going

Note

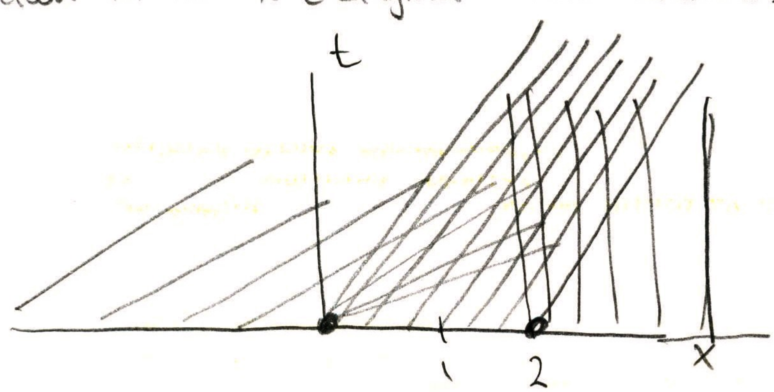
$$\frac{d\upsilon(X(t),t)}{dt} = 0 \quad \text{if} \quad \frac{dX}{dt} = \upsilon(X(t),t) \; *$$

$$* \Rightarrow \frac{dX(0)}{dt} = \upsilon_0(X(0)) = \begin{cases} 2 & X(0)<0 \\ 1 & 0<X(0)<2 \\ 0 & X(0)>2 \end{cases}$$

Then

$$X(t) = \begin{cases} 2t + x_0 & x_0 < 2 \\ t + x_0 & 0<x_0<2 \\ x_0 & x_0>2 \end{cases}$$

Drawn in the x-t diagram these characteristics look like



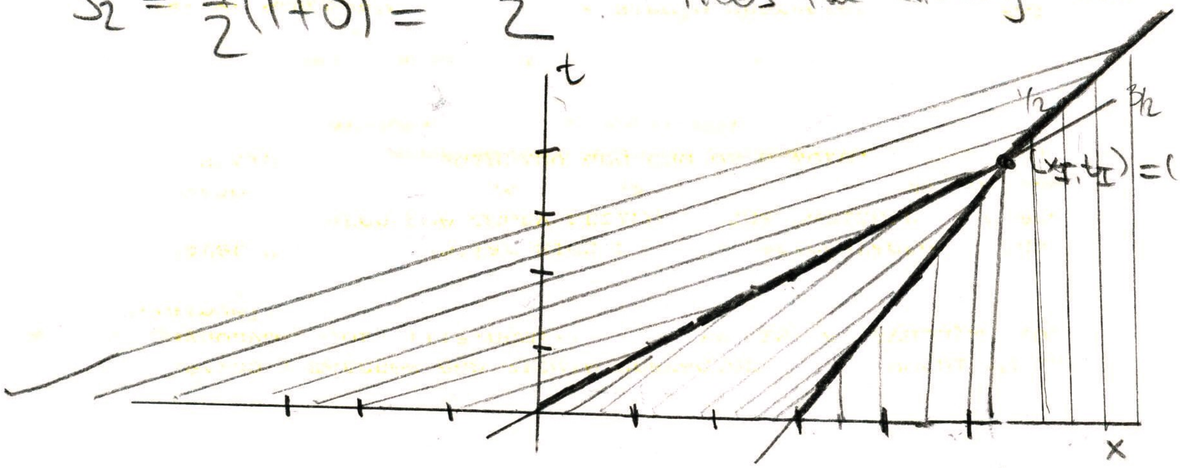
Now this preliminary drawing shows that two shocks form immediatly one at $x=0$ + one at $x=2$

Now The 1st shock that starts at $x=0$ has $U=2$ on its back w/ 1 in front of it Thus the shock speed is

$$S_1 = \frac{1}{2}(1+2) = 3/2$$

The 2nd shock starting at $x=2$ begins w/ speed

$S_2 = \frac{1}{2}(1+0) = \frac{1}{2}$ Thus the drawing now looks like



Shock intersects at $\quad x_1 = \frac{3}{2}t_1 \quad x_2 = \frac{1}{2}t_2 + 2$

$$\frac{3}{2}t_I = \frac{1}{2}t_I + 2 \Rightarrow t_I = 4$$

$$x_I = 6$$

From this point on the shock is fed w/ $u=2$ from the left & $u=0$ from the Right

$$S_3 = \frac{1}{2}(2+0) = 1$$

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Ex 4.1 If $(\rho_\ell + \rho_r) < \rho_{max}$ & w/ initial condition of $\Rightarrow s > 0$

$$\rho(x,0) = \begin{cases} \rho_\ell & x < 0 \\ \rho_r & x > 0 \end{cases}$$

pick $\rho_r = 0$ & $\rho_\ell = \frac{\rho_{max}}{2}$ Then $s = U_{max}\left(1 - \frac{1}{2}\right) = \frac{U_{max}}{2}$ speed of shock

Characteristic speeds are $f'\left(\rho_\ell = \frac{\rho_{max}}{2}\right) = U_{max}\left(1 - \frac{2\rho_{max}}{2\rho_{max}}\right) = 0$

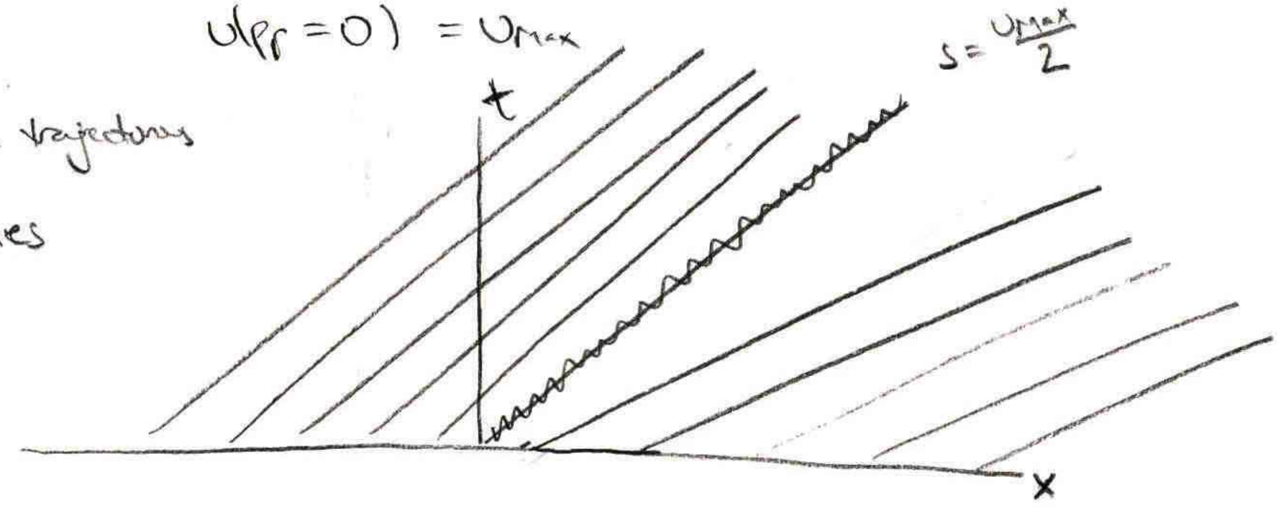
$$f'(\rho_r = 0) = U_{max}$$

Particle speeds are

$$U\left(\rho_\ell = \frac{\rho_{max}}{2}\right) = U_{max}\left(1 - \frac{1}{2}\right) = \frac{U_{max}}{2}$$

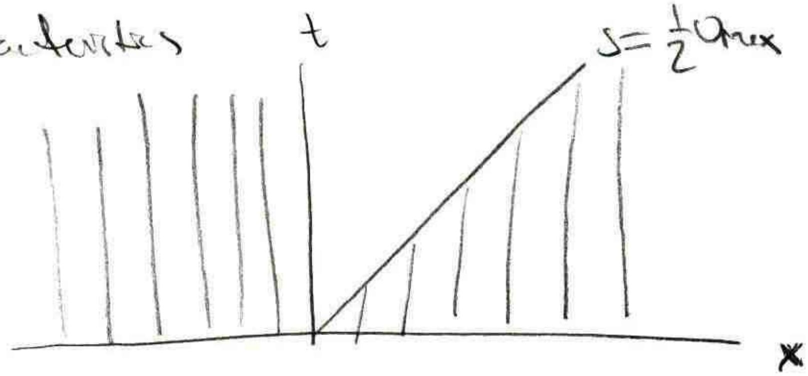
$$U(\rho_r = 0) = U_{max}$$

Then vehicle trajectories look like:
plotting velocities



This seems strange. Are you sure it's correct?

Plot characteristics



The reason for this strange results is that you don't have initial conditions that correspond to a shock. The requirement for a shock is that $\rho_\ell < \rho_r$

Thus we need an initial condition where $\rho_\ell + \rho_r < \rho_{max}$

pick $\rho_r = 2\rho_\ell$ Then $3\rho_\ell < \rho_{max} \Rightarrow \rho_\ell < \frac{\rho_{max}}{3}$

pick $\rho_\ell = \frac{\rho_{max}}{6}$, $\rho_r = \frac{\rho_{max}}{3}$

Then particle speeds $U(\rho) = U_{max}(1 - \rho/\rho_{max})$

$$U(\rho_\ell = \frac{\rho_{max}}{6}) = U_{max}(1 - 1/6) = 5/6\, U_{max}$$

$$U(\rho_r = \frac{\rho_{max}}{3}) = U_{max}(1 - \frac{1}{3}) = \frac{2}{3} U_{max} = \frac{4}{6} U_{max}$$

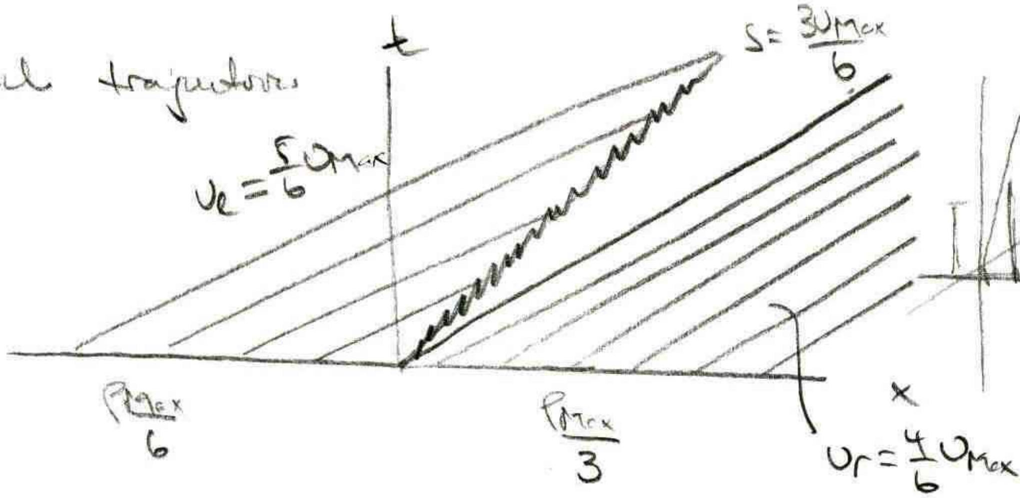
Then characteristic speeds

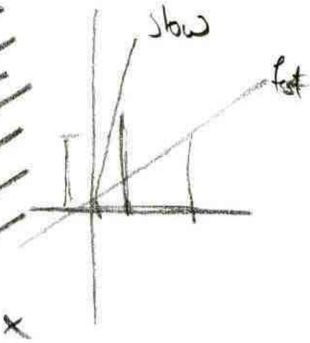
$$C_\ell = U_{max}\left(1 - \frac{2\rho_{max}}{6\rho_{max}}\right) = \frac{2}{3} U_{max} = \frac{4}{6} U_{max}$$

$$C_r = U_{max}\left(1 - \frac{2}{3}\right) = \frac{1}{3} U_{max} = \frac{2}{6} U_{max}$$

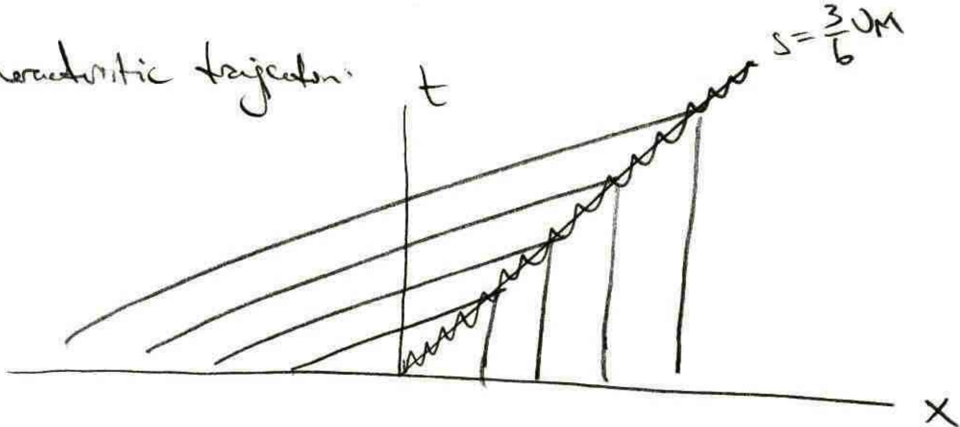
$$s = v_{max}\left(1 - \left(\frac{\rho_{max}}{6} + \frac{2\rho_{max}}{6}\right)\Big/\rho_{max}\right) = v_{max}\left(1 - \frac{1}{2}\right) = \frac{v_{max}}{2} = \frac{3v_{max}}{6}$$

Then particle trajectories





Then the characteristic trajectories



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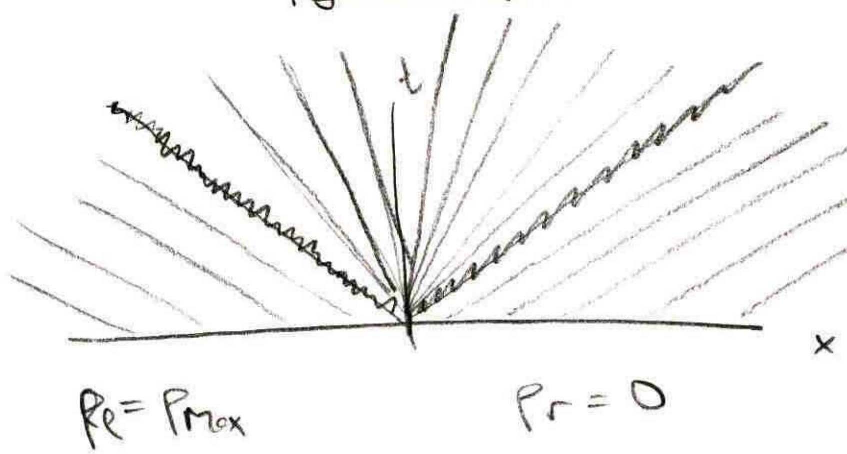
(Ex 4,2)



Characteristic sketch

Then since we are dealing w/ scalar equations $\exists$ one wave (a shock or a rarefaction wave that can propagate from this Riemann problem & separating constant states.

For a shock the requirement of $f'(\rho_\ell) > s > f'(\rho_r) \Rightarrow f'(\rho_\ell) > f'(\rho_r)$

$\Leftrightarrow \rho_\ell < \rho_r$ since this is not satisfied we cannot have a propagating shock. Since we now know that we have a rarefaction fan, the edges will move w/ a speed of the characteristics

$$C_\ell = U_{max}(-1) = -U_{max}$$

$$C_r = U_{max}$$

Now to find the particle paths know that when $\rho_\ell = \rho_{max}$

$U_\ell = 0$. When $\rho = \rho_r = 0$ $U_r = U_{max}$

Inside the fan

$$\frac{dx}{dt} = \text{"slope" of chord to characteristic}$$

$$\frac{dx_{path}}{dt} = v_{particle} = v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

Told that $\frac{x}{t}$ = slope in x-t plane = characteristic speed at that point.

$$= f'(\rho) = v_{max}\left(1 - 2\frac{\rho}{\rho_{max}}\right)$$

$$\Rightarrow 1 - \left(\frac{x}{t v_{max}}\right) = \frac{2\rho}{\rho_{max}}$$

$$\frac{\rho}{\rho_{max}} = \frac{1}{2}\left[1 - \left(\frac{x}{t v_{max}}\right)\right]$$ pt in space

$$\frac{dx}{dt} = v_{max}\left[1 - \frac{1}{2}\left[1 - \frac{x}{t v_{max}}\right]\right] = v_{max}\left[\frac{1}{2} + \frac{x}{2 t v_{max}}\right]$$

$$= \frac{v_{max}}{2}\left[1 + \frac{x}{t v_{max}}\right]$$

Now given $x_0 < 0$ car does not move until it hits edge of Rarefaction fan at $t = \frac{-x_0}{v_{max}}$

Thus solve $\frac{dx}{dt} = \frac{v_M}{2} + \frac{x}{2t}$

$$t\frac{dx}{dt} = \frac{v_M}{2}t + \frac{x}{2}$$

$$t\frac{dx}{dt} - \frac{x}{2} = \frac{v_M}{2}t$$

linear eq for x 1st solve homogenous then particular. Homogenous eq is

$$t\frac{dx}{dt} - \frac{x}{2} = 0$$

let $x = C_1 t^p$

$$C_1 p \cancel{t} t^{p-1} - \frac{C_1 t^p}{2} = 0 \Rightarrow p = \frac{1}{2} \Rightarrow x = C_1 t^{1/2}$$

Alternatively this is a seperable eq. Thus Now find a particular eq

Try $x = C_2 t$

$$\underbrace{t C_2 - \frac{C_2 t}{2}}\overset{\text{set}}{=} \frac{v_m}{2} t$$

$$\frac{C_2}{2} t = \frac{v_m}{2} t \Rightarrow C_2 = v_m$$

The solution to this D.E is

$$x(t) = C_1 t^{1/2} + v_{max} t$$

Initial condition is $x_0 = C_1\left(\frac{-x_0}{v_{max}}\right)^{1/2} + v_{max}\left(\frac{-x_0}{v_{max}}\right)$

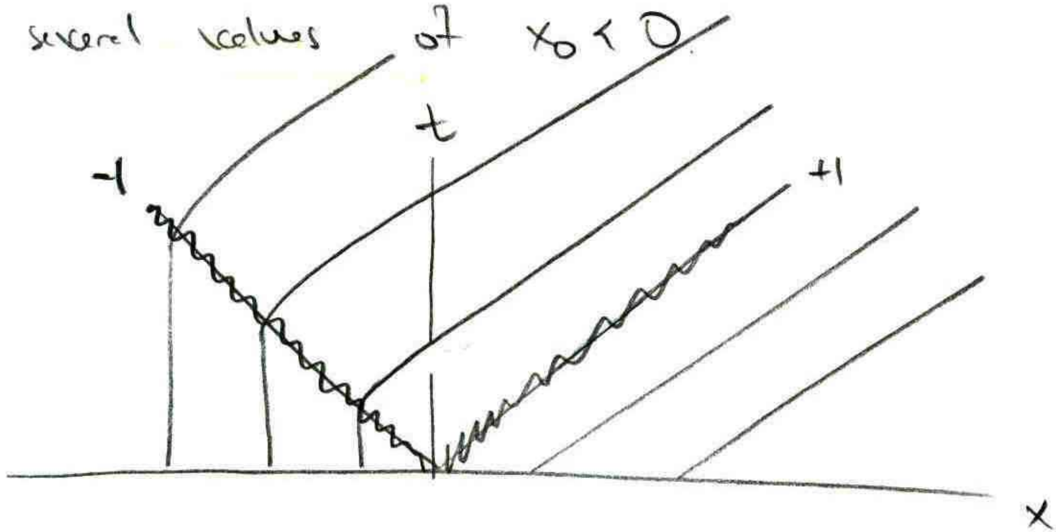
$$2x_0\left(\frac{v_{max}}{-x_0}\right)^{1/2} = C_1$$

Then $x(t) = 2x_0\left(+\frac{v_{max}}{-x_0}\right)^{1/2} t^{1/2} + v_{max} t$

$$x(t) = U_{max} t + 2x_0 \left(\frac{t U_{max}}{-x_0} \right)^{1/2}$$

As a test I'll look at this path w/ $U_{max} = +1$

for several values of $x_0 < 0$



Check this w/ plot.

Ex 4.3

$$\rho(x,t) = \rho_l = \rho_{max} \quad \text{if} \quad x < c_l t \Rightarrow x < -u_{max} t$$

$$\rho(x,t) = \rho_r = 0 \quad \text{if} \quad x > c_r t \Rightarrow x > u_{max} t$$

$$\text{\& } \rho(x,t) = \frac{\rho_{max}}{2}\left[1 - \left(\frac{x}{t u_{max}}\right)\right] \qquad -u_{max} t < x < u_{max} t$$

Since

$$U(\rho) = U_{max}\left(1 - \frac{\rho}{\rho_{max}}\right)$$

$$U(x,t) = U_{max}\left(1 - \frac{\rho(x,t)}{\rho_{max}}\right) = U_{max}\left(1 - \frac{1}{2}\left(1 - \left(\frac{x}{t U_{max}}\right)\right)\right)$$

$$= \frac{U_{max}}{2}\left[1 + \frac{x}{t U_{max}}\right]$$

Ex 4.4

$$v(t) = \frac{dx_{particle}(t)}{dt} \quad \text{w/ } x_{particle}(t) \text{ from Exercise}$$

$$v(t) = u_{max} + \cancel{2}x_0\left(\frac{1}{\cancel{2}}\right)\left(\frac{t u_{max}}{-x_0}\right)^{-1/2}\left(\frac{u_{max}}{-x_0}\right)$$

$$= u_{max} - u_{max}\left(\frac{t u_{max}}{-x_0}\right)^{-1/2} = u_{max}\left(1 - \left(\frac{t u_{max}}{-x_0}\right)^{-1/2}\right)$$

Note: $\lim_{t \to +\infty} v(t) = u_{max}$

4.2 + 4.5 are

$$u(\rho) = u_{max}(1 - \rho/\rho_{max})$$

$$f'(\rho) = u_{max}(1 - 2\rho/\rho_{max})$$

Then $c = f'(\rho_0) - u(\rho_0)$

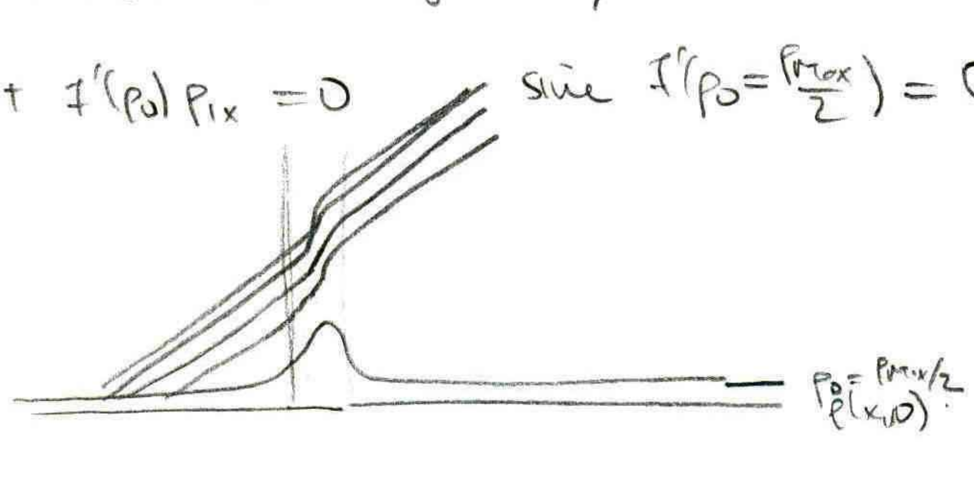
$$= \cancel{u_{max}} - \frac{2u_{max}\rho_0}{\rho_{max}} - \cancel{u_{max}} - \frac{u_{max}\rho_0}{\rho_{max}}$$

$$= -\frac{u_{max}\rho_0}{\rho_{max}} \quad \text{eq } 4.14$$

Ex 4.5 Information about what is happening propagates backwards

Ex 4.6 small disturbances are governed by

$\rho_{1t} + f'(\rho_0)\rho_{1x} = 0$ since $f'(\rho_0 = \frac{\rho_{max}}{2}) = 0$



Since $J'(\rho_0 = \frac{\rho_{max}}{2}) = 0$ the location of the disturbance does not move.

Ex 4.7 This is simply taylor expansion about the average of the two states $\rho^* = \frac{1}{2}(\rho_L + \rho_r)$

See also habermans book. There is an example there also.

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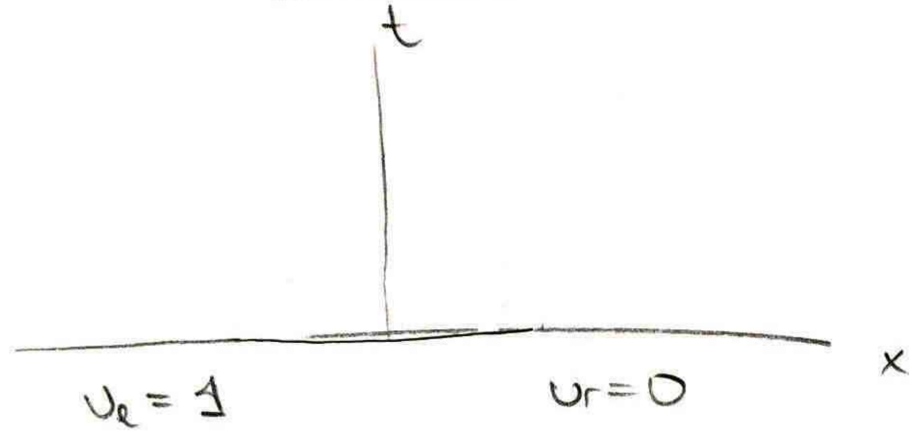
8-10-01 1

Ex 4.8



$f'(u_l)$

$$f(u) = \frac{u^2}{u^2 + \frac{1}{2}(1-u)^2} = \frac{1}{1 + \frac{1}{2}\left(\frac{1}{u} - 1\right)^2}$$

$$f'(u) = \frac{1}{\left(1 + \frac{1}{2}\left(\frac{1}{u}-1\right)^2\right)^2} \left(\frac{1}{u} - 1\right)^3 \left(\frac{-1}{u^2}\right)$$

$$= -\frac{f(u)^2}{u^2} \frac{(1-u)^3}{u^3} = -\frac{(1-u)^3}{u^5} f(u)^2$$

$f'(1) = 0$

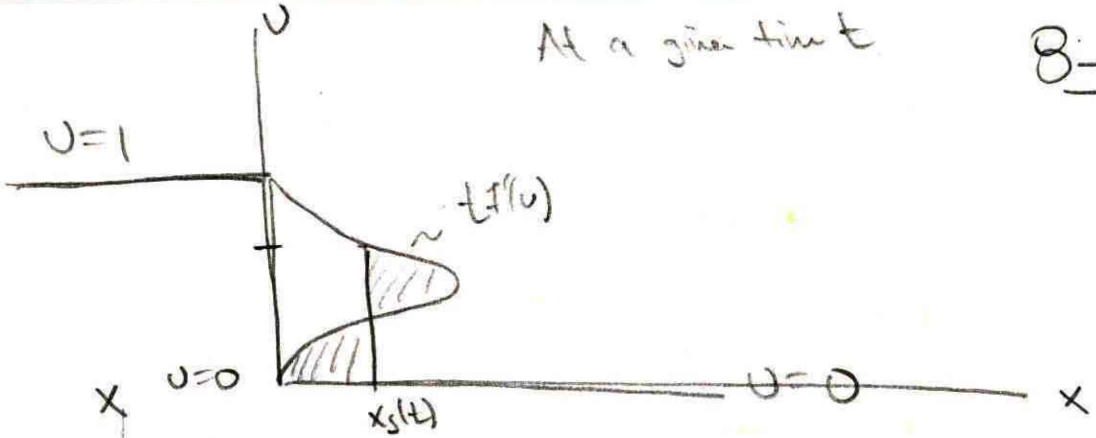
$f'(0) = 0$

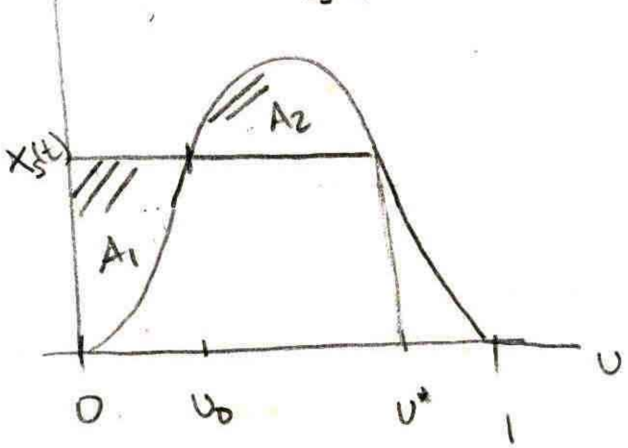
Then for an arbitrary $0 < u < 1$ $\quad f'(u) < 0$

Assuming result from LeVeque that profile of the bulge is

$t f'(u)$ turned side ways

At a given time t





Now if $A_1 = A_2$ then

$$\int_0^{u^*} \left(x_s(t) - t f'(u) \right) du = 0$$

$$\Rightarrow \quad x_s(t)\, u^* - t \int_0^{u^*} f'(u)\, du = 0$$

$$x_s(t)\, u^* - t\left(f(u^*) - 0\right) = 0$$

$$x_s(t) = \frac{t f(u^*)}{u^*}$$

So that the shock speed $S = \dfrac{f(u^*) - f(0)}{u^* - 0}$ = Rankine-Hugoniot

We still must derive an equation to determine u^*. How do?

Ex 4.9 When f is convex or concave $f''(u) > 0$

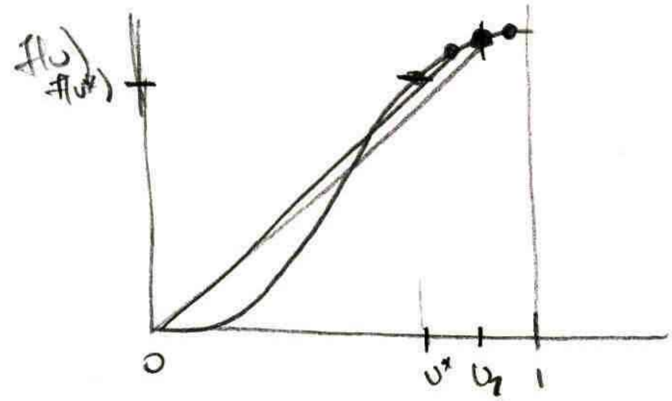
Then sound speed is monotone & thus ~~cond~~ ~~If~~

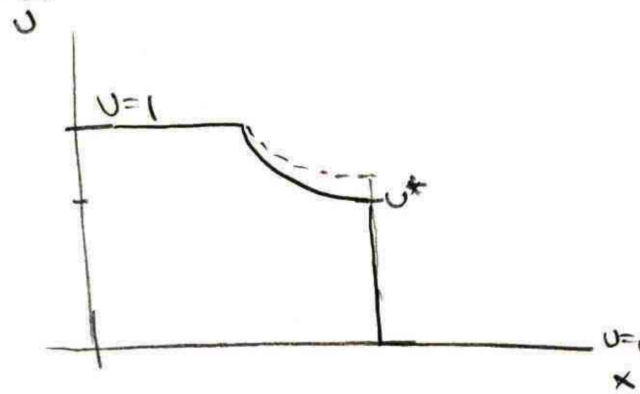
can be inverted uniquely.

Then also the convex hull construction gives either a single line or the fn itself. See last statement on pg 50 LeVeque

Ex 4.10 eq 3.46 is

$$\frac{f(u) - f(u_l)}{u - u_l} \geq s \geq \frac{f(u) - f(u_r)}{u - u_r} \quad \forall\ u \text{ between } u_l \,\&\, u_r$$





If shock goes above u^* Then $u_l = 1$ $u_r = 0$

So $$\frac{f(u) - f(1)}{u - 1} \overset{?}{\geq} s \overset{?}{\geq} \frac{f(u) - 0}{u}$$

$$\frac{f(u)}{u-1} \overset{?}{\geq} s \overset{?}{\geq} \frac{f(u)}{u}$$

this is not satisfied since s is the slope to point $U_?$ from O

& a point below $U_?$ w/ $\mathcal{F}$ smaller has a greater slope

& thus $\frac{\mathcal{F}(U)}{U} \geq S$ $\rightarrow\leftarrow$

PDQ 52 LeVeque

from Whitham:

Eng: $\rho \frac{De}{Dt} + p \frac{\partial u_i}{\partial x_i} = 0.$

If we 1D: $\rho(e_t + ue_x) + pu_x = 0$

Mass

$\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0.$

let E = total energy = kenetic + potential

$= \frac{1}{2}\rho u^2 + \rho e$

"$E_t + (v(E+p))_x = 0$"

MMt:

$\rho \frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = \cancel{\rho F_i}$

Then ~~$E_t = \rho_t u^2 + 2\rho u u_t + \rho_t e + \rho e_t$~~

~~$(v(E+p))_x = (\frac{1}{2}\rho u^3 + \rho u e + up)_x$~~

1D:

$\rho_t + u\rho_x + \rho u_x = 0$

$\rho u_t + \rho u u_x + p_x = 0.$

$(\rho e)_t - \rho_t e + \rho u e_x + p u_x = 0$

~~Take MMt~~ $\Rightarrow$

$\Rightarrow$ ~~$(E - \frac{1}{2}\rho u^2)_t - \rho_t e + \rho u e_x + pu_x$~~

~~$(pu)_x - p_x u = 0.$~~

~~$E_t - \frac{1}{2}\rho_t u^2 - \frac{2}{2}\rho u u_t - \rho_t e + \rho u e_x + (pu)_x - p_x u = 0.$~~

~~$-(\frac{1}{2}u^2 + e)\rho_t - u(\rho u_t) + \rho u e_x - p_x u + (pu)_x = 0.$~~

|| MASS

~~$(= u\rho_x - \rho u_x)$~~

~~$E_t + (u(E+\frac{1}{2}u^2))$~~

Start w/ eq: $\rho e_t + \rho u e_x + p u_x = 0$

$$\Rightarrow \quad (\rho e)_t - \rho_t e + u((\rho e)_x - \rho_x e) + (pu)_x - p_x u = 0$$

then

$$(\rho e)_t + (u\rho e)_x - u_x(\rho e) - \rho_t e - u\rho_x e - p_x u + (pu)_x = 0.$$

let $\rho e = E - \frac{1}{2}\rho u^2$

$$E_t - \left(\tfrac{1}{2}\rho u^2\right)_t + (uE)_x + (pu)_x - \left(\tfrac{1}{2}\rho u^3\right)_x - u_x \rho e - \rho_t e - u\rho_x e - p_x u = 0.$$

$$\Rightarrow \quad E_t + (u(E+p))_x = \left(\tfrac{1}{2}\rho u^2\right)_t + \underline{\rho_t} e + \left(\tfrac{1}{2}\rho u^3\right)_x + \underline{u_x}\rho e + u\underline{\rho_x} e + p_x u$$

3 - terms vanish by cons of mass

$$\Rightarrow \quad E_t + (u(E+p))_x = \tfrac{1}{2}u^2 \rho_t + \tfrac{1}{2}u^2(\rho u)_x + u u_t \rho + \tfrac{1}{2}\rho u \, 2 u u_x + p_x u$$

$= 0$

1st 2 terms. By cons. of mass.

$$E_t + (u(E+p))_x = u(\rho u_t + \rho u u_x + p_x) = 0$$

By cons of mmt.

Pg 54 LeVeque

Derive jump conditions for Gas Dynamics

From LeVeque & show consistant w/ 10.3c Currie

i.e. $$\frac{1}{2}u_1^2 + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} = \frac{1}{2}u_2^2 + \frac{\gamma}{\gamma-1}\frac{P_2}{\rho_2}$$

Consider LeVeque conservation eqs

$$\rho_t + (\rho u)_x = 0$$

$$(\rho u)_x + (\rho u^2 + P)_x = 0$$

$$E_t + (u(E+p))_x = 0$$

Now: jump conditions relative to stationary shock are

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2$$

} same as Currie

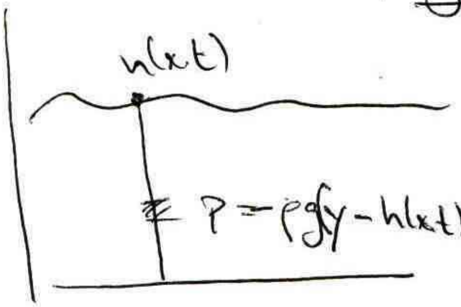
$$u_1(E_1 + p_1) = u_2(E_2 + p_2)$$

✓

$$\cancel{u_1 \rho_1}\left(\frac{E_1}{\rho_1} + \frac{P_1}{\rho_1}\right) = \cancel{u_2\rho_2}\left(\frac{E_2}{\rho_2} + \frac{P_2}{\rho_2}\right) \quad \text{By Mass}$$

$$\Rightarrow \left(\underline{e_1} + \frac{1}{2}u_1^2 + \underline{\frac{P_1}{\rho_1}}\right) = e_2 + \frac{1}{2}u_2^2 + \frac{P_2}{\rho_2}.$$

$$e_1 + \frac{P_1}{\rho_1} = \frac{1}{\gamma-1}\frac{P_1}{\rho_1} + \frac{P_1}{\rho_1} = \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} \quad ✓.$$



$$-\rho g\int_0^{h(x,t)}(y-h)\,dy = -\rho g\left(\frac{y^2}{2}-hy\right)\Big|_0^h = -\rho g\left(-\frac{h^2}{2}\right) = \frac{h^2\rho g}{2}$$

$\Rightarrow$ ÷ by ρ

$$(hv)_t + \left(hv^2 + \frac{h^2 g}{2}\right)_x = 0$$

$$h_t v + v_t h + \cancel{h_x v^2} + \cancel{2} v v_x h + h h_x g = 0$$

||
By Mass

$$(-v_x\cancel{h} - \cancel{h_x v})v + \cdots$$

$$\Rightarrow \; v_t\cancel{h} + v v_x \cancel{h} + \cancel{h} h_x g = 0.$$

$$v_t + \left(\frac{v^2}{2} + gh\right)_x = 0. \qquad + \quad \text{Mass} \quad h_t + (vh)_x = 0.$$

$\phi = gh$ $\qquad g \rightarrow (gh)_t + (vhg)_x = 0$

$$\Rightarrow \quad \begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} \frac{v^2}{2} + \phi \\ v\phi \end{pmatrix}_x = \bar{0}.$$

Pg 60 LeVeque

$$V = R^{-1} U$$

$$V(x,0) = R^{-1} U_0(x)$$

get $V_p(x,0)$ $\quad p = 1, 2, \dots, m$.

$$U(x,t) = \sum_{p=1}^{m} V_p(x - \lambda_p t, 0)\, r_p = \sum_{p \neq i} c_p r_p + V_i(x - \lambda_i t, 0)\, r_i$$

$$= R \left(\bar{c} \quad \bar{c} = \begin{cases} c_p & p \neq i \\ V_i & p = i \end{cases} \right) = R \bar{V}(x - \lambda_i t)$$

$$= U_0(x - \lambda_i t)$$

Ex 6.1

pg 61 LeVeque

eval $A = \pm c$

$$\begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = +c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$-v_2 = cv_1 \qquad \text{evector } \bar{v} = \begin{pmatrix} 1 \\ -c \end{pmatrix}$$

$$-c^2 v_1 = cv_2$$

$$\begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -c\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad \text{evecto } \bar{v} = \begin{pmatrix} 1 \\ +c \end{pmatrix}$$

let $R = \dfrac{1}{\sqrt{1+c^2}}\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}$; $R^{-1} = \dfrac{\sqrt{1+c^2}}{2}\begin{pmatrix} 1 & 1/c \\ 1 & -1/c \end{pmatrix}$

Check $\Lambda = \begin{pmatrix} -c & 0 \\ 0 & c \end{pmatrix} \overset{?}{=} R^{-1}AR = \frac{1}{2}\begin{pmatrix} 1 & 1/c \\ 1 & -1/c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}$ ✓

Thus let $V = R^{-1}U$ $\qquad$ $U_t + AU_x = 0$

$$ARR^{-1}U_x = 0$$

Then system becomes $\Rightarrow$

$$V_t + \Lambda V_x = 0$$

or $\quad V_{1_t} - cV_{1x} = 0 \quad \Rightarrow \quad V_1(x,t) = V_1^0(x+ct)$

& $\quad V_{2_t} + cV_{2x} = 0 \quad \Rightarrow \quad V_2(x,t) = V_2^0(x-ct)$

get this V_1^0 & V_2^0 $\qquad$ $\bar{V}^0 = R^{-1}\bar{U}^0 = R^{-1}\begin{pmatrix} U_0'(x) \\ U_1(x) \end{pmatrix}$

$$\begin{pmatrix} V_1^0 \\ V_2^0 \end{pmatrix} = \frac{\sqrt{1+c^2}}{2}\begin{pmatrix} U_0'(x) + \frac{U_1(x)}{c} \\ U_0'(x) - \frac{U_1(x)}{c} \end{pmatrix}$$

Thus,
$$V_1(x,t) = \frac{\sqrt{1+c^2}}{2}\left(U_0'(x+ct) + \frac{1}{c}U_1(x+ct)\right)$$
$$V_2(x,t) = \frac{\sqrt{1+c^2}}{2}\left(U_0'(x+ct) - \frac{1}{c}U_1(x+ct)\right)$$

Now: $u(x,t) = Rv(x,t)$

$$\Rightarrow \begin{pmatrix} v(x,t) \\ w(x,t) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ c & -c \end{pmatrix}\begin{pmatrix} u_0'(x+ct) + \frac{1}{c}u_1(x+ct) \\ u_0'(x-ct) - \frac{1}{c}u_1(x-ct) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} u_0'(x_1) + u_0'(x_2) + \frac{1}{c}(u_1(x_1) - u_1(x_2)) \\ cu_0'(x_1) - cu_0'(x_2) + u_1(x_1) + u_1(x_2) \end{pmatrix}$$

Then $\quad u_x = v \;=\; \frac{1}{2}(\;\cdots\cdots\;)$

$\quad\quad\;\; u_t = w \;=\; \frac{1}{2}(\qquad\qquad)$

$$u = \frac{1}{2}\left[\int (u_0'(x+ct) + u_0'(x-ct))\,dx + \frac{1}{c}\int (u_1(x+ct) - u_1(x-ct))\,dx\right] + C(t)$$

$u = x+ct$

$du = dx$

$$= \frac{1}{2}\left[u_0(x+ct) + u_0(x-ct) + \frac{1}{c}\left(\int (u_1(x+ct) - u_1(x-ct))\,\underset{d(x+ct)}{dx}\right)\right] \; c(t)$$

Then — see back of this page for ~~better~~ explanation.

$$u_t = \frac{1}{2}\left[c u_0'(x+ct) - c u_0'(x-ct) + \frac{1}{\cancel{c}}\left(\cancel{c} u_1(x+ct) + \cancel{c} u_1(x-ct)\right)\right] + C'(t)$$

$$\equiv w(x,t) \qquad \text{if} \quad C'(t) = 0$$

$$\Rightarrow C(t) = \text{const ind of } t$$

$\therefore$

Ex 6.2

Pg 61 LeVeque

$$v_t = u_{xt} = \phi_x$$

$$w_t = \phi_y$$

$$\phi_t = c^2 (v_x + w_y)$$

$$\therefore \begin{pmatrix} v \\ w \\ \phi \end{pmatrix}_t + \begin{pmatrix} -\phi \\ 0 \\ -c^2 v \end{pmatrix}_x + \begin{pmatrix} 0 \\ -\phi \\ -c^2 w \end{pmatrix}_y = 0$$

$$\begin{pmatrix} v \\ w \\ \phi \end{pmatrix}_t + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -c^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \\ \phi \end{pmatrix}_x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -c^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ w \\ \phi \end{pmatrix}_y = 0$$

Ex 6.3 — pg 61 LeVeque

Assume $\begin{pmatrix} u \\ \phi \end{pmatrix}^0 = \begin{pmatrix} u_0(x) \\ \phi_0(x) \end{pmatrix}$

Introduce char var $\begin{pmatrix} \bar{u} & 1 \\ \bar{\phi} & \bar{u} \end{pmatrix} \longrightarrow$ e.values $\{\bar{u} - \sqrt{\bar{\phi}}, \bar{u} + \sqrt{\bar{\phi}}\}$

e.vectors $\{\{-\frac{1}{\sqrt{\bar{\phi}}}, 1\}, \{\frac{1}{\sqrt{\bar{\phi}}}, 1\}\}$

$$\Lambda = \begin{pmatrix} \bar{u} - \sqrt{\bar{\phi}} & 0 \\ 0 & \bar{u} + \sqrt{\bar{\phi}} \end{pmatrix}$$

$$R = \frac{1}{\sqrt{1 + \frac{1}{\bar{\phi}}}} \begin{pmatrix} -\frac{1}{\sqrt{\bar{\phi}}} & \frac{1}{\sqrt{\bar{\phi}}} \\ 1 & 1 \end{pmatrix}$$ ✓ed

$$R^{-1} = \frac{\sqrt{1 + \frac{1}{\bar{\phi}}}}{2} \begin{pmatrix} -\sqrt{\bar{\phi}} & 1 \\ \sqrt{\bar{\phi}} & 1 \end{pmatrix}$$

Define $v = R^{-1} u$. Then, eq for v_1 + v_2 decouple, To ~~become~~: $\partial_t(V_1) + (\bar{U} - \sqrt{\bar{\Phi}})\partial_x(V_1) = 0$

$$\partial_t(V_2) + (\bar{U} + \sqrt{\bar{\Phi}})\partial_x(V_2) = 0$$

w/ Initial fns of: $\begin{pmatrix} V_1^0 \\ V_2^0 \end{pmatrix} = \frac{\sqrt{1 + 1/\bar{\Phi}}}{2} \begin{pmatrix} -\sqrt{\bar{\Phi}}\, U_0(x) + \phi_0(x) \\ \sqrt{\bar{\Phi}}\, U_0(x) + \phi_0(x) \end{pmatrix}$

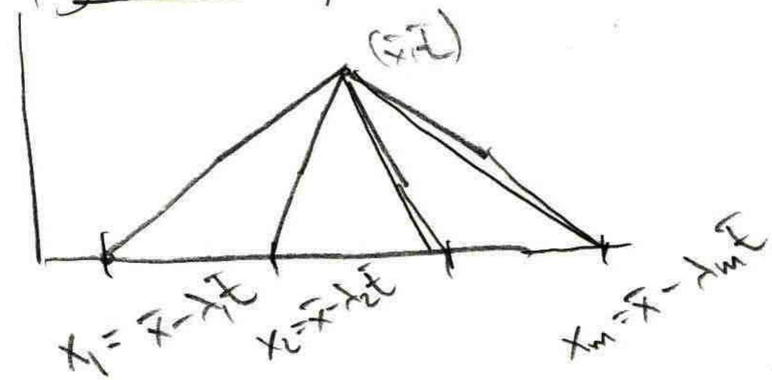
& soln: $V_1(x,t) = \frac{\sqrt{1 + 1/\bar{\Phi}}}{2}\left(-\sqrt{\bar{\Phi}}\, U_0\left(x - (\bar{U} - \sqrt{\bar{\Phi}})t\right) + \phi_0\left(x - (\bar{U} - \sqrt{\bar{\Phi}})t\right)\right)$

$$V_2(x,t) = \frac{\sqrt{\phantom{1 + 1/\bar{\Phi}}}}{2}\left(\sqrt{\bar{\Phi}}\, U_0\left(x - (\bar{U} + \sqrt{\bar{\Phi}})t\right) + \phi_0\left(x - (\bar{U} + \sqrt{\bar{\Phi}})t\right)\right)$$

Then $\bar{U} = \begin{pmatrix} U(x,t) \\ \phi(x,t) \end{pmatrix} = R\begin{pmatrix} V_1(x,t) \\ V_2(x,t) \end{pmatrix} =$

$$= \frac{1}{2}\begin{pmatrix} U_0(x_1) + U_0(x_2) - \frac{1}{\sqrt{\bar{\Phi}}}\phi_0(x_1) + \frac{1}{\sqrt{\bar{\Phi}}}\phi_0(x_2) \\ -\sqrt{\bar{\Phi}}\, U_0(x_1) + \sqrt{\bar{\Phi}}\, U_0(x_2) + \phi_0(x_1) + \phi_0(x_2) \end{pmatrix}$$

Pg 61 Lavegue



If sol smooth at $\{x_1, \ldots, x_m\}$ sol will be

smooth at $(\bar{x},\bar{t})$

Pg 63. LeVeque

$\rho,\ P,\ v,\ E = E(p,\rho)$

$$E = \frac{P}{\gamma - 1} + \frac{1}{2}\rho v^2$$

$\bar{q} = \begin{pmatrix} \rho \\ v\rho \\ E \end{pmatrix}$ state for polytropic gas.

$$f(u) = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{pmatrix} = \begin{pmatrix} u_2 \\ u_2^2/u_1 + p(u) \\ \frac{u_2}{u_1}(u_3 + p(u)) \end{pmatrix}$$

$$f' = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{u_2^2}{u_1^2} + \frac{\partial p}{\partial u_1} & \frac{2u_2}{u_1} + \frac{\partial p}{\partial u_2} & \frac{\partial p}{\partial u_3} \\ -\frac{u_2}{u_1^2}(u_3 + p(u)) + \frac{u_2}{u_1}\left(\frac{\partial p}{\partial u_1}\right) & \frac{u_3 + p(u)}{u_1} + \frac{u_2}{u_1}\left(\frac{\partial p}{\partial u_2}\right) & \frac{u_2}{u_1}\left(1 + \frac{\partial p}{\partial u_3}\right) \end{pmatrix}$$

$$\frac{\partial p}{\partial u_1} = \ ?$$ get using eq of state:

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho v^2$$

$$\Rightarrow \quad p = (\gamma - 1)\left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1} \right)$$

$$\frac{\partial p}{\partial u_1} = (\gamma - 1)\left(\frac{1}{2}\frac{u_2^2}{u_1^2} \right)$$

$$\frac{\partial p}{\partial u_2} = (\gamma - 1)\left(-\frac{u_2}{u_1} \right) \quad ; \quad \frac{\partial p}{\partial u_3} = (\gamma - 1).$$

$$\therefore \quad f'(u) = \begin{pmatrix} 0 & 1 & 0 \\ -v^2 + \frac{1}{2}(\gamma-1)v^2 & 2v - (\gamma-1)v & (\gamma-1) \\ (\gamma - \ )\frac{v^2}{2} & (3-\gamma)v & \end{pmatrix}$$

$$(f'(u))_{31} = -\frac{v}{\rho}(E+p) + v\left(\frac{1}{2}(\gamma-1)v^2\right)$$

$$= \frac{(\gamma-1)}{2}v^3 - \frac{(E+p)v}{\rho} \quad \checkmark$$

$$(f'(u))_{32} = \frac{(E+p)}{\rho} - v^2(\gamma-1) \quad \checkmark$$

$$(f'(u))_{33} = v(1+\gamma-1) = \gamma v \quad \checkmark$$

entering 6.30 into Ma & Asking for Eivalues gives

$E = \frac{1}{2}\rho v^2 + \rho e$

$\frac{E}{\rho} = \frac{1}{2}v^2 + e$

$$v,\; v \pm \frac{\sqrt{(\gamma-1)\rho(2e+2p-v^2\rho)}}{\sqrt{2}\,\rho}$$

putting e in gives

$$v,\; v \pm \sqrt{\frac{\gamma p}{\rho}}$$

As claimed !!

Ex 6.4 $\rho(x,t) = \hat{\rho}(x - \bar{v}t)$

$$v = \bar{v}$$
$$p = \bar{p}$$

Euler eq 5.6.

$$\begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix}_x = 0$$

$$\Rightarrow \quad -\bar{v}\,\partial_\xi \hat{\rho}(\xi) + \partial_\xi \hat{\rho}(\xi)\,\bar{v} = 0 \quad ✓$$

$$* \quad \bar{v}\,\partial_\xi \hat{\rho}\,(-\bar{v}) + \bar{v}^2 \partial_\xi \hat{\rho} + 0 = 0 \quad ✓.$$

$$+ \quad \partial_t E + \bar{v}\,\partial_x E = 0 \quad ??$$

Ex 6.4 pg 63 LeVeque

$$\bar{p} = \hat{K} \hat{\rho}^{\gamma}(x - \bar{v} t)$$

1D Euler energy eq is $E_t + (v(E + p))_x = 0$

But for polytropic gas $E = \frac{p}{\gamma - 1} + \frac{1}{2} \rho v^2$

$$= \frac{\bar{p}}{\gamma - 1} + \frac{1}{2} \hat{\rho}(x - \bar{v} t) \bar{v}^2$$

$$\therefore \quad \partial_t E = -\frac{1}{2} \partial_\xi \hat{\rho} \; \bar{v}^3$$

$$v(E + p) = \bar{v}\left(\left(\frac{\bar{p}}{\gamma - 1} + \frac{1}{2} \hat{\rho}(x - \bar{v} t) \bar{v}^2\right) + \bar{p}\right)$$

$$\therefore \quad \partial_x \downarrow \quad = \frac{1}{2} \partial_\xi \hat{\rho} \; \bar{v}^3$$

✓

isentropic? No if so then $\bar{p} = \hat{k}\hat{\rho}^{\gamma}$ w/ same $\hat{k}$ but $\bar{p}$ is a constant & $\hat{\rho}($ is not depends on $x - \bar{v}t$.

Ex 6.5

Pg 64 LeVeque

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \rho \\ \rho v \\ s \end{pmatrix}$$

$$u_{1_t} + u_{2_x} = 0$$

$$u_{2_t} + \left(\frac{u_2^2}{u_1} + p(u_1, u_3)\right)_x = 0$$

$$u_{3_t} + \frac{u_2}{u_1} u_{3x} = 0$$

Assume
$$u_1 = \bar{u}_1 + u(x,t)$$
$$u_2 = \bar{u}_2 + v(x,t)$$
$$u_3 = \bar{u}_3 + w(x,t)$$

$$\frac{u_2^2}{u_1} = \frac{\bar{u}_2^2 + 2\bar{u}_2 v + v^2}{\bar{u}_1 + u}$$

$$\approx \frac{1}{\bar{u}_1}\left(1 - \frac{u}{\bar{u}_1}\right)\left(\bar{u}_2^2 + 2\bar{u}_2 v + v^2\right)$$

$$\frac{u_2}{u_1} \approx \frac{1}{\bar{u}_1}\left(1 - \frac{u}{\bar{u}_1}\right)\left(\bar{u}_2 + v\right)$$

$$\longrightarrow \frac{\bar{u}_2}{\bar{u}_1}$$ only order term needed.

$$\left(\frac{u_2^2}{u_1}\right)_x = \left(\frac{1}{\bar{u}_1}\left(2\bar{u}_2 v - \frac{\bar{u}_2^2}{\bar{u}_1} u\right)\right)_x = \frac{1}{\bar{u}_1}\left(2\bar{u}_2 v_x - \frac{\bar{u}_2^2}{\bar{u}_1} u_x\right)$$

$$p(u_1, u_3) \simeq p(\bar{u}_1, \bar{u}_3) + \left.\frac{\partial p}{\partial \rho}\right|_s (\bar{u}_1, \bar{u}_3) u + \left.\frac{\partial p}{\partial S}\right|_\rho (\bar{u}_1, \bar{u}_3) w + O(u \cdot w)$$

$$\partial_x p \approx \left.\frac{\partial p}{\partial \rho}\right|_s u_x + \left.\frac{\partial p}{\partial S}\right|_\rho w_x$$

Putting everything together get

$$u_t + v_x = 0$$

$$v_t + 2\frac{\bar{u}_2}{\bar{u}_1} v_x - \frac{\bar{u}_2^2}{\bar{u}_1^2} u_x + \left.\frac{\partial p}{\partial \rho}\right|_s u_x + \left.\frac{\partial p}{\partial S}\right|_\rho w_x = 0$$

$$w_t + \frac{\bar{u}_2}{\bar{u}_1} w_x = 0.$$

or:

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_t + \begin{pmatrix} 0 & 1 & 0 \\ -\frac{\bar{u}_2^2}{\bar{u}_1^2} + \frac{\partial p}{\partial \rho}\Big|_s & 2\frac{\bar{u}_2}{\bar{u}_1} & \frac{\partial p}{\partial s}\Big|_\rho \\ 0 & 0 & \frac{\bar{u}_2}{\bar{u}_1} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = 0$$

want to determine e.values of this matrix: ↑

get $\frac{\bar{u}_2}{\bar{u}_1}$, $\frac{\bar{u}_2}{\bar{u}_1} \pm \sqrt{\frac{\partial p}{\partial \rho}\Big|_s}$

$\Rightarrow$ $\bar{v}$, $\bar{v} \pm \sqrt{\frac{\partial p}{\partial \rho}\Big|_s}$ $\quad \bar{v}$ velocity we are lin around

Ex 6.6 Pg 64 LeVeque

$$\begin{pmatrix}\rho\\ \rho v\end{pmatrix}_t + \begin{pmatrix}\rho v\\ \rho v^2 + \hat{k}\rho^r\end{pmatrix}_x = 0 \qquad \hat{k} = ke^{S/c_v}$$

$u_1 = \rho$

$u_2 = \rho v$

$$\Rightarrow \quad u_t + \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + \hat{k}u_1^r \end{pmatrix}_x = 0$$

$$f'(u) = \begin{pmatrix} 0 & 1 \\ -\frac{u_2^2}{u_1^2} + r\hat{k}u_1^{r-1} & \frac{2u_2}{u_1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -v^2 + r\hat{k}\rho^{r-1} & 2v \end{pmatrix}$$

evalues $\quad \lambda^2 - (2v)\lambda + (v^2 - r\hat{k}\rho^{r-1}) = 0$

$$\lambda = \frac{2v \pm \sqrt{4v^2 - 4(v^2 - r\hat{k}\rho^{r-1})}}{2}$$

$$\lambda = \nu \pm \sqrt{r \hat{k} \rho^{r-1}} = \nu \pm \sqrt{\frac{r}{\rho} \hat{k} \rho^{r}} = \nu \pm \sqrt{\frac{rp}{\rho}}$$

Pg 64 LeVeque

Ex 6.7

$$\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v^2/2 + \phi \\ v\phi \end{pmatrix}_x = 0$$

$v = \bar{v} + v_1$ $\qquad v^2/2 = \frac{1}{2}(\bar{v}^2 + 2\bar{v}v_1 + v_1^2)$

$\phi = \bar{\phi} + \phi_1$ $\qquad v\phi = \bar{v}\bar{\phi} + \bar{v}\phi_1 + \bar{\phi}v_1 + \phi_1 v_1$

$$\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}_t + \begin{pmatrix} \bar{v}v_{1x} + \phi_{1x} \\ \bar{v}\phi_{1x} + \bar{\phi}v_{1x} \end{pmatrix} = 0$$

$$\parallel$$

$$\begin{pmatrix} \bar{v} & 1 \\ \bar{\phi} & \bar{v} \end{pmatrix}\begin{pmatrix} v_1 \\ \phi_1 \end{pmatrix}_x \quad \checkmark$$

$\bar{q} = \begin{pmatrix} v \\ \phi \end{pmatrix}$ $\qquad f'(u)\Big|_{(\bar{v},\bar{\phi})} = \begin{pmatrix} \bar{v} & 1 \\ \bar{\phi} & \bar{v} \end{pmatrix} \quad \checkmark$

sound speed find evalue of $F'(u)$

$= \bar{u} - \sqrt{\bar{\phi}} \quad \& \quad \bar{u} + \sqrt{\bar{\phi}} \quad \Rightarrow \quad$ sound speed $= \sqrt{\bar{\phi}}$.

pg 65 Leveque

$$u(x,t) = \sum_{p=1}^{P(x,t)} \beta_p r_p + \sum_{p=P(x,t)+1}^{m} \beta_p r_p + \sum_{p=P(x,t)+1}^{m} (\alpha_p - \beta_p) r_p$$

$$= u_r - \sum_{\lambda_p \geq x/t} (\beta_p - \alpha_p) r_p \qquad \Longleftrightarrow \qquad P(x,t) \text{ max } p \ni \lambda_p < \frac{x}{t}$$

Sim.

$$u(x,t) = \sum_{p=1}^{P(x,t)} (\beta_p - \alpha_p) r_p + u_l$$

$$= u_l + \sum_{\lambda_p < x/t} (\beta_p - \alpha_p) r_p$$

Pg 67 LeVeque

$$U_m - U_\ell = \beta_1 r_1 + \alpha_2 r_2 - (\alpha_1 r_1 + \alpha_2 r_2)$$

$$= (\beta_1 - \alpha_1) r_1$$

$$U_r - U_m = \beta_1 r_1 + \beta_2 r_2 - (\beta_1 r_1 + \alpha_2 r_2)$$

$$= (\beta_2 - \alpha_2) r_2$$

)))

Pg 71 LeVeque

$$f(\tilde{u}_p(\xi)) - f(\hat{u}) = s_p(\xi)\left(\tilde{u}_p(\xi) - \hat{u}\right)$$

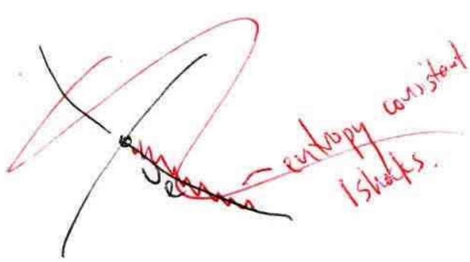
$\frac{d}{d\xi}$ d'sd $\xi = 0$

$$f'(\hat{u})\,\tilde{u}_p'(0) = s_p'(\xi)(\cdot 0) + s_p(\xi)(\tilde{u}_p'(0))$$

$$= s_p(0)\,\tilde{u}_p'(0)$$

$$\begin{pmatrix} \rho \\ m \end{pmatrix}_t + \begin{pmatrix} m \\ \frac{m^2}{\rho} + a^2\rho \end{pmatrix}_x = 0$$

$$+ \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & 2\frac{m}{\rho} \end{pmatrix} \begin{pmatrix} \rho \\ m \end{pmatrix}_x = 0$$



pg 71 LeVeque

$$\begin{pmatrix} \rho \\ m \end{pmatrix}_t + \begin{pmatrix} & \end{pmatrix}\begin{pmatrix} \rho \\ m \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \times \quad \text{can't linearize}$$

$$f(u) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + a^2\rho \end{pmatrix}$$

$$f'(u) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & \frac{2m}{\rho} \end{pmatrix}$$

$$\tilde{u}_1(\xi, \hat{u}) = \begin{pmatrix} \tilde{\rho} \\ \hat{m} \end{pmatrix} = \begin{pmatrix} \hat{\rho}(1+\xi) \\ \hat{m}(1+\xi) - a\xi\sqrt{1+\xi} \end{pmatrix} \qquad s_1(\xi; \hat{u}) = \frac{\hat{m}}{\hat{\rho}} - a\sqrt{1+\xi}$$

$$= \underbrace{\begin{pmatrix} \hat{\rho} \\ \hat{m} \end{pmatrix}}_{=\hat{u}} + \xi \begin{pmatrix} \hat{\rho} \\ \hat{m} - a\sqrt{1+\xi} \end{pmatrix}$$

$$\frac{\partial \tilde{U}_1}{\partial \xi}(\xi;\hat{U}) = \begin{pmatrix} \hat{p} \\ \hat{m} - a\hat{p}\sqrt{1+\xi} \end{pmatrix} + \xi \begin{pmatrix} 0 \\ -\dfrac{a\hat{p}}{2\sqrt{1+\xi}} \end{pmatrix}$$

$$\frac{\partial \tilde{U}_1(0,\hat{U})}{\partial \xi} = \begin{pmatrix} \hat{p} \\ \hat{m} - a\hat{p} \end{pmatrix} = \hat{p}\begin{pmatrix} 1 \\ \frac{\hat{m}}{\hat{p}} - a \end{pmatrix} \propto r_1(\hat{U})$$

$$\frac{\partial \tilde{U}_2}{\partial \xi}(0) = \left.\begin{pmatrix} \hat{p} \\ \hat{m} + a\hat{p}\sqrt{1+\xi} \end{pmatrix}\right|_{\xi=0} = \hat{p}\begin{pmatrix} 1 \\ \frac{\hat{m}}{\hat{p}} + a \end{pmatrix}$$

$$\rho_t + m_x = 0$$

$$m_t + (m^2/\rho + a^2\rho)_x = 0$$

$$\vec{U} = \begin{pmatrix} \rho \\ m \end{pmatrix}$$

$$f'(u) = \begin{pmatrix} 0 & 1 \\ -\frac{m^2}{\rho^2} + a^2 & 2\frac{m}{\rho} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{pmatrix}$$

eigenvalues $\Rightarrow$ solve

$$|\lambda I - f'(u)| = 0$$

$$\begin{vmatrix} \lambda & -1 \\ -a^2 + \frac{m^2}{\rho^2} & -\frac{2m}{\rho} + \lambda \end{vmatrix} = 0$$

$$= -\frac{2m}{\rho}\lambda + \lambda^2 + -a^2 + \frac{m^2}{\rho^2} = 0$$

$$\Rightarrow \lambda^2 - \frac{2m}{\rho}\lambda + \frac{m^2}{\rho^2} - a^2 = 0$$

$$\cancel{\lambda^2 - \frac{2m}{\rho}\lambda + \frac{m^2}{\rho^2}}$$

$$\left(\lambda - \frac{m}{\rho}\right)^2 - a^2 = 0$$

$$\cancel{\lambda = \ldots} \pm a$$

$$\therefore \left(\lambda - \frac{m}{\rho}\right)^2 = a^2$$

$$\lambda - \frac{m}{\rho} = \pm a$$

$$\lambda = \frac{m}{\rho} \pm a$$

Then eigenvectors

$$\tilde{m} - \hat{m} = s(\tilde{\rho} - \hat{\rho})$$

State ahead is $\hat{\rho}, \hat{m}$

$$\frac{\tilde{m}^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s(\tilde{m} - \hat{m})$$

2 eqs & 3 unknowns $(\tilde{\rho}, \tilde{m}, s)$ solve for $\tilde{m}$ & s in terms of $\tilde{\rho}$
(why? $\tilde{\rho}$ appears linearly in eqs?)

1st eq solve for $\tilde{m} = \hat{m} + s(\tilde{\rho} - \hat{\rho})$ put into 2nd eq

$$\frac{\left(\hat{m} + s(\tilde{\rho} - \hat{\rho})\right)^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s\left(s(\tilde{\rho} - \hat{\rho})\right)$$

$$\Rightarrow \frac{\hat{m}^2 + 2\hat{m}(\tilde{\rho} - \hat{\rho})s + (\tilde{\rho} - \hat{\rho})^2 s^2}{\tilde{\rho}} + a^2\tilde{\rho} - \frac{\hat{m}^2}{\hat{\rho}} - a^2\hat{\rho} = s^2(\tilde{\rho} - \hat{\rho})$$

$$\Rightarrow (\tilde{\rho} - \hat{\rho})^2 s^2 + 2\hat{m}(\tilde{\rho} - \hat{\rho})s + \hat{m}^2 + a^2\tilde{\rho}^2 - \frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} - a^2\tilde{\rho}\hat{\rho} = s^2(\tilde{\rho} - \hat{\rho})\tilde{\rho}$$

$$\Rightarrow \left[(\tilde{\rho} - \hat{\rho})^2 - \tilde{\rho}(\tilde{\rho} - \hat{\rho})\right] s^2 + 2\hat{m}(\tilde{\rho} - \hat{\rho})s + \hat{m}^2 + a^2\tilde{\rho}^2 - \frac{\hat{m}^2\tilde{\rho}}{\hat{\rho}} - a^2\tilde{\rho}\hat{\rho} = 0$$

$$(\tilde{\rho} - \hat{\rho})\left[\cancel{\tilde{\rho}} - \hat{\rho} - \cancel{\tilde{\rho}}\right] s^2 + \ \ldots$$

$$\Rightarrow -\hat{\rho}(\tilde{\rho} - \hat{\rho}) s^2 + \ \ldots$$

$$S = -2\hat{m}(\tilde{p}-\hat{p}) \pm \sqrt{4\hat{m}^2(\tilde{p}-\hat{p})^2 + 4\hat{p}(\tilde{p}-\hat{p})}$$

Could use quadratic eq here

I'm going to complete the square

$$\Rightarrow \quad S^2 - \frac{2\hat{m}}{\hat{p}} S - \frac{1}{\hat{p}(\tilde{p}-\hat{p})}\left[\hat{m}^2 + a^2\hat{p}^2 - \left(\frac{\hat{m}^2\tilde{p}}{\hat{p}} + a^2\tilde{p}\hat{p}\right)\right] = 0$$

like how nice this term is

Take middle term $\div$ by 2 & square $\Rightarrow \dfrac{\hat{m}^2}{\hat{p}^2}$

$$\Rightarrow \quad S^2 - \frac{2\hat{m}}{\hat{p}} S + \frac{\hat{m}^2}{\hat{p}^2} - \frac{\hat{m}^2}{\hat{p}^2} - \frac{1}{\hat{p}(\tilde{p}-\hat{p})}\left[\hat{m}^2 + a^2\hat{p}^2 - \left(\frac{\hat{m}^2\tilde{p}}{\hat{p}} + a^2\tilde{p}\hat{p}\right)\right] = 0$$

$$\Rightarrow \quad \left(S - \frac{\hat{m}}{\hat{p}}\right)^2 - \frac{\hat{m}^2}{\hat{p}^2} - \frac{1}{\hat{p}(\tilde{p}-\hat{p})}\left[\hat{m}^2\left(1 - \frac{\tilde{p}}{\hat{p}}\right) + a^2\tilde{p}(\hat{p}-\hat{p})\right] = 0$$

$$\frac{\hat{m}^2}{\hat{p}}(\hat{p}-\tilde{p})$$

$$\Rightarrow \quad \left(S - \frac{\hat{m}}{\hat{p}}\right)^2 - \cancel{\frac{\hat{m}^2}{\hat{p}^2}} + \cancel{\frac{m^2}{\hat{p}^2}} - \frac{a^2\tilde{p}}{\hat{p}} = 0$$

$$s = \frac{\hat{m}}{\hat{p}} \pm a\sqrt{\frac{\tilde{p}}{\hat{p}}} \qquad \text{eq 7.10.}$$

Then $$\tilde{m} = \hat{m} + (\tilde{p} - \hat{p})\left(\frac{\hat{m}}{\hat{p}} \pm a\sqrt{\frac{\tilde{p}}{\hat{p}}}\right)$$

$$= \frac{\hat{m}}{\hat{p}}(\cancel{\hat{p}} + \tilde{p} - \cancel{\hat{p}}) \pm a(\tilde{p} - \hat{p})\sqrt{\frac{\tilde{p}}{\hat{p}}}$$

$$= \frac{\tilde{p}}{\hat{p}}\hat{m} \pm a(\tilde{p} - \hat{p})\sqrt{\frac{\tilde{p}}{\hat{p}}} \qquad \text{eq 7.9}$$