Ex 1: $\frac{d}{dx}(e^{-x^2}) = -2x e^{-x^2} \quad \ldots$

Thm 2:

$$g^{(p)}(y) = \int_{-\infty}^{+\infty} f(x)(-2\pi i x)^p e^{-2\pi i x y}\,dx$$

$$= \cancelto{0}{f(x)(-2\pi i x)^p \frac{e^{-2\pi i x y}}{(-2\pi i y)}\Bigg|_{-\infty}^{+\infty}} - \frac{1}{(-2\pi i y)}\int_{-\infty}^{+\infty} \frac{d}{dx}\{f(x)(-2\pi i x)^p\} e^{-2\pi i x y}\,dx$$

$$= \frac{-1}{(-2\pi i y)}\left[\cancelto{0}{\frac{\frac{d}{dx}\{f(x)(-2\pi i x)^p\} e^{-2\pi i x y}}{(-2\pi i y)}\Bigg|_{-\infty}^{+\infty}} - \frac{1}{(-2\pi i y)}\int_{-\infty}^{+\infty} \frac{d^2}{dx^2}\{f(x)(-2\pi i x)^p\} e^{\cdots}\,dx \right]$$

$$= \frac{(-1)^2}{(-2\pi i y)^2}\int_{-\infty}^{+\infty} \frac{d^2}{dx^2}\{f(x)(-2\pi i x)^p\} e^{-2\pi i x y}\,dx$$

$$= \cdots = \frac{(-1)^N}{(-2\pi i y)^N}\int_{-\infty}^{+\infty} \frac{d^N}{dx^N}\{(-2\pi i x)^p f(x)\} e^{-2\pi i x y}\,dx$$

$$\therefore \quad |g^{(p)}(y)| \le \frac{(2\pi)^p}{(2\pi y)^N}\int_{-\infty}^{+\infty} \left| \frac{d^N}{dx^N}\{x^p f(x)\}\right| dx$$

$$= \frac{(2\pi)^{p-N}}{|y|^N}\int_{-\infty}^{+\infty} \cdots \quad = O(|y|^{-N})$$

Thm 4:

$$\mathcal{F}\{e^{-\epsilon x^2} e^{-2\pi i x y}\} = \int_{-\infty}^{\infty} e^{-\epsilon x^2} g(-x) e^{-2\pi i x y}\, dx$$

Then $\quad f(y) = \int_{-\infty}^{\infty} e^{-\epsilon x^2}$

$$\left(\frac{\pi}{\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\pi^2 (y-t)^2/\epsilon} f(t)\, dt$$

$$-\left(\frac{\pi}{\epsilon}\right)^{1/2} f(y) \int_{-\infty}^{\infty} e^{-\pi^2 (y-t)^2/\epsilon}\, dt$$

$$\cancel{v^2 = \pi^2 (y-t)^2/\epsilon} \Rightarrow \cancel{v = \pm \frac{\pi (y-t)}{\epsilon^{1/2}}}$$

$$\cancel{dv = \pm \frac{\pi}{\epsilon^{1/2}}} (-dt)$$

$$\Rightarrow \cancel{\int_{u=+\infty}^{-\infty} e^{-u^2}}$$

Do shift 1st $\quad v = y - t \quad y$ fixed

$$dv = -dt$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi^2 v^2/\epsilon}\, dv = 2\int_0^{\infty} e^{-\pi^2 v^2/\epsilon}\, dv$$

$$u^2 = \frac{\pi^2 v^2}{\epsilon} \Rightarrow u = \frac{\pi}{\epsilon^{1/2}} v \qquad dv = \frac{\epsilon^{1/2}}{\pi} du$$

$$\Rightarrow 2\int_0^{\infty} e^{-u^2}\left(\frac{\pi}{\epsilon^{1/2}}\right)^{-1} du = \cancel{\frac{2}{\pi}(\epsilon^{1/2})\int_0^{\infty} e^{-u^2} du}$$

$$= \frac{2\epsilon^{1/2}}{\pi}\underbrace{\int_0^{\infty} e^{-u^2} du}_{\frac{\sqrt{\pi}}{2}} = \frac{\epsilon^{1/2}}{\sqrt{\pi}}.$$

$$\therefore \left(\frac{\pi}{\epsilon}\right)^{1/2}\int_{-\infty}^{\infty} e^{-\pi^2 (y-t)^2/\epsilon}\, dt = 1.$$

$$\therefore \left| \int_{-\infty}^{\infty} e^{-\epsilon t^2} g(-t) e^{-2\pi i t y}\, dt - \left(\frac{\pi}{\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} f(y) e^{-\pi(y-t)^2/\epsilon}\, dt \right|$$

$$=$$

1st integral $\displaystyle\int_{-\infty}^{\infty} e^{-\epsilon t^2} g(-t) e^{-2\pi i t y}\, dt$

$$\|$$

$$\int_{-\infty}^{\infty} f(x) e^{+2\pi i x t}\, dx$$

$$= \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{\infty} e^{\epsilon t^2} f(x) e^{2\pi i (xt - ty)}\, dt\, dx$$

Do t integral

$$= \int_{x=-\infty}^{\infty} f(x) \int_{t=-\infty}^{\infty} e^{-\epsilon t^2} e^{2\pi i(x-y)t}\,dt\,dx$$

$$e^{-\epsilon\left[t^2 - \frac{2\pi i}{\epsilon}(x-y)t\right]}\,dt$$

$$e^{-\epsilon\left[t^2 - \frac{2\pi i}{\epsilon}(x-y)t - \frac{\pi^2}{\epsilon^2}(x-y)^2\right] - \frac{\pi^2}{\epsilon}(x-y)^2}$$

Then

$$\int_{x=-\infty}^{\infty} f(x) e^{-\frac{\pi^2}{\epsilon}(x-y)^2} \underbrace{\int_{-\infty}^{\infty} e^{-\epsilon\left[t - \frac{\pi i}{\epsilon}(x-y)\right]^2}\,dt}$$

$$\int_{-\infty}^{\infty} e^{-\epsilon t^2}\,dt = \frac{1}{\sqrt{\epsilon}} \int_{-\infty}^{+\infty} e^{-u^2}\,du$$

$$u^2 = \epsilon t^2$$
$$u = \sqrt{\epsilon}\, t$$

3c

$$\therefore = \frac{1}{\sqrt{\epsilon}} \int_{-\infty}^{\infty} f(x) e^{-\frac{\pi^2}{\epsilon}(x-y)^2} \sqrt{\pi}\, dx$$

$$= \frac{\sqrt{\pi}}{\sqrt{\epsilon}} \int_{-\infty}^{\infty} f(x) e^{-\frac{\pi^2}{\epsilon}(x-y)^2} dx = \mathcal{F}\{e^{-\epsilon x^2} g(-x)\}$$

Then $\left| \mathcal{F}\{e^{-\epsilon x^2} g(-x)\} - f(y)\left(\frac{\pi}{\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\pi^2/\epsilon (y-t)^2} dt \right|$

$$= \left| \left(\frac{\pi}{\epsilon}\right)^{1/2} \left[\int_{-\infty}^{\infty} f(t) e^{-\frac{\pi^2}{\epsilon}(t-y)^2} dt - \int_{-\infty}^{\infty} f(y) e^{-\frac{\pi^2}{\epsilon}(y-t)^2} dt \right] \right|$$

$$= \left| \left(\frac{\pi}{\epsilon}\right)^{1/2} \int_{-\infty}^{\infty} \{f(t) - f(y)\} e^{-\frac{\pi^2}{\epsilon}(y-t)^2} dt \right|$$

$\leq \left(\frac{\pi}{\epsilon}\right)^{1/2}$ As: $|f(t) - f(y)| \leq f'(\xi)\,|t-y|$

$\xi \in (t, y)$

3d

$$= \left(\frac{\pi}{\epsilon}\right)^{1/2} \max_{\xi\in\mathbb{R}} |f'(\xi)| \int_{-\infty}^{\infty} |y-t| e^{-\frac{\pi^2}{\epsilon}(y-t)^2} dt$$

$$= \left(\right)^{1/2} \max |f'(\xi)| \underbrace{\int_{-\infty}^{\infty} |t| e^{-\pi^2/\epsilon\, t^2} dt}$$

$$= 2\int_0^{\infty} t e^{-\pi^2/\epsilon\, t^2} dt$$

$$u = \frac{\pi^2}{\epsilon} t^2$$

$$du = \frac{2\pi^2}{\epsilon} t\, dt$$

$$= 2\, \frac{\epsilon}{2\pi^2} \int_0^{\infty} du\, e^{-u} = \frac{\epsilon}{\pi^2} \cdot 1$$

$$\therefore \longrightarrow \left(\frac{\pi}{\epsilon}\right)^{1/2} \cdot \frac{\epsilon}{\pi^2} \longrightarrow O(\sqrt{\epsilon}) \xrightarrow{\epsilon\to 0} 0$$

Thm 5: pg 16

L.H.S.

$$\int_{-\infty}^{\infty} g_1(y) \int_{-\infty}^{\infty} f_2(x) e^{-2\pi i x y}\, dx\, dy$$ Def of $g_2(y)$ = ...

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(y) f_2(x) e^{-2\pi i x y}\, dx\, dy$$

exchange order of integration

$$= \int_{-\infty}^{\infty} f_2(x) \int_{-\infty}^{\infty} g_1(y) e^{-2\pi i x y}\, dy\, dx$$

$$= \cancel{\int_{-\infty}^{\infty} f_2(x) \int_{-\infty}^{\infty} g_1(-y) e^{2\pi i x y}\, dy\, dx}$$

~~F.T. of $g_1(-y)$~~

~~F.T. of $g(y)$~~

$f(x) \xrightarrow{F.T.} g(y)$

$g(-x) \xrightarrow{F.T.} f(y)$

$\therefore g(x) \xrightarrow{F.T.} f(-y)$

But $f_1(x) = \int_{-\infty}^{\infty} g_1(y) e^{2\pi i x y} \, dy$

$\therefore f_1(-x) = \int_{-\infty}^{\infty} g_1(y) e^{-2\pi i x y} \, dy$

$\therefore$ Above

$= \int_{-\infty}^{\infty} f_2(x) f_1(-x) \, dx$ ✓

<u>Pg 17</u>

$\left| \int_{-\infty}^{\infty} e^{-ux^2} \left(\frac{u}{\pi}\right)^{1/2} F(x) \, dx - F(0) \right|$

$= \left| \int_{-\infty}^{\infty} e^{-ux^2} \left(\frac{u}{\pi}\right)^{1/2} (F(x) - F(0)) \, dx \right|$ iff

$\int_{-\infty}^{\infty} e^{-ux^2} \left(\frac{u}{\pi}\right)^{1/2} dx = 1$

Check: $\int_{-\infty}^{\infty} \cdots \, dx = 2\int_0^{\infty} \cdots$

$u^2 = nx^2$ $\quad u\,du = nx\,dx \Rightarrow du = \frac{nx}{\sqrt{n}\,x}dx = \sqrt{n}\,dx$

$$\Rightarrow \left(\frac{n}{\pi}\right)^{1/2} 2\int_0^{\infty} e^{-u^2} \frac{1}{\sqrt{n}}\,du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}du = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1 \quad \checkmark$$

$$\left| \int_{-\infty}^{\infty} e^{-nx^2}\left(\frac{n}{\pi}\right)^{1/2} \{F(x) - F(0)\}\,dx \right|$$

$0 < \xi < x$.

$F(x) - F(0) = F'(\xi)(x - 0)$ By Mean value Thm

$\therefore \quad |F(x) - F(0)| \leq \text{Max}|F'(x)|\,|x|$.

$$\therefore \text{ Above } \leq \max_{x\in\mathbb{R}} |F'(x)| \left| \int_{-\infty}^{\infty} \left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2}|x|\,dx \right|$$

$$= \max|F'|\left(\frac{n}{\pi}\right)^{1/2} 2 \left| \int_0^\infty e^{-nx^2} x \, dx \right|$$

$$u = nx^2$$
$$du = 2nx\,dx$$

$$= \max|F'| \cancel{2}\left(\frac{n}{\pi}\right)^{1/2} \frac{1}{\cancel{2}n} \underbrace{\int_0^\infty e^{-u} \, du}$$

$$-e^{-u} \Big|_0^\infty = -1(0-1) = \underline{1}$$

$$= \max|F'| \frac{1}{\sqrt{n\pi}} \xrightarrow{n \to \infty} 0$$

<u>pg 18</u>

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(y)\, G(y)\, dy = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(\underbrace{\int_{-\infty}^{\infty} f_n(\xi) e^{-2\pi i y \xi} d\xi} \right) G(y)\, dy$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(\xi) \overbrace{\int_{-\infty}^{\infty} e^{-2\pi i y \xi} G(y)\, dy}^{\text{F.T. of}}\, d\xi$$

5

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(\xi) F(\xi)\, d\xi$$ limit exists + ind. of representation

if $$G(y) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x y}\, dx$$

Then $$F(x) = \int_{-\infty}^{\infty} G(y) e^{2\pi i x y}\, dy$$

$$\therefore \int_{-\infty}^{\infty} G(y) e^{-2\pi i x y}\, dy = F(-x)$$

Ex 7:

F.T. $\delta(x) \to 1$

$$\cancel{\int_{-\infty}^{\infty} \delta(x) e^{-2\pi i x y}\, dx} = 1.$$

~~[illegible]~~ limit of F.T. sequence

seq $$\left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2} = f_n(x)$$

$$\text{F.T. } f_n(x) = \int_{-\infty}^{\infty} \left(\frac{n}{\pi}\right)^{1/2} e^{-nx^2} e^{-2\pi i x y}\, dx$$

$$= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-n\left[x^2 + 2\pi i y \cancel{x} /n\right]} dx$$

$$x^2 + 2\pi i y x/n + \left(\frac{\pi i y}{n}\right)^2 = x^2 + 2\pi i y x - \frac{\pi^2 y^2}{n^2}$$

$$= \left(x + \frac{\pi i y}{n}\right)^2$$

$$= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-n\left[(x + \pi y i)^2 + \frac{\pi^2 y^2}{n^2}\right]} dx$$

$$= \left(\frac{n}{\pi}\right)^{1/2} e^{-\pi^2 y^2/n} \int_{-\infty}^{\infty} e^{-n[(x+\pi i y)^2)} dx$$

$$= \quad \cdot\cdot \quad \int_{-\infty}^{\infty} e^{-nx^2} dx \qquad \underline{\text{Check}}$$

$$= \cancel{\left(\frac{n}{\pi}\right)^{1/2}} e^{-\pi^2 y^2/n} \qquad \cancel{\frac{1}{\sqrt{n}}} \cancel{\sqrt{\pi}} = e^{-\pi^2 y^2/n}$$

$$\int_{-\infty}^{\infty} \delta^{(n)}(x) F(x) dx = (-1) \int_{-\infty}^{\infty} \delta^{(n-1)}(x) F'(x)$$

$$= (-1)^2 \int_{-\infty}^{\infty} \delta^{(n-2)}(x) F''(x)$$

$$= \cdots \quad = (-1)^n \int \delta(x) F^{(n)}(x)$$

$$= (-1)^n F^{(n)}(0)$$

<u>Pg 20</u>

$$- \int_{-\infty}^{\infty} \frac{d}{dx}\{F(x)\phi(x)\} f(x)\, dx + \int_{-\infty}^{\infty} F(x)\phi'(x) f(x) dx$$

= By 16 1st identity.

$$\int_{-\infty}^{\infty} F(x)\phi(x)f'(x)\, dx + \int_{-\infty}^{\infty} F(x)\phi'(x)f(x)$$

$$= \int_{-\infty}^{\infty} F(x)\{\phi(x)f'(x) + \phi'(x)f(x)\}\, dx$$

$$\therefore \int_{-\infty}^{\infty} F(x) \frac{d}{dx}\{\phi(x) f(x)\} dx = \int_{-\infty}^{\infty} F(x) \{\phi(x) f'(x) + \phi'(x) f(x)\} dx$$

$$\therefore \frac{d}{dx}\{\phi(x) f(x)\} = \phi(x) f'(x) + \phi'(x) f(x)$$

Thm 7:

$f(x)$ g.f. w/ F.T. $g(y)$

F.T. ~~of~~ $f(ax+b)$

$f_n(x)$ $\qquad g_n(y) =$ F.T. of $f_n(x)$

~~$\therefore$~~ F.T., of $f_n(ax+b)$ is $|a|^{-1} e^{2\pi i b y/a} g_n(y/a)$

By Thm 3.

As $|a|^{-1} e^{2\pi i b y/a}$ is a fairly good fn By def 6:

$\Leftrightarrow |a|^{-1} e^{2\pi i b y/a} g(y/a)$ is the ass.

generalised fn

$f'(t)$

<u>Def 6:</u>

$$f_n'(t) \xrightarrow{F.T.} 2\pi i y\, g_n(y)$$

$2\pi i y$ is a fairly good fn

$\therefore$ $2\pi i y\, g_n(y) \longrightarrow 2\pi i y\, g(y)$

$f(y)$ is F.T. $g(-x)$

Seq $f_n(x)$ define $f(x)$

$f_n \xrightarrow{F.T.} g_n(y)$ define $g(y)$

But now $g_n(y)$ define $g(-y)$

But by Thm 4

F.T. of $g_n(-x)$ is $f_n(y)$

$\therefore$ F.T. of $g(-x)$ is $f(y)$

Ex 9:

F.T. of $\delta(x-c)$ $\quad a=1 \quad b=-c$

is $e^{-2\pi i c y}\, 1 = \underbrace{e^{-2\pi i c y}}_{g(y)}$

S - 10.
is AP 10

Then by Thm 7.

F.T. $e^{+2\pi i c y} \xrightarrow{\text{F.T.}} \delta(x-c)$

F.T. $e^{2\pi i c x} \xrightarrow{\text{F.T.}} \delta(y-c)$

Thm 8:

Don't follow.

$$F_1(x) = \int_{-\infty}^{x} F(x)\,dx$$ good iff $$\int_{-\infty}^{\infty} F(x)\,dx = 0.$$

$$F_1(x) = O(|x|^{-N}) \qquad |x| \to \infty \quad \forall N.$$

$$F_2 = \int_{-\infty}^{x} \left\{ F(x) - \frac{e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} F(t)\,dt \right] dx$$

As

$$\int_{-\infty}^{\infty} \left\{ F(x) - \frac{e^{-x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} F(t)\,dt \right] dx = 0 \quad \checkmark$$

RHS. of 19

$$\Rightarrow \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\frac{e^{-x^2}}{\sqrt{\pi}} F(t)\,dt\,dx} + \int_{-\infty}^{\infty} f(x)F(x) - \underbrace{\frac{e^{-x^2}}{\sqrt{\pi}} f(x) \int_{-\infty}^{\infty} F(t)\,dt}\,dx$$

$$= \int_{-\infty}^{\infty} f(x)F(x)\,dx$$

$$\therefore \int_{-\infty}^{\infty} f(x)F(x)\,dx = C\int_{-\infty}^{\infty} F(t)\,dt + \int_{-\infty}^{\infty} f(x)F_2'(x)\,dx$$

$$= C\int_{-\infty}^{\infty} F(t)\,dt - \cancel{\int_{-\infty}^{\infty} f'(x)F_2(x)\,dx}^{\,0}$$

$$\therefore \int_{-\infty}^{\infty} f(x)F(x)\,dx = C\int_{-\infty}^{\infty} F(x)\,dx$$

$\Rightarrow \quad f(x) = C\,\mathbb{1}$ $\qquad$ $\mathbb{1}$ = gen fn 1

$\mathbb{1}_n = e^{-x^2/n^2}$

<u>Thm 9:</u> $\cancel{f(y) - y g(y) = 0}$

$\cancel{f'(y) - y g'(y) + g(y) = 0}$

$\Rightarrow \therefore \cancel{f(y) = C}$

$\Rightarrow \cancel{y(g'(y) + y g(y))} = \cancel{y g(y) = C}$

~~F.T. $f'(x)$ is $2\pi i y g(y)$~~

$\therefore$ ~~F.T. $\frac{f'(x)}{2\pi i}$ is $y g(y)$~~

~~if $y g(y) \equiv 0$~~

$\Rightarrow$ $\cancel{f} = 0$ $f'(x) = 0 \xrightarrow{\text{Thm 8}} f(x) = C$

F.T.

$2\pi i y g(y) = 0$

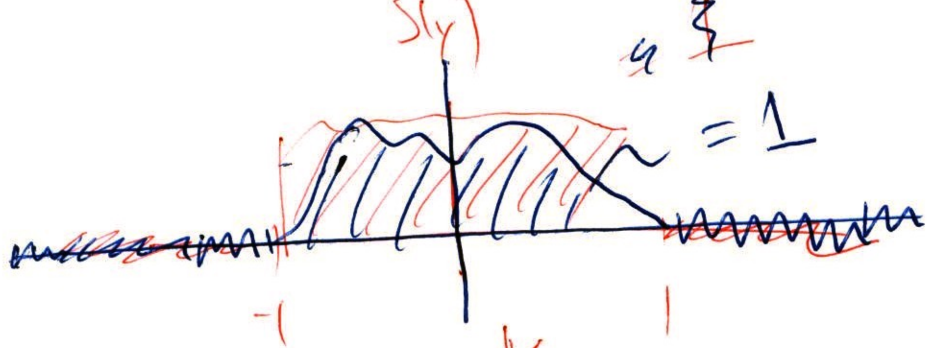
Now $y g(y)$

$G(y) = y G_1(y) + G(0) e^{-y^2}$

$$\int_{-\infty}^{\infty} g(y) G(y)\, dy = G(0) \int e^{-y^2} g(y)\, dy + \cancel{\int_{-\infty}^{\infty} y G_1(y) g(y)\, dy}^{\,0}$$

$= G(0) C$

$\Rightarrow \quad g(y) = C \delta(y)$ ✓



$$Ce^{-1/(1-y^2)}$$

$$|f_n^{(p)}(x)| = \left|\int_{-\infty}^{\infty} f(t)(-n)^p S^{(p)}[n(t-x)]\, n e^{-t^2/n^2}\, dt\right|$$

$$\leq n^{p+1} \operatorname{Max} |S^{(p)}(y)|$$

Ex 10:

$$\int_{-\infty}^{\infty} \frac{d\,\mathrm{sgn}\,x}{dx} F(x)\, dx = -\int_{-\infty}^{\infty} \mathrm{sgn}\,x\, F'(x)\, dx$$

$$= -\int_{0}^{\infty} F'(x)\, dx + \int_{-\infty}^{0} F'(x)\, dx$$

$$= -F(x)\Big|_{0}^{\infty} + F(x)\Big|_{-\infty}^{0} = 2F(0)$$

Thm II:

$$g(y) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x)\,dx \quad \text{Then } f(x) = \int_{-\infty}^{\infty} e^{2\pi i x y} g(y)\,dy$$

+ g(y) must be + $g(y \to \pm\infty) = O(|y|^{-2})$

∴ $(1+y^2)^{-1} g(y)$ is certainly absolutely integrable.

$$\int_{-\infty}^{\infty} g(y) G(y)\,dy = \int_{-\infty}^{\infty} f(x) F(-x)\,dx$$

Ex II:

$\delta(x) = 0$ for $0 < x < \infty$ + $-\infty < x < 0$

$$\int_{-\infty}^{\infty} \delta(x) F(x)\,dx = 0 \quad \text{As} \quad F(x) \equiv 0$$

outside $a < x < b$

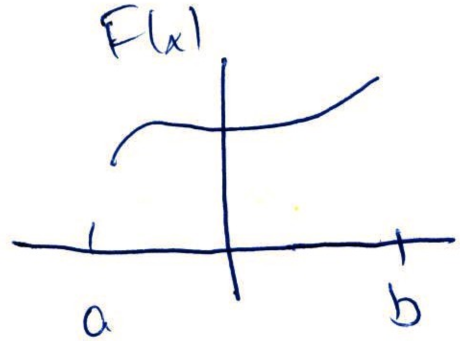
Thm 12:

$$\int_{-\infty}^{\infty} f'(x)F(x)\,dx = -\int_{-\infty}^{\infty} f(x)\,F'(x)\,dx$$

$$= -\int_a^b h(x)F'(x)\,dx = -\Big[h(x)F(x)\Big|_a^b$$

$$- \int_a^b h'(x)F(x)\,dx\Big]$$

$=$



if $a = -\infty$

As $\int_a^b h(x)F(x)\,dx$ exists

By def B.

Then

$$\lim_{x\to a} h(x)F(x) = 0.$$

if $a \neq -\infty$ Then $\lim_{x\to a} \frac{h(x)F(x)}{x-a} = \pm\infty$ $\rightarrow\leftarrow$

But $\left(\frac{F(x)}{x-a}\right)$ is a good fn

f vanishes outside $a < x < b$

f $\therefore$ By Def B $\int_a^b h(x) \frac{F(x)}{(x-a)}\,dx$ exists

Ex 16: $\delta'(x)$ is even

Thm 15:

$$\lim_{t\to c} \int_{-\infty}^{\infty} f_t'(x)F(x)\,dx = \lim_{t\to c} -\int_{-\infty}^{\infty} f_t(x)F'(x)\,dx = -\int_{-\infty}^{\infty} f(x)F'(x)\,dx$$

$$= -\left[\cancelto{0}{f(x)F(x)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f'(x)F(x)\,dx \right]$$

$$= \int_{-\infty}^{\infty} f'(x)F(x)\,dx$$

$$\cancel{\frac{d}{dx} \lim_{\epsilon} = 2\delta(x)}$$

$$\cancel{\frac{d}{dx}\left(\lim_{\epsilon\to 0} |x|^{\epsilon} \operatorname{sgn} x\right) = 2\delta(x)}$$

$$\cancel{= \frac{d}{dx}\left\{ \operatorname{sgn} x \lim_{\epsilon\to 0} |x|^{\epsilon} \right\} = 2\delta(x)}$$

ϵ let $f_\epsilon(x) = |x|^{\epsilon} \operatorname{sgn} x$.

Then $f_\epsilon(x) \xrightarrow{\epsilon\to 0} f$

$\therefore$ By Thm 15:

$$f_\epsilon'(x) \longrightarrow f'(x)$$

$$\therefore \quad \frac{d}{dx} f_\epsilon(x) = \frac{d}{dx}\left(|x|^{\epsilon} \operatorname{sgn} x\right)$$

$$= \epsilon |x|^{\epsilon-1} \operatorname{sgn}^2 x + |x|^{\epsilon}\, 2\delta(x)$$

$$= \epsilon |x|^{\epsilon-1} + |x|^{\epsilon}\, 2\delta(x)$$

$$\lim_{\epsilon \to 0} |x|^{\epsilon} \operatorname{sgn} x = \operatorname{sgn} x$$

$$\frac{d}{dx} \rightarrow \qquad\qquad = 2\delta(x)$$

$$\lim_{\epsilon \to 0} \left(\epsilon |x|^{\epsilon - 1} \operatorname{sgn}^2 x + \underbrace{|x|^{\epsilon} 2\delta(x)}_{|0|^{\epsilon} = 0} \right)$$

$\therefore$

$$\lim_{\epsilon \to 0} \epsilon |x|^{\epsilon - 1} = 2\delta(x)$$

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1

$$\left| \int_{-\infty}^{\infty} (|x|^{\epsilon} - 1) \operatorname{sgn} x \, F(x) \, dx \right|$$

$$\leq \cancel{\int_{-\infty}^{\infty} |F(x)| \, |1 + |x|^{\epsilon}| \, dx}$$

Note:

$$|\tanh z| \leq |z| \qquad \tanh z = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

$$z = \log |x|^{\epsilon/2}$$

$$\tanh(\log f)$$

$$= \frac{f - f^{-1}}{f + f^{-1}} \cdot \frac{f}{f}$$

Then

$$\cancel{|\tanh z|}$$

$$= \frac{f^2 - 1}{f^2 + 1}.$$

if $f = |x|^{\epsilon/2}$

$$= \frac{|x|^{\epsilon} - 1}{|x|^{\epsilon} + 1}$$

Then

$$\left| \frac{|x|^{\epsilon} - 1}{|x|^{\epsilon} + 1} \right| \leq \log |x|^{\epsilon/2}$$

Thus $\left| |x|^{\epsilon} - 1 \right| \le \frac{\epsilon}{2} \log|x| \, \left| |x|^{\epsilon} + 1 \right|$

$\therefore$

$$\left| \int_{-\infty}^{\infty} (|x|^{\epsilon} - 1) \operatorname{sgn} x \, F(x) \, dx \right|$$

$$\le \frac{\epsilon}{2} \int_{-\infty}^{\infty} \left| \log|x| \right| \left| |x|^{\epsilon} + 1 \right| |F(x)| \, dx = O(\epsilon)$$

Pr: $|\tanh z| \le |z|$

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Ex 2:

Pr: $\phi(x)\delta(x) = \phi(0)\delta(x)$

let $F(x)$ be a good f

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)F(x)\,dx = \phi(0)F(0) = \phi(0)\int_{-\infty}^{\infty}\delta(x)F(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \phi(x)\delta(x)F(x)\,dx = \int_{-\infty}^{\infty}\phi(0)\delta(x)F(x)\,dx$$

Then $\phi(x)\delta(x) = \phi(0)\delta(x)$

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$$\frac{d}{dx}|x|^\alpha = \alpha|x|^{\alpha-1}\operatorname{sgn}x \,.\quad ✓.$$

$$\frac{d}{dx}|x|^\alpha \operatorname{sgn}x$$

$$= \alpha|x|^{\alpha-1} + \underbrace{|x|^\alpha 2\delta(x)}_{= 0 \text{ if } \underline{\alpha > 0} \text{ from}}$$

~~[illegible]~~:

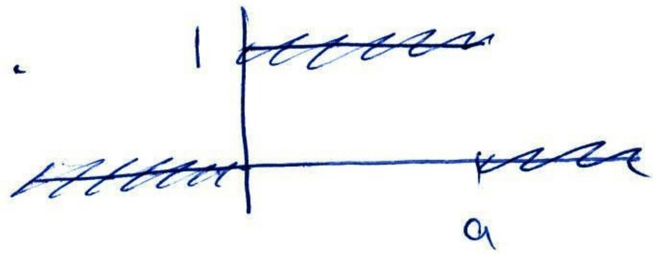
$$\frac{d}{dx}x^\alpha H(x) = \alpha x^{\alpha-1}H(x) + \underbrace{x^\alpha\delta(x)}_{= 0 \text{ by Ex } 2.}$$

As: $\frac{d}{dx}H(x) = 1\,\delta(x)$

$$|x|^\alpha = \frac{1}{\alpha+1}\frac{d}{dx}\left\{|x|^{\alpha+1}\operatorname{sgn}x\right\}$$

$$\int_0^a x^\alpha F(x)\,dx$$

$$x^\alpha \left[\underbrace{H(x) - H(x-a)}_{\text{ploted}}\right].$$



$$\therefore \int_{-\infty}^{\infty} x^\alpha \{H(x) - H(x-a)\} F(x)\,dx = \int_0^a x^\alpha F(x)\,dx$$

$$\frac{d}{dx}\{f(x)H(x-a)\} = \frac{d}{dx}\{(f(x) - f(a) + f(a))H(x-a)\}$$

$$= \frac{d}{dx}\{f(x) - f(a)\}H(x-a) + f(a)\,\delta(x-a).$$

$$= f'(x)H(x-a) + f(a)\,\delta(x-a)$$

$$x^\alpha H(x) - x^\alpha H(x-a) =$$

$$x^\alpha H(x) \equiv \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)} \frac{d^n}{dx^n}\{x^{\alpha+n}H(x)\}$$

eq (5) w/ $f = x^{\alpha+1}$ gives

$$\frac{d}{dx}\{x^{\alpha+1}H(x-a)\} = (\alpha+1)x^\alpha H(x-a) + x^{\alpha+1}\Big|_a \delta(x-a)$$

$$= (\alpha+1)x^\alpha H(x-a) + a^{\alpha+1}\delta(x-a)$$

$$\frac{d^2}{dx^2}\{x^{\alpha+2}H(x-a)\} \equiv \frac{d}{dx}\Big\{(\alpha+2)x^{\alpha+1}H(x-a) + a^{\alpha+2}\delta(x-a)\Big\}$$

$$= (\alpha+2)(\alpha+1)x^\alpha H(x-a) + (\alpha+2)a^{\alpha+1}\delta(x-a)$$

$$+ a^{\alpha+2}\delta^{(1)}(x-a)$$

Th

$$\frac{d^n}{dx^n}\{x^{\alpha+n}H(x-a)\} = (\alpha+n)(\alpha+n-1)\cdots(\alpha+2)(\alpha+1)x^\alpha H(x-a)$$

$$+ (\alpha+n)(\alpha+n-1)\cdots(\alpha+2)a^{\alpha+1}\delta(x-a)$$

$$+ (\alpha+n)(\alpha+n-1)\cdots(\alpha+3)a^{\alpha+2}\delta'(x-a)$$

$$+ (\alpha+n)(\alpha+n-1)\cdots(\alpha+k)\, a^{\alpha+k-1}\, \delta^{k-2}(x-a)$$

$$+ \cdots$$

$$+ \quad a^{\alpha+n}\, \delta^{n-1}(x-a)$$

$$\therefore \quad x^{\alpha} H(x-a) = \frac{1}{(\alpha+n)(\alpha+n-1)\cdots(\alpha+2)(\alpha+1)} \left\{ \cdots\cdots \right\}$$

$$= \frac{1}{(\alpha+n)(\alpha+n-1)\cdots(\alpha+2)(\alpha+1)} \frac{d^n}{dx^n}\left\{ x^{\alpha+n} H(x-a) \right\}$$

$$- \frac{1}{(\alpha+1)} a^{\alpha+1}\, \delta(x-a) - \frac{1}{(\alpha+2)(\alpha+1)} a^{\alpha+2}\, \delta'(x-a)$$

$$- \cdots - \frac{1}{(\alpha+k-1)(\alpha+k-2)\cdots(\alpha+2)(\alpha+1)} a^{\alpha+k-1}\, \delta^{k-2}(x-a)$$

$$- \cdots - \frac{1 \quad a^{\alpha+n}}{(\alpha+n)(\alpha+n-1)\cdots(\alpha+2)(\alpha+1)} \delta^{(n-1)}(x-a)$$

$$\therefore \quad x^{\alpha} H(x) - x^{\alpha} H(x-a)$$

$$= \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)} \frac{d^n}{dx^n}\left\{ x^{\alpha+n} H(x) - x^{\alpha+n} H(x-a) \right\}$$

$$+ \frac{a^{\alpha+1}}{\alpha+1}\,\delta(x-a) + \frac{a^{\alpha+2}}{(\alpha+2)(\alpha+1)}\,\delta'(x-a) + \cdots$$

$$+ \frac{a^{\alpha+n}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\,\delta^{n-1}(x-a) \qquad \text{into } (4)$$

$$\Rightarrow \quad \int_0^a x^{\alpha} F(x)\,dx = \int_{-\infty}^{\infty} x^{\alpha}\left\{H(x) - H(x-a)\right\} F(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\prod_{k=1}^{n}(\alpha+k)} \frac{d^n}{dx^n}\left\{ x^{\alpha+n}\left(H(x) - H(x-a)\right)\right\} dx$$

$$+ \frac{a^{\alpha+1}}{\alpha+1} F(a) + \frac{a^{\alpha+2}}{(\alpha+2)(\alpha+1)} \int_{-\infty}^{\infty} \delta'(x-a)\,F(x)\,dx$$

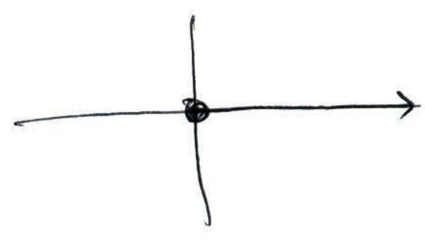
$$+ \cdots + \frac{a^{\alpha+n}}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)} \int_{-\infty}^{\infty} \delta^{(n-1)}(x-a)\, F(x)\, dx$$

$$= \frac{1}{\prod_{k=1}^{n}(\alpha+k)} \int_{-\infty}^{\infty} \frac{d^n}{dx^n}\left\{x^{\alpha+n}\big(H(x)-H(x-a)\big)\right\} dx$$

$$+ \frac{a^{\alpha+1}}{\alpha+1} F(a) - \frac{a^{\alpha+2}}{(\alpha+2)(\alpha+1)} F'(a) + \frac{a^{\alpha+3}}{(\alpha+3)(\alpha+2)(\alpha+1)} F''(a)$$

$$- \cdots + \frac{a^{\alpha+n}(-1)^{n-1} F^{(n-1)}(a)}{\prod_{k=1}^{n}(\alpha+k)}$$

$$\Gamma(\alpha+1) = \int_{0}^{\infty} x^{\alpha} e^{-x}\, dx$$



$$\Gamma(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x}\, dx$$

$$\int_{0}^{\infty} x^{\alpha} e^{-ax}\, dx = a^{-\alpha-1} \int_{0}^{\infty} z^{\alpha} e^{-z}\, dx = a^{-\alpha-1}\alpha!$$

$$z = ax \qquad dz = a\,dx \qquad x = z/a$$

F.T. $x^\alpha H(x)$ $\alpha > -1$

$$\lim_{t\to 0} x^\alpha e^{-xt} H(x) = x^\alpha H(x)$$

$$\left|\int_{-\infty}^{\infty} \left(x^\alpha e^{-xt} H(x) F(x) - x^\alpha H(x) F(x)\right) dx\right|$$

$$= \left|\int_0^\infty \left(x^\alpha e^{-xt} F(x) - x^\alpha F(x)\right) dx\right|$$

$$= \left|\int_0^\infty (1-e^{-xt})\, x^\alpha F(x)\, dx\right| = \left|\int_0^\infty \frac{(1-e^{-xt})}{x^{3/2}}\, x^{\alpha+3/2} F(x)\, dx\right|$$

$$\le A \int_0^\infty \frac{1-e^{-xt}}{x^{3/2}}\, dx$$

$\alpha > -1$

$\alpha + 3/2 > +1/2$

$\therefore$ $\lim_{x\to 0} \frac{(1-e^{-xt})}{x^{3/2}} = \cancel{\lim_{x\to 0} \frac{t e^{-xt}}{\frac{3}{2} x^{1/2}}}$

$\cancel{5/2 > 1}$ $\frac{0}{0}$

$$\lim_{x\to 0} \frac{(1-e^{-xt})}{x^{3/2}} = \lim_{x\to 0} \frac{te^{-xt}}{3/2\, x^{1/2}}$$

$$= \mathcal{O}(x^{-1/2}) \to \cancel{\infty} \quad \text{conv. at } x=0 \checkmark.$$

Now:

$$\int_0^\infty \frac{1-e^{-xt}}{x^{3/2}}\,dx \qquad \begin{array}{l} u = xt \\ du = dx \cdot t \end{array}$$

$$= t^{3/2} \int_0^\infty \frac{1-e^{-u}}{u^{3/2}} \frac{du}{t} = t^{1/2} \int_0^\infty \frac{1-e^{-u}}{u^{3/2}}\,du.$$

How do $\int_0^\infty \frac{1-e^{-u}}{u^{3/2}}\,du = ?$

$$\int_0^\infty u^{-3/2}\,du - \underbrace{\int_0^\infty e^{-u} u^{-3/2}\,du}$$

$$\cancel{u^{-1/2}\Big|_0^\infty}$$

$$1-e^{-u} = \sum_{k=1}^\infty \frac{(-1)^k u^k}{k!}$$

$$\int u\,dv = uv - \int v\,du$$

$$u = 1 - e^{-u} \qquad v = -2u^{-1/2}$$

$$dv = u^{-3/2} \qquad du = e^{-u}$$

$$\int_0^\infty \frac{1-e^{-u}}{u^{3/2}}\,du = -2u^{-1/2}(1-e^{-u})\Big|_0^\infty + 2\int_0^\infty u^{-1/2}e^{-u}\,du$$

$$= 0 + 2\lim_{u\to 0}\left[\frac{(1-e^{-u})}{u^{1/2}}\right]$$

$$+ 2\int_0^\infty e^{-u}u^{-1/2}\,du$$

$$= 2\,\cancel{\frac{e^{-u}\to 0}{\tfrac{1}{2}u^{-1/2}}} + 2\int_0^\infty e^{-u}u^{-1/2}\,du \qquad \frac{0}{0}$$

Ex 22:

F.T. of $\dfrac{1}{(x-c_1-ic_2)^m}$ $c_2 \neq 0$ is

$$2\pi i\, H(-c_2 y)\, \operatorname{sgn} c_2\, \frac{(-2\pi i y)^{m-1}}{(m-1)!}\, e^{-2\pi i(c_1+ic_2)}$$

Ex 23

P

$c_2=-1$ $c_2=+1$

$$\frac{x^5}{x^4-1} = x + \frac{1/4}{x+1} + \frac{1/4}{x-1} - \frac{1/4}{x+i} - \frac{1/4}{x-i}$$

$$x \longrightarrow (2\pi i)^{-1}\delta^{(1)}(y)$$

$$\tfrac{1}{4}(-\pi i)\operatorname{sgn} y \cdot e^{2\pi i(1)y}$$

$$\tfrac{1}{4}(-\pi i)\operatorname{sgn} y\, e^{-2\pi i y}$$

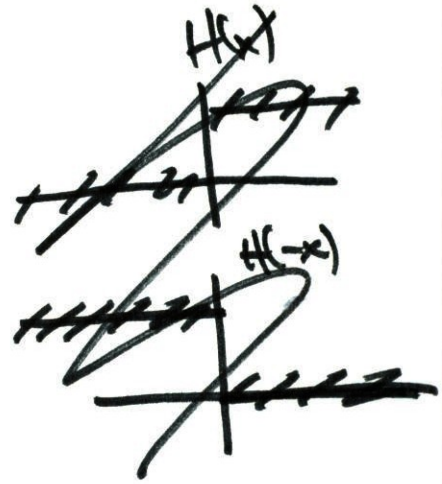
$$-\tfrac{1}{4}\left[2\pi i H(y)(-1)\, e^{-2\pi i y(-i)}\right]$$

$$-\frac{1}{4}\left[2\pi i H(-y)(1)\, e^{-2\pi i y (i)}\right]$$

$$= \frac{i\,\delta^{(1)}(y)}{2\pi} + \frac{-i\pi}{4}\left\{e^{2\pi i y} + e^{-2\pi i y}\right\} \operatorname{sgn} y$$

$$-\frac{1}{4}(2\pi i)\left\{-H(-y)\,e^{-+2\pi y} + H(y)\,e^{2\pi y}\right\}$$

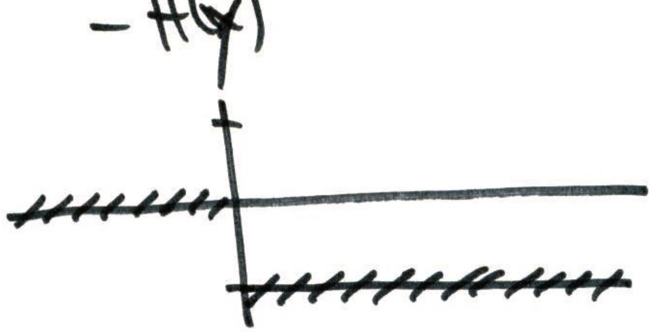
$$= \frac{i\,\delta^{(1)}(y)}{2\pi} - \frac{i\pi}{2}\operatorname{sgn} y \cos 2\pi y$$

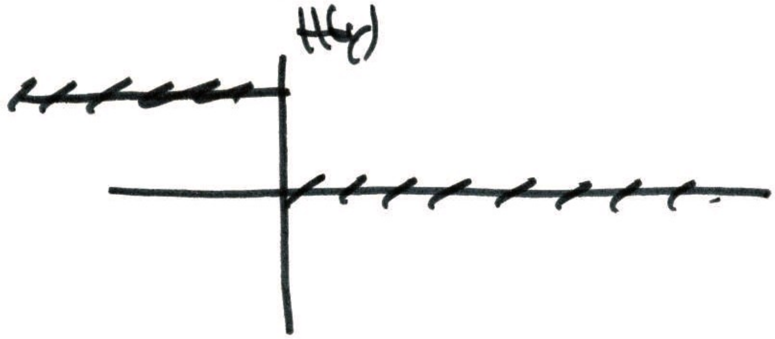


$$-\frac{\pi i}{2}\left\{\cancel{e^{-2\pi|y|}}\right\}$$

$$\therefore\ \forall y \Rightarrow e^{-2\pi|y|}$$

$$\Rightarrow \left(\begin{array}{l} + \\ \Rightarrow \ x<0 \\ e^{2\pi y} \\ x>0 \\ e^{-2\pi y} \end{array} \right)$$





$$\left\{-H(y)e^{-2\pi y} + H(y)e^{2\pi y}\right\} = \begin{cases} e^{2\pi y} & y<0 \\ -e^{-2\pi y} & y>0 \end{cases}$$

$$= e^{-2\pi|y|}$$

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Ex 10:

$$\frac{1}{(x+4)^2(x+1)^2} =$$

Thm 16:

$$g(y) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x y}\, dx$$

Note

$$-\int_{-\infty}^{\infty} f\left(x + \frac{1}{2y}\right) e^{-2\pi x i y}\, dx$$

if $u = x + \frac{1}{2y} \Rightarrow x = u - \frac{1}{2y}$

$du = dx$

$$= -\int_{-\infty}^{\infty} f(u)\, e^{-2\pi i y \left(u - \frac{1}{2y}\right)}\, du$$

$$= -\int_{-\infty}^{\infty} f(u)\, e^{-2\pi i y u}\, du \, \underbrace{e^{i\pi}}_{-1} = \int_{-\infty}^{\infty} f(u)\, e^{-2\pi i y u}\, du$$

$$= g(y)$$

$$|g(y)| = \left| \frac{1}{2} \int_{-\infty}^{\infty} [f(x) - f(x+\tfrac{1}{2y})] e^{-2\pi i x y} dx \right|$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} |f(x) - f(x+\tfrac{1}{2y})| dx \xrightarrow{|y| \to \infty} 0$$

<u>Ex 24</u>:

$f(x) = O(|x - x_m|^{\beta_m}) \quad x \to x_m \quad \beta_m > -1$

$m = 1, \dots, M$

$f(x) = O(|x|^{\beta_0}) \quad |x| \to +\infty \quad \beta_0 < -1$

<u>Ex 25</u>: $\quad f(x) = \dfrac{\operatorname{sgn} x}{|x^2-1|^{1/4} |x^2+1|^{1/2}}$

$$= \frac{\operatorname{sgn} x}{|x-1|^{1/4} |x+1|^{1/4} |x+i|^{1/2} |x-i|^{1/2}}$$

$\beta_i = -1/2 \quad \beta_i > -1 \quad \checkmark$

$f(x) = O(|x|^{-1/2}) = O(|x|^{2})$

$g(y) \to 0 \qquad g(y) = O(h(y)) \qquad y \to c$

$g(y) = o(h(y))$

~~[illegible]~~ pg 48 Lighthill

Thm 17: gen fn $f(x)$ abs. integrable in $(-\infty, \infty)$

+ $g(y)$ is F.T. $g(y) \to 0 \quad |y| \to +\infty$

Try to weaken this thm. by Breaking up

Abs integ $(-\infty, \infty)$ into every finite

int $(-R, R)$ + $(-\infty, -R)$ + (R, ∞)

Note: Must have Abs integ on $(-R, R)$

look at Table F.T.

$$|x|^{\alpha} \longrightarrow \left(2\cos\frac{\pi}{2}(\alpha+1)\right)\alpha!\,(2\pi|y|)^{-\alpha-1}$$

but

$\alpha < 1 \qquad$ F.T. $\longrightarrow 0 \qquad + \quad |x|^{\alpha}$ Abs inte

$\alpha - 1 < 0 \qquad\qquad\qquad\qquad (-R, R)$ ✓

4

$$|x|^{\alpha}\,\mathrm{sgn}\,x \xrightarrow{F.T.} \left\{-2i\sin\frac{\pi}{2}(\alpha+1)\right\}\alpha! \cdot (2\pi|y|)^{-\alpha-1}\,\mathrm{sgn}\,y \longrightarrow 0$$

✓

$$|x|^{\alpha}H(x) \xrightarrow{F.T.} \qquad \longrightarrow 0$$

✓

$$x^{n} \longrightarrow \qquad \longrightarrow 0$$

✓

$$x^{n}\,\mathrm{sgn}\,x \longrightarrow \qquad \longrightarrow 0$$

✓

$$x^{n}H(x) \Longrightarrow \qquad \longrightarrow 0$$

✓

$$x^{-m} \xrightarrow{F.T.} \qquad \longrightarrow + \infty$$

Not Abs integrable if $m > 0$ + m integer

$m > 0 \Rightarrow m = 1, 2, 3, \ldots$

$$|x|^\alpha \log|x| \xrightarrow{\text{F.T.}} \quad \mathcal{O} \quad ✓$$

~~$\alpha \geq 1$~~ $\alpha > -1$

$$\psi(n) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \underset{n\to\infty}{\sim} \underline{\log|n|}$$

$$= -\gamma + \sum_{k=1}^{n} \frac{1}{k} = -\gamma + H_n.$$

$$|x|^\alpha \log|x| \operatorname{sgn} x \xrightarrow{\text{F.T}} \quad \mathcal{O} \quad ✓$$

~~$\alpha \geq 1$~~ $\alpha > -1$

~~$\alpha \geq 1 > 0$~~ $-\alpha < 1$

$-\alpha - 1 < 0$

$$\lim_{x\to 0} x^n \log|x| \qquad n > 0$$

$$\lim_{x\to 0} \frac{\log|x|}{x^{-n}} = \lim_{x\to 0} \frac{\cancel{1/|x|} \cancel{\operatorname{sgn} x}}{-n x^{\cancel{-n-1}}} \quad \frac{-\infty}{\infty}$$

$$= \lim_{x\to 0} -\frac{1}{n} x^n = 0 \ ✓.$$

Claim Abs conv. on $(-R, R)$.

$n=0$ Not true then we don't have $g(y) \xrightarrow{|y| \to +\infty} 0$

$$\int_0^5 x^n \log x \, dx$$ how show conv?

Claim $\log|x| \overset{?}{\leq} |x|^{-1}$ $\underline{x \to 0}$

Thus: we can conclude to generate a generalized R-L. ~~[illegible]~~ Lemma that we are going to require Absolute integrability $(-R, R)$ $\forall R$.

It turns out that we don't need Absolutly integrable on $(-\infty, -R)$ + (R, ∞) but what is needed

But we can't just drop any end pt conditions $(\pm\infty)$

$\underline{\text{eg}}$: $\quad e^{ix^2} \xrightarrow{F.T} \left(\frac{1+i}{\sqrt{\pi}}\right) e^{-i\pi^2 \xi^2}$

$\not\to 0$ ossilates

$y \to \infty$

$\underline{\text{Pf}}$: Show limit $e^{(i-\epsilon)x^2} \xrightarrow{\epsilon \to 0} e^{ix^2}$

As gen fur. if $F(x)$ good fn

$\underline{\text{Then}}$: $\left| \int_{-\infty}^{\infty} e^{ix^2} F(x) \left(e^{-\epsilon x^2} - 1 \right) dx \right|$

$$\leq \int_{-\infty}^{\infty} |F(x)| \left| e^{-\epsilon x^2} - 1 \right| dx$$

But by MV.T.

$$e^{-\epsilon x^2} - 1 = f(\epsilon x^2) - f(0)$$

w/ $f(x) = \cancel{e^{x^2}}\ e^{-x}$

M.V.T. on $\underline{\epsilon}$

$$e^{-\epsilon x^2} - 1 = f(\epsilon) - f(0) = f'(\xi)(\epsilon - 0)$$

$$0 < \xi < \epsilon$$

w/ $f(\epsilon) = e^{-\epsilon x^2}$

$$f' = -x^2 e^{-\epsilon x^2}$$

$$|f'(\epsilon)| \le |x^2||e^{-\epsilon x^2}|$$

$$\therefore \quad |e^{-\epsilon x^2} - 1| \le |\epsilon| x^2 e^{-\epsilon x^2}$$

$\therefore$

$$\Big| \qquad \Big| \le |\epsilon| \int_{-\infty}^{\infty} |F(x)| x^2 e^{-\epsilon x^2} dx \underset{\epsilon \to 0}{=} O(|\epsilon|)$$

$$\to 0$$

$\cancel{u = \epsilon x^2}$

$u = \cancel{\epsilon} x^2$ $\qquad \int_{-\infty}^{\infty} |F(\frac{u^{1/2}}{\epsilon^{1/2}})| \frac{u}{\epsilon} e^{-u}$

$x^2 = \frac{u}{\epsilon} \Rightarrow x = \sqrt{\frac{u}{\epsilon}}$

$dx = \frac{1}{2} \frac{u^{-1/2}}{\sqrt{\epsilon}} du = \cancel{\frac{1}{2} \epsilon^{-1/2} x} \, du$

$\Rightarrow \cancel{du} = \frac{2\epsilon^{1/2}}{x}$

$$= \cancel{\sqrt{t}\int_{-\infty}^{\infty} F\left(\frac{U^{1/2}}{t^{1/2}}\right) \frac{U}{t} e^{-U} \frac{1}{2\sqrt{t}} U^{-1/2} dU}$$

$$\left| e^{-tx^2} - 1 \right| \le \operatorname{Max} \left| 1 - e^{-\xi} \right|$$

$$\begin{pmatrix} 0 \le \xi \le tx^2 & t>0 \\ -tx^2 \le \xi \le 0 & t<0 \end{pmatrix}$$

$$\int_0^\infty |F(x)| \left| e^{-tx^2} - 1 \right| dx + \int_{-\infty}^0 |F(x)| \left| e^{-tx^2} - 1 \right| dx$$

$$+ \int_{+\infty}^0 |F(-x)| \left| e^{-tx^2} - 1 \right| (-dx)$$

$$+ \int_0^\infty |F(-x)|$$

$\therefore$ LHS

$$\le \int_0^\infty |F(x) + F(-x)| \left| e^{-tx^2} - 1 \right| dx$$

Def 20:

Thm 18:

Ex 29: $g(y) \overset{F.T.}{=}$ of $|x|^{\nu} J_{\nu}(|x|) = f(x)$

$$J_{\nu}(|x|) \xrightarrow{|x|\to 0} \frac{1}{2^{\nu}\nu!}|x|^{\nu} \qquad \nu \geq 0$$

$$J_{\nu}(|x|) \xrightarrow{|x|\to 0} \frac{2^{\nu}}{(-\nu)!}|x|^{-\nu} \qquad 0 \leq \nu \leq \frac{1}{2}$$

Then $|x|^{\nu} J_{\nu}(|x|) \longrightarrow \dfrac{1}{2^{\nu}\nu!}|x|^{2\nu} \qquad \nu \geq 0$

$$|x|^{\nu} J_{\nu}(|x|) \longrightarrow \frac{2^{\nu}}{(-\nu)!}$$

Def 21:

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$e^{ikx}|x|^{\beta}$ $\qquad$ $|x|^{\beta}$ $\qquad$ (R,∞) conv.

$\beta < -1$

$F_1(x)$ w/ $\beta > -1$

Ex 29: $f(x) = |x|^{\nu} J_{\nu}(|x|)$ cont

& At $x=0$ $\sim O(|x|^{2\nu})$

$\therefore$ $f(x)$ Abs integ over every finite int.

if $2\nu > -1 \Rightarrow \nu > -1/2$ ✓

is $|x|^{\nu} J_{\nu}(|x|)$ well behaved at ∞?

$$J_{\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k |x|^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}$$

$$\Rightarrow |x|^{\nu} J_{\nu}(|x|) = \sum_{k=0}^{\infty} \frac{(-1)^k |x|^{2k+2\nu}}{k!\,\Gamma(k+\nu+1)}$$

if

$$|x|^{\nu} J_{\nu}(|x|) = \frac{|x|^{\nu-1/2}}{\sqrt{2\pi}} \sum_{n=0}^{N-1} \frac{(\nu+n-1/2)!}{n!\,(\nu-n-1/2)!\,(2i|x|)^n}$$

$$\cdot \left\{ (-1)^n e^{i(|x|-\frac{\nu}{2}\pi-\frac{\pi}{4})} + e^{-i(|x|-\frac{\nu}{2}\pi-\frac{\pi}{4})} \right\}$$

$$+ O\left(|x|^{\nu-1/2+N}\right)$$

Then if $\cancel{\nu-1/2+N<-1}$ $\cancel{N>\nu+1/2}$

$\cancel{-N<-\nu-1/2}$

$\cancel{\nu-1/2-N}$

$\nu-1/2-N < \nu-1/2-\nu-1/2$

$=-1$

Want $\nu-1/2-N<-1$

or $N>\nu+1/2$

Then $\sum$ part is lin combination

of fns like (9) + $\therefore$

if $N > \nu + 1/2$ we have Absolute integrability up to ∞

$\therefore$ $g(y) \to 0$ As $|y| \to +\infty$

if

$\nu > -1$ $\qquad \lim_{\epsilon \to 0} \{ e^{-\epsilon|x|} f(x) \} = f(x)$

good $F(x)$

$$\left| \int_{-\infty}^{\infty} \left\{ |x|^\nu J_\nu(|x|) - e^{-\epsilon|x|} |x|^\nu J_\nu(|x|) \right\} F(x)\, dx \right|$$

$$= \left| \int_{-\infty}^{\infty} \left(1 - e^{-\epsilon|x|}\right) |x|^\nu J_\nu(|x|) F(x)\, dx \right|$$

But $1 - e^{-\epsilon|x|} \leq$

$$1-e^{-\epsilon|x|} = 1-\sum_{k=0}^{\infty}\frac{(-1)^k\epsilon^k|x|^k}{k!}$$

$$= -\sum_{k=1}^{\infty}\frac{(-1)^k\epsilon^k|x|^k}{k!} = \epsilon|x| - \sum_{k=2}^{\infty}\frac{(-1)^k\epsilon^k|x|}{k!}$$

$$\therefore\quad 1-e^{-\epsilon|x|} = \epsilon|x| + \frac{f^{(2)}(\xi)}{2!} \qquad 0<\xi<|x|$$

$$f(x) = 1-e^{-\epsilon|x|}$$

$$f'(x) = +\epsilon e^{-\epsilon|x|}\,\mathrm{sgn}\,x$$

$$f'' = -\epsilon^2 e^{-\epsilon|x|} - 2\delta(x)(-\epsilon)e^{-\epsilon|x|}$$

$$f''(\xi) = -\epsilon^2 e^{-\epsilon|\xi|} < 0$$

$$\therefore\quad \cancel{1+e^{-\epsilon|x|} \geq \epsilon|x|} \qquad \epsilon|x| \geq 1-e^{-\epsilon|x|}$$

$$\epsilon|x| - \frac{\epsilon^2|x|^2}{2!}$$

$$\therefore\quad \epsilon\int_{-\infty}^{\infty}|x||x|^{\nu}|J_\nu(|x|)||F(x)|\,dx = O(\epsilon).$$

If $\nu > -1$

F.T. of $e^{-\epsilon|x|} f(x)$ is *

$$\frac{2^{\nu}(\nu-\frac{1}{2})!}{\{1+(\epsilon+2\pi i y)^2\}^{\nu+\frac{1}{2}}\sqrt{\pi}} + \frac{2^{\nu}(\nu-\frac{1}{2})!}{\{1+(\epsilon-2\pi i y)^2\}^{\nu+\frac{1}{2}}\sqrt{\pi}}$$

$\therefore$ If $\epsilon \to 0$

$$g(y) \longrightarrow \frac{2^{\nu+1}(\nu-\frac{1}{2})!}{(1-4\pi^2 y^2)^{\nu+\frac{1}{2}}\sqrt{\pi}}$$

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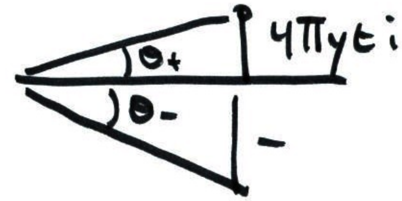
$$(1+(\epsilon \pm 2\pi i y)^2)^{\nu+1/2} = e^{(\nu+1/2)\left[\log|1+(\epsilon\pm 2\pi i y)^2| + i\,\mathrm{Arg}(\cdots\cdots)\right]}$$

if $|y| < \frac{1}{4\pi^2}$ Then As $\epsilon \xrightarrow{\epsilon>0} 0$.

$$1+\epsilon^2 \pm 4\pi i y\epsilon + -4\pi^2 y^2$$



$$\mathrm{Arg}_+(\quad) \to 0$$

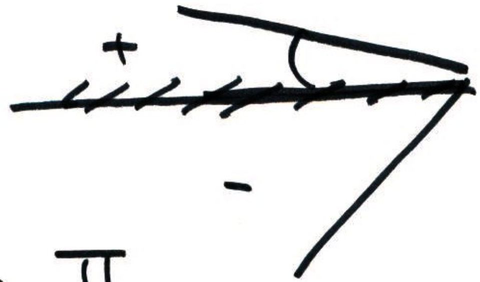
$$\& \ \mathrm{Arg}_-(\quad) \to 0$$

$=$ $\qquad\qquad 4\pi^2 y^2$

if $|y| > \frac{1}{2\pi}$

for $+$ $\quad 1-4\pi^2 y^2 < 0$

$$1-4\pi^2y^2+\epsilon^2 \pm 4\pi y\epsilon i$$



$\operatorname{Arg}(\quad) \to \pi$

$\operatorname{Arg}(\quad) \to -\pi$

$\therefore + \longrightarrow e^{(\nu+1/2)\pi i} e^{\log|\cdots|}$

$- \longrightarrow e^{-(\nu+1/2)\pi i}.$

$$\sum = i e^{\nu\pi i} + \underbrace{e^{-\frac{\pi i}{2}}}_{-i} e^{-\nu\pi i}$$

$$= i(e^{\nu\pi i} - e^{-\nu\pi i})$$

$$= 2i^2 \sin\nu\pi = \underline{-2\sin\nu\pi.}$$

$$f(x) - F_1(x) = O(|x|^{2\nu+2})$$

$$\Big\downarrow N \text{ der.}$$

$$O(|x|^{2\nu+2-N}) \qquad \text{Abs integ.}$$

over $x_0 = 0$ iff $2\nu+2-N \geq$

Abs integ over $(-\infty, \infty)$
iff
$2\nu + 2 - N > -1$
$2\nu + 3 > N$.

??

Then $g(y) = G_1(y) + O(|y|^{-N})$

$G_1(y)$ = F.T. of $\dfrac{|x|^{2\nu}}{\nu!\,2^\nu}$

$$= \frac{1}{\nu!\,2^\nu}\left[\frac{2\cos\frac{\pi}{2}(2\nu+1)\,(2\nu)!}{(2\pi|y|)^{2\nu+1}}\right] + o(|y|^{-N})$$

Also:
$$2\nu! = \nu!\,(\nu-\tfrac{1}{2})!\,2^{2\nu}\,\pi^{-1/2}$$

$$\therefore \Rightarrow \quad \frac{2}{\cancel{\nu!}\,\cancel{2^{\nu}}}\,\cancel{\nu!}\,(\nu-\tfrac{1}{2})!\,2^{\cancel{2\nu}\,\nu}\,\pi^{-1/2}$$

$$\cdot\ \underbrace{\cos\left(\pi\nu+\frac{\pi}{2}\right)}_{-\sin\pi\nu}$$

$$\Rightarrow \quad -\frac{2^{\nu+1}\,(\nu-\frac{1}{2})!\,\sin\pi\nu}{(2\pi|y|)^{2\nu+1}\sqrt{\pi}} \qquad \checkmark$$

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$$f(x) = \frac{|x|\,|x+1|^{1/2}}{|x-1|^{1/2}} \qquad x = 0, -1, +1$$

$x = 0$

$$f(x) = |x| \left[\frac{|x+1|}{|x-1|} \right]^{1/2}$$

$$= |x| \left[1 + \left(\frac{\operatorname{sgn}(x+1) \cdot 1}{|x-1|} - \frac{\operatorname{sgn}(x-1)\,|x+1|}{|x-1|^2} \right)\Bigg|_{x=} |x|^2 \right]$$

$$= |x| + O(|x|^2)$$

$x = -1$

$$f(x) = |x+1|^{1/2} \left[\frac{1}{2^{1/2}} + O(|x+1|) \right]$$

$$= \frac{|x+1|^{1/2}}{\sqrt{2}} + O(|x+1|^{3/2})$$

$$|x| \longrightarrow \frac{2\cos\pi}{2\pi^2|y|^2} = \cancel{2} \quad \frac{-2}{4\pi^2|y|^2}$$

$$= \frac{-2}{4\pi^2 y^2}$$

$$\frac{|x+1|^{1/2}}{\sqrt{2}} \longrightarrow \frac{1}{\sqrt{2}}\, 2\cos$$

$$|x| = x \operatorname{sgn} x$$

$$\frac{\sqrt{2}}{|x-1|^{1/2}} \longrightarrow$$

$$\frac{5}{2\sqrt{2}}|x-1|^{1/2}\operatorname{sgn}(x-1) \longrightarrow$$

Ex 32:

$X = 0$

$f(x) = |x| \Big[$

$X = -1$

$$f(x) = |x+1|^{1/2} \left[\frac{|x|}{|x-1|^{1/2}} \right]$$

$$= |x+1|^{1/2} \left[\frac{1}{2^{1/2}} + \left(\frac{\operatorname{sgn}(x) \cdot 1}{|x-1|^{1/2}} - \frac{1}{2} \frac{|x|}{|x-1|^{3/2}} \operatorname{sgn}(x-1) \right) \Bigg|_{x=-1} (x+1) + O((x-1)^2) \right]$$

$$= |x+1|^{1/2} \left[\frac{1}{\sqrt{2}} + \left(\frac{-1}{\sqrt{2}} - \frac{1}{2} \frac{(1)}{2^{3/2}} \right) (x+1)^{2} + O((x-1)^2) \right]$$

$$= |x+1|^{1/2} \left[\frac{1}{\sqrt{2}} + O(|x+1|) \right]$$

$x = 1$

$$f(x) = \frac{1}{|x-1|^{1/2}}\left[|x| \, |x+1|^{1/2} \right]$$

$$= \frac{1}{|x-1|^{1/2}}\left[\sqrt{2} + \left[\operatorname{sgn} x \, |x+1|^{1/2} + |x| \tfrac{1}{2} |x+1|^{-1/2} \operatorname{sgn}(x+1) \right]_{x=1} (x-1) + O((x-1)^2) \right]$$

$$= \frac{1}{|x-1|^{1/2}}\left[\sqrt{2} + \left(\sqrt{2} + \frac{1}{2\sqrt{2}} \right)(x-1) + O(\;)^2 \right]$$

$$= \frac{\sqrt{2}}{|x-1|^{1/2}} + \frac{5}{2\sqrt{2}} \frac{(x-1)}{|x-1|^{1/2}} + O((x-1)^{3/2})$$

$$\|$$

$$+ \frac{5}{2\sqrt{2}} \frac{|x-1| \operatorname{sgn}(x-1)}{|x-1|^{1/2}} + O((\;)^{3/2})$$

$$+ \frac{5}{2\sqrt{2}} |x-1|^{1/2} \operatorname{sgn}(x-1)$$

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$$g(y) = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x y}\, dx$$

$$\frac{1}{2i} g(y) =$$

$$g(y) = \int_{-\infty}^{\infty} F(|x|) \operatorname{sgn} x \, e^{-2\pi i x y}\, dx = \int_{-\infty}^{0} F(-x)(-1) e^{-2\pi i x y}\, dx$$

$$+ \int_{0}^{+\infty} F(x)(1) e^{-2\pi i x y}\, dx$$

$$= -\int_{0}^{\infty} F(x) e^{2\pi i x y}\, dx + \int_{0}^{\infty} F(x) e^{-2\pi i x y}\, dx$$

$$= \int_{0}^{\infty} F(x)\left(- 2i \sin(2\pi i x y)\right) dx$$

$$= -2i \int_{0}^{\infty} F(x) \sin(2\pi i x y)\, dx$$

$$\frac{1}{1-x^4} = \frac{1}{(1-x^2)(1+x^2)}$$

$$\| $$

$$\frac{1}{(1-x)(1+x)(\quad)}$$

But $f(x) = \cdots \quad H(x)H(1-x)$

is zero at -1

Thus only 2 sing are $0, 1$

At 1

$$\frac{H(1-x)}{(1-x)^{1/2}} \left\{ \underbrace{\frac{\cosh x \, H(x)}{(1+x)^{1/2}(1+x^2)^{1/2}}}_{\text{taylor around } x=1} \right\} = \left(\quad\right)\left\{ \frac{\cosh x \, H(x)}{((1+x)(1+x^2))^{1/2}} \right\}$$

Ex 11: $x=0$ sing to $O(|y|^{-3})$

$$\int_{-\infty}^{\infty} e^{-|x|} e^{-2\pi i x y}\, dx =$$

Expand $e^{-|x|} = \sum_{k=0}^{\infty} \frac{(-1)^k |x|^k}{k!}$

Then Must have 3 deriv. Also integ over D.

$$e^{-|x|} = 1 - |x| + \frac{|x|^2}{2!} - \frac{|x|^3}{3!} + O(|x|^4) \quad x \to 0.$$

if ~~\#\#\#~~ I take $F_1(x) = 1 - |x| + \frac{|x|^2}{2!}$ $\underline{2,1,0}$

Then $e^{-|x|} - F_1(x) = O(|x|^3)$ $x \to 0$

& 3 derivatives of $O(|x|^3) \longrightarrow O(1)$

& $g(y) = G_1(y) + O(|y|^{-3})$

w/

$$G(y) = \delta(y) - 2(1!)(2\pi i y)^{-2} + \frac{(-2\pi i)^{-2}}{2!}\delta^{(2)}(y)$$

$$\therefore \quad g(y) = \delta(y) + \frac{2}{4\pi^2 y^2} + \frac{-1}{2(4\pi^2)}\delta^{(2)}(y) + O(|y|^{-3})$$

$$= \delta(y) + \frac{1}{2\pi^2 y^2} - \frac{\delta^{(2)}(y)}{8\pi^2} + O(|y|^{-3})$$

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Ex 12:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x y}}{1+|x|^3}\,dx$$

F.T. of $\frac{1}{1+|x|^3}$ sing $x=0$ only.

$$\frac{1}{1+|x|^3} = \sum_{k=0}^{\infty} |x|^{3k} = 1 + |x|^3 + |x|^6 + \cdots$$

Then $f(x) = \frac{1}{1+|x|^3} = \sum_{k=0}^{N} |x|^{3k} + \underbrace{O(|x|^{3N+3})}_{\phi}$

has $3N+3$ Abs integ. derivatives

$\therefore$

$$g(y) = \sum_{k=0}^{N} g_k(y) + o(|y|^{-3N+3})$$

$$= \sum_{k=0,2,4,\dots}^{\text{"}N\text{" "}N-1\text{"}} + \sum_{k=1,3,5}^{\text{"}N\text{", "}N-1\text{"}}$$

depending on if N is odd or even.

<u>Now:</u> $|x|^{3k} = x^{3k}\,\mathrm{sgn}\,x$

if $k=$ even $\quad |x|^{3k} = x^{3k}$ (a)

if $k=$ odd $\quad |x|^{3k} = x^{3k}\,\mathrm{sgn}\,x$ (b)

$$(a) \xrightarrow{F.T.} (-2\pi i)^{-3k}\,\delta^{(3k)}(y)$$

$$(b) \xrightarrow{F.T.} 2(3k)!\,(2\pi i y)^{-3k-1}$$

$$\therefore \quad g(y) = \sum_{k=0,2,4}^{\text{"}N,N-1\text{"}} (-2\pi i)^{-3k}\,\delta^{(3k)}(y)$$

$$+ \sum_{k=1,3,5,\ldots}^{\text{"}N,N-1\text{"}} 2(3k)!\,(2\pi i y)^{-3k-1} + o\left(|y|^{-3N-3}\right)$$

$$= \sum_{k=0}^{\text{"}\frac{N}{2},\frac{N-1}{2}\text{"}} (-2\pi i)^{-6k}\,\delta^{(6k)}(y) + \sum_{k=0}^{\text{"}\frac{N-1}{2},\frac{N-2}{2}\text{"}} 2(3(2k+1))!\,(2\pi i y)^{-3(2k+1)-1}$$

$$+ o\left(|y|^{-3N-3}\right).$$

Prob 4: look at

$\frac{1}{1+|x|^{3/2}}$ sing at $x=0$ only.

taylor expand $\Rightarrow \frac{1}{1+|x|^{3/2}} = \sum_{k=0}^{\infty} (-1)^k |x|^{3k/2}$

Must have 3 derivatives of the remainder

Absolutely integrable $\quad \frac{3k}{2} - N > 0 \quad w/ \; N = 4$

$$\Rightarrow k > \frac{2N}{3} = \frac{8}{3} \cong 2.66$$

take $k=3$.

$\therefore$ use

$$\frac{1}{1+|x|^{3/2}} = 1 - |x|^{3/2} + |x|^3 + O(|x|^{9/2})$$

Check again $\quad \frac{9}{2} - 4 = \frac{9}{2} - \frac{8}{2} = \frac{1}{2}$ Abs integ over 0 ✓ yes

Then

$$g(y) = \text{F.T. of } \frac{1}{1+|x|^{3/2}}$$

$$= G_1(y) + O(|y|^{-4})$$

$G_1(y) = $ F.T. of $1 \cdot -|x|^{3/2} + |x|^3$

$$1 \xrightarrow{\text{F.T. from table}} (-2\pi i)^{-0}\delta(y) = \delta(y)$$

$$-|x|^{3/2} \longrightarrow -2\cos\left(\tfrac{\pi}{2}(3/2+1)\right)(3/2)!\,(2\pi|y|)^{-3/2-1}$$

$$|x|^3 = (x \operatorname{sgn} x)^3 = x^3 \operatorname{sgn} x$$

$$= 2(3!)(2\pi i y)^{-4}$$

$$\therefore\ G_1(y) = \delta(y) - \frac{2\Gamma(5/2)\cos\left(\frac{5\pi}{4}\right)}{(2\pi|y|)^{5/2}}$$

$$+ \frac{\cancel{12}^{\,3}}{4\cancel{16}\pi^4 y^4}$$

$$\therefore\ g(y) = \delta(y) - \frac{2\Gamma(5/2)\cos\left(\frac{5\pi}{4}\right)}{(2\pi|y|)^{5/2}} + \frac{3}{4\pi^4 y^4} + O(|y|^{-4})$$

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Ex 14:

$$\int_0^\infty (1-x)^{-1} \cos 2\pi xy \, dx$$

$$\text{"} \int_{-\infty}^{\infty} \frac{H(x)}{1-x} e^{-2\pi i xy} \, dx \text{"} = \text{F.T.}$$

F.T. of $\quad \dfrac{H(x)}{1-|x|} e^{-2\pi i xy}$