

Ex 7.1

Pg 73 Helge
$$\begin{pmatrix} 1 \\ \phi \end{pmatrix} + \begin{pmatrix} \sqrt{1/2} + \phi \\ \sqrt{\phi} \end{pmatrix} = 0$$

$$U = \begin{pmatrix} 1 \\ \phi \end{pmatrix}; f(U) = \begin{pmatrix} \sqrt{1/2} + \phi \\ \sqrt{\phi} \end{pmatrix} \quad f'(U) = \begin{pmatrix} 1 & 1 \\ \phi & 1 \end{pmatrix}$$

eigenvalues $\sqrt{1 - \sqrt{\phi}}$, $\sqrt{1 + \sqrt{\phi}}$

eigenvectors $\vec{e}_1 = \begin{pmatrix} -1/\sqrt{\phi} \\ 1 \end{pmatrix}$, $\vec{e}_2 = \begin{pmatrix} 1/\sqrt{\phi} \\ 1 \end{pmatrix}$

To satisfy the Rankine-Hugoniot $F(U) - F(\tilde{U}) = S(U - \tilde{U})$

$$\Rightarrow \frac{1}{2}U^2 + \phi - \frac{1}{2}\tilde{U}^2 - \phi^2 = S(U - \tilde{U})$$

$$+ \frac{1}{2}U^2 - \frac{1}{2}\tilde{U}^2 = S(U - \tilde{U})$$

Using Fa

Left
Shots

$$= 27 = 47 + \frac{\sqrt{2}(\phi - 1)}{\sqrt{\phi + 2}}$$

$$S = 47 - \frac{\sqrt{2}\phi}{\sqrt{\phi + 2}}$$

def most
simple
solving for

$$27 + S$$

Right
Shots

$$= 21 = 47 + \frac{\sqrt{2}(-\phi + 2)}{\sqrt{\phi + 2}}$$

$$S = 47 + \frac{\sqrt{2}\phi}{\sqrt{\phi + 2}}$$

$$\text{Let } 2\phi = \phi(1+\epsilon)$$

$$27 = 47 - \frac{\sqrt{2\phi}}{\sqrt{2+\epsilon}}$$

$$S = 47 - \frac{\sqrt{2\phi}(1+\epsilon)}{\sqrt{2+\epsilon}}$$

$$27 = 47 + \frac{\sqrt{2\phi}\epsilon}{\sqrt{2+\epsilon}}$$

$$S = 47 + \frac{\sqrt{2\phi}(1+\epsilon)}{\sqrt{2+\epsilon}}$$

Then

$$C_1(z; \psi) = \begin{pmatrix} 27 & 2\phi \\ \phi & 27 \end{pmatrix} = \begin{pmatrix} \langle \gamma & \gamma \rangle \\ \phi & \langle \phi \rangle \end{pmatrix} + \sqrt{z} \begin{pmatrix} -\frac{\sqrt{2\phi}}{\sqrt{2+\sqrt{z}}} \\ \phi \end{pmatrix}$$

Check $\frac{\partial C_1(0; \psi)}{\partial \sqrt{z}} = \begin{pmatrix} -\frac{\sqrt{2\phi}}{\sqrt{2}} \\ \phi \end{pmatrix} = \begin{pmatrix} +\left(-\frac{1}{\sqrt{\phi}}\right) \\ \phi \end{pmatrix} = \phi \begin{pmatrix} -\frac{1}{\sqrt{\phi}} \\ 1 \end{pmatrix}$

$\propto C_1(\psi) \checkmark$

$$C_2(z; \psi) = \begin{pmatrix} 27 & 2\phi \\ \phi & 27 \end{pmatrix} = \begin{pmatrix} \langle \gamma & \gamma \rangle \\ \phi & \langle \phi \rangle \end{pmatrix} + \sqrt{z} \begin{pmatrix} \frac{\sqrt{2\phi}}{\sqrt{2+\sqrt{z}}} \\ \phi \end{pmatrix}$$

Then $\frac{\partial C_2(0; \psi)}{\partial \sqrt{z}} = \begin{pmatrix} \frac{1}{\sqrt{\phi}} \\ \phi \end{pmatrix} = \phi \begin{pmatrix} \frac{1}{\sqrt{\phi}} \\ 1 \end{pmatrix} \propto C_2(\psi) \checkmark$

(Ex 7.1) $\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v^2/2 + \phi \\ v\phi \end{pmatrix}_x = 0$

$$\tilde{U} = \begin{pmatrix} v \\ \phi \end{pmatrix} \quad \tilde{F}(\tilde{U}) = \begin{pmatrix} v^2/2 + \phi \\ v\phi \end{pmatrix} \quad \tilde{F}'(\tilde{U}) = \begin{pmatrix} v & 1 \\ \phi & v \end{pmatrix}$$

eigenvalues $\lambda_1 = v - \sqrt{\phi} \quad \lambda_2 = v + \sqrt{\phi}$

e.vectors $r_1 = \begin{pmatrix} -1/\sqrt{\phi} \\ 1 \end{pmatrix} \quad r_2 = \begin{pmatrix} +1/\sqrt{\phi} \\ 1 \end{pmatrix}$

To satisfy the Rankine-Hugoniot eqs $\tilde{F}(\tilde{U}) - \tilde{F}(\tilde{U}) = s(\tilde{U} - \tilde{U})$

$$\frac{\hat{v}^2}{2} + \hat{\phi} - \frac{\tilde{v}^2}{2} - \tilde{\phi} = s(\hat{v} - \tilde{v})$$

$$\hat{v}\hat{\phi} - \tilde{v}\tilde{\phi} = s(\hat{\phi} - \tilde{\phi})$$

$$(1) \Rightarrow \frac{\hat{v}^2}{2} - \frac{\tilde{v}^2}{2} + \hat{\phi} - \tilde{\phi} = s(\hat{v} - \tilde{v}) \text{ mult by } (\hat{\phi} - \tilde{\phi})$$

$$(\hat{\phi} - \tilde{\phi}) \left[\frac{1}{2}(\hat{v}^2 - \tilde{v}^2) + (\hat{\phi} - \tilde{\phi}) \right] = (\hat{v} - \tilde{v})(\hat{v}\hat{\phi} - \tilde{v}\tilde{\phi})$$

$$(\hat{\phi} - \tilde{\phi}) \left[\frac{\hat{v} + \tilde{v}}{2} + \frac{\hat{\phi} - \tilde{\phi}}{\hat{v} - \tilde{v}} \right] = \hat{v}\hat{\phi} - \tilde{v}\tilde{\phi}$$

Since w/ RMA we solved for $\tilde{v}$ in terms of $\tilde{\phi}$ try to solve for

$\hat{\phi}$ in terms of $\tilde{v}$.

$$\Rightarrow (\hat{\phi} - \tilde{\phi}) \bar{v} + \frac{(\hat{\phi} - \tilde{\phi})^2}{\hat{v} - \tilde{v}} = \underbrace{\hat{v} \hat{\phi} + \tilde{v} (\hat{\phi} - \tilde{\phi} - \hat{\phi})}$$

$$\begin{aligned} \text{w/ } \bar{v} &= \frac{1}{2}(\tilde{v} + \hat{v}) &= \hat{v} \hat{\phi} - \tilde{v} \hat{\phi} + \tilde{v} (\hat{\phi} - \tilde{\phi}) \\ & &= (\hat{v} - \tilde{v}) \hat{\phi} + \tilde{v} (\hat{\phi} - \tilde{\phi}) \end{aligned}$$

$$\Rightarrow \frac{1}{(\hat{v} - \tilde{v})} (\hat{\phi} - \tilde{\phi})^2 + \underbrace{(\bar{v} - \tilde{v}) (\hat{\phi} - \tilde{\phi})} - (\hat{v} - \tilde{v}) \hat{\phi} = 0$$

$$\frac{\tilde{v} + \hat{v} - 2\tilde{v}}{2}$$

$$\frac{\hat{v} - \tilde{v}}{2}$$

$$\Rightarrow (\hat{\phi} - \tilde{\phi})^2 + \frac{(\hat{v} - \tilde{v})^2 (\hat{\phi} - \tilde{\phi})}{2} - (\hat{v} - \tilde{v})^2 \hat{\phi} = 0$$

$$\Rightarrow (\tilde{\phi} - \hat{\phi})^2 - \frac{A}{2} (\tilde{\phi} - \hat{\phi}) - A \hat{\phi} = 0$$

$$\tilde{\phi} - \hat{\phi} = \frac{\frac{A}{2} \pm \sqrt{\frac{A^2}{4} + 4A}}{2}$$

$$= \frac{A \pm \sqrt{A^2 + 16A}}{4}$$

QM Lecture

$$\left(\frac{a}{\sqrt{p_r}} + \frac{a}{\sqrt{p_r}}\right)z^2 + \left(\frac{m_r}{p_r} - \frac{m_r}{p_r}\right)z - a(\sqrt{p_r} + \sqrt{p_r}) = 0$$

$$z = -\left(\frac{m_r}{p_r} - \frac{m_r}{p_r}\right) \pm \sqrt{\left(\frac{m_r}{p_r} - \frac{m_r}{p_r}\right)^2 + 4a(\sqrt{p_r} + \sqrt{p_r})\left(\frac{a}{\sqrt{p_r}} + \frac{a}{\sqrt{p_r}}\right)}$$

$$2a\left(\frac{1}{\sqrt{p_r}} + \frac{1}{\sqrt{p_r}}\right)$$

$\Rightarrow$ Take + & get positive root!

Think β_{fm} is just simply another parametrization of $\tilde{p}$

$$i.e. \quad \tilde{p} = \tilde{p}(1 + \beta_{fm}) \quad -\tilde{p} < \tilde{\xi} < \infty$$

The two equations written as I think they should be would give

$$S_1(\tilde{\xi}, u_m) = \frac{u_m}{f_m} - a \sqrt{1 + \tilde{\xi}} < \frac{u_m}{f_m} \quad \tilde{\xi} > -1$$

$$S_2(\tilde{\xi}, u_m) = \frac{u_m}{f_m} + a \sqrt{1 + \tilde{\xi}} > \frac{u_m}{f_m} \quad \tilde{\xi} > -1$$

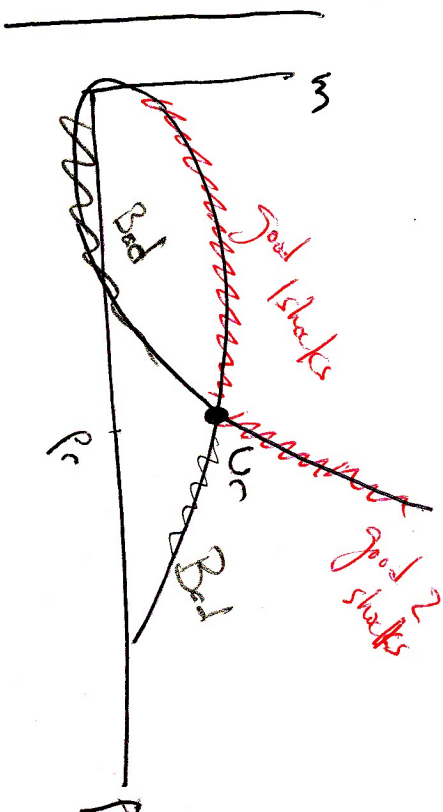
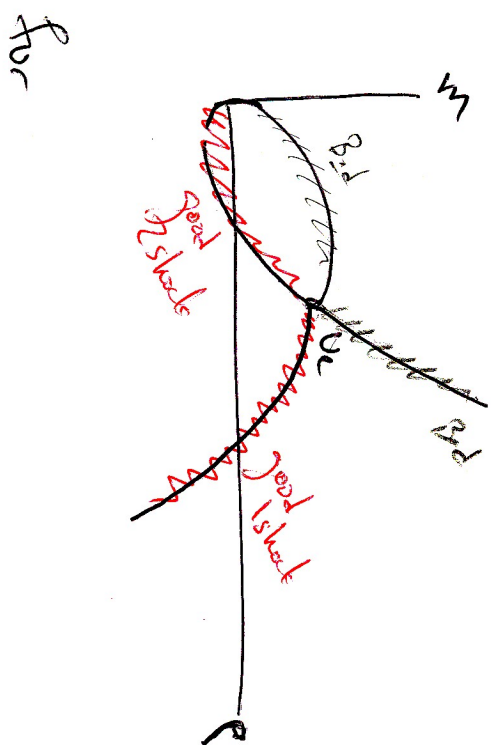
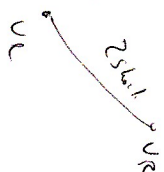
$$\therefore S_1 < S_2 \quad \forall \quad \tilde{\xi}_1, \tilde{\xi}_2 > -1.$$

P9776Vigw

$$S = \frac{m}{\rho_e} + a \sqrt{\frac{\rho_r}{\rho_e}} = \frac{m}{\rho_r} + a \sqrt{\frac{\rho_e}{\rho_r}}$$

for speed of 2 shock

if $\rho_r > \rho_e$ for a 1 shock (entropy set shock)
for 2 shock $\rho_r < \rho_e$



Idea: ^{maybe} if field is gen. non linear $\nabla f(u) \circ r_p(u) \neq 0$
 $\forall u$. Then we have shocks in that field

if $\nabla f(u) \circ r_p(u) \equiv 0$ linear degenerate
 then we have a contact discontinuity.

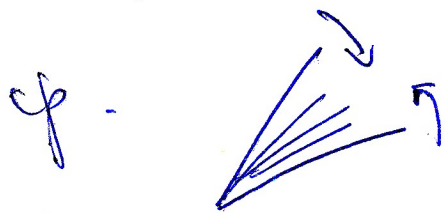
if $\nabla f(u) \circ r_p(u)$ changes sign on U .

Then we have a rarefaction wave?

Wrong. Euler eqs 2 fields gen. non linear
 1 lin. degenerate

But true for weak shocks?

$f(u_e) = f(u_r)$ like a rarefaction wave closes



Maybe w/ weak enough $u_e \rightarrow u_r$ can you

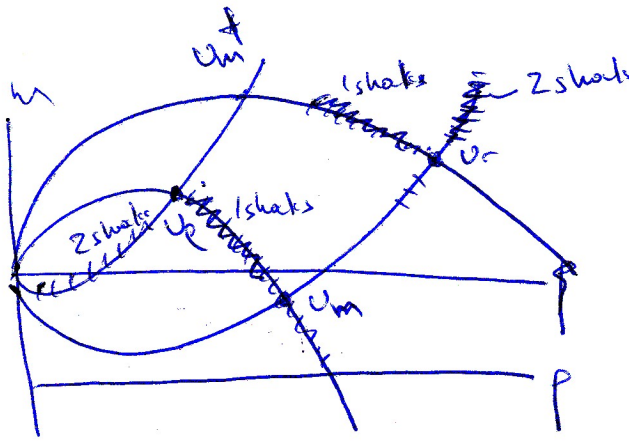
be assumed shocks? ~~No singular eq w/~~

Yes I just looking for weak solutions.

Idea: If

2

Fig 7.2



Thus u_m & u_m^* are "weak" solutions to the Riemann problem.

Hatched marks show states that can be connected by an entropy sd shock. Notice

Now at u_m^* as shock speeds are not compatible

Thus the ~~flow is not~~ ~~correct~~ ~~solution~~ if we wanted to enforce 2 shocks would be in Region II

Fig 7.4

other 3 reg

$\Rightarrow$

Shock - Fan

Fan - Shock

Fan - Fan.

(Ex 7.2) eq 5.38 is

$$\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v^2/2 + \phi \\ v\phi \end{pmatrix}_x = 0 \quad \text{w/ } \phi = gh$$

Smooth flows

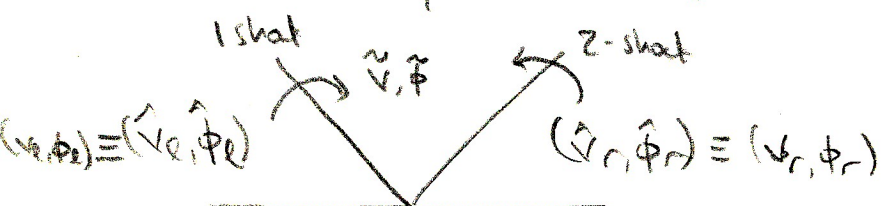
$$\Rightarrow \begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v & 1 \\ \phi & v \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix}_x = 0$$

Let's solve the Riemann problem consisting of $\Sigma_e = (v_e, \phi_e) + \Sigma_r = (v_r, \phi_r)$ considering for the time being the case where only shocks are present we get a weak solution to the Riemann problem consisting of only shocks. The Rankine-Hugoniot equations between a fixed state $(\hat{v}, \hat{\phi})$ to a state behind the shock of $(\tilde{v}, \tilde{\phi})$ or from Ex 7.1

$$\text{Left shocks} \quad \tilde{v} = \hat{v} + \frac{\sqrt{2}(\hat{\phi} - \tilde{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}}$$

$$\text{Right shocks} \quad \tilde{v} = \hat{v} - \frac{\sqrt{2}(\hat{\phi} - \tilde{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}}$$

To find a weak solution to our equations consisting of only shocks we find an intersection of the R-H eqs from a left-state $(\hat{v}_e, \hat{\phi}_e)$ & from the R-H eq from a right-state $(\hat{v}_r, \hat{\phi}_r)$



Then satisfy

$$\tilde{V} = V_e + \frac{\sqrt{Z'}(\phi_e - \tilde{\phi})}{\sqrt{\phi_e + \tilde{\phi}}}$$

$$\tilde{V} = V_r - \frac{\sqrt{Z'}(\phi_r - \tilde{\phi})}{\sqrt{\phi_r + \tilde{\phi}}}$$

Setting these two equations equal we get

$$V_e + \frac{\sqrt{Z'}(\phi_e - \tilde{\phi})}{\sqrt{\phi_e + \tilde{\phi}}} = V_r - \frac{\sqrt{Z'}(\phi_r - \tilde{\phi})}{\sqrt{\phi_r + \tilde{\phi}}}$$

$$\sqrt{Z'} \left[\frac{\phi_e - \tilde{\phi}}{\sqrt{\phi_e + \tilde{\phi}}} + \frac{\phi_r - \tilde{\phi}}{\sqrt{\phi_r + \tilde{\phi}}} \right] = (V_r - V_e)$$

$$(\phi_e - \tilde{\phi})\sqrt{\phi_r + \tilde{\phi}} + (\phi_r - \tilde{\phi})\sqrt{\phi_e + \tilde{\phi}} = \frac{\sqrt{\phi_e + \tilde{\phi}}\sqrt{\phi_r + \tilde{\phi}}}{\sqrt{Z'}} (V_r - V_e)$$

Solve this for $\tilde{\phi}$. Is this the simplest expression? Perhaps the expression for the height or potential $\tilde{\phi}$ is simpler in terms of $\tilde{V}$. Idea is that if $\phi_e, \phi_r > 0$ this equation can be solved for $\tilde{\phi}$.

↓ consider which expression is simpler ...

Solving on Ma Page 72. let $\epsilon = \frac{\tilde{\phi} - \hat{\phi}}{\hat{\phi}}$... we get

4

$$\tilde{v} = \hat{v} \pm \frac{\sqrt{2}(\hat{\phi} - \tilde{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}}$$

$$S = \hat{v} \mp \frac{\sqrt{2}\tilde{\phi}}{\sqrt{\hat{\phi} + \tilde{\phi}}}$$

we are parametrized by

$\tilde{\phi}$ let

$$\tilde{\phi} = \hat{\phi}(1 + \epsilon)$$

$$\tilde{\phi} = \hat{\phi}(1 + \epsilon)$$

Then $\epsilon = 0 \Rightarrow \tilde{\phi} = \hat{\phi}$

↓ we get

$$\tilde{v} = \hat{v} \pm \frac{\sqrt{2}(\hat{\phi} - \hat{\phi} - \hat{\phi}\epsilon)}{\sqrt{\hat{\phi} + \hat{\phi} + \hat{\phi}\epsilon}}$$

$$S = \hat{v} \mp \frac{\sqrt{2}\hat{\phi}(1 + \epsilon)}{\sqrt{2\hat{\phi} + \hat{\phi}\epsilon}}$$

$$\Rightarrow \hat{v} = \hat{v} \pm \frac{\sqrt{2}(-\hat{\phi})}{\sqrt{2}\sqrt{\hat{\phi}}\sqrt{1+\frac{3}{2}}}$$

$$S = \hat{v} \mp \frac{\sqrt{2}\hat{\phi}(1+\frac{3}{2})}{\sqrt{2}\sqrt{\hat{\phi}}\sqrt{1+\frac{3}{2}}}$$

$$\Rightarrow \hat{v} = \hat{v} \mp \frac{\sqrt{\hat{\phi}}}{\sqrt{1+\frac{3}{2}}}$$

$$\downarrow S = \hat{v} \mp \frac{\sqrt{\hat{\phi}}(1+\frac{3}{2})}{\sqrt{1+\frac{3}{2}}}$$

~~WAAAAA~~ To determine the correct signs we need

~~then WAAAAA~~ to look at the problem

$$\Rightarrow f(v) = \begin{pmatrix} v & 1 \\ \phi & v \end{pmatrix} \quad \text{whose evales are}$$

$$\lambda_1 = v - \sqrt{\phi}$$

$$\lambda_2 = v + \sqrt{\phi}$$

$$r_1 = \begin{pmatrix} -1/\sqrt{\phi} \\ 1 \end{pmatrix}$$

$$r_2 = \begin{pmatrix} 1/\sqrt{\phi} \\ 1 \end{pmatrix}$$

Now $U = \begin{pmatrix} \hat{V} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \hat{V} \mp \frac{\sqrt{\hat{\phi}} \tau}{\sqrt{1+\tau/2}} \\ \hat{\phi}(1+\tau) \end{pmatrix}$

$U(\hat{U}, \tau) = \dots$

Then $\left. \frac{\partial U(\hat{U}, \tau)}{\partial \tau} \right|_{\tau=0} = \begin{pmatrix} \mp \sqrt{\hat{\phi}} \\ \hat{\phi} \end{pmatrix} = \hat{\phi} \begin{pmatrix} \mp 1/\sqrt{\hat{\phi}} \\ 1 \end{pmatrix}$

$S(\hat{U}, \tau) \Big|_{\tau=0} = \hat{V} \mp \sqrt{\hat{\phi}}$

Thus must have $\left. \frac{\partial U_1(\tau; \hat{U})}{\partial \tau} \right|_{\tau=0} \propto r_1$

+ $\left. \frac{\partial U_2(\tau; \hat{U})}{\partial \tau} \right|_{\tau=0} \propto r_2$

Then let $U_1(\tau; \hat{U}) = \begin{pmatrix} \hat{V} - \frac{\sqrt{\hat{\phi}} \tau}{\sqrt{1+\tau/2}} \\ \hat{\phi}(1+\tau) \end{pmatrix}$

$S_1(\hat{U}, \tau) = \hat{V} - \frac{\sqrt{\hat{\phi}}(1+\tau)}{\sqrt{1+\tau/2}}$

$U_2(\tau; \hat{U}) = \begin{pmatrix} \hat{V} + \frac{\sqrt{\hat{\phi}} \tau}{\sqrt{1+\tau/2}} \\ \hat{\phi}(1+\tau) \end{pmatrix}$

$S_2(\hat{U}, \tau) = \hat{V} + \frac{\sqrt{\hat{\phi}}(1+\tau)}{\sqrt{1+\tau/2}}$

↓ we have the correct

We are alright if ~~ϕ~~ $\phi_l + \phi_r > 0$

7

As we have $\sqrt{\phi}$ everywhere

An intermediate state can be found by connecting

a v_l to a v_m by a 1 shock +

v_r to a v_m by a 2 shock

$$\text{Thus w/ } - \left(\tilde{v} = \hat{v} + \frac{\sqrt{2}(\tilde{\phi} - \hat{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}} \right)$$

~~scribbled out text~~

$$v_m = v_l - \frac{\sqrt{2}(\phi_m - \phi_l)}{\sqrt{\phi_l + \phi_m}}$$

+ w/ +

$$v_m = v_r + \frac{\sqrt{2}(\phi_m - \phi_r)}{\sqrt{\phi_r + \phi_m}}$$

$\therefore$ we get

we get

$$\phi_l - \sqrt{2}(\phi_m)$$

$$\phi_m =$$

Setting equal + solving for ϕ_m

$$\Rightarrow V_e - \sqrt{2'} \frac{(\phi_m - \phi_e)}{\sqrt{\phi_e + \phi_m}} = V_r + \sqrt{2'} \frac{(\phi_m - \phi_r)}{\sqrt{\phi_m + \phi_r}}$$

$$\frac{V_e - V_r}{\sqrt{2'}} = \frac{(\phi_m - \phi_r)}{\sqrt{\phi_m + \phi_r}} + \frac{\phi_m - \phi_e}{\sqrt{\phi_m + \phi_e}}$$

■ Page 79 LeVeque

In[8]:= Eigensystem[{{v, 1}, {φ, v}}]

Out[8]= $\left\{ \left\{ v - \sqrt{\phi}, v + \sqrt{\phi} \right\}, \left\{ \left\{ -\frac{1}{\sqrt{\phi}}, 1 \right\}, \left\{ \frac{1}{\sqrt{\phi}}, 1 \right\} \right\} \right\}$

In[1]:= eqs = $\left\{ \frac{\tilde{v}^2}{2} + \tilde{\phi} - \frac{\hat{v}^2}{2} - \hat{\phi} == s (\tilde{v} - \hat{v}), \tilde{\phi} \tilde{v} - \hat{\phi} \hat{v} == s (\tilde{\phi} - \hat{\phi}) \right\};$

In[4]:= Solve[eqs, {ṽ, φ̃}] // Simplify

Out[4]= $\left\{ \left\{ \tilde{\phi} \rightarrow \hat{\phi}, \tilde{v} \rightarrow \hat{v} \right\}, \left\{ \tilde{\phi} \rightarrow \frac{1}{4} (s - \hat{v}) \left(s - \hat{v} + \sqrt{s^2 - 2 s \hat{v} + \hat{v}^2 + 8 \hat{\phi}} \right), \tilde{v} \rightarrow \frac{1}{2} \left(3 s - \hat{v} - \sqrt{s^2 - 2 s \hat{v} + \hat{v}^2 + 8 \hat{\phi}} \right) \right\}, \left\{ \tilde{\phi} \rightarrow -\frac{1}{4} (s - \hat{v}) \left(-s + \hat{v} + \sqrt{s^2 - 2 s \hat{v} + \hat{v}^2 + 8 \hat{\phi}} \right), \tilde{v} \rightarrow \frac{1}{2} \left(3 s - \hat{v} + \sqrt{s^2 - 2 s \hat{v} + \hat{v}^2 + 8 \hat{\phi}} \right) \right\} \right\}$

In[5]:= Solve[eqs, {ṽ, s}] // Simplify

Out[5]= $\left\{ \left\{ \tilde{v} \rightarrow \hat{v} + \frac{\sqrt{2} (\hat{\phi} - \tilde{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}}, s \rightarrow \hat{v} - \frac{\sqrt{2} \tilde{\phi}}{\sqrt{\hat{\phi} + \tilde{\phi}}} \right\}, \left\{ \tilde{v} \rightarrow \hat{v} + \frac{\sqrt{2} (-\hat{\phi} + \tilde{\phi})}{\sqrt{\hat{\phi} + \tilde{\phi}}}, s \rightarrow \hat{v} + \frac{\sqrt{2} \tilde{\phi}}{\sqrt{\hat{\phi} + \tilde{\phi}}} \right\} \right\}$

In[6]:= Solve[eqs, {φ̃, s}] // Simplify

Out[6]= $\left\{ \left\{ \tilde{\phi} \rightarrow \frac{1}{4} \left(\hat{v}^2 + 4 \hat{\phi} + \hat{v} \left(\hat{v} - \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) + \hat{v} \left(-2 \hat{v} + \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) \right), s \rightarrow \frac{1}{4} \left(\hat{v} + 3 \hat{v} - \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) \right\}, \left\{ \tilde{\phi} \rightarrow \frac{1}{4} \left(\hat{v}^2 + 4 \hat{\phi} + \hat{v} \left(\hat{v} + \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) - \hat{v} \left(2 \hat{v} + \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) \right), s \rightarrow \frac{1}{4} \left(\hat{v} + 3 \hat{v} + \sqrt{\hat{v}^2 + 16 \hat{\phi} - 2 \hat{v} \hat{v} + \hat{v}^2} \right) \right\} \right\}$

It looks like the second expression is the cleanest...

In[12]:= D[$\left\{ \hat{v} + \sqrt{\hat{\phi}} \frac{\xi}{\sqrt{1 + \frac{\xi}{2}}}, \hat{\phi} (1 + \xi) \right\}, \xi] /. \{\xi \rightarrow 0\}$

D[$\left\{ \hat{v} - \sqrt{\hat{\phi}} \frac{\xi}{\sqrt{1 + \frac{\xi}{2}}}, \hat{\phi} (1 + \xi) \right\}, \xi] /. \{\xi \rightarrow 0\}$

Out[12]= $\{\sqrt{\hat{\phi}}, \hat{\phi}\}$

Out[13]= $\{-\sqrt{\hat{\phi}}, \hat{\phi}\}$

In[17]:= Solve[$v_1 - \sqrt{2} \frac{(\phi_m - \phi_1)}{\sqrt{\phi_1 + \phi_m}} == v_r + \sqrt{2} \frac{(\phi_m - \phi_r)}{\sqrt{\phi_r + \phi_m}}, \phi_m] // Simplify$

②

$$\lambda_1 = v - \sqrt{\phi}$$

$$\Delta \lambda_1 = \begin{pmatrix} 1 \\ -\frac{1}{2\sqrt{\phi}} \end{pmatrix}$$

$$\Delta \lambda_0 \cdot \begin{pmatrix} 1 \\ -\frac{1}{2\sqrt{\phi}} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{\sqrt{\phi}} \\ 1 \end{pmatrix} = -\frac{1}{\sqrt{\phi}} - \frac{1}{2\sqrt{\phi}} = -\frac{3}{2\sqrt{\phi}} \neq 0$$

$$\lambda_2 = v + \sqrt{\phi}$$

$$\Delta \lambda_2 = \begin{pmatrix} 1 \\ \frac{1}{2\sqrt{\phi}} \end{pmatrix}$$

$$\Delta \lambda_2 \cdot \begin{pmatrix} 1 \\ \frac{1}{2\sqrt{\phi}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{\phi}} \\ 1 \end{pmatrix} = \frac{3}{2\sqrt{\phi}} \neq 0$$

③

entropy and

$$\lambda_p(u_e) > S > \lambda_p(u_r)$$

For one shock

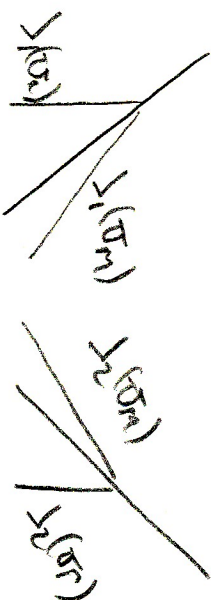
$$u_e - \sqrt{\phi_e} > u_r - \sqrt{\phi_r}$$

$$u_e - u_r > \sqrt{\phi_r} - \sqrt{\phi_e}$$

For two shocks

$$u_e + \sqrt{\phi_e} > u_r + \sqrt{\phi_r}$$

$$u_e - u_r > \sqrt{\phi_r} - \sqrt{\phi_e}$$



Ex 7.3

$$v_t + [vg(v, \phi)]_x = 0$$

$$\phi_t + [\phi g(v, \phi)]_x = 0$$

$$\Rightarrow v_t + \cancel{[vg(v, \phi)]_x} + v_x g(v, \phi) + v g_v(v, \phi) v_x + v g_\phi(v, \phi) \phi_x = 0$$

$$\phi_t + \phi_x g(v, \phi) + \phi g_v v_x + \phi g_\phi \phi_x = 0$$

$$\Rightarrow \begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} g + v g_v & v g_\phi \\ \phi g_v & g + \phi g_\phi \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

eigenvalues, eigenvectors ...

w/ $g(v, \phi) = \phi^2$ system becomes

$$\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} v\phi^2 \\ \phi^3 \end{pmatrix}_x = 0$$

$$\begin{pmatrix} v \\ \phi \end{pmatrix}_t + \begin{pmatrix} \phi^2 & 2v\phi \\ 0 & 3\phi^2 \end{pmatrix} \begin{pmatrix} v \\ \phi \end{pmatrix}_x = 0$$

eigenvalues

$$\begin{vmatrix} 1 - \phi^2 & -2v\phi \\ 0 & 1 - 3\phi^2 \end{vmatrix} = 0$$

$$(\lambda - \phi^2)(\lambda - 3\phi^2) = 0 \quad \lambda = \phi^2 \quad \lambda = 3\phi^2 \quad \checkmark$$

eigenvalues: $\lambda_1 = \phi^2$

$$(A - \lambda I)\vec{v} = 0$$

$$\begin{pmatrix} 0 & -2v\phi \\ 0 & -2\phi^2 \end{pmatrix} \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} = 0 \quad \checkmark$$

$$r_{11} = \text{arb}, \quad r_{12} = 0 \quad \vec{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 3\phi^2$$

$$\begin{pmatrix} 2\phi^2 & -2v\phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_{21} \\ r_{22} \end{pmatrix} = \vec{0}$$

$$\phi r_{21} - v r_{22} = 0 \quad \text{let } r_{21} = v \quad r_{22} = \phi$$

$$\vec{r}_2 = \begin{pmatrix} v \\ \phi \end{pmatrix}$$

① linearly degenerate $\Rightarrow \nabla \lambda_i \circ \vec{r}_i = 0 \quad \forall (v, \phi)$

$$\nabla \lambda_1 = (\lambda_v, \lambda_\phi) = \vec{0} \quad \therefore \nabla \lambda_1 \circ \vec{r}_1 = 0 \quad \forall (v, \phi)$$

$$\nabla \lambda_2 = (\lambda_v, \lambda_\phi) = (1, 1) \quad \therefore \nabla \lambda_2 \circ \vec{r}_2 = v + \phi \neq 0 \quad \forall (v, \phi)$$

② Hougho basis $f(\tilde{u}) - f(\hat{u}) = s(\tilde{u} - \hat{u}) \quad \hat{u} \text{ fixed } \tilde{u} \text{ variable}$

$$\Rightarrow \begin{pmatrix} \tilde{v} \tilde{\phi}^2 \\ \tilde{\phi}^3 \end{pmatrix} - \begin{pmatrix} \hat{v} \hat{\phi}^2 \\ \hat{\phi}^3 \end{pmatrix} = s \begin{pmatrix} \tilde{v} - \hat{v} \\ \tilde{\phi} - \hat{\phi} \end{pmatrix}$$

$$\tilde{v}\tilde{\phi}^2 - \hat{v}\hat{\phi}^2 = s(\tilde{v} - \hat{v})$$

$$\tilde{\phi}^3 - \hat{\phi}^3 = s(\tilde{\phi} - \hat{\phi}) \Rightarrow (\tilde{\phi} - \hat{\phi})(\tilde{\phi}^2 + \tilde{\phi}\hat{\phi} + \hat{\phi}^2) = (\tilde{\phi} - \hat{\phi})s \quad \checkmark$$

Putting this expression in 1st eq for s gives

$$\tilde{v}\tilde{\phi}^2 - \hat{v}\hat{\phi}^2 = (\tilde{v} - \hat{v})(\tilde{\phi}^2 + \tilde{\phi}\hat{\phi} + \hat{\phi}^2) \Leftarrow \text{linear in } \tilde{v}! \therefore \text{Solve for } \tilde{v}.$$

$$\tilde{v}(\cancel{\tilde{\phi}^2} - \cancel{\hat{\phi}^2} - \tilde{\phi}\hat{\phi} - \hat{\phi}^2) - \hat{v}\cancel{\hat{\phi}^2} = -\hat{v}(\tilde{\phi}^2 + \tilde{\phi}\hat{\phi} + \cancel{\hat{\phi}^2})$$

$$-\tilde{v}\hat{\phi}(\tilde{\phi} + \hat{\phi}) = -\hat{v}\tilde{\phi}(\tilde{\phi} + \hat{\phi})$$

$$\tilde{v}\hat{\phi} = \hat{v}\tilde{\phi} \quad \Rightarrow \quad \tilde{v} = \frac{\hat{v}}{\hat{\phi}} \tilde{\phi} \quad \text{let } \tilde{\phi} = \hat{\phi}(1+i) \quad ?$$



$$\vec{U}_2 = \begin{pmatrix} \tilde{v} \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \hat{v}/\hat{\phi}(1+i)\hat{\phi} \\ (1+i)\hat{\phi} \end{pmatrix} = \begin{pmatrix} \hat{v} \\ \hat{\phi} \end{pmatrix} + i \begin{pmatrix} \hat{v} \\ 0 \end{pmatrix} \quad \perp \text{ line}$$

Is the other direction horizontal? $\vec{U}_3 = \begin{pmatrix} \hat{v}(1+i) \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \hat{v} \\ \hat{\phi} \end{pmatrix} + i \begin{pmatrix} \hat{v} \\ 0 \end{pmatrix}$

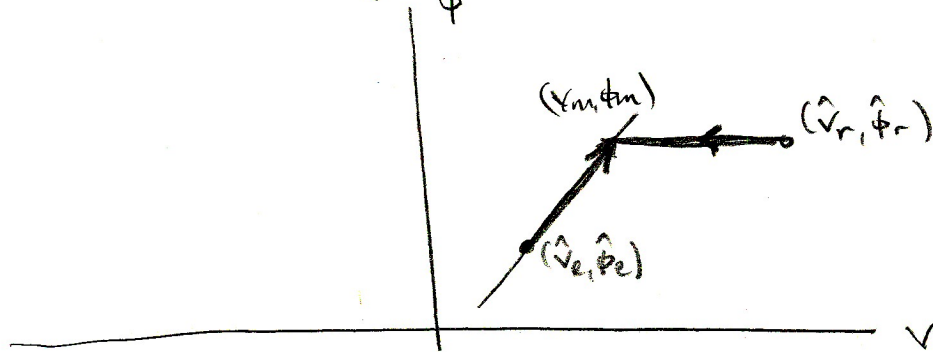
Check each line is also the integral curve for the corresponding

$$\frac{d\vec{U}_1}{d\hat{\phi}} = \begin{pmatrix} \hat{v} \\ 0 \end{pmatrix} \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Yes}$$

$$\frac{d\vec{U}_2}{d\hat{\phi}} = \begin{pmatrix} \hat{v} \\ \hat{\phi} \end{pmatrix} \propto \begin{pmatrix} \hat{v} \\ \hat{\phi} \end{pmatrix} \Big|_{(v,\phi)=(\hat{v},\hat{\phi})} \quad \text{Yes}$$

③ obtain Riemann solution w/ one shock & one contact disc

Assuming shock is along $\hat{U}_2 + \phi$ & CD is along $\hat{U}_1$.

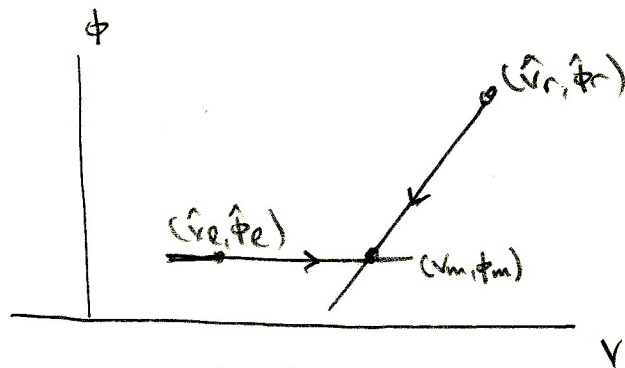


If left state is connected to middle state through a shock
 & Right state is connected to the middle state through a CD
 See picture above

Then eqs are

$$v_m = \frac{\hat{v}_e}{\hat{\phi}_e} \phi_m + \phi_m = \hat{\phi}_r$$

$$\therefore v_m = \left(\frac{\hat{\phi}_r}{\hat{\phi}_e} \right) \hat{v}_e$$



If left state is connected to middle state by a CD
 & Right state is connected to the middle state through a shock
 See picture above. Then eqs are

$$\phi_m = \hat{\phi}_e$$

$$v_m = \frac{\hat{v}_r}{\hat{\phi}_r} \phi_m \Rightarrow v_m = \left(\frac{\hat{\phi}_e}{\hat{\phi}_r} \right) \hat{v}_r$$

8-10-a 5

Entropy condition requires.

$$\lambda_p(u_e) \geq \lambda_p(u_r)$$

$$\phi_e^2 \geq \phi_r^2 \quad \Rightarrow \quad \phi_e \geq \phi_r.$$

Thus the pictures drawn above are increased for the requirements needed above to satisfy the entropy condition.

~~U~~
~~+~~

~~A~~
~~G~~

11

~~2~~
~~2~~

(H
-

~~2~~
~~2~~

A₂)

Ex 7.4

$$p_t + (pv)_x = 0$$

$$v_t + \left(\frac{v^2}{2}\right)_x = 0$$

$$\Rightarrow p_t + v p_x + p v_x = 0$$

$$v_t + v v_x = 0$$

①

$$\Rightarrow \begin{pmatrix} p \\ v \end{pmatrix}_t + \begin{pmatrix} v & p \\ 0 & v \end{pmatrix} \begin{pmatrix} p \\ v \end{pmatrix}_x = \vec{0}$$

eigen values

$$\begin{vmatrix} \lambda - v & -p \\ 0 & \lambda - v \end{vmatrix} = 0 \quad \lambda = v$$

eigen vectors

$$\begin{pmatrix} 0 & -p \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \vec{0} \quad \vec{r}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

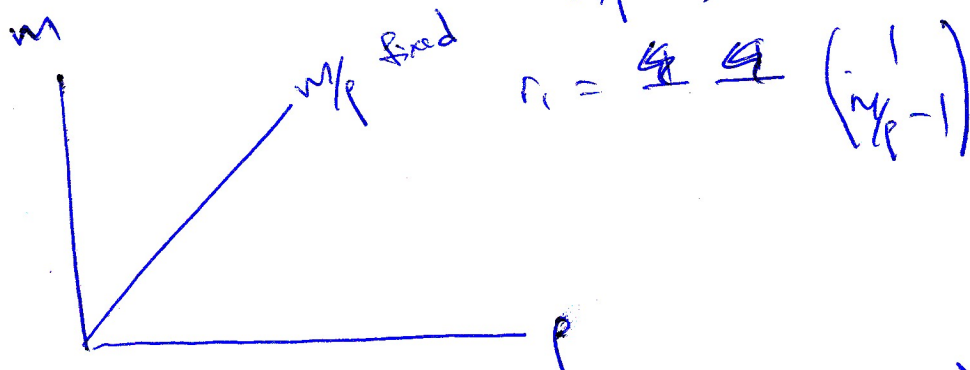
② In $p_t + v p_x = -p v_x$ w/ v_x known p is still unknown?

When v is discontinuous v_x is undefined & one must apply to a weak formulation.

would like to see how $\nabla \lambda(u) \cdot r_p \neq 0$ becomes in the newness...

for isothermal eqs $r_1(u) = \begin{pmatrix} 1 \\ m/p - a \end{pmatrix}$ $A_1 = \frac{m}{p} - 1$

if $a \equiv 1$ $r_1(u) = \begin{pmatrix} 1 \\ m/p - 1 \end{pmatrix}$



As Characteristic field is gen non linear. $\lambda(u)$ varies monotonically along integral curve of r_1 .

8.4 - 8.7 show that integral curves $\Rightarrow$ rarefaction waves something

$u_l = w(\xi_1)$ $u_r = w(\xi_2)$ both lie on the same integral curve

$$\xi = \lambda_p(w(\xi))$$

But if $x = \lambda(w)t$ $\Rightarrow \xi = \lambda(w)$

$\Rightarrow \lambda(w) = \lambda(w(\xi))$ & $w = \text{constant along } \frac{x}{t} = \lambda$

No... idea is integral waves & depend on 2
~~genuinely~~ genuinely non linear $\lambda(u)$ ~~monotonically~~

Monotonically varying along entire integral wave.

linearly degenerate $\lambda(u) = \text{constant}$ on

integral wave

general non linearly case $\lambda(u)$ can be increasing
then decreasing along integral wave

$$\xi = \lambda_p(w(\xi))$$

$$\begin{aligned} \partial_\xi &= 1 = \nabla \lambda_p(w(\xi)) \circ w'(\xi) \\ &= \alpha(\xi) \nabla \lambda_p(w(\xi)) \circ r_p(w(\xi)) \end{aligned}$$

$\Rightarrow$ ~~is~~ ∇ given near line

$$\alpha(\xi) = \frac{1}{\nabla \lambda_p(w(\xi)) \circ r_p(w(\xi))}$$

$$\Rightarrow w'(\xi) = \frac{r_p(w(\xi))}{\nabla \lambda_p(w(\xi)) \circ r_p(w(\xi))}$$

$$w(\xi_1) = u_e$$

$$\xi_1 = \lambda_p(u_e)$$

$$\xi_2 = \lambda_p(u_r)$$

Ex 8.2 : (7.7) $r_1 = \begin{pmatrix} 1 \\ w/p - a \end{pmatrix}$ $r_2 = \begin{pmatrix} 1 \\ w/p + a \end{pmatrix}$

$$(7.19) \quad \nabla \lambda_1 \circ r_1 = -a/p$$

$$\nabla \lambda_2 \circ r_2 = a/p$$

$$W = \begin{pmatrix} p \\ m \end{pmatrix}$$

$$\frac{1 \text{ Rarefaction}}{p=1}$$

$$\begin{pmatrix} p'(\xi) \\ m'(\xi) \end{pmatrix} = \frac{1}{(-a/p(\xi))} \begin{pmatrix} 1 \\ \frac{m(\xi)}{p(\xi)} - a \end{pmatrix} = \begin{pmatrix} -p/a \\ -\frac{m}{a} + p \end{pmatrix}$$

$$p' = -p/a$$

$$p(\xi_1) = p_e$$

$$m' = p - m/a$$

$$m(\xi_1) = m_e$$

$$\xi = \xi_1 \longrightarrow \xi_2$$

$$\xi_1 = \lambda_p(u_e) < \lambda_p(u_r) = \xi_2$$

For rare waves still must get from slow speed to quick speed but we must now λ 's are continuous rather than discrete...

entropy condition for shocks $\Rightarrow$

$$\lambda(u_e) > s > \lambda(u_r) \Rightarrow \text{conditions on}$$

Shall also eq $p=1,2$ 2 conditions on $p_e < p_r$
 $p_e > p_r$

Selects 2 of the lines from the 4.

Then For Fans $\underbrace{\xi \rightarrow \xi_1}_{u_l} < \underbrace{\xi_2}_{u_r}$ Must increase

3

$\Rightarrow$ 2 conditions on $p_l < p_r$ or $p_l > p_r$ Selects 2 of the lines from the γ

Riemann problem $\wedge$ 1 wave from u_l

w/ 2-wave from u_r .

$$U = \begin{pmatrix} p \\ pV \\ pE \end{pmatrix}$$

Do know that $\tilde{U}$ have lin deg, cannot have shocks or fans.

Special case of Pg 90 LeVeque 6.32

$$\begin{aligned} p(x,t) &= \hat{p}(x - \bar{v}t) \\ v &= \bar{v} \\ p &= \bar{p} \end{aligned}$$

w/ $\hat{p}$ step fn + $\bar{v} = v$.

$$(5.21) \Rightarrow e = c_v T = \frac{c_v}{R} \left(\frac{p}{\rho} \right) = \frac{p}{(\gamma-1)\rho}$$

If p 's don't change but ρ

P 's the same & p 's diff $\Rightarrow T$'s different. 4

$$(5.23) \quad S = C_v \log(P/P^r) + \underline{S_0}$$

p 's jump $\Rightarrow S$ jumps $\checkmark$.

isothermal T constant } $\Rightarrow$ if p jumps so must P (by above)
isentropic S const } $\Rightarrow \therefore$ no contact discontinuity

(p, v, P) As v & P don't jump across contact discontinuity. As we have contact discontinuity

variables $(p, v, P) \Rightarrow$ Back into 2D space.

project at variable that changes f .

$$\omega(\xi) = \alpha(\xi) \circ p(\omega(\xi))$$

$$\xi = \gamma_p(\omega(\xi))$$

$$1 = \nabla \gamma_p(\omega(\xi)) \circ \omega'(\xi) = \alpha(\xi)$$

$$\omega'(\xi) = \begin{pmatrix} p'(\xi) \\ m(\xi) \end{pmatrix} = \frac{\begin{pmatrix} m'(\xi)/p'(\xi) - a \end{pmatrix}}{\nabla(m/p - a) \circ \begin{pmatrix} 1 \\ m/p - a \end{pmatrix}} = \frac{\begin{pmatrix} m'(\xi) \\ p'(\xi) \end{pmatrix} - a}{\begin{pmatrix} -m/p^2, 1/p \end{pmatrix} \circ \begin{pmatrix} 1 \\ m/p - a \end{pmatrix}}$$

$$= \frac{\begin{pmatrix} m' \\ p \end{pmatrix} - a}{\begin{pmatrix} -m/p^2 + m/p^2 - a/p \end{pmatrix}} = \begin{pmatrix} -p/a \\ -m/a + p \end{pmatrix}$$

$$-pe \xi_1 + m_e = +a$$

$$\text{But } \xi_1 = \gamma_1(u_e) = \frac{m_e}{pe} - a \quad \swarrow \text{makes this true}$$

$$f'(z) = -f(z)/a$$

$$m'(z) = f(z) - \frac{m(z)}{a}$$

$$\underline{\text{I.C.}} \quad m(z_1) = u_z = \begin{pmatrix} f_e \\ m_e \end{pmatrix} = \begin{pmatrix} f_e \\ m_e \end{pmatrix}$$

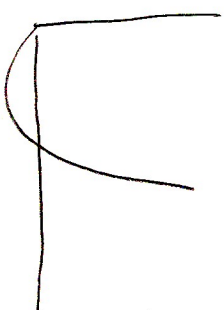
$$\text{opt } 8.13 + 8.15$$

$$8.13 \Rightarrow -a \log \frac{f}{f_e} + z_1 = z \quad \text{put in 1st eq in 8.15}$$

$$= m = \left(-a f_e \log \left(\frac{f}{f_e} \right) + m_e \right) \frac{f}{f_e} = \frac{m_e f}{f_e} - a f_e \log \left(\frac{f}{f_e} \right)$$

Take $u_z = (u_1, 1)$ + $a=1$ + plot.

If > do for know which f_e is the correct one?



Ex 2 - reflections:

Pg 85 LeVeque

4

$$\begin{aligned} w'(z) &= \frac{r_2(w(z))}{\nabla \chi_2(w(z)) \circ r_2(w(z))} = \frac{\begin{pmatrix} 1 \\ w/p + a \end{pmatrix}}{\begin{pmatrix} -w/p^2 \\ 1/p \end{pmatrix} \circ \begin{pmatrix} 1 \\ w/p + a \end{pmatrix}} = \frac{\begin{pmatrix} \end{pmatrix}}{-\cancel{w/p^2} + \cancel{w/p} + \frac{a}{p}} \\ &= \begin{pmatrix} p/a \\ w/p + p \end{pmatrix} \end{aligned}$$

Solve: $f'(z) = \frac{r(z)}{a}$ $f(z_1) = p$

$$m'(z) = \frac{m(z)}{a} + f(z) \quad m(z_1) = m_e$$

$$\begin{aligned} \Rightarrow f(z) &= p e^{(z-z_1)/a} \\ m(z) &= e^{(z-z_1)/a} [p e^z + m_e - p e^{z_1}] = (p e^{(z-z_1)} + m_e) e^{(z-z_1)/a} \end{aligned}$$

$$\text{Now } \xi_1 = \lambda_2(\alpha) = \frac{m_e}{p_e} + \alpha \Rightarrow m_e = p_e(\xi_1 - \alpha)$$

$$\begin{aligned} \therefore m(\xi) &= (p_e(\xi - \xi_1) + p_e(\xi_1 - \alpha)) e^{(\xi - \xi_1)/\alpha} \\ &= p_e(\xi - \alpha) e^{(\xi - \xi_1)/\alpha} \end{aligned}$$

$$B_1 \quad p(\xi) = \xi - \xi_1 = \alpha \log\left(\frac{p}{p_e}\right)$$

$$\therefore m(\xi) = \left(p_e \alpha \log\left(\frac{p}{p_e}\right) + m_e \right) \frac{\xi}{p_e} = \frac{p m_e}{p_e} + \alpha p \log\left(\frac{p}{p_e}\right)$$

Ex 0.2

What's wrong?

pg 35 lecture

$$\frac{dM}{dP} = \frac{M}{P} - a \log\left(\frac{P}{P_0}\right) - a \frac{P}{P_0}$$

$$\frac{dM}{dP} = \frac{M}{P} - a P = \lambda(P_0) = \frac{M}{P} - a$$

No???

$$\frac{dM}{dP} = \frac{M}{P} + a \left[\log\left(\frac{P}{P_0}\right) + \frac{P}{P_0} \right]$$

$$\frac{dM}{dP} = \frac{M}{P} + a P$$

Ex 8.3

Pg 86 Vol 9

$$\frac{2z}{(z^2 - 1)} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{z-1} + \frac{1}{z+1}$$

$$\frac{2z}{(z^2 - 1)} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{z-1} + \frac{1}{z+1}$$

$$\frac{2z}{(z^2 - 1)} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{z-1} + \frac{1}{z+1}$$

$$\frac{2z}{(z^2 - 1)} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{z-1} + \frac{1}{z+1}$$

$$\frac{2z}{(z^2 - 1)} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{z-1} + \frac{1}{z+1}$$

man

$$\frac{2}{\omega} \left| \frac{\omega}{2\pi} \right|$$

"

$$T_1 \left[\frac{1}{2} \left(\frac{3}{2} \right)^{3/2} - \frac{1}{4} \left(\frac{3}{2} \right)^{3/2} + \frac{M}{100} \left(\frac{3}{2} \right)^{3/2} \right]$$

"

$$T_1 \left[\frac{1}{2} \left(\frac{3}{2} \right)^{3/2} - \frac{1}{4} \left(\frac{3}{2} \right)^{3/2} + \frac{M}{100} \left(\frac{3}{2} \right)^{3/2} \right]$$

"

$$T_1 \left[\frac{1}{2} \left(\frac{3}{2} \right)^{3/2} - \frac{1}{4} \left(\frac{3}{2} \right)^{3/2} + \frac{M}{100} \left(\frac{3}{2} \right)^{3/2} \right]$$

"

from B16 + B19

$$\frac{d^3}{d^3 p} = \frac{m^3}{p^3} + \alpha \left[\log\left(\frac{p}{p_c}\right) + p_c \right]$$

$$\frac{d^3}{d^3 p} = + \alpha \left[\frac{1}{p} \right]$$

Same at $p = p_c$ ✓

$$\frac{d^3}{d^3 p} = + \frac{\alpha}{p^2}$$

Not same at $p = p_c$

2 Port Feeder waves:

Pg 871, Neque

$$\left\{ \begin{aligned} m_m &= \frac{f_m m_l}{f_l} - a f_m \log\left(\frac{f_m}{f_l}\right) \end{aligned} \right.$$

$$m_m = \frac{f_m m_l}{f_l} + a f_m \log\left(\frac{f_m}{f_l}\right)$$

Solving w/ 17a: $f_m = 0$ or

$$f_m = \pm \sqrt{f_l f_r} \exp\left\{ \frac{1}{2a} \left(\frac{m_l}{f_l} - \frac{m_r}{f_r} \right) \right\}$$

Take only positive
one side + one cancellation. Pic 8.1

$$f_m = \frac{f_m m_l}{f_l} + a \sqrt{\frac{f_m}{f_l}} (f_m - f_l) \quad \text{Solve}$$

$$m_m = \frac{f_m m_l}{f_l} + a f_m \log\left(\frac{f_m}{f_l}\right) \quad \text{22(u)} \quad \text{22(u)} \quad \text{22(u)}$$

(Ex 8.4) This is simply an initial condition to the isothermal eqs.

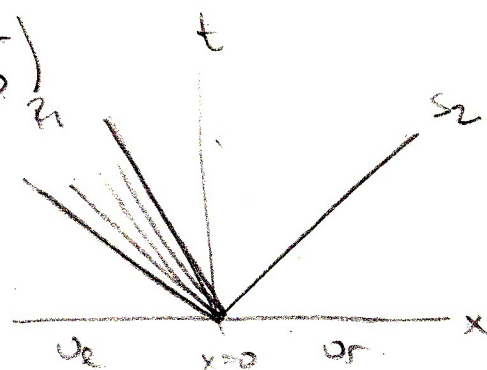
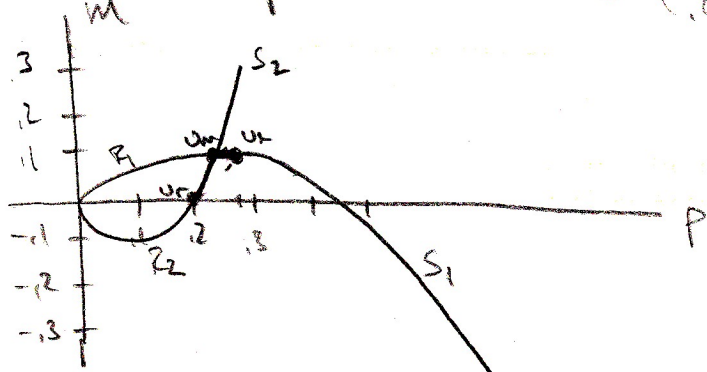
Thus we can solve this in several ways 1) Use LeVeque's Clew pack codes w/ this initial condition 2) compute the solution to each Riemann problem by hand & develop the wave profiles. What I'll do is qualitatively sketch the solution to these two Riemann problems & qualitatively give the flow profile

$$\bar{U} = \begin{pmatrix} P \\ m \end{pmatrix}$$

1st Riemann problem

$$U_L = \begin{pmatrix} .23 \\ .09 \end{pmatrix}$$

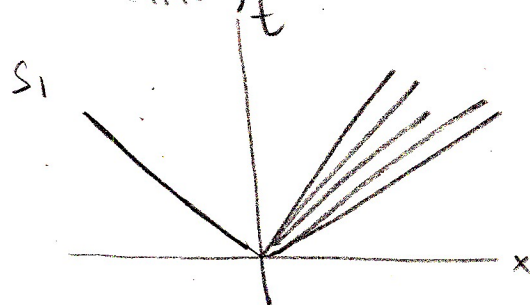
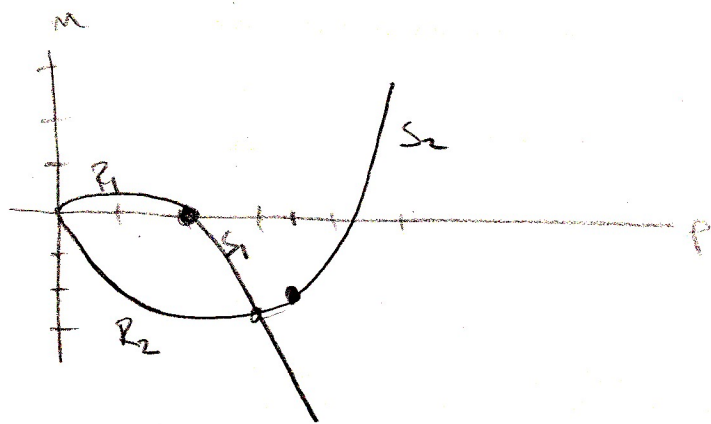
$$U_R = \begin{pmatrix} .2 \\ 0 \end{pmatrix}$$



2nd Riemann problem

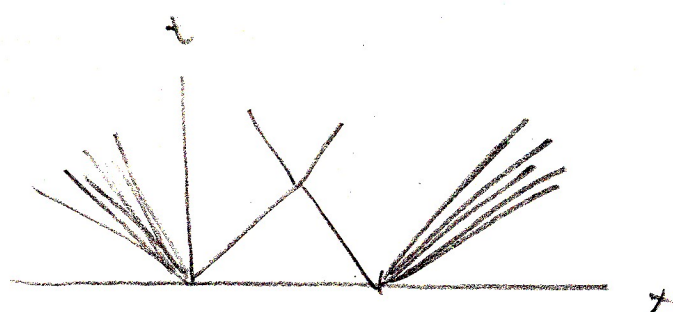
$$U_L = \begin{pmatrix} .2 \\ 0 \end{pmatrix}$$

$$U_R = \begin{pmatrix} .35 \\ .193 \end{pmatrix}$$



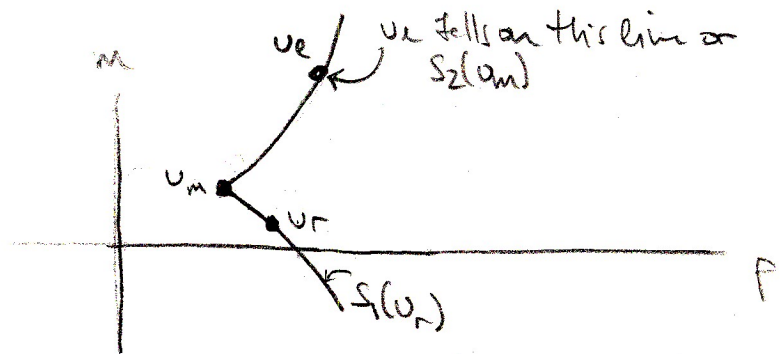
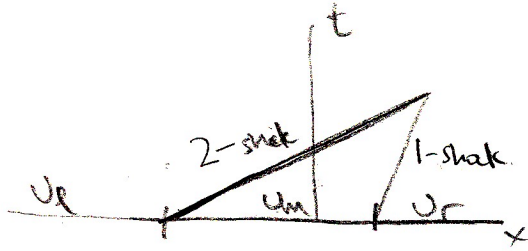
Total problem looks like

It starts catch up w/ Rarefaction fan wave till interaction to work at



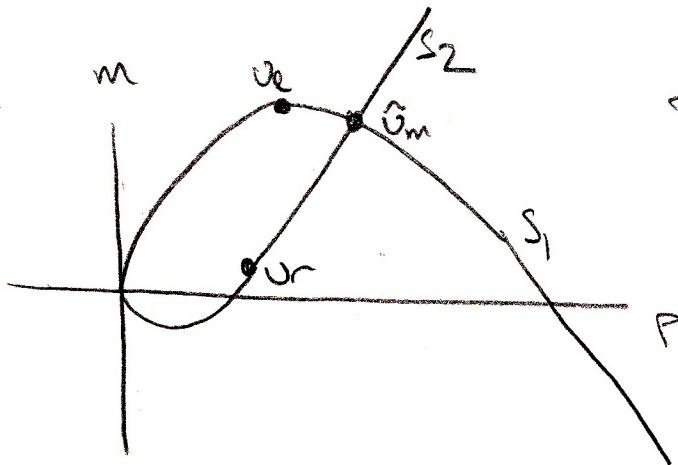
Ex 8.5

① When two shocks of different families collide say 1 & 2 shock



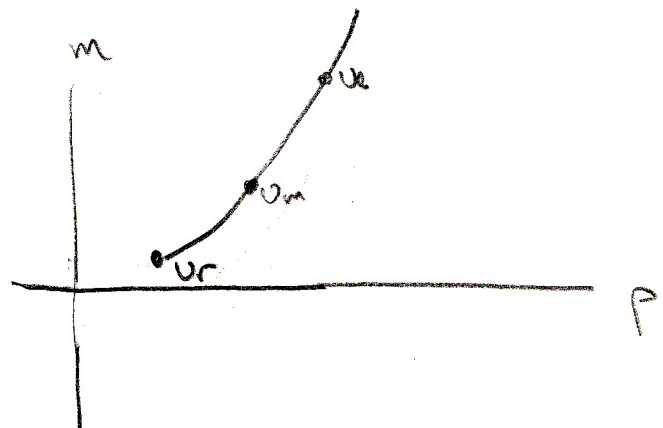
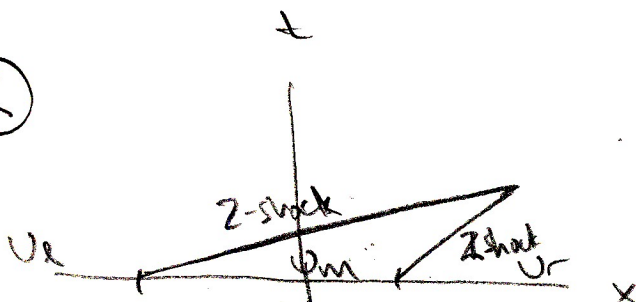
Through u_m draw $S_2(u_m)$ + also u_m must be on $S_1(u_r)$

Now draw Hugoniot locus for Riemann problem consisting of $u_l + u_r$



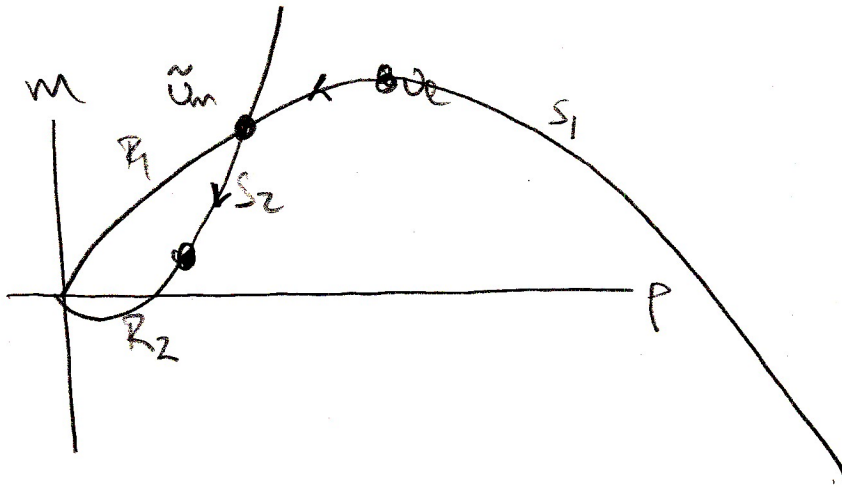
To get $\tilde{u}_m$ must pass through two shocks

②



look at the Riemann problem connecting states $u_l + u_r$

8-13-01 1



Middle state is from a reflection far & then a shock

Pg 89 LeVoyre

$$E_{\text{tot}} = \gamma \gamma + \frac{\sqrt{(v \cdot c)} \rho (2E + 2p - v^2 \rho)}{\sqrt{2} \rho}$$

$$1. \{E + \frac{p}{v-1} + \frac{p}{2} v^2\}$$

$$= \{ \gamma, \gamma + c, \gamma - c \} \quad c = \sqrt{\frac{p}{\rho}}$$

LeVoyre:

$$\begin{pmatrix} \gamma/2 \\ \gamma/4 \\ 1 \end{pmatrix} = 2\gamma^{-2} \begin{pmatrix} 1 \\ \gamma \\ \frac{1}{2}\gamma^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ \gamma \\ \frac{1}{2}\gamma^2 \end{pmatrix} = \gamma \begin{pmatrix} 1 \\ \gamma \\ \frac{1}{2}\gamma^2 \end{pmatrix} ; \begin{pmatrix} \gamma/2 \\ \gamma/4 \\ 1 \end{pmatrix} = \gamma \begin{pmatrix} \gamma/2 \\ \gamma/4 \\ 1 \end{pmatrix}$$

$$\nabla_{\lambda_2}^{\lambda_2} = \nabla_{\lambda_1}^{u_2} = \left(-\frac{u_2}{u_1^2}, \lambda_1, 0\right) = \left(-\frac{1}{p}, \frac{1}{p}, 0\right)$$

$$\nabla_{\lambda_2} \circ \nabla_{\lambda_2} = \left(-\frac{1}{p}\right)(1) + \frac{1}{p}(u) + 0 = 0 \checkmark$$

Pf to LeYoyue

$$F(u) - F(\hat{u}) = S(u - \hat{u})$$

$$\begin{aligned} \text{etc} \quad p u - \hat{p} \hat{u} &= S(p - \hat{p}) \end{aligned}$$

$$S = u = \hat{u} \Rightarrow S(p - \hat{p}) = S(p - \hat{p}) \quad \checkmark \quad \text{if } p = \hat{p}$$

$$\frac{1}{2} S^2(p - \hat{p}) + p - \hat{p} = S^2(p - \hat{p}) \leftarrow \text{Not satisfied!!}$$

$$S(E - \hat{E}) + S(p - \hat{p}) = S(E - \hat{E})$$

$$C - C \Rightarrow$$

$$(p - p) \begin{pmatrix} 1 \\ 1 \\ +2 \end{pmatrix} = (p - p) \begin{pmatrix} 1 \\ 1 \\ +2 \end{pmatrix}$$

Pg 100 LeVeque

$$U(x, t+h) = U(x, t) - kA U_x(x, t) + \frac{1}{2} k^2 A^2 U_{xx}(x, t) + O(k^3)$$

want an stable method for deriv

$$\text{ie. } U_x \approx AU(x-2h, t) + BU(x-h, t) + CU(x, t) \quad \text{want } O(h^2)$$

expand RHS in Taylor

$$= (A+B+C)U(x, t) + (-2A-B)U_{xx}h$$

$$+ \frac{1}{2}(4A+B)U_{xxx}h^2 - O(h^3)$$

$$A+B+C=0 \quad A = \frac{1}{2h}$$

$$-2A-B = \frac{1}{h} \quad B = -2h$$

$$\frac{1}{2}(4A+B) = 0 \quad C = \frac{3}{2h}$$

$$U_x \approx \frac{1}{2h}(3U_j'' - 4U_{j-1}'' + U_{j-2}'')$$

accuracy.
Most for 3 terms.

for 1st deriv
to get 2nd order
accuracy want all
but $O(h^3)$ to
be correct

work αv^2 accuracy for v_x term

$$v_x \approx A(v_x - 2h, t) + B(v_x - h, t) + C(v_x, t)$$

exp in day 10

All but αv^3
model res

$$A+B+C=0$$

$$-2A-B=0$$

$$\frac{1}{2}(A+B) = \frac{1}{\sqrt{2}}$$

$$\begin{aligned} A &= \frac{1}{\sqrt{2}} \\ B &= -\frac{1}{\sqrt{2}} \\ C &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$v_x \approx \frac{1}{\sqrt{2}}(v_j^s - 2v_{j-1}^s + v_{j+2}^s) \quad \checkmark$$

Pg 113 LeVeque

$$(10.47) = V_t + \Delta V_x = 0$$

~~LeVeque's~~ $\Downarrow$ ~~$V_j^{n+1} = V_j^n + \frac{k}{\Delta x} (V_{j+1}^n - V_{j-1}^n)$~~

$$V_j^{n+1} = V_j^n - \frac{k}{\Delta x} A^+ (V_j^n - V_{j-1}^n) - \frac{k}{\Delta x} A^- (V_{j+1}^n - V_j^n)$$

$$V = Q^{-1} U$$

$\therefore$ mult by Q

$$U_j^{n+1} = U_j^n - \frac{k}{\Delta x} (Q A^+ Q^{-1}) (U_j^n - U_{j-1}^n) - \frac{k}{\Delta x} A^- (U_{j+1}^n - U_j^n)$$

$\parallel$

$$A^+$$

Pg 117 LeVeque

$$L_t(u_t) = u_t + Au_x + \frac{1}{2}(ku_{tt} - \frac{h^2}{\epsilon} u_{xx}) + O(h^2)$$

Now Assume $\underbrace{\hspace{10em}}_{=0} O(k)$

$$u_t \equiv -Au_x - \frac{1}{2}(ku_{tt} - \frac{h^2}{\epsilon} u_{xx})$$

$$= -A(u_t - \frac{1}{2}ku_{tt} - \frac{h^2}{\epsilon} u_{xx})$$

$$= -A \left(-Au_x - \frac{1}{2}(ku_{tt} - \frac{h^2}{\epsilon} u_{xx}) \right) - \frac{1}{2}ku_{tt} + \frac{h^2}{\epsilon} u_{xx} + O(k^2)$$

$$= A^2 u_{xx} + O(k)$$

Thus put in 11.2 to get $u_t + Au_x + \frac{1}{2}(kA^2 - \frac{h^2}{\epsilon})u_{xx} + O(k^2) = 0$

pg 117 LeVeque

Why do we need to change to only 1 time der.

" " " get rid of 2nd time der? Can still do

Run stab analysis on 11.2

No!!

Let $q = u = e^{i(kx - \omega t)}$ put in 11.2

$$\Rightarrow -i\omega + Aik + \frac{1}{2} \left(\Delta t (-i\omega)^2 - \frac{\Delta x^2}{\Delta t} (ik)^2 \right) = 0$$

ω now quadratic eq

$$-i\omega + Aik + \frac{-\Delta t \omega^2}{2} + \frac{\Delta x^2 k^2}{2\Delta t} = 0$$

But in solving for ω there should be an approx made to limit to the case in 11.3 or eq before that.

Scalar Case I

Pg 118 LeVeque

$$A = 1$$

$$\frac{v}{s} = \frac{1}{2}$$

$$v = \frac{1}{100}$$

$$D_{\text{max}} - F = v \frac{2}{2} \left(1 - \frac{1}{4} \right) = \frac{3v}{4}$$

$$D_{\text{bound}} = \frac{v}{2} \left(1 - \frac{1}{2} \right) = \frac{v}{4}$$

Ex 11.1

$$u_j^{n+1} = u_j^n - \frac{k}{2h} A (u_{j+1}^n - u_{j-1}^n)$$

$$\frac{1}{k} (u_{j+1}^{n+1} - u_j^n) + \frac{A}{2h} (u_{j+1}^n - u_{j-1}^n) = 0$$

$$\Rightarrow \frac{1}{k} (u(x, t+k) - u(x, t)) + \frac{A}{2h} (u(x+h, t) - u(x-h, t)) = 0$$

$$\Rightarrow \partial_t u + A \partial_x u + \frac{1}{2} k \partial_t^2 u = O(k^2)$$

$$= \partial_t u + A \partial_x u = -\frac{1}{2} k \partial_t^2 u + O(k^2)$$

This method will be unstable if k/h because it is the advection diffusion equation w/ coefficient in front of $\partial_t^2 u$ Negative

$$\frac{d}{dt} \left(\frac{1}{2} \dot{x}^T U + A^T \dot{x} U \right) = -\frac{1}{2} k A^T \dot{x}^2 + \alpha \dot{x}^2$$

 $\frac{d}{dt}$
 $\frac{d}{dt}$
 $\frac{d}{dt}$
 11.4

9/19 lectures

LM:

$$\frac{1}{k}(u_{j+1}^n - u_j^n) + \frac{1}{2h}A(u_j^n - u_{j-1}^n) - \frac{k}{2h^2}A^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = 0$$

$$\Rightarrow \frac{1}{k}(u(x,t+k) - u(x,t)) + \frac{A}{2h}(u(x+h,t) - u(x-h,t)) - \frac{kA^2}{2h^2}(u(x+h,t) - 2u(x,t) + u(x-h,t)) = 0$$

$$\Rightarrow \partial_t u + A \partial_x u = -\frac{k}{2}(\partial_t^2 u - A^2 \partial_x^2 u) - \frac{k^2}{6} \partial_t^3 u - \frac{1}{24}A^2 \partial_x^3 u - \frac{1}{24}A^2 \cancel{\partial_x \partial_t^2 u} + O(k^3) *$$

If $\partial_t u = -A \partial_x u$

$$\Rightarrow \partial_t^2 u = -A(\partial_x^2 u) = A(-A \partial_x^2 u) = A^2 \partial_x^2 u$$

+ $O(t)$ term drops.

2

$$= \partial_t^2 u + A \partial_x^2 u = \frac{\sqrt{2}}{6}$$

$$\partial_t^3 u = -A^3 \partial_x^3 u.$$

If $\partial_t^2 u + A \partial_x^2 u = 0 + O(k^2) = u_{tt} = -A^3 u_{xxx} + O(k^2)$

Then $\partial_t^2 u + A \partial_x^2 u = \left(\frac{k^2}{6} A^3 - \frac{\sqrt{2}}{6} A^2 \right) u_{xxx}$

$$= \frac{\sqrt{2}}{6} A \left(\frac{k^2}{\sqrt{2}} A^2 - I \right) u_{xxx} + O(k^3)$$

$$u_t + A u_x + \frac{k}{2} (u_{tt} - A^2 u_{xx}) = \frac{\sqrt{2}}{6} A \left(\frac{k^2}{\sqrt{2}} A^2 - I \right) u_{xxx} + O(k^3)$$

can disregard that term to get

$$\partial_t U + A \partial_x U = O(k^2)$$

$$\Rightarrow \partial_t^3 U = -A^3 \partial_x^3 U + O(k^2)$$

$$\vdash \partial_t U + A \partial_x U = \frac{1}{6} A^3 U_{xxx} - \frac{1}{6} A^2 U_{xxx} + O(k^3)$$

$$= \frac{1}{6} A \left(\frac{1}{6} A^2 A^2 - I \right) U_{xxx} + O(k^3).$$

B.M:

Pg 119 to 120

$$\frac{1}{k}(u_j^{n+1} - u_j^n) + \frac{1}{2h}(3u_j^n - 4u_{j-1}^n + u_{j-2}^n) - \frac{k}{2h^2}A^2(u_j^n - 2u_{j-1}^n + u_{j-2}^n) = 0$$

$$\Rightarrow u_t + Au_x + \frac{1}{2}(u_{tt} - A^2 u_{xx})k + \frac{1}{6}u_{ttt}k^2 + \frac{1}{2}Au_{xxx}hk$$

$$- \frac{1}{6}Au_{xxx}h^2 = O(k^3)$$

Same reasoning As before $u_{tt} - A^2 u_{xx} = O(k^2)$ p 71

$$u_t + Au_x = O(k^2)$$

$$u_{tt} - A^2 u_{xx} = O(k^2)$$

$$u_{ttt} + A^3 u_{xxx} = O(k^2)$$

$$\Rightarrow u_t + Au_x = \frac{h^2}{6}A(2I - 3k_1 + \frac{k^2}{h^2}A^2)u_{xxx}$$

Lemma: $\psi_t \text{ a.o. } x = \mu \psi_{xxx} \Rightarrow \frac{d}{dt} \hat{\psi}(k, t) + a i k^3 \hat{\psi}(k, t) = (i k^3 \mu) \hat{\psi}(k, t)$

$$\psi(x, t) = \int_{-\infty}^{\infty} \hat{\psi}(k, t) e^{i k x} dk$$

$\Downarrow$

$$\frac{d}{dt} \hat{\psi} = (a i k^3 - i k^3 \mu) \hat{\psi}(k, t)$$

$$\hat{\psi}(k, t) = \hat{\psi}(k) e^{-i(a k^3 + \mu k^3)t}$$

$$\psi(x, t) = \int_{-\infty}^{\infty} \hat{\psi}(k) e^{i k x} e^{-i(a k^3 + \mu k^3)t} dk$$

$$= \int_{-\infty}^{\infty} \hat{\psi}(k) e^{i k (x - a t - \mu k^2 t)} dk$$

$$\phi(\xi) = -\frac{1}{\xi}(-a\xi - \mu\xi^2 + t) = a + \mu\xi^2$$

Peak speed

$$\phi'(\xi) = \frac{d(a\xi + \mu\xi^3)}{d\xi}$$

"into speed"

$$= a + 3\mu\xi^2$$

$$\mu = \frac{2t}{\xi}$$

$$v = v_{ax} \left| \frac{dx}{ds} \right| \quad \text{if } \text{Scale} = \left| \frac{dx}{ds} \right|$$

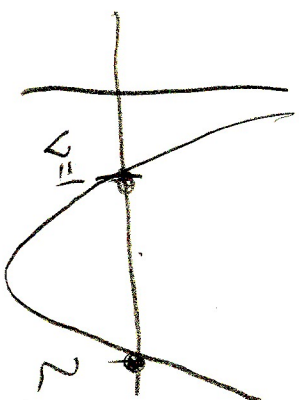
$$\mu = \frac{1}{2} a (\sqrt{2} - 1)$$

$$\left[\begin{array}{l} e^{ik(x-ut)} \\ \omega = u k \\ \frac{d\omega}{dk} \\ u k(x - \textcircled{\frac{3}{2}t}) \end{array} \right] \quad \text{Peak speed}$$

BWD:

$$u = \frac{1}{2} \alpha (2 - 3v + v^2)$$

$$= \frac{1}{2} \alpha (2 - v)(1 + v)$$



$$u + \alpha u_x = 0 \quad u(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

$$u_t + \alpha u_x = D u_{xx}$$

!!
Fourier method

$$\text{let } u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(k,t) e^{ikx} dk$$

$$\hat{u} + \alpha i k \hat{u} = D(-k^2) \hat{u} \rightarrow \tilde{u}(k,t) = \tilde{u}(k) e^{(-k^2 D - \alpha i k)t}$$

$$= u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(k) e^{(-k^2 D - \alpha i k)t + i k x} dk = \int_{-\infty}^{\infty} \tilde{u}(k) e^{ik(x - \alpha t) - k^2 D t} dk$$

$$U(x,0) = \int_{-\infty}^{\infty} A(k) e^{ikx - i\cancel{E}Dt} dk = u_0(x)$$

(Note: A red arrow points from the crossed-out term to a red '0')

Mult by $\frac{1}{2\pi} e^{-ik'x}$ + integrate from x goes $(-\infty, \infty)$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t) e^{i(k-k')x} dt dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_0(x) e^{-ik'x} dx$$

Do 1 more

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t) \delta(t-t') dt = \frac{1}{2\pi} \int_{-\infty}^{\infty}$$

$$\Rightarrow \hat{g}(k') = \int_{-\infty}^{\infty} u_0(x) e^{-ik'x} dx$$

But now $u_0(x)$ is heavy side fun. How do?

Pg 121 LeVeque

$$u_t + au_x = D u_{xx}$$

let $u = u(x-at, t)$

$$u_t = u_{\xi_1}(-a) + u_{\xi_2}$$

$$u_x = u_{\xi_1}$$

$$u_{xx} = u_{\xi_1 \xi_1} \quad \text{p.d.m}$$

$$\Rightarrow u_{\xi_2} = D u_{\xi_1 \xi_1} \quad \text{but Diff eq. in } \xi_1 + \xi_2$$

Pg 123 L. Negro

$$C_{j+1}^n = C_j^n - \frac{1}{5} C_j^n (C_j^n - C_{j-1}^n)$$

$$C_j^0 = \begin{cases} 1 & j < 0 \\ 0 & j \geq 0 \end{cases}$$

then for C_j^1 when $j < 0$

$$= 1 - \frac{1}{5} (1)(1-1) = 1$$

when $j \geq 1$ $C_j^1 = 0$

when $j = 0$ $C_0^1 = 0 - \frac{1}{5} 0 = 0$

$$\therefore C_j^1 = \begin{cases} 1 & j < 0 \\ 0 & j \geq 1 \end{cases} = C_j^0$$

Pg 123 LKaguo

Ex 121

$$\frac{1}{L}(y_{j+1}^n - y_j^n) + \frac{1}{L}y_j^n(y_j^n - y_{j-1}^n) = 0$$

$$= \frac{1}{L}(y(x, t+k) - y(x, t)) + \frac{1}{L}y(x, t)(y(x, t) - y(x, t-h)) = 0$$

$$\Rightarrow y_t + y \partial_x y = 0(x, t)$$

eq to 121 + 122

LE

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{k}{2h}(f(u_{j+1}^n) - f(u_{j-1}^n))$$

$$f(u_j^n) - f(u_j^n)$$

$$= u_j^n + \frac{1}{2}(u_j^n - 2u_j^n + u_{j+1}^n) - \frac{k}{2h}(\quad)$$

$$= u_j^n + \frac{k}{h} \left[\frac{1}{2} \frac{h}{k} (u_{j+1}^n - u_j^n) - \frac{1}{2} \frac{h}{k} (u_j^n - u_{j-1}^n) \right]$$

$$+ \frac{1}{2}(f(u_j^n) - f(u_{j+1}^n)) - \frac{1}{2}(f(u_j^n) - f(u_{j-1}^n)) \Big]$$

$$= u_j^n + \frac{k}{h} [f(u_j^n, u_{j+1}^n) - f(u_{j-1}^n, u_j^n)]$$

$$3) \quad f(u_j^n, u_{j+1}^n) = \frac{h}{2k}(u_{j+1}^n - u_j^n) + \frac{1}{2}(f(u_j^n) - f(u_{j+1}^n))$$

Pg 126 lecture

$$12.16 = F(5,5) = 0 + \frac{1}{2}(2F(5)) \quad \checkmark$$

$$|F(x,w) - F(5)| = \left| \frac{1}{2}(x-w) + \frac{1}{2}(F(x) + F(w)) - F(5) \right|$$

$\frac{1}{2}(2F(5))$

$$= \left| \frac{1}{2}(x-5) + \frac{1}{2}(5-w) + \frac{1}{2}(F(x) - F(5)) + \frac{1}{2}(F(w) - F(5)) \right|$$

if smooth $\Rightarrow$ Lipschitz.

$$\leq \frac{1}{2} \frac{1}{2}(x-5) + |5-w| + L_1 |F(x-5)| + L_2 |w-5|$$

$$\leq C \max\{1, 1, 1\}$$

Ex 23

for AV ; $F' = A$

PJ127 LeVeque

12.25

$$U_j^{n+1} = U_j^n - \frac{k}{\Delta x} (AV_{j+1}^n - AV_{j-1}^n) + \frac{k^2}{\Delta x^2} A^2 (U_{j+1}^n - U_j^n - U_j^n + U_{j-1}^n)$$

reduces to 12.24 for $F(u) = Au$.

Show 2nd order accurate w smooth sol.
(scalar case)

$$\frac{U_j^{n+1} - U_j^n}{k} = -\frac{1}{\Delta x} (F(U_{j+1}^n) - F(U_{j-1}^n)) + \frac{k}{\Delta x^2} \left[F' \left(\frac{1}{2} (U_j^n + U_{j+1}^n) \right) (F(U_{j+1}^n) - F(U_{j-1}^n)) \right]$$

$$- F' \left(\frac{1}{2} (U_j^n + U_{j-1}^n) \right) (F(U_j^n) - F(U_{j-1}^n))$$

expand RHS in Taylor

$$= -\frac{1}{\Delta x} \left[f(u(x+h, t)) - f(u(x, t)) \right] + \frac{k}{\Delta x^2}$$

$$u_{j+1}^{n+1} = u_j^n - \frac{k}{s} \left[\frac{f(u_{j+1}^n)}{2} - \frac{k}{2h} A_{j+1/2} (f(u_{j+1}^n) - f(u_j^n)) \right]$$

$$- \left[\frac{f(u_j^n)}{2} - \frac{k}{2h} A_{j-1/2} (f(u_j^n) - f(u_{j-1}^n)) \right]$$

$$= u_j^n - \frac{k}{s} \left[\frac{1}{2} (f(u_{j+1}^n) + f(u_j^n)) - \frac{k}{2h} A_{j+1/2} (f(u_{j+1}^n) - f(u_j^n)) \right]$$

$$- \left[\frac{1}{2} (f(u_j^n) + f(u_{j-1}^n)) - \frac{k}{2h} A_{j-1/2} (f(u_j^n) - f(u_{j-1}^n)) \right]$$

$$f(u_j^n, u_{j+1}^n) = \frac{1}{2} (f(u_j^n) + f(u_{j+1}^n)) - \frac{k}{2h} A_{j+1/2} (f(u_{j+1}^n) - f(u_j^n))$$

Check flow is constant

$$F(0,0) = F(0) \quad \checkmark$$

$$|F(v,w) - F(0)| = \left| \frac{1}{2}(F(v) + F(w)) - F(0) \right| + \frac{k}{2n} |A_{j+\frac{1}{2}}| |F(w) - F(v)|$$

//

$$\frac{k}{2n} |A_{j+\frac{1}{2}}| |F(w) - F(0) + F(0) - F(v)|$$

$$\leq \frac{1}{2} |F(v) - F(0)| + \frac{1}{2} |F(w) - F(0)| + \dots$$

$$\leq C_1 |F(v) - F(0)| + C_2 |F(w) - F(0)| \quad \text{if } j \text{ Bound on}$$

$$A_j^{-1/2}.$$

As F is sup cont each difference can be

bounded by $k_1 \max(|v-0|, |w-0|) + F(v,w)$ is Lipschitz cont

Show 2nd order accurate on $u_t + f'(u)u_x = 0$

$$= u_x = -(f''(u)u_x + f''(u)u_x^2)$$

$$= u_{txx} = -(f''(u)u_x u_{xx} + f''(u)u_{xxx} + f'''(u)u_x^3 + 2f''(u)u_x u_{xx})$$

$$= -(3f''(u)u_x u_{xx} + f''(u)u_{xxx} + f'''(u)u_x^3)$$

$$u_t = (-f'(u)u_x)_t$$

$$= -[f''(u)u_t u_x + f'(u)u_{xt}]$$

from above

$$= -[f''(u)(-f'(u)u_x^2) + f'(u)[-f''(u)u_x - f''(u)u_x^2]]$$

$$= -[-2f''(u)f'(u)u_x^2 - f'(u)^2 u_{xx}]$$

$$= 2f''(u)f'(u)u_x^2 + f'(u)^2 u_{xx}$$

Showing 2nd order accuracy is as simple as putting
 Approx into ∇a + checking that All coefficients $h + k$
 are ≥ 2 . They are when we use the fact that
 $u_t + f(u)u_x = 0$. Done on ∇a .

1226

$\mathbb{H} = A0$ Reduces to 1224?

$$v_{j+1/2}^{n+1/2} = \frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{F}{2\Delta x} A (v_{j+1}^n - v_j^n)$$

Then

$$v_j^{n+1} = v_j^n - \frac{F}{\Delta x} A \left[\frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{F}{2\Delta x} A (v_{j+1}^n - v_j^n) \right]$$

$$= \frac{1}{2}(v_{j-1}^n + v_j^n) + \frac{F}{2\Delta x} A (v_j^n - v_{j-1}^n)$$

$$= v_j^n - \frac{F}{2\Delta x} A (v_{j+1}^n - v_{j-1}^n) + \frac{F^2}{2\Delta x} A^2 (v_{j+1}^n - 2v_j^n + v_{j-1}^n)$$

$$= 1224 //$$

conservative? yes by 2nd eq in 1226. flow fn given by

$$F(v_j^n; v_{j+1}^n) = F(v_{j+n}^{n+1/2}) = F\left(\frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{h}{2\Delta t}(F(v_{j+1}^n) - F(v_j^n))\right)$$

$$F(v, v) = F(v) \quad \checkmark$$

$$|F(v, w) - F(v)| = \left| F\left(\frac{1}{2}(v+w) - \frac{h}{2\Delta t}(F(w) - F(v))\right) - F(v) \right|$$

$$v = v + \xi$$

$$F(v) = F(v) + \xi F'(v) + O(\xi^2)$$

$$w = v + \eta$$

$$F(w) = F(v) + \eta F'(v) + O(\eta^2)$$

} As F is assumed
Smooth

$$\frac{1}{2}(v+w) = v + \frac{1}{2}(\xi + \eta)$$

$$F(w) - F(v) = F'(v)(\eta - \xi) + O(\xi^2) + O(\eta^2)$$

$$= \left(f(5) + \frac{1}{2}(3+7) - \frac{K}{2h} f'(5)(7-3) + O(\epsilon^2) + O(\eta^2) \right) - f(5)$$

$$\Rightarrow \left(f(5) + \frac{1}{2} \left(1 - \frac{K}{h} f'(5) \right) 3 + \frac{1}{2} \left(1 + \frac{K}{h} f'(5) \right) 7 + O(\epsilon^2) + O(\eta^2) \right)$$

$$- f(5) \Big|$$

$$\left| \frac{K}{h} f'(5) \right| < \Delta \quad B_1 \Gamma L$$

$$\begin{aligned} & \left| \cancel{f(5)} + f'(5) \frac{1}{2} \left(1 - \frac{K}{h} f'(5) \right) 3 + f(5) \frac{1}{2} \left(1 + \frac{K}{h} f'(5) \right) 7 \right. \\ & \left. + O(\epsilon^2) + O(\eta^2) - \cancel{f(5)} \right| \end{aligned}$$

$$\leq \frac{|f'(c)|}{2} \left(1 + \frac{k}{n} |f'(c)|\right) \max(|\xi|, |\eta|)$$

||

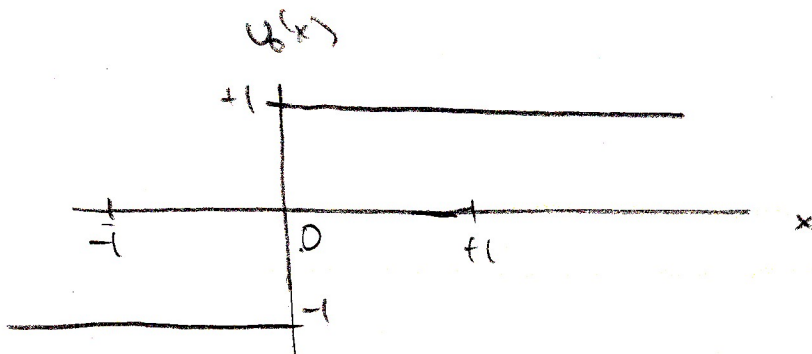
$$\max(|n-c|, |m-c|) \quad \checkmark$$

Show 2nd order: ...

Ex 12.4 Flux 12.51

$$F(v, w) = \begin{cases} f(v) & \text{if } \frac{f(v) - f(w)}{v - w} \geq 0 \\ f(w) & \text{if } \frac{f(v) - f(w)}{v - w} < 0 \end{cases}$$

$$u_j^{n+1} = u_j^n - \frac{k}{h} [F(u_j^n, u_{j+1}^n) - F(u_{j-1}^n, u_j^n)] \quad \text{w/ Numerical flux given by choice}$$

Burgers eq means $f(u) = u^2/2$

Discretizing w/ cell Averaging gives $u_j^0 = \begin{cases} -1 & j < 0 \\ 0 & j = 0 \\ +1 & j > 0 \end{cases}$ Assuming that a cell straddles the origin,

$$\text{Then } \frac{f(v) - f(w)}{v - w} = \frac{1}{2} \frac{(v-w)(v+w)}{v-w} = \frac{1}{2} (v+w)$$

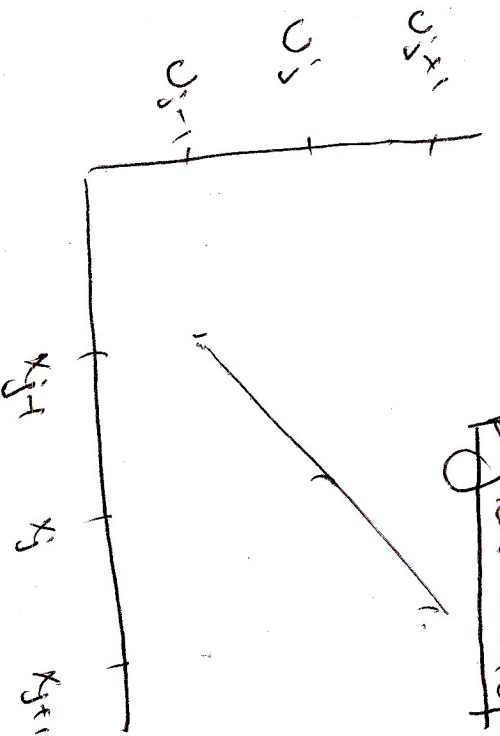
So Numerical scheme becomes

$$F(u_j^n, u_{j+1}^n) = \begin{cases} (u_j^n)^2/2 & \text{if } \frac{1}{2}(u_j^n + u_{j+1}^n) \geq 0 \\ (u_{j+1}^n)^2/2 & \text{if } \frac{1}{2}(u_j^n + u_{j+1}^n) < 0 \end{cases}$$

$$\text{let } j \leq -2 \quad F(u_j^n, u_{j+1}^n) = \frac{1}{2} \quad + \quad F(u_{j-1}^n, u_j^n) = \frac{1}{2}$$

$$\text{let } j = -1 \quad F(u_j^n, u_{j+1}^n) = 0 \quad + \quad F(\dots) = \frac{1}{2}$$

Ex 13.6 Lagrange



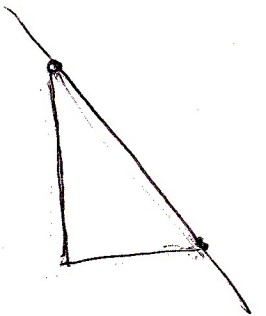
$$f(x) = \left(\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{x_j - x_{j-1}} \right) (x - x_{j-1}) + u_{j-1}^{(n)}$$

$$u_j^{(n+1)} = f(x_j - \alpha k) = \frac{1}{2h} \left[\frac{u_{j+1}^{(n)} - u_{j-1}^{(n)}}{2h} (h - \alpha k) + 2h u_{j-1}^{(n)} \right]$$

$$= \frac{1}{2h} \left[(h - \alpha k) u_{j+1}^{(n)} + (h + \alpha k) u_{j-1}^{(n)} \right]$$

P3 136 lecture

u_j
 u_{j-1}



1 h 1
 x_{j-1} x_j

up wind

$$f(x) = \left(\frac{u_j^s - u_{j-1}^s}{h} \right) (x - x_j) + u_j^s$$

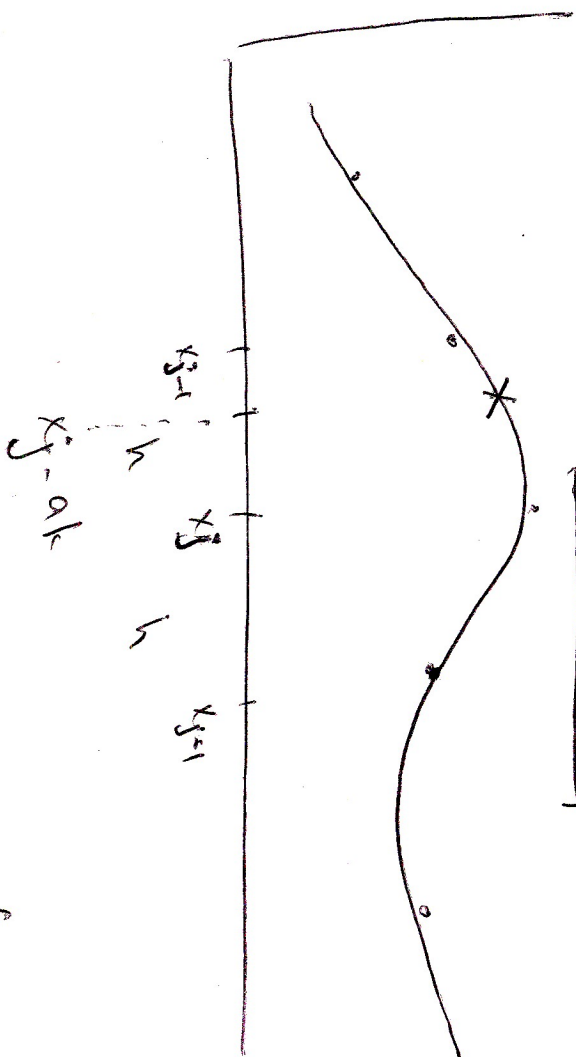
evaluate at $x_j - \alpha \Delta t$

$$u_{j-1}^{n+1} = \left(\frac{u_j^n - u_{j-1}^n}{h} \right) (-\alpha \Delta t) + u_j^n$$

$$= \frac{1}{h} \left[(h - \alpha \Delta t) u_j^n + \alpha \Delta t u_{j-1}^n \right]$$

PS 137 LeVeque

Ex 13.1



Fit a quadratic to the points $(x_{j-1}, y_{j-1}^n), (x_j, y_j^n)$

(x_{j+1}, y_{j+1}^n)

$$ax_{j-1}^2 + bx_{j-1} + c = y_{j-1} \quad \text{etc}$$

$$\begin{bmatrix} x_{j-1}^2 & x_{j-1} & 1 \\ x_j^2 & x_j & 1 \\ x_{j+1}^2 & x_{j+1} & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_{j-1} \\ y_j \\ y_{j+1} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sqrt{\frac{1}{2}} \begin{pmatrix} C_{j+1}^s \\ C_j^s \end{pmatrix}$$

$$= C_j^s - \frac{1}{2} A (C_{j+1}^s - C_{j-1}^s)$$

$$+ \frac{1}{2} A^2 (C_{j+1}^s - 2C_j^s + C_{j-1}^s)$$

$$a(x - x_j^*)^2 + b(x - x_j^*) + c = U_j'$$

$$a h^2 - b h + c = U_{j-1} \quad j-1$$

$$\cancel{a h^2} + \cancel{b h} + c = U_j \quad j$$

$$a h^2 + b h + c = U_{j+1} \quad j+1$$

$$a = \frac{U_{j+1} - 2U_j + U_{j-1}}{2h^2}, \quad b = \frac{U_{j+1} - U_{j-1}}{2h}$$

4 get LM when put into

$$a(x - x_j^*)^2 + b(x - x_j^*) + c$$

Beam Memory:

$$a(x - x_{j-1})^2 + \cancel{b(x - x_{j-1})} + c = U_j$$

$$\Rightarrow C = U_{j-1}$$

$$a h^2 - b h = U_{j-2} - U_{j-1}$$

$$j-2$$

$$a h^2 + b h = U_j - U_{j-1}$$

evaluating the parabola etc that is just doing
lex-meritoff Ages at $j \rightarrow j-1$.

Brain Memory:

$$a(x-x_j)^2 + b(x-x_j) + c = U_j^1$$

$$j=2: \quad 4M^2 a - 2M b + c = U_{j-2}^1$$

$$j-1: \quad M^2 a - M b + c = U_{j-1}^1$$

$$j: \quad c = U_j^1$$

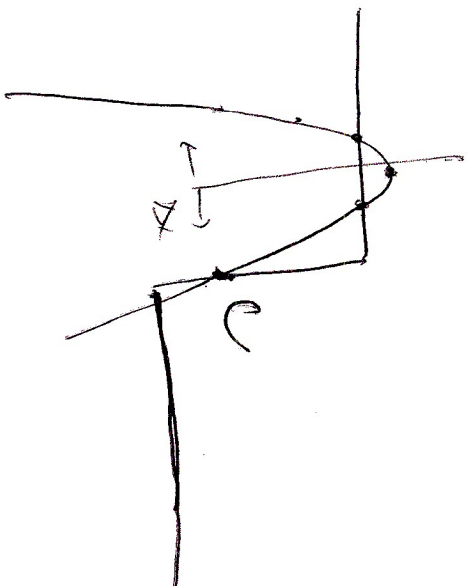
$a \rightarrow$

$b \rightarrow$

Then

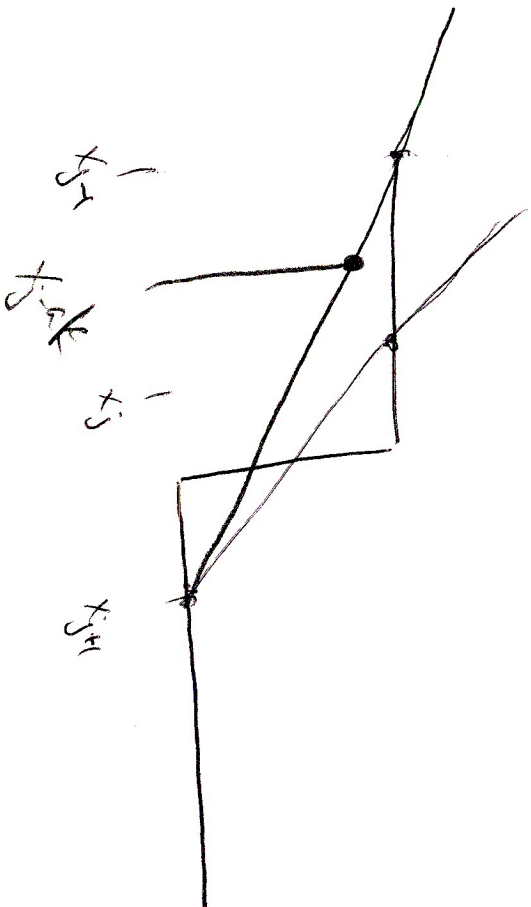
$$U_j^{n+1} = f(x_j - a_k) = U_j^n - a_k \left(\frac{3U_j^n - 4U_{j-1}^n + U_{j-2}^n}{2h} \right)$$

$$a_k^2 \left(\frac{U_{j+1} - 2U_j + U_{j-2}}{2h^2} \right) \quad \checkmark$$



If a discontinuity is present then fitting a parabola to 3 points one of which is point "lower" then the others like pt c then for small changes to the left & right give drastic changes to the y values along this parabola. Thus there is a greater change that the point U_{j+1} will be above

or below the line behind the discount.



Both points have Fredholm
the linear interpolation does
not have quite the same
change in values.

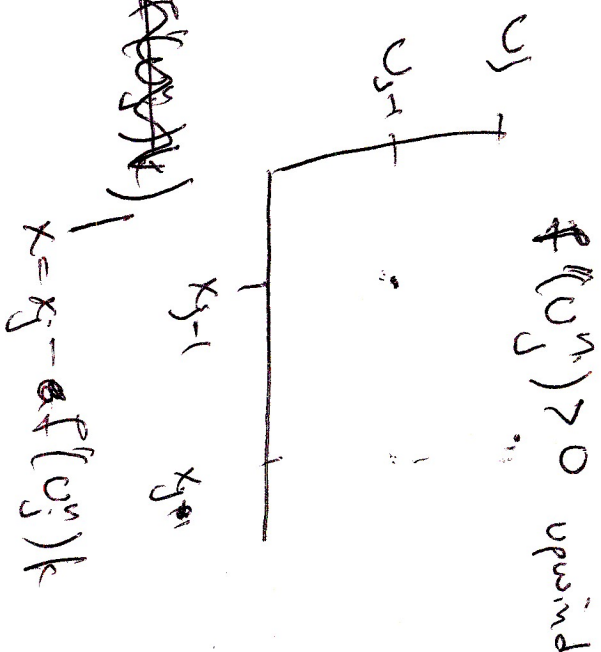
Pg 137 LeVeque

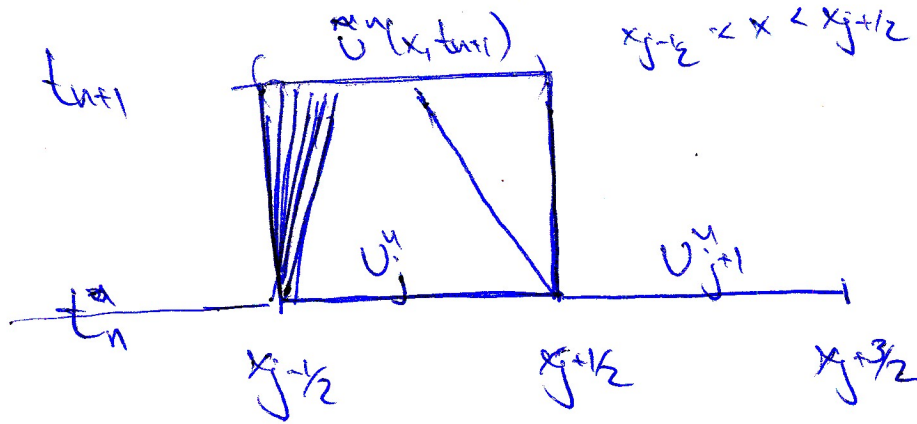
$$\text{Approx } U_j^{n+1} = f(x_j - f'(u_j^n) \tau)$$

$$= \left(\frac{u_j^n - u_{j-1}^n}{\tau} \right) \left(\cancel{x_j - x_j} \right) + u_j^n$$

$$= \frac{1}{\tau} \left[(u_j^n - u_{j-1}^n) u_j^n + f'(u_j^n) \tau u_{j-1}^n \right]$$

$$= u_j^n - \frac{\tau}{\tau} f'(u_j^n) (u_j^n - u_{j-1}^n)$$





$\hat{u}^n(x, t)$ require $\eta(u)$ be convex...

.....
If $\delta \rightarrow 0 \rightarrow \delta \rightarrow 0 \Rightarrow$

|

$$F(u_e, u_r) = f(u^*(u_e, u_r)) = \dots \quad \text{B.22}$$

To fix use entropy set. want to implement

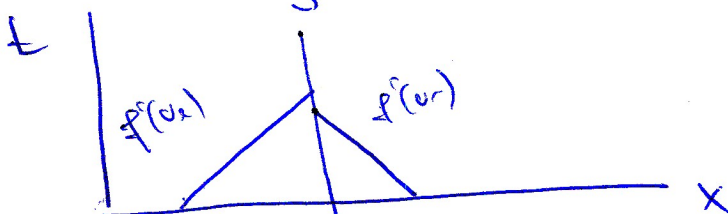
Goddard method in scalar case

for $u_t + f(u)_x = 0$ scalar case

f convex

have s

$$f(u_e) \geq s \geq f(u_r)$$



either shock or fan

3

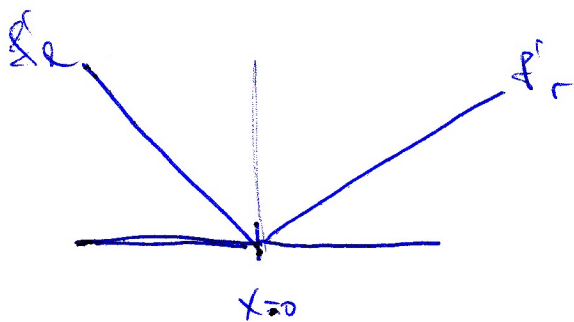
→ Then we want to evaluate at $x=0$

Thus if Both f'_l & f'_r to Right of $x=0$

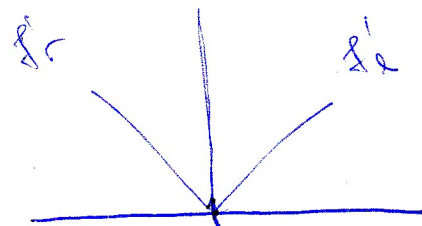
$$\Rightarrow u^* = u_l$$

if Both f'_l & f'_r to left of $x=0 \Rightarrow u^* = u_r$

$$f'_l < 0 < f'_r$$



$$f'_r < 0 < f'_l$$



= shock

Fan $u^* = u_s$ u same on both in a fan

Pg 142 lecture

$$u_{j+1}^s = u_j^s - \frac{1}{N} [F(u_j^s, u_{j+1}^s) - F(u_{j+1}^s, u_j^s)]$$

Putting B.15 in Above gives

$$u_{j+1}^s = u_j^s - \frac{1}{N} \left[\frac{1}{2} A \left(u_j^s + u_{j+1}^s \right) - \left(u_{j+1}^s + u_j^s \right) \right. \\ \left. - \frac{1}{2} A \left[\left(u_j^s - u_{j+1}^s \right) - \left(u_{j+1}^s - u_j^s \right) \right] \right]$$

$$= u_j^s - \frac{1}{N} A \left(u_{j+1}^s - u_{j+1}^s \right) + \frac{1}{2N} A \left[u_{j+1}^s - 2u_j^s - u_{j+1}^s \right]$$

I get -

Pg 143 LeVeque

$$\eta \left(\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{\tilde{u}^n(x, t_{n+1})}{h} dx \right) \leq \int \eta \left(\frac{\tilde{u}^n(x, t_{n+1})}{h} \right) dx$$

$$\text{at } \tilde{u} = \frac{x}{h} \parallel$$

$$\eta \left(\int_{\frac{x_{j-1/2}}{h}}^{\frac{x_{j+1/2}}{h}} \tilde{u}^n(\eta, t_{n+1}) d\eta \right) \leq$$

$$\int_{\frac{x_{j-1/2}}{h}}^{\frac{x_{j+1/2}}{h}} \eta(\tilde{u}^n(\eta, t_{n+1})) d\eta$$

$\parallel$

$$\xi = \frac{x}{h}$$

$$\frac{1}{h} \int_{x_{j-1/2}}^{x_{j+1/2}} \eta(\tilde{u}^n(x, t_{n+1})) dx$$

PS 145 LeVeque

Ex B.2

1) $f(u_L), f'(u_r) > 0$

if $u_L < u_r$

Then $\min_{u_L \leq u \leq u_r} f(u) = f(u_L)$ ✓

if $u_L > u_r$

$\max_{u_L \leq u \leq u_r} f(u) = f(u_L)$ ✓

2) $f'(u_L), f'(u_r) \leq 0$

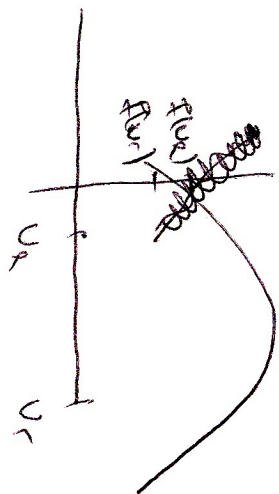
if $u_L < u_r$

$\min_{u_L \leq u \leq u_r} f(u) = f(u_r)$ ✓

or $u_L > u_r$

$\max_{u_L \leq u \leq u_r} f(u) = f(u_r)$ ✓

$$3) \quad f'(u_2) > 0 > f'(u_1)$$

 $f(u)$


$$\frac{[f]}{[u]} > 0$$

$$= [f] > 0 \quad \text{As } [u] > 0$$

$$f(u_1) > f(u_2)$$

8. when $u_2 < u_1$

$$\lim_{[u] \rightarrow 0} f = f(u_2) \quad \checkmark$$

when $u_2 > u_1$ then $[u] < 0 \therefore [f] < 0$

$$\max_{[u], [u_2]} f = f(u_2) \quad \checkmark$$

$$\frac{[f]}{[u]} < 0 \quad + \quad u_2 < u_1 \quad = \quad [f] < 0$$

$$\lim_{[u] \rightarrow 0} f = f(u_1) \quad \checkmark$$

$$\frac{[f]}{[u]} < 0 \quad + \quad u_2 > u_1 \quad = \quad [f] > 0$$

$$f(u_1) > f(u_2)$$

$$\max_{[u], [u_2]} f = f(u_1) \quad \checkmark$$

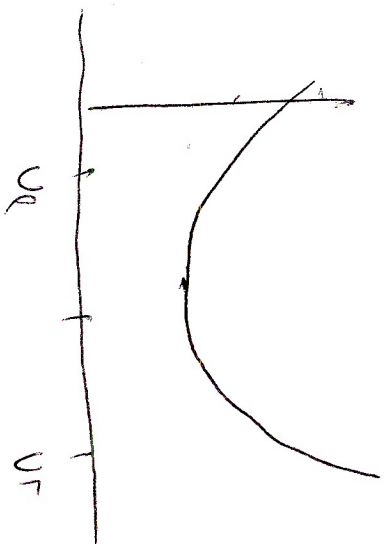
41

$$f'(u_L) < 0 < f'(u_R)$$

$$\text{if } u_L < u_R$$

$$\min_{[u_L, u_R]} f = f(u_S)$$

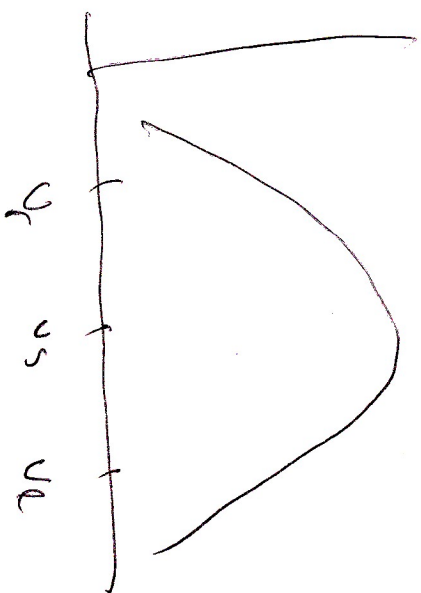
$$\text{if } f'(u_S) = 0. \quad \checkmark$$



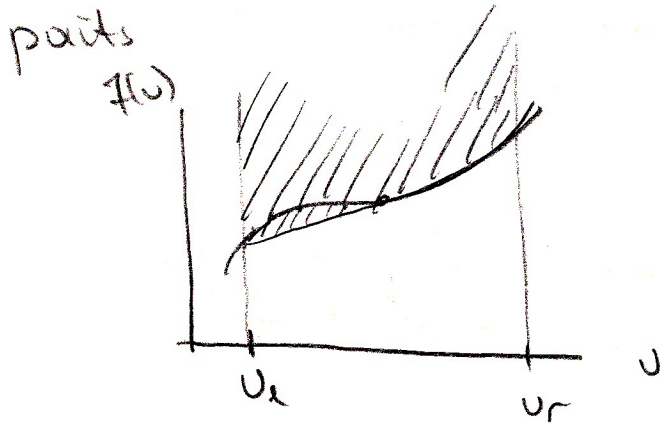
$$\text{if } u_R > u_L$$

$$\max_{[u_L, u_R]} f = f(u_S)$$

$$\text{if } f'(u_S) = 0. \quad \checkmark$$



Ex 13.3 The convex hull solution to the scalar hyperbolic systems is the following idea/method since $f(u)$ is a scalar it can be plotted from u_l to u_r if $u_l \leq u_r$ or from u_r to u_l if $u_r < u_l$. Assuming $u_l < u_r$ we plot $f(u)$ between these



we then form the convex hull of all points Above the graph $f(u)$

Then straight lines represent shocks & curved portions represent fans

Here you see a shock & a fan. If $u_l > u_r$ find the convex hull of points above the graph $f=f(u)$.

Thus eq 13.24 simply takes max's or min's depending on the direction of u_l & u_r as in the convex-hull construction.